Smooth Manifolds

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1 Topological spaces

1.1 Topological spaces

A topological space X is a set with a qualitative notion of closeness which is formalized by the language of open sets.

(In a metric space, the notion of closeness is quantitative.)

A function $f : X \to Y$ between topological spaces is continuous if it maps points close to X to points close to Y, or more precisely, if inverse image of any open set in Y is open in X.

As examples,

$$\mathbb{R}, \quad S^1, \quad \mathbb{R}^2, \quad \mathbb{R}^2 \setminus \{0\}$$

are topological spaces, and

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2, \qquad f : \mathbb{R} \to S^1,\ f(x) = e^{2\pi i x}$$

are continuous maps.

Two topological spaces X and Y are homeomorphic if there is a continuous map $f : X \to Y$ which is a bijection and whose inverse is also continuous.

For instance, e^x is a homeomorphism from $\mathbb{R}$ to $(0, \infty)$ with inverse given by $\log(x)$.

1.2 Categories of interest

Informally, the category of topological spaces consists of

- all topological spaces $X, Y, \ldots$,
- all continuous maps $X \to Y$ for any topological spaces X and Y.

We denote this category by Top.

Formally, a category consists of objects and morphisms. There is a notion of isomorphism of objects in any category.

In the category of topological spaces, objects are topological spaces and morphisms are continuous maps. An isomorphism in this category is precisely a homeomorphism.

We are interested in the following categories.

- topological spaces (where we can talk of closeness)
- topological manifolds (which look locally like euclidean spaces)
- smooth manifolds (where we can talk of tangent spaces)
- riemannian manifolds (where we can talk of geodesics and curvature)

Each category is obtained from the previous one by imposing either more conditions or more structure.

The category of topological spaces is the most basic.

Note that there is a more basic category, namely that of sets, which underlies topological spaces.

In the category of sets, objects are sets (possibly infinite) and morphisms are ordinary functions.

So for two topological spaces to be homeomorphic, their underlying sets must be bijective.

1.3 Review of standard definitions

Let X be a topological space.

- X is Hausdorff if for any distinct points $x, y \in X$, there exist neighborhoods of x and y which are disjoint. (Recall that a neighborhood of a point x in X is an open set U of X containing x.)
- X is compact if every open cover of X has a finite subcover.
- X is second countable if it has a countable basis. (That is, X has a countable collection of open sets with the property that each open set of X is a union of elements in this collection.)

Connected and path-connected spaces

A topological space X is

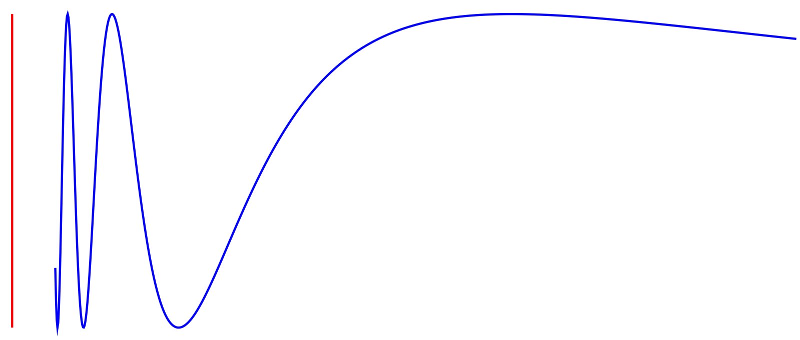
- connected if it cannot be written as a disjoint union of two nonempty open sets,
- path-connected if there is a path joining any two points of X. (A path from x to y is a continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.)

Recall that path-connected implies connected. The converse is not true in general.

Example (The topologist's sine curve)**.** Consider the set

$$\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ y = \sin(1/x)\} \\ \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$$

with subspace topology from $\mathbb{R}^2$.



This is the topologist's sine curve.

It is connected but not path-connected.

Locally euclidean spaces

A topological space X is locally euclidean if every point of X has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n.

The open n-ball of radius r around x_0 is

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}.$$

By translation and scaling, $B(x_0, r)$ and $B(0, 1)$ are homeomorphic. Further, the map

$$B(0,1) \to \mathbb{R}^n, \qquad y \mapsto \frac{y}{1 - \|y\|}$$

is a homeomorphism with inverse $z \mapsto \frac{z}{1+\|z\|}$.

Thus X is locally euclidean iff every point of X has a neighborhood homeomorphic to an open n-ball for some n.

Proposition 1. *A point in a topological space can never have two neighborhoods, one homeomorphic to* $\mathbb{R}^n$ *and the other to* $\mathbb{R}^m$ *for* $m \neq n$.

Equivalently, an open subset of $\mathbb{R}^n$ *cannot be homeomorphic to an open subset of* $\mathbb{R}^m$ *for* $m \neq n$.

The previous result is a consequence of the invariance of domain theorem of Brouwer:

Theorem 1. *If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $f(U)$ is open and f induces a homeomorphism between U and $f(U)$.*

2 Topological manifolds

We now look at a class of topological spaces which locally look like euclidean spaces.

These are called topological manifolds.

They bring us a step closer to objects on which we can do geometry.

2.1 Topological manifolds

Definition 2. A topological manifold is a topological space M which is second countable, Hausdorff and locally euclidean.

A morphism $M \to N$ between topological manifolds is a continuous map from M to N.

This defines the category of topological manifolds. We denote it by $\mathsf{TopManifold}$.

The category of topological manifolds is obtained from the category of topological spaces by imposing conditions on the object (morphisms have not changed):

A topological space is either a topological manifold or not. The formal way to say this is:

The category of topological manifolds is a full subcategory of the category of topological spaces.

Note that the notion of isomorphism in this category is homeomorphism.

We say that a topological manifold has dimension n if it is locally n-euclidean at all points. That is, every point of X has a neighborhood homeomorphic to $\mathbb{R}^n$.

In this situation, we talk of a topological n-manifold.

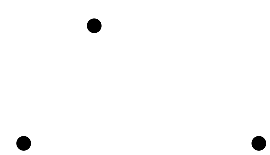
Note that a connected topological manifold always has a dimension.

Also observe that any open set in a topological n-manifold is a topological n-manifold (with the subspace topology).

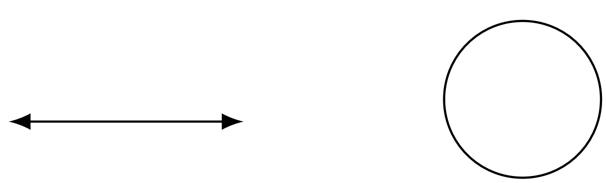
2.2 Examples

Here are some examples of topological manifolds listed according to dimension.

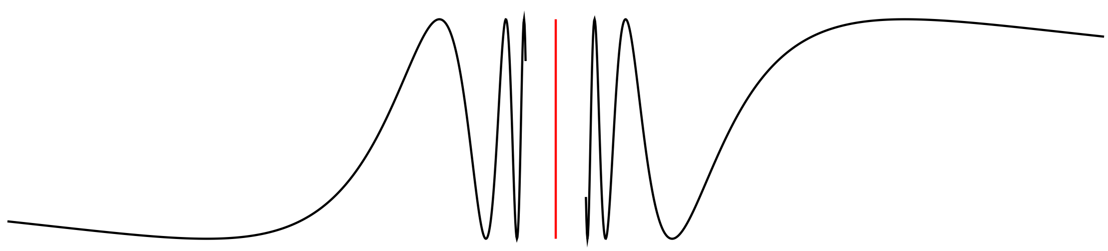
- 0 : a point, or more generally, any discrete countable set of points.



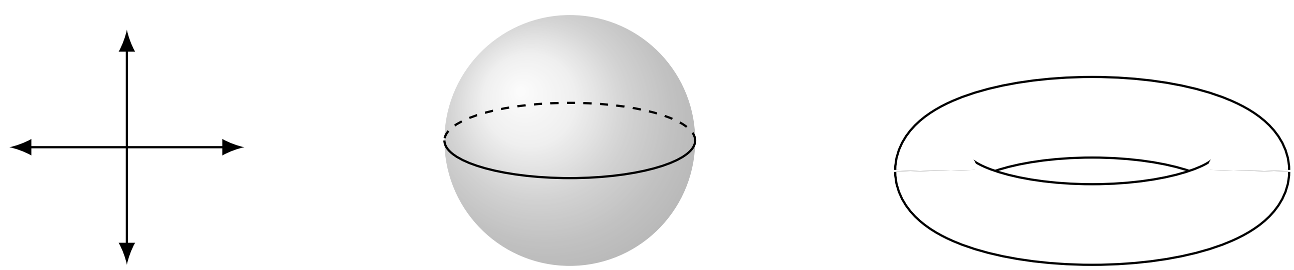
- 1 : $\mathbb{R}^1$ (also called a line), an open interval, circle, countable disjoint unions of any of these. (An open interval is homeomorphic to the line, so it need not be listed separately.)



A somewhat wild example (with two connected components) is the set $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0$ and $y = \sin(1/x)\}$.



- 2 : $\mathbb{R}^2$ (also called a plane), any open set in the plane, sphere, torus, or more generally the surface with k holes, the real projective plane, the Klein bottle.



A surface is a compact 2-dimensional topological manifold. Surfaces have been completely classified.

- 3: $\mathbb{R}^3$, the three dimensional torus $S^1 \times S^1 \times S^1$, any open set in $\mathbb{R}^3$, for instance, the complement of a knot.

  Are the complement of the unknot and the trefoil homeomorphic?

Some general higher-dimensional examples to keep in mind are $\mathbb{R}^n$, S^n, the n-dimensional torus which is the n-fold product $S^1 \times \cdots \times S^1$.

Example (Covering spaces)**.** Let $X \twoheadrightarrow Y$ be a finite-sheeted covering map. In this situation, if one of them is a topological manifold, then so is the other.

For example, the map $S^n \twoheadrightarrow \mathbb{R}P^n$ which identifies the antipodal points on S^n is a double cover, that is, a two-sheeted covering map. So it follows that $\mathbb{R}P^n$ is a topological n-manifold. This is the n-dimensional real projective space.

It is fairly common to have nice examples of countably infinite-sheeted covering spaces.

For instance, consider the covering maps

$$f : \mathbb{R} \twoheadrightarrow S^1, \quad f(x) = e^{2\pi i x}$$

or

$$f : \mathbb{C} \twoheadrightarrow \mathbb{C} \setminus \{0\}, \quad f(z) = e^z.$$

All spaces involved are topological manifolds.

2.3 Products and coproducts

Let us discuss the initial object, terminal object, product and coproduct in the category of topological manifolds.

- Initial object: The empty set $\emptyset$ is a topological manifold. By convention, it has dimension -1. There is a unique continuous map from $\emptyset$ to any topological manifold. So it is an initial object.

- Terminal object: There is a unique continuous map from every topological manifold to any one-point space. So any one-point space is a terminal object. They are all homeomorphic. A one-point space is a topological manifold of dimension 0.

- Product: The cartesian product $M \times N$ of a topological n-manifold M and a topological m-manifold N is a topological $(n+m)$-manifold under the product topology.

  There are canonical projections $M \times N \to M$ and $M \times N \to N$.

  For instance, $S^1 \times S^1$ (torus), $S^1 \times \mathbb{R}$ (infinite cylinder), $S^2 \times \mathbb{R}$, $S^1 \times S^2$, and so on.

- Coproduct: The disjoint union $M \coprod N$ is the coproduct of M and N.

  There are canonical inclusions $M \to M \coprod N$ and $N \to M \coprod N$.

  For instance, the disjoint union of a 2-sphere and a line in $\mathbb{R}^3$.

  Note that the connected components of a topological manifold can have different dimensions.

Proposition 2. *If a topological manifold is connected, then it is also path-connected.*

Proof. Consider the path components of the topological manifold. These are equivalence classes of the relation: $x \sim y$ if there is a path joining x and y.

Since a topological manifold is locally euclidean, each path component is open.

If there is more than one path component, then any one of them and the union of the remaining would disconnect the space. (Note that the union of the remaining is an open set.)

So there is only one path component, that is, the topological manifold is path-connected. $\square$

Proposition 3. *A continuous bijection between topological manifolds of the same dimension is a homeomorphism. (That is, the inverse is necessarily continuous.)*

This can be proved using the invariance of domain Theorem 1.

In this regard, recall that a continuous bijection between topological spaces may not be a homeomorphism.

We state another important result.

Theorem 3. *A compact topological manifold can be embedded in euclidean space, that is, it is homeomorphic to a subspace of euclidean space.*

For instance, S^1 can be embedded in $\mathbb{R}^2$. Similarly, orientable surfaces can be viewed as subspaces of $\mathbb{R}^3$. These embeddings provide a concrete way to visualize these manifolds.

The real projective plane or the Klein bottle are nonorientable surfaces which embed in $\mathbb{R}^4$ but not in $\mathbb{R}^3$. So it is not as easy to visualize them.

3 Smooth manifolds

Smooth manifolds provide a nice class of objects to do differential calculus.

The smooth manifolds that one studies in a multivariable calculus class are usually open subsets of euclidean spaces.

A general smooth manifold is built by gluing such euclidean subsets.

3.1 Smooth maps between euclidean spaces

Let U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$.

A map $f : U \to V$ is smooth if partial derivatives of f of all orders exist and are continuous.

A smooth function on U is a smooth map $U \to \mathbb{R}$.

Note that composite of smooth maps is again smooth.

Proposition 4. *Let U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$, and let $f : U \to V$ be any map. Then the following are equivalent.*

1. *f is a smooth map.*
2. *f has the form*

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

   where each f_i is a smooth function on U.

3. *For every smooth function g on V, the composite $g \circ f$ is a smooth function on U.*

Proof. (1) $\iff$ (2). Clear from the definition.

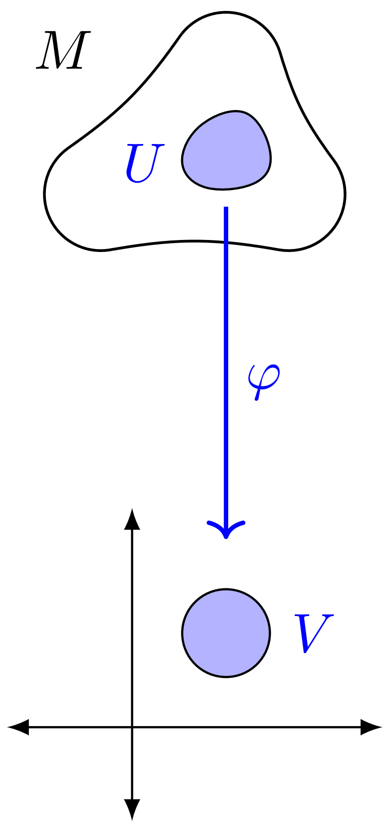
(1) $\implies$ (3). Use the fact that composite of smooth maps is smooth.

(3) $\implies$ (2). Take g to be any coordinate function on $\mathbb{R}^m$ (which is clearly smooth). Then $g \circ f$ is one of the f_i. So the f_i are all smooth. $\square$

3.2 Smooth manifolds

Let M be a topological n-manifold.

A chart on M is a pair (U, φ), where U is an open set in M and $\varphi : U \to V$ is a homeomorphism with V being an open set in $\mathbb{R}^n$.



Suppose (U, φ) is a chart and U' is an open set contained in U.

Then restricting φ to U' yields a chart (U', φ').

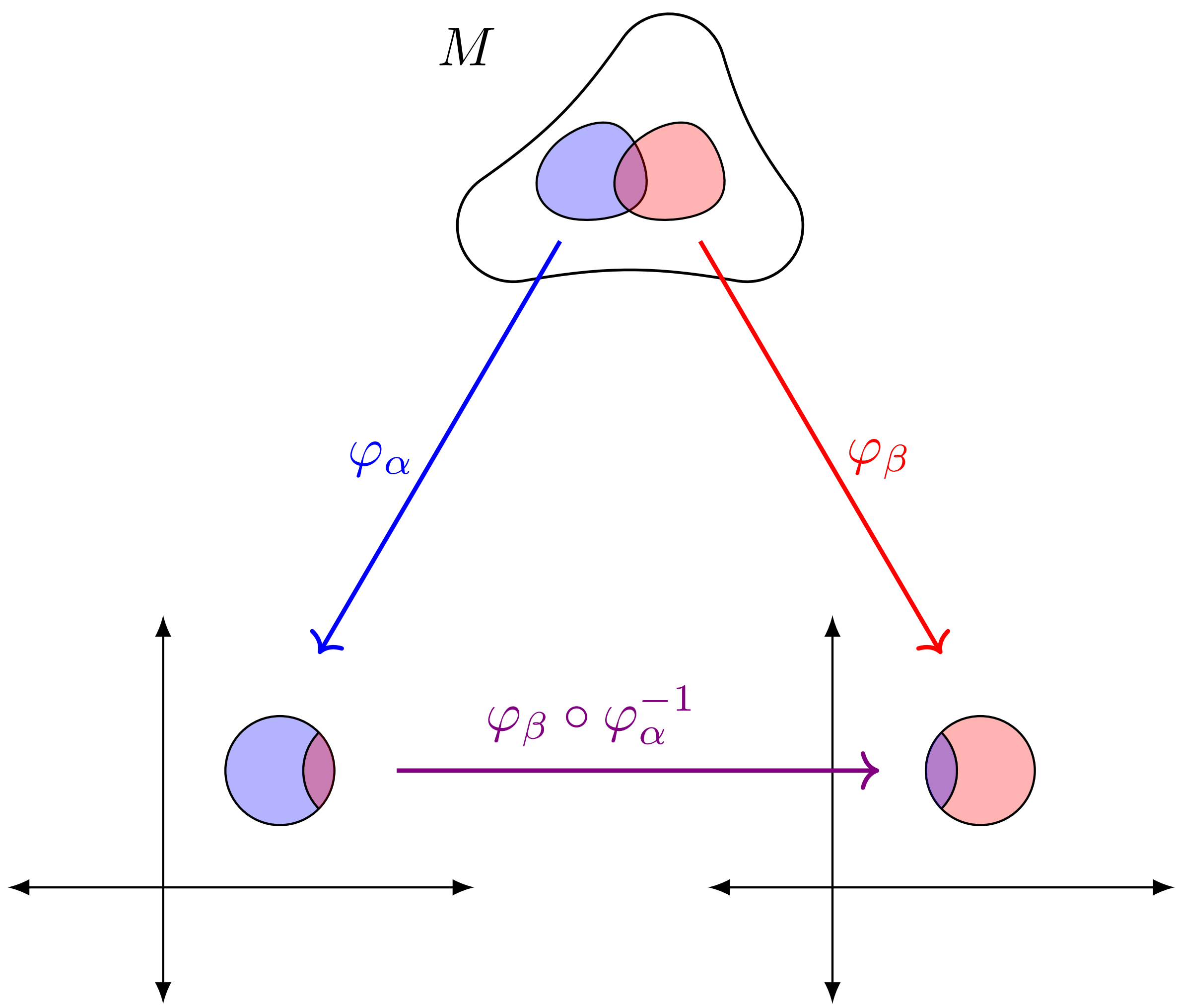
Since φ is a homeomorphism, it will take open sets to open sets, so the image of U' will be an open set V' in $\mathbb{R}^n$.

An atlas or a smooth structure on M is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M such that the U_α cover M and they are mutually compatible:

For any pair of indices α and β,

$$f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a smooth map (from an open set of $\mathbb{R}^n$ to another open set of $\mathbb{R}^n$).


M
φ_α
φ_β
$\varphi_\beta \circ \varphi_\alpha^{-1}$

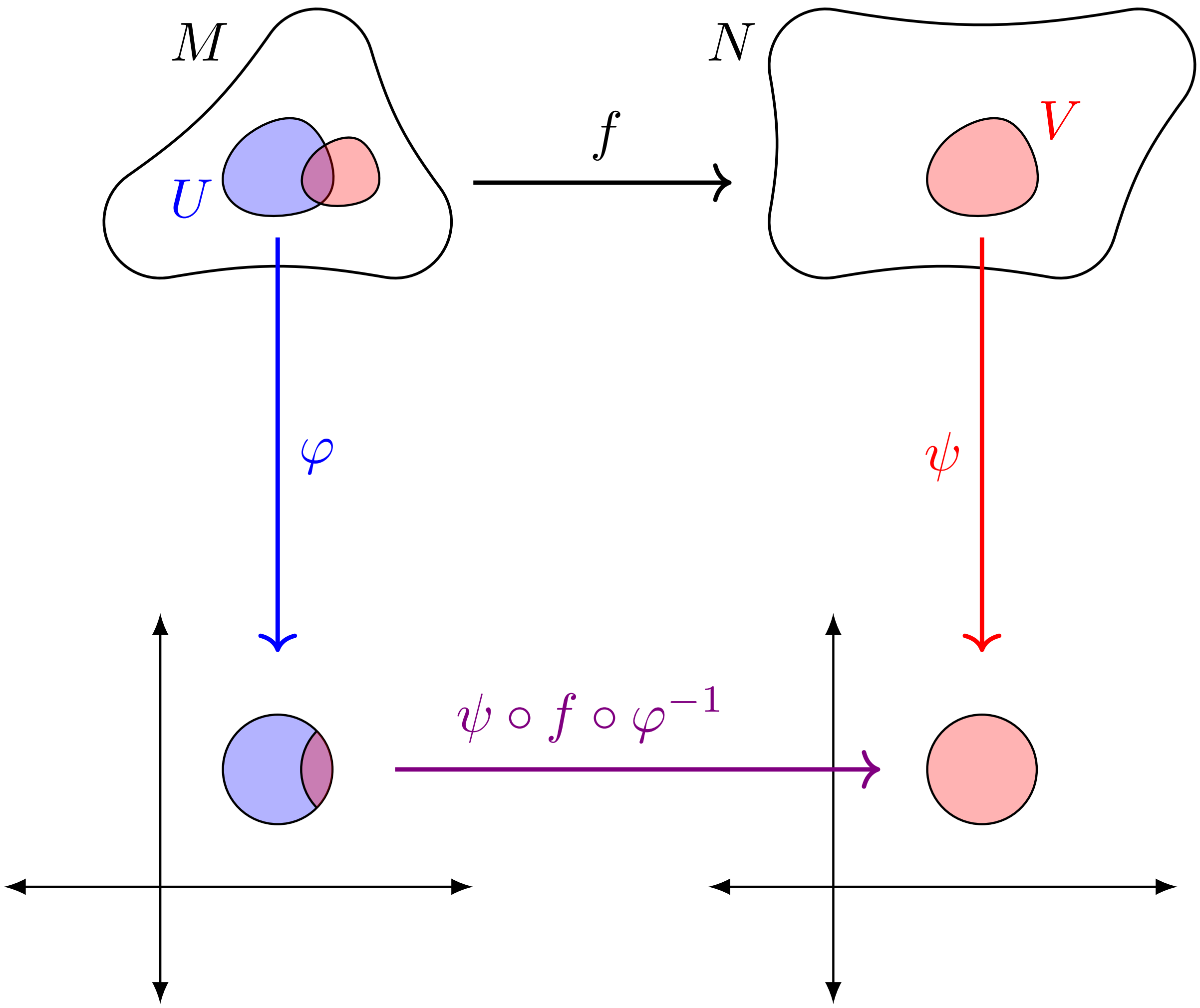
If the connected components of M have different dimensions, then an atlas on M is defined to be an atlas on each of its pure pieces.

Definition 4. A smooth manifold is a topological manifold M equipped with an atlas.

A smooth map $f : M \to N$ between smooth manifolds (not necessarily of the same dimension) is a continuous map $f : M \to N$ such that for any chart (U, φ) on M and (V, ψ) on N, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth.


M
N
f
U
V
φ
ψ
ψ ∘ f ∘ φ⁻¹

Check that the composite of smooth maps is smooth: If $f : M \to N$ and $g : N \to P$ are smooth, then so is $g \circ f : M \to P$.

This defines the category of smooth manifolds: objects are smooth maniolds and morphisms are smooth maps.

We denote this category by $\mathsf{Manifold}$.

The notion of isomorphism in this category is called a diffeomorphism. Explicitly, a smooth map $f : M \to N$ is a diffeomorphism if it is a homeomorphism, and f^{-1} is smooth.

Proposition 5. *Any open set V of a smooth n-manifold M is also a smooth n-manifold.*

Proof. For the smooth structure on V, we take charts obtained by intersecting the charts of M with V. This is alright because charts can be restricted. Since charts of M cover M, their restrictions to V cover V, so we indeed have an atlas on V. $\square$

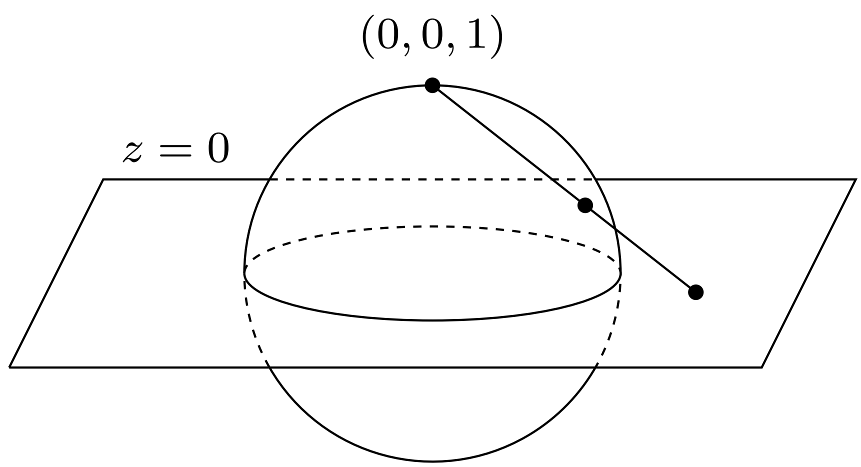
3.3 Examples

The examples of topological manifolds that we considered can be turned into smooth manifolds. We explain some of them.

Example (Euclidean spaces)**.** $\mathbb{R}^n$ with the atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id})$ is a smooth n-manifold.

Similarly any open set in $\mathbb{R}^n$ is a smooth n-manifold.

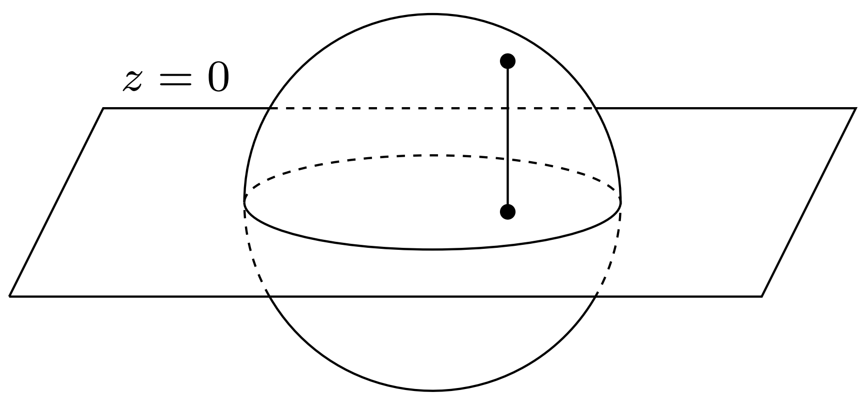
Example (Spheres). The n-sphere S^n with the atlas consisting of two charts (S^n minus the north pole, stereographic projection from the north pole) and (S^n minus the south pole, stereographic projection from the south pole) is a smooth n-manifold.



To check this, write down formulas for the stereographic projections and check that the change of coordinates is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$.

Another method to construct an atlas on S^n is as follows. We explain the case $n = 2$.

For the open set $z > 0$ on the sphere, we take the projection $\varphi(x, y, z) = (x, y)$. The inverse map is $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$.



Use similar charts for the open sets $z < 0$, $x > 0$, $x < 0$, $y > 0$, $y < 0$.

These six charts define an atlas.

Call the first atlas $\mathcal{A}_1$ and the second atlas $\mathcal{A}_2$. Then the identity map $(S^n, \mathcal{A}_1)$ and $(S^n, \mathcal{A}_2)$ is a diffeomorphism.

This is equivalent to saying that the charts from the two atlases are also compatible with each other. In this sense, it does not matter which atlas we use.

We can also combine the charts from the two atlases in different ways to construct other atlases but that will not change the diffeomorphism class.

The canonical inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map.

3.4 Products and coproducts

The product and coproduct works in a manner similar to the category of topological manifolds. The additional ingredient is to see how charts work.

- Initial object: The empty set is a smooth manifold (of dimension -1 by convention). It is the initial object in the category.
- Terminal object: The one-point space is a 0-dimensional smooth manifold. It is the terminal object in the category.

- Product: If $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are two smooth manifolds, then so is

  $$(M_1 \times M_2, \mathcal{A}_1 \times \mathcal{A}_2),$$

  where

  $$\mathcal{A}_1 \times \mathcal{A}_2 = \{(U_1 \times U_2, \varphi_1 \times \varphi_2) \mid (U_i, \varphi_i) \in \mathcal{A}_i\}.$$

  This is the product in the category.

- Coproduct: Similarly, if $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are two smooth manifolds, then so is

$$(M_1 \coprod M_2, \mathcal{A}_1 \coprod \mathcal{A}_2),$$

where $\mathcal{A}_1 \coprod \mathcal{A}_2$ is the disjoint union of the two atlases.

This is the coproduct in the category.

3.5 How many smooth structures?

A smooth manifold is obtained from a topological manifold by imposing more structure.

So the question here is not whether a topological manifold is a smooth manifold.

Rather it is: How many different smooth structures does a topological manifold carry?

Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are called compatible if all charts in $\mathcal{A}_1$ are compatible with all charts in $\mathcal{A}_2$ (in either order), or equivalently, if the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also an atlas.

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible iff the identity map $(M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$ is a diffeomorphism. (We saw this in the example of S^n above.)

So given an atlas $\mathcal{A}$ on M, one can pass to the maximal atlas compatible with $\mathcal{A}$ by adding all charts compatible with the charts in $\mathcal{A}$, and this does not change the diffeomorphism class of M.

We state some results below about the existence and uniqueness of smooth structures.

Proposition 6. *If $n \leq 3$, then every topological n-manifold possesses a smooth structure. Further, this structure is unique up to diffeomorphism.*

As a concrete example, consider $\mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. This defines a smooth structure on $\mathbb{R}$ consisting of a single chart $(\mathbb{R}, \varphi)$. This chart will be compatible with $(\mathbb{R}, \mathrm{id})$ iff φ is a diffeomorphism. (It is very easy to construct homeomorphisms $\mathbb{R} \to \mathbb{R}$ which are not diffeomorphisms.)

The smooth structures defined by the charts $(\mathbb{R}, \varphi)$ and $(\mathbb{R}, \mathrm{id})$ are diffeomorphic. Why?

Proposition 7. *If $n \geq 4$, then there exists a topological n-manifold with no smooth structure, and also a topological n-manifold with non-diffeomorphic smooth structures.*

Interesting fact: $\mathbb{R}^n$ has only one smooth structure up to diffeomorphism if $n \neq 4$, while $\mathbb{R}^4$ has uncountably many different (non-diffeomorphic) smooth structures.

3.6 Algebra of smooth functions

Let M be a smooth manifold.

A smooth function on M is a smooth map $M \to \mathbb{R}$.

Explicitly, $f : M \to \mathbb{R}$ is smooth if for every chart (U, φ), the composite

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$$

is smooth.

We denote the set of smooth functions on M by $C(M)$.

Proposition 8. *If f and g are smooth functions on M, then so are $f + g$, fg and cf for any real number c.*

Proof. We can use the argument given in multivariable calculus. Suppose f and g are smooth. Then

$$M \xrightarrow{(f,g)} \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a composite of smooth maps, hence smooth. (The second map could be addition or multiplication.) $\square$

This says that $C(M)$ is a commutative $\mathbb{R}$-algebra.

Proposition 9. *Suppose M and N are smooth manifolds, and $f : M \to N$ is any map. Then*

f is smooth iff

For every $g \in C(N)$, $g \circ f \in C(M)$.

Proof. This follows from Proposition 4. $\square$

The association of $C(M)$ to M is functorial:

If $f : M \to N$ is a smooth map, then there is an induced algebra morphism $C(N) \to C(M)$ which sends $h : N \to \mathbb{R}$ to $h \circ f : M \to \mathbb{R}$.

Thus we have a contravariant functor

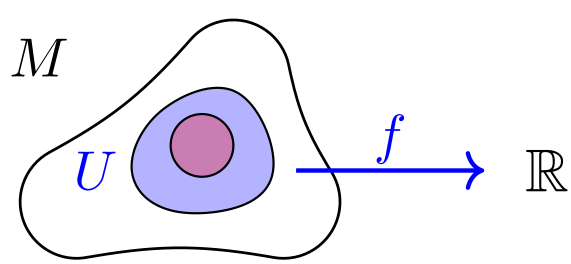
$$C : \mathsf{Manifold} \to \mathsf{Alg}^{\mathsf{co}}$$

from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras.

Lemma 1. *Let U be an open set of a smooth manifold M. Let $f : U \to \mathbb{R}$ be a smooth map. Then there exists a nonempty open set V whose closure is contained in U and a smooth function g on M such that*

$$g(p) = \begin{cases} f(p) & \text{if } p \in V, \\ 0 & \text{if } p \notin U. \end{cases}$$

Further, the same V can be chosen for each f.



Lemma 2. *Let M be a smooth manifold. Any point $p \in M$ determines an algebra morphism*

$$C(M) \to \mathbb{R}, \qquad f \mapsto f(p).$$

Conversely, every algebra morphism from $C(M)$ to $\mathbb{R}$ arises in this manner.

The first part is clear.

The second part requires some work.

Theorem 5. *The functor C from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras is full and faithful.*

That is, for any smooth manifolds M and N, the canonical map

$$\mathsf{Manifold}(M, N) \to \mathsf{Alg}(C(N), C(M))$$

is a bijection.

This implies that $C(M)$ determines M up to diffeomorphism.

It follows that the category of smooth manifolds is equivalent to a full subcategory of the category of commutative $\mathbb{R}$-algebras.

Proof. For injective:

Suppose $f, g : M \to N$ with $f(p) \neq g(p)$ for some $p \in M$.

Pick a smooth function h on N which is 1 at $f(p)$ and 0 at $g(p)$. This is possible by Lemma 1.

Then $h \circ f \neq h \circ g$ since their values differ at p.

So f and g do induce different algebra morphisms $C(N) \to C(M)$.

For surjective: Observe that the case when M is a point and N is arbitrary follows from Lemma 2.

Now let M be arbitrary. Suppose we are given $\varphi : C(N) \to C(M)$. Let $p \in M$. Evaluation at p yields a morphism $C(M) \to \mathbb{R}$. Precomposing by φ yields a morphism $C(N) \to \mathbb{R}$. By Lemma 2, this must be evaluation at some $q \in N$.

Define $f : M \to N$ by $f(p) = q$. It follows that f induces φ.

Further from Proposition 9, f is smooth. $\square$

4 Submanifolds

5 Tangent bundle

6 Cotangent bundle

7 Tensors