Optimization Modeling I (4/3)

Learning Objectives:

- Verbally articulating the decision, objective, and constraints in a given business problem.
- Encapsulate the decision using suitable decision variables.
- Expressing the objective and each side of each constraint as linear functions of the decision variables.
- Writing mathematical formulations using the correct notation.

Here are the three in-class exercises we did and the solutions. Pay close attention to the framework where we approach each problem, by first articulating in English, then expressing in math (writing out numbers, no summations).

As an optional final step, you can express the mathematical formulation in a generalizable form, with data contained in variables and summation statements. This last form makes it most convenient to code in Python and scale up to larger data sets.

Exercise 1

Amazon.com is expanding its business by launching a physical store in West LA. As the manager, you need to select which bestsellers to carry at the store's grand opening. The following table provides the list of Top 10 Bestsellers in Literature & Fiction, along with their genres. Note that some bestsellers belong to more than one genre.

Rank \ Genre	Literary	Sci-Fi	Romance	Thriller
1	√			
2		√		√
3			√	√
4	√		√	
5	√			
6			√	
7		√		
8				√
9	√	√		
10			√	

You wish to carry the minimum number of bestsellers, while ensuring that there are at least two bestsellers in each genre. Formulate this as an optimization problem.

Modeling Process

What is the decision (in words)?
Which bestsellers to carry.

What type of decision variables do you need to encode it and how many variables of each type? What does each variable mean?
Ten binary decision variables, one for each book. Each decision variable denotes whether to carry a particular book (yes/no).

Define the decision variables:
For each book $i \in \{1, 2, \cdots, 10\}$, let x_i be a binary decision variable, such that $x_i = 1$ if we carry book, and $x_i = 0$ otherwise.

What is the objective (in words)?

To minimize the total number of bestsellers carried.

Express the objective as a linear function of decision variables.

The total number of bestsellers carried is $x_1 + x_2 + \cdots + x_{10}$.

What are the constraints (in words)? How many are there of each kind?

For each of the four genres, we have a constraint specifying that we need to carry at least two books of that genre.

For each kind of constraint, what are the LHS and RHS (in words)?

For each genre we have,

$$\text{\# of books carried of this genre} \geq 2$$

Express the constraints as linear functions of decision variables.

$$
\begin{array}{rl}
\text{(Literary)} & x_1 + x_4 + x_5 + x_9 \geq 2 \\
\text{(Sci-Fi)} & x_2 + x_7 + x_9 \geq 2 \\
\text{(Romance)} & x_3 + x_4 + x_7 + x_{10} \geq 2 \\
\text{(Thriller)} & x_2 + x_3 + x_8 \geq 2
\end{array}
$$

Mathematical Formulation

Decision variables: For each book $i \in \{1, 2, \cdots, 10\}$, let x_i be a binary decision variable, such that $x_i = 1$ if we carry book , and $x_i = 0$ otherwise.

$$
\begin{array}{rl}
\text{Minimize:} & x_1 + x_2 + \cdots + x_{10} \\
\text{subject to:} & \\
\text{(Literary)} & x_1 + x_4 + x_5 + x_9 \geq 2 \\
\text{(Sci-Fi)} & x_2 + x_7 + x_9 \geq 2 \\
\text{(Romance)} & x_3 + x_4 + x_7 + x_{10} \geq 2 \\
\text{(Thriller)} & x_2 + x_3 + x_8 \geq 2 \\
& x_i \in \{0, 1\} \text{ for every book } i \in \{1, 2, \cdots, 10\}.
\end{array}
$$

(Optional) Mathematical Formulation in Generalizable Form

Data:

- I: the set of books.
- J: the set of genres.
- a_{ij}: a binary variable denoting whether book i is of genre j. (These corresponds to the checkmarks in the original question.)
- q_j: how many books do we need of genre j.

Decision variables: For each book $i \in I$, let x_i be a binary decision variable, such that $x_i = 1$ if we carry book , and $x_i = 0$ otherwise.

$$
\begin{array}{rl}
\text{Minimize:} & \displaystyle\sum_{i \in I} x_i \\
\text{subject to:} & \\
\text{(Enough books in genre)} & \displaystyle\sum_{i \in I} a_{ij} x_i \geq q_j \text{ for each genre } j \in J. \\
& x_i \in \{0, 1\} \text{ for every book } i \in I.
\end{array}
$$

Exercise 2

Tom Burke, owner of Burke Construction, has promised to complete five projects this winter. Burke Construction has 10 workers that will work 40 hours a week for 12 weeks this winter. Since this is a limited workforce, Tom knows that he will not be able to complete all of his construction projects without subcontracting some of them. In the table below, he has estimated the amount of labor hours required by each project, and the profit to his company.

Project	1	2	3	4	5
Labor hours required	1300	950	1000	1400	1600
Profit (if done by own company)	$30,000	$10,000	$26,000	$18,000	$20,000
Profit (if subcontracted)	$6,000	$2,000	$8,000	$9,000	$4,500

To maximize profit, which jobs should Tom schedule for his company, and which should be subcontracted? Formulate an optimization problem. Assume that projects cannot be partially subcontracted; that is, a project will be completed entirely by either Burke Construction or the subcontractor.

Modeling Process

What is the decision (in words)?
Which projects to schedule for own company. (The other projects will be subcontrated).
What type of decision variables do you need to encode it and how many variables of each type? What does each variable mean?
5 binary decision variables, one for each project. The decision variable for a project denotes whether Burke will do it within his own company.
Define the decision variables:
For each project $i \in \{1, 2, \cdots, 5\}$, let x_i be a binary decision variable that equals 1 if Burke does the project internally and 0 otherwise.
What is the objective (in words)?
Maximize the total profit, which is the sum of the profit from projects done by own company and the profit from subcontracted projects.
Express the objective as a linear function of decision variables.
The following function measures profit in thousands of dollars.

$$30x_1 + 10x_2 + 26x_3 + 18x_4 + 20x_5 + 6(1 - x_1) + 2(1 - x_2) + 8(1 - x_3) + 9(1 - x_4) + 4.5(1 - x_5)$$

What are the constraints (in words)? How many are there of each kind?
The constrained is not to use more labor than what we have.
For each kind of constraint, what are the LHS and RHS (in words)?

$$\text{\# of hours of labor used within the company} \leq \text{\# of hours available}$$

Express the constraints as linear functions of decision variables.

$$1300x_1 + 950x_2 + 1000x_3 + 1400x_4 + 1600x_5 \leq 4800$$

Mathematical Formulation

Decision variables: For each project $i \in \{1, 2, \cdots, 5\}$, let x_i be a binary decision variable that equals 1 if Burke does the project internally and 0 otherwise.

Maximize $30x_1 + 10x_2 + 26x_3 + 18x_4 + 20x_5 + 6(1 - x_1) + 2(1 - x_2) + 8(1 - x_3) + 9(1 - x_4) + 4.5(1 - x_5)$

subject to:

(Labor) $1300x_1 + 950x_2 + 1000x_3 + 1400x_4 + 1600x_5 \leq 4800$

$$x_i \in \{0, 1\} \text{ for each project } i.$$

(Optional) Mathematical Formulation in Generalizable Form

Data:

- I: the set of projects
- a_i: the profit if Burke does project i within the company.
- b_i: the profit if Burke subcontracts project i.
- l_i: the number of hours of labor required by project i for doing it internally.
- L: the total number of hours labor within the company.

Decision variables: For each project $i \in I$, let x_i be a binary decision variable that equals 1 if Burke does the project internally and 0 otherwise.

$$\text{Maximize} \quad \sum_{i \in I}(a_i x_i + b_i(1 - x_i))$$

subject to:

(Labor) $$\sum_{i \in I} l_i x_i \leq L$$

$$x_i \in \{0, 1\} \text{ for each project } i.$$

Exercise 3

The city of Metropolis is divided into nine districts and is considering seven possible sites to place emergency vehicles. The table below shows the time (minutes) it takes for an emergency vehicle to travel from each district to each site. (The column labels are sites and row labels are districts.)

District \ Row	1	2	3	4	5	6	7
1	5	3	4	3	8	9	0
2	3	6	5	4	8	0	3
3	4	3	6	8	10	3	2
4	6	0	2	7	3	2	5
5	2	8	2	5	0	6	8
6	2	6	4	0	7	3	5
7	0	12	5	5	5	7	2
8	10	9	0	2	3	5	7
9	2	4	5	7	3	4	5

Formulate an optimization problem to find the minimum number of sites so that all districts are within three minutes of an emergency vehicle.

Modeling Process

What is the decision (in words)?

Which sites to place emergency vehicles.

What type of decision variables do you need to encode it and how many variables of each type? What does each variable mean?

7 binary variables, one for each site, denoting whether or not we place an emergency vehicle there.

Define the decision variables:

For each site $i \in \{1, 2, \cdots, 7\}$, let x_i be a binary variable denoting whether we place an emergency vehicle at this site.

What is the objective (in words)?

Minimize the number of sites at which we place emergency vehicles.

Express the objective as a linear function of decision variables.

The number sites at which we place emergency vehicles is:

$$x_1 + x_2 + \cdots + x_7$$

What are the constraints (in words)? How many are there of each kind?

For each of the 9 districts, we need there to be at least one site that is within three minutes at which we placed emergency vehicles.

For each kind of constraint, what are the LHS and RHS (in words)?

For each district, we need

$$\text{\# of sites within three minutes with emergency vehicle} \geq 1$$

Express the constraints as linear functions of decision variables.

$$
\begin{aligned}
&\text{(District 1)} & x_2 + x_4 + x_7 &\geq 1 \\
&\text{(District 2)} & x_1 + x_6 + x_7 &\geq 1 \\
&\text{(District 3)} & x_2 + x_6 + x_7 &\geq 1 \\
&\text{(District 4)} & x_2 + x_3 + x_5 + x_6 &\geq 1 \\
&\text{(District 5)} & x_1 + x_3 + x_5 &\geq 1 \\
&\text{(District 6)} & x_1 + x_4 + x_6 &\geq 1 \\
&\text{(District 7)} & x_1 + x_7 &\geq 1 \\
&\text{(District 8)} & x_3 + x_4 + x_5 &\geq 1 \\
&\text{(District 9)} & x_1 + x_5 &\geq 1
\end{aligned}
$$

Mathematical Formulation

Decision variables: For each site $i \in \{1, 2, \cdots, 7\}$, let x_i be a binary variable denoting whether we place an emergency vehicle at this site.

$$\begin{aligned}
\text{Minimize} \quad & x_1 + x_2 + \cdots + x_7 \\
\text{subject to:} & \\
\text{(District 1)} \quad & x_2 + x_4 + x_7 \geq 1 \\
\text{(District 2)} \quad & x_1 + x_6 + x_7 \geq 1 \\
\text{(District 3)} \quad & x_2 + x_6 + x_7 \geq 1 \\
\text{(District 4)} \quad & x_2 + x_3 + x_5 + x_6 \geq 1 \\
\text{(District 5)} \quad & x_1 + x_3 + x_5 \geq 1 \\
\text{(District 6)} \quad & x_1 + x_4 + x_6 \geq 1 \\
\text{(District 7)} \quad & x_1 + x_7 \geq 1 \\
\text{(District 8)} \quad & x_3 + x_4 + x_5 \geq 1 \\
\text{(District 9)} \quad & x_1 + x_5 \geq 1 \\
& x_i \in \{0,1\} \text{ for each site } i.
\end{aligned}$$

(Optional) Mathematical Formulation in Generalizable Form

Data:

- I: the set of sites.
- J: the set of districts.
- a_{ij}: a binary variable for whether site i is within three minutes of district j. (This data can be obtained from the distance table by checking whether each entry is no more than 3.)

Decision variables: For each site $i \in I$, let x_i be a binary variable denoting whether we place an emergency vehicle at this site.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{i \in I} x_i \\
\text{subject to:} & \\
\text{(Close by site exists)} \quad & \sum_{i \in I} a_{ij} x_i \geq 1 \qquad \text{for each district } j. \\
& x_i \in \{0,1\} \quad \text{for each site } i.
\end{aligned}$$