## Solution to Homework 1

**Learning Objectives Assessed:**

- Identify potential applications of optimization in daily life and in future career.
- Identify the decision variables, constraints, and objective of an optimization problem, and articulate the tradeoffs.
- Read mathematical notations for optimization.
- Identifying local and global optimum.

## Question 1.

Alice is interviewing for an internship at Optimized Financial Services, which applies analytics to help individuals and firms make financial decisions. Her interviewer is on a team that is developing a software to help tax payers decide when to invest in a retirement account, how much to invest in, and what types of account to invest in. The types of accounts include IRA, Roth IRA, 401(k), and Roth 401(k). Each account has contribution limits set by the Internal Revenue Services (IRS) and has various types of tax benefits. (If you are unfamiliar with these types of accounts, see **this article** for a comparison. See also **this Wikipedia table** for a detailed comparison of the four types, taking into account the **recent changes under the GOP tax reform**) Help Alice to answer the following questions from the interviewer?

**a)** How can optimization be used to help a client make retirement account investment decisions? Describe a possible choice of:

- Decision variables.
- Objective.
- Constraints. (There are many constraints due to IRS rules, so limit your response to at most 5 constraints.)

You do not have to use mathematical notation, but can use English to describe each of the above. But try to be as precise in your language as you can.

**b)** Briefly describe one tradeoff present in the above optimization problem.

**c)** Is this an appropriate context to apply optimization? Why or why not?

### Sample Solution

This is an open ended modeling question with many possible valid solutions. The goal is to provide an opportunity for you to practice applying the conceptual framework of optimization to a realistic setting. Optimize Financial Services (OFS) is a real company, and they currently do use sophisticated optimization models to advise clients on IRA and 401(k) investment decisions.

If you are interested in what they actually do, you can read a paper they wrote on the exact problem posted here: https://pubsonline.informs.org/doi/pdf/10.1287/inte.2016.0849 Their solution approach is based on linear programming, which we will cover in the second half of the course. To handle uncertainty in life expectancy and investment returns, they solve the optimization under multiple scenarios to study the robustness of their solution under various possibilities. (This is the same principal as that of sensitivity analysis in decision trees.) They also customize the objectives and constraints based on each client's desired spending levels (to maintain a certain standard of living), as well as income, employment situation and age.

**a)** In the actual OFS implementation, their choice of **decision variables** is the amount in each year:

- to contribute to each of the possible types of accounts.
- to spend for consumption
- to convert from tax-deferred (i.e. IRA) to tax-free (i.e. Roth) accounts.
- to withdraw from each account (this is called a distribution)

Note that they have a decision variable for each of the above in each future year, as they are not only deciding what to do now, but deciding on a future plan of investment, based on expected income and returns.

Possible **objectives** include:

- minimize the shortfall from the client's desired spending level per year.
- maximize the net present value (NPV) of the final estate value of the client (amount the client can pass on to their descendents).
- maximize the value of the account at retirement age.

In fact, as described in the paper above, OFS first minimizes the spending shortfall to prioritize maintaining standard of living, then among solutions that achieve the minimal shortfall, they find one that maximizes the NPV of the estate value. In other words, they solve two optimization models in sequence: the first minimizes shortfall; the second maximizes estate value subject to an additional constraint of achieving the minimal shortfall.

The **constraints** include:

- Respecting IRS contribution, distribution, and conversion limits on various types of accounts each year. (There are many specific constraints in this category, as seen in the articles in the question text.)
- Not spending more each year than one's income minus taxes and social security contributions, plus distributions (withdrawal from investment accounts), minus contributions to the accounts.
- One year's expected account value is equal to the previous year's expected value plus expected investment returns and planned contributions minus the planned distributions.
- Actual withdrawal amount is equal to planned distribution amount minus expected taxes and penalties.
- Actual estate value that can be passed on to descendants is determined by expected left over asset value minus tax penalties.

Again, there are many valid ways of applying optimization to this problem, so your answer does not have to be the same as the above.

**b)** There are many tradeoffs in the optimization problem.

- Consuming more now or saving more for retirement.
- Higher quality of life versus saving more to pass on to descendants. (This is related to the above but different in that the beneficiary above is self, while the beneficiary here is descendants.)
- Paying taxes now (via a Roth account) versus paying taxes later (via a normal account).
- The flexibility of an IRA account versus the obtaining possible employer match in a 401(k) account.

**c)** This is an appropriate context to apply optimization because:

- The decision variables and objectives and constraints can be clearly defined and quantified.
- The current practice (using simple rules of thumbs) can be greatly improved. (When the current solution is good, then there's less of a need to optimize, but when the current solution is bad, there's more room to optimize.)

- Tax rules are relatively stable over time. (There was a tax reform this year but this kind of things happen very rarely and the effect of the change on retirement investment may not be large; one can also reoptimize when there is a change coming.)

Some have argued that this may not be appropriate because of uncertainty in future parameters, such as income and investment returns. While uncertainties do make optimization less reliable, it is possible to deal with these via considering a range of parameters and finding solutions that is robust to multiple possibilities (i.e. sensitivity analysis), which is what OFS currently does. **It is not uncertainty that invalidates optimization, but vagueness.** The monetary decision variables and objectives here can be clearly defined.

Others have argued that it is not appropriate because each client's need is different. However, this is also not a barrier because optimization can be customized to each particular person, which is also what OFS does.

## Question 2.

Proctor & Gamble (P&G) has a multi-billion dollar fabric-care business that oversees a global portfolio of products, including brands such as Tide, Dash, and Gain. Components of the business include:

- Sourcing raw material from vendors;
- Developing the right mix of raw materials to formulate its products;
- Choosing where and how to manufacture its products;
- Marketing its products;
- Deciding on pricing and production quantities.

Pick three of the above areas and formulate an optimization problem related with each area, stating the decision variables, objective, and constraints, and describe one tradeoff in the optimization. You can state everything in English, without using mathematical notation, but try to be precise in your language.

### Sample Solution

Again, this is a modeling question so there are many possible valid solutions.

#### Sourcing

**Decision variables:**

- For each material, which vendors to use.
- How much to source from each.

**Objective:**

- Minimize total cost (material + shipping + other costs).

**Constraints:**

- Obtaining the required amount of material within the required time frame.
- Maintain a certain level of quality and reliability for each material (can be estimated based on vendor historic data).
- Satisfying corporate social responsibility committments of the company.

*Tradeoffs:*

- Cost versus quality/reliability.
- Cost versus speed (oversea vendors may be cheaper but local vendors can ship faster).
- Cost versus environmental ratings of the vendor.

### Development

**Decision variable:**

- What proportion of each material to in the formula for a certain product.

**Objective:**

- Maximize quality of the product.

**Constraints:**

- Staying within a budget constraint in terms of average cost.
- Respecting regulatory rules regarding the safe use of certain material.

*Tradeoffs:*

- Cost versus quality.
- Cost versus regulatory compliance.
- Better quality in one aspect versus another.

### Manufacturing

**Decision variable:**

- Which countries/cities to build plants.
- What manufacturing technology to use.

**Objective:**

- Minimize long-term average cost.

**Constraints:**

- Short-term budget limits on investment of new equipments.
- A set of feasible manufacturing technology that provides the necessary quality and reliability.
- Regulatory rules on taxes and labor in various locations.

*Tradeoffs:*

- Cost versus quality/reliability.
- Higher current cash flow versus investing in better technology.
- Plants in developed countries with high quality versus plants in developing countries with lower costs.
- Labor intensive versus capital intensive technologies.

### Marketing

**Decision variables:**

- How much to spend on advertising in each channel (TV, Internet, Coupons, etc).
- What discounts and promotional strategies to apply.

**Objective:**

- Maximize expected long-term profit.

**Constraints:**

- Short-term cash flow constraints.
- Long-term advertising budget.

*Tradeoffs:*

- Short term profit loss versus long term increased market share.
- Relative benefits of spending money on attracting one segment of customers versus attracting another.
- Saving more money to invest on product quality versus spending that money on promotions.

### Strategy:

**Decision variables:**

- Price of each product.
- Quantity to pre-order for each product.

**Objective:**

- Maximize long-term profit

**Constraints:**

- Capacity constraints at vendors and manufaturing plants.

*Tradeoffs:*

- Higher market share from lower prices versus higher profit margins from higher prices.
- Risking stock out from lower production quantities versus risking over-supply from higher quantities.
- Focusing resources producing products for one customer segment versus another.

## Question 3.

Consider optimizing your decision of what you would do after graduation. What would be your decision variables, objectives, domain/constraints? What are some of the trade-offs?

*This is a very personal question and there's no right/wrong answers. I hope this course helps you to better reflect on your decision and choose a good path for yourself.*

## Question 4.

Let $f(x)$, $g(x)$, and $h(x)$ be three real-valued functions (meaning that they map $\mathbb{R} \to \mathbb{R}$.) Consider the following optimization problem.

$$\max_{x \in \mathbb{R}} f(x)$$

$$\text{Subject to the constraint: } g(x) \geq 0$$

Suppose that an optimal solution is $x = 5$, with optimal objective value $f(5) = 13$. Suppose also that $h(5) = 0$.

Now, if we add the additional constraint

$$h(x) = 0$$

to the above optimization problem, would the optimal objective value still necessarily be 13? Why or why not?

### Solution

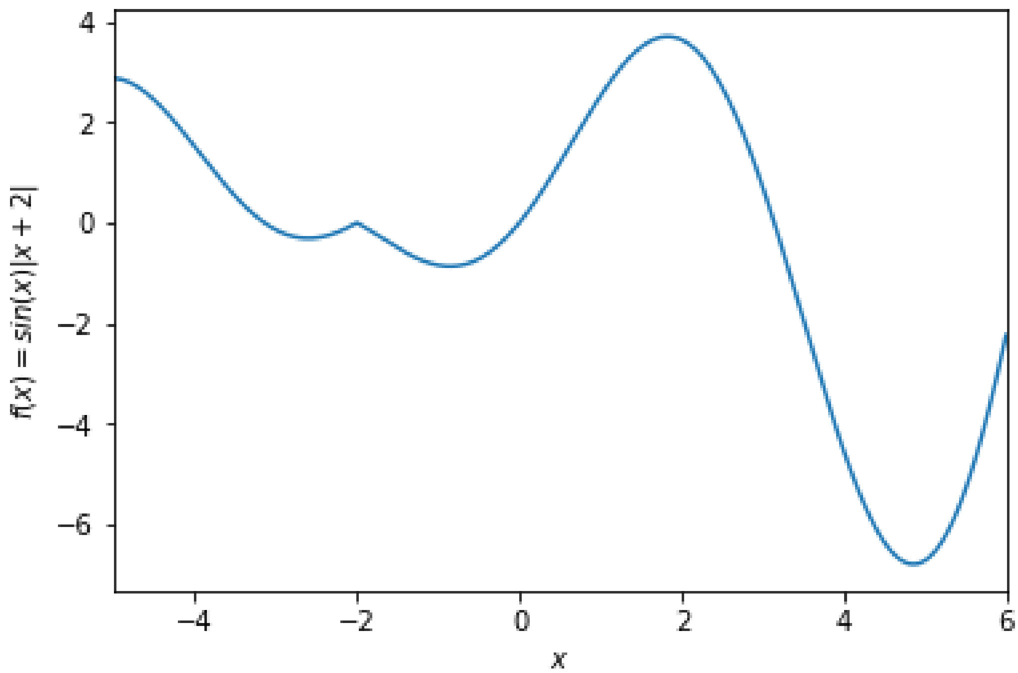
The optimal objective is necessarily 13, because:

- When we add a constraint, it only makes the optimization harder, so the optimal objective of the new problem can be no more than 13.
- An objective value of 13 is achievable using $x = 5$, because this satisfies both the constraints $g(x) \geq 0$ and $h(x) = 0$. It satisfies the first because it is a feasible solution to the first optimization problem. It satisfies the second because $h(5) = 0$ by assumption.

## Question 5.

Consider the following function $f(x)$ under the domain $x \in [-5, 6]$. For your convenience, the graph of the function is given below (ignore the Python code for now). Label on the graph all of the local and global maxima, as well as all of the local and global minima. (You can either label them on a printout of the graph directly or sketch the graph yourself and label it there.)

```
[1]: import numpy as np
     import matplotlib.pyplot as plt
     x=np.arange(-5,6.01,0.01)
     def f(x):
         return np.sin(x)*np.abs(x+2)
     plt.plot(x,f(x))
     plt.xlabel('$x$')
     plt.ylabel('$f(x)=sin(x)|x+2|$')
     plt.xlim((-5,6))
     plt.show()
```




### Solution

There are three local minima and four local maxima:

- The three local minima occur at the three "valleys" in the above graph. One is when $x$ is between $-4$ and $-2$, another when $x$ is between $-2$ and $0$, and the last is when $x$ is between 4 and 6.
- One local maximum occurs at the cusp at $x = -2$, and another is at the peak around $x = 2$. **There are two other local maxima, which many have missed, and they are at the boundary of the domain** at $x = -5$ and $x = 6$.

The global minimum occurs at the valey when $x$ is between 4 and 6, and the global maximum occur at the peak around $x = 2$.