MA 556 PRESENTATION

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Problem 2

Problem 2 of Notes Chapter 2

For any field $\mathbb{K}$ and $\mathbb{K}$-algebra A, we have an inclusion $\mathbb{K} \hookrightarrow A$.
Show that:
For any derivation D of A, we have $D(c) = 0$ for any scalar $c \in \mathbb{K}$.
Is the same true for a derivation D of A into a bimodule M?

Problem 2 (Contd.)

Recall the definition of a derivation on an algebra A.

Definition of Derivation

For any $\mathbb{K}$-algebra A, and a bimodule M, a linear map $D : A \to M$ is a derivation iff

1. $D(a+b) = D(a) + D(b)$
2. $D(ca) = cD(a)$ for all scalars c
3. $D(ab) = a.D(b) + D(a).b$

Problem 2 (Contd.)

In a field $\mathbb{K}$ and $\mathbb{K}$-algebra A, we attempt to calculate $D(c)$ for $c \in \mathbb{K}$, c being a scalar. Our inclusion map is $\mathbb{K} \hookrightarrow A$.

$$\begin{aligned}
D(c) &= D(c \times \mathbb{I}) && \text{(property of } \mathbb{I}\text{)}\\
&= cD(\mathbb{I}) && \text{(definition of scalar)}\\
&= cD(\mathbb{I} \times \mathbb{I}) && (\mathbb{I} \times \mathbb{I} = \mathbb{I})\\
&= c(D(\mathbb{I}) \times \mathbb{I} + \mathbb{I} \times D(\mathbb{I})) && \text{(Leibniz Rule)}\\
&= c(D(\mathbb{I}) + D(\mathbb{I})) && \text{(property of } \mathbb{I}\text{)}\\
&= 2cD(\mathbb{I})
\end{aligned}$$

Thus,

$$\begin{aligned}
&D(c) = cD(\mathbb{I}) = 2cD(\mathbb{I})\\
\Rightarrow &D(c) = 0
\end{aligned}$$

A similar argument works on bimodules. Derivation of any scalar will be 0.

Problem 3

Problem 3 of Notes Chapter 2

Write down an explicit formula for the Lie bracket on the space of derivations of $\mathbb{R}[x]$.

Problem 3 (Contd.)

We have the algebra $\mathbb{R}[x]$ generated by x. Now, we can send x to any **polynomial** function $a(x)$ so that that derivation is given by

$$D = a(x)\frac{d}{dx}$$

Now consider two derivations $D = a(x)\frac{d}{dx}$ and $E = b(x)\frac{d}{dx}$ so that

$$\begin{aligned}
[D, E]\psi &= (D \circ E)\psi - (E \circ D)\psi \\
&= a(x)\frac{d}{dx}\left(b(x)\frac{d\psi}{dx}\right) - b(x)\frac{d}{dx}\left(a(x)\frac{d\psi}{dx}\right) \\
&= ab\frac{d^2\psi}{dx^2} + a\frac{db}{dx}\frac{d\psi}{dx} - b\frac{da}{dx}\frac{d\psi}{dx} - ba\frac{d^2\psi}{dx^2} \\
&= a\frac{db}{dx}\frac{d\psi}{dx} - b\frac{da}{dx}\frac{d\psi}{dx}
\end{aligned}$$

Problem 3 (Contd.)

Thus,

$$[D, E] = \left(a(x)\frac{db(x)}{dx} - b(x)\frac{da(x)}{dx} \right) \frac{d}{dx}$$

Taylor's Theorem

Lemma 2.9 of Notes Chapter 2

For any smooth function f on an open set in $\mathbb{R}^n$ containing 0,

$$f(x) = f(0) + \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

$$f(x) = f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j} x_i x_j \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tx)dt$$

Taylor's Theorem (Contd.)

We will derive a generalized version of the Taylor's Theorem about 0 with remainder in $\mathbb{R}^n$. Taylor's Theorem in a $\mathbb{R}$ helps to approximate the value of functions at certain points. In higher dimensions, it will help us approximate the value of a function.
We will assume familiarity with the concepts of limits, continuity, derivatives, partial derivatives, directional derivatives from MA 105 (or more recently, MA 111). We will apply the product, quotient and chain rules of regular derivatives directly. We will also presume familiarity with the fundamental theorem of Calculus.
First off, we start with some notations. We will use the Einstein summation notation for ease. Let us define the tensors

$$x^{\otimes k}_{a_1 a_2 \cdots a_k} = x_{a_1} x_{a_2} \cdots x_{a_k}$$

$$f^{(k)}_{a_1 a_2 \cdots a_k}(x) = \frac{\partial^k f(x)}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_k}}$$

Taylor's Theorem (Contd.)

Also, for simplicity, let

$$x^{\otimes k} f^{(k)}(x) = f^{(k)}(x) x^{\otimes k} = x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(x) \quad \text{(summation notation)}$$

We define

$$g^k_{a_1 a_2 \cdots a_k}(x) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} f^{(k)}_{a_1 a_2 \cdots a_k}(tx) dt$$
$$h^k(x) = g^k_{a_1 a_2 \cdots a_k}(x) x^{\otimes k}_{a_1 a_2 \cdots a_k}$$

Observe that

$$h^1(x) = g^1_{a_1}(x) x_{a_1} = \int_0^1 x_{a_1} f'_{a_1}(tx) dt = f(x) - f(0) \quad \text{(FTC)} \quad (1)$$

Taylor's Theorem (Contd.)

Now,

$$h^k(x) - h^{k+1}(x) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(tx) - \frac{(1-t)^k}{(k)!} x^{\otimes (k+1)}_{a_1 a_2 \cdots a_{k+1}} f^{(k+1)}_{a_1 a_2 \cdots a_{k+1}}(tx) dt$$

Note that we can use chain rule and product rule to show that

$$\frac{d}{dt}\left(\frac{(1-t)^k}{(k)!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(tx)\right) = -\frac{(1-t)^{k-1}}{(k-1)!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(tx) + \frac{(1-t)^k}{(k)!} x^{\otimes (k+1)}_{a_1 a_2 \cdots a_{k+1}} f^{(k+1)}_{a_1 a_2 \cdots a_{k+1}}(tx)$$

Taylor's Theorem (Contd.)

Thus,

$$h^k(x) - h^{k+1}(x) = -\int_0^1 \frac{d}{dt}\left(\frac{(1-t)^k}{(k)!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(tx)\right) dt$$

$$= \frac{1}{k!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(0) \tag{FTC}$$

Summing k from 1 to k in the above equation

$$h^1(x) - h^{k+1}(x) = \sum_{k=1}^{n} \frac{1}{k!} x^{\otimes k}_{a_1 a_2 \cdots a_k} f^{(k)}_{a_1 a_2 \cdots a_k}(0)$$

Substituting the values of the tensors

$$h^1(x) - h^{k+1}(x) = \sum_{k=1}^{n} \sum_{a_1 a_2 \cdots a_k} \frac{1}{k!} x_{a_1} x_{a_2} \cdots x_{a_k} \frac{\partial^k f}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_k}}(0) \tag{2}$$

Taylor's Theorem (Contd.)

Following from the definition

$$h^{k+1}(x) = x^{\otimes(k+1)}_{a_1 a_2 \cdots a_{k+1}} g^{k+1}_{a_1 a_2 \cdots a_{k+1}}(x) = \int_0^1 \frac{(1-t)^k}{(k)!} x^{\otimes(k+1)}_{a_1 a_2 \cdots a_{k+1}} f^{(k+1)}_{a_1 a_2 \cdots a_{k+1}}(tx)dt$$

$$= \sum_{a_1, a_2, \cdots a_{k+1}} \left[x_{a_1} x_{a_2} \cdots x_{a_{k+1}} \int_0^1 \frac{(1-t)^k}{(k)!} \frac{\partial^{k+1}}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_{k+1}}} f(tx)dt \right] \tag{3}$$

Substituting 1 and 3 in 2, we get

$$f(x) = f(0) + \sum_{k=1}^{n} \sum_{a_1 a_2 \cdots a_k} \frac{1}{k!} x_{a_1} x_{a_2} \cdots x_{a_k} \frac{\partial^k f}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_k}}(0)$$

$$+ \sum_{a_1, a_2, \cdots a_{k+1}} \left[x_{a_1} x_{a_2} \cdots x_{a_{k+1}} \int_0^1 \frac{(1-t)^k}{(k)!} \frac{\partial^{k+1}}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_{k+1}}} f(tx)dt \right]$$

The cases in the question correspond to $k = 0$ and $k = 1$ respectively.

THANK YOU!