Project 1: Finite difference methods

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1 Finite difference methods in 1D

1.1 Local Truncation Error

First let's write out the $\widetilde{D}_0\left(\kappa(x_j)\widetilde{D}_0 u_j\right)$ operator explicitly:

$$\begin{aligned}\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right) &= \widetilde{D}_0\left(\kappa(x)\frac{u(x+h/2)-u(x-h/2)}{h}\right)\\ &= \kappa(x+h/2)\frac{u(x+h)-u(x)}{h^2} - \kappa(x-h/2)\frac{u(x)-u(x-h)}{h^2}\end{aligned} \tag{1}$$

We do the Taylor expansion for κ at the point x:

$$\kappa(x+h/2) = k(x) + \kappa'(x)\frac{h}{1} + \kappa''(x)\frac{h^2}{8} + \kappa^{(3)}(x)\frac{h^3}{48} + o(h^3) \tag{2}$$

similarly:

$$\kappa(x-h/2) = k(x) - \kappa'(x)\frac{h}{1} + \kappa''(x)\frac{h^2}{8} - \kappa^{(3)}(x)\frac{h^3}{48} + o(h^3) \tag{3}$$

We do Taylor expansion for u at the point x as well:

$$u(x+h) - u(x) = u'(x)h + u''(x)\frac{h^2}{2} + u^{(3)}\frac{h^3}{6} + u^{(4)}(x)\frac{h^4}{24} + o(h^4) \tag{4}$$

$$u(x) - u(x-h) = u'(x)h - u''(x)\frac{h^2}{2} + u^{(3)}\frac{h^3}{6} - u^{(4)}(x)\frac{h^4}{24} + o(h^4) \tag{5}$$

Then we can have the expansion of $\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right)$:

$$\begin{aligned}\widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x))\right) &= \kappa(x)\left[u''(x) + u^{(4)}\frac{h^2}{12} + o(h^2)\right]\\ &+ \frac{h}{2}\kappa'(x)\left[\frac{2u'(x)}{h} + \frac{u^{(3)}(x)h}{3} + o(h)\right]\\ &+ \frac{h^2}{8}\kappa'(x)\left[u''(x) + o(1)\right]\\ &= \kappa(x)u''(x) + \kappa'(x)u'(x) + \left[\frac{\kappa(x)u^{(4)}(x)}{12} + \frac{\kappa'(x)u^{(3)}(x)}{6} + \frac{\kappa''(x)u''(x)}{8}\right]h^2 + o(h^2)\end{aligned} \tag{6}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = \kappa(x)u''(x) + \kappa'(x)u'(x)$, the local truncation error is:

$$\begin{aligned}\text{LTE} &= \widetilde{D}_0\left(\kappa(x)\widetilde{D}_0 u(x)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) \\ &= \left[\frac{\kappa(x)u^{(4)}(x)}{12} + \frac{\kappa'(x)u^{(3)}(x)}{6} + \frac{\kappa''(x)u''(x)}{8}\right]h^2 + o(h^2)\end{aligned} \tag{7}$$

Thus this is a second-order scheme.

1.2 Local Truncation Error

Denote $\kappa(x_j - h/2)$ as $k_{j-1/2}$ and $\kappa(x_j + h/2)$ as $k_{j+1/2}$, we can convert the operator to a matrix:

$$\mathbf{A} = \begin{bmatrix} -k_{1+1/2} - k_{1-1/2} & k_{1+1/2} & 0 & \dots & 0 \\ k_{2-1/2} & -k_{2+1/2} - k_{2-1/2} & k_{2+1/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & k_{M-1/2} & -k_{M+1/2} - k_{M-1/2} \end{bmatrix} \frac{1}{h^2} \tag{8}$$

I pick $u(x) = sin(\pi x), x \in [0, 2]$ as the input function, see Fig. 1a. I compute f analytically:

$$f(x) = k'(x)u'(x) + k(x)u''(x) \tag{9}$$

I also confirmed the approximation of f given by $\vec{f}_approx = A\vec{u}$, to make sure it is correct, see Fig. 1bc.

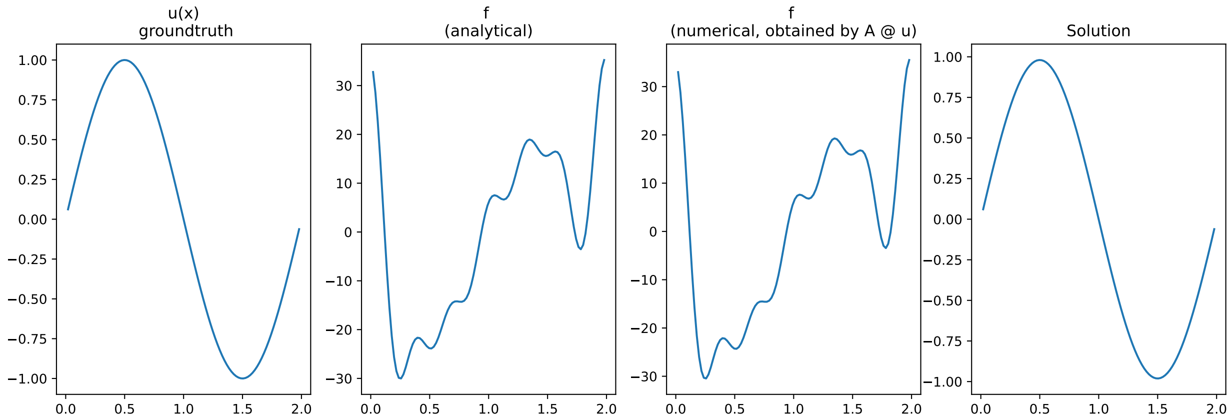

Figure 1: Results generated from my solution programming: (a) the input $u(x)$, (b) the analytical f computed by differentiating $u(x)$, (c) the numerical approximation of f, and (d) the solution of the ODE given by solving the linear system.

1.3 Solving

Please see Fig.2 for how the error and LTEs related to the steps of discretization. Both of those curves has a slope of -2.

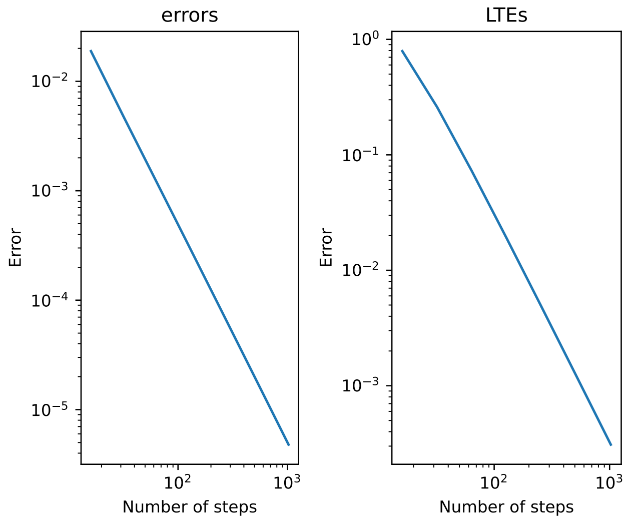

Figure 2: How the errors and the LTEs changes with the steps of discretization.

2 Finite difference methods in 2/3D

2.1 Discretization

We just need to discretize the operator in two directions as we do in Problem 1:

$$\begin{aligned} u_{xx}(x_i, y_i) &= \frac{1}{h^2}\left[k_{i-1/2}u(x_{i-1}, y_i) - (k_{i+1/2} - k_{i-1/2})u(x_i, y_i) - k_{i+1/2}u(x_{i+1}, y_i)\right] \\ u_{yy}(x_i, y_i) &= \frac{1}{h^2}\left[k_{i-1/2}u(x_i, y_{i-1}) - (k_{i+1/2} - k_{i-1/2})u(x_i, y_i) - k_{i+1/2}u(x_i, y_{i+1}\right] \end{aligned} \tag{10}$$

The stencil is:

$$\begin{bmatrix} & & k_{i-1/2} \\ k_{2-1/2} & -2(k_{2+1/2} - k_{2-1/2}) & k_{2+1/2} \\ & k_{2+1/2} & \end{bmatrix} \frac{1}{h^2} \tag{11}$$

Note that I count x, y using matrix coordinate, where the origin is at the top left corner.