Metric spaces and Topological spaces

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1 Metric spaces

Roughly speaking, metric spaces are spaces in which there is a notion of distance.

The notion of continuity intuitively means that nearby points map to nearby points. This can be formalized using the concept of metric spaces.

There are many variants of this notion such as uniform continuity, Lipschitz continuity, weak contraction.

1.1 Metric spaces

A metric space is a set X equipped with a map

$$d : X \times X \to \mathbb{R}$$

such that for all $x, y, z \in X$,

$$d(x,x) = 0 \quad \text{and} \quad d(x,y) > 0 \text{ if } x \neq y,$$
$$d(x,y) = d(y,x),$$
$$d(x,z) \leq d(x,y) + d(y,z).$$

The function d is called a distance function or a metric on X, and $d(x,y)$ is called the distance from x to y.

Elements of X are usually called points.

In words:

The first property says that the distance from a point to itself is 0, and to any other point is strictly greater than 0.

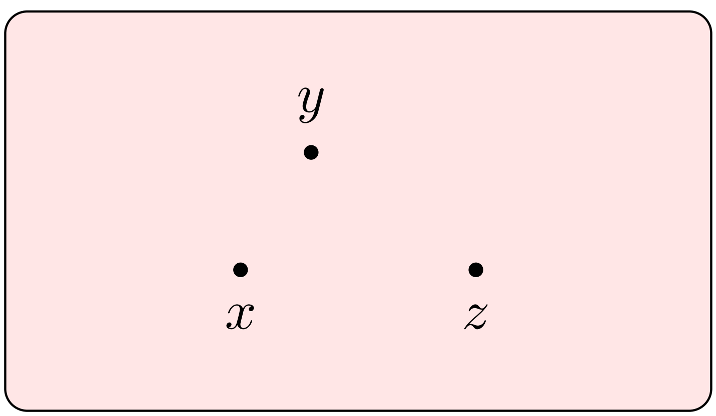
The second property says the distance function is symmetric.

The last property is referred to as the triangle inequality motivated from the fact that the sum of two sides of a triangle is greater than the third.

Thus, a metric space is a pair (X, d), where X is a set and d is a metric on X.

It is often convenient to keep d implicit and denote the metric space simply by X.

A metric space can be pictured as follows.



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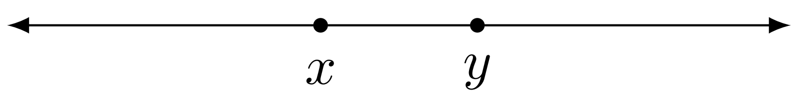
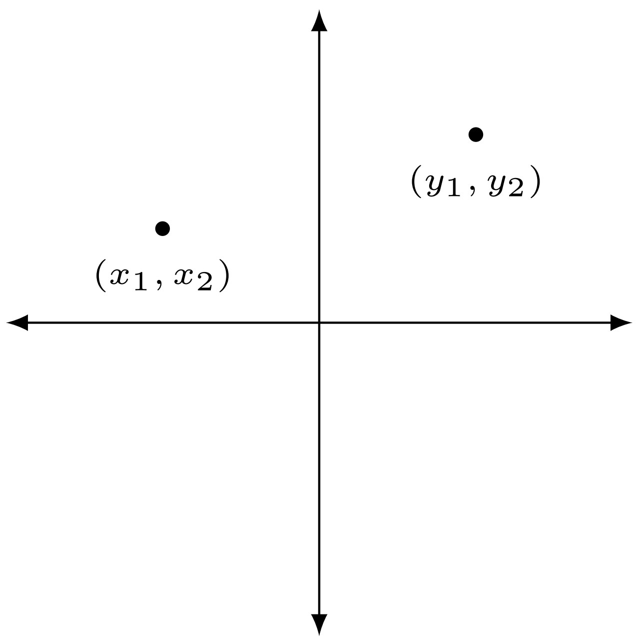
Example (Euclidean metrics). The set of real numbers $\mathbb{R}$ with $d(x, y) := |x - y|$ is a metric space.



Any subset of a metric space is a metric space with the induced metric.

For instance, the set of rational numbers $\mathbb{Q}$ with $d(x, y) := |x - y|$ is a metric space, the metric being induced from $\mathbb{R}$.

Consider $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, the cartesian product of $\mathbb{R}$ with itself.



How do we define the distance between points (x_1, x_2) and (y_1, y_2)?

There are many interesting choices for a metric on $\mathbb{R}^2$:

The euclidean metric

$$d_2((x_1, x_2), (y_1, y_2)) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2},$$

the diamond metric

$$d_1((x_1, x_2), (y_1, y_2)) := |x_1 - y_1| + |x_2 - y_2|,$$

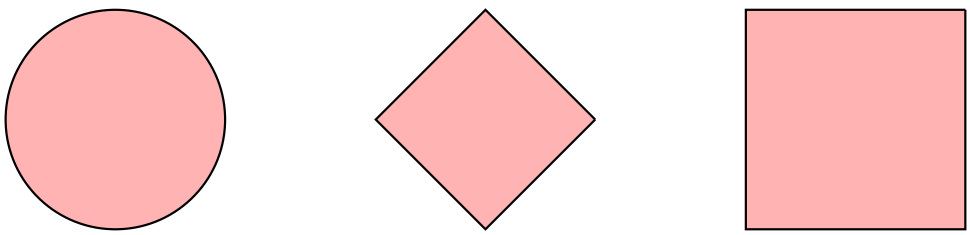
the square metric

$$d_\infty((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

are three of the most commonly considered metrics.

One can check that these are indeed metrics.

For these metrics, the set of points at distance 1 from the origin form a circle, diamond, square, respectively.



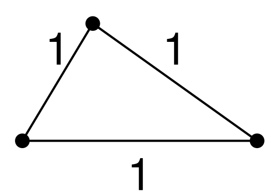
These definitions generalize in an obvious manner to $\mathbb{R}^n$, the n-fold cartesian product of $\mathbb{R}$.

There is an even more general context for these metrics where they are called l^2, l^1, l^∞, respectively.

Example (Discrete metric). Any set X can be equipped with the metric

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

An illustration is shown below.



This is called the discrete metric on X.

1.2 Continuity

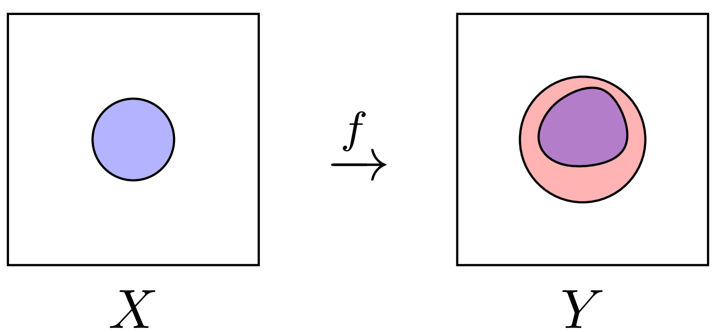
Suppose X and Y are metric spaces.

A function $f : X \to Y$ is continuous if for any point $x_0 \in X$, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \text{ implies } d(f(x), f(x_0)) < \epsilon.$$

The d on the left refers to the metric on X, while the one on the right refers to the metric on Y.

The condition says that points within distance δ of x_0 map to within distance ϵ of $f(x_0)$.



For $X = Y = \mathbb{R}$ with the usual metric, this definition agrees with the usual definition of continuous map from $\mathbb{R}$ to $\mathbb{R}$.

Let Metric denote the category whose objects are metric spaces, and whose morphisms are continuous maps.

The composition rule is given by usual composition of functions.

So we need to check that the composite of two continuous maps is continuous:

Lemma 1. *For metric spaces X, Y, Z, if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Proof. We show that $g \circ f$ is continuous at each $x_0 \in X$.

Let $\epsilon > 0$ be given.

By continuity of g at $f(x_0)$, there exists a $\gamma > 0$ such that

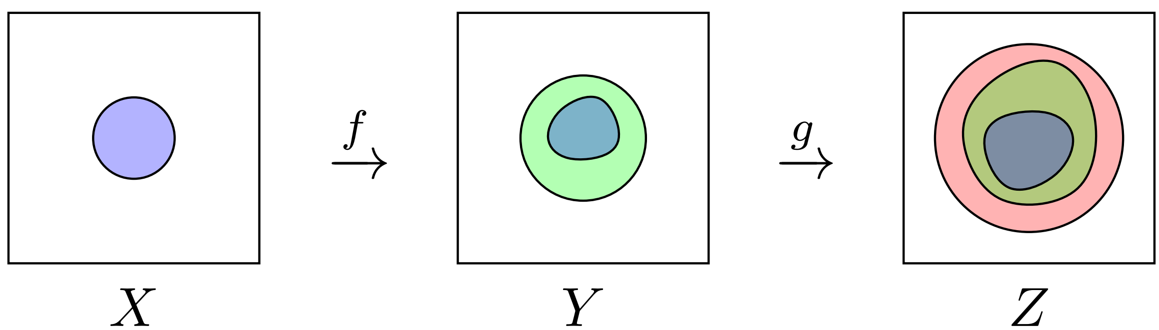
$$d(y, f(x_0)) < \gamma \text{ implies } d(g(y), g(f(x_0))) < \epsilon.$$

By continuity of f at x_0, there exists a $\delta > 0$ such that

$$d(x, x_0) < \delta \text{ implies } d(f(x), f(x_0)) < \gamma.$$

Combining the two yields the required implication.

The argument is illustrated below.



□

The identity map $X \to X$ is clearly continuous, and it serves as the identity morphism id_X in the category.

Since metric spaces are sets with structure, there is a forgetful functor $\mathsf{Metric} \to \mathsf{Set}$.

1.3 Uniform continuity

A function $f : X \to Y$ between metric spaces is uniformly continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \text{ implies } d(f(x), f(y)) < \epsilon.$$

It is clear that a uniformly continuous function is continuous, but the converse is not true in general.

The term uniform is used because given ϵ, a δ can be chosen independent of x_0.

Lemma 2. *The composite of two uniformly continuous maps is uniformly continuous.*

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous. We show that $g \circ f$ is uniformly continuous. Let $\epsilon > 0$ be given. By uniform continuity of g, there exists a $\gamma > 0$ such that

$$d(f(x), f(y)) < \gamma \text{ implies } d(g(f(x)), g(f(y))) < \epsilon.$$

Now by uniform continuity of f, there exists a $\delta > 0$ such that

$$d(x, y) < \delta \text{ implies } d(f(x), f(y)) < \gamma.$$

Combining the two yields the required implication. $\square$

Let Metric_u denote the category whose objects are metric spaces, and morphisms are uniformly continuous maps.

Since any uniformly continuous map is also continuous, Metric_u is a subcategory of Metric (in which we have kept all objects, but thrown away some morphisms).

Note: Being uniformly continuous is a property of a continuous map.

1.4 Lipschitz continuity

A function $f : X \to Y$ between metric spaces is Lipschitz continuous if there exists a $K > 0$ such that

$$d(f(x), f(y)) \leq K\, d(x, y)$$

for all $x, y \in X$.

If the above holds, then K is called a Lipschitz constant for f.

Observe that the composite of two Lipschitz continuous maps is Lipschitz continuous:

With notations as before, for all $x, y \in X$,

$$d(g(f(x)), g(f(y))) \leq K' \, d(f(x), f(y)) \leq KK' \, d(x, y),$$

where K is a Lipschitz constant for f, and K' is a Lipschitz constant for g.

Thus, we obtain the category Metric_L whose objects are metric spaces and morphisms are Lipschitz continuous maps.

1.5 Weak contraction

A function $f : X \to Y$ between metric spaces is a weak contraction or a short map or a nonexpansive map if

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in X$.

It is clear that the composite of two weak contractions is a weak contraction.

Thus, we obtain the category Metric_{wc} whose objects are metric spaces and morphisms are weak contractions.

A weak contraction is Lipschitz continuous, a Lipschitz continuous map is uniformly continuous, a uniformly continuous map is continuous.

In other words, we have functors

$$\mathsf{Metric}_{wc} \to \mathsf{Metric}_{L} \to \mathsf{Metric}_{u} \to \mathsf{Metric}.$$

All inclusions are proper, that is, there are continuous maps which are not uniformly continuous, and so on.

Inverse image of a function

Let $f : X \to Y$ be a function between sets X and Y.

Define for $A \subseteq X$ and for $B \subseteq Y$,

$$f(A) := \{y \in Y \mid f(x) = y \text{ for some } x \in A\}$$
$$f^{-1}(B) := \{x \in X \mid f(x) = y \text{ for some } y \in B\}.$$

(We emphasize that $f^{-1}(B)$ is a notation, and we are not assuming here that f is a bijection. This may be confusing, but this notation is standard.)

For any $B \subseteq Y$,

$$f^{-1}(B^c) \;=\; f^{-1}(B)^c.$$

For $\{B_j\}$ a nonempty family of subsets of Y,

$$f^{-1}(\bigcup_j B_j) \;=\; \bigcup_j f^{-1}(B_j),$$

$$f^{-1}(\bigcap_j B_j) \;=\; \bigcap_j f^{-1}(B_j).$$

For $f : X \to Y$ and $g : Y \to Z$ and $U \subseteq Z$,

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)).$$

2 Topological spaces

Roughly speaking, topological spaces are spaces in which there is a qualitative notion of closeness.

This is formalized through the notion of open sets.

This is in contrast to metric spaces where we have a quantitative notion of closeness.

2.1 Topological spaces

A topological space is a set X equipped with a collection τ of subsets of X (that is, τ is a subset of 2^X) such that the following axioms hold.

- $\emptyset$ and X are in τ.
- the union of elements in any subset of τ is in τ.
- the intersection of elements in any finite subset of τ is in τ.

The collection τ is called a topology on X, and its elements are called open sets.

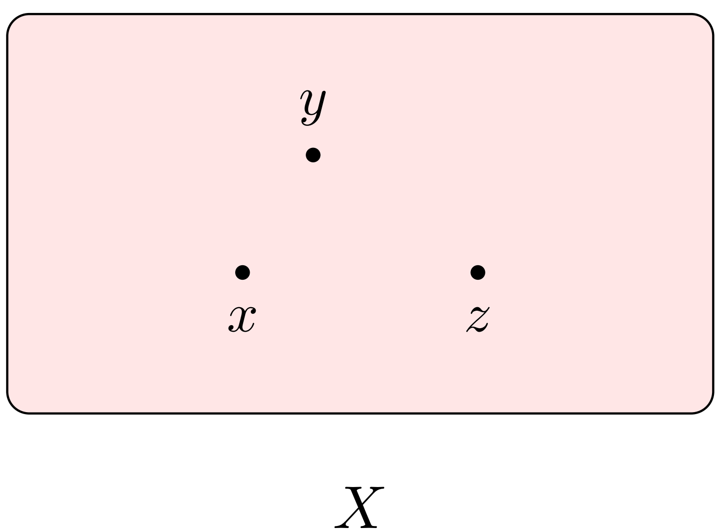
The above axioms can be rephrased using the language of open sets as follows.

- The empty set and the full space are open.
- Arbitrary union of open sets is open.
- Finite intersection of open sets is open.

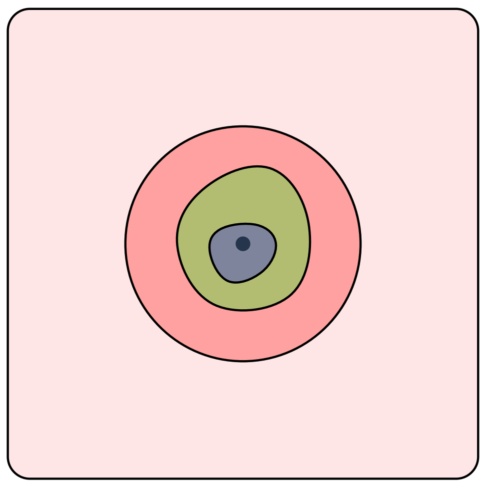
Thus, a topological space is a pair (X, τ) consisting of a set and a topology on it. It is often convenient to simply write X and keep the topology τ implicit.

This is similar to the practice for metric spaces where the metric d is kept implicit.

A topological space can be pictured similar to a metric space.



The difference is that instead of distance between points we now have open sets containing points.



The picture shows smaller and smaller open sets containing a point. These indicate how close we can get to that point.

Example. Let X be any set.

The collection of all subsets of X is a topology on X, that is, $\tau = 2^X$. It is called the discrete topology.

The collection $\tau = \{\emptyset, X\}$ is also a topology on X. It is called the indiscrete topology.

In general, given any subset S of 2^X, there is a smallest topology τ on X which contains S:

The discrete topology on X contains S.

Now take the (nonempty) intersection of all topologies which contains S.

Check that this is a topology on X.

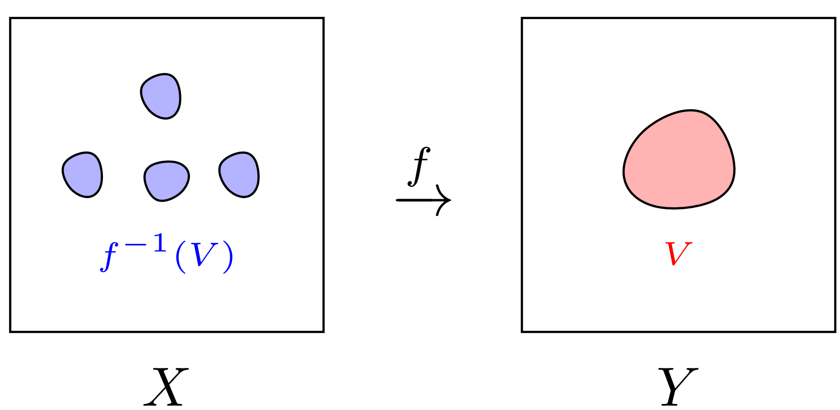
By construction, this is the smallest topology on X which contains S.

In this situation, τ is called the topology generated by S.

2.2 Continuity

Suppose X and Y are topological spaces.

A function $f : X \to Y$ is continuous if for any open set V of Y, $f^{-1}(V)$ is an open set of X.



Let Top denote the category whose objects are topological spaces, and whose morphisms are continuous maps.

The composition rule is given by usual composition of functions.

We check:

Lemma 3. *For topological spaces X, Y, Z, if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Proof. Suppose U is open in Z.

Since g is continuous, $g^{-1}(U)$ is open in Y.

Now since f is continuous, $f^{-1}(g^{-1}(U))$ is open in X.

But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, and the result follows. $\square$

By construction, we have a forgetful functor
$\mathsf{Top} \to \mathsf{Set}$.

Now, a topological space is a set with more structure, which is why we use the term forgetful functor.

However, continuity is a property of a function.

Hence this functor is faithful: distinct continuous maps have distinct underlying functions.

A continuous map $f : X \to Y$ between topological spaces is a homeomorphism if it is an isomorphism in the category Top.

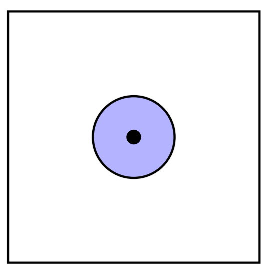
Explicitly, $f : X \to Y$ is a homeomorphism if there is a continuous map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

2.3 Underlying topology of a metric space

Let X be a metric space. The subset

$$B(x,r) := \{y \in X \mid d(x,y) < r\}$$

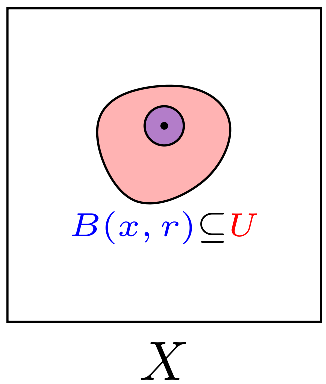
is called the open ball with center at x and radius r.



X

Note that if $r \leq s$, then $B(x,r) \subseteq B(x,s)$.

A subset U of X is open if for every $x \in U$, there exists a $r > 0$ such that $B(x, r) \subseteq U$.



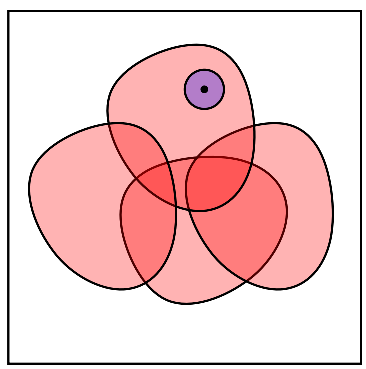
By the triangle inequality, one can see that any open ball is indeed open. This explains the terminology "open ball".

Lemma 4. *The collection of open sets of a metric space X defines a topology on X.*

Proof. We need to verify the axioms for a topology.

- Clearly, $\emptyset$ and X are open.
- Suppose $\{U_i\}_{i \in I}$ is an arbitrary collection of open sets. (The indexing set I may not be finite or even countable.) Then the union of the U_i is also open:

Suppose x is in this union. Then it must belong to at least one member of the collection. Say $x \in U_j$. By openness of U_j, there exists a $r > 0$ such that $B(x, r) \subseteq U_j$. Since the union is larger than U_j, the ball $B(x, r)$ is also a subset of the union.



- Similarly, argue for intersections. Why do we have to restrict to a finite indexing set now?

$\square$

Thus, to every metric space (X, d), one can associate the topological space (X, τ_d), where τ_d is the collection of open sets of (X, d).

We call τ_d the underlying topology of d.

Lemma 5. *Suppose X and Y are metric spaces and $f : X \to Y$ is a function. Then f is continuous iff for every open set V in Y, $f^{-1}(V)$ is an open set of X.*

Proof. Forward implication. Let V be an open set in Y. We want to show that $f^{-1}(V)$ is an open set of X. Take $x_0 \in f^{-1}(V)$. Hence $f(x_0) \in V$. Since V is open, choose $\epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq V$. Continuity at x_0 yields a $\delta > 0$ such that

$$B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon)) \subseteq f^{-1}(V).$$

Thus, $f^{-1}(V)$ is open in X.

Backward implication. We show that f is continuous at each $x_0 \in X$. We are given $\epsilon > 0$. Consider $B(f(x_0), \epsilon)$. This is an open set in Y. By hypothesis, its inverse image is open in X and contains x_0. Therefore, there exists a $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon))$. This is precisely the continuity condition. $\square$

Proposition 1. *There is a full and faithful functor*

$$\mathsf{Metric} \to \mathsf{Top}. \tag{1}$$

Proof. This is a consequence of Lemmas 4 and 5: The functor sends a metric space (X, d) to (X, τ_d), and a continuous function f between two metric spaces to f itself. Denoting this functor by $\mathcal{F}$, the map

$$\mathsf{Metric}(X, Y) \to \mathsf{Top}(\mathcal{F}(X), \mathcal{F}(Y)), \qquad f \mapsto f$$

is clearly a bijection. This is why $\mathcal{F}$ is full and faithful. $\square$

A topological space (X, τ) is called metrizable if there exists a metric d on X whose underlying topology is τ.

Thus Metric is equivalent to the full subcategory of metrizable topological spaces.

Example (Standard topology on euclidean space)**.** Let us begin with $\mathbb{R}$. It is a metric space under the usual metric. So it has an underlying topology.

A set is open in this topology if it can be written as a union of open intervals. In particular, any open interval is an open set.

In fact, any open interval is an instance of an open ball in the metric (with center at the midpoint of the interval).

Observe that

$$\bigcap_{n\geq 1} (\frac{-1}{n}, \frac{1}{n}) = \{0\}.$$

Now $\{0\}$ is not an open set.

Hence this identity shows that an arbitrary intersection (even countable intersection) of open sets is not open in general.

In general, $\mathbb{R}^n$ has an underlying topology wrt the euclidean metric.

We call it the standard topology on $\mathbb{R}^n$.

A set is open in this topology if it can be written as a union of open balls.

What happens if we use the diamond metric or square metric instead?

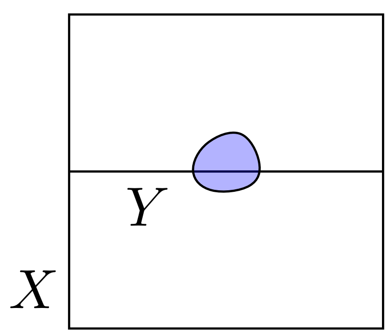
2.4 Subspace topology

Suppose (X, τ) is a topological space. Let Y be any subset of X. This can be viewed as an injective map

$$i : Y \hookrightarrow X.$$

We now put a topology on Y.

Call a subset V of Y open if there exists an open set U in X such that $V = U \cap Y$.



Let τ_Y denote the collection of these open sets. That is,

$$\tau_Y = \{V \subseteq Y \mid V = U \cap Y \text{ for some } U \in \tau\}.$$

We claim that τ_Y is a topology on Y. It is called the subspace topology.

Observe that the inclusion map i is continuous with this topology on Y.

- First, $\emptyset$ and Y are open since

$$\emptyset = \emptyset \cap Y \quad \text{and} \quad Y = X \cap Y,$$

and $\emptyset$ and X are open in X.

- To verify that arbitrary union of open sets is open, suppose that for each $i \in I$, we are given a $V_i \in \tau_Y$. Then $V_i = U_i \cap Y$ for some $U_i \in \tau$. Now

$$\bigcup_{i\in I} V_i = \bigcup_{i\in I} U_i \cap Y = \big(\bigcup_{i\in I} U_i\big) \cap Y.$$

Since τ is a topology, $\cup_{i\in I} U_i \in \tau$, and hence $\cup_{i\in I} V_i \in \tau_Y$ as required.

- Similarly, the identity

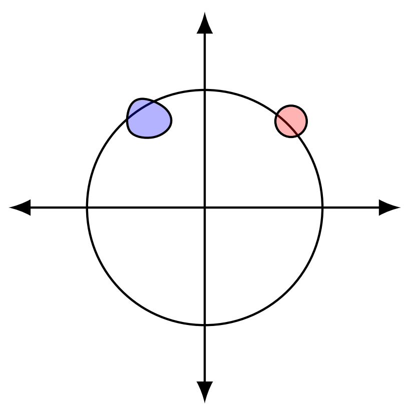
$$V_1 \cap \cdots \cap V_k = (U_1 \cap Y) \cap \cdots \cap (U_k \cap Y) = (U_1 \cap \cdots \cap U_k) \cap Y$$

shows that finite intersections of open sets is open.

Example (Spheres)**.** The unit circle is defined by

$$S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

It is a subset of $\mathbb{R}^2$. We view it as a topological space with the subspace topology.



More generally, the unit n-sphere defined by

$$S^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is a topological space with subspace topology inherited from $\mathbb{R}^{n+1}$.

Example. Any open interval (a, b) with $a < b$ is a topological space with subspace topology inherited from $\mathbb{R}$. By translation and scaling, we see that (a, b) is homeomorphic to $(-1, 1)$. Moreover,

$$(-1, 1) \to \mathbb{R}, \qquad y \mapsto \frac{y}{1 - |y|}$$

is an homeomorphism. The formula for the inverse is $z \mapsto \frac{z}{1+|z|}$. Thus, any open interval (a, b) with $a < b$ is homeomorphic to $\mathbb{R}$.

Similarly, e^x is a homeomorphism from $\mathbb{R}$ to $(0, \infty)$ with inverse given by $\log(x)$.

More generally: The open n-ball of radius r around x_0 is

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}.$$

We give it the subspace topology from $\mathbb{R}^n$.

By translation and scaling, we see that $B(x_0, r)$ and $B(0, 1)$ are homeomorphic. Further, the map

$$B(0,1) \to \mathbb{R}^n, \qquad y \mapsto \frac{y}{1 - \|y\|} \tag{2}$$

is an homeomorphism. The formula for the inverse is $z \mapsto \frac{z}{1+\|z\|}$.

2.5 Quotient topology

Suppose (X, τ) is a topological space. Let Y be a set and

$$p : X \twoheadrightarrow Y$$

a surjective map. This is the same as specifying an equivalence relation on X whose equivalence classes are elements of Y. We now put a topology on Y.

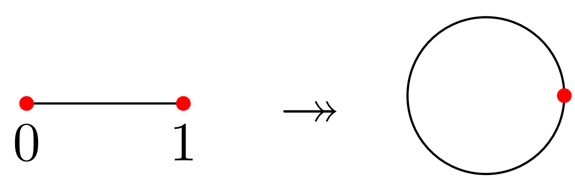
Call a subset V of Y open if $p^{-1}(V)$ is open in X. This is called the quotient topology.

Observe that the map p is continuous with this topology on Y.

Example (Circle)**.** Let $X = [0, 1]$ with subspace topology from $\mathbb{R}$. Now consider the quotient space Y obtained from X by identifying the points 0 and 1.

Can we identify Y with some known topological space?

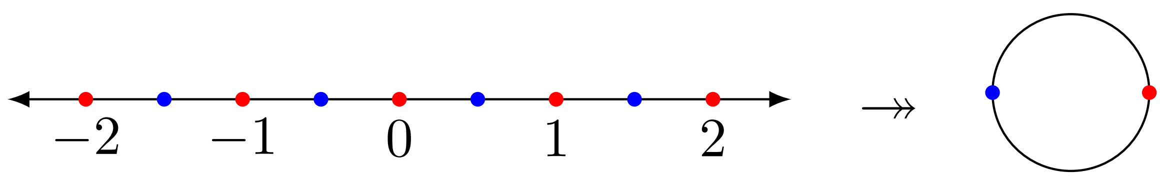
Yes, $Y = S^1$ with the quotient map illustrated below.



A related idea is to view the circle as a quotient space of the real line via the map

$$p : \mathbb{R} \twoheadrightarrow S^1, \qquad x \mapsto e^{2\pi i x}$$

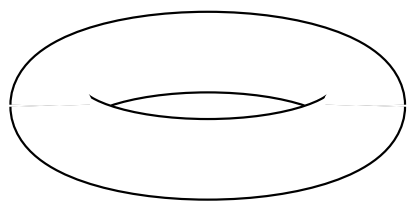
as illustrated below.



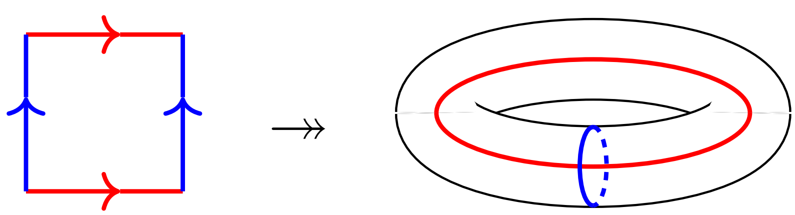
This is an example of what is called a covering map.

The group of integers $\mathbb{Z}$ acts on $\mathbb{R}$ by translation $m \cdot x := x + m$ and orbits of this action are precisely points of S^1.

Example (Torus)**.** Let $X = [0,1] \times [0,1]$ with subspace topology from $\mathbb{R}^2$. Now identify the two horizontal lines on the boundary $(x,0) \sim (x,1)$ and the two vertical lines on the boundary $(0,y) \sim (1,y)$. The resulting quotient space Y is called the torus.

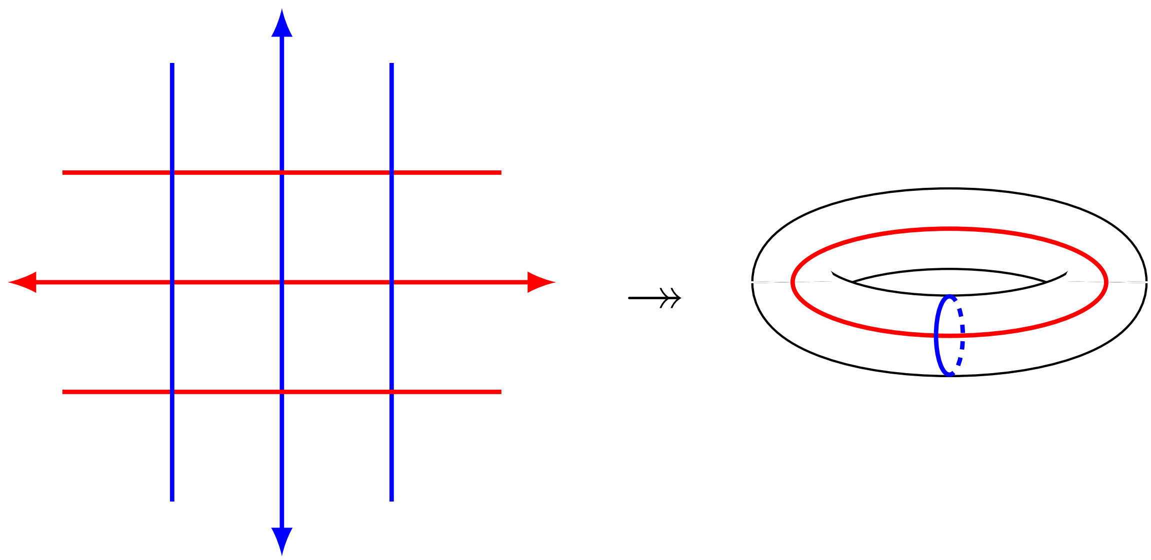


The quotient map is illustrated below.



The torus can also be interpreted as $S^1 \times S^1$.

Following the idea from the previous example, the torus can be viewed as a quotient of $\mathbb{R}^2$ in which we identify two points whenever they differ by an integer in either coordinate:

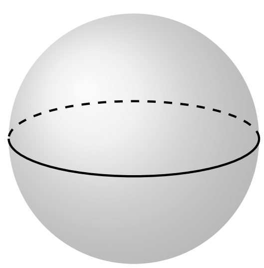


This is another example of a covering map.

The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translation $(m,n) \cdot (x,y) := (x+m, y+n)$ and orbits of this action are points of the torus.

Example (Real projective space)**.** Let $\mathbb{R}P^2$ denote the quotient of S^2 obtained by identifying antipodal points, that is, $x \sim -x$.

This is called the real projective plane.



$$\twoheadrightarrow \quad \mathbb{R}P^2$$

More generally: Recall the n-sphere S^n. Let $\mathbb{R}P^n$ denote the quotient of S^n obtained by identifying antipodal points.

This is called the n-dimensional real projective space.

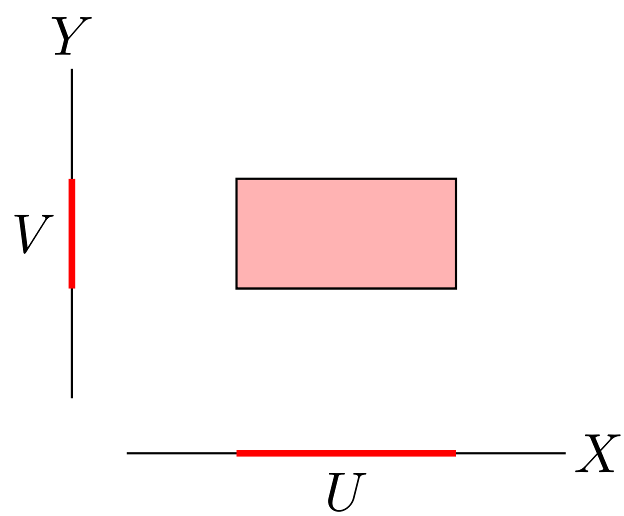
The quotient map $S^n \twoheadrightarrow \mathbb{R}P^n$ is also a covering map.

The group $\mathbb{Z}_2$ with two elements acts on S^n with the nontrivial element acting by $x \mapsto -x$ and orbits of this action are points of $\mathbb{R}P^n$.

2.6 Categorical product. Product topology

Let X and Y be topological spaces. We define a topology on the cartesian product $X \times Y$ as follows.

An open rectangle in $X \times Y$ is a subset of the form $U \times V$, where U is an open set in X and V is an open set in Y. An illustration is shown below.



Note that intersection of two open rectangles is again an open rectangle.

A subset W of $X \times Y$ is open if it can be written as the union of open rectangles.

In other words, for each point $w \in W$, there exists an open rectangle $U \times V$ such that $U \times V \subseteq W$ and $w \in U \times V$.

In particular, any open rectangle is an open set.

This defines a topology on $X \times Y$ called the product topology. Let us check the axioms.

- $\emptyset$ and $X \times Y$ are open rectangles, hence open.

- Suppose for each $i \in I$, W_i is open. Then each W_i is an union of open rectangles, and hence so is $\cup_{i \in I} W_i$. This shows that the latter is open.

- Suppose W and W' are open. Then write $W = \cup_i U_i \times V_i$ and $W' = \cup_j U'_j \times V'_j$. Hence

$$W \cap W' = \big(\bigcup_i U_i \times V_i\big) \cap \big(\bigcup_j U'_j \times V'_j\big) = \bigcup_{i,j} (U_i \cap U'_j) \times (V_i \cap V'_j)$$

is a union of open rectangles, and open.

The canonical projections

$$\pi_1 : X \times Y \to X \quad \text{and} \quad \pi_2 : X \times Y \to Y$$

are continuous:

For any open set U in X, $\pi_1^{-1}(U) = U \times Y$, which is an open rectangle and hence open in $X \times Y$.

This shows that π_1 is continuous. The argument for π_2 is similar.

Suppose Z is another topological space, and $f : Z \to X$ and $g : Z \to Y$ are continuous maps. Then we claim that

$$h : Z \to X \times Y, \qquad h(z) = (f(z), g(z))$$

is continuous. The main observation is that for any open rectangle

$$h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$$

is open in Z (being the intersection of two open sets).

In general, any open set in the product topology can be written as $W = \cup_i U_i \times V_i$, and

$$h^{-1}(W) = h^{-1}(\bigcup_i U_i \times V_i) = \bigcup_i h^{-1}(U_i \times V_i)$$

is open in Z (being the union of open sets).

Conclusion: $X \times Y$ is the categorical product in Top.

Example. Consider $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the product topology (with usual topology on $\mathbb{R}$). Does this coincide with the standard topology on $\mathbb{R}^2$?

Let

$$f : Z \to \mathbb{R}^2, \qquad f(z) = (f_1(z), f_2(z)).$$

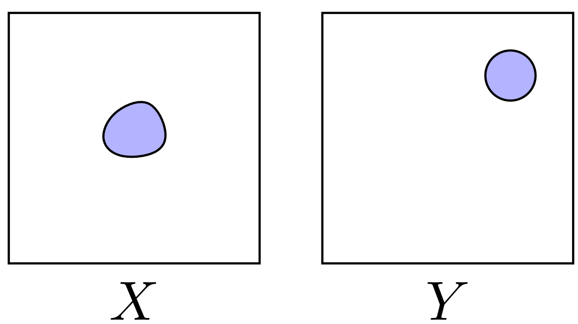
Then f is continuous iff f_1 and f_2 are continuous.

Does the product topology on the torus $S^1 \times S^1$ coincide with the topology that we have defined?

2.7 Categorical coproduct

For topological spaces X and Y, their categorical coproduct is given by their disjoint union $X \coprod Y$.

By definition, a subset in the disjoint union is open if its intersection with each piece is open.



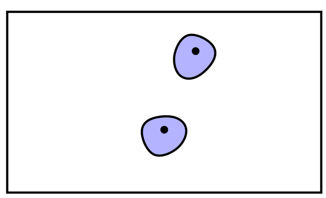
There are canonical inclusions $X \to X \coprod Y$ and $Y \to X \coprod Y$ which are both continuous.

For any topological space Z and continuous maps $f : X \to Z$ and $g : Y \to Z$, the map $f \coprod g : X \coprod Y \to Z$ is continuous.

2.8 Hausdorff topological spaces

A neighborhood of a point x in a topological space X is an open set in X containing x.

A topological space X is Hausdorff if for any distinct points $x, y \in X$, there exist neighborhoods of x and y which are disjoint (that is, their intersection is empty).



This is a property of a topological space.

It says that distinct points can be separated by neighborhoods.

Example. Metric spaces are Hausdorff: Given $x, y \in X$, the balls $B(x, r)$ and $B(y, r)$ do not intersect whenever $0 < r < \frac{1}{2}d(x, y)$. This follows from the triangle inequality.

The indiscrete topology on a set with at least two points is not Hausdorff (since any open set containing a point is necessarily the full space).

3 References

The standard reference for topology is:

1. James R. Munkres, Topology, Prentice Hall, Inc., Upper Saddle River, NJ, 2000

More books are listed in my notes on Real Analysis.