Quiz 2 (15 Minutes)

The following version is for the 12:30pm section. In the 5pm section, the choices were shuffled so that the lettering is different, so that the correct answers are **Q1:** A) **Q2:** E).

Quesiton 1.

Consider the following production planning LP.

$$\begin{array}{ll} \text{Maximize} & 2X + Y \\ \text{subject to:} & \\ \text{(Material 1)} & X + 2Y \leq 6 \\ \text{(Material 2)} & 3Y \leq 6 \\ \text{(Non-negativity)} & X, Y \geq 0 \end{array}$$

Solve the LP graphically. **What is the optimal value of decision variable X?**
A) 0
B) 2
C) 4
Answer D) 6
E) None of the above.

Solution

First draw the feasible region (see video of lecture on 3/29), and note that the direction of the objective function is $(+2, +1)$, meaning that it points two units to the right in the X direction and one unit up in the Y direction (assuming that X is horizontal and Y is vertical). Hence, the optimal solution is the intersection of the lines $Y = 0$ and $X + 2Y = 6$, which implies that at the optimum solution, $X = 6$.

Many people incorrectly chose $X = 2$, assuming that the solution is at the intersection of the lines $X + 2Y = 6$ and $3Y = 6$. However, there are actually four lines that define the feasible region: these two plus $X = 0$ and $Y = 0$. The region bounded by these four constraints has four corners. Which of these corners is optimal depends on the direction of the optimal objective. Because in this case the direction of the optimal objective is mostly to the right (in the positive X direction), the corner that is furthest in direction of the optimal objective is $(X, Y) = (6, 0)$. For the same constraints, the optimal solution might be different if the direction of the optimal objective were different.

Such multiple choice questions involving graphical solutiosn may show up in Exam 2. You can review this topic by viewing the lecture video for 3/1. Please come to office hour if you have trouble.

Question 2.

Consider the following LP, which is the same as the above except for a different objective function.

$$\begin{array}{ll} \text{Maximize} & 2X + 10Y \\ \text{subject to:} & \\ \text{(Material 1)} & X + 2Y \leq 6 \\ \text{(Material 2)} & 3Y \leq 6 \\ \text{(Non-negativity)} & X, Y \geq 0 \end{array}$$

The optimal solution is $(X, Y) = (2, 2)$, and the shadow price of the material 1 constraint is 2.00, which is valid whenever the right hand side (RHS) is between 4 and infinity. (Currently the RHS is 6.) **Which of the following statements is FALSE?**

A) At the optimal solution, the material 1 constraint is binding.

B) At the optimal solution, the material 2 constraint is binding.

Answer C) Removing 3 units of material 1 would decrease the optimal objective by 6.

D) If someone offers 50 units of material 1 for a total cost of 90, then it is profitable to purchase them and make up the cost through increased production.

E) Regardless of how many additional units of material 1 we obtain, we could still improve the objective function by having more material 1, even though the total amount of material 2 remains at 6.

Solution

To identify the set of binding constraints, we check whether the left hand side (LHS) equals the RHS. The constraints are:

$$\begin{aligned} X + 2Y &\leq 6 \\ 3Y &\leq 6 \\ X &\geq 0 \\ Y &\geq 0 \end{aligned}$$

The two sides are equal for the first two inequalities, but not equal for the last two. So the material 1 and material 2 constraints are both binding.

Statement **C)** is wrong because the current RHS of the material 1 constraint is 6, and if we reduce it by 3, then the new RHS would be 3. However, 3 is outside the allowable range $[4, \infty]$ (see problem text). Hence, we cannot use the shadow price of 2 in this case. (In fact, by the geometry of LPs, the shadow price is guaranteed to be greater when RHS is less than 4, so removing 3 units of material 1 would decrease the optimal objective by strictly more than 6.)

An alternative way to see that **C)** is wrong would be to graphically solve the LP with the constraint $X + 2Y \leq 3$. The optimal solution in that case would be $(X, Y) = (0, 1.5)$, with the optimal objective being 15, which is 9 less than the current objective value of $2 \times 2 + 10 \times 2 = 24$.

Statement D) is correct because if the shadow price is 2, then we should be willing to pay 2 per unit for extra units. Moreover, the allowable range includes the new RHS of $6 + 50 = 56$, so using the shadow price is valid here. Since the unit price of $90/50 = 1.8 < 2$, we are willing to pay 90 for 50 extra units of material 1.

Statement E) is correct because the shadow price for material 1 is valid for arbitrarily large RHS, as the allowable range includes infinity. This means that material 1 is of value independent of material 2. Another way to see this is that having more material 1 would always allow us to produce more units of X, even though the amount of material 2 stays the same.