Smooth functions and tangent vectors

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1 Algebra of smooth functions

Let M be a smooth manifold.

A smooth function on M is a smooth map $M \to \mathbb{R}$.

Explicitly, $f : M \to \mathbb{R}$ is smooth if for every chart (U, φ), the composite

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$$

is smooth.

We denote the set of smooth functions on M by $C(M)$.

Proposition 1. *If f and g are smooth functions on M, then so are $f+g$, fg and cf for any real number c.*

Proof. We can use the argument given in multivariable calculus. Suppose f and g are smooth. Then

$$M \xrightarrow{(f,g)} \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a composite of smooth maps, hence smooth. (The second map could be addition or multiplication.) $\square$

This says that $C(M)$ is a commutative $\mathbb{R}$-algebra.

Proposition 2. *Suppose M and N are smooth manifolds, and $f : M \to N$ is any map. Then*

f is smooth iff

For every $g \in C(N)$, $g \circ f \in C(M)$.

Proof. This follows from Proposition **??**. $\square$

The association of $C(M)$ to M is functorial:

If $f : M \to N$ is a smooth map, then there is an induced algebra morphism $C(N) \to C(M)$ which sends $h : N \to \mathbb{R}$ to $h \circ f : M \to \mathbb{R}$.

Thus we have a contravariant functor

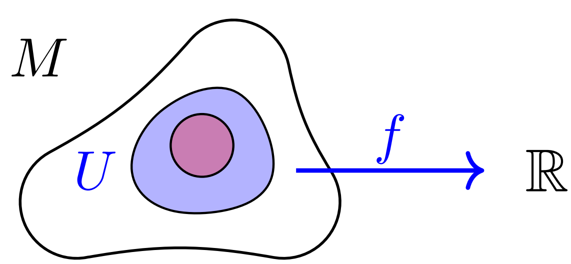
$$C : \mathsf{Manifold} \to \mathsf{Alg}^{\mathsf{co}}$$

from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras.

Lemma 1. *Let U be an open set of a smooth manifold M. Let $f : U \to \mathbb{R}$ be a smooth map. Then there exists a nonempty open set V whose closure is contained in U and a smooth function g on M such that*

$$g(p) = \begin{cases} f(p) & \text{if } p \in V, \\ 0 & \text{if } p \notin U. \end{cases}$$

Further, the same V can be chosen for each f.



Lemma 2. *Let M be a smooth manifold. Any point $p \in M$ determines an algebra morphism*

$$C(M) \to \mathbb{R}, \qquad f \mapsto f(p).$$

Conversely, every algebra morphism from $C(M)$ to $\mathbb{R}$ arises in this manner.

The first part is clear.

The second part requires some work.

Theorem 1. *The functor C from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras is full and faithful.*

That is, for any smooth manifolds M and N, the canonical map

$$\mathsf{Manifold}(M, N) \to \mathsf{Alg}(C(N), C(M))$$

is a bijection.

This implies that $C(M)$ determines M up to diffeomorphism.

It follows that the category of smooth manifolds is equivalent to a full subcategory of the category of commutative $\mathbb{R}$-algebras.

Proof. For injective:

Suppose $f, g : M \to N$ with $f(p) \neq g(p)$ for some $p \in M$.

Pick a smooth function h on N which is 1 at $f(p)$ and 0 at $g(p)$. This is possible by Lemma 1.

Then $h \circ f \neq h \circ g$ since their values differ at p.

So f and g do induce different algebra morphisms $C(N) \to C(M)$.

For surjective: Observe that the case when M is a point and N is arbitrary follows from Lemma 2.

Now let M be arbitrary. Suppose we are given $\varphi : C(N) \to C(M)$. Let $p \in M$. Evaluation at p yields a morphism $C(M) \to \mathbb{R}$. Precomposing by φ yields a morphism $C(N) \to \mathbb{R}$. By Lemma 2, this must be evaluation at some $q \in N$.

Define $f : M \to N$ by $f(p) = q$. It follows that f induces φ.

Further from Proposition 2, f is smooth. $\square$

2 Derivations of an algebra

We define the notion of derivation of an algebra.

The space of all derivations of an algebra carries the structure of a Lie algebra.

We make this explicit for the algebra of polynomials.

2.1 Derivations of an algebra

Let $\mathbb{k}$ be a field. Suppose A is any $\mathbb{k}$-algebra.

A derivation of A is a $\mathbb{k}$-linear map $D : A \to A$ such that

$$D(aa') = D(a)a' + aD(a'). \tag{1}$$

Here aa', $D(a)a'$ and $aD(a')$ are products taken in A.

Observe that a derivation is completely determined once it is specified on the generators of the algebra.

Example 1 (Algebra of polynomials)**.** The space of polynomials in one variable $\mathbb{k}[x]$ is a $\mathbb{k}$-algebra, with product being the usual multiplication of polynomials.

The derivative operator

$$\frac{d}{dx} : \mathbb{k}[x] \to \mathbb{k}[x]$$

is a derivation. The condition (1) is the Leibniz rule.

What are all derivations of $\mathbb{k}[x]$?

Since x is the generator of the algebra, any derivation is completely determined by its value on x.

Now where can we send x to?

There is no restriction on where x can go. Suppose it goes to the polynomial $a(x)$.

Then the resulting derivation of $\mathbb{k}[x]$ is given by

$$a(x)\frac{d}{dx} : \mathbb{k}[x] \to \mathbb{k}[x], \qquad a(x)\frac{d}{dx}(f(x)) := a(x)f'(x).$$

More generally, the space of polynomials in n variables $\mathbb{k}[x_1, \dots, x_n]$ is a $\mathbb{k}$-algebra.

Now any derivation is completely determined by its values on $x_1, \dots, x_n$.

Say x_i goes to the polynomial $a_i(x_1, \dots, x_n)$ for each i.

Then the resulting derivation of $\mathbb{k}[x_1, \dots, x_n]$ is given by

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i} : \mathbb{k}[x_1, \dots, x_n] \to \mathbb{k}[x_1, \dots, x_n].$$

Now consider $\Bbbk = \mathbb{R}$, the field of real numbers.

Here, instead of $\mathbb{R}[x]$, we could take the larger algebra $C(\mathbb{R})$ of all smooth functions on $\mathbb{R}$.

A derivation of $C(\mathbb{R})$ is also of the form $a(x)\frac{d}{dx}$, where $a(x)$ is now allowed to be any smooth function on $\mathbb{R}$.

More generally, any derivation of $C(\mathbb{R}^n)$ is of the form $\sum_i a_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}$, where $a_i(x_1, \ldots, x_n)$ is any smooth function on $\mathbb{R}^n$.

2.2 Lie algebra of derivations

Let $\mathrm{Der}(A)$ denote the set of all derivations of A.

Suppose D and E are derivations. Then so are $D + E$ and cD for any scalar c. Thus $\mathrm{Der}(A)$ is a vector space over $\Bbbk$.

Further,

$$[D, E] := D \circ E - E \circ D, \tag{2}$$

called the Lie bracket of D and E, is a derivation.

Why is $D \circ E$ is not a derivation?

$$\begin{aligned} DE(aa') &= D(E(a)a' + aE(a')) \\ &= D(E(a)a') + D(aE(a')) \\ &= DE(a)a' + E(a)D(a') + D(a)E(a') + aDE(a'). \end{aligned}$$

The bracket operation turns $\mathrm{Der}(A)$ into a Lie algebra, that is, the bracket operation satisfies antisymmetry and the Jacobi identity:

$$[D, E] = -[E, D]$$

and

$$[[D, E], F] + [[E, F], D] + [[F, D], E] = 0.$$

Example 2 (Algebra of polynomials)**.** For the algebra of polynomials in Example 1, the Lie bracket works as follows.

For

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad E = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

$$[D, E] = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j}.$$

In particular,

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

The same formulas hold for the algebra of smooth functions on $\mathbb{R}^n$.

The only difference is that a_i and b_i are smooth functions on $\mathbb{R}^n$ instead of polynomials.

2.3 Derivations into a bimodule

Let A be a $\Bbbk$-algebra and M an A-bimodule.

A derivation of A into M is a $\Bbbk$-linear map $D : A \to M$ such that

$$D(aa') = D(a) \cdot a' + a \cdot D(a'), \tag{3}$$

where we use $\cdot$ to denote both the left and right actions of A on M.

We recover (1) when we take $M = A$ with the left and right actions of A on A given by the product in A.

We let $\mathrm{Der}(A, M)$ denote the space of all derivations of A into M. It is a vector space over $\Bbbk$.

2.4 Derivations wrt an algebra morphism

Let $\varphi : A \to B$ be an algebra morphism.

Then B is an A-bimodule with left and right actions given by

$$a \cdot b := \varphi(a)b \quad \text{and} \quad b \cdot a := b\varphi(a),$$

with products taken in B.

Thus, we can talk of derivations of A into B.

Explicitly, $D \in \mathrm{Der}(A, B)$ if

(4) $$D(aa') = D(a)\varphi(a') + \varphi(a)D(a')$$

with products in the rhs taken in B.

This is a specialization of (3).

If $D \in \mathrm{Der}(A)$, then $\varphi \circ D \in \mathrm{Der}(A, B)$:

$$\begin{aligned}
\varphi(D(aa')) &= \varphi(D(a)a' + aD(a')) \\
&= \varphi(D(a)a') + \varphi(aD(a')) \\
&= \varphi(D(a))\varphi(a') + \varphi(a)\varphi(D(a')).
\end{aligned}$$

Example 3 (Algebra of polynomials)**.** Fix $p \in \mathbb{k}^n$, and let

$$\mathrm{ev}_p : \mathbb{k}[x_1, \dots, x_n] \to \mathbb{k}, \qquad f \mapsto f(p)$$

be the algebra morphism given by evaluation at p. Then a derivation of $\mathbb{k}[x_1, \dots, x_n]$ into $\mathbb{k}$ wrt this morphism is given by

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{x=p} : \mathbb{k}[x_1, \dots, x_n] \to \mathbb{k}$$

for some scalars a_i.

For $\mathbb{k} = \mathbb{R}$, any derivation of $C(\mathbb{R}^n)$ wrt evaluation at a point $p \in \mathbb{R}^n$ is of the form $\sum_i a_i \frac{\partial}{\partial x_i}\big|_{x=p}$ for some real numbers a_i.

3 Tangent vectors and derivations

We have already stated that smooth manifolds allow us to do calculus.

We now take an important step in that direction by explaining how to define tangent vectors in a smooth manifold.

It is linked to the notion of derivations of the algebra of smooth functions on that smooth manifold.

3.1 Tangent vectors in a smooth manifold

Let M be a smooth n-manifold and let $p \in M$ be any point.

A tangent vector at p is a linear map

$$X_p : C(M) \to \mathbb{R}$$

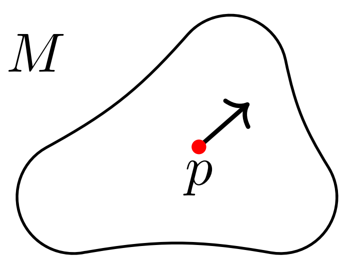
such that for all smooth functions f, g on M,

(5) $$X_p(fg) = X_p(f)g(p) + f(p)X_p(g).$$

(Here fg denotes the pointwise product of f and g.)

Thus, a tangent vector at p is the same as a derivation in the sense of (4) wrt the algebra morphism $C(M) \to \mathbb{R}$ given by evaluation at p.

The following is a way to visualize a tangent vector.



Let T_pM denote the set of all tangent vectors at p.

Note that if X_p and Y_p are tangent vectors, then so are $X_p + Y_p$ and cX_p for any real number c with

$$(X_p + Y_p)(f) := X_p(f) + Y_p(f)$$

and

$$(cX_p)(f) := cX_p(f).$$

Thus T_pM is a vector space over $\mathbb{R}$.

The zero tangent vector is the map which sends all smooth functions to 0.

3.2 Tangent vectors in euclidean space

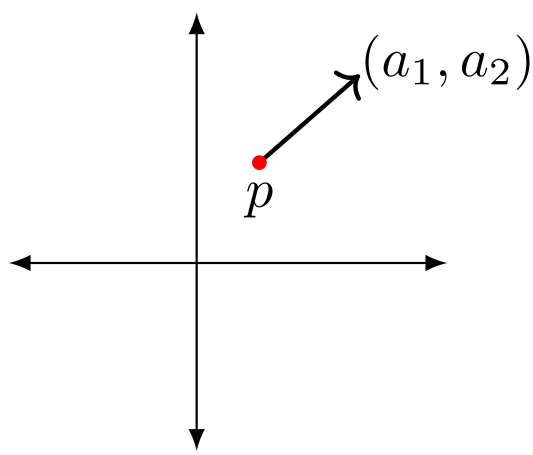
Let $M = \mathbb{R}^n$ and p be any point in it.

Then for any n-tuple $(a_1, \ldots, a_n)$ of real numbers,

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{x=p} : C(\mathbb{R}^n) \to \mathbb{R} \tag{6}$$

is a tangent vector at p. It sends f to $\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}\big|_{x=p}$.

This is a directional derivative.



We now show that all tangent vectors are of this form.

Proposition 3. *Any tangent vector at any point in* $\mathbb{R}^n$ *is given by a unique directional derivative.*

In particular, the tangent space at any point in $\mathbb{R}^n$ *is* n*-dimensional.*

Proof. Let X_p be any tangent vector. We let p be the origin for convenience.

Set $a_i := X_p(x_i)$. We claim that $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$.

Let

$$Y_p := X_p - \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}.$$

Then, by construction, $Y_p(x_i) = 0$ for all i.

Now any smooth function f can be written in the form $f = f(0) + x_1 g_1 + \cdots + x_n g_n$ for some (not necessarily unique) choice of smooth functions g_i. Hence,

$$\begin{aligned} Y_p(f) &= Y_p(f(0) + x_1 g_1 + \cdots + x_n g_n) \\ &= Y_p(x_1 g_1) + \cdots + Y_p(x_n g_n). \end{aligned}$$

By (5), we see that $Y_p(x_i g_i) = 0$, so $Y_p(f) = 0$. Thus Y_p is the zero tangent vector as required. $\square$

3.3 Tangent spaces of a smooth manifold

Lemma 3. *Let M be a smooth n-manifold and p be a point on it.*

If two smooth functions f and g on M agree on a neighborhood of p, then $X_p(f) = X_p(g)$ for any tangent vector X_p.

Proof. By linearity, it suffices to show that $X_p(f) = 0$ whenever f is zero in a neighborhood of p. Suppose this is the case.

Pick a smooth function h which is 0 outside this neighborhood, but has value 1 at p. So $fh \equiv 0$. Now apply (5):

$$0 = X_p(fh) = X_p(f)h(p) + f(p)X_p(h) = X_p(f).$$

$\square$

This result says that we can restrict to any open submanifold U of M containing p, and the tangent space at p does not change:

$$T_p(U) = T_p(M).$$

Now pick any chart (U, φ) containing p. Then there is an isomorphism of algebras

$$C(U) \to C(\varphi(U)), \quad f \mapsto f \circ \varphi^{-1}.$$

The inverse map sends g to $g \circ \varphi$.

Since the notion of a tangent vector is defined entirely in terms of algebra of smooth functions, it follows that the tangent spaces $T_p(U)$ and $T_{\varphi(p)}(\varphi(U))$ are canonically isomorphic.

Since by Lemma 3,

$$T_p(U) = T_p(M) \quad \text{and} \quad T_{\varphi(p)}(\varphi(U)) = T_{\varphi(p)}(\mathbb{R}^n),$$

it follows from Proposition 3 that $T_p(M)$ is n-dimensional.

By abuse of notation, we allow ourselves to let $\frac{\partial}{\partial x_i}$ denote tangent vectors at p keeping the isomorphism implicit.

3.4 Change of coordinates

Let us now see how a tangent vector transforms under change of coordinates.

Lemma 4. *Let M be a smooth n-manifold and p be a point on it. Let $(x_1, \ldots, x_n)$ coming from (U, φ) and $(u_1, \ldots, u_n)$ coming from (V, ψ) be two coordinate systems containing a point p. Then*

$$\frac{\partial}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial}{\partial x_i}. \tag{7}$$

All partials are evaluated at the point p.

The map $\varphi \circ \psi^{-1}$ expresses the x_i in terms of the u_j, and it is the partial derivatives of this map that are being calculated in $\frac{\partial x_i}{\partial u_j}$.

Proof. To check the lemma, we need to show that for any smooth function f on M,

$$\frac{\partial(f \circ \psi^{-1})}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}.$$

This is the familiar chain rule (in euclidean spaces). $\square$

This gives us an alternative way to define a tangent vector:

Lemma 5. *A tangent vector at a point p in M is an assignment to every coordinate system a n-tuple of real numbers such that if $(a_1, \ldots, a_n)$ is assigned to $(x_1, \ldots, x_n)$ and $(b_1, \ldots, b_n)$ to $(u_1, \ldots, u_n)$, then*

$$a_i = \sum_{j=1}^{n} b_j \frac{\partial x_i}{\partial u_j}.$$

Proof. The calculation here is

$$\sum_{j=1}^{n} b_j \frac{\partial}{\partial u_j} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_j \frac{\partial x_i}{\partial u_j} \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} b_j \frac{\partial x_i}{\partial u_j} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

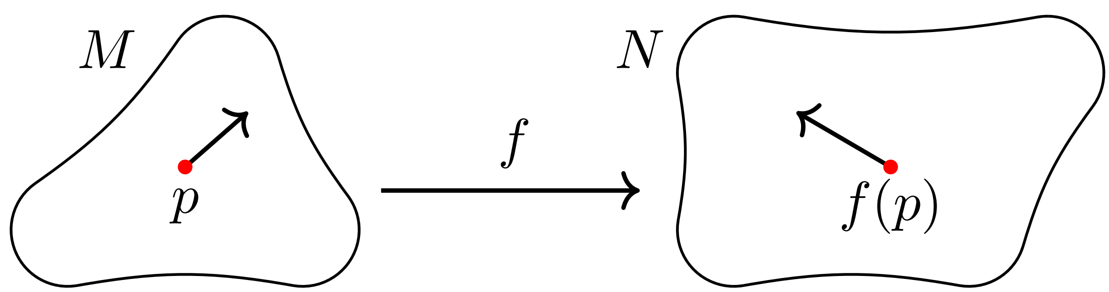
$\square$

3.5 Linear maps between tangent spaces

Suppose $f : M \to N$ is a smooth map.

Then for any point $p \in M$, there is an induced linear map

(8) $f_* : T_pM \to T_{f(p)}N, \quad f_*(X_p)(g) := X_p(g \circ f).$



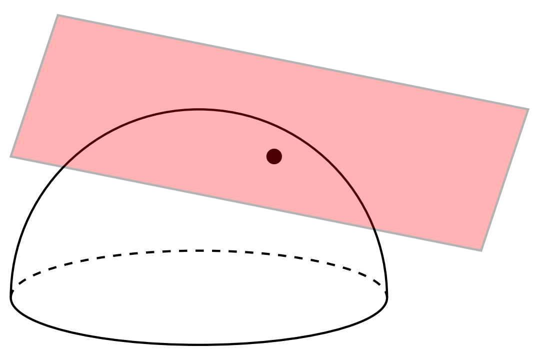
Suppose $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$.

Then the tangent spaces at any point can be identified with the space itself.

In this case, one may check that f_* is the familiar derivative of f. In the canonical bases for the tangent spaces, it is the $m \times n$ jacobian matrix of partial derivatives of f.

Since the general case reduces to this one after choosing charts, it is correct to think of f_* as the derivative of f.

While learning multivariable calculus, a tangent space is explained by drawing a surface in $\mathbb{R}^3$ such as a spherical cap, and then drawing a plane that is tangent to the surface at a given point.



There is a smooth inclusion i of the surface in $\mathbb{R}^3$. So i_* maps the tangent space of the surface at p into the tangent space of $\mathbb{R}^3$ at $i(p)$ which is again $\mathbb{R}^3$.

It is this affine subspace of $\mathbb{R}^3$ that is being shown in the picture.

Recall that we defined rank of a smooth map $f : M \to N$ at a point $p \in M$ by using charts and then looking at the rank of the jacobian matrix of the induced map between euclidean spaces.

Observe that it is the same as the rank of the linear map $f_* : T_pM \to T_{f(p)}N$.

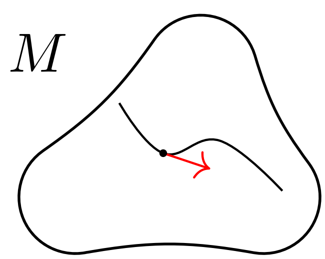
3.6 Tangent vectors from curves

Let I be some open interval containing zero. So I is a 1-dimensional manifold.

A smooth map $f : I \to M$ is called a smooth curve in M.

Consider the standard tangent vector $\frac{\partial}{\partial x}$ in I at the origin.

The image $f_*(\frac{\partial}{\partial x})$ is a tangent vector in M at $f(0)$.



It is easy to see that all tangent vectors can be obtained in this manner by differentiating curves.

4 Tangent bundle and vector fields

The space of all tangent vectors (at all points) is called the tangent bundle. It carries a natural topology, and even further a smooth structure, so it is a smooth manifold. Its dimension is twice that of the original manifold.

We also discuss vector fields. A vector field is a choice of a tangent vector at each point of the smooth manifold such that the choice varies smoothly with the point. This can also be expressed by saying that a vector field is a section of the tangent bundle.

4.1 Tangent bundle

Let M be any smooth manifold.

We now proceed to construct a smooth manifold TM called the tangent bundle of M.

As a set

$$TM := \bigsqcup_{p \in M} T_pM.$$

We say that T_pM is the fiber of TM over p.

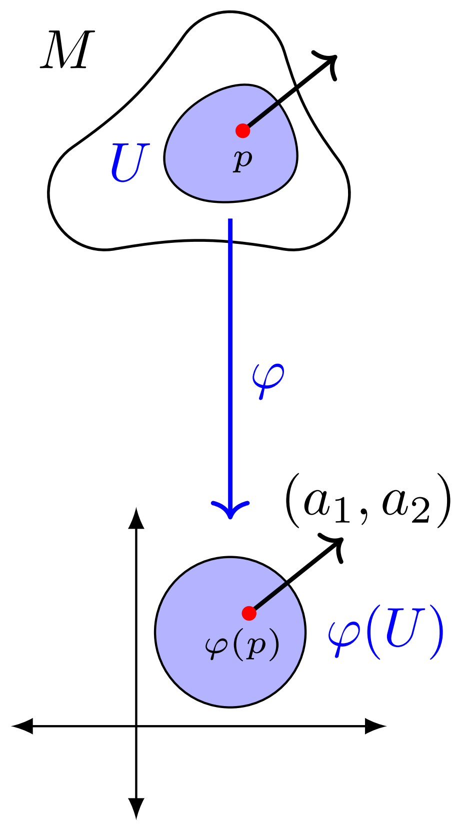
Note that there is a canonical map $\pi : TM \twoheadrightarrow M$. There is also an inclusion map $M \hookrightarrow TM$ which sends p to the zero tangent vector at p.

For (U, φ) a chart on M, there is an injective map

(9)

$$\varphi(U) \times \mathbb{R}^n \hookrightarrow TM, \quad (\varphi(p), a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{x=\varphi(p)}$$

with p varying over points in U.



Define the topology of TM to be such that this map is a homeomorphism on to its image for all charts.

In other words, a set in TM is open if its inverse image under the map (9) is open in $\varphi(U) \times \mathbb{R}^n$ for all charts.

Proposition 4. *Let M be a smooth manifold. The tangent bundle TM is a smooth manifold. If M has dimension n, then TM has dimension $2n$.*

An atlas for TM is constructed from the atlas on M as follows.

Each chart (U, φ) on M yields a chart on TM, where the open set is $\pi^{-1}(U)$ and the map is

$$\pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

(The first map is essentially the inverse of (9).)

We need to check that these charts are compatible.
The reason in informal terms is:

A smooth map between euclidean spaces is automatically smooth on the tangent spaces, that is, “nearby” tangent vectors in the domain map to “nearby” tangent vectors in the codomain.

4.2 Tangent bundle functor

It is convenient to write elements of TM as pairs (p, v), where p is a point of M and v is a tangent vector at p.

If $f : M \to N$ is a map of smooth manifolds, then there is an induced smooth map

$$(f, f_*) : TM \to TN$$

which sends (p, v) to $(f(p), f_*(v))$ with the map f_* as in (8).

Moreover, the association of TM to M is functorial, that is, we have a functor

$$T : \mathsf{Manifold} \to \mathsf{Manifold} \tag{10}$$

from the category of smooth manifolds to itself.

Thus, for smooth maps $f : M \to N$ and $g : N \to P$ and $p \in M$, the diagram of linear maps

$$\begin{tikzcd} T_p(M) \arrow[r, "f_*"'] \arrow[rr, bend left, "(g\circ f)_*"] & T_{f(p)}(N) \arrow[r, "g_*"'] & T_{g(f(p))}(P) \end{tikzcd}$$

commutes.

This is in fact the chain rule that we learn in multivariable calculus stated in the generality of smooth manifolds.

Let us make the comparison between multivariable calculus and our present setting more precise.

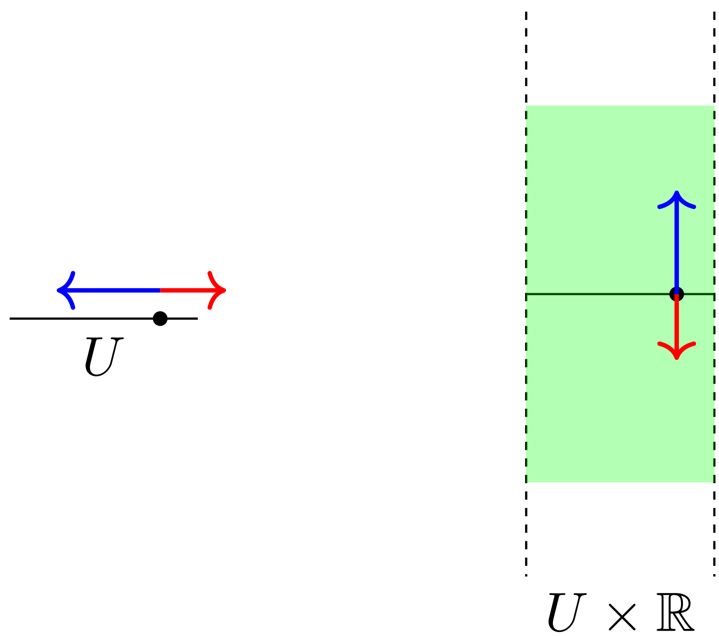
In multivariable calculus, the category that we deal with is the one whose objects are open sets in euclidean spaces and morphisms are smooth maps.

This is a full subcategory of the category of smooth manifolds.

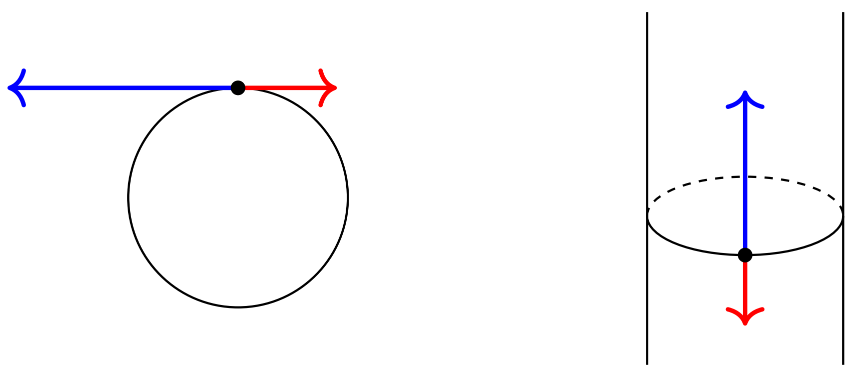
The tangent bundle functor restricts to this full subcategory and this is what we implicitly learn in multivariable calculus.

4.3 Examples

- Let U be any open subset of $\mathbb{R}^n$. Then the tangent bundle $TU = U \times \mathbb{R}^n$. An illustration for $n = 1$ is shown below.



- The tangent bundle of the circle S^1 is diffeomorphic to the infinite cylinder $S^1 \times \mathbb{R}^1$. The picture illustrates how tangent vectors to the circle can be vertically straightened.



- The tangent bundle of the sphere S^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

  This is a consequence of the hairy ball theorem:

  It is not possible to choose a smoothly varying nonzero tangent vector at each point of S^2.

4.4 Vector fields

Let M be a smooth manifold and $C(M)$ its algebra of smooth functions.

A smooth vector field on M is a linear map $X : C(M) \to C(M)$ such that

$$X(fg) = X(f)g + fX(g). \tag{11}$$

In the rhs, we are multiplying the functions $X(f)$ and g in the first term, and f and $X(g)$ in the second term.

In other words, a vector field is a derivation of $C(M)$ in the sense of (1).

Suppose X is a smooth vector field on M. Set

$$X_p(f) := X(f)(p),$$

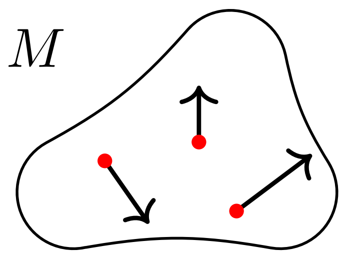
the function $X(f)$ evaluated at p. Equivalently, X_p is the composite map

$$C(M) \xrightarrow{X} C(M) \xrightarrow{\mathrm{ev}_p} \mathbb{R},$$

the second map being evaluation at p.

A comparison with (5) shows that X_p is a tangent vector at p.

The following is a way to visualize a vector field.



Is a vector field on M then the same as a choice of a tangent vector at each point of M?

Almost.

The only extra ingredient is that for any f, $X(f)$ is a smooth function. This means that the tangent vectors are varying smoothly with p.

This can be formalized by saying:

Proposition 5. *A smooth vector field on M is the same as a smooth section of the tangent bundle TM, that is, a smooth map $M \hookrightarrow TM$ such that the composite $M \hookrightarrow TM \twoheadrightarrow M$ is the identity.*

The zero section is always a smooth section of the tangent bundle. This vector field assigns the zero tangent vector at each point.

In local coordinates $(x_1, \dots, x_n)$, a smooth vector field can be written as

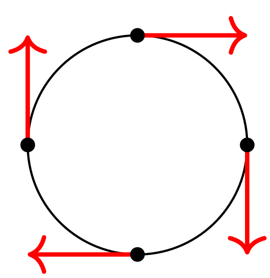
(12) $$\sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i},$$

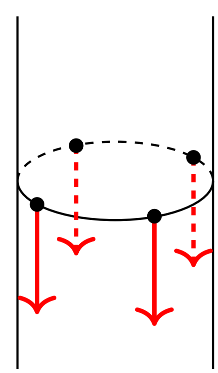
where each a_i is smooth function.

Compare and contrast with (6).

Example 4 (Vector fields on the circle)**.** A vector field on S^1 consists of a choice of a tangent vector at each point on S^1. It can also be viewed as a section of the tangent bundle which is $S^1 \times \mathbb{R}$.

An illustration is shown below with all vectors of the same length and pointing in the clockwise direction.





4.5 Lie algebra of vectors fields

Let $\Gamma(TM)$ denote the set of all smooth vector fields. It is a Lie algebra with bracket defined by

$$[X, Y] := X \circ Y - Y \circ X. \tag{13}$$

This is a special case of the general construction (2).

In local coordinates $(x_1, \ldots, x_n)$, for

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

$$[X, Y] = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j}.$$

In particular,

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

This is the same calculation as in Example 2.

4.6 Parallelizable manifolds

Manifolds with trivial tangent bundle are called parallelizable.

Taking products preserves this property. Since S^1 is parallelizable, it follows that the torus $S^1 \times S^1$ and all higher dimensional tori are parallelizable.

Lie groups are parallelizable.

We point out that S^1 is a Lie group under the product

$$S^1 \times S^1 \to S^1, \qquad (e^{i\theta}, e^{i\theta'}) \mapsto e^{i(\theta+\theta')}.$$

We mention that S^3 is also a Lie group. It is the group of unit quaternions. So it is also parallelizable.

5 Cotangent bundle and one-forms

We discuss the cotangent bundle of a smooth manifold. This is a construction dual to that of the tangent bundle with tangent vectors replaced by cotangent vectors.

Similarly, one-forms are sections of the cotangent bundle just as vector fields are sections of the tangent bundle.

A new ingredient is an operation which takes smooth functions to one-forms. This map is called the exterior derivative.

5.1 Dual vector space

Let $\mathbb{k}$ be a field.

For any vector space V over $\mathbb{k}$, its dual space V^* consists of linear maps $f : V \to \mathbb{k}$. The latter are also called linear functionals on V.

If V is finite-dimensional, then so is V^* and further its dimension equals that of V.

If $(e_1, \ldots, e_n)$ is an ordered basis of V, then its dual basis $(f_1, \ldots, f_n)$ is the ordered basis of V^* defined by

$$f_i(e_j) := \delta_{ij},$$

which is 1 if $i = j$ and zero otherwise.

If $f : V \to W$ is a linear map, then we have an induced linear map $f^* : W^* \to V^*$ which sends $g : W \to \mathbb{k}$ to $g \circ f : V \to \mathbb{k}$.

5.2 Cotangent vectors

Recall that T_pM denotes the tangent space of M at the point p.

Let T_p^*M denote the vector space dual to T_pM.

Explicitly,

$$T_p^*M = \{w : T_pM \to \mathbb{R} \mid w \text{ is a linear map}\}.$$

This is called the cotangent space of M at p.

An element of T_p^*M is called a cotangent vector at p.

In a chart on M with coordinates $(x_1, \ldots, x_n)$, recall that $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ is an ordered basis of T_pM (with the partials evaluated at p).

Its dual basis is denoted $(dx_1, \ldots, dx_n)$.

5.3 Change of coordinates

Let us now see how a cotangent vector transforms under change of coordinates.

Lemma 6. *Let $(x_1, \ldots, x_n)$ coming from (U, φ) and $(u_1, \ldots, u_n)$ coming from (V, ψ) be two coordinate systems containing a point p. Then*

$$du_j = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} dx_i. \tag{14}$$

Proof. Evaluating on $\frac{\partial}{\partial u_k}$ and using the transformation of a tangent vector given by (7), we are reduced to the matrix identity

$$\delta_{jk} = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} \frac{\partial x_i}{\partial u_k}.$$

□

This gives us an alternative way to define a cotangent vector at a point p:

Pick a cotangent vector $\sum_{i=1}^{n} a_i dx_i$ in each coordinate system $(x_1, \ldots, x_n)$ such that different choices transform into one another by formula (14).

5.4 Linear maps between cotangent spaces

Suppose $f : M \to N$ is a smooth map. Then for any point p in M, there is an induced linear map

(15)
$$f^* : T^*_{f(p)}N \to T^*_p M,$$

obtained by dualizing (8).

5.5 Cotangent bundle

Let M be any smooth manifold.

There is a smooth manifold T^*M called the cotangent bundle of M.

Its construction parallels that of the tangent bundle TM.

As a set

$$T^*M := \bigsqcup_{p\in M} T_p^*M.$$

We say that T_p^*M is the fiber of T^*M over p.

For (U, φ) a chart on M, there is an injective map

(16)

$$\varphi(U) \times \mathbb{R}^n \hookrightarrow T^*M, \qquad (\varphi(p), a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i dx_i$$

with p varying over points in U.

Define the topology of T^*M to be such that this map is a homeomorphism on to its image for all charts.

In other words, a set in T^*M is open if its inverse image under the map (16) is open in $\varphi(U) \times \mathbb{R}^n$ for all charts.

Proposition 6. *The cotangent bundle T^*M is a smooth manifold. If M has dimension n, then T^*M has dimension $2n$.*

An atlas for T^*M is constructed from the given atlas on M as follows. Each chart (U, φ) on M yields a chart on T^*M, where the open set is $\pi^{-1}(U)$ and the map is

$$\pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

(The first map is essentially the inverse of (16).)

It is convenient to write elements of T^*M as pairs (p, w), where p is a point of M and w is a cotangent vector at p.

Note that there is a canonical map $\pi : T^*M \twoheadrightarrow M$ which sends (p, w) to p. There is also an inclusion map $M \hookrightarrow T^*M$ which sends p to $(p, 0)$, the zero cotangent vector at p.

5.6 1-forms

A smooth 1-form ω on a smooth manifold M is a choice of a cotangent vector ω_p at each point p of M such that this choice varies smoothly with p.

More formally, a smooth 1-form on M is a smooth section of the cotangent bundle T^*M, that is, a smooth map $M \hookrightarrow T^*M$ such that the composite $M \hookrightarrow T^*M \twoheadrightarrow M$ is the identity.

Let $\Gamma(T^*M)$ denote the set of all smooth 1-forms.

In local coordinates $(x_1, \ldots, x_n)$, a smooth 1-form can be written as

(17) $$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n) dx_i,$$

where each a_i is smooth function.

1-forms and vector fields are dual notions in the following sense.

Suppose we have a smooth 1-form ω and a smooth vector field X. Then they can be “contracted” to obtain a smooth function f:

$$f(p) := \omega_p(X_p).$$

We will use the shorthand $f = \omega(X)$.

In fact, a choice of a cotangent vector at each point is smooth precisely if its contraction with every smooth vector field yields a smooth function.

Lemma 7. *A smooth 1-form is the same as a map of $C(M)$-modules from $\Gamma(TM)$ to $C(M)$.*

Proof. We saw how a 1-form ω yields a map from $\Gamma(TM)$ to $C(M)$ by sending a vector field X to the function $\omega(X)$.

Conversely suppose we are given such a map α. Then we construct the 1-form ω by letting

$$\omega_p(X_p) := \alpha(X)(p),$$

where X is any vector field whose tangent vector at p is X_p. The value is independent of this choice. Why? $\square$

5.7 Exterior derivative on functions

A 0-form on M is the same as a smooth function on M.

There is a map from 0-forms to 1-forms called the exterior derivative. It is given by

$$d : C(M) \to \Gamma(T^*M), \qquad f \mapsto df, \tag{18}$$

where the 1-form df is defined by

$$df(X) := X(f). \tag{19}$$

Suppose (U, φ) is a chart on M with coordinate system $(x_1, \dots, x_n)$. Then for any smooth function f, the 1-form df restricted to U is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i. \tag{20}$$

To see this, evaluate df on the basis vectors $\frac{\partial}{\partial x_i}$. By (19),

$$df(\frac{\partial}{\partial x_i}) = \frac{\partial f}{\partial x_i}$$

which yields (20).

6 Bundle of tensors and tensor fields

We have discussed the tangent and cotangent bundles of a smooth manifold.

Taking tensor product of the tangent bundle r times and of the cotangent bundle s times, one obtains the bundle of (r, s)-tensors.

Sections of this bundle are (r, s)-tensor fields.

This construction is a special case of tensor product of vector bundles, which we will discuss separately.

6.1 Bundle of (r, s)-tensors

Let r and s be any nonnegative integers.

Let M be a smooth manifold.

Recall that TM is the tangent bundle of M and T^*M is the cotangent bundle of M.

$$TM := \bigsqcup_{p\in M} T_pM \quad \text{and} \quad T^*M := \bigsqcup_{p\in M} T_p^*M.$$

Using the tangent and cotangent bundles of M, one can construct the bundle of (r, s)-tensors

$$TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M$$

$$:= \bigsqcup_{p \in M} T_pM \otimes \cdots \otimes T_pM \otimes T_p^*M \otimes \cdots \otimes T_p^*M$$

with r factors of T_pM and s factors of T_p^*M.

It is a smooth manifold, with its smooth structure constructed in the obvious manner.

As special cases,

- the bundle of $(1, 0)$-tensors is the tangent bundle,
- the bundle of $(0, 1)$-tensors is the cotangent bundle.

By convention, the 0-th tensor power of any $\mathbb{R}$-vector space is $\mathbb{R}$.

What would be the bundle of $(0, 0)$-tensors?

6.2 (r, s)-tensors

A (r, s)-tensor at the point $p \in M$ is an element of

$$T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}.$$

This vector space is the fiber over p.

Thus a (r, s)-tensor at p is an element in the fiber over p.

As special cases,

- a $(0, 0)$-tensor at p is a real number (also called a scalar),
- a $(1, 0)$-tensor at p is a tangent vector at p,
- a $(0, 1)$-tensor at p is a cotangent vector at p.

6.3 (r, s)-tensor fields

A (r, s)-tensor field is a choice of a (r, s)-tensor at each point p which varies smoothly with p.

As special cases,

- a $(0, 0)$-tensor field is a smooth function (also called a scalar field),
- a $(1, 0)$-tensor field is a smooth vector field,
- a $(0, 1)$-tensor field is a smooth 1-form.

This can be said more formally using the language of sections.

Recall that smooth vector fields and one-forms can be written in local coordinates using formulas (12) and (17).

Similarly, locally, in a chart (U, φ) on M with coordinates $(x_1, \ldots, x_n)$, a $(2, 0)$-tensor field can be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \ldots, x_n) \, \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j},$$

and a $(0, 2)$-tensor field as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \ldots, x_n) \, dx_i \otimes dx_j,$$

where each a_{ij} is a smooth function.

We now give another equivalent way to define a (r, s)-tensor field.

Lemma 8. *A (r, s)-tensor field is a map of $C(M)$-modules*

$$\underbrace{\Gamma(TM) \otimes \cdots \otimes \Gamma(TM)}_{s} \otimes \underbrace{\Gamma(T^*M) \otimes \cdots \otimes \Gamma(T^*M)}_{r}$$
$$\downarrow$$
$$C(M).$$

All tensor products are taken over the algebra $C(M)$.

Note very carefully that r and s have switched positions.

When $r = 0$ and $s = 1$, Lemma 8 reduces to Lemma 7.

Proof. The lhs above can be identified with

$$\Gamma(\underbrace{TM \otimes \cdots \otimes TM}_{s} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{r}),$$

the space of sections of the bundle of (s, r)-tensors.

A map of $C(M)$-modules from this space to $C(M)$ (sections of the trivial line bundle over M) is the same as an element of

$$\Gamma((\underbrace{TM \otimes \cdots \otimes TM}_{s} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{r})^*),$$

the space of sections of the dual of the the bundle of (s, r)-tensors which is the bundle of (r, s)-tensors.

(The duality switches tangent and cotangent.) $\square$

7 Differential forms

There is particular interest surrounding the exterior powers of the cotangent bundle of a smooth manifold.

This is the bundle of differential forms.

The space of sections of this bundle has the structure of a differential graded algebra.

7.1 Bundle of k-forms

Let k be any nonnegative integer. Similar to the bundle of (r, s)-tensors, one can construct the bundle

$$\wedge^k(T^*M) := \bigsqcup_{p\in M} \wedge^k(T_p^*M)$$

of smooth k-forms.

The fiber over p is the k-th exterior power of the cotangent space at p.

Note that when k exceeds the dimension of M, the fibers are the zero space.

When $k = 0$, the fibers are $\mathbb{R}$.

The bundle of smooth k-forms is a smooth manifold.

7.2 k-forms

There is no particular term used for a k-fold wedge of cotangent vectors. This would be the analogue of a (r, s)-tensor.

The analogue of a (r, s)-tensor field is a smooth k-form. Formally, this is a section of $\wedge^k(T^*M)$. We denote the set of all smooth k-forms by $\Omega^k(M)$. This is a vector space. Note that

$$\Omega^0(M) = C(M) \quad \text{and} \quad \Omega^1(M) = \Gamma(T^*M).$$

Locally, in a chart with coordinates $(x_1, \ldots, x_n)$, a smooth k-form can be written as

$$\sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 \ldots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{21}$$

How would a smooth k-form transform under a change of coordinates?

To go from $(u_1, \ldots, u_n)$ to $(x_1, \ldots, x_n)$, use formula (14) on each du_{i_j} occurring in the wedge $du_{i_1} \wedge \cdots \wedge du_{i_k}$.

Linearity in each factor of the wedge can then be used to write (21) as a sum of the wedges of the dx_{i_j}.

7.3 Bundle of differential forms

One can put together the bundles of k-forms as k varies between 0 and n, where n is the dimension of M:

(22) $$\wedge(T^*M) := \bigoplus_{k=0}^{n} \wedge^k(T^*M).$$

This is the bundle of differential forms.

The notation means that the fiber at each point p is the direct sum (as vector spaces) of the fibers at p of each of the $n+1$ bundles in the rhs.

In other words, the fiber at p is the exterior algebra of T_p^*M.

7.4 Differential forms

A differential form is a section of the bundle (22). The vector space of all differential forms is denoted $\Omega(M)$.

Observe that

$$\Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M). \tag{23}$$

Thus a differential form can be uniquely decomposed as a sum of a 0-form, a 1-form, and so forth.

In local coordinates, it can be written as a sum of terms of the form (21), one for each k.

For instance,

$$f(x_1, x_2)+g_1(x_1, x_2)\,dx_1+g_2(x_1, x_2)\,dx_2+h(x_1, x_2)\,dx_1\wedge dx_2$$

is the general expression for a differential form on a surface in local coordinates (x_1, x_2).

7.5 Algebra of differential forms

Recall that the fibers of $\wedge(T^* M)$ are exterior algebras. By operating fiberwise, one can multiply forms.

More formally, the vector space $\Omega(M)$ is a graded commutative algebra.

The term graded means that the product of a k-form and a ℓ-form is a $(k + \ell)$-form.

The product of $\Omega(M)$ (just like the exterior algebra) is denoted by $\wedge$.

In local coordinates, for instance,

$$(g_1\, dx_1 + g_2\, dx_2) \wedge (h_1\, dx_1 + h_2\, dx_2) = (g_1 h_2 - g_2 h_1)\, dx_1 \wedge dx_2.$$

The term commutative is to be understood here in the signed sense, that is,

$$\omega \wedge \omega' = (-1)^{kk'} \omega' \wedge \omega,$$

where ω is a k-form and ω' is a k'-form. This comes from the fact that the exterior algebra is signed commutative.

For instance, in the above calculation, if we multiply in the other order, then we get the negative of the rhs above.

7.6 Exterior derivative

Recall the exterior derivative d from 0-forms to 1-forms (18). It extends to the algebra of differential forms as follows.

Theorem 2. *There is a unique sequence of $\mathbb{R}$-linear maps*

$$d : \Omega^k(M) \to \Omega^{k+1}(M), \qquad k = 0, 1, 2, \ldots$$

such that

1. *for $k = 0$, the map d agrees with* (18)*,*
2. *$d^2 = 0$,*
3. *$d(w \wedge w') = dw \wedge w' + (-1)^k w \wedge dw'$ for $w \in \Omega^k(M)$.*

In local coordinates $(x_1, \ldots, x_n)$, the map d is given by

(24)

$$d(a_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial a_{i_1\ldots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Proof. Let us start with $k = 1$. Suppose ω is a 1-form. Write it locally as $a_1 dx_1 + \cdots + a_n dx_n$. Properties (2) and (3) force

$$d\omega = da_1 \wedge dx_1 + \cdots + da_n \wedge dx_n,$$

and the da_i have been defined by property (1). In general, observe that these three properties force (24). So the uniqueness assertion is clear.

To show that d is well-defined, one needs to check that changing coordinates yields the same local formula but in the other coordinate system. We leave this as an exercise.

Next one needs to check that properties (2) and (3) hold. It suffices to check them in each chart using the local formula (24).

For instance,

$$d^2(f) = d\big(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = 0.$$

The terms with $i = j$ clearly do not contribute. For $i \neq j$, the terms involving $dx_i \wedge dx_j$ and $dx_j \wedge dx_i$ cancel.

Appreciate that this happens because we are in an exterior algebra, as opposed to say the symmetric algebra. $\square$

In view of the above properties, we say that $\Omega(M)$ is a differential graded algebra, the product is $\wedge$ and the differential is d.

Note that the product is constructed fiberwise but not d which is saying how the section values are changing as we move across fibers.

The notions of gradient, curl, divergence that we learn in multivariable calculus are instances of the exterior derivative d.

For $M = \mathbb{R}^3$,

$$0\text{-forms} \xrightarrow{\text{gradient}} 1\text{-forms} \xrightarrow{\text{curl}} 2\text{-forms} \xrightarrow{\text{divergence}} 3\text{-forms.}$$

The fact that

$$\text{curl}(\text{gradient}) = 0 \quad \text{and} \quad \text{divergence}(\text{curl}) = 0$$

are instances of the property $d^2 = 0$.

7.7 Forms as alternating maps

The following extends Lemma 7.

Lemma 9. *There is a canonical linear isomorphism between $\Omega^k(M)$ and alternating $C(M)$-module maps from the k-fold tensor product of $\Gamma(TM)$ to $C(M)$.*

Explicitly, for 1-forms ω_i and vector fields X_j,

$$(\omega_1 \wedge \cdots \wedge \omega_k)(X_1 \otimes \cdots \otimes X_k) = \det(\omega_i(X_j)).$$

For $k = 2$,

(25)

$$(\omega_1 \wedge \omega_2)(X_1 \otimes X_2) = \omega_1(X_1)\omega_2(X_2) - \omega_1(X_2)\omega_2(X_1).$$

Note that this is consistent with the identity

$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$.

Lemma 10. *For any smooth k-form ω, and vector fields $X_1, \ldots, X_{k+1}$,*

(26)

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

where the hat over X_i indicates that X_i is deleted from the sequence.

In particular, for a one-form ω,

(27)

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

In an alternative approach, one can define the exterior derivative by (26) and then establish Theorem 2.

7.8 Functoriality

Proposition 7. *If $f : M \to N$ is a smooth map, then it induces a morphism $f^* : \Omega(N) \to \Omega(M)$ of differential graded algebras.*

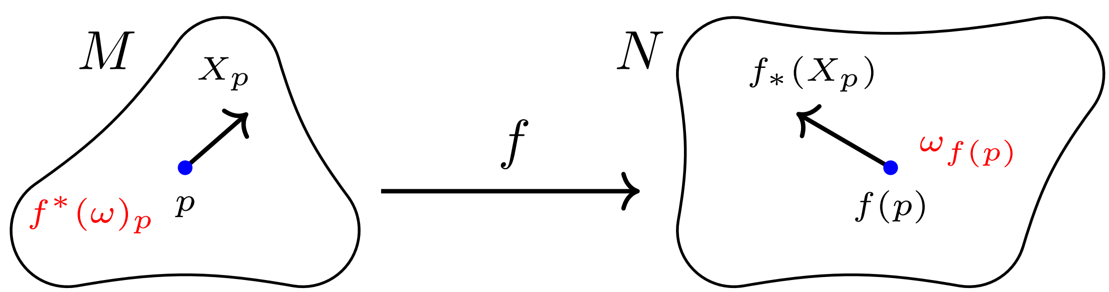
Proof. Given a differential form ω on N, it can be pulled back to a differential form $f^*(\omega)$ on M via the map f. For this, we can employ Lemma 9.

Let us elaborate on the case $k = 1$.

For ω a one-form on N, define a one-form $f^*(\omega)$ on M by

$$f^*(\omega)_p(X_p) := \omega_{f(p)}(f_*(X_p)).$$

An illustration is shown below.



Further, one may check that f^* is an algebra morphism, and f^* commutes with the exterior derivatives on M and N. In other words, f^* is a morphism of differential graded algebras. $\square$

Thus Ω is a contravariant functor

$$\mathsf{Manifold} \to \mathsf{dgAlg}$$

from the category of smooth manifolds to the category of differential graded algebras.

8 Tensor, shuffle, symmetric, exterior algebras

We discuss some important algebras associated to a vector space listed below.

- Tensor algebra
- Shuffle algebra
- Symmetric algebra (Two versions)
- Exterior algebra (Two versions)

8.1 Tensor product of vector spaces

Fix a field $\mathbb{k}$.

The category of vector spaces Vec has vector spaces over $\mathbb{k}$ as objects and $\mathbb{k}$-linear maps as morphisms.

The zero space is the initial as well as the terminal object.

The product and coproduct are both given by direct sum.

Let V and W be vector spaces over $\mathbb{k}$. An important construction is the tensor product $V \otimes W$. Elements of this vector space are formal sums

$$\sum_{i=1}^{n} a_i(v_i \otimes w_i), \qquad a_i \in \mathbb{k}, v_i \in V, w_i \in W$$

subject to the relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$
$$w \otimes (v_1 + v_2) = w \otimes v_1 + w \otimes v_2,$$
$$a(v \otimes w) = av \otimes w = v \otimes aw.$$

If V has basis $\{e_1, \ldots, e_m\}$ and W has basis $\{f_1, \ldots, f_n\}$, then $V \otimes W$ has basis $\{e_i \otimes f_j\}_{ij}$.

The tensor product is functorial. If $f : V \to V'$ and $g : W \to W'$ are linear maps, then there is an induced linear map $f \otimes g : V \otimes W \to V' \otimes W'$.

Note that the tensor product is neither the product nor the coproduct in Vec, but it defines a monoidal structure on Vec.

Monoids wrt the tensor product are called algebras. More explicitly, an algebra is a vector space A equipped with linear maps

$$A \otimes A \to A \quad \text{and} \quad \Bbbk \to A$$

subject to associativity and unitality axioms.

8.2 Tensor algebra and shuffle algebra

Let V be a vector space over the field $\mathbb{k}$.

The tensor algebra of V is

$$\mathcal{T}(V) := \bigoplus_{k \geq 0} V^{\otimes k},$$

where $V^{\otimes k}$ denotes the k-fold tensor product of V.

The product is concatenation of tensors:

$$(u_1 \otimes \cdots \otimes u_m) \otimes (v_1 \otimes \cdots \otimes v_n) \mapsto u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n.$$

Note that $\mathcal{T}(V)$ is a graded algebra, the grading is given by the number of tensor factors.

The product is not commutative.

If V is finite-dimensional with basis $(e_1, \ldots, e_n)$, then elements of $\mathcal{T}(V)$ are noncommutative polynomials in the e_i, with product being the usual multiplication of polynomials. The standard notation for this is $\Bbbk\langle e_1, \ldots, e_n \rangle$.

A related construction is that of the shuffle algebra of a vector space:

$$\mathcal{T}^{\vee}(V) := \bigoplus_{k \geq 0} V^{\otimes k}.$$

It has the same underlying vector space as $\mathcal{T}(V)$, but the product is different. It is the sum over all ways to shuffle the two tensors.

For example:

$$(u \otimes v) \otimes w \mapsto u \otimes v \otimes w + u \otimes w \otimes v + w \otimes u \otimes v.$$

Note that $\mathcal{T}^{\vee}(V)$ is also a graded algebra, the grading is given by the number of tensor factors. Further, the product is commutative.

There is a signed version of this product, which keeps track of the parity of the shuffle.

For example:

$$(u \otimes v) \otimes w \mapsto u \otimes v \otimes w - u \otimes w \otimes v + w \otimes u \otimes v.$$

This defines the signed shuffle algebra, which we denote by $\mathcal{T}_{-1}^{\vee}(V)$.

The product is now graded commutative.

8.3 Symmetric algebra (Coinvariant and invariant versions)

Consider the symmetrization

$$V^{\otimes k} \to V^{\otimes k}, \quad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The sum is over all permutations on k letters.

For example,

$$u \otimes v \mapsto u \otimes v + v \otimes u.$$

Adding over all $k \geq 0$, we obtain a map

$$\kappa : \mathcal{T}(V) \to \mathcal{T}^{\vee}(V). \tag{28}$$

This is called the norm map.

Proposition 8. *The norm map is a morphism of graded algebras, that is, symmetrization relates concatenation to shuffling.*

Proof. Exercise. □

The norm map is far from being an isomorphism; this is of significance.

Let $\mathcal{S}^\vee(V)$ denote its image and $\mathcal{S}(V)$ denote its coimage.

This yields the following commutative diagram of graded algebras.

$$
\begin{tikzcd}
\mathcal{T}(V) \arrow[r, "\kappa"] \arrow[d, two heads] & \mathcal{T}^\vee(V) \\
\mathcal{S}(V) \arrow[r, "\cong"'] & \mathcal{S}^\vee(V) \arrow[u, hook]
\end{tikzcd}
\tag{29}
$$

The algebras $\mathcal{S}(V)$ and $\mathcal{S}^{\vee}(V)$ are isomorphic.

Either one is called the symmetric algebra on V.

Elements of $\mathcal{S}(V)$ are tensors such as $u \otimes v$ subject to the relation

$$u \otimes v = v \otimes u$$

while elements of $\mathcal{S}^{\vee}(V)$ are symmetric tensors such as

$$u \otimes v + v \otimes u.$$

If V is finite-dimensional with basis $(e_1, \ldots, e_n)$, then elements of $\mathcal{S}(V)$ are commutative polynomials in the e_i, with product being the usual multiplication of polynomials.

The standard notation for this is $\mathbb{k}[e_1, \ldots, e_n]$, and is commonly referred to as the polynomial algebra.

The quotient map $\mathcal{T}(V) \twoheadrightarrow \mathcal{S}(V)$ is

$$\mathbb{k}\langle e_1, \ldots, e_n\rangle \twoheadrightarrow \mathbb{k}[e_1, \ldots, e_n], \qquad e_i \mapsto e_i.$$

There is another way of describing the symmetric algebras using invariants and coinvariants:

Consider the left action of the symmetric group S_k on $V^{\otimes k}$ where the element $\sigma \in \mathrm{S}_k$ acts by

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The symmetric algebra of V is

$$\mathcal{S}(V) = \bigoplus_{k \geq 0} (V^{\otimes k})_{\mathrm{S}_k} \quad \text{and} \quad \mathcal{S}^{\vee}(V) = \bigoplus_{k \geq 0} (V^{\otimes k})^{\mathrm{S}_k},$$

where $(-)_{\mathrm{S}_k}$ denotes the space of S_k-coinvariants, and $(-)^{\mathrm{S}_k}$ denotes the space of S_k-invariants.

8.4 Exterior algebra (Coinvariant and invariant versions)

Consider the antisymmetrization

$$V^{\otimes k} \to V^{\otimes k}, \quad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)},$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

For example,

$$u \otimes v \mapsto u \otimes v - v \otimes u.$$

Adding over all $k \geq 0$, we obtain a morphism of algebras

$$\kappa_{-1} : \mathcal{T}_{-1}(V) \to \mathcal{T}^{\vee}_{-1}(V). \tag{30}$$

This is the signed norm map.

The signed norm map is also far from being an isomorphism.

Let $\Lambda^{\vee}(V)$ denote its image and $\Lambda(V)$ denote its coimage.

This yields the following commutative diagram of graded algebras.

$$\begin{tikzcd} \mathcal{T}_{-1}(V) \arrow[r, "\kappa_{-1}"] \arrow[d, two heads] & \mathcal{T}_{-1}^{\vee}(V) \\ \Lambda(V) \arrow[r, "\cong"'] & \Lambda^{\vee}(V) \arrow[u, hook] \end{tikzcd} \tag{31}$$

The algebras $\Lambda(V)$ and $\Lambda^{\vee}(V)$ are isomorphic.

Either one is called the exterior algebra on V. The term Grassmann algebra is also used.

Elements of $\Lambda(V)$ are wedges such as $u \wedge v$ subject to the relation

$$u \wedge v = -v \wedge u.$$

(It is standard to use wedge instead of tensor to distinguish this quotient from $\mathcal{S}(V)$.)

Elements of $\Lambda^{\vee}(V)$ are antisymmetric tensors such as

$$u \otimes v - v \otimes u.$$

The isomorphism between the two exterior algebras antisymmetrizes the wedge.

$$\Lambda(V) \to \Lambda^{\vee}(V), \qquad u \wedge v \mapsto u \otimes v - v \otimes u.$$

If V is finite-dimensional with basis $(e_1, \ldots, e_n)$, then $\Lambda(V)$ has a basis given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n}.$$

Thus $\Lambda(V)$ is a finite dimensional graded algebra:

$$\Lambda(V) = \bigoplus_{k=1}^{n} \Lambda^k(V).$$

The k-th component is called the k-th exterior power of V, its dimension is $\binom{n}{k}$.

For any finite-dimensional vector space W, there is a canonical linear isomorphism

$$W \to \mathrm{Hom}(W^*, \Bbbk) = (W^*)^*, \quad w \mapsto (f \mapsto f(w)).$$

Now take $W = (\Lambda^\vee)^k(V)$.

First note that $(\Lambda^\vee)^k(V)^* \cong \Lambda^k(V^*)$.

Further, a linear map from $\Lambda^k(V^*)$ to $\Bbbk$ is the same as a linear map

$$V^* \otimes \cdots \otimes V^* \to \Bbbk$$

(with k tensor factors) which is alternating.

Alternating means that the value of the map changes sign if the two adjacent tensor factors are switched.

By precomposing by the isomorphism from $\Lambda^k(V)$ to $(\Lambda^\vee)^k(V)$:

Lemma 11. *For a finite-dimensional vector space V, there is a canonical linear isomorphism between $\Lambda^k(V)$ and alternating linear maps from the k-fold tensor product of V^* to $\mathbb{k}$.*

Explicitly,

$$(v_1 \wedge \cdots \wedge v_k)(f_1 \otimes \cdots \otimes f_k) = \det(f_j(v_i)).$$

(We need to antisymmetrize the wedge and then evaluate.)

8.5 q-shuffle algebra

Let q be any scalar.

One can define an algebra $\mathcal{T}_q^\vee(V)$, whose product is given by q-shuffling:

The coefficient of the shuffle is q power the number of interchanges of the shuffle.

This is the q-shuffle algebra.

For $q = 1$, we recover the shuffle algebra, and for $q = -1$, we recover the signed shuffle algebra.

Next, one can then define the q-norm map $\kappa_q : \mathcal{T}(V) \to \mathcal{T}_q^\vee(V)$ by using q-symmetrization.

This map is an isomorphism if q not a root of unity.

Observe that, for $q = 0$, the map is in fact the identity.