Solution to Additional Practice Problems (from 22-Optimization Modeling II)

The solutions to exercise 4 and 9 appeared in the course notes to 24-Modeling with Auxiliary Decision Variables.

Exercise 5

The following table provides the shipping cost for one-pound packages, from 7 of Amazon's fulfillment centers (FC) to 4 regions.

Region	1	2	3	4	5	6	7
A. Kings County, NY	20.25	7.70	24.59	23.26	7.69	7.70	7.69
B. Los Angeles County, CA	18.43	23.30	7.69	7.69	24.16	22.12	24.91
C. King County, WA	21.28	24.18	7.70	17.67	23.91	22.98	24.57
D. Harris County, TX	7.69	7.70	18.73	7.71	18.79	7.70	19.47

A shipping cost of \$10 or less indicates that the package will be transported via ground shipping; otherwise, it will be transported via air shipping.

For a certain item that weights a pound, Amazon would like to stock it in as few FCs as possible, while guaranteeing that it can fulfill demand in all 4 regions via ground shipping. Moreover,

- the item must be stocked in at least one of FCs 5 or 7;
- the item cannot be stocked in FC 4 unless it is also stocked in FC 1;
- if the item is stocked in FC 2, then it cannot also be stocked in FC 3.

Formulate an optimization problem to find the minimum number of FCs needed.

Solution

This problem is like Exercise 1 from Session 21.

The decision is to select which FCs to use to stock the item. The objective is to minimize the number of FCs selected. A key constraint is to make sure that for each region, there is at least one FC selected that can ship to the region in under \$10.

Decision Variables: Let $X_1, \cdots X_7$ denote whether to use each FC. (binary)

$$\begin{aligned}
\text{Min.} \quad & X_1 + X_2 + \cdots + X_7 \\
\text{s.t.} \quad & X_2 + X_5 + X_6 + X_7 \geq 1 \\
& X_3 + X_4 \geq 1 \\
& X_3 \geq 1 \\
& X_1 + X_2 + X_4 + X_6 \geq 1 \\
& X_5 + X_7 \geq 1 \\
& X_4 \leq X_1 \\
& X_2 + X_3 \leq 1
\end{aligned}$$

The abstract formulation of this problem is also like exercise 1 (see course notes to Session 21), except for the there special constraints, which you would not need to generalize.

Exercise 6

Your software company has launched a new Analytics product. As sales manager, you are planning to promote the product by sending salesforce to software conventions running concurrently in Los Angeles, Saint Louis, and Detroit.

You have 6 representatives available at each of your Little Rock and Urbana Branches. You would like to send at least 2 to the Los Angeles convention, 5 to the Saint Louis convention, and at least 4 to the Detroit convention.

Roundtrip airfare between the locations are as follows:

	Los Angeles	St. Louis	Detroit
Little Rock	250	150	200
Urbana	300	200	150

Formulate an optimization problem to allocate your sales force so as to minimize total airfare.

Solution

The decision is how many representative to send from each branch to each convention. The objective is to minimize total airfare. The constraints are to not use more representative than available at each site, and to send enough representives to each convention.

DV: Let X_{ij} denote how many representatives to send from branch i to convention j. (Integer)

Denote Little Rock and Urbana by 1 and 2 respectively, and the three cities by L, S, and D respectively.

$$
\begin{aligned}
\text{Min} \quad & 250X_{1L} + 150X_{1S} + 200X_{1D} + 300X_{2L} + 200X_{2S} + 150X_{2D} \\
\text{s.t.} \quad & \\
& X_{1L} + X_{1S} + X_{1D} \leq 6 \\
& X_{2L} + X_{2S} + X_{2D} \leq 6 \\
& X_{1L} + X_{2L} \geq 2 \\
& X_{1S} + X_{2S} \geq 5 \\
& X_{1D} + X_{2D} \geq 4 \\
& X_{ij} \geq 0 \qquad \text{for all } i \text{ and } j.
\end{aligned}
$$

If you did not specify the X_{ij}'s to be integers when defining, then you need to write

$$X_{ij} \in \mathbb{Z} \qquad \text{for all } i \text{ and } j.$$

The abstract formulation of this problem is the same as the transportation problem defined in 17-LP Modeling in the section in Index Notation, except that the variables would need to be integers.

Exercise 7

SALS Marketing Inc. is developing an advertising campaign for a large consumer goods corporation. SALS has promised a plan that will yield the highest possible "exposure rating," a measure of the ability to reach the appropriate demographic group and generate demand. The options for advertisements with their respective costs (per unit of advertising) and per-unit exposure ratings are given in the table below.

Category	Subcategory	Cost/Unit	Exposure/Unit
Magazines	Literary	$7,500	15000
	News	$10,000	22500
	Topical	$15,000	24000
Newspapers	Major Evening	$3,000	75000
	Major Morning	$2,000	37500
Television	Morning	$20,000	275000
	Midday	$10,000	180000
	Evening	$60,000	810000
Radio	Morning	$15,000	180000
	Midday	$15,000	17000
	Evening	$10,000	16000

Of course, certain restrictions exist for the advertising campaign. The client corporation has budgeted $800,000 for the campaign, but to restrict overexposure to any particular audience it wants no more than $300,000 put into any one category (Magazine, Newspaper, etc.). Also, to ensure a broad range of exposure, at least $100,000 must be spent in each category.

Formulate this problem as a LP or MIP.

Solution 1

The decision is how many units of each kind of add to include in the plan. The objective is to maximize the total exposure rating. The constraints are not to spend more than $800 K in total and to spend between $100 K and $300 K in each category.

DV:

- M_L, M_N, M_T: how many units of literary, news, and topical magazines to include. (Integer)
- N_E, N_M: analogous decision variables for newspapers. (Integer)
- T_M, T_D, T_E: analogous decision variables for television. (Integer)
- R_M, R_D, R_E: analogous decision variables for radio. (Integer)

(Since the problem was unclear about whether fractional units are allowed, it is also correct if you specify the above as continuous.)

For convenience, we measure money in thousands of dollars.

$$\begin{aligned}
\text{Max} \quad & 15M_L + 22.5M_N + 24M_T + 75N_E + 37.5N_M + \\
& 275T_M + 180T_D + 810T_E + 180R_M + 17R_D + 16R_E \\
\text{s.t.} \quad & \\
& 100 \le 7.5M_L + 10M_N + 15M_T \le 300 \\
& 100 \le 3N_E + 2N_M \le 300 \\
& 100 \le 20T_M + 10T_D + 60T_E \le 300 \\
& 100 \le 15R_M + 15R_D + 10R_E \le 300 \\
& 7.5M_L + 10M_N + 15M_T + 3N_E + 2N_M + 20T_M + \\
& 10T_D + 60T_E + 15R_M + 15R_D + 10R_E \le 800 \\
& M_L, M_N, M_T, \cdots, R_E \ge 0
\end{aligned}$$

Solution 2:

The concrete formulation above is quite tedious, with many variable names. The following abstract formulation simplifies the notation.

Data:

- I: the set of all ad options.
- T: the set of categories
- I_t: the set of options of category $t \in T$.
- c_i: the unit cost of option $i \in I$.
- e_i: the unit exposure rating of option $i \in I$.

DV: Let x_i be the number of units of ad option $i \in I$ to include in the plan. (Integer)

$$
\begin{aligned}
\text{Max} \quad & \sum_{i \in I} e_i x_i \\
\text{s.t.} \quad & \\
& 100 \leq \sum_{i \in I_t} c_i x_i \leq 300 && \text{for each category } t \in T. \\
& \sum_{i \in I} c_i x_i \leq 800 \\
& x_i \geq 0 && \text{for all } i.
\end{aligned}
$$

Exercise 8

Hospital administrators must schedule nurses so that the hospital's patients are provided with adequate care. At the same time, in the face of tighter competition in the health care industry, they must pay careful attention to keeping costs down.

From historical records, administrators estimated the minimum number of nurses to have on hand for the various times of the day and days of the week, as shown in the following table.

Shift	Time	Minimum number of nurses needed
1	Midnight-4am	5
2	4am-8am	12
3	8am-noon	14
4	noon-4pm	8
5	4pm-8pm	14
6	8pm-Midnight	10

Nurses work 8 hours a day in two consecutive shifts. As a result, in each shift, there are two types of nurses: those that started in the previous shift (and are now working their second shift), and those that just started in this shift (and will be working in the next shift as well).

Formulate an optimization problem that determines the minimum total number of nurses required to fulfill the schedule above.

Solution

The decision is how many nurses to *start* at each shift. The objective is to minimize the total number of nurses. The constraint is that for each shift, the number of nurses who either start in that shift or started in the previous shift must be at least the required minimum.

DV: Let $X_1, X_2, \cdots, X_6$ denote the number of nurses that start at the beginning of each shift. (Integer)

$$
\begin{aligned}
\text{Min.} \quad & X_1 + X_2, \cdots + X_6 \\
\text{s.t.} \quad & \\
& X_6 + X_1 \geq 5 \\
& X_1 + X_2 \geq 12 \\
& X_2 + X_3 \geq 14 \\
& X_3 + X_4 \geq 8 \\
& X_4 + X_5 \geq 14 \\
& X_5 + X_6 \geq 10 \\
& X_i \geq 0 \qquad \text{for all } i.
\end{aligned}
$$

Solution 2

If there were more shifts than 6, the above concrete formulation can be generalized to the following abstract formulation.

Data:

- n: the number of shifts.
- q_i the number of nurses needed in shift i.

DV: Let X_i denote the number of nurses that start at the beginning of shift i. (Integer)

$$
\begin{aligned}
\text{Min.} \quad & \sum_{i=1}^{n} X_i \\
\text{s.t.} \quad & \\
& X_n + X_1 \geq q_1 \\
& X_{i-1} + X_i \geq q_i \quad \text{for every } i \in \{2, 3, \cdots, n\}. \\
& X_i \geq 0 \quad \text{for all } i.
\end{aligned}
$$