Session 19: Abstract Formulation I

Example (Concrete formulation from last session's Q1)

Amazon.com is expanding its business by launching a physical store in West LA. As the manager, you need to select which bestsellers to carry at the store's grand opening. The following table provides the list of Top 10 Bestsellers in Literature & Fiction, along with their genres. Note that some bestsellers belong to more than one genre.

Rank \ Genre	Literary	Sci-Fi	Romance	Thriller
1	√			
2		√		√
3			√	√
4	√		√	
5	√			
6			√	
7		√		
8				√
9	√	√		
10			√	

You wish to carry the minimum number of bestsellers, while ensuring that there are at least two bestsellers in each genre. Formulate this as an optimization problem.

Decision variables: Let x_i denote whether to carry book i, where $i \in \{1, 2, \cdots, 10\}$. (Binary)

Objective and Constraints:

$$\begin{aligned} \text{Minimize:} \quad & x_1 + x_2 + \cdots + x_{10} \\ \text{subject to:} \quad & \\ \text{(Literary)} \quad & x_1 + x_4 + x_5 + x_9 \geq 2 \\ \text{(Sci-Fi)} \quad & x_2 + x_7 + x_9 \geq 2 \\ \text{(Romance)} \quad & x_3 + x_4 + x_7 + x_{10} \geq 2 \\ \text{(Thriller)} \quad & x_2 + x_3 + x_8 \geq 2 \end{aligned}$$

Step 3a. Use variables to encode all input data

Data:

- I: the set of books.
- J: the set of genres.
- a_{ij}: a binary data variable (not decision variable) denoting whether book i is of genre j. (These corresponds to the checkmarks in the original question.)
- q_j: how many books do we need of genre j.

Step 3b. Formulate the LP/MIP in terms of only data and decision variables

Decision Variables: For each book $i \in I$, let x_i denote whether to carry the book. (Binary)
Objective and constraints:

$$\begin{aligned} \text{Minimize:} \quad & \sum_{i \in I} x_i \\ \text{subject to:} \quad & \\ \text{(Enough books in genre)} \quad & \sum_{i \in I} a_{ij} x_i \geq q_j \quad \text{for each genre } j \in J. \end{aligned}$$

Alternative encoding of data

Data:

- I: the set of books.
- J: the set of genres.
- I_j: the set of books of genre j.
- q_j: how many books we need of genre j.

Decision Variables: For each book $i \in I$, let x_i be a decision variable denoting whether to carry the book. (Binary)
Objective and constraints:

$$\begin{aligned} \text{Minimize:} \quad & \sum_{i \in I} x_i \\ \text{subject to:} \quad & \\ \text{(Enough books in genre)} \quad & \sum_{i \in I_j} x_i \geq q_j \quad \text{for each genre } j \in J. \end{aligned}$$

Q1 (Emergency Vehicle Location)

The city of Metropolis is divided into nine districts and is considering seven possible sites to place emergency vehicles. The table below shows the time (minutes) it takes for an emergency vehicle to travel from each district to each site. (The column labels are sites and row labels are districts.)

District \ Row	1	2	3	4	5	6	7
1	5	3	4	3	8	9	0
2	3	6	5	4	8	0	3
3	4	3	6	8	10	3	2
4	6	0	2	7	3	2	5
5	2	8	2	5	0	6	8
6	2	6	4	0	7	3	5
7	0	12	5	5	5	7	2
8	10	9	0	2	3	5	7
9	2	4	5	7	3	4	5

Formulate an optimization problem to find the minimum number of sites so that all districts are within three minutes of an emergency vehicle, then generalize this to be able to handle arbitrary data of a similar format.

Step 1. Identify the decision, objective, and constraints in English

Decision:

Objective:

Constraints:

Step 2. Formulate the optimization as linear expressions of decision variables

Decision variables:

Objective and constraints:

Step 3a. Use variables to encode all input data

Data:

Step 3b. Formulate the LP/MIP in terms of only data and decision variables

Decision variables:

Objective and constraints:

Q2 (Investment Planning Revisited)

Recall the investment planning problem from last session, with the following concrete formulation:

Decision Variables:

- Let x_A, x_B, x_C, x_D, x_E denote how much to invest in each of the five options. (continuous)
- Let y_0, y_1, y_2, y_3 denote the cash at hand at the end of each year. (Continuous)

Objective and constraints:

$$
\begin{aligned}
\text{Maximize:} \quad & y_3 \\
\text{subject to:} \quad & \\
\text{(Limit on investment)} \quad & x_A, x_B, x_C, x_D, x_E \leq 75000 \\
\text{(Cash flow in year 0)} \quad & 10000 - x_A - x_C - x_D = y_0 \\
\text{(Cash flow in year 1)} \quad & y_0 + 0.5x_A + 1.2x_C - x_B = y_1 \\
\text{(Cash flow in year 2)} \quad & y_1 + x_A + 0.5x_B - x_E = y_2 \\
\text{(Cash flow in year 3)} \quad & y_2 + x_B + 1.9x_D + 1.5x_E = y3 \\
\text{(Non-negative investments)} \quad & x_A, x_B, x_C, x_D, x_E \geq 0 \\
\text{(Non-negative cash flow)} \quad & y_0, y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Use data variables to represent all numbers and re-formulate the above into an abstract formuation, which can handle arbitrarily many investment options and arbitrarily many years. The parameters 75000 and 10000 should also be represented using parameter variables L and C. The objective should be to maximize the cash on hand at the end of year m.

Step 3a. Use variables to encode all input data

Data:

Step 3b. Formulate the LP/MIP in terms of only data and decision variables

Decision variables:

Objective and constraints: