Project 1: Finite difference methods

Anka Chen

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1 Problem 1: Fourier Spectral Methods

1.1 Question 1

We select the trail space V_N as:

$$V_N = span\{\Phi_l(x) = e^{ik2\pi x}, x \in [0, 1] \mid |l| <= N\} \tag{1}$$

Since it's a Fourier-Galerkin scheme the trial space is the same as the test space. For each basis $\Phi_k(x)$ of V_N, we have:

$$\Phi'_k(x) = ik2\pi\Phi_k(x) \tag{2}$$
$$\Phi''_k(x) = -k^2 4\pi^2\Phi_k(x) \tag{3}$$
$$\Phi_l(x)\Phi_k(x) = \Phi_{k+l}(x) \tag{4}$$

$sin(2\pi x)$ can represented by basis of V_N:

$$sin(2\pi x) = \frac{1}{2i}(\Phi_1(x) - \Phi_{-1}(x)) \tag{5}$$

Assume the weak solution given by Fourier-Galerkin scheme is $u_N(x) = \sum_{|l|\leq N} \hat{u}_l\Phi_l(x)$ Thus,

$$sin(2\pi x)u_x(x) = \frac{1}{2i}(\Phi_1(x) - \Phi_{-1}(x))i2\pi \sum_{|l|\leq N} l\hat{u}_l\Phi_l(x) \tag{6}$$
$$= \pi \sum_{|l|\leq N} l\hat{u}_l(\Phi_{l+1}(x) - \Phi_{l-1}(x)) \tag{7}$$

Now we can write the discrete scheme as:

$$\frac{d}{dt}\hat{\mathbf{u}} = -A\hat{\mathbf{u}} + \frac{1}{2}\hat{D}_2\hat{\mathbf{u}} \tag{8}$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & \pi N & 0 & 0 & 0 \\ \pi(-N+1) & 0 & -\pi(-N+1) & \dots & 0 \\ 0 & \pi(-N+2) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \pi N & 0 \end{bmatrix} \tag{9}$$

and

$$\hat{D}_2 = diag(-4\pi^2N^2, -4\pi^2(N-1)^2, \dots, -4\pi^2N^2) \tag{10}$$

1.2 Question 2

The fully discrete scheme is:

$$\hat{\mathbf{u}}^{n+1} = (I + k(-A + \frac{1}{2}\hat{D}_2))\hat{\mathbf{u}}^n \tag{11}$$

so for stability, say in the 2-norm, we require,

$$\|(I + k(-A + \frac{1}{2}\hat{D}_2))^n\| \leq 1 \tag{12}$$

Using submutiplicativity of the norm, this is ensured with,

$$\|I + k(-A + \frac{1}{2}\hat{D}_2)\| \leq 1 \tag{13}$$

Denote $H = (-A + \frac{1}{2}\hat{D}_2)$, this requires:

$$\|I + k\lambda_i(H)\| \leq 1 \tag{14}$$

which is equivalent to:

$$k|\lambda_{\max(H)}| \leq 2 \tag{15}$$

The time step needs to satisfy:

$$k \leq \frac{2}{|\lambda_{\max(H)}|} \tag{16}$$

I don't have sufficient knowledge to analytically compute H's singular value. But numerical analysis shows that $|\lambda_{\max(H)}| \sim 20N^2$. Thus $k < \frac{1}{10N^2}$. To simulate the same period of time, the computational time grows cubically with N, where N^2 comes from the number of time steps and another N comes from the computation of $H\hat{\mathbf{u}}$.

1.3 Question 3

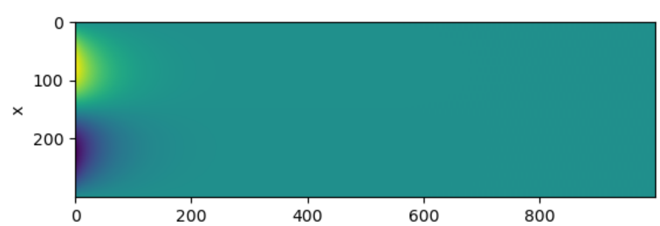

Figure 1: Plot of the result.

The explicit method is faster in computing each step but it requires a lot of time steps to be stable. In contrast, the implicit integrator is slower to compute but it requires less time step to be stable.

2 Problem 3: Legendre spectral methods

2.1 Formulation of Legendre-Galerkin

The boundary function is homogeneous. We apply an essential treatment of boundary conditions. We choose the trial/test space as:

$$P_{N,0} = \text{span}\{q_n \mid j = 2, \ldots, N\} \tag{17}$$

where

$$q_n(x) = p_n(x) - p_n(-1)\frac{1-x}{2} - p(1)\frac{1+x}{2}, n \geq 2 \tag{18}$$

where p_n are Legendre polynomials.

As to our problem, we let:

$$u(x) = \sum_{j=2}^{N} \lambda_j q_j \tag{19}$$

Since $(\frac{u^2}{2})_x = uu_x$, we have:

$$(\frac{u^2}{2})_x = (\sum_{j=2}^{N} \lambda_j q_j)(\sum_{k=2}^{N} \lambda_k q_k') \tag{20}$$

thus:

$$< (\frac{u^2}{2})_x, q_l >= \sum_{j=2}^{N} \sum_{k=2}^{N} < \lambda_j q_j \lambda_k q_k', q_l > \tag{21}$$

This already looks way too complicated. As to the term vu_{xx}, we have:

$$< u_x x, q_l >= - < u_x, q_l' >= \sum_{j=2}^{N} \lambda_j < q_j', q_l' > \tag{22}$$

Here we've obtained the discrete scheme for Legendre-Galerkin method, which is:

$$\frac{d}{dt}\mathbf{u} = -(A + vM)\mathbf{u} \tag{23}$$

References