

# Project 1: Finite difference methods

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## 1 Problem 1: Fourier Spectral Methods

### 1.1 Question 1

We select the trial space  $V_N$  as:

$$V_N = \text{span}\{\Phi_l(x) = e^{ik2\pi x}, x \in [0, 1] \mid |l| \leq N\} \quad (1)$$

Since it's a Fourier-Galerkin scheme the trial space is the same as the test space. For each basis  $\Phi_k(x)$  of  $V_N$ , we have:

$$\Phi'_k(x) = ik2\pi\Phi_k(x) \quad (2)$$

$$\Phi''_k(x) = -k^2 4\pi^2 \Phi_k(x) \quad (3)$$

$$\Phi_l(x)\Phi_k(x) = \Phi_{k+l}(x) \quad (4)$$

$\sin(2\pi x)$  can be represented by basis of  $V_N$ :

$$\sin(2\pi x) = \frac{1}{2i}(\Phi_1(x) - \Phi_{-1}(x)) \quad (5)$$

Assume the weak solution given by Fourier-Galerkin scheme is  $u_N(x) = \sum_{|l| \leq N} \hat{u}_l \Phi_l(x)$ . Thus,

$$\sin(2\pi x)u_x(x) = \frac{1}{2i}(\Phi_1(x) - \Phi_{-1}(x))i2\pi \sum_{|l| \leq N} l \hat{u}_l \Phi_l(x) \quad (6)$$

$$= \pi \sum_{|l| \leq N} l \hat{u}_l (\Phi_{l+1}(x) - \Phi_{l-1}(x)) \quad (7)$$

Now we can write the discrete scheme as:

$$\frac{d}{dt} \hat{\mathbf{u}} = -A \hat{\mathbf{u}} + \frac{1}{2} \hat{D}_2 \hat{\mathbf{u}} \quad (8)$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & \pi N & 0 & 0 & 0 \\ \pi(-N+1) & 0 & -\pi(-N+1) & \dots & 0 \\ 0 & \pi(-N+2) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \pi N & 0 \end{bmatrix} \quad (9)$$

and

$$\hat{D}_2 = \text{diag}(-4\pi^2 N^2, -4\pi^2(N-1)^2, \dots, -4\pi^2 N^2) \quad (10)$$

## 1.2 Question 2

The fully discrete scheme is:

$$\hat{\mathbf{u}}^{n+1} = (I + k(-A + \frac{1}{2}\hat{D}_2))\hat{\mathbf{u}}^n \quad (11)$$

so for stability, say in the 2-norm, we require,

$$\|(I + k(-A + \frac{1}{2}\hat{D}_2))^n\| \leq 1 \quad (12)$$

Using submultiplicativity of the norm, this is ensured with,

$$\|I + k(-A + \frac{1}{2}\hat{D}_2)\| \leq 1 \quad (13)$$

Denote  $H = (-A + \frac{1}{2}\hat{D}_2)$ , this requires:

$$\|I + k\lambda_i(H)\| \leq 1 \quad (14)$$

which is equivalent to:

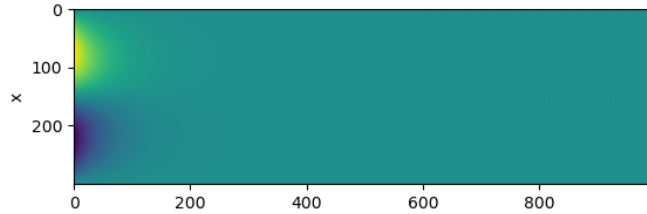
$$k|\lambda_{\max}(H)| \leq 2 \quad (15)$$

The time step needs to satisfy:

$$k \leq \frac{2}{|\lambda_{\max}(H)|} \quad (16)$$

I don't have sufficient knowledge to analytically compute  $H$ 's singular value. But numerical analysis shows that  $|\lambda_{\max}(H)| \sim 20N^2$ . Thus  $k < \frac{1}{10N^2}$ . To simulate the same period of time, the computational time grows cubically with  $N$ , where  $N^2$  comes from the number of time steps and another  $N$  comes from the computation of  $H\hat{\mathbf{u}}$ .

## 1.3 Question 3



**Figure 1:** Plot of the result.

The explicit method is faster in computing each step but it requires a lot of time steps to be stable. In contrast, the implicit integrator is slower to compute but it requires less time step to be stable.

## 2 Problem 3: Legendre spectral methods

### 2.1 Formulation of Legendre-Galerkin

The boundary function is homogeneous. We apply an essential treatment of boundary conditions. We choose the trial/test space as:

$$P_{N,0} = \text{span}\{q_n \mid j = 2, \dots, N\} \quad (17)$$

where

$$q_n(x) = p_n(x) - p_n(-1)\frac{1-x}{2} - p_n(1)\frac{1+x}{2}, n \geq 2 \quad (18)$$

where  $p_n$  are Legendre polynomials.