# Project 1: Finite difference methods

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## 1 Finite difference methods in 1D

### 1.1 Local Truncation Error

First let's write out the $\widetilde{D}_0\left(\kappa(x_j)\widetilde{D}_0 u_j\right)$ operator explicitly:

$$\begin{aligned}\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right) &= \widetilde{D}_0\left(\frac{u(x+h/2)-u(x-h/2}{h}\right)\\ &= \kappa(x+h/2)\frac{u(x+h)-u(x)}{h^2} + \kappa(x-h/2)\frac{u(x)-u(x-h)}{h^2}\end{aligned} \tag{1}$$

We do the Taylor expansion for $\kappa$ at the point $x$:

$$\kappa(x+h/2) = k(x) + \kappa'(x)\frac{h}{1} + \kappa''(x)\frac{h^2}{8} + \kappa^{(3)}(x)\frac{h^3}{48} + o(h^3) \tag{2}$$

similarly:

$$\kappa(x-h/2) = k(x) - \kappa'(x)\frac{h}{1} + \kappa''(x)\frac{h^2}{8} - \kappa^{(3)}(x)\frac{h^3}{48} + o(h^3) \tag{3}$$

We do Taylor expansion for $u$ at the point $x$ as well:

$$u(x+h) - u(x) = u'(x)h + u''(x)\frac{h^2}{2} + u^{(3)}\frac{h^3}{6} + u^{(4)}(x)\frac{h^4}{24} + o(h^4) \tag{4}$$

$$u(x) - u(x-h) = u'(x)h - u''(x)\frac{h^2}{2} + u^{(3)}\frac{h^3}{6} - u^{(4)}(x)\frac{h^4}{24} + o(h^4) \tag{5}$$

Then we can have the expansion of $\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right)$:

$$\begin{aligned}\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right) &= \kappa(x)\left[u''(x) + u^{(4)}\frac{h^2}{12} + o(h^2)\right]\\ &+ \frac{h}{2}\kappa'(x)\left[\frac{2u'(x)}{h} + \frac{u^{(3)}(x)h}{3} + o(h)\right]\\ &+ \frac{h^2}{8}\kappa'(x)\left[u''(x) + o(1)\right]\\ &= \kappa(x)u''(x) + \kappa'(x)u'(x) + \left[\frac{\kappa(x)u^{(4)}(x)}{12} + \frac{\kappa'(x)u^{(3)}(x)}{6} + \frac{\kappa''(x)u''(x)}{8}\right]h^2 + o(h^2)\end{aligned} \tag{6}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = \kappa(x)u''(x) + \kappa'(x)u'(x)$, the local truncation error is:

$$\begin{aligned}\text{LTE} &= \widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) \\ &= \left[\frac{\kappa(x)u^{(4)}(x)}{12} + \frac{\kappa'(x)u^{(3)}(x)}{6} + \frac{\kappa''(x)u''(x)}{8}\right]h^2 + o(h^2)\end{aligned} \tag{7}$$

Thus this is a second-order scheme.

## 1.2 Local Truncation Error

Denote $\kappa(x_j - h/2)$ as $k_{j-1/2}$ and $\kappa(x_j + h/2)$ as $k_{j+1/2}$, we can convert the operator to a matrix:

$$\mathbf{A} = \begin{bmatrix} -k_{1+1/2} - k_{1-1/2} & k_{1+1/2} & 0 & \dots & 0 \\ k_{2-1/2} & -k_{2+1/2} - k_{2-1/2} & k_{2+1/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & k_{M-1/2} & -k_{M+1/2} - k_{M-1/2} \end{bmatrix} \frac{1}{h^2} \tag{8}$$

I pick $u(x) = sin(\pi x), x \in [0, 2]$ as the input function, see Fig. 1a. I compute $f$ analytically:

$$f(x) = k'(x)u'(x) + k(x)u''(x) \tag{9}$$

I also confirmed the approximation of $f$ given by $\vec{f_a}pprox = A\vec{u}$, to make sure it is correct, see Fig. 1bc.

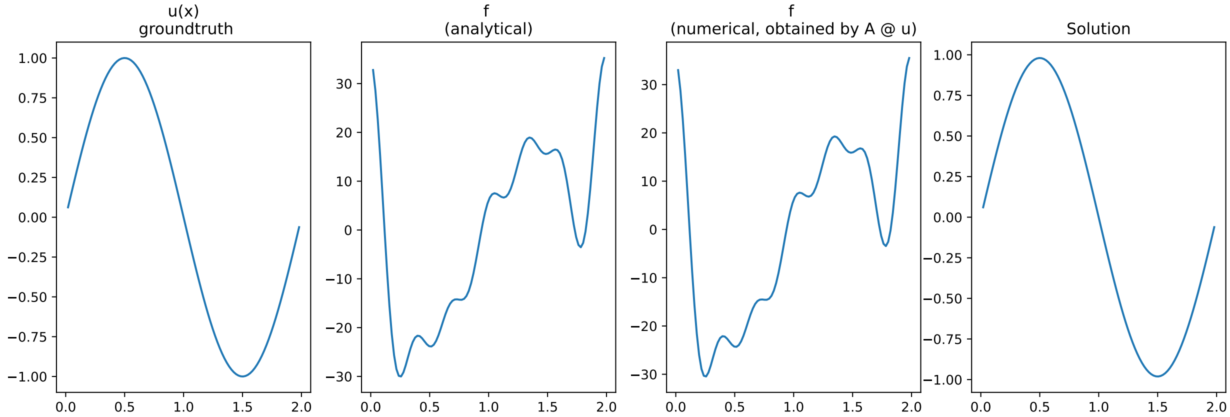


**Figure 1:** Results generated from my solution programming: (a) the input $u(x)$, (b) the analytical $f$ computed by differentiating $u(x)$, (c) the numerical approximation of $f$, and (d) the solution of the ODE given by solving the linear system.

## 1.3 Solving

Please see Fig.2 for how the error related to the steps of discretization.

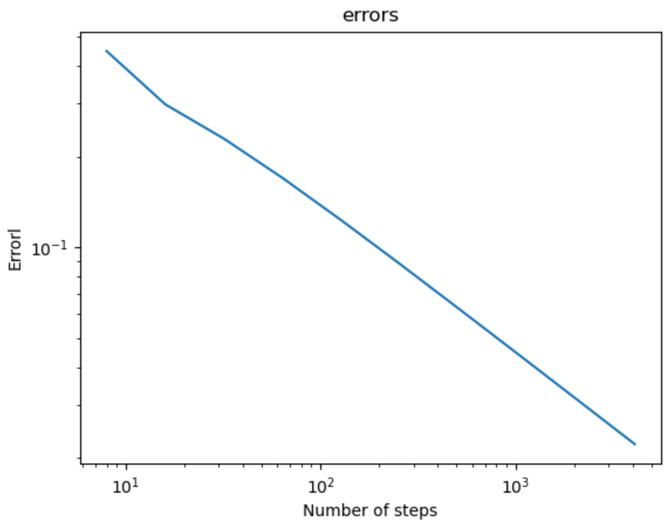


**Figure 2:** Caption