Connections and curvature

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1 Covariant derivatives wrt vector fields

Recall that one can differentiate a smooth function f wrt a vector field X on a manifold M. It yields the smooth function that we have denoted $X(f)$.

In local coordinates, this is the directional derivative.

Now a smooth function f on M is the same as a smooth section of the trivial line bundle on M.

So a natural question arises:

Can one differentiate a smooth section of any vector bundle E wrt a vector field X on M?

The answer in general is no.

To make this idea work, we need to postulate it as an extra requirement on E.

This is the notion of a covariant derivative.

It was introduced by Koszul around 1950.

Definition 1. Let $\pi : E \to M$ be a smooth vector bundle. A covariant derivative on E is a family of maps

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$

one for each smooth vector field X on M, such that

$$\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s), \tag{1a}$$

$$\nabla_{fX}(s) = f\nabla_X(s), \tag{1b}$$

$$\nabla_X(s_1 + s_2) = \nabla_X(s_1) + \nabla_X(s_2), \tag{1c}$$

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s. \tag{1d}$$

We call $\nabla_X(s)$ the covariant derivative of the section s wrt the vector field X.

Note very carefully that ∇_X is not $C(M)$-linear due to the extra term $X(f)s$ in (1d). However, it is $\mathbb{R}$-linear.

Let (A, M) be an object of the category AlgMod.

Explicitly, A is an algebra and M is an A-module.

A derivation of (A, M) is a pair (D, D') such that $D : A \to A$ is a derivation of the algebra A, and $D' : M \to M$ is a linear map satisfying

$$D'(am) = aD'(m) + D(a)m.$$

In our context, for a manifold M, the pair (X, ∇_X) is a derivation of $(C(M), \Gamma(E))$.

Example 1 (Trivial covariant derivative)**.** Consider the trivial line bundle $E = M \times \mathbb{R}$.

Recall that $\Gamma(E) = C(M)$.

Define a covariant derivative on E by

$$\nabla_X(f) := X(f). \tag{2}$$

Condition (1d) takes the form

$$\nabla_X(fg) = f\nabla_X(g) + \nabla_X(f)g.$$

This is the familiar derivation property of vector fields.

More generally, consider the trivial bundle $E = M \times \mathbb{R}^d$.

In this case, $\Gamma(E)$ is the free rank d module over $C(M)$. Thus, a smooth section of this vector bundle is a d-tuple $(f_1, \ldots, f_d)$ of smooth functions on M.

Define a covariant derivative on E by

$$\nabla_X(f_1, \ldots, f_d) := (Xf_1, \ldots, Xf_d). \tag{3}$$

The conditions of a covariant derivative clearly hold.

We call this the trivial covariant derivative.

Example 2 (Tangent bundle of euclidean space)**.** Recall that the tangent bundle of $\mathbb{R}^n$ is trivial. More precisely, $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ be the canonical frame vector field.

By the previous example, we have the trivial covariant derivative (∇_X) on $T\mathbb{R}^n$. Explicitly,

$$\nabla_X\big(f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}\big) = X(f_1)\frac{\partial}{\partial x_1} + \cdots + X(f_n)\frac{\partial}{\partial x_n}. \tag{4}$$

This is an instance of (3).

Example 3 (Tangent bundle of submanifolds of euclidean space)**.** Let M be a submanifold of $\mathbb{R}^n$. Then we can define a covariant derivative on the tangent bundle TM as follows.

Suppose X and Y are vector fields on M.

Extend them to vector fields on $\mathbb{R}^n$, and continue to denote them by X and Y.

Then $\nabla_X(Y)$ is a vector field on $\mathbb{R}^n$ (as in the previous example), but it will not be a vector field on M in general.

So decompose this vector field into its tangential and normal components wrt M.

Taking the tangential component of $\nabla_X(Y)$ defines a covariant derivative on M.

One needs to check that this does not depend on the choice of the extensions.

Covariant derivative is a local notion:

Lemma 1. *Suppose* (∇_X) *is a covariant derivative on a smooth vector bundle* E. *Then if for some open set* U *either* $X|_U = 0$ *or* $s|_U = 0$, *then*

$$(\nabla_X(s))(p) = 0$$

for all $p \in U$.

In other words, $\nabla_X(s)(p)$ only depends on the values of X and s in a neighborhood of p.

Proof. Suppose $s|_U = 0$. Choose a smooth function f such that $f(p) = 1$ and f is zero outside U. Then $fs \equiv 0$. So

$$0 = \nabla_X(fs) = f\nabla_X(s) + X(f)s$$

Now evaluate the rhs at p. Since $s(p) = 0$ and $f(p) = 1$, we obtain $(\nabla_X(s))(p) = 0$ as required.

The argument for $X|_U = 0$ is similar and left as an exercise for you. $\square$

For an open set U of M, recall that E_U is the smooth vector bundle obtained by restricting E to U.

A covariant derivative (∇_X) on E restricts to a covariant derivative on E_U, which we denote by (∇_X^U). Explicitly, for a vector field X on U and a section s of E_U,

$$\nabla_X^U(s)(p) := \nabla_{X'}(s')(p),$$

where X' is a vector field on M which extends X, and s' is a section of E which extends s.

Lemma 2. *The value of* $\nabla_X(s)$ *at a point* p *depends only on* X_p *(and not on the values of* X *in a neighborhood of* p*).*

Proof. By Lemma 1, to calculate $\nabla_X(s)$ we can work in an open set U containing the point p.

Let $(X_1, \ldots, X_n)$ be a tuple of linearly independent vector fields on U (where $n = \dim M$). Write $X = f_1 X_1 + \cdots + f_n X_n$. Then, by (1a) and (1b),

$$\nabla_X(s) = f_1 \nabla_{X_1}(s) + \cdots + f_n \nabla_{X_n}(s).$$

So its value at p only depends on $f_1(p), \ldots, f_n(p)$ and the fixed vector fields X_i.

Alternatively, in view of (1a), we need to show that if $X_p = 0$, then $\nabla_X(s)(p) = 0$.

The key step is to write $X = f_1 X_1 + \cdots + f_n X_n$, where the f_i vanish at p.

Now use the formula

$$\nabla_X(s) = f_1 \nabla_{X_1}(s) + \cdots + f_n \nabla_{X_n}(s).$$

$\square$

For the trivial covariant derivative on the trivial line bundle, Lemma 2 says that $X(f)(p)$ only depends on X_p. This is clear since $X(f)(p) = X_p(f)$.

2 Connections on vector bundles

We saw how a covariant derivative allows us to differentiate a section of a vector bundle wrt a vector field.

We now provide a more abstract way of understanding a covariant derivative by using the concept of a bundle-valued 1-form.

This is the notion of a connection.

2.1 Connection on a vector bundle

Recall that T^*M is the cotangent bundle of M, and a section of this bundle is a 1-form on M.

Also recall that one can take the tensor product of two vector bundles.

The discussion below makes use of the bundle $T^*M \otimes E$, where E is an arbitrary but fixed vector bundle.

A section of this bundle is called an E-valued 1-form on M.

Definition 2. Let $\pi : E \to M$ be a smooth vector bundle.

A connection on E is a $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that

$$\nabla(fs) = f\nabla(s) + df \otimes s \tag{5}$$

for any smooth section s of E and smooth function f on M.

The $\mathbb{R}$-linearity says that

$$\nabla(s_1+s_2) = \nabla(s_1)+\nabla(s_2) \quad \text{and} \quad \nabla(cs) = c\nabla(s).$$

Recall that the functor of smooth sections Γ is a strong monoidal functor wrt tensor products.

In particular,

$$\Gamma(T^*M \otimes E) \;\cong\; \Gamma(T^*M) \otimes_{C(M)} \Gamma(E).$$

Hence, a section of $T^*M \otimes E$ can be written as a sum of tensor products of a section of T^*M and a section of E.

The action of $C(M)$ can be used on either term of the tensor product.

That is, if the section is $\omega \otimes s$, then the action of f can be written as $f\omega \otimes s$ or $\omega \otimes fs$.

Warning. *We point out that*

$$\Gamma(T^*M) \otimes_{C(M)} \Gamma(E) \neq \Gamma(T^*M) \otimes_{\mathbb{R}} \Gamma(E).$$

Many books while writing the tensor product do not explicitly specify whether the tensor product is over $C(M)$ *or over* $\mathbb{R}$*.*

2.2 Equivalence between connections and covariant derivatives

Proposition 1. *For E a smooth vector bundle, a connection on E is the same as a covariant derivative on E.*

Proof. Suppose ∇ is a connection on E.

For any vector field X, there is a map

$$\nabla_X : \Gamma(E) \to \Gamma(E), \qquad \nabla_X(s) := \nabla(s)(X),$$

where $\nabla(s)(X)$ is the contraction of X with the 1-form appearing in $\nabla(s)$.

To illustrate, if $\nabla(s) = \omega \otimes t$, then $\nabla_X(s) = \omega(X)t$.

We claim that this defines a covariant derivative:

(1c) follows since ∇ is $\mathbb{R}$-linear,

(1d) follows from (5).

If ω is a 1-form, then

$$\omega(X+Y) = \omega(X)+\omega(Y) \quad \text{and} \quad \omega(fX) = f\omega(X),$$

so (1a) and (1b) automatically follow.

Thus, a connection leads to a covariant derivative.

Conversely, the connection can be reconstructed from the covariant derivative as follows.

Let $X_1, \ldots, X_n$ be a frame vector field on U. Define

$$\nabla^U : \Gamma(E_U) \to \Gamma(T^*U \otimes E_U), \quad s \mapsto \sum_{i=1}^{n} X_i^* \otimes \nabla_{X_i}(s),$$

where the X_i^* denote the basis of 1-forms dual to the X_i.

By (1c) and (1d) (applied to constant functions), this is a $\mathbb{R}$-linear map.

Further, the ∇^U patch together, that is, ∇^U and ∇^V agree on $U \cap V$.

This means that if $(Y_1, \ldots, Y_n)$ is another frame vector field on U, then

$$\sum_{i=1}^{n} X_i^* \otimes \nabla_{X_i}(s) = \sum_{j=1}^{n} Y_j^* \otimes \nabla_{Y_j}(s).$$

To check this, we evaluate both sides on Y_j. Writing $Y_j = \sum_i f_i X_i$, the evaluation of the lhs is

$$\sum_{i=1}^{n} f_i \nabla_{X_i}(s) = \nabla_{\sum_i f_i X_i}(s) = \nabla_{Y_j}(s),$$

which is the evaluation of the rhs.

Note that we used (1a) and (1b).

This yields the connection ∇, with (1d) translating to (5). $\square$

Example 4 (Trivial connection)**.** We now connect to Example 1.

Consider the trivial line bundle $E = M \times \mathbb{R}$. We saw that $\nabla_X(f) := Xf$ is a covariant derivative on E. Since E is the trivial line bundle, $\Gamma(E) = C(M)$ and $\Gamma(T^*M \otimes E) = \Gamma(T^*M)$, which is the space of 1-forms.

Observe that the associated connection on E is

$$\nabla(f) = df. \tag{6}$$

In other words, ∇ coincides with the exterior derivative d from 0-forms to 1-forms.

We call this the trivial connection on E.

More generally, consider the trivial bundle $E = M \times \mathbb{R}^d$.

Let us write $(e_1, \ldots, e_d)$ for the canonical frame field.
That is, $e_1(p) = (p, (1, 0, \ldots, 0))$, and so on.

We saw that

$$\nabla_X \big(\sum_{j=1}^{d} f_j e_j \big) := \sum_{j=1}^{d} X(f_j)\, e_j$$

is a covariant derivative on E.

The associated connection can be written as

$$\nabla \big(\sum_{j=1}^{d} f_j e_j \big) = \sum_{j=1}^{d} df_j \otimes e_j . \tag{7}$$

We call this the trivial connection on E.

The special case (4) of the tangent bundle on $\mathbb{R}^n$ can be written as

$$\nabla\big(f_1\frac{\partial}{\partial x_1}+\cdots+f_n\frac{\partial}{\partial x_n}\big) \\ = d(f_1)\otimes\frac{\partial}{\partial x_1}+\cdots+d(f_n)\otimes\frac{\partial}{\partial x_n}. \tag{8}$$

3 Connections and matrix-valued one-forms

Recall that a connection ∇ on a vector bundle E specifies a map from sections of E to E-valued 1-forms.

Using a local frame field of E, we will now see how a connection is determined locally by a matrix-valued 1-form.

This is called the connection form.

It can be equivalently expressed using a family of locally defined smooth functions which are called Christoffel symbols of the connection.

3.1 Connection form

Let ∇ be a connection on a rank d smooth vector bundle E over a smooth n-manifold M.

Choose a local frame field $(s_1, \ldots, s_d)$ on an open set U of M which trivializes E.

Write

$$\nabla^U(s_j) = \sum_{k=1}^{d} \theta_k^j \otimes s_k \tag{9}$$

for unique smooth 1-forms θ_k^j on U.

The $d \times d$ matrix (θ_k^j) of 1-forms is called the connection form.

It determines ∇^U completely, since by (5), for $s = \sum_j f_j s_j$,

$$\nabla^U(s) = \sum_{j=1}^{d} f_j \nabla^U(s_j) + df_j \otimes s_j.$$

It is customary to write this as $\theta(s) + d(s)$, but it is important to note that θ and d are not defined globally.

If the bundle is trivial, then they are.

3.2 Christoffel symbols

Even more explicitly, we may assume that $(x_1, \ldots, x_n)$ is a coordinate system on U, and write

$$\theta_k^j = \sum_{i=1}^{n} \Gamma_{ij}^k dx_i, \tag{10}$$

for unique smooth functions Γ_{ij}^k on U.

The Γ_{ij}^k are called the Christoffel symbols of the connection. Here i varies between 1 and n, while j and k vary between 1 and d.

In terms of covariant derivatives,

$$\nabla_{\frac{\partial}{\partial x_i}}(s_j) = \sum_{k=1}^{d} \Gamma_{ij}^k s_k. \tag{11}$$

3.3 Change of frame fields

Let $(s_1, \ldots, s_d)$ and $(s'_1, \ldots, s'_d)$ be two local frame fields on U with connection forms θ_k^j and $(\theta')_k^j$, respectively.

$$\nabla^U(s_j) = \sum_{k=1}^{d} \theta_k^j \otimes s_k$$

$$\nabla^U(s'_j) = \sum_{k=1}^{d} (\theta')_k^j \otimes s'_k.$$

We want to understand the relation between the two connection forms.

Write

$$s'_i = \sum_j P^i_j s_j \quad \text{and} \quad s_i = \sum_j (P^{-1})^i_j s'_j. \tag{12}$$

Here P^i_j are smooth functions on U. They define a $d \times d$ matrix $P = (P^i_j)$.

It is invertible at all points in U and $(P^{-1})^i_j$ are the entries of the inverse matrix P^{-1}.

We calculate:

$$
\begin{aligned}
\nabla(s_i') &= \nabla(\sum_j P^i_j s_j) = \sum_j \nabla(P^i_j s_j) \\
&= \sum_j P^i_j \nabla(s_j) + d(P^i_j) \otimes s_j \\
&= \sum_j P^i_j \sum_k \theta^j_k \otimes s_k + \sum_j d(P^i_j) \otimes s_j \\
&= \sum_k \sum_j P^i_j \theta^j_k \otimes s_k + \sum_k d(P^i_k) \otimes s_k \\
&= \sum_k \big(\sum_j P^i_j \theta^j_k + d(P^i_k) \big) \otimes s_k \\
&= \sum_k \big(\sum_j P^i_j \theta^j_k + d(P^i_k) \big) \otimes \big(\sum_l (P^{-1})^k_l s_l' \big) \\
&= \sum_k \sum_l \big(\sum_j P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l \big) \otimes s_l' \\
&= \sum_l \big(\sum_j \sum_k P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l \big) \otimes s_l'.
\end{aligned}
$$

This yields

$$(\theta')^i_l = \sum_j \sum_k P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l.$$

This formula can be written compactly in matrix notation as

$$\theta' = P\theta P^{-1} + (dP)P^{-1}. \tag{13}$$

This is an identity of $d \times d$ matrices whose entries are 1-forms on U.

3.4 Space of all connections

Recall that the space of sections $\Gamma(E)$ of any vector bundle E over M is a module over $C(M)$.

The term $f\nabla(s)$ is the action of f on the section $\nabla(s)$.

Due to the additional term $df \otimes s$ in (5), a connection is not a map of $C(M)$-modules.

However, the difference of two connections is:

If ∇_1 and ∇_2 are two connections on E, then $\nabla_1 - \nabla_2$ is a map of $C(M)$-modules:

$$\begin{aligned}(\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs) \\ &= (f\nabla_1(s) + df \otimes s) - (f\nabla_2(s) + df \otimes s) \\ &= f\nabla_1(s) - f\nabla_2(s) \\ &= f(\nabla_1(s) - \nabla_2(s)).\end{aligned}$$

Hence, $\nabla_1 - \nabla_2$ can be viewed as a 1-form on M with values in the bundle $\mathrm{End}(E) = E \otimes E^*$:

$$(\nabla_1 - \nabla_2) \in \Gamma(T^*M \otimes \mathrm{End}(E)).$$

Conversely, if ∇ is a connection on E and θ is a 1-form on M with values in $\mathrm{End}(E)$, then $\nabla + \theta$ is a connection on E:

$$\begin{aligned}(\nabla + \theta)(fs) &= \nabla(fs) + \theta(fs) \\ &= f\nabla(s) + df \otimes s + f\theta(s) \\ &= f(\nabla + \theta)(s) + df \otimes s.\end{aligned}$$

Proposition 2. *Suppose* ∇^0 *is a connection on a smooth vector bundle* E.

Then every connection ∇ *on* E *can be uniquely written in the form* $\nabla = \nabla^0 + \theta$, *where* θ *is an* $\mathrm{End}(E)$*-valued* 1*-form on* M.

Example 5 (Connections on the trivial bundle)**.** Recall from Example 4 that when E is a trivial bundle, it carries a canonical connection which is called the trivial connection.

Thus, we can take ∇^0 to be the trivial connection, and then by Proposition 2, every connection is given by ∇^0 plus a matrix-valued 1-form on M.

Let us spell this out.

What are all connections on the trivial line bundle $E = M \times \mathbb{R}$?

Claim: A connection on E can be uniquely written as

$$\nabla(f) = f\omega + df \tag{14}$$

for a smooth 1-form ω.

To see this, suppose θ is a map $C(M) \to \Gamma(T^*M)$ of $C(M)$-modules. Then θ is determined by its value on the constant function 1, and this value can be arbitrary. Thus $\theta(f) = f\omega$, where $\omega := \theta(1)$.

The Christoffel symbols of the connection are present in ω. In local coordinates, if we write $\omega = \sum_{i=1}^{n} g_i dx_i$, then $\Gamma_{i1}^{1} = g_i$. In particular, the Christoffel symbols of the trivial connection are zero.

More generally, let us consider the trivial bundle $E = M \times \mathbb{R}^d$. Let us write $(e_1, \ldots, e_d)$ for the canonical frame field. Any connection on E is uniquely given by

$$\nabla(s) = \sum_{j=1}^{d} f_j \nabla(e_j) + df_j \otimes e_j,$$

with

$$\nabla(e_j) = \sum_{k=1}^{d} \theta_k^j \otimes e_k,$$

where θ_k^j are arbitrary smooth 1-forms on M. In this case, θ and d are globally defined. Recall that θ is a $(E^* \otimes E)$-valued 1-form. In the above notation,

$$\theta = \sum_{j,k} \theta_k^j \otimes e_j^* \otimes e_k.$$

4 Operations on connections

4.1 Direct sum and tensor product

Suppose ∇ is a connection on E, and ∇' is a connection on E', both smooth vector bundles over M.

Then we get a connection ∇'' on $E \oplus E'$ by setting

(15) $$\nabla''_X(s, s') := (\nabla_X(s), \nabla'_X(s')).$$

Similarly, we get a connection ∇'' on $E \otimes E'$ by setting

(16) $$\nabla''_X(s \otimes s') := \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s').$$

4.2 Duals

Suppose ∇ is a connection on E. Then we get a connection ∇'' on E^* by setting

(17) $$\nabla''_X(\alpha)(s) := X(\alpha(s)) - \alpha(\nabla_X(s)).$$

Here α is a section of E^* and s is a section of E.

Equation (17) can be rewritten

$$\nabla_X(\alpha(s)) = \alpha(\nabla_X(s)) + \nabla_X(\alpha)(s),$$

where all covariant derivatives are denoted by ∇_X. If we call $\alpha(s)$ the contraction of α and s, then the above equation says that ∇_X commutes with contractions.

5 Differential forms on a vector bundle

We have studied differential forms on a manifold M.

We now tensor the bundle of differential forms on M
with any vector bundle E over M.

Taking sections yields the notion of an E-valued
differential form on M.

The connection now plays the role of the exterior derivative.

We recover the classical setup by taking E to be the
trivial line bundle with trivial connection.

5.1 Differential forms on a manifold

Let M be a manifold.

Recall that $\wedge^k(T^*M)$ denotes the k-th exterior power of the cotangent bundle T^*M.

A section of $\wedge^k(T^*M)$ is a k-form on M. The space of all k-forms on M is denoted $\Omega^k(M)$.

Taking direct sum over k yields the algebra of differential forms $\Omega(M)$.

5.2 Differential forms on a vector bundle

Now suppose E is a vector bundle over M.

We consider the $C(M)$-module

$$\Omega^k(E) := \Gamma(\wedge^k(T^*M) \otimes E).$$

An element of this space is called

- an E-valued k-form on M, or
- a k-form on M with values in E, or
- simply a k-form on E.

Taking direct sum over k yields the space $\Omega(E)$.

We note that $\Omega(E)$ is a bimodule over the algebra $\Omega(M)$:

The left and right actions (both denoted by $\wedge$) are

$$\begin{aligned} \omega \wedge (\omega' \otimes s) &:= (\omega \wedge \omega') \otimes s, \\ (\omega \otimes s) \wedge \omega' &:= (\omega \wedge \omega') \otimes s. \end{aligned} \tag{18}$$

Since we implicitly identify $E \otimes \mathrm{triv} \cong E$, many different looking notations specify the same element:

$$\omega \wedge (1 \otimes s) = \omega \wedge s = \omega \otimes s, \quad f \wedge s = f \otimes s = fs.$$

Note that $\Omega(\mathrm{triv}) = \Omega(M)$ and it is a bimodule over itself with left and right actions given by left and right multiplication in $\Omega(M)$.

5.3 Gauge exterior derivative

In the language of k-forms on a vector bundle E, a connection on E is a $\mathbb{R}$-linear map

$$\nabla : \Omega^0(E) \to \Omega^1(E)$$

satisfying (5).

Theorem 3. *Suppose ∇ is a connection on a smooth vector bundle E. Then there is a unique sequence of $\mathbb{R}$-linear maps*

$$\nabla : \Omega^k(E) \to \Omega^{k+1}(E), \qquad k = 0, 1, 2, \ldots$$

such that for $k = 0$, ∇ is the given connection, and

$$\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^k \omega \wedge \nabla(s) \tag{19}$$

for any smooth k-form ω on M, and any smooth section s of E.

The term $\omega \wedge \nabla(s)$ refers to the left action of ω on $\nabla(s)$ as defined in (18), where ∇ is the connection that we started with. Note that for $k = 0$, (19) reduces to (5).

Proof. The uniqueness assertion is clear since terms of the form $\omega \otimes s$ linearly span $\Omega^k(E)$, and the rhs of (19) is known.

One needs to check that (19) is well-defined, that is,

$$\nabla((\omega_1 + \omega_2) \otimes s) = \nabla(\omega_1 \otimes s) + \nabla(\omega_2 \otimes s),$$
$$\nabla(\omega \otimes (s_1 + s_2)) = \nabla(\omega \otimes s_1) + \nabla(\omega \otimes s_2),$$
$$\nabla(f\omega \otimes s) = \nabla(\omega \otimes fs).$$

The first two are clear; we check the last one.

$$\begin{aligned}\nabla(f\omega \otimes s) &= d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= (df \wedge \omega) \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s).\end{aligned}$$

$$\begin{aligned}\nabla(\omega \otimes fs) &= d\omega \otimes fs + (-1)^k \omega \wedge \nabla(fs) \\ &= f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s) + (df \wedge \omega) \otimes s.\end{aligned}$$

So both sides are equal as required. $\square$

The extension of ∇ is sometimes called the exterior covariant derivative. In some contexts, it is called the gauge exterior derivative.

Note that if E is the trivial line bundle $M \times \mathbb{R}$, and ∇ is the trivial connection given by the exterior derivative d, then the above extension of ∇ is the exterior derivative d on higher differential forms on M.

Recall that the exterior derivative satisfies $d^2 = 0$. However, in general, $\nabla \circ \nabla \neq 0$; this is of significance as we will see later.

Lemma 3. *Suppose ∇ is a connection on a smooth vector bundle E.*

For any smooth k-form θ on E and smooth function f on M,

$$\nabla(f\theta) = df \wedge \theta + f\nabla(\theta). \tag{20}$$

Proof. It suffices to check for $\theta = \omega \otimes s$.

$$\begin{aligned}\nabla(f\omega \otimes s) &= d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= (df \wedge \omega) \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= df \wedge \theta + f\nabla(\theta).\end{aligned}$$

$\square$

More generally, by the same calculation:

Lemma 4. *Suppose ∇ is a connection on a smooth vector bundle E.*

For any smooth k-form ω on M and smooth m-form θ on E,

$$\begin{aligned} \nabla(\omega \wedge \theta) &= d\omega \wedge \theta + (-1)^k \omega \wedge \nabla(\theta) \\ \nabla(\theta \wedge \omega) &= \nabla(\theta) \wedge \omega + (-1)^m \theta \wedge d\omega. \end{aligned} \tag{21}$$

There is one identity for the left action and one for the right action.

5.4 Forms as alternating maps

Lemma 5. *For a smooth vector bundle E over M, there is a canonical linear isomorphism between $\Omega^k(E)$ and alternating $C(M)$-module maps from the k-fold tensor product of $\Gamma(TM)$ to $\Gamma(E)$.*

Explicitly, for 1-forms ω_i, a section s, and vector fields X_j,

$$(\omega_1 \wedge \cdots \wedge \omega_k \otimes s)(X_1 \otimes \cdots \otimes X_k) = \det(\omega_i(X_j))s.$$

Lemma 6. *Suppose ∇ is a connection on a smooth vector bundle E. For any smooth k-form θ on E, and smooth vector fields $X_1, \ldots, X_{k+1}$,*

(22) $$\nabla(\theta)(X_1, \ldots, X_{k+1})$$
$$= \sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{X_i}(\theta(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

where the hat over X_i indicates that X_i is deleted from the sequence.

6 Curvature

We discuss curvature.

This notion makes sense on any vector bundle with a connection.

Recall that connection and covariant derivative are equivalent.

Hence the notion of curvature can be formulated in either language.

We explain both approaches independently.

6.1 Curvature of a connection

Definition 4. Let ∇ be a connection on a smooth vector bundle $\pi : E \to M$.

The curvature of ∇ is the map

$$R := \nabla \circ \nabla : \Omega^0(E) \longrightarrow \Omega^2(E).$$

This is the composite map

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E).$$

The connection is called flat if $R = 0$, that is, $R(s) = 0$ for every smooth section s of E.

Lemma 7. *The curvature R of a connection ∇ is a map of $C(M)$-modules. That is,*

$$R(fs) = fR(s) \quad \textit{and} \quad R(s_1{+}s_2) = R(s_1){+}R(s_2).$$

Proof. The second identity is clear. For the first, we calculate using (5), (19) and (20):

$$\begin{aligned}\nabla(\nabla(fs)) &= \nabla(f\nabla(s) + df \otimes s) \\ &= (df \wedge \nabla(s) + f\nabla(\nabla(s))) + (d^2 f \otimes s - df \wedge \nabla(s)) \\ &= f\nabla(\nabla(s)).\end{aligned}$$

□

Thus, curvature is a map of $C(M)$-modules

$$\Gamma(E) \to \Gamma(\wedge^2(T^*M) \otimes E),$$

and hence, it can be viewed as an element of

$$\Gamma(\wedge^2(T^*M) \otimes E^* \otimes E).$$

Thus, curvature is a 2-form on M with values in $\mathrm{End}(E)$.

(Recall that we had a similar statement for the difference of two connections.)

6.2 Matrix of curvature 2-forms

Choose a local frame field $(s_1, \ldots, s_d)$ on an open set U of M which trivializes E. Write

(23) $$R(s_j) = \sum_{k=1}^{d} R_k^j \otimes s_k,$$

or equivalently,

(24) $$R = \sum_{j,k=1}^{d} R_k^j \otimes s_j^* \otimes s_k,$$

where R_k^j are smooth 2-forms on U.

(These depend on the choice of the local frame field.)

Let us express them in terms of the connection form (9):

$$
\begin{aligned}
\nabla \circ \nabla(s_j) &= \sum_{k=1}^{d} \nabla(\theta_k^j \otimes s_k) \\
&= \sum_{k=1}^{d} d\theta_k^j \otimes s_k - \sum_{k=1}^{d} \sum_{l=1}^{d} \theta_k^j \wedge \theta_l^k \otimes s_l \\
&= \sum_{k=1}^{d} \left(d\theta_k^j - \sum_{m=1}^{d} \theta_m^j \wedge \theta_k^m \right) \otimes s_k \\
&= \sum_{k=1}^{d} R_k^j \otimes s_k.
\end{aligned}
$$

Thus,

$$R^j_k = d\theta^j_k - \sum_{m=1}^{d} \theta^j_m \wedge \theta^m_k. \tag{25}$$

This can be expressed in short by $R = d\theta - \theta \wedge \theta$.

Some books will write $R = d\theta + \theta \wedge \theta$. Note here that formula (25) can be rewritten as

$$R^j_k = d\theta^j_k + \sum_{m=1}^{d} \theta^m_k \wedge \theta^j_m. \tag{26}$$

Also some books will write θ^k_j for θ^j_k and R^k_j for R^j_k.

6.3 Change of frame fields

Let $(s_1, \ldots, s_d)$ and $(s'_1, \ldots, s'_d)$ be two local frame fields on U with curvature forms R^j_k and $(R')^j_k$, respectively. Then

$$R' = PRP^{-1}, \tag{27}$$

where P is the matrix for change of frame fields (12).

Compare and contrast with (13). If we were to do a change of frame fields for a difference of two connections, then the formula would be as in (27).

Some books will say: R transforms as a tensor.

6.4 Flat connections on the trivial bundle

Recall that the trivial connection on the trivial line bundle $M \times \mathbb{R}$ is nothing but the exterior derivative d. Since $d^2 = 0$, the trivial connection is flat.

Consider the rank d trivial bundle E on M. Formula (26) shows that flat connections on E correspond to solutions of the equation

$$d\theta_k^j + \sum_{m=1}^{d} \theta_k^m \wedge \theta_m^j = 0. \tag{28}$$

This is called the Maurer-Cartan equation.

We need to solve for the θ_k^j.

Of course, the trivial connection is flat since $\theta^j_k \equiv 0$ in that case.

For $d = 1$, the Maurer-Cartan equation simply says that $d\theta = 0$.

So flat connections on the trivial line bundle on M correspond to closed 1-forms on M.

6.5 Curvature of a covariant derivative

Let us now look at curvature from the perspective of covariant derivatives.

Definition 5. Let (∇_X) be a covariant derivative on a smooth vector bundle $\pi : E \to M$.

The curvature of (∇_X) is the map which assigns to every pair of smooth vector fields (X, Y) on M the operator

$$R(X, Y) : \Gamma(E) \longrightarrow \Gamma(E)$$

defined by

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{29}$$

Some formal consequences are discussed below.

Lemma 8. *For any smooth vector fields X and Y, the curvature operator $R(X, Y)$ in* (29) *is a map of $C(M)$-modules. That is,*

$$R(X, Y)(fs) = fR(X, Y)(s),$$
$$R(X, Y)(s_1 + s_2) = R(X, Y)(s_1) + R(X, Y)(s_2).$$

Proof. We check the first identity.

Using (1d),

$$\nabla_X \nabla_Y (fs) = \nabla_X (f \nabla_Y (s) + Y(f)s)$$
$$= f\nabla_X \nabla_Y (s) + X(f)\nabla_Y (s) + Y(f)\nabla_X (s) + XY(f)s.$$

It follows that

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fs) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(s) + [X,Y](f)s.$$

Thus, the commutator of ∇_X and ∇_Y fails to be a map of $C(M)$-modules.

Observe how this defect is corrected by the term $\nabla_{[X,Y]}$.

$$\nabla_{[X,Y]}(fs) = f\nabla_{[X,Y]}(s) + [X,Y](f)s.$$

$\square$

Lemma 9. *The curvature operator* $R(X, Y)$ *in* (29) *is* $C(M)$*-linear in* X *and in* Y*. That is,*

$$R(fX, Y) = fR(X, Y),$$
$$R(X_1 + X_2, Y) = R(X_1, Y) + R(X_2, Y),$$
$$R(X, Y_1 + Y_2) = R(X, Y_1) + R(X, Y_2).$$

In addition,

$$R(X, Y) = -R(Y, X).$$

Proof. We explain the first identity, the rest are clear.

Recall that the bracket of vector fields behaves as follows under scalar multiplication by smooth functions.

$$[fX, gY] = fX(g)Y - gY(f)X + fg[X, Y].$$

For symmetry purposes, we have multiplied in both coordinates.

Using this formula, and (1b),

$$\begin{aligned} &R(fX, gY) \\ &\quad = f\nabla_X(g\nabla_Y) - g\nabla_Y(f\nabla_X) - fX(g)\nabla_Y + gY(f)\nabla_X \\ &\qquad\qquad - fg\nabla_{[X,Y]}. \end{aligned}$$

Now use (1d) and note how the unwanted terms $fX(g)\nabla_Y$ and $gY(f)\nabla_X$ cancel out to give

$$R(fX, gY) = fgR(X, Y).$$

$\square$

Some books will say: R is tensorial wrt X, Y and s.

Lemma 10. *The value of $R(X,Y)(s)(p)$ only depends on X_p and Y_p (and not on the values of X and Y at other points of M).*

Proof. Due to the antisymmetry relation $R(X,Y) = -R(Y,X)$, it suffices to prove the result in one of the two coordinates.

Accordingly, suppose $X_p = 0$.

As in the proof of Lemma 2, write $X = f_1 X_1 + \cdots + f_n X_n$, where the f_i vanish at p. Now using Lemma 9,

$$R(X,Y)(s)(p) = \sum_i f_i(p) R(X_i, Y)(s)(p) = 0.$$

□

Lemma 11. *The value of* $R(X, Y)(s)(p)$ *only depends on* $s(p)$ *(and not on the values of* s *at other points of* M*).*

Proof. Suppose $s(p) = 0$.

Write $s = \sum_i f_i s_i$ with $f_i(p) = 0$.

Now using Lemma 8,

$$R(X, Y)(s)(p) = \sum_i f_i(p) R(X, Y)(s_i)(p) = 0.$$

□

6.6 Equivalence of the two approaches

Suppose (θ_k^j) is the connection 1-form of ∇. We know from formula (26) that

$$R(s_j) = \sum_{k=1}^{d} \Big(d\theta_k^j + \sum_{m=1}^{d} \theta_k^m \wedge \theta_m^j \Big) \otimes s_k.$$

Hence,

$$R(X,Y)(s_j) = \sum_{k=1}^{d} \Big(d\theta_k^j(X,Y) + \sum_{m=1}^{d} (\theta_k^m \wedge \theta_m^j)(X,Y) \Big) s_k.$$

Now

$$d\theta_k^j(X,Y) = X(\theta_k^j(Y)) - Y(\theta_k^j(X)) - \theta_k^j([X,Y]),$$

and

$$(\theta_k^m \wedge \theta_m^j)(X,Y) = \theta_k^m(X)\theta_m^j(Y) - \theta_k^m(Y)\theta_m^j(X).$$

Further,

$$\nabla_X(s_j) = \sum_k \theta_k^j(X) s_k,$$

from which we have

$$\nabla_X \nabla_Y(s_j) = \sum_{k=1}^{d} \big(X(\theta_k^j(Y)) + \sum_{m=1}^{d} \theta_k^m(X)\theta_m^j(Y) \big) s_k,$$

and a similar expression for $\nabla_Y \nabla_X(s_j)$.

Putting all this information together, we deduce

$$R(X,Y)(s_j) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(s_j),$$

as required.

We could have done this calculation earlier and turned Definition 5 into a proposition.

Lemma 9 would then immediately follow since we are contracting the pair of vector fields on a 2-form.

Similarly, Lemma 8 would follow from Lemma 7.