Homework 8 (due Tue. 4/17 before class)

Name:

This homework set is composed of two practice exams with the same format and difficulty as the upcoming Exam 2 (on Tue. 4/24). **To help you prepare for the exam, you should complete this homework set using pencil/pen and scan it to submit it on Blackboard. (Free scanning is available at all USC libraries.)** You should also time yourself to see how long each type of question takes you and do additional practice if needed to improve efficiency.

Questions 1-4 is one complete sample exam 2, and questions 5-8 is another sample exam 2. Just like the actual exam, each sample exam is intended to be completed in 75 minutes.

Exam 2 Sample A (100 Points)

Q1. Multiple Choice (10 points)

i) Consider the following linear program, which has an unique optimal solution.

$$\text{Maximize:} \quad 3X - Y$$
$$\text{subject to:}$$
$$2X + Y \leq 10$$
$$X + 2 \leq Y$$
$$X \leq 2$$

What is the optimal value of decision variable Y?
   A) 2
   B) 4
   C) 6
   D) 10
   E) None of the above

ii) Suppose that X and Y are positive **input parameters (data with known numerical values)** and Z and W are **decision variables**, which of the following **CANNOT be the objective function** of a linear program (LP)?
   A) $\frac{X}{Y}Z + \frac{Y}{X}W$
   B) $(X - Y)(Z + X)$
   C) $W + XZ$
   D) WZ
   E) XY

Q2. Concrete Modeling (25 points)

A food factory requires 2000 tons of canola oil every month as a raw ingredient. The price of canola oil fluctuates from month to month due to market conditions. The predicted prices for the next six months are as follows

	Apr	May	Jun	Jul	Aug	Sep
Price ($) per ton	150	160	180	170	180	160

The factory's supplier for canola oil delivers it on the first day of every month, and charges the prices above. The factory can decide how much oil to buy each month from the supplier. The factory can also store up to 1000 tons of oil for future use. (The current inventory before the shipment in April is zero.) Formulate an optimization problem to decide how much canola oil to buy for each of the six months in order to minimize the total purchase cost over these six months.

a) What is the decision (in words)? (2 points)

b) What is the objective (in words)? (2 points)

c) Describe any three constraints (in words). (6 points)

d) Write a concrete LP formulation using the correct mathematical notation, specifying the decision variables, the objective function and all constraints. (20 points)

Q3. Abstract Modeling (35 points)

Ebony is an ambitious master's student who would like to maximize the number of extra-curricular business analytics projects she takes part of this year. However, projects may conflict with one another. The following graph summarizes the conflicts. (For example, project A conflicts with B, C and D, but projects B and D can be done together.)
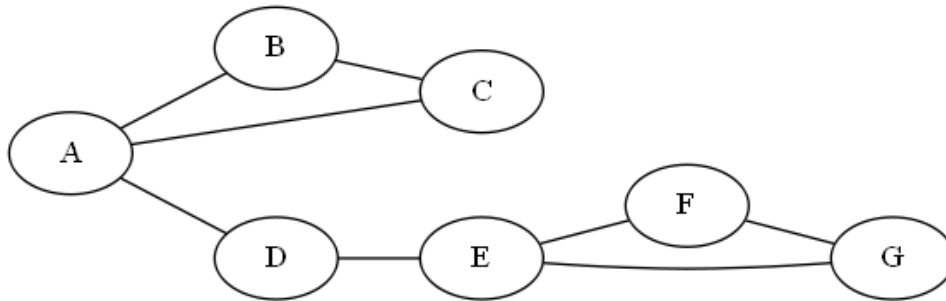
Beside the conflict above,

- Project A is a prerequisite to project F (meaning that pursuing F requires also pursuing A.)
- Project B is a prerequisite to project G.

Formulate an optimization problem to help her decide which projects to pursue.
a) What is the decision (in words)? (2 points)

b) What is the objective (in words)? (2 points)

c) Except for the prerequisite constraints, describe any one constraint (in words). (2 points)

d) Write a concrete MIP formulation using the correct mathematical notation, specifying the decision variables, the objective function and all constraints. (15 points)

e) Suppose that I is the set of projects. Define data variables to encode all inputs needed when generalizing the above MIP formulation to arbitrarily large data sets. (4 points)

f) Write an abstract MIP formulation using the data variables you defined in part e), specifying the decision variables, the objective function and all constraints. (10 points)

Q4. Gurobi Coding (25 points)

The following MIP is used to assign students into project teams in this course, to balance the overall characteristics of each team.

Data:

- I: set of students.
- n: number of teams
- $J = \{1, 2, \cdots, n\}$: set of teams.
- K: set of characteristics.
- a_{ik}: student i's value for characteristics k.
- w_k: the weight for characteristics k in the objective.
- L_k: the best lower bound for the sum of characteristic k for any team.
- U_k: the best upper bound for the sum of characteristics k for any team.

For example, suppose that $n = 2$, and the data is stored in an excel file called `data.xlsx` with two sheets. The sheet named `Sheet 1` encodes I, K and a_{ik}'s. In the below, $I = \{A, B, C, D, E, F\}$, and $K = \{Person, Male, Programmer, Math, Speaking\}$.

	A	B	C	D	E	F
1	Names	Person	Male	Programmer	Math	Speaking
2	A	1	1	1	1	1
3	B	1	1	0	0	0
4	C	1	0	0	1	1
5	D	1	0	0	1	0
6	E	1	0	0	0	1
7	F	1	1	1	0	1
8						

The sheet named `Sheet 2` encodes the w_k, L_k and U_k for each characteristic k.

	A	B	C	D	E	F
1		Person	Male	Programmer	Math	Speaking
2	Weights	10	3	5	2	2
3	L	3	1	1	1	2
4	U	3	2	1	2	2
5						

Decision variables:

- x_{ij} : whether to assign student i to team j. (Binary)
- s_k, and t_k : auxilliary variables for each characteristic k. (Continuous)

MIP:

$$\text{Minimize:} \quad \sum_{k \in K} w_k(s_k + t_k)$$

subject to:

(Every person assigned) $\quad \sum_{j \in J} x_{ij} = 1 \quad$ For each person $i \in I$.

(Team balance) $\quad L_k - s_k \leq \sum_{i \in I} a_{ik} x_{ij} \leq U_k + t_k \quad$ For each team $j \in J$ and each $k \in K$.

(Binary) $\quad x_{ij} \in \{0, 1\} \quad$ for all i and j.

(Non-negativity) $\quad s_k, t_k \geq 0 \quad$ for all k.

Complete the following code to implement the above MIP. You should store the answer in a DataFrame named `answer` **with one column called** `Team`**: the row index is the name of each person and the column** `Team` **stores the team number** j **to which the person is assigned.**

```python
import pandas as pd
sheet1=pd.read_excel('data.xlsx',sheet_name='Sheet 1',index_col=0)
sheet2=pd.read_excel('data.xlsx',sheet_name='Sheet 2',index_col=0)
n=2
# Write your code below
```

Exam 2 Sample B (100 Points)

Q5. Multiple Choice (10 points)

i) Consider the following linear program.

$$\text{Minimize:} \quad Y$$
$$\text{subject to:}$$
$$\text{(C1)} \quad X + 2Y \geq 12$$
$$\text{(C2)} \quad X \leq 10$$
$$\text{(C3)} \quad Y \leq 5$$
$$\text{(C4)} \quad X \geq 0$$
$$\text{(C5)} \quad Y \geq 0$$

What are the binding constraints?
A) C1 and C2.
B) C2 and C3.
C) C1 and C3.
D) C3 and C4.
E) C2 and C5.

ii) Suppose there are two factories A and B, and two demand centers 1 and 2. Goods are produced in the two factories and shipped to the demand centers. The following LP minimizes the total transportation cost (measured in thoursands of dollars) subject to respecting the capacity of each factory and fulfilling all demand.

$$\text{Minimize:} \quad 5X_{A1} + 3X_{A2} + 4X_{B1} + 7X_{B2}$$
$$\text{subject to:}$$

(Capacity A)	$X_{A1} + X_{A2} \leq 10$
(Capacity B)	$X_{B1} + X_{B2} \leq 8$
(Demand 1)	$X_{A1} + X_{B1} \geq 9$
(Demand 2)	$X_{A2} + X_{B2} \geq 8$

$$X_{A1}, X_{A2}, X_{B1}, X_{B2} \geq 0$$

The shadow price of the (Capacity B) constraint is -1, which is valid when the RHS is between 7 and 9. The shadow price of the (Demand 1) constraint is 5, which is valid when the RHS is between 8 and 10. **Which of the following statements is TRUE?**

A) Increasing the capacity of factory B by 2 units would decrease the transportation cost by 2 units.

B) Increasing the demand requirement at center 1 would reduce the transportation cost.

C) Increasing the demand requirement of center 1 by one unit would change the optimal value of decision variables X_{A1} or X_{B1}.

D) If we simultaneously reduce the demand requirement at center 1 to 8 and increase the capacity of factory B to 9, then the total transportation cost would decrease by 6 units.

E) If it is possible to pay another company 6 thousand dollars in order to satisfy one unit of demand at center 1, then we should pay them to do it.

Q6. Concrete Modeling (30 points)

USC would like to protect every intersection around USC by stationing security staff, so that every intersection either has a staff stationed, or neighbors another intersection that has a staff stationed. For example, in the following sample map with 10 intersections, the neighbors of intersection A are B and E. So if a staff is stationed at intersection A, then intersections A, B, and E are all protected.
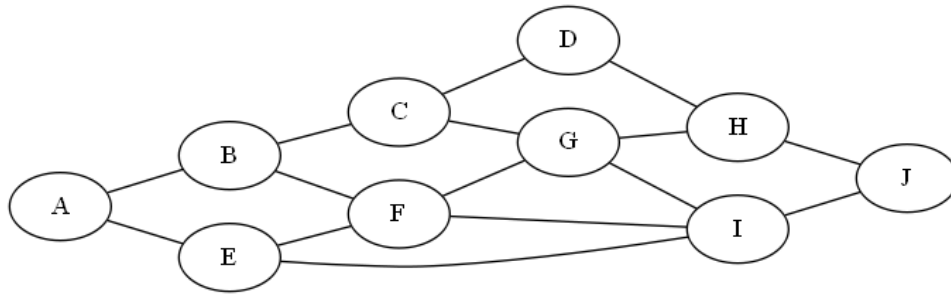
For the above map, formulate an optimization problem (LP or MIP) to minimize the number of staff needed, subject to protecting all 10 intersections, as well as satisfying the following constraints:

- Staff cannot be stationed at both intersections A and B.
- At least three of E, F, G and H must have a staff directly stationed.
- No one can be stationed at intersection J.
- There must be at least one staff who is stationed at an intersection where he/she can simultaneously protect both intersections C and H.

a) What is the decision (in words)? (3 points)

b) What is the objective (in words)? (2 points)

c) Other than the constraints listed in bullets above, describe any one constraint (in words). (3 points)

d) Write a concrete LP formulation using the correct mathematical notation, specifying the decision variables, the objective function and all constraints. (22 points)

Q7. Abstract Modeling (35 points)

Luke would like to optimize his diet to minimize cost while satisfying minimal nutritional requirements. The first table below shows his favorite foods, as well as the nutrition information per serving and cost per servint. The second table shows the requirement on his total daily in-take of each nutrient.

Foods	Calories	Protein	Fat	Sodium	Costs
1. ice cream	330	8	15	180	$1.59
2. chicken	420	32	10	300	$2.89
3. pizza	320	15	20	820	$1.99
4. fries	380	4	19	270	$1.89
5. macaroni	320	12	10	830	$2.09
6. milk	100	8	2.5	125	$0.89
7. salad	320	31	2	123	$2.49

	Calories	Protein	Fat	Sodium
Lower bound	1800	91	0	0
Upper bound	2200	9999	65	1779

For example, having one serving of ice cream and one serving of chicken would not give him enough calories, but having 6 servings of ice cream would, but that would be too much fat. Luke is okay with fractional servings, but he would like to make sure that the number of servings of each item is either zero, or between 1 and 3. (For example, 0 or 1.2 servings of pizza are okay, but 0.2 or 3.2 are not.) Formulate a LP or MIP to help him decide his optimal diet.

a) What is the decision (in words)? (2 points)

b) What is the objective (in words)? (2 points)

c) Describe any constraint (in words). (2 points)

d) Write a concrete MIP formulation using the correct mathematical notation, specifying the decision variables, the objective function and all constraints. (15 points)

e) Suppose that I is the set of projects. Define data variables to encode all inputs needed when generalizing the above MIP formulation to arbitrarily large data sets. (4 points)

f) Write an abstract MIP formulation using the data variables you defined in part e), specifying the decision variables, the objective function and all constraints. (10 points)

Q8. Gurobi Coding (25 points)

Consider the following LP for optimizing how to best payoff one's credit card debts (assuming that one stops using these cards from now on).

Data:

- I: the set of credit cards you carry.
- n: the number of months to optimize.
- $T = \{0, 1, \cdots, n-1\}$: the set of months.
- r_i: the monthly interest rate of card i.
- B_i: the current balance on credit card i.
- C: the total cash allocated each month to pay the debts.

The set of cards I, current balance B_i and interest rate r_i are stored in the following Excel sheet called `data.xlsx`. The other data (n and C) are supplied in the Python code below.

	A	B	C
1	**Credit Card**	**Balance**	**Monthly Rate**
2	Saks Fifth Avenue	$20,000	0.005
3	Bloomingdale's	$50,000	0.010
4	Macy's	$40,000	0.015

Decision variables:

- x_{it}: how much to pay off credit card $i \in I$ in month $t \in T$.
- y_{it}: the balance at the end of month t on credit card i.

LP:

$$\text{Minimize:} \quad \sum_{i \in I} \sum_{t \in T} x_{it} + \sum_{i \in I} y_{i(n-1)} \qquad \text{(total payment + final month's balance)}$$

subject to:

$$\text{(Maximum payment each month)} \qquad \sum_{i \in I} x_{it} \leq C \quad \text{for each month } t \in T.$$

$$\text{(Initial cash flow)} \qquad (1 + r_i)B_i - x_i = y_{i0} \quad \text{for each card } i.$$

$$\text{(Cash flow)} \qquad (1 + r_i)y_{i(t-1)} - x_i = y_{it} \quad \text{for each card } i, \text{ each month } t \geq 1.$$

$$\text{(Non-negativity)} \qquad x_{it}, y_{it} \geq 0 \quad \text{for all } i, t.$$

Implement the above LP by completing the following code. You should store the output in a DataFrame named `answer` **where the row indices are the months** T**, and the column names are the names of the credit cards** I**. In each cell, you should indicate the payment** x_{it} **to that credit card in that month.**

```python
[ ]: import pandas as pd
     data=pd.read_excel('data.xlsx',index_col=0)
     n=36
     C=5000

     # Write your Python code below
```