An introduction to differential geometry

Swapneel Mahajan

Department of Mathematics
Indian Institute of Technology Mumbai
Powai, Mumbai 400 076
India
E-mail address: swapneel@math.iitb.ac.in
URL: http://www.math.iitb.ac.in/~swapneel

Contents

Prerequisites.

- You should be familiar with linear algebra, multivariable calculus, point set topology, basic algebra including modules and tensor products, and basic language of category theory. Some exposure to algebraic topology is useful.

References. Here is a list of useful references, which is by no means exhaustive.

- For topology, munkres [20], armstrong [2], janich [12, 5], warner [27],
- For geometry on surfaces, pressley [23], thorpe [25], docarmo [7],
- For riemannian geometry in general, docarmo [8, 9], chavel [6], boothby [4], morita [18], lee [14], petersen [22], jost [13], spivak [24],
- For a unique treatment of manifolds, nestruev [21],
- For informal expositions, berger [3], thurston [26]
- For category theory, maclane [15].
- Wikipedia is a good source for getting a birds-eye-view of many of the concepts discussed in these notes.

Pick a book that suits you. To understand the subject matter, it is not necessary to understand each and every sentence written in a particular book.

A number of exercises are included in the notes (many with complete solutions). Apart from the above sources, the exercises are mainly borrowed from a variety of sources which I have not kept track of.

CHAPTER 1

Manifolds

1.1. Topological spaces

You are assumed to be familiar with this concept. An informal discussion is given below.

1.1.1. Topological spaces. A topological space X is a set with a qualitative notion of closeness which is formalized by the language of open sets. (In a metric space, the notion of closeness is quantitative.) A function $f : X \to Y$ between topological spaces is continuous if it maps points close to X to points close to Y. Precise definitions are given in [20].

As examples,

$$\mathbb{R}, \quad S^1, \quad \mathbb{R}^2, \quad \mathbb{R}^2 \setminus \{0\}$$

are topological spaces, and

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2, \qquad f : \mathbb{R} \to S^1,\ f(x) = e^{2\pi i x}$$

are continuous maps.

Two topological spaces X and Y are homeomorphic if there is a continuous map $f : X \to Y$ which is a bijection and whose inverse is also continuous. For instance, e^x is a homeomorphism from $\mathbb{R}$ to $(0, \infty)$ with inverse given by $\log(x)$.

1.1.2. Categories of interest. Informally, the category of topological spaces consists of

- all topological spaces $X, Y, \ldots$,
- all continuous maps $X \to Y$ for any topological spaces X and Y.

We denote this category by Top.

Formally, a category consists of objects and morphisms. There is a notion of isomorphism of objects in any category. Refer Mac Lane.

In the category of topological spaces, objects are topological spaces and morphisms are continuous maps. An isomorphism in this category is precisely a homeomorphism. Topology is the study of this category.

We are interested in the following categories.

- topological spaces (where we can talk of closeness)
- topological manifolds (which look locally like euclidean spaces)
- smooth manifolds (where we can talk of tangent spaces)
- riemannian manifolds (where we can talk of geodesics and curvature)

Each category is obtained from the previous one by imposing either more conditions or more structure. The category of topological spaces is the most basic. Note that there is a more basic category, namely that of sets, which underlies topological spaces. In the category of sets, objects are sets (possibly infinite) and morphisms are ordinary functions. So for two topological spaces to be homeomorphic, their underlying sets must be bijective.

1.1.3. Review of standard definitions. Let X be a topological space.

- X is *Hausdorff* if for any distinct points $x, y \in X$, there exist neighborhoods of x and y which are disjoint. (Recall that a *neighborhood* of a point x in X is an open set U of X containing x.)
- X is *compact* if every open cover of X has a finite subcover.
- X is *second countable* if it has a countable basis. (That is, X has a countable collection of open sets with the property that each open set of X is a union of elements in this collection.)

Connected and path-connected spaces. A topological space X is

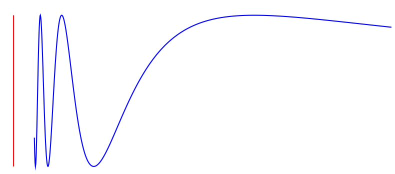
- *connected* if it cannot be written as a disjoint union of two nonempty open sets,
- *path-connected* if there is a path joining any two points of X. (A path from x to y is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.)

Recall that path-connected implies connected. The converse is not true in general.

Example 1.1 (The topologist's sine curve). Consider the set

$$\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ y = \sin(1/x)\} \cup \{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$$

with subspace topology from $\mathbb{R}^2$.



The red line is the part of the y-axis between -1 and 1. The blue line is the graph of the function $y = \sin(1/x)$. (It oscillates rapidly as it approaches the y-axis.)

This is the topologist's sine curve. It is connected but not path-connected.

Locally euclidean spaces. A topological space X is *locally euclidean* if every point of X has a neighborhood homeomorphic to $\mathbb{R}^n$ for some n.

The open n-ball of radius r around x_0 is

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}.$$

By translation and scaling, we see that $B(x_0, r)$ and $B(0,1)$ are homeomorphic. (The scaling aspect here ties in with the qualitative notion of closeness.) Further, check that the map

$$B(0,1) \to \mathbb{R}^n, \qquad y \mapsto \frac{y}{1 - \|y\|}$$

is a homeomorphism. The formula for the inverse is $z \mapsto \frac{z}{1+\|z\|}$. Thus X is locally euclidean iff every point of X has a neighborhood homeomorphic to an open n-ball for some n.

Proposition 1.2. *A point in a topological space can never have two neighborhoods, one homeomorphic to $\mathbb{R}^n$ and the other to $\mathbb{R}^m$ for $m \neq n$. Equivalently, an open subset of $\mathbb{R}^n$ cannot be homeomorphic to an open subset of $\mathbb{R}^m$ for $m \neq n$.*

This is a consequence of the invariance of domain theorem of Brouwer:

Theorem 1.3. *If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $f(U)$ is open and f induces a homeomorphism between U and $f(U)$.*

PROOF. See [19, Theorem 36.5]. □

1.2. Topological manifolds

We now look at a class of topological spaces which locally look like euclidean spaces. These are called topological manifolds. They bring us a step closer to objects on which we can do geometry.

1.2.1. Topological manifolds.

Definition 1.4. A *topological manifold* is a topological space M which is second countable, Hausdorff and locally euclidean.

A morphism $M \to N$ between topological manifolds is a continuous map from M to N.

This defines the category of topological manifolds. We denote it by TopManifold. It is obtained from the category of topological spaces by imposing conditions on the object (morphisms have not changed): A topological space is either a topological manifold or not. The formal way to say this is: The category of topological manifolds is a full subcategory of the category of topological spaces. Note that the notion of isomorphism in this category is homeomorphism. (A simple example: A set is either finite or not. So the category of finite sets is a full subcategory of the category of sets.)

We say that a topological manifold has dimension n if it is locally n-euclidean at all points. That is, every point of X has a neighborhood homeomorphic to $\mathbb{R}^n$. In this situation, we talk of a topological n-manifold. Note that a connected topological manifold always has a dimension.

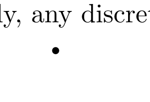
Also observe that any open set in a topological n-manifold is a topological n-manifold (with the subspace topology).

Remark 1.5. The most crucial assumption is locally euclidean. The Hausdorff and second countable conditions are put to avoid "pathological" examples, they get used in the proofs of some results. There are other contexts where these assumptions are dropped (and instead replaced by something else such as paracompactness).

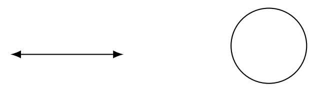
Remark 1.6. One can also define a topological manifold as a sheaf of continuous functions on a topological space such that every point has a neighborhood isomorphic to the sheaf of continuous functions on $\mathbb{R}^n$.

1.2.2. Examples. Here are some examples of topological manifolds listed according to dimension.

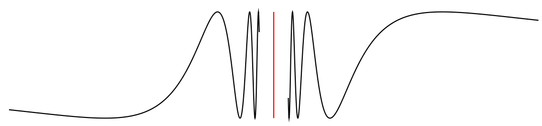
- 0 : a point, or more generally, any discrete countable set of points.



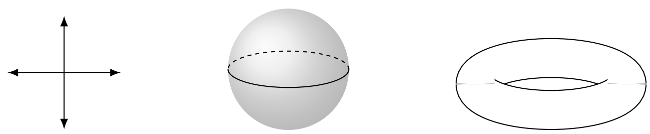
- 1 : $\mathbb{R}^1$ (also called a line), an open interval, circle, countable disjoint unions of any of these. (An open interval is homeomorphic to the line, so it need not be listed separately.)



A somewhat wild example (with two connected components) is the set $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y = \sin(1/x)\}$.



- 2 : $\mathbb{R}^2$ (also called a plane), any open set in the plane, sphere, torus, or more generally the surface with k holes, the real projective plane, the Klein bottle.

  A *surface* is a compact 2-dimensional topological manifold. Surfaces have been completely classified [2, Chapter 7]. A stimulating discussion is given in [28, Chapter 5].
- 3: $\mathbb{R}^3$, the three dimensional torus $S^1 \times S^1 \times S^1$, any open set in $\mathbb{R}^3$, for instance, the complement of a knot. Are the complement of the unknot and the trefoil homeomorphic? Many nice examples with illustrations are given by Weeks [28].

  Compact topological 3-manifolds are much harder to classify. Pioneering work of Thurston.

Classification of compact topological manifolds in higher dimensions is perhaps not possible because they are so many of them. Some general examples to keep in mind are $\mathbb{R}^n$, S^n, the n-dimensional torus which is the n-fold product $S^1 \times \cdots \times S^1$.

Example 1.7 (Covering spaces). Let $X \twoheadrightarrow Y$ be a covering map. Then X and Y are locally homeomorphic. So if one of them is locally euclidean, then so is the other. Suppose in addition that the covering map is finite-sheeted. Then if one of them is Hausdorff (second countable), then so is the other. So in this situation, if one of them is a topological manifold, then so is the other.

For example, the map $S^n \twoheadrightarrow \mathbb{R}P^n$ which identifies the antipodal points on S^n is a double cover, that is, a two-sheeted covering map. So it follows that $\mathbb{R}P^n$ is a topological n-manifold. This is called the n-dimensional real projective space.

It is fairly common to have nice countably infinite-sheeted covering spaces, where Hausdorff and second countable is not an issue. For instance, consider the covering maps $f : \mathbb{R} \twoheadrightarrow S^1,\ f(x) = e^{2\pi i x}$ or $f : \mathbb{C} \twoheadrightarrow \mathbb{C} \setminus \{0\},\ f(z) = e^z$. All spaces involved are topological manifolds.

1.2.3. Products and coproducts. Let us discuss the initial object, terminal object, product and coproduct in the category of topological manifolds.

- Initial object: The empty set $\emptyset$ is a topological manifold. By convention, it has dimension -1. There is a unique continuous map from $\emptyset$ to any topological manifold. So it is an initial object.

- Terminal object: There is a unique continuous map from every topological manifold to any one-point space. So any one-point space is a terminal object. They are all homeomorphic. A one-point space is a topological manifold of dimension 0.
- Product: The cartesian product $M \times N$ of a topological n-manifold M and a topological m-manifold N is a topological $(n+m)$-manifold under the product topology. There are canonical projections $M \times N \to M$ and $M \times N \to N$.
  The product construction gives many interesting examples: $S^1 \times S^1$ (torus), $S^1 \times \mathbb{R}$ (infinite cylinder), $S^2 \times \mathbb{R}$, $S^1 \times S^2$, and so on.
- Coproduct: The disjoint union $M \coprod N$ is the coproduct of M and N. There are canonical inclusions $M \to M \coprod N$ and $N \to M \coprod N$. For instance, the disjoint union of a 2-sphere and a line in $\mathbb{R}^3$. Note that the connected components of a topological manifold can have different dimensions.

1.2.4. Additional properties. We discuss some features of topological manifolds, some are stated without proof.

Proposition 1.8. *The connected components of a topological manifold are both open and closed, and they are countable in number.*

Proof. Connected components of a topological space are always closed. If the space is a topological manifold, then a connected component is also open: Take any point x in a connected component. Then any euclidean neighborhood of x being connected must lie entirely in this component. The component is then the union of all such euclidean neighborhoods, and hence is open.

Pick a countable basis of open sets. We can do this because the space is second countable. In each connected component, there must be at least one element from this basis. So the number of connected components must be countable. □

Proposition 1.9. *The connected and path-connected components of a topological manifold coicide. In particular, if a topological manifold is connected, then it is also path-connected.*

Proof. Since a manifold is locally euclidean, it has good local properties, for instance, it is locally path-connected. In a locally path-connected topological space, the connected and path-connected components coincide (see for instance [20, Theorem 25.5]) and the result follows.

Here is a direct argument for the second claim. Consider the path components of the topological manifold. These are equivalence classes of the relation: $x \sim y$ if there is a path joining x and y. Since a topological manifold is locally euclidean, each path component is open. If there is more than one path component, then any one of them and the union of the remaining would disconnect the space. (Note that the union of the remaining is an open set.) So there is only one path component, that is, the topological manifold is path-connected. □

Proposition 1.10. *A continuous bijection between topological manifolds of the same dimension is a homeomorphism. (That is, the inverse is necessarily continuous.)*

What happens if the manifolds are not of the same dimension?

Proof. Suppose $f : M \to N$ is a continuous bijection. We only need to show that the map f is open, that is, it takes open sets to open sets. Suppose U is any open set in M. Then any $x \in U$ is contained in an euclidean neighborhood U_x such that $f(U_x)$ is contained in an euclidean neighborhood of $f(x)$. By the invariance of domain Theorem 1.3, $f(U_x)$ is open in N. Now $f(U)$ is the union of the $f(U_x)$ as x varies over the points of U, so it is open as well. □

In this regard, recall that a continuous bijection between topological spaces may not be a homeomorphism, but take a look at [20, Theorem 26.6].

Proposition 1.11. *A topological manifold is metrizable.*

Proof. See [4, Theorem 3.6, Chapter 1]. □

Also recall that a compact metric space is second countable and Hausdorff. Thus:

a compact topological manifold is the same as a compact metrizable space which is locally euclidean.

Munkres on page 227 in exercise 3 says that:

a compact topological manifold is the same as a compact Hausdorff space which is locally euclidean.

Theorem 1.12. *A compact topological manifold can be embedded in euclidean space, that is, it is homeomorphic to a subspace of euclidean space.*

Proof. See [20, Theorem 36.2]. □

For instance, S^1 can be embedded in $\mathbb{R}^2$. Similarly, orientable surfaces can be viewed as subspaces of $\mathbb{R}^3$. These embeddings provide a concrete way to visualize these manifolds. The real projective plane or the Klein bottle are nonorientable surfaces which embed in $\mathbb{R}^4$ but not in $\mathbb{R}^3$. So it is not as easy to visualize them.

Proposition 1.13. *A connected topological manifold is homogeneous, that is, given any points $x, y \in M$, there is a homeomorphism $M \to M$ which takes x to y.*

1.3. Smooth manifolds

Smooth manifolds provide a nice class of objects to do differential calculus. The smooth manifolds that one studies in a multivariable calculus class are usually open subsets of euclidean spaces. A general smooth manifold is built by gluing such euclidean subsets.

1.3.1. Smooth maps between euclidean spaces. Let U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$. A map $f : U \to V$ is *smooth* if partial derivatives of f of all orders exist and are continuous. A *smooth function* on U is a smooth map $U \to \mathbb{R}$. Note that composite of smooth maps is again smooth.

Proposition 1.14. *Let U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$, and let $f : U \to V$ be any map. Then the following are equivalent.*

(1) *f is a smooth map.*
(2) *f has the form*

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

where each f_i is a smooth function on U.

(3) *For every smooth function g on V, the composite $g \circ f$ is a smooth function on U.*

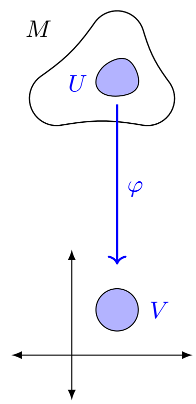
PROOF. (1) $\iff$ (2). Clear from the definition.

(1) $\implies$ (3). Use the fact that composite of smooth maps is smooth.

(3) $\implies$ (2). Take g to be any coordinate function on $\mathbb{R}^m$ (which is clearly smooth). Then $g \circ f$ is one of the f_i. So the f_i are all smooth. $\square$

The notion of a smooth map between euclidean spaces will get used in the definition of a smooth manifold, as also in the definition of a smooth map between smooth manifolds, as we will now see.

1.3.2. Smooth manifolds. Let M be a topological n-manifold. A *chart* on M is a pair (U, φ), where U is an open set in M and $\varphi : U \to V$ is a homeomorphism with V being an open set in $\mathbb{R}^n$.

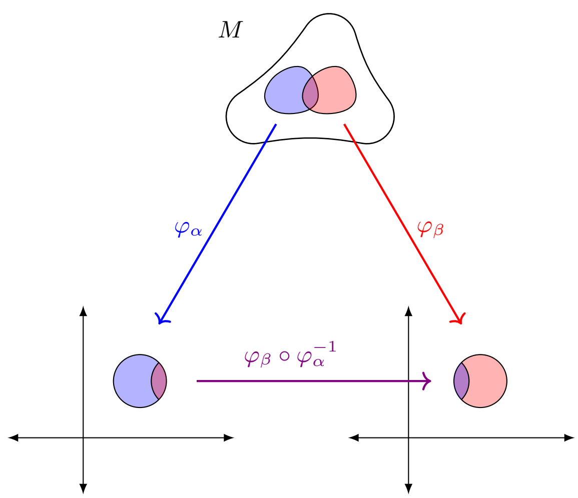


Suppose (U, φ) is a chart and U' is an open set contained in U. Then restricting φ to U' yields a chart (U', φ'). Since φ is a homeomorphism, it will take open sets to open sets, so the image of U' will be an open set V' in $\mathbb{R}^n$.

An *atlas* or a *smooth structure* on M is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M such that the U_α cover M and they are mutually compatible: For any pair of indices α and β,

$$f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a smooth map (from an open set of $\mathbb{R}^n$ to another open set of $\mathbb{R}^n$).



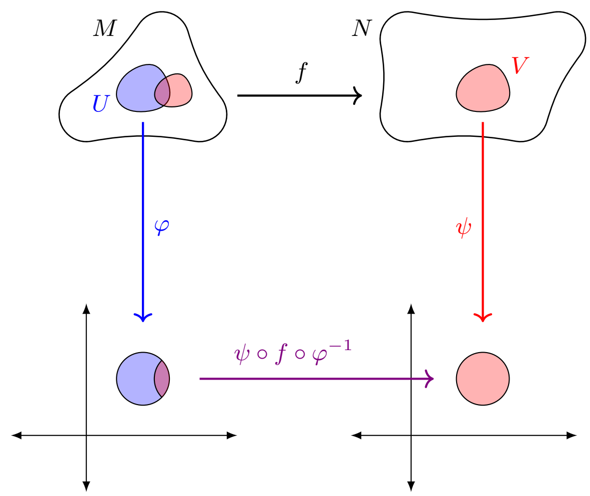
If the connected components of M have different dimensions, then an atlas on M is defined to be an atlas on each of its pure pieces.

Definition 1.15. A *smooth manifold* is a topological manifold M equipped with an atlas.

A smooth map $f : M \to N$ between smooth manifolds (not necessarily of the same dimension) is a continuous map $f : M \to N$ such that for any chart (U, φ) on M and (V, ψ) on N, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth.



Check that the composite of smooth maps is smooth: If $f : M \to N$ and $g : N \to P$ are smooth, then so is $g \circ f : M \to P$. This defines the category of smooth manifolds: objects are smooth maniolds and morphisms are smooth maps. We denote this category by $\mathsf{Manifold}$. The notion of isomorphism in this category is called a *diffeomorphism*. Explicitly, a smooth map $f : M \to N$ is a diffeomorphism if it is a homeomorphism, and f^{-1} is smooth.

We can talk of the dimension of a connected smooth manifold (in the same way as that of a connected topological manifold.)

Proposition 1.16. *Any open set V of a smooth n-manifold M is also a smooth n-manifold.*

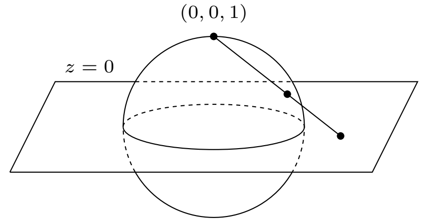
Proof. For the smooth structure on V, we take charts obtained by intersecting the charts of M with V. This is alright because charts can be restricted. Since charts of M cover M, their restrictions to V cover V, so we indeed have an atlas on V. $\square$

Remark 1.17. One can also define a smooth manifold as a sheaf of continuous function on a topological space such that every point has a neighborhood which is isomorphic to the sheaf of smooth functions on some open set in $\mathbb{R}^n$.

1.3.3. Examples. The examples of topological manifolds that we considered can be turned into smooth manifolds. We explain some of them.

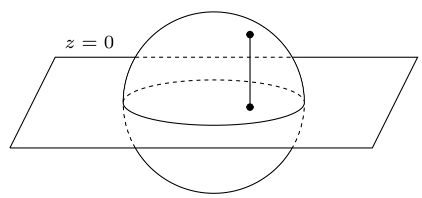
Example 1.18 (Euclidean spaces). $\mathbb{R}^n$ with the atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id})$ is a smooth n-manifold. Similarly any open set in $\mathbb{R}^n$ is a smooth n-manifold.

Example 1.19 (Spheres). The n-sphere S^n with the atlas consisting of two charts (S^n minus the north pole, stereographic projection from the north pole) and (S^n minus the south pole, stereographic projection from the south pole) is a smooth n-manifold.



To check this, write down formulas for the stereographic projections and check that the change of coordinates is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$.

Another method to construct an atlas on S^n is as follows. We explain the case $n = 2$. For the open set $z > 0$ on the sphere, we take the projection $\varphi(x, y, z) = (x, y)$. The inverse map is $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$.



Use similar charts for the open sets $z < 0$, $x > 0$, $x < 0$, $y > 0$, $y < 0$. These six charts define an atlas. See the front cover of [8].

Call the first atlas $\mathcal{A}_1$ and the second atlas $\mathcal{A}_2$. Then the identity map $(S^n, \mathcal{A}_1)$ and $(S^n, \mathcal{A}_2)$ is a diffeomorphism. This is equivalent to saying that the charts from the two atlases are also compatible with each other. In this sense, it does not matter which atlas we use. We can also combine the charts from the two atlases in different ways to construct other atlases but that will not change the diffeomorphism class.

The canonical inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map.

1.3.4. Products and coproducts. The product and coproduct works in a manner similar to the category of topological manifolds. The additional ingredient is to see how charts work.

- Initial object: The empty set is a smooth manifold (of dimension -1 by convention). It is the initial object in the category.
- Terminal object: The one-point space is a 0-dimensional smooth manifold. It is the terminal object in the category.
- Product: If $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are two smooth manifolds, then so is $(M_1 \times M_2, \mathcal{A}_1 \times \mathcal{A}_2)$, where

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{(U_1 \times U_2, \varphi_1 \times \varphi_2) \mid (U_i, \varphi_i) \in \mathcal{A}_i\}.$$

  This is the product in the category.

- Coproduct: Similarly, if $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are two smooth manifolds, then so is $(M_1 \coprod M_2, \mathcal{A}_1 \coprod \mathcal{A}_2)$, where $\mathcal{A}_1 \coprod \mathcal{A}_2$ is the disjoint union of the two atlases. This is the coproduct in the category.

1.3.5. How many smooth structures? A smooth manifold is obtained from a topological manifold by imposing more structure. So the question here is not whether a topological manifold is a smooth manifold. Rather it is: How many different smooth structures does a topological manifold carry? (An analogous example: A group is obtained from a set by imposing more structure. So the question is: How many different group structures does a set carry?)

Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are called compatible if all charts in $\mathcal{A}_1$ are compatible with all charts in $\mathcal{A}_2$ (in either order), or equivalently, if the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also an atlas. Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible iff the identity map $(M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$ is a diffeomorphism. (We saw this in the example of S^n above.) So given an atlas $\mathcal{A}$ on M, one can pass to the maximal atlas compatible with $\mathcal{A}$ by adding all charts compatible with the charts in $\mathcal{A}$, and this does not change the diffeomorphism class of M.

We state some results below about the existence and uniqueness of smooth structures.

Proposition 1.20. *If $n \leq 3$, then every topological n-manifold possesses a smooth structure. Further, this structure is unique up to diffeomorphism.*

As a concrete example, consider $\mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. This defines a smooth structure on $\mathbb{R}$ consisting of a single chart $(\mathbb{R}, \varphi)$. This chart will be compatible with $(\mathbb{R}, \mathrm{id})$ iff φ is a diffeomorphism. (It is very easy to construct homeomorphisms $\mathbb{R} \to \mathbb{R}$ which are not diffeomorphisms.)

However, the smooth structures on $\mathbb{R}$ defined above are diffeomorphic: Consider the smooth structures arising from say two homeomorphisms φ and ψ. Then the maps $\psi^{-1} \circ \varphi : \mathbb{R} \to \mathbb{R}$ and $\varphi^{-1} \circ \psi : \mathbb{R} \to \mathbb{R}$ are smooth (Why?) and inverses of each other.

Proposition 1.21. *If $n \geq 4$, then there exists a topological n-manifold with no smooth structure, and also a topological n-manifold with non-diffeomorphic smooth structures.*

Regarding the first claim, the first example was constructed by Kervaire in 1960. Regarding the second claim, the first example was constructed by Milnor in 1956.

Interesting fact: $\mathbb{R}^n$ has only one smooth structure up to diffeomorphism if $n \neq 4$, while $\mathbb{R}^4$ has uncountably many different (non-diffeomorphic) smooth structures. Work of Kirby, Freedman, Donaldson, Taubes.

1.4. Submanifolds

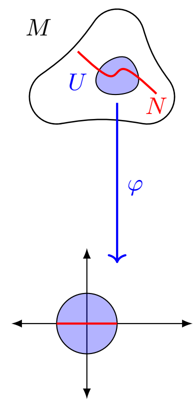
We now discuss submanifolds. They are nice subsets of a smooth manifold in the sense that they are themselves smooth manifolds, and their smooth structure is induced from that of the ambient manifold. A common method to construct submanifolds is by the implicit function theorem. (Books differ in their usage of the term submanifold.)

1.4.1. Submanifolds.

Definition 1.22. Let M be a smooth n-manifold. A subset N of M is called a *submanifold* if there is an integer $k \leq n$ such that the following property is satisfied.

For each $y \in N$, there is a chart (U, φ) (compatible with the charts) of M such that $y \in U$ and $\varphi(y) = 0$ and

$$\varphi(U) \cap \left(\mathbb{R}^k \times \{(0, \ldots, 0)\}\right) = \varphi(U \cap N).$$



We refer to (U, φ) as a preferred chart relative to N.

Remark 1.23. The reason for writing "compatible with the charts" is that M may not have enough charts. For instance, recall that the smooth structure of $\mathbb{R}^n$ was defined using only one chart.

Many authors define a smooth structure on a manifold to be a maximal atlas. This way one deals with a smaller equivalent category. The above issue does not arise in this setting since all possible charts have been included.

A submanifold (as suggested by the terminology) inherits a smooth structure of dimension k given by the charts $(U \cap N, \varphi|_{U \cap N})$ with (U, φ) being the preferred charts. Further, the inclusion map $N \hookrightarrow M$ is smooth. We leave these checks as exercises.

Example 1.24 (Open submanifolds). Let M be a smooth manifold. If N is an open subset of M, then it is a submanifold of the same dimension as M. We call N an *open submanifold* of M. For instance:

- knot complements. These are open submanifolds of $\mathbb{R}^3$.
- The set of all matrices $\mathrm{M}(n, \mathbb{R})$ of size n with real entries is a smooth manifold of dimension n^2. For its smooth structure, we take a single chart which identifies each matrix entry to a coordinate of $\mathbb{R}^{n^2}$. The general linear group $\mathrm{GL}(n, \mathbb{R})$ consisting of invertible matrices of size n is an open submanifold of $\mathrm{M}(n, \mathbb{R})$.

1.4.2. Rank of a smooth map. A smooth map f from an open set U of $\mathbb{R}^n$ to $\mathbb{R}^m$ has rank k at a point $x \in U$ if the $m \times n$ jacobian matrix of partial derivatives of f at x has rank k. Recall that for $f = (f_1, \ldots, f_m)$, the jacobian matrix is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

For example, $f(x,y) = x^2 + y^2$ has jacobian matrix $(2x, 2y)$ which has rank 0 at the origin and rank 1 at all other points.

Definition 1.25. A smooth map $f : M \to N$ has rank k at a point $x \in M$ if for some charts (U, φ) and (V, ψ) with $x \in U$ and $f(x) \in V$, the map $\psi \circ f \circ \varphi^{-1}$ has rank k at $\varphi(x)$.

Since diffeomorphisms preserve ranks, this property does not depend on the particular charts chosen. Spell this out, if not clear.

1.4.3. Inverse and implicit function theorems.

Theorem 1.26 (Inverse function theorem). *Let U be an open set of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a smooth map. Let f have rank n at a point $a \in U$ (that is, the jacobian matrix at a is invertible). Then there exists a neighborhood V of a such that $f(V)$ is open in $\mathbb{R}^n$ and $f : V \to f(V)$ is a diffeomorphism.*

Comments on the proof. Consider $n = 1$. The rank n condition means that $f'(a) \neq 0$. Wlog suppose $f'(a) > 0$. Then f is increasing in a neighborhood V of a. So $f : V \to f(V)$ is a bijection, and the inverse f^{-1} can be shown to be smooth. (If $f'(a) = 0$, then f may not be increasing. Consider for instance $f(x) = x^2$. However, it is possible for f to be increasing even if $f'(a) = 0$. Consider for instance $f(x) = x^3$.)

A proof for the case $n > 1$ can be found in many books, for instance, see [4, Chapter II, Theorem 6.4]. We omit it. $\square$

Theorem 1.27 (Implicit function theorem). *Suppose U is an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ with $n \geq m$ is a smooth map of rank m for all points of U. Then $f^{-1}(b)$ is a $(n-m)$-dimensional submanifold of $\mathbb{R}^n$ for each $b \in f(U)$.*

Proof. Fix $b \in f(U)$. Let a be any point such that $f(a) = b$. We now construct a preferred chart containing a relative to $f^{-1}(b)$. Assume wlog that the last m columns of the jacobian matrix at a are linearly independent. So the same holds in a neighborhood U' of a. Now define

$$\varphi : U' \to \mathbb{R}^n, \qquad \varphi(x_1, \dots, x_n) = (x_1, \dots, x_{n-m}, f(x)).$$

Then φ has rank n since its jacobian matrix has the form

$$\left(\begin{array}{c|c} \mathrm{I} & 0 \\ \hline \multicolumn{2}{c}{\text{jacobian matrix of } f} \end{array}\right).$$

For example: For $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2 + y^2$, we take $\varphi(x,y) = (x, x^2 + y^2)$. The jacobian matrix of φ is

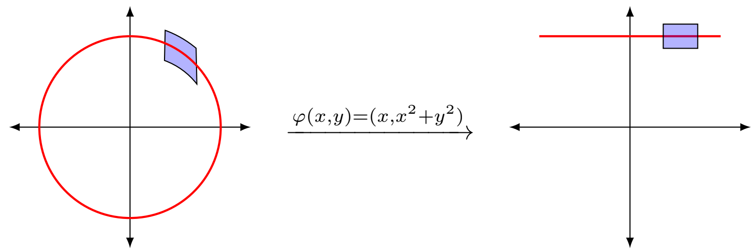
$$\begin{pmatrix} 1 & 0 \\ 2x & 2y \end{pmatrix}$$

which has rank two whenever $y \neq 0$. For $b = 1$, this choice of φ works for any a on the unit circle with nonzero y-coordinate.

Going back to the general case: Apply the inverse function theorem to φ at the point a. So there exists a neighborhood V of a (contained in U') such that $\varphi(V)$ is open in $\mathbb{R}^n$ and $\varphi : V \to \varphi(V)$ is a diffeomorphism. Observe that (V, φ) is a preferred chart around a since

$$\varphi(V) \cap (\mathbb{R}^{n-m} \times b) = \varphi(V \cap f^{-1}(b)).$$

(Minor point: φ does not take a to the origin, so we need to do a translation.) In our example, we see the following picture.



In this way each point $a \in f^{-1}(b)$ has a preferred chart, so it is indeed a submanifold of M and it has dimension $n-m$. □

Remark 1.28. The above proof also shows that: Each point of $f^{-1}(b)$ has a chart

$$(V \cap f^{-1}(b), \varphi|_{f^{-1}(b)})$$

where $\varphi|_{f^{-1}(b)}$ projects onto some subset of $n-m$ coordinates of $\mathbb{R}^n$. By reordering coordinates, we may assume that these are the first $n-m$ coordinates; the inverse map φ^{-1} (called a parametrization) is then of the form $(x_1, \dots, x_{n-m}) \mapsto (x_1, \dots, x_{n-m}, \ , \dots, \ )$ with the last m coordinates being functions of the first $n-m$ coordinates. Thus, the submanifold locally is a graph of some function. This function is implicitly defined by f whence the name of the theorem.

Example 1.29 (Spheres). We illustrate the implicit function theorem with S^n, the n-sphere. (The example in the proof was the case $n=1$.) It is defined by the equation $x_1^2 + \cdots + x_{n+1}^2 = 1$. Consider the function

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad (x_1, \dots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2.$$

The jacobian matrix of f is the $1 \times (n+1)$ matrix given by $(2x_1, \dots, 2x_{n+1})$. Thus f has rank 1 at all points other than the origin where it has rank 0. The n-sphere is $f^{-1}(1)$, so by the implicit function theorem, it is a n-dimensional submanifold of $\mathbb{R}^{n+1}$. To get the parametrizations, we need to express one coordinate as a function of the remaining n coordinates in the equation $x_1^2 + \cdots + x_{n+1}^2 = 1$. This cannot be done globally, that is, we require more than one chart. We could for instance restrict to $x_{n+1} > 0$, and then write

$$x_{n+1} = \sqrt{1 - x_1^2 - \cdots - x_n^2}.$$

We could use $2n$ charts to cover S^n. This was one of the two atlases on S^n that we discussed in Example 1.19.

1.4.4. Lie groups. We discuss some examples similar to $\mathrm{GL}(n, \mathbb{R})$. The similarity is in the fact that apart from being smooth, they also have a group structure. Such objects are called Lie groups.

- $\mathrm{SL}(n, \mathbb{R})$, the set of all matrices of size n of determinant 1.
- $\mathrm{O}(n)$, the set of all orthogonal matrices of size n. Recall A is an orthogonal matrix if $A^t A = A A^t = \mathrm{id}$, that is, the inverse of A is its transpose.
- $\mathrm{SO}(n)$, the set of all orthogonal matrices of size n of determinant 1. Recall that an orthogonal matrix has determinant either $+1$ or -1. The smooth manifold $\mathrm{O}(n)$ has two connected components, and the component containing the identity matrix is precisely $\mathrm{SO}(n)$.

In each case, understand how the implicit function theorem applies. Determine the dimensions of these smooth manifolds. Can you construct explicit parametrizations?

Remark 1.30. To any category C with finite products, one can associate the category of group objects in C. Here are some examples of categories with finite products and the standard names that are employed for the corresponding group objects.

Category with finite products	Category of group objects
sets	groups
topological spaces	topological groups
smooth manifolds	Lie groups
affine varieties	algebraic groups
affine schemes	affine group schemes
schemes	group schemes

As is evident, the motivation for the terminology "groups objects" comes from the example of sets.

1.4.5. Immersion and embedding. Let $f : M \to N$ be a smooth map. We say that:

- f is an *immersion* if the rank of f at any point $p \in M$ equals the dimension of M at p.
- f is an *embedding* if f is an immersion, and $f : M \to f(M)$ is a homeomorphism, with $f(M)$ given the subspace topology from N.

Proposition 1.31. *If $f : M \to N$ is an embedding, then $f(M)$ is a submanifold of N, and $f : M \to f(M)$ is a diffeomorphism.*

Conversely, if M is a submanifold of N, then the inclusion map $M \hookrightarrow N$ is an embedding.

Proof. Use implicit function theorem 1.27. See [18, Theorem 1.37] for details. □

The Klein bottle can be immersed in $\mathbb{R}^3$ but not embedded in $\mathbb{R}^3$.

1.4.6. Whitney embedding theorem.

Theorem 1.32. *If M is a compact smooth n-manifold, then M is diffeomorphic to a submanifold of $\mathbb{R}^q$ for some q.*

Proof. See for instance, [4, Chapter V, Theorem 4.6] or [18, Theorem 1.38]. □

More generally, the Whitney embedding theorem says that any smooth n-manifold (not necessarily compact) is diffeomorphic to a submanifold of $\mathbb{R}^{2n}$. If, in addition, the smooth manifold is orientable, then it can be embedded in $\mathbb{R}^{2n-1}$.

This suggests an alternative approach to the category of smooth manifolds. A smooth manifold is a pair $(M, \mathbb{R}^n)$ where M is a submanifold of $\mathbb{R}^n$ in the sense of Definition 1.22. (A chart on $\mathbb{R}^n$ is defined to be a pair (U, φ) where φ is a diffeomorphism onto its image. See relevant exercise.)

This approach is taken by Milnor [16] and Guillemin and Pollack [10]. It is similar to the way affine varieties are defined.

Problems

(1) Show by elementary means that $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ are not homeomorphic. This is a special case of Proposition 1.2.
(2) Show that: Every subspace of a secound countable topological space is second countable. Every subspace of a Hausdorff topological space is Hausdorff.
(3) Show that a topological space X is locally n-euclidean iff every point $x \in X$ has a neighborhood homeomorphic to an open set of $\mathbb{R}^n$.
(4) Consider the three conditions on a topological space in the definition of a topological manifold. Think of examples which satisfy two of those conditions but not the third.
(5) The topologist's sine curve is connected but not path-connected, so it cannot be a topological manifold. Which of the three defining conditions does it violate?
(6) Show that a topological n-manifold has a countable basis of open sets each homeomorphic to $\mathbb{R}^n$.
(7) What is the minimum number of open 2-balls (also called open dics) do we need to cover the torus? Why cannot we cover the sphere S^n by a single chart?
(8) Show that a diffeomorphism $f : M \to N$ is a homeomorphism in which the atlas on M when transferred to N via f is compatible with the given atlas on N. Deduce that if $f : M \to N$ is a diffeomorphism and the atlases on M and N are maximal, then the charts in these maximal atlases are in bijection with each other.
(9) Show that a chart (U, φ) on $\mathbb{R}^n$ is compatible with its usual smooth structure iff φ is a diffeomorphism onto its image.
(10) Give an example of a smooth homeomorphism between two smooth manifolds which is not a diffeomorphism.
(11) Give an example of an injective smooth map which is not an immersion.
(12) Is the open n-ball diffeomorphic to $\mathbb{R}^n$ (with their usual smooth structures)?
(13) Show that the figure 8 immersion of $\mathbb{R}$ into $\mathbb{R}^2$ is not a submanifold.
(14) Show that the graph of $|x|$ is not a submanifold of $\mathbb{R}^2$.
(15) Answer the questions in Section 1.4.4.

CHAPTER 2

Smooth functions and tangent vectors

2.1. Algebra of smooth functions

We now relate the category of smooth manifolds and the category of commutative $\mathbb{R}$-algebras by a functor which sends a smooth manifold to its algebra of smooth functions. This functor is full and faithful.

Let M be a smooth manifold. A smooth function on M is a smooth map $M \to \mathbb{R}$. Explicitly, $f : M \to \mathbb{R}$ is smooth if for every chart (U, φ), the composite $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth. We denote the set of smooth functions on M by $C(M)$.

Proposition 2.1. *If f and g are smooth functions on M, then so are $f + g$, fg and cf for any real number c.*

Proof. We can use the argument given in multivariable calculus. Suppose f and g are smooth. Then

$$M \xrightarrow{(f,g)} \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a composite of smooth maps, hence smooth. (The second map could be addition or multiplication.) $\square$

This says that $C(M)$ is a commutative $\mathbb{R}$-algebra.

Proposition 2.2. *Suppose M and N are smooth manifolds, and $f : M \to N$ is any map. Then f is smooth iff For every $g \in C(N)$, $g \circ f \in C(M)$.*

Proof. This follows from Proposition 1.14. $\square$

The association of $C(M)$ to M is functorial: If $M \to N$ is a smooth map, then there is an induced algebra morphism $C(N) \to C(M)$. Thus we have a contravariant functor

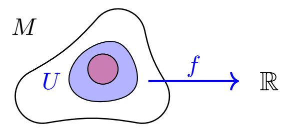
$$C : \mathsf{Manifold} \to \mathsf{Alg}^{\mathsf{co}} \tag{2.1}$$

from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras.

Lemma 2.3. *Let U be an open set of a smooth manifold M. Let $f : U \to \mathbb{R}$ be a smooth map. Then there exists a nonempty open set V whose closure is contained in U and a smooth function g on M such that*

$$g(p) = \begin{cases} f(p) & \text{if } p \in V, \\ 0 & \text{if } p \notin U. \end{cases}$$

Further, the same V can be chosen for each f.



Proof. Fix a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ which is 1 on the closed unit ball, and 0 outside the open ball of radius 2. Pick any compatible chart (U', φ) for M with U' lying inside U and such that $\varphi(U')$ includes a big ball around the origin. Now take V to be φ^{-1} of the open unit ball, and g to be $(h(f \circ \varphi^{-1})) \circ \varphi$ on U' and zero outside. □

Thus, any smooth function defined on an open set of M can be extended to a smooth function on all of M provided we cut down on the domain a little bit.

Lemma 2.4. *Let M be a smooth manifold. Any point $p \in M$ determines an algebra morphism*

$$C(M) \to \mathbb{R}, \qquad f \mapsto f(p).$$

Conversely, every algebra morphism from $C(M)$ to $\mathbb{R}$ arises in this manner.

Proof. The first part is clear. Both parts are stated in [17, Problem 1-C]. The second part requires some work. Let $C(M) \to \mathbb{R}$ be an algebra morphism. Let I denote its kernel, it is a maximal ideal in $C(M)$ of codimension 1. We claim that there is a unique point $p \in M$ on which all functions in I evaluate to zero, and this then is the required point. We explain the case when M is compact. Suppose there is no such point. For each $f \in C(M)$, let U_f be the set of points where f does not vanish. Then the U_f cover M. Pick a finite subcover, and say it arises from the functions $f_1, \dots, f_n$. Then $f_1^2 + \cdots + f_n^2 > 0$ on M and hence it is invertible in $C(M)$. But it belongs to I and maps to zero under the algebra morphism which is a contradiction. We leave the general case as an exercise. □

Theorem 2.5. *The functor C from the category of smooth manifolds to the category of commutative $\mathbb{R}$-algebras is full and faithful. That is, for any smooth manifolds M and N, the canonical map*

$$\mathsf{Manifold}(M, N) \to \mathsf{Alg}(C(N), C(M))$$

is a bijection.

Proof. For injective: Suppose $f, g : M \to N$ with $f(p) \neq g(p)$ for some $p \in M$. Pick a smooth function h on N which is 1 at $f(p)$ and 0 at $g(p)$. This is possible by Lemma 2.3. Then $h \circ f \neq h \circ g$ since their values differ at p. So f and g do induce different algebra morphisms $C(N) \to C(M)$.

For surjective: Observe that the case when M is a point and N is arbitrary follows from the converse in Lemma 2.4. Now let M be arbitrary. Suppose we are given $\varphi : C(N) \to C(M)$. Let $p \in M$. Evaluation at p yields a morphism $C(M) \to \mathbb{R}$. Precomposing by φ yields a morphism $C(N) \to \mathbb{R}$. By the converse in Lemma 2.4, this must be evaluation at some $q \in N$. Define $f : M \to N$ by $f(p) = q$. It follows that f induces φ. Further from Proposition 2.2, f is smooth. □

The above result implies that $C(M)$ determines M up to diffeomorphism. It follows that the category of smooth manifolds is equivalent to a full subcategory of the category of commutative $\mathbb{R}$-algebras. Is there any characterization of this subcategory? There are related ideas in [21]. Compare with the Nullstellensatz.

2.2. Derivations of an algebra

We define the notion of derivation of an algebra. The space of all derivations of an algebra carries the structure of a Lie algebra. We make this explicit for the algebra of polynomials.

2.2.1. Derivations of an algebra. Let $\Bbbk$ be a field. Suppose A is any $\Bbbk$-algebra. A *derivation* of A is a $\Bbbk$-linear map $D : A \to A$ such that

$$D(aa') = D(a)a' + aD(a'). \tag{2.2}$$

Here aa', $D(a)a'$ and $aD(a')$ are products taken in A.

Observe that a derivation is completely determined once it is specified on the generators of the algebra.

Example 2.6 (Algebra of polynomials). The space of polynomials in one variable $\Bbbk[x]$ is a $\Bbbk$-algebra, with product being the usual multiplication of polynomials. The derivative operator

$$\frac{d}{dx} : \Bbbk[x] \to \Bbbk[x]$$

is a derivation. The condition (2.2) is the Leibniz rule. What are all derivations of $\Bbbk[x]$? Since x is the generator of the algebra, any derivation is completely determined by its value on x. Now where can we send x to? There is no restriction on where x can go. Suppose it goes to the polynomial $a(x)$. Then the resulting derivation of $\Bbbk[x]$ is given by

$$a(x)\frac{d}{dx} : \Bbbk[x] \to \Bbbk[x], \qquad a(x)\frac{d}{dx}(f(x)) := a(x)f'(x).$$

(A formal reason why the value of x is unrestricted is that $\Bbbk[x]$ is the free $\Bbbk$-algebra on x.)

More generally, the space of polynomials in n variables $\Bbbk[x_1, \dots, x_n]$ is a $\Bbbk$-algebra. Now any derivation is completely determined by its values on $x_1, \dots, x_n$. Say x_i goes to the polynomial $a_i(x_1, \dots, x_n)$ for each i. Then the resulting derivation of $\Bbbk[x_1, \dots, x_n]$ is given by

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n)\frac{\partial}{\partial x_i} : \Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1, \dots, x_n].$$

Now consider $\Bbbk = \mathbb{R}$, the field of real numbers. Here, instead of $\mathbb{R}[x]$, we could take the larger algebra $C(\mathbb{R})$ of all smooth functions on $\mathbb{R}$. A derivation of $C(\mathbb{R})$ is also of the form $a(x)\frac{d}{dx}$, where $a(x)$ is now allowed to be any smooth function on $\mathbb{R}$. More generally, any derivation of $C(\mathbb{R}^n)$ is of the form $\sum_i a_i(x_1, \dots, x_n)\frac{\partial}{\partial x_i}$, where $a_i(x_1, \dots, x_n)$ is any smooth function on $\mathbb{R}^n$. We show this in Section 2.4.

2.2.2. Lie algebra of derivations. Let $\mathrm{Der}(A)$ denote the set of all derivations of the algebra A. Suppose D and E are derivations. Then so are $D + E$ and cD for any scalar c. Thus $\mathrm{Der}(A)$ is a vector space over $\Bbbk$. Further,

$$[D, E] := D \circ E - E \circ D, \tag{2.3}$$

called the Lie bracket of D and E, is a derivation. Do this calculation. It also shows why $D \circ E$ is not a derivation in general:

$$\begin{aligned} DE(aa') = D(E(a)a' + aE(a')) &= D(E(a)a') + D(aE(a')) \\ &= DE(a)a' + E(a)D(a') + D(a)E(a') + aDE(a'). \end{aligned}$$

The bracket operation turns $\mathrm{Der}(A)$ into a Lie algebra, that is, the bracket operation satisfies antisymmetry and the Jacobi identity:

$$[D,E] = -[E,D] \quad \text{and} \quad [[D,E],F] + [[E,F],D] + [[F,D],E] = 0.$$

Example 2.7 (Algebra of polynomials). For the algebra of polynomials in Example 2.6, the Lie bracket works as follows. For

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad E = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$
$$[D,E] = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j}.$$

In particular,

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

The same formulas hold for the algebra of smooth functions on $\mathbb{R}^n$. The only difference is that a_i and b_i are smooth functions on $\mathbb{R}^n$ instead of polynomials.

2.2.3. Derivations into a bimodule. We can generalize the notion of derivation as follows. Let A be a $\Bbbk$-algebra and M an A-bimodule. A *derivation* of A into M is a $\Bbbk$-linear map $D : A \to M$ such that

$$D(aa') = D(a) \cdot a' + a \cdot D(a'), \tag{2.4}$$

where we use $\cdot$ to denote both the left and right actions of A on M.

We recover (2.2) when we take $M = A$ with the left and right actions of A on A given by the product in A.

We let $\mathrm{Der}(A, M)$ denote the space of all derivations of A into M. It is a vector space over $\Bbbk$.

2.2.4. Derivations wrt an algebra morphism. Let $\varphi : A \to B$ be an algebra morphism. Then B is an A-bimodule with left and right actions given by

$$a \cdot b := \varphi(a)b \quad \text{and} \quad b \cdot a := b\varphi(a),$$

with products taken in B. Thus, we can talk of derivations of A into B. Explicitly, $D \in \mathrm{Der}(A, B)$ if

$$D(aa') = D(a)\varphi(a') + \varphi(a)D(a') \tag{2.5}$$

with products in the rhs taken in B. This is a specialization of (2.4).

Also observe that if $D \in \mathrm{Der}(A)$, then $\varphi \circ D \in \mathrm{Der}(A, B)$:

$$\begin{aligned} \varphi(D(aa')) = \varphi(D(a)a' + aD(a')) &= \varphi(D(a)a') + \varphi(aD(a')) \\ &= \varphi(D(a))\varphi(a') + \varphi(a)\varphi(D(a')). \end{aligned}$$

Example 2.8 (Algebra of polynomials). Fix $p \in \Bbbk^n$, and let

$$\mathrm{ev}_p : \Bbbk[x_1, \dots, x_n] \to \Bbbk, \qquad f \mapsto f(p)$$

be the algebra morphism given by evaluation at p. Then a derivation of $\Bbbk[x_1, \dots, x_n]$ into $\Bbbk$ wrt this morphism is given by

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{x=p} : \Bbbk[x_1, \dots, x_n] \to \Bbbk$$

for some scalars a_i.

For $\Bbbk = \mathbb{R}$, any derivation of $C(\mathbb{R}^n)$ wrt evaluation at a point $p \in \mathbb{R}^n$ is of the form $\sum_i a_i \frac{\partial}{\partial x_i}\big|_{x=p}$ for some real numbers a_i. We show this in Section 2.3.

2.3. Tangent vectors and derivations

We have already stated that smooth manifolds allow us to do calculus. We now take an important step in that direction by explaining how to define tangent vectors in a smooth manifold. It is linked to the notion of derivations of the algebra of smooth functions on that smooth manifold. Recall that any smooth manifold is determined by its algebra of smooth functions. So it is reasonable that one can capture geometric concepts such as tangent vectors through algebraic concepts such as derivations. Derivations are reviewed in Section 2.2.

2.3.1. Tangent vectors in a smooth manifold. Let M be a smooth n-manifold and let $p \in M$ be any point. A tangent vector at p is a linear map

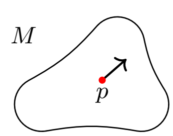
$$X_p : C(M) \to \mathbb{R}$$

such that for all smooth functions f, g on M,

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g). \tag{2.6}$$

(Here fg denotes the pointwise product of the functions f and g.) Thus, a tangent vector at p is the same as a derivation in the sense of (2.5) of $C(M)$ into $\mathbb{R}$ for the algebra morphism $C(M) \to \mathbb{R}$ given by evaluation at p.

The following is a way to visualize a tangent vector.



The justification for this picture will become clear in the subsequent discussion.

Let T_pM denote the set of all tangent vectors at p. Note that if X_p and Y_p are tangent vectors, then so are $X_p + Y_p$ and cX_p for any real number c with

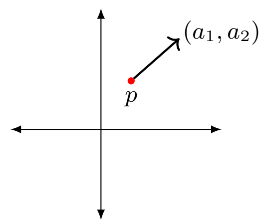
$$(X_p + Y_p)(f) := X_p(f) + Y_p(f) \quad \text{and} \quad (cX_p)(f) := cX_p(f).$$

Thus T_pM is a vector space over $\mathbb{R}$. The zero tangent vector is the map which sends all smooth functions to 0.

2.3.2. Tangent vectors in euclidean space. Consider the smooth manifold $M = \mathbb{R}^n$ and let p be any point in it. Then for any n-tuple $(a_1, \ldots, a_n)$ of real numbers,

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{x=p} : C(\mathbb{R}^n) \to \mathbb{R} \tag{2.7}$$

is a tangent vector at p. It sends f to $\sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}\big|_{x=p}$, the partial derivatives being evaluated at p. This is a directional derivative. The identity (2.6) is the familiar Leibniz rule.



We now show that all tangent vectors are of this form. A related lemma is stated below.

Lemma 2.9 (Taylor theorem with remainder). *For any smooth function f on an open set in $\mathbb{R}^n$ containing 0,*

$$f(x) = f(0) + \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt,$$

$$f(x) = f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j} x_i x_j \int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(tx)dt$$

in some open ball around 0.

We leave this as an exercise.

Proposition 2.10. *Any tangent vector at any point in $\mathbb{R}^n$ is given by a unique directional derivative. In particular, the tangent space at any point in $\mathbb{R}^n$ is n-dimensional.*

Proof. Let X_p be any tangent vector. We let p be the origin for convenience. Set $a_i := X_p(x_i)$. We claim that $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$. Let

$$Y_p := X_p - \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

Then, by construction, $Y_p(x_i) = 0$ for all i. Now by the first formula in Lemma 2.9, any smooth function f can be written in the form $f = f(0) + x_1 g_1 + \cdots + x_n g_n$ for some (not necessarily unique) choice of smooth functions g_i. Hence,

$$Y_p(f) = Y_p(f(0) + x_1 g_1 + \cdots + x_n g_n) = Y_p(x_1 g_1) + \cdots + Y_p(x_n g_n).$$

By (2.6), we see that $Y_p(x_i g_i) = 0$, so $Y_p(f) = 0$. Thus Y_p is the zero tangent vector as required. $\square$

Remark 2.11. If we were dealing with polynomial functions instead of smooth, then we could use that the polynomial algebra is generated by the coordinate functions x_i. So it immediately follows that: if $Y_p(x_i) = 0$, then $Y_p(f) = 0$ for all f.

2.3.3. Tangent spaces of a smooth manifold.

Lemma 2.12. *Let M be a smooth n-manifold and p be a point on it. If two smooth functions f and g on M agree on a neighborhood of p, then $X_p(f) = X_p(g)$ for any tangent vector X_p.*

PROOF. By linearity, it suffices to show that $X_p(f) = 0$ whenever f is zero in a neighborhood of p. Suppose this is the case. Pick a smooth function h which is 0 outside this neighborhood, but has value 1 at p. So $fh \equiv 0$. Now apply (2.6):

$$0 = X_p(fh) = X_p(f)h(p) + f(p)X_p(h) = X_p(f).$$

□

This result says that we can restrict to any open submanifold U of M containing p, and the tangent space at p does not change:

$$T_p(U) = T_p(M).$$

Now pick any chart (U, φ) containing p. Then there is an isomorphism of algebras

$$C(U) \to C(\varphi(U)), \quad f \mapsto f \circ \varphi^{-1}.$$

The inverse map sends g to $g \circ \varphi$. Since the notion of a tangent vector is defined entirely in terms of algebra of smooth functions, it follows that the tangent spaces $T_p(U)$ and $T_{\varphi(p)}(\varphi(U))$ are canonically isomorphic. Since by Lemma 2.12,

$$T_p(U) = T_p(M) \quad \text{and} \quad T_{\varphi(p)}(\varphi(U)) = T_{\varphi(p)}(\mathbb{R}^n),$$

it follows from Proposition 2.10 that $T_p(M)$ is n-dimensional. By abuse of notation, we allow ourselves to let $\frac{\partial}{\partial x_i}$ denote tangent vectors at p keeping the isomorphism implicit.

2.3.4. Change of coordinates. Let us now see how a tangent vector transforms under change of coordinates.

Lemma 2.13. *Let M be a smooth n-manifold and p be a point on it. Let $(x_1, \dots, x_n)$ coming from (U, φ) and $(u_1, \dots, u_n)$ coming from (V, ψ) be two coordinate systems containing a point p. Then*

$$\frac{\partial}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial}{\partial x_i}. \tag{2.8}$$

All partials are at the point p. To avoid cumbersome notation, this has not been written explicitly. The map $\varphi\psi^{-1}$ expresses the x_i in terms of the u_j, and it is the partial derivatives of this map that are being calculated in $\frac{\partial x_i}{\partial u_j}$.

PROOF. To check the lemma, we need to show that for any smooth function f on M,

$$\frac{\partial(f \circ \psi^{-1})}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}.$$

This is the familiar chain rule (in euclidean spaces). □

This gives us an alternative way to define a tangent vector:

Lemma 2.14. *A tangent vector at a point p in M is an assignment to every coordinate system a n-tuple of real numbers such that if $(a_1, \ldots, a_n)$ is assigned to the coordinate system $(x_1, \ldots, x_n)$ and $(b_1, \ldots, b_n)$ is assigned to $(u_1, \ldots, u_n)$, then*

$$a_i = \sum_{j=1}^{n} b_j \frac{\partial x_i}{\partial u_j}.$$

Proof. The calculation here is

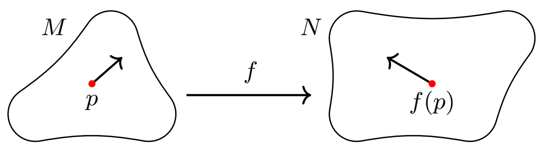
$$\sum_{j=1}^{n} b_j \frac{\partial}{\partial u_j} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_j \frac{\partial x_i}{\partial u_j} \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} b_j \frac{\partial x_i}{\partial u_j} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

□

2.3.5. Linear maps between tangent spaces. Suppose $f : M \to N$ is a smooth map. Then for any point $p \in M$, there is a natural induced linear map

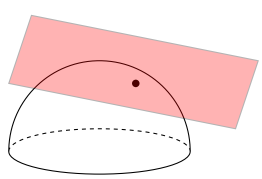
$$f_* : T_pM \to T_{f(p)}N, \qquad f_*(X_p)(g) := X_p(g \circ f). \tag{2.9}$$

It is straightforward to check that the derivation property (2.6) holds for $f_*(X_p)$.



Suppose $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$. Then the tangent spaces at any point can be identified with the space itself. In this case, check that f_* is the familiar derivative of f. In the canonical bases for the tangent spaces, it is the $m \times n$ jacobian matrix of partial derivatives of f. Since the general case reduces to this one after choosing charts, it is correct to think of f_* as the derivative of f.

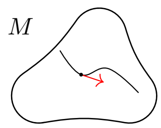
Remark 2.15. While learning multivariable calculus, a tangent space is explained by drawing a surface in $\mathbb{R}^3$ such as a spherical cap, and then drawing a plane that is tangent to the surface at a given point.



In the context of our present discussion, what is happening is the following. There is a smooth inclusion i of the surface in $\mathbb{R}^3$. So i_* maps the tangent space of the surface at p into the tangent space of $\mathbb{R}^3$ at $i(p)$ which is again $\mathbb{R}^3$. It is this affine subspace of $\mathbb{R}^3$ that is being shown in the picture.

Rank of a smooth map $f : M \to N$ at a point $p \in M$ was defined in Definition 1.25. Observe that it is the same as the rank of the linear map (2.9).

2.3.6. Tangent vectors from curves. Let I be some open interval containing zero. So I is a 1-dimensional manifold. A smooth map $f : I \to M$ is called a *smooth curve* in M. Consider the standard tangent vector $\frac{\partial}{\partial x}$ in I at the origin. The image $f_*(\frac{\partial}{\partial x})$ is a tangent vector in M at $f(0)$.



It is easy to see that all tangent vectors can be obtained in this manner by differentiating curves. (Of course, different curves can yield the same tangent vector.)

2.4. Tangent bundle and vector fields

The space of all tangent vectors (at all points) of a smooth manifold is called the tangent bundle of that manifold. It carries a natural topology, and even further a smooth structure, so it is a smooth manifold. Its dimension is twice that of the original manifold.

We also discuss vector fields on a smooth manifold. A vector field is a choice of a tangent vector at each point of the smooth manifold such that the choice varies smoothly with the point. This can also be expressed by saying that a vector field is a section of the tangent bundle.

2.4.1. Tangent bundle. Let M be any smooth manifold. We now proceed to construct a smooth manifold TM called the *tangent bundle* of M. As a set

$$TM := \bigsqcup_{p \in M} T_pM.$$

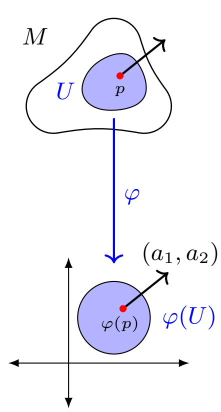
We say that T_pM is the *fiber* of TM over p.

Note that there is a canonical map $\pi : TM \twoheadrightarrow M$. There is also an inclusion map $M \hookrightarrow TM$ which sends p to the zero tangent vector at p.

For (U, φ) a chart on M with coordinates $(x_1, \ldots, x_n)$, there is an injective map

$$\varphi(U) \times \mathbb{R}^n \hookrightarrow TM, \qquad (\varphi(p), a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\bigg|_{x=\varphi(p)} \tag{2.10}$$

with p varying over points in U.



Define the topology of TM to be such that this map is a homeomorphism on to its image for all charts. In other words, a set in TM is open if its inverse image under the map (2.10) is open in $\varphi(U) \times \mathbb{R}^n$ for all charts.

Proposition 2.16. *Let M be a smooth manifold. The tangent bundle TM is a smooth manifold. If M has dimension n, then TM has dimension $2n$.*

An atlas for TM is constructed from the atlas on M as follows. Each chart (U, φ) on M yields a chart on TM, where the open set is $\pi^{-1}(U)$ and the map is

$$\pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

(The first map is essentially the inverse of (2.10).) We need to check that these charts are compatible. The reason in informal terms is: A smooth map between euclidean spaces is automatically smooth on the tangent spaces, that is, "nearby" tangent vectors in the domain map to "nearby" tangent vectors in the codomain.

2.4.2. Tangent bundle functor.

It is convenient to write elements of TM as pairs (p, v), where p is a point of M and v is a tangent vector at p. If $f : M \to N$ is a map of smooth manifolds, then there is an induced smooth map

$$(f, f_*) : TM \to TN$$

which sends (p, v) to $(f(p), f_*(v))$ with the map f_* as in (2.9). Moreover, the association of TM to M is functorial, that is, we have a functor

$$T : \mathsf{Manifold} \to \mathsf{Manifold} \tag{2.11}$$

from the category of smooth manifolds to itself. Thus, for smooth maps $f : M \to N$ and $g : N \to P$ and $p \in M$, the diagram of linear maps

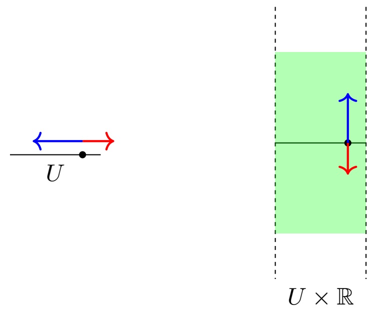
$$\begin{tikzcd} T_p(M) \arrow[r, "f_*"'] \arrow[rr, bend left, "(g\circ f)_*"] & T_{f(p)}(N) \arrow[r, "g_*"'] & T_{g(f(p))}(P) \end{tikzcd}$$

commutes. This is in fact the chain rule that we learn in multivariable calculus stated in the generality of smooth manifolds.

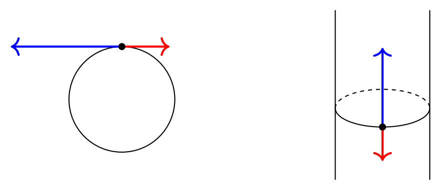
Remark 2.17. Let us make the comparison between multivariable calculus and our present setting more precise. In multivariable calculus, the category that we deal with is the one whose objects are open sets in euclidean spaces and morphisms are smooth maps. This is a full subcategory of the category of smooth manifolds. The tangent bundle functor restricts to this full subcategory and this is what we implicitly learn in multivariable calculus.

2.4.3. Examples.

- Let U be any open subset of $\mathbb{R}^n$. Then the tangent bundle $TU = U \times \mathbb{R}^n$. An illustration for $n = 1$ is shown below.



- The tangent bundle of the circle S^1 is diffeomorphic to the infinite cylinder $S^1 \times \mathbb{R}^1$. The picture illustrates how tangent vectors to the circle can be vertically straightened.



- The tangent bundle of the sphere S^2 is *not* diffeomorphic to $S^2 \times \mathbb{R}^2$. This is a consequence of the hairy ball theorem: It is not possible to choose a smoothly varying nonzero tangent vector at each point of S^2.

2.4.4. Vector fields. Let M be a smooth manifold and $C(M)$ its algebra of smooth functions. A *smooth vector field* on M is a linear map $X : C(M) \to C(M)$ such that

$$X(fg) = X(f)g + fX(g). \tag{2.12}$$

In the rhs, we are multiplying the functions $X(f)$ and g in the first term, and f and $X(g)$ in the second term. In other words, a smooth vector field is a derivation of $C(M)$ in the sense of (2.2).

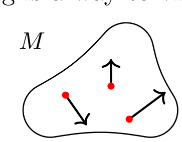
Suppose X is a smooth vector field on M. Set

$$X_p(f) := X(f)(p),$$

the function $X(f)$ evaluated at p. Equivalently, X_p is the composite map

$$C(M) \xrightarrow{X} C(M) \xrightarrow{\mathrm{ev}_p} \mathbb{R},$$

the second map being evaluation at p. A comparison with (2.6) shows that X_p is a tangent vector at p. The following is a way to visualize a vector field.



Is a smooth vector field on M then the same as a choice of a tangent vector at each point of M? Almost. The only extra ingredient is that for any f, $X(f)$ is a *smooth* function. This means that the tangent vectors are varying smoothly with p. This can be formalized by saying:

Proposition 2.18. *A smooth vector field on M is the same as a smooth section of the tangent bundle TM, that is, a smooth map $M \hookrightarrow TM$ such that the composite $M \hookrightarrow TM \twoheadrightarrow M$ is the identity.*

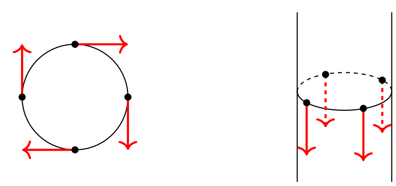
The zero section is always a smooth section of the tangent bundle. This vector field assigns the zero tangent vector at each point.

In local coordinates $(x_1, \dots, x_n)$, a smooth vector field can be written as

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}, \tag{2.13}$$

where each a_i is smooth function. Compare and contrast with (2.7).

Example 2.19 (Vector fields on the circle). Let us go back to the circle. A vector field on S^1 consists of a choice of a tangent vector at each point on S^1. It can also be viewed as a section of the tangent bundle which is $S^1 \times \mathbb{R}$. An illustration is shown below with all vectors of the same length and pointing in the clockwise direction.



What happens in the bundle picture when the vectors point in the anticlockwise direction? You can also imagine vector fields where the lengths of the vectors vary and even become zero at some points. In that case, the section would intersect with $S^1 \times 0$, the copy of S^1 in $S^1 \times \mathbb{R}$.

2.4.5. Lie algebra of vector fields. Let $\Gamma(TM)$ denote the set of all smooth vector fields on M, that is, the set of all smooth sections of TM. It is a Lie algebra with bracket defined by

$$[X, Y] := X \circ Y - Y \circ X. \tag{2.14}$$

This is a special case of the general construction (2.3).

In local coordinates $(x_1, \dots, x_n)$, for

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

$$[X, Y] = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j}.$$

In particular,

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

This is the same calculation as in Example 2.7 which was the case $M = \mathbb{R}^n$. It boils down to computing the bracket $[X, Y]$ on the coordinate functions $x_1, \dots, x_n$.

2.4.6. Parallelizable manifolds. The tangent bundle of a smooth n-manifold M is called *trivial* if there is a diffeomorphism $TM \to M \times \mathbb{R}^n$ which is identity on M and a linear isomorphism in each fiber. Manifolds with trivial tangent bundle are called *parallelizable*. Taking products preserves this property. Since S^1 is parallelizable, it follows that the torus $S^1 \times S^1$ and all higher dimensional tori are parallelizable.

Lie groups are parallelizable. We can take any basis for the tangent space at the identity, and then move it around using the group structure. This gives us n linearly independent smooth sections, where n is the dimension of the Lie group. These sections can be used to obtain a global trivialization of the bundle. (See Proposition 3.15.)

Among compact connected surfaces (without boundary) the only parallelizable one is the torus (which is therefore the only one admitting the structure of a Lie group).

The answer for spheres is as follows.

Theorem 2.20 (Adams, 1962). *The n-sphere S^n is parallelizable iff $n = 1, 3, 7$.*

S^1 and S^3 (the group of unit quaternions) are Lie groups. So they are parallelizable by the preceding discussion. Some argument needs to be given for S^7. As expected, it is much harder to prove that S^n is not parallelizable in the remaining cases. The $n = 2$ case was mentioned before.

2.5. Cotangent bundle and one-forms

We discuss the cotangent bundle of a smooth manifold. This is a construction dual to that of the tangent bundle with tangent vectors replaced by cotangent vectors. Similarly, one-forms are sections of the cotangent bundle just as vector fields are sections of the tangent bundle. A new ingredient is an operation which takes smooth functions to one-forms. This map is called the exterior derivative.

2.5.1. Dual vector space. Let $\Bbbk$ be a field. For any vector space V over $\Bbbk$, its dual vector space V^* consists of linear maps $f : V \to \Bbbk$. The latter are also called linear functionals on V. If V is finite-dimensional, then so is V^* and further its dimension equals that of V. More precisely, if $(e_1, \dots, e_n)$ is an ordered basis of a finite-dimensional vector space V, then its dual basis $(f_1, \dots, f_n)$ is the ordered basis of V^* defined by

$$f_i(e_j) := \delta_{ij},$$

which is 1 if $i = j$ and zero otherwise.

If $f : V \to W$ is a linear map, then we have an induced linear map $f^* : W^* \to V^*$ which sends $g : W \to \Bbbk$ to $g \circ f : V \to \Bbbk$.

2.5.2. Cotangent vectors. Recall that T_pM denotes the tangent space of M at the point p. Let T_p^*M denote the vector space dual to T_pM. Explicitly,

$$T_p^*M = \{w : T_pM \to \mathbb{R} \mid w \text{ is a linear map}\}.$$

This is called the *cotangent space* of M at p. An element of T_p^*M is called a *cotangent vector* at p.

In a chart on M with coordinates $(x_1, \dots, x_n)$, recall that $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is an ordered basis of T_pM (with the partials evaluated at p). Its dual basis is denoted $(dx_1, \dots, dx_n)$.

2.5.3. Change of coordinates. Let us now see how a cotangent vector transforms under change of coordinates.

Lemma 2.21. *Let $(x_1, \dots, x_n)$ coming from (U, φ) and $(u_1, \dots, u_n)$ coming from (V, ψ) be two coordinate systems containing a point p. Then*

$$du_j = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} dx_i. \tag{2.15}$$

PROOF. We show that the evaluation of the cotangent vectors (in the lhs and rhs above) on any tangent vector coincide. Evaluating on $\frac{\partial}{\partial u_k}$ and using the transformation of a tangent vector given by (2.8), we are reduced to the matrix identity

$$\delta_{jk} = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} \frac{\partial x_i}{\partial u_k}.$$

□

This gives us an alternative way to define a cotangent vector at a point p: Pick a cotangent vector $\sum_{i=1}^{n} a_i dx_i$ in each coordinate system $(x_1, \dots, x_n)$ such that different choices transform into one another by formula (2.15).

Observation 2.22. To express du_j in terms of the dx_i, we need to differentiate the u_j wrt the x_i. For tangent vectors, it is the other way round, we need to differentiate the x_i wrt the u_j.

2.5.4. Linear maps between cotangent spaces. Suppose $f : M \to N$ is a smooth map. Then for any point p in M, there is an induced linear map

$$f^* : T^*_{f(p)}N \to T^*_p M, \tag{2.16}$$

obtained by dualizing (2.9).

2.5.5. Cotangent bundle. Let M be any smooth manifold. There is a smooth manifold T^*M called the *cotangent bundle* of M. Its construction parallels that of the tangent bundle TM. As a set

$$T^*M := \bigsqcup_{p \in M} T^*_p M.$$

We say that $T^*_p M$ is the *fiber* of T^*M over p.

If (U, φ) is a chart on M with coordinates $(x_1, \dots, x_n)$, then there is an injective map

$$\varphi(U) \times \mathbb{R}^n \hookrightarrow T^*M, \qquad (\varphi(p), a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} a_i dx_i \tag{2.17}$$

with p varying over points in U. Define the topology of T^*M to be such that this map is a homeomorphism on to its image for all charts. In other words, a set in T^*M is open if its inverse image under the map (2.17) is open in $\varphi(U) \times \mathbb{R}^n$ for all charts.

Proposition 2.23. *The cotangent bundle T^*M is a smooth manifold. If M has dimension n, then T^*M has dimension $2n$.*

An atlas for T^*M is constructed from the given atlas on M as follows. Each chart (U, φ) on M yields a chart on T^*M, where the open set is $\pi^{-1}(U)$ and the map is

$$\pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

(The first map is essentially the inverse of (2.17).)

It is convenient to write elements of T^*M as pairs (p, w), where p is a point of M and w is a cotangent vector at p.

Note that there is a canonical map $\pi : T^*M \twoheadrightarrow M$ which sends (p, w) to p. There is also an inclusion map $M \hookrightarrow T^*M$ which sends p to $(p, 0)$, the zero cotangent vector at p.

Remark 2.24. Every smooth manifold carries smoothly varying nondegenerate bilinear forms on the tangent spaces. So the tangent and cotangent bundles are diffeomorphic. However this diffeomorphism depends on the choice of the bilinear forms. So identifying the two is not advisable, unless the bilinear form is a part of the data as for riemannian manifolds.

In view of (2.16), a smooth map $M \to N$ does not induce a smooth map $T^*M \to T^*N$ (or even a map in the other direction). So T^* is not a functor. How do we resolve this?

2.5.6. 1-forms. A smooth 1-form ω on a smooth manifold M is a choice of a cotangent vector ω_p at each point p of M such that this choice varies smoothly with p. More formally, a *smooth 1-form* on M is a smooth section of the cotangent bundle T^*M, that is, a smooth map $M \hookrightarrow T^*M$ such that the composite $M \hookrightarrow T^*M \twoheadrightarrow M$ is the identity. Let $\Gamma(T^*M)$ denote the set of all smooth 1-forms.

In local coordinates $(x_1, \dots, x_n)$, a smooth 1-form can be written as

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n) dx_i, \tag{2.18}$$

where each a_i is smooth function.

1-forms and vector fields are dual notions in the following sense. Suppose we have a smooth 1-form ω and a smooth vector field X. Then they can be "contracted" to obtain a smooth function f:

$$f(p) := \omega_p(X_p).$$

We will use the shorthand $f = \omega(X)$. In fact, a choice of a cotangent vector at each point is smooth precisely if its contraction with every smooth vector field yields a smooth function.

Lemma 2.25. *For a smooth manifold M, a smooth 1-form on M is the same as a map of $C(M)$-modules from $\Gamma(TM)$ to $C(M)$.*

Proof. We saw how a 1-form ω yields a map from $\Gamma(TM)$ to $C(M)$ by sending a vector field X to the function $\omega(X)$. Conversely suppose we are given such a map. Then we construct the 1-form ω by letting $\omega_p(X_p) := \omega(X)(p)$, where X is any vector field whose tangent vector at p is X_p. The value is independent of this choice. Why? $\square$

2.5.7. Exterior derivative on functions. A 0-form on M is the same as a smooth function on M. There is a map from 0-forms to 1-forms called the *exterior derivative*. It is given by

$$d : C(M) \to \Gamma(T^*M), \qquad f \mapsto df, \tag{2.19}$$

where the 1-form df is defined by

$$df(X) := X(f). \tag{2.20}$$

Suppose (U, φ) is a chart on M with coordinate system $(x_1, \ldots, x_n)$. Then for any smooth function f, the 1-form df restricted to U is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i. \tag{2.21}$$

To see this, evaluate df on the basis vectors $\frac{\partial}{\partial x_i}$. By (2.20),

$$df(\frac{\partial}{\partial x_i}) = \frac{\partial f}{\partial x_i}$$

which yields (2.21).

Observe from (2.21) that the exterior derivative d applied to the coordinate function x_i is indeed dx_i, so the notations are consistent. Using this observation, Lemma 2.21 can also be deduced by applying (2.21) to $f = u_j$.

2.6. Bundle of tensors and tensor fields

In preceding sections, we discussed the tangent and cotangent bundles of a smooth manifold. Taking tensor product of the tangent bundle r times and of the cotangent bundle s times, one obtains the bundle of (r, s)-tensors. Sections of this bundle are (r, s)-tensor fields.

This construction is a special case of tensor product of vector bundles, which we will discuss later.

2.6.1. Bundle of (r, s)-tensors. Let r and s be any nonnegative integers. Let M be a smooth manifold. Using the tangent and cotangent bundles of M, one can construct the bundle of (r, s)-tensors

$$TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M := \bigsqcup_{p \in M} T_pM \otimes \cdots \otimes T_pM \otimes T_p^*M \otimes \cdots \otimes T_p^*M$$

with r factors of T_pM and s factors of T_p^*M. It is a smooth manifold, with its smooth structure constructed in the obvious manner.

Warning. Some books call this the bundle of (s, r)-tensors instead of (r, s)-tensors.

As special cases,

- the bundle of $(1, 0)$-tensors is the tangent bundle,
- the bundle of $(0, 1)$-tensors is the cotangent bundle.

By convention, the 0-th tensor power of any $\mathbb{R}$-vector space is $\mathbb{R}$. What would be the bundle of $(0, 0)$-tensors?

2.6.2. (r, s)-tensors. A (r, s)-tensor at the point $p \in M$ is an element of

$$T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}.$$

This vector space is the fiber over p. Thus a (r, s)-tensor at p is an element in the fiber over p. As special cases,

- a $(0, 0)$-tensor at p is a real number (also called a scalar),
- a $(1, 0)$-tensor at p is a tangent vector at p,
- a $(0, 1)$-tensor at p is a cotangent vector at p.

2.6.3. (r, s)-tensor fields. A (r, s)-tensor field is a choice of a (r, s)-tensor at each point p which varies smoothly with p. As special cases,

- a $(0, 0)$-tensor field is a smooth function (also called a scalar field),
- a $(1, 0)$-tensor field is a smooth vector field,
- a $(0, 1)$-tensor field is a smooth 1-form.

This can be said more formally using the language of sections.

Recall that smooth vector fields and one-forms can be written in local coordinates using formulas (2.13) and (2.18). Similarly, locally, in a chart (U, φ) on M with coordinates $(x_1, \dots, x_n)$, a $(2, 0)$-tensor field can be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \dots, x_n) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j},$$

and a $(0, 2)$-tensor field as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \dots, x_n)\, dx_i \otimes dx_j,$$

where each a_{ij} is a smooth function.

We now give another equivalent way to define a (r, s)-tensor field. Many books give it as a definition. The statement and proof may not make complete sense right now, but it will after we discuss vector bundles.

Lemma 2.26. *A (r, s)-tensor field is a map of $C(M)$-modules*

$$\underbrace{\Gamma(TM) \otimes \cdots \otimes \Gamma(TM)}_{s} \otimes \underbrace{\Gamma(T^*M) \otimes \cdots \otimes \Gamma(T^*M)}_{r} \to C(M).$$

All tensor products are taken over the algebra $C(M)$. Note very carefully that r and s have switched positions.

Proof. The lhs above can be identified with

$$\Gamma(\underbrace{TM \otimes \cdots \otimes TM}_{s} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{r}),$$

the space of sections of the bundle of (s, r)-tensors. A map of $C(M)$-modules from this space to $C(M)$ (sections of the trivial line bundle over M) is the same as an element of

$$\Gamma((\underbrace{TM \otimes \cdots \otimes TM}_{s} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{r})^*),$$

the space of sections of the dual of the the bundle of (s, r)-tensors which is the bundle of (r, s)-tensors. (The duality switches tangent and cotangent.) $\square$

When $r = 0$ and $s = 1$, Lemma 2.26 reduces to Lemma 2.25.

2.7. Differential forms

There is particular interest surrounding the exterior powers of the cotangent bundle of a smooth manifold. This is the bundle of differential forms. The space of sections of this bundle has the structure of a differential graded algebra.

2.7.1. Bundle of k-forms. Let k be any nonnegative integer. Similar to the bundle of (r, s)-tensors, one can construct the bundle

$$\wedge^k(T^*M) := \bigsqcup_{p \in M} \wedge^k(T_p^*M)$$

of smooth k-forms. The fiber over p is the k-th exterior power of the cotangent space at p. Note that when k exceeds the dimension of M, the fibers are the zero space. When $k = 0$, the fibers are $\mathbb{R}$.

The bundle of smooth k-forms is a smooth manifold.

2.7.2. k-forms. There is no particular term used for a k-fold wedge of cotangent vectors which is the analogue of a (r, s)-tensor.

The analogue of a (r, s)-tensor field is a smooth k-form. Formally, this is a section of $\wedge^k(T^*M)$. We denote the set of all smooth k-forms by $\Omega^k(M)$. This is a vector space. Note that

$$\Omega^0(M) = C(M) \quad \text{and} \quad \Omega^1(M) = \Gamma(T^*M).$$

Locally, in a chart with coordinates $(x_1, \dots, x_n)$, a smooth k-form can be written as

$$\sum_{i_1 < i_2 < \dots < i_k} a_{i_1 \dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}. \tag{2.22}$$

How would a smooth k-form transform under a change of coordinates? To go from $(u_1, \dots, u_n)$ to $(x_1, \dots, x_n)$, use formula (2.15) on each du_{i_j} occurring in the wedge $du_{i_1} \wedge \dots \wedge du_{i_k}$. Linearity in each factor of the wedge can then be used to write (2.22) as a sum of the wedges of the dx_{i_j}.

2.7.3. Bundle of differential forms. One can put together the bundles of k-forms as k varies between 0 and n, where n is the dimension of M:

$$\wedge(T^*M) := \bigoplus_{k=0}^{n} \wedge^k(T^*M). \tag{2.23}$$

This is the bundle of differential forms. The notation means that the fiber at each point p is the direct sum (as vector spaces) of the fibers at p of each of the $n+1$ bundles in the rhs. In other words, the fiber at p is the exterior algebra of T_p^*M.

2.7.4. Differential forms. A *differential form* is a section of the bundle (2.23). The vector space of all differential forms is denoted $\Omega(M)$. Observe that

$$\Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M). \tag{2.24}$$

Thus a differential form can be uniquely decomposed as a sum of a 0-form, a 1-form, and so forth. In local coordinates, it can be written as a sum of terms of the form (2.22), one for each k. For instance,

$$f(x_1, x_2) + g_1(x_1, x_2)\, dx_1 + g_2(x_1, x_2)\, dx_2 + h(x_1, x_2)\, dx_1 \wedge dx_2$$

is the general expression for a differential form on a surface in local coordinates (x_1, x_2).

2.7.5. Algebra of differential forms. Recall that the fibers of $\wedge(T^*M)$ are exterior algebras. By operating fiberwise, one can multiply forms. More formally, the vector space $\Omega(M)$ is a graded commutative algebra. The algebra is over $C(M)$. The term graded means that the product of a k-form and a ℓ-form is a $(k+\ell)$-form. The product of $\Omega(M)$ (just like the exterior algebra) is denoted by $\wedge$. In local coordinates, for instance,

$$(g_1\,dx_1 + g_2\,dx_2) \wedge (h_1\,dx_1 + h_2\,dx_2) = (g_1h_2 - g_2h_1)\,dx_1 \wedge dx_2.$$

The term commutative is to be understood here in the signed sense, that is,

$$\omega \wedge \omega' = (-1)^{kk'}\omega' \wedge \omega,$$

where ω is a k-form and ω' is a k'-form. This comes from the fact that the exterior algebra is signed commutative. For instance, in the above calculation, if we multiply in the other order, then we get the negative of the rhs above.

2.7.6. Exterior derivative. Recall the exterior derivative d from 0-forms to 1-forms (2.19). It extends to the algebra of differential forms as follows.

Theorem 2.27. *For a smooth manifold M, there is a unique sequence of $\mathbb{R}$-linear maps*

$$d : \Omega^k(M) \to \Omega^{k+1}(M), \qquad k = 0, 1, 2, \ldots$$

such that

(1) *for $k = 0$, the map d agrees with* (2.19),
(2) *$d^2 = 0$,*
(3) *$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$ for $\omega \in \Omega^k(M)$.*

In local coordinates $(x_1, \ldots, x_n)$, the map d is given by

$$d(a_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial a_{i_1\ldots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.25}$$

Proof. Let us start with $k = 1$. Suppose ω is a 1-form. Write it locally as $a_1 dx_1 + \cdots + a_n dx_n$. Properties (2) and (3) force

$$d\omega = da_1 \wedge dx_1 + \cdots + da_n \wedge dx_n,$$

and the da_i have been defined by property (1). In general, observe that these three properties force (2.25). So the uniqueness assertion is clear.

To show that d is well-defined, one needs to check that changing coordinates yields the same local formula but in the other coordinate system. We leave this as an exercise. Next one needs to check that properties (2) and (3) hold. It suffices to check them in each chart using the local formula (2.25). For instance,

$$d^2(f) = d\Big(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = 0.$$

The terms with $i = j$ clearly do not contribute. For $i \neq j$, the terms involving $dx_i \wedge dx_j$ and $dx_j \wedge dx_i$ cancel. Appreciate that this happens because we are in an exterior algebra, as opposed to say the symmetric algebra. $\square$

In view of the above properties, we say that $\Omega(M)$ is a differential graded algebra, the product is $\wedge$ and the differential is d. Note that the product is constructed fiberwise but not d which is saying how the section values are changing as we move across fibers.

Remark 2.28. The notions of gradient, curl, divergence that we learn in multivariable calculus are instances of the exterior derivative d. For $M = \mathbb{R}^3$,

$$\text{0-forms} \xrightarrow{\text{gradient}} \text{1-forms} \xrightarrow{\text{curl}} \text{2-forms} \xrightarrow{\text{divergence}} \text{3-forms}.$$

The fact that curl(gradient) $= 0$ and divergence(curl) $= 0$ are instances of the property $d^2 = 0$.

2.7.7. Forms as alternating maps. The following extends Lemma 2.25.

Lemma 2.29. *For a smooth manifold M, there is a canonical linear isomorphism between $\Omega^k(M)$ and alternating $C(M)$-module maps from the k-fold tensor product of $\Gamma(TM)$ to $C(M)$.*

Proof. Lemma A.2 explains what is happening fiberwise. For the global case, see the discussion after Corollary 3.28. $\square$

Explicitly, for 1-forms ω_i and vector fields X_j,

$$(\omega_1 \wedge \cdots \wedge \omega_k)(X_1 \otimes \cdots \otimes X_k) = \det(\omega_i(X_j)).$$

For $k = 2$,

$$(\omega_1 \wedge \omega_2)(X_1 \otimes X_2) = \omega_1(X_1)\omega_2(X_2) - \omega_1(X_2)\omega_2(X_1). \tag{2.26}$$

Note that this is consistent with the identity $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$.

Lemma 2.30. *For any smooth k-form ω, and smooth vector fields $X_1, \dots, X_{k+1}$,*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}), \end{aligned} \tag{2.27}$$

where the hat over X_i indicates that X_i is deleted from the sequence.

Proof. Calculate locally. This is left as an exercise. $\square$

In particular, for a one-form ω,

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{2.28}$$

In an alternative approach, one can define the exterior derivative by (2.27) and then establish Theorem 2.27.

2.7.8. Functoriality. To a smooth manifold M, we have associated another smooth manifold $\wedge(TM)$. But this does not define an endofunctor on the category of smooth manifolds. How does one resolve this? The issue is the same as with the cotangent bundle construction.

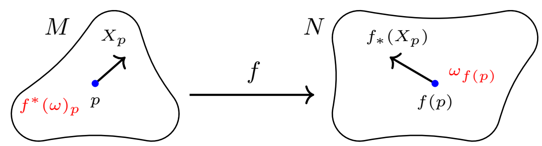
Now by looking at sections of $\wedge(TM)$, we obtained a differential graded algebra $\Omega(M)$? Here the functoriality is more clear:

Proposition 2.31. *If $f : M \to N$ is a smooth map, then it induces a morphism $f^* : \Omega(N) \to \Omega(M)$ of differential graded algebras.*

PROOF. Given a differential form ω on N, it can be pulled back to a differential form $f^*(\omega)$ on M via the map f. For this, we can employ Lemma 2.29. Let us elaborate on the case $k = 1$. For ω a one-form on N, define a one-form $f^*(\omega)$ on M by

$$f^*(\omega)_p(X_p) := \omega_{f(p)}(f_*(X_p)).$$

An illustration is shown below.



Further, one may check that f^* is an algebra morphism, and f^* commutes with the exterior derivatives on M and N. In other words, f^* is a morphism of differential graded algebras. □

Thus Ω is a contravariant functor

$$\mathsf{Manifold} \to \mathsf{dgAlg}$$

from the category of smooth manifolds to the category of differential graded algebras.

Problems

(1) Show that $C(M)$ is a finite-dimensional vector space iff M has dimension zero and it contains finitely many points.
 In this case, describe the algebra $C(M)$ directly (without reference to M). What are all algebra morphisms $C(M) \to \mathbb{R}$? Do the same when M has countably many points.

(2) For any field $\Bbbk$ and $\Bbbk$-algebra A, we have an inclusion $\Bbbk \hookrightarrow A$. Show that: For any derivation D of A, we have $D(c) = 0$ for any scalar $c \in \Bbbk$. Is the same true for a derivation D of A into a bimodule M?

(3) Write down an explicit formula for the Lie bracket on the space of derivations of $\mathbb{R}[x]$.

(4) Let c be a curve on M. In local coordinates $(x_1, \dots, x_n)$, write $c(t) = (c_1(t), \dots, c_n(t))$. Show that

$$c_*(\frac{d}{dt}) = \sum_i \dot{c}_i(t)\frac{\partial}{\partial x_i},$$

with the dot on top indicating derivative wrt t.

(5) Suppose X_p is a tangent vector at p on a manifold M. Let f be a smooth function on M. Show that $X_p(f)$ only depends on the values of f along any curve passing through p whose tangent vector at p is X_p.

(6) Recall that $\mathrm{SL}(2, \mathbb{R})$ is a smooth 3-manifold. Describe its tangent space at the point $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What happens at other points? Can you generalize to arbitrary n?

(7) Show that: If $f : M \to N$ is a diffeomorphism, then for any point p on M, the map $f_* : T_pM \to T_{f(p)}N$ is a linear isomorphism.

(8) A functor is *cartesian* if it preserves categorical products. Show that the tangent bundle functor (2.11) is cartesian. Explicitly, for manifolds M and N, there is a natural isomorphism
$$T(M \times N) \cong TM \times TN.$$
Deduce that tori are parallelizable.

(9) A smooth manifold is *orientable* if it admits an atlas such that all change of coordinates have positive determinant. Show that the tangent bundle is always orientable.

(10) Find a concrete description of the tangent bundle of real projective space $\mathbb{R}P^n$. Search for more examples of tangent bundles.

(11) Is the tangent bundle to S^2 equal to $\mathbb{R} \times SO(3)$? An element of $SO(3)$ can take any tangent vector to any tangent vector of the same length uniquely.

(12) Show that: If X is a smooth vector field and f a smooth function on M, then fX is a smooth vector field on M. This turns $\Gamma(TM)$ into a $C(M)$-module. Moreover,
$$[fX, gY] = fX(g)Y - gY(f)X + fg[X, Y]. \tag{2.29}$$

(13) Show that: For $n \geq 1$, the smooth vector field
$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \cdots + x^{2n} \frac{\partial}{\partial x_{2n-1}} - x^{2n-1} \frac{\partial}{\partial x_{2n}}$$
on $\mathbb{R}^{2n}$ restricts to a nowhere zero smooth vector field on S^{2n-1}.

(14) For a tangent vector at a point p on M, show that there exists a smooth vector field X on M such that X_p is the given tangent vector.

Suppose X is a smooth vector field on an open set U of M. Can it always be extended to a smooth vector field on all of M?

(15) Check directly that: For smooth functions f and g on M,
$$d(fg) = (df)g + f(dg),$$
where d is the exterior derivative defined in (2.19). Moreover, d is a derivation in the sense of (2.4) for $A = C(M)$. Also explain how the above identity is a special case of property (3) in Theorem 2.27.

(16) Prove formula (2.27).

(17) Take an embedded curve α in $\mathbb{R}^2$. Take a 1-form ω on $\mathbb{R}^2$. It pulls back to a 1-form on the curve α (viewed as a submanifold of $\mathbb{R}^2$). A parametrization of α allows us to write it as a 1-form on $\mathbb{R}$. Elaborate on these statements.

(18) Construct a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ such that $d\omega = 0$ but ω is not of the form df for any smooth function f on $\mathbb{R}^2 \setminus \{0\}$.

(19) Consider the smooth 2-form
$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$$
on $\mathbb{R}^{2n}$. Compute the n-th exterior power ω^n.

CHAPTER 3

Vector bundles

3.1. Smooth vector bundles

We have met the tangent bundle, cotangent bundle, more generally, the bundle of (r, s)-tensors, and the bundle of differential forms on a manifold M. They share a common feature, namely, they are constructed by assigning a vector space to each point p of M. For instance, in the case of the tangent bundle, this is the tangent space T_pM.

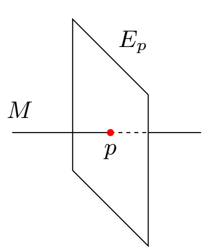
This idea can be formalized in the notion of a vector bundle.

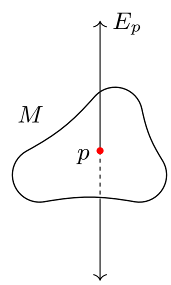
3.1.1. Smooth vector bundles.

Definition 3.1. Let M be a smooth manifold. A *smooth vector bundle* over M of rank k is a smooth manifold E equipped with a smooth map $\pi : E \to M$ such that for each $p \in M$,

(1) $\pi^{-1}(p) =: E_p$ has the structure of a k-dimensional vector space,
(2) there is a neighborhood U and a diffeomorphism
$$\varphi_U : E_U \to U \times \mathbb{R}^k,$$
with $E_U := \pi^{-1}(U)$ such that for each point q in U, its restriction to $\pi^{-1}(q)$ is a linear isomorphism $\varphi_q : E_q \to \{q\} \times \mathbb{R}^k$.

The second property is called local triviality of the bundle.





A smooth vector bundle of rank $k = 1$ is called a *line bundle*.

Notation 3.2. We denote a smooth vector bundle by a triple (E, π, M). Related terminology:

- The manifold E is called the *total space*,
- the manifold M is called the *base space*,
- the map π is called the *canonical projection*,
- the vector space $\pi^{-1}(p)$ is called the *fiber* over p,
- the map φ_U is called a *local trivialization* of E over U.

Equivalently, we may denote a smooth vector bundle by $\pi : E \to M$. It is also common to refer to a smooth vector bundle simply by its total space E, with M and π understood.

Definition 3.3. A *smooth bundle map* between smooth vector bundles $\pi : E \to M$ and $\pi' : E' \to M'$ is a pair (f, g) of smooth maps $f : M \to M'$ and $g : E \to E'$ such that $\pi' \circ g = f \circ \pi$, that is, the diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd} \tag{3.1}$$

commutes, and $g : E_p \to E'_{f(p)}$ is a linear map for all $p \in M$.

This defines the category of smooth vector bundles: objects are smooth vector bundles and morphisms are smooth bundle maps. We denote it by Bundle.

There is a functor $\mathsf{Bundle} \to \mathsf{Manifold}$ which sends (E, π, M) to E. There is also another functor which sends (E, π, M) to M. Moreover, π specifies a natural transformation between the two functors, with (3.1) an instance of the naturality diagram.

Remark 3.4. One can also consider more general vector bundles, where the base space M and the total space are only required to be topological spaces, with smooth replaced by continuous (and diffeomorphism replaced by homeomorphism).

Example 3.5 (Tangent bundle). Let M be a smooth n-manifold. Then the tangent bundle TM constructed in Section 2.4 is a smooth vector bundle over M of rank n. The fiber over p is the tangent space T_pM. Recall that TM was built out of the various tangent spaces using local trivializations.

In fact, the construction of TM from M is functorial and we have a functor

$$T : \mathsf{Manifold} \to \mathsf{Bundle} \tag{3.2}$$

from the category of smooth manifolds to the category of smooth vector bundles. This is an improvement over what we said earlier in (2.11). This can be understood as a commutative diagram of functors

$$\begin{tikzcd} & \mathsf{Bundle} \arrow[dr] & \\ \mathsf{Manifold} \arrow[ur, "T"] \arrow[rr, "T"'] & & \mathsf{Manifold}. \end{tikzcd}$$

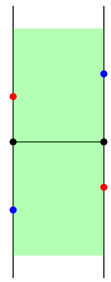
The unlabeled functor sends a smooth vector bundle to its total space.

Related examples are the cotangent bundle, bundle of (r, s)-tensors, bundle of differential forms mentioned in Sections 2.5, 2.6, 2.7.

Example 3.6 (Trivial bundle). Let M be a smooth n-manifold. Then $M \times \mathbb{R}^k$ is a smooth vector bundle over M of rank k. This is called the *product bundle*. A smooth vector bundle isomorphic to the product bundle is called the *trivial bundle*.

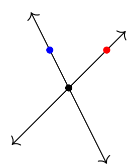
We refer to the product bundle for $k = 0$ as the *zero bundle*. Note that $\mathbb{R}^0 = \{0\}$ is the zero vector space. We may also write the zero bundle as (M, id, M). This is the only vector bundle where the total space equals the base space.

Example 3.7 (Mobius bundle). The infinite Mobius strip is a line bundle over S^1. The total space is defined as the quotient of $[0,1] \times \mathbb{R}$ by $(0,t) \sim (1,-t)$. The base space is the quotient of $[0,1]$ by $0 \sim 1$, which is the circle S^1. The canonical projection is the projection on the first coordinate.



This bundle is not trivial, that is, not isomorphic to $S^1 \times \mathbb{R}$. Why?

Example 3.8 (Canonical line bundle). The previous example can be generalized as follows. Recall that the real n-dimensional projective space $\mathbb{R}P^n$ consists of lines through the origin in $\mathbb{R}^{n+1}$.



Consider the space γ_n consisting of pairs (ℓ, v), where ℓ is a line through the origin in $\mathbb{R}^{n+1}$ and v is a point on ℓ. This is a line bundle over $\mathbb{R}P^n$, with canonical projection $\pi(\ell, v) = \ell$. It is called the *canonical line bundle*.

3.1.2. Smooth vector bundles over a fixed base. One can fix a smooth manifold M and consider the subcategory Bundle_M of smooth vector bundles over M: objects are smooth vector bundles with base space M, and morphisms are smooth bundle maps which are identity on M as shown below.

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \tag{3.3}$$

We refer to these as smooth bundle maps over M.

Lemma 3.9. *Let E and F be smooth vector bundles over M of rank m and n, respectively. A smooth bundle map $E \to F$ over M is a family of linear maps $E_p \to F_p$ for each $p \in M$ such that in any trivialization U of both E and F, this yields a smoothly varying family of $n \times m$ matrices (A_p) for $p \in U$.*

Proof. Clear. □

In the category Bundle_M, the notion of isomorphism can be characterized as follows.

Lemma 3.10. *Consider two smooth vector bundles E and E' over M. A smooth bundle map $E \to E'$ over M is an isomorphism of bundles iff it is a linear isomorphism on each fiber.*

Many books take this as a definition (and do not define a general morphism). The argument below is also given in [11, Chapter 3, Theorem 2.5] in the setting of topological spaces and continuous maps.

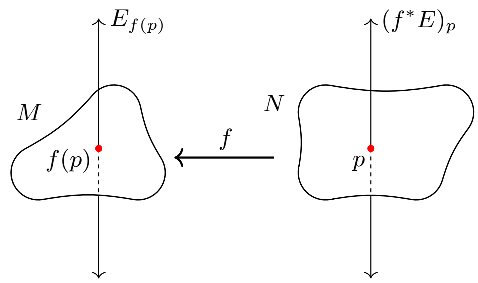
Proof. An isomorphism of bundles is clearly a linear isomorphism on each fiber. Conversely, suppose we have a morphism $u : E \to E'$ of bundles over M which is a linear isomorphism on each fiber. Then define $v : E' \to E$ by inverting the isomorphism in each fiber. We need to check that v is a morphism. For this, we make use of the criterion given by Lemma 3.9. Let U be any open set of M which trivializes both E and E'. Write u as $(p, x) \mapsto (p, f_p(x))$, where $p \in U$ and $x \in \mathbb{R}^k$ (with k being the rank of E and E'). By hypothesis, f_p is an invertible $k \times k$ matrix, whose entries vary smoothly with p. Note that v is given by $(p, x) \mapsto (p, f_p^{-1}(x))$. Since the inverse map $\mathrm{GL}_k \to \mathrm{GL}_k$ is a smooth map, it follows that v is smooth, as required.

An alternative approach would be to use the inverse function theorem. □

3.1.3. Pullback bundle. Suppose $E \to M$ is a smooth vector bundle, and $f : N \to M$ is a smooth map. Then one can construct a smooth vector bundle over N, denoted f^*E, whose fibers are defined by

$$(f^*E)_p := E_{f(p)}.$$

This is called the *pullback bundle* of E via f.



To complete the construction, one needs to specify the local trivializations of f^*E:

$$(f^*E)_U \to U \times \mathbb{R}^k.$$

Take any trivialization V of E, and put $U := f^{-1}(V)$. Then for each $q \in U$, we have $f(q) \in V$, so the trivialization V gives an identification of $E_{f(q)}$ with $\mathbb{R}^k$.

A little humor: If you are on N, then you look at M via f and start calling the fibers of M your own.

Example 3.11. We mention a couple of special cases:

- If N is an open submanifold of M with f the inclusion map, then f^*E is the restriction of E to N.
- If f is a constant map with $f(p) = q$ for all $p \in N$, then $f^*E = N \times E_q$ is a trivial bundle.

Observe that as a set, f^*E is the subset of $N \times E$ consisting of those pairs (p, x) such that $f(p) = \pi(x)$. The canonical projection $\pi : f^*E \to N$ is projection on the first coordinate. Let us denote projection on the second coordinate by $\tilde{f} : f^*E \to E$. These maps fit into the commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M. \end{tikzcd} \tag{3.4}$$

Thus, we have a bundle map from f^*E to E.

Moreover, any bundle map $g : E' \to E$ factors through the pullback bundle f^*E via the map $f : N \to M$ on the base manifolds as illustrated below.

$$\begin{tikzcd} E' \arrow[drr, bend left, "g"] \arrow[dr, dashed] \arrow[ddr, bend right, "\pi"'] & & \\ & f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ & N \arrow[r, "f"'] & M. \end{tikzcd}$$

The induced bundle map $E' \to f^*E$ is over N. (This is why we see a triangle instead of a square.)

3.1.4. Sub and quotient vector bundles. Suppose $E \to F$ is a smooth bundle map over M. We say E is a *subbundle* of F if the map from E to F is injective. It follows that for each $p \in M$, E_p is a linear subspace of F_p.

We say F is a *quotient bundle* of E if the map from E to F is surjective. It follows that for each $p \in M$, F_p is a linear quotient space of E_p.

Suppose $f : E \to F$ is a smooth bundle map over M. Then $\ker(f)$ is a subbundle of E and $\operatorname{coker}(f)$ is a quotient bundle of F.

Remark 3.12. In comparison to vector bundles, there is a more flexible notion called fiber bundles, where the fiber is only required to be a smooth manifold (without any linear structure). For example, the torus $S^1 \times S^1$ is an example of a trivial fiber bundle whose base space is S^1 and the fiber over each point is also S^1. For more details, see [11], [14, Chapter 6].

3.2. Sections of a smooth vector bundle

Let M be a smooth manifold. We consider the functor of smooth sections. It goes from the category of vector bundles over M to the category of modules over $C(M)$. We show that it is full and faithful. We mention that this induces a categorical equivalence between vector bundles over M and projective modules over $C(M)$. This restricts to an equivalence between trivial vector bundles over M and free modules over $C(M)$.

3.2.1. Smooth sections. A *smooth section* of the smooth vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$.

The set of all smooth sections, denoted $\Gamma(E)$, is a $\mathbb{R}$-vector space: For smooth sections s, s' and a real number c,

$$(s + s')(p) := s(p) + s'(p), \quad (cs)(p) := cs(p).$$

Here we need to know that $s + s'$ and cs are smooth maps. These are evident in $U \times \mathbb{R}^k$ and we have imposed "diffeomorphism" and linearity on each fiber in the local triviality requirement, so the vector space operations are smooth.

In fact, the space of smooth sections $\Gamma(E)$ is a module over the $\mathbb{R}$-algebra $C(M)$: For a smooth function f and a smooth section s, define the smooth section fs by

$$(fs)(p) := f(p)s(p),$$

where $f(p)$ is a scalar, $s(p)$ is a vector and $f(p)s(p)$ uses the vector space structure of E_p.

Example 3.13. The space of smooth sections of the trivial line bundle $M \times \mathbb{R}$ is $C(M)$ (viewed as a module over itself). More generally, the space of smooth sections of the trivial bundle $M \times \mathbb{R}^k$ is the free $C(M)$-module of rank k.

Example 3.14. A smooth section of the tangent bundle is a vector field, of the cotangent bundle is a 1-form, of the bundle of (r, s)-tensors is a (r, s)-tensor field, of the bundle of differential forms is a differential form.

3.2.2. Functor of smooth sections. The assignment of $\Gamma(E)$ to E yields a functor

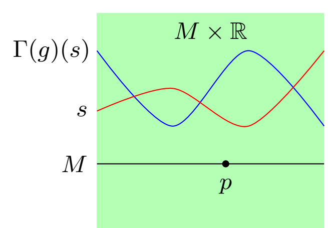
$$\Gamma : \mathsf{Bundle}_M \to \mathsf{Mod}_{C(M)} \tag{3.5}$$

from the category of smooth vector bundles over M to the category of $C(M)$-modules. On morphisms, it works as follows. A smooth bundle map $g : E \to F$ over M induces a map $\Gamma(g) : \Gamma(E) \to \Gamma(F)$ of $C(M)$-modules:

$$(\Gamma(g)(s))(p) := g(s(p)).$$

(Recall that for $C(M)$-modules A and B, a morphism from A to B is a map $\varphi : A \to B$ such that $\varphi(a + a') = \varphi(a) + \varphi(a')$ and $\varphi(fa) = f\varphi(a)$ for all $a, a' \in A$ and $f \in C(M)$.)

An illustration when E and F are the trivial line bundles over M is shown below.



The map g operates within each fiber by multiplication by a scalar; the fact that g is smooth means that this scalar varies smoothly with the point in M. The map g tells us how to transform the sections.

You can also imagine a picture for $E = F = M \times \mathbb{R}^2$. Now the fibers are planes, and g is a family of smoothly varying 2×2 matrices which specify how to transform the sections.

3.2.3. Zero and nonvanishing sections. Let $\pi : E \to M$ be a smooth vector bundle. The section s such that $s(p) = 0$, the zero vector in E_p, for all p is called the *zero section*.

A smooth section s such that $s(p) \neq 0$ for all p is called *nonzero* or *nonvanishing*. Note that a trivial bundle has many nonzero smooth sections. However, there are smooth vector bundles with no nonzero smooth sections. For instance, the tangent bundle of the sphere S^2 has no nonzero smooth sections. This is a nontrivial result called the hairy ball theorem (mentioned earlier as well). It implies that the tangent bundle TS^2 is not trivial.

3.2.4. Trivial vector bundles and frame fields. Suppose $s_1, \dots, s_n$ are smooth sections of E. Let U be any open set of M. We say that $s_1, \dots, s_n$ are linearly independent over U if $s_1(p), \dots, s_n(p)$ are linearly independent vectors in the fiber E_p for each $p \in U$. If $n = \operatorname{rank}(E)$, then we say that $(s_1, \dots, s_n)$ is a *frame field* over U. If U is unspecified, then it is understood that $U = M$, that is, the linear independence is over all of M.

Proposition 3.15. *A smooth vector bundle $E \to M$ of rank k is trivial iff it admits k linearly independent smooth sections.*

Proof. Forward implication. Consider the trivial bundle over M of rank k, namely, $M \times \mathbb{R}^k$. Let $(e_1, \dots, e_k)$ be any ordered basis of $\mathbb{R}^k$. Then the s_i defined by $s_i(p) := (p, e_i)$ are k linearly independent smooth sections.

Backward implication. Suppose $s_1, \dots, s_k$ are k linearly independent smooth sections. Define

$$h : M \times \mathbb{R}^k \to E, \qquad h(p, a_1, \dots, a_k) := a_1 s_1(p) + \dots + a_k s_k(p).$$

Since the sections s_i and the vector space operations on E are smooth, the map h is smooth. Since the s_i are linearly independent, it is a linear isomorphism in each fiber. By Lemma 3.10, this is enough to get an isomorphism of bundles. □

The local triviality property of vector bundles implies that local frame fields always exist, that is, given any point $p \in M$, there exists a frame field over some open set U containing p.

3.2.5. Equivalence between vector bundles and projective modules. We now proceed to show that the functor of smooth sections is full and faithful. Here we encounter the following result.

Lemma 3.16. *Let E be a smooth vector bundle over M and let $p \in M$. Then:*

1. *For any $v \in E_p$, there is a smooth section $s \in \Gamma(E)$ such that $s(p) = v$.*
2. *If s is a smooth section vanishing at p, then there exist smooth functions f_i vanishing at p and smooth sections s_i such that $s = \sum_{i=1}^n f_i s_i$ for some n.*

The first part says that the total space of a vector bundle is completely covered by its sections.

Proof. Let U be a neighborhood of p which trivializes E, and choose a smooth function f on M which is zero outside U and 1 at p.

For part (1): Let t be a smooth section of $E|_U$ with $t(p) = v$. Then $s = ft$ works.

For part (2): Pick a local frame field $(s_1, \dots, s_n)$ over U. Write $s = \sum_{i=1}^n g_i s_i$, where g_i are smooth functions on U. Since $s(p) = 0$, by linear independence, we obtain $g_i(p) = 0$. Multiplying by f^2, we can write

$$f^2 s = \sum_{i=1}^n (f g_i)(f s_i)$$

where fg_i are smooth functions on M, and fs_i are smooth sections of E. Hence

$$s = (1 - f^2)s + \sum_{i=1}^{n} (fg_i)(fs_i).$$

It remains to note that $1 - f^2$ and the fg_i all vanish at p. □

Proposition 3.17. *The functor Γ is full and faithful. That is, for any smooth vector bundles E and F over M, the canonical map*

$$\mathsf{Bundle}_M(E, F) \xrightarrow{\cong} \mathsf{Mod}_{C(M)}(\Gamma(E), \Gamma(F))$$

is a bijection.

Proof. For injectivity: This holds because a vector bundle is covered by its smooth sections. More formally: Suppose $f : E \to F$ is a bundle map. Then f can be recovered from $\Gamma(f)$: For $v \in E_p$, to know $f(v)$, we pick a smooth section s whose value at p is v (possible by Lemma 3.16, item (1)). Then $f(v)$ is the value of $\Gamma(f)(s)$ at p.

For surjectivity: Suppose $\varphi : \Gamma(E) \to \Gamma(F)$ is a map of $C(M)$-modules. We need to construct $f : E \to F$ such that $\Gamma(f) = \varphi$. We proceed as above. For $v \in E_p$, we pick a smooth section s whose value at p is v, and define $f(v)$ to be the value of $\varphi(s)$ at p. By Lemma 3.16, item (2), this does not depend on the particular choice of s. It remains to check that f is smooth and linear on each fiber. Suppose E is a trivial bundle, and $s_1, \dots, s_k$ is the corresponding frame field for E. Observe that

$$f(p, a_1, \dots, a_k) = a_1\varphi(s_1)(p) + \cdots + a_k\varphi(s_k)(p),$$

which is clearly smooth. For the general case, pick a neighborhood U of p which trivializes E, and smooth sections $s_1, \dots, s_k$ of E whose restriction to U is a local frame field. (Why is this possible?) Then f restricted to U is again given by the above formula, hence smooth. The formula is also clearly linear on the fiber over p. □

By general category theory, any full and faithful functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ induces an equivalence between C and the subcategory of D which is the image of $\mathcal{F}$. For this result to be effective, one then needs a good description of this subcategory.

For the functor Γ, it is asking: Which $C(M)$-modules are of the form $\Gamma(E)$ for some vector bundle E? The solution is given by the Serre-Swan theorem stated (but not proved) below.

Theorem 3.18 (Serre-Swan). *For a smooth manifold M, the functor of smooth sections Γ induces an equivalence between the category of smooth vector bundles over M, and the category of finitely generated and projective $C(M)$-modules.*

Proof. We need to show that for any vector bundle E, $\Gamma(E)$ is a finitely generated and projective $C(M)$-module, and conversely that every finitely generated and projective $C(M)$-module is of the form $\Gamma(E)$ for some vector bundle E. This is proved in [21, Theorem 11.32]. □

Remark 3.19. Observe that Γ restricts to an equivalence between the full subcategory of trivial bundles over M and the full subcategory of finitely generated free modules over $C(M)$. This is straightforward. We mention that there is a way to deduce the Serre-Swan theorem from this simpler statement by making use of Karoubi envelopes which is a general construction in category theory.

3.3. Operations on vector spaces

Recall that there are many standard operations on vector spaces such as direct sum, tensor product, duality, and so on. We explain the functoriality of these constructions. Moreover, when the base field is $\mathbb{R}$, these functors are smooth in a sense that we make precise.

3.3.1. Operations on vector spaces. Fix a field $\Bbbk$. Suppose V and W are vector spaces over $\Bbbk$. Then one can construct many other vector spaces out of them such as

- the dual vector space V^*
- the direct sum $V \oplus W$
- the tensor product $V \otimes W$
- the internal hom $\mathrm{Hom}(V, W)$ consisting of all linear maps from V to W, and in particular $\mathrm{End}(V) := \mathrm{Hom}(V, V)$.

If V is finite dimensional, then there is a canonical isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$, and in particular, $\mathrm{End}(V) \cong V^* \otimes V$.

3.3.2. Functoriality wrt linear maps. The above constructions are functorial in the sense that we now explain. Let Vec denote the category of vector spaces over $\Bbbk$. Let $\mathsf{Vec}^{\mathrm{op}}$ denote its opposite category.

For nonnegative integers m and n, define

$$\mathsf{Vec}^{m,n} := (\mathsf{Vec}^{\mathrm{op}} \times \cdots \times \mathsf{Vec}^{\mathrm{op}}) \times (\mathsf{Vec} \times \cdots \times \mathsf{Vec})$$

to be the cartesian product of m copies of $\mathsf{Vec}^{\mathrm{op}}$, and n copies of Vec. Explicitly, an object is a $(m + n)$-tuple

$$(V_1, \dots, V_m, V_1', \dots, V_n'),$$

where each V_i and each V_j' is a vector space. A morphism

$$(V_1, \dots, V_m, V_1', \dots, V_n') \longrightarrow (W_1, \dots, W_m, W_1', \dots, W_n')$$

is a $(m + n)$-tuple $(f_1, \dots, f_m, f_1', \dots, f_n')$, where each f_i is a linear map from W_i to V_i, and each f_j' is a linear map from V_j' to W_j'.

Direct sum is a functor

$$\mathsf{Vec} \times \mathsf{Vec} \to \mathsf{Vec}, \qquad (V, W) \mapsto V \oplus W,$$

tensor product is also a functor of the same form

$$\mathsf{Vec} \times \mathsf{Vec} \to \mathsf{Vec}, \qquad (V, W) \mapsto V \otimes W,$$

duality is a functor

$$\mathsf{Vec}^{\mathrm{op}} \to \mathsf{Vec}, \qquad V \mapsto V^*,$$

internal hom (wrt the tensor product) is a functor

$$\mathsf{Vec}^{\mathrm{op}} \times \mathsf{Vec} \to \mathsf{Vec}, \qquad (V, W) \mapsto \mathrm{Hom}(V, W).$$

For any vector spaces U, V, W, there is a natural bijection

$$\mathsf{Vec}(U \otimes V, W) \xrightarrow{\cong} \mathsf{Vec}(U, \mathrm{Hom}(V, W)). \tag{3.6}$$

This is the reason for the term "internal hom".

We point out that the assignment $V \mapsto \mathrm{End}(V)$ is not functorial in V.

3.3.3. Functoriality wrt linear isomorphisms. Let $\mathsf{Vec}_\times$ denote the subcategory of Vec whose objects are vector spaces and morphisms are linear isomorphisms. For a nonnegative integer k, define

$$\mathsf{Vec}_\times^k := \mathsf{Vec}_\times \times \cdots \times \mathsf{Vec}_\times$$

to be the k-fold cartesian product of $\mathsf{Vec}_\times$. An object is a k-tuple $(V_1, \ldots, V_k)$ with each V_i a vector space, and a morphism $(V_1, \ldots, V_k) \to (W_1, \ldots, W_k)$ is a k-tuple $(f_1, \ldots, f_k)$ with each f_i a linear isomorphism from V_i to W_i.

Here we do not ever need to consider the opposite category of $\mathsf{Vec}_\times$ since it is isomorphic to $\mathsf{Vec}_\times$. For instance, duality is a functor

$$\mathsf{Vec}_\times \to \mathsf{Vec}_\times, \qquad V \mapsto V^*.$$

For a linear isomorphism $V \to W$, there is an induced linear isomorphism $V^* \to W^*$, which is the *inverse* of the dual linear map $W^* \to V^*$.

Also note that the assignment $V \mapsto \mathrm{End}(V)$ is now functorial in V. That is, we have a functor

$$\mathsf{Vec}_\times \to \mathsf{Vec}_\times, \qquad V \mapsto \mathrm{End}(V).$$

Given an invertible matrix A, we often change it to PAP^{-1}, where P is a change of basis matrix. This is a manifestation of this functoriality.

This is an advantage of working with isomorphisms as opposed to linear maps.

3.3.4. Smooth functors wrt linear isomorphisms. Now we specialize to $\Bbbk := \mathbb{R}$. Also we assume that vector spaces are finite-dimensional, though we continue to use the same notations for the categories. We say that a functor

$$\mathcal{F} : \mathsf{Vec}_\times^k \to \mathsf{Vec}_\times \tag{3.7}$$

is *smooth* if $\mathcal{F}(f_1, \ldots, f_k)$ varies smoothly with the f_i. This means that for $V = \mathcal{F}(V_1, \ldots, V_k)$ and $W = \mathcal{F}(W_1, \ldots, W_k)$,

$$\mathrm{GL}(V_1, W_1) \times \cdots \times \mathrm{GL}(V_k, W_k) \to \mathrm{GL}(V, W)$$

is a smooth map of manifolds. (The set of linear isomorphisms $\mathrm{GL}(V, W)$ can be identified with the general linear group, and hence is a smooth manifold.)

The operations on vector spaces mentioned above are all functorial and smooth. For instance, the direct sum functor

$$\mathsf{Vec}_\times \times \mathsf{Vec}_\times \to \mathsf{Vec}_\times, \qquad (V, W) \mapsto V \oplus W$$

is smooth. Explicitly, the map on the Hom sets is

$$\mathrm{GL}_m \times \mathrm{GL}_n \to \mathrm{GL}_{m+n}, \qquad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

which is clearly smooth. Similarly, the map on the Hom sets for the duality functor is

$$\mathrm{GL}_n \to \mathrm{GL}_n, \qquad A \mapsto (A^t)^{-1} = (A^{-1})^t$$

which is smooth.

3.3.5. Smooth functors wrt linear maps. Instead of the above setting, one can also work with smooth functors

$$\mathcal{F} : \mathsf{Vec}^{m,n} \to \mathsf{Vec}. \tag{3.8}$$

Smoothness is defined in a similar manner: For $V = \mathcal{F}(V_1, \dots, V_m, V'_1, \dots, V'_n)$ and $W = \mathcal{F}(W_1, \dots, W_m, W'_1, \dots, W'_n)$, we require that

$$\mathrm{Hom}(W_1, V_1) \times \cdots \times \mathrm{Hom}(V'_n, W'_n) \to \mathrm{Hom}(V, W)$$

is a smooth map of manifolds.

Since a functor preserves isomorphisms and taking inverse is a smooth operation on invertible matrices, a smooth functor in the sense of (3.8) induces a smooth functor in the sense of (3.7) for $k = m + n$.

3.4. Operations on smooth vector bundles

We show how standard operations on vector spaces discussed in Section 3.3 such as direct sum, tensor product, and so on lead to corresponding operations on smooth vector bundles. The key property that we require of these operations is that they are smooth wrt either linear maps or linear isomorphisms.

3.4.1. Operations on vector bundles (Linear isomorphism version).

Proposition 3.20. *Suppose $E_1, \dots, E_k$ are smooth vector bundles over M, and $\mathcal{F}$ is a smooth functor of k variables as in (3.7). Then there is a smooth vector bundle over M, denoted $\mathcal{F}(E_1, \dots, E_k)$, whose fibers are*

$$\mathcal{F}(E_1, \dots, E_k)_p := \mathcal{F}(E_{1p}, \dots, E_{kp}).$$

This result is also discussed in [17, Theorem 3.6 and Remark 3].

Proof. We give a sketch. Let us take $k = 2$. The general case is no more complicated. Let E and F be smooth vector bundles over M of ranks m and n, respectively. Let the fibers of $\mathcal{F}(E, F)$ be defined as above, that is,

$$\mathcal{F}(E, F)_p := \mathcal{F}(E_p, F_p).$$

At the moment, $\mathcal{F}(E, F)$ is only a family of vector spaces indexed by points of M. To turn it into a smooth vector bundle, we proceed as follows.

Let U be any open set of M which trivializes both E and F. Thus, we have

$$E_U \xrightarrow{\cong} U \times \mathbb{R}^m \quad \text{and} \quad F_U \xrightarrow{\cong} U \times \mathbb{R}^n.$$

For each $p \in U$, this yields linear isomorphisms

$$\mathcal{F}(E_p, F_p) \xrightarrow{\cong} \mathcal{F}(\mathbb{R}^m, \mathbb{R}^n).$$

Taking union over all $p \in U$, we obtain the bijection

$$\mathcal{F}(E, F)_U \xrightarrow{\cong} U \times \mathcal{F}(\mathbb{R}^m, \mathbb{R}^n).$$

We use these as charts, as U varies, to define a topology on $\mathcal{F}(E, F)$ and also a smooth structure and local trivializations on $\mathcal{F}(E, F)$. We need to check compatibility between charts. Suppose we pick some other trivialization over U for E (and for F). Then the two trivializations of E relate by a family of (invertible) $m \times m$ matrices A_p varying smoothly with $p \in U$. Similarly, the two trivializations of F relate by a smoothly varying family B_p of (invertible) $n \times n$ matrices. Observe that the change of coordinates of $\mathcal{F}(E,F)_U$ is given by the family $\mathcal{F}(A_p, B_p)$, so by smoothness of $\mathcal{F}$, this family is smooth as required. □

As special cases of Proposition 3.20: If E and F are smooth vector bundles over M, we can talk of the smooth vector bundles

$$E \oplus F, \quad \mathrm{Hom}(E,F), \quad E \otimes F, \quad \mathrm{End}(E), \quad E^*,$$

and so on. This is because the corresponding operations on vector spaces do yield smooth functors. We also point out that since smooth vector bundles are of finite rank, the usage of operations involving E^* does not require any special care.

Most books, instead of saying things this way, go over one construction such as $E \oplus F$ in detail, and say that the rest is similar.

Remark 3.21. Proposition 3.20 also shows that once we construct the tangent bundle TM of a manifold M, we automatically get access to the cotangent bundle T^*M and, more generally, the bundle of (r,s)-tensors.

Remark 3.22. Fix a vector space V of dimension k. A smooth vector bundle modeled on V is a triple (E,π,M) as before, except that for local triviality, we use

$$\pi^{-1}(U) \cong U \times V.$$

In other words, a smooth vector bundle of rank k is a smooth vector bundle modeled on $\mathbb{R}^k$. This language is very convenient. For instance, if E and F are smooth vector bundles over M modeled on V and W, respectively, then $E \otimes F$ is a smooth vector bundle over M modeled on $V \otimes W$, and so on.

3.4.2. Operations on vector bundles (Linear maps version).

Proposition 3.23. *Suppose $E_1, \dots, E_{m+n}$ are smooth vector bundles over M, and $\mathcal{F}$ is a smooth functor of $m+n$ variables as in (3.8). Then there is a smooth vector bundle over M, denoted $\mathcal{F}(E_1, \dots, E_{m+n})$, whose fibers are*

$$\mathcal{F}(E_1, \dots, E_{m+n})_p := \mathcal{F}(E_{1p}, \dots, E_{m+n\,p}).$$

This result is also discussed in [14, Theorem 6.53].

Proof. Repeat the previous proof. To illustrate a distinctive point, take $m = n = 1$. Then the change of coordinates of $\mathcal{F}(E,F)$ is given by the family $\mathcal{F}(A_p^{-1}, B_p)$. But inverse is a smooth operation, so this is indeed a smooth family. The reason this was not visible in the proof of Proposition 3.20 is because we discarded $\mathsf{Vec}_\times^{\mathrm{op}}$, and we did this precisely by making use of the inverse.

Alternatively, we may directly apply Proposition 3.20 by recalling that a functor $\mathcal{F}$ as in (3.8) yields a functor as in (3.7). □

3.4.3. Functoriality of operations. For nonnegative integers m and n, define

$$\mathsf{Bundle}_M^{m,n} := (\mathsf{Bundle}_M^{\mathrm{op}} \times \cdots \times \mathsf{Bundle}_M^{\mathrm{op}}) \times (\mathsf{Bundle}_M \times \cdots \times \mathsf{Bundle}_M)$$

to be the cartesian product of m copies of $\mathsf{Bundle}_M^{\mathrm{op}}$ (the opposite category) and n copies of Bundle_M.

Proposition 3.24. *Suppose $\mathcal{F}$ is a smooth functor of $m+n$ variables as in (3.8). Then $\mathcal{F}$ induces a functor*

$$\mathsf{Bundle}_M^{m,n} \to \mathsf{Bundle}_M, \qquad (E_1, \dots, E_{m+n}) \mapsto \mathcal{F}(E_1, \dots, E_{m+n}).$$

PROOF. You would have noticed that verification that a map is smooth is very similar to checking that charts on a manifold define a smooth structure. So this argument is similar to the argument in Proposition 3.23. Working with $m = n = 1$, suppose $E' \to E$ and $F \to F'$ are smooth bundle maps over M. Then over a common local trivialization U, we have two smoothly varying families of matrices (A_p) and (B_p) indexed by $p \in U$. The resulting family for $\mathcal{F}(E, F) \to \mathcal{F}(E', F')$ is $\mathcal{F}(A_p, B_p)$, which is smooth by smoothness of $\mathcal{F}$. Note that here A_p and B_p are not necessarily square matrices (forget invertible), and here we see the reason for working with a functor smooth wrt linear maps. $\square$

We now look at some consequences of Proposition 3.24. There is a direct sum functor
$$\mathsf{Bundle}_M \times \mathsf{Bundle}_M \to \mathsf{Bundle}_M, \qquad (E, F) \mapsto E \oplus F.$$
This means that if we have smooth bundle maps $E \to E'$ and $F \to F'$, then there is an induced smooth bundle map $E \oplus F \to E' \oplus F'$.

The same is true for the tensor product functor
$$\mathsf{Bundle}_M \times \mathsf{Bundle}_M \to \mathsf{Bundle}_M, \qquad (E, F) \mapsto E \otimes F.$$
It defines a monoidal structure, thus we have a monoidal category $(\mathsf{Bundle}_M, \otimes)$. The unit object is the trivial line bundle over M. (We mention that $(\mathsf{Bundle}_M, \oplus)$ is also a monoidal category, with the unit object being the zero bundle over M. In fact, it is (co)cartesian, since direct sum is the (co)product in Bundle_M.)

By precomposing the tensor product functor with the diagonal, one obtains the tensor square functor
$$\mathsf{Bundle}_M \to \mathsf{Bundle}_M, \qquad E \mapsto E \otimes E.$$
Similarly, one can consider higher tensor powers, or symmetric, or exterior tensor powers.

The internal hom functor is
$$\mathsf{Bundle}_M^{\mathrm{op}} \times \mathsf{Bundle}_M \to \mathsf{Bundle}_M, \qquad (E, F) \mapsto \mathrm{Hom}(E, F).$$
Note that the association $E \mapsto \mathrm{End}(E)$ is not functorial wrt bundle maps but is functorial wrt bundle maps which are isomorphisms. A similar remark applies more generally when we take tensor product of some copies of E with some copies of E^*. A concrete instance is the bundle of (r, s)-tensors.

We also point out that in the above discussion, functoriality is wrt bundle maps over a fixed base manifold M.

3.4.4. Examples of naturally isomorphic vector bundles. For any smooth vector bundles E and F over M, there are natural isomorphisms of bundles
$$\mathrm{Hom}(E, F) \cong E^* \otimes F, \quad (E \oplus F)^* \cong E^* \oplus F^*, \quad (E \otimes F)^* \cong E^* \otimes F^*. \tag{3.9}$$
In particular, if $F = \mathrm{triv}$, the trivial line bundle, then
$$\mathrm{Hom}(E, \mathrm{triv}) \cong E^*. \tag{3.10}$$
The isomorphisms are defined fiberwise by making use of the corresponding natural isomorphisms of vector spaces. The extra issue is that of smoothness. For this, one can use Lemma 3.9.

For any smooth vector bundles E, F, G over M, there is a natural bijection
$$\mathsf{Bundle}_M(E \otimes F, G) \xrightarrow{\cong} \mathsf{Bundle}_M(E, \mathrm{Hom}(F, G)), \tag{3.11}$$

thus justifying the usage of the term internal hom. This bijection is defined using (3.6) on each fiber.

3.4.5. Algebra bundles. A monoid in $(\mathsf{Bundle}_M, \otimes)$ is a smooth vector bundle E over M, equipped with a smooth bundle map $E \otimes E \to E$ over M which is associative and unital. One may call this an *algebra bundle*. For instance, for any smooth vector bundle E, $\mathrm{End}(E)$ is an algebra bundle.

This term does appear in the literature (perhaps with different meanings), for instance, see the discussion by Lee [14, Section 6.6]. Warning: His definition is more rigid and requires an extra condition.

3.5. Interactions of the functor of smooth sections with operations

We have seen many interesting operations on smooth vector bundles. Now recall the Serre-Swan Theorem 3.18 which says that the functor of smooth sections Γ induces an equivalence between categories of smooth vector bundles over M and finitely generated projective modules over $C(M)$. Thus to every operation on smooth vector bundles over M, there is a corresponding operation on finitely generated projective modules over $C(M)$. We must try to describe these explicitly. It is also reasonable to expect that some of them would extend to the bigger category of all $C(M)$-modules. We focus here on the direct sum, tensor product and its internal hom.

3.5.1. Direct sums. We know that any equivalence of categories must preserve products and coproducts. Since direct sum is the product and coproduct in Bundle_M, the corresponding operation in $\mathsf{Mod}_{C(M)}$ under Γ must also be direct sum, which is the product and coproduct in that category. Thus:

Proposition 3.25. *For any smooth vector bundles E and F over M, there is a natural isomorphism*

$$\Gamma(E) \oplus \Gamma(F) \xrightarrow{\cong} \Gamma(E \oplus F) \tag{3.12}$$

of $C(M)$-modules.

It is also easy to see this directly. Explicitly: If s is a smooth section of E and t is a smooth section of F, then the element (s,t) of $\Gamma(E) \oplus \Gamma(F)$ maps to the smooth section of $E \oplus F$ whose value at p is $(s(p), t(p))$.

The above result says that Γ is a strong monoidal functor wrt direct sums. Background information on monoidal functors can be found in [1, Chapter 3].

3.5.2. Tensor products and internal hom. Now let us look at tensor products.

Theorem 3.26. *For any smooth vector bundles E and F over M, there is a natural isomorphism*

$$\Gamma(E) \otimes_{C(M)} \Gamma(F) \xrightarrow{\cong} \Gamma(E \otimes F)$$

of $C(M)$-modules.

Proof. Construction of the map is easy. If s is a smooth section of E and t is a smooth section of F, then the element $s \otimes t$ of $\Gamma(E) \otimes_{C(M)} \Gamma(F)$ maps to the smooth section of $E \otimes F$ whose value at p is $s(p) \otimes t(p)$. If E and F are both trivial, then so is $E \otimes F$, and it is straightforward to verify that the above map is an isomorphism.

The general case is quite involved. A complete argument is given in [21, Theorem 11.39]. It makes use of the Serre-Swan Theorem 3.18. □

This result says that Γ is a strong monoidal functor wrt tensor products. The unit is given by the identity map

$$C(M) \to \Gamma(\text{triv}),$$

where triv is the trivial line bundle over M.

Proposition 3.27. *For any smooth vector bundles E and F over M, there is a natural isomorphism*

$$\Gamma(\text{Hom}(E, F)) \cong \mathsf{Mod}_{C(M)}(\Gamma(E), \Gamma(F)) \tag{3.13}$$

of $C(M)$-modules.

Proof. By Theorem 3.26, Γ preserves tensor products, so it must also preserve the internal homs wrt the two tensor products, one on vector bundles over M and the other on $C(M)$-modules. This is precisely the claim made in this result.

One can also establish this result directly. First observe that there is a natural isomorphism

$$\mathsf{Bundle}_M(E, F) \xrightarrow{\cong} \Gamma(\text{Hom}(E, F)),$$

and then use Proposition 3.17. $\square$

Corollary 3.28. *Suppose E and F are smooth vector bundles over M, and $\varphi : \Gamma(E) \to \Gamma(F)$ is a map of $C(M)$-modules. Then φ can be viewed as an element of $\Gamma(F \otimes E^*)$.*

Proof. This follows from (3.9) and (3.13). One can also proceed directly as follows.

Let U be an open set of M which trivializes E, and let $(s_1, \dots, s_d)$ be the corresponding frame field. Then the required element, when restricted to U, is

$$\sum_{i=1}^{d} \varphi(s_i) \otimes s_i^*.$$

Using $C(M)$-linearity, one can check that this is independent of the choice of the frame field. $\square$

The above analysis of Γ has many payoffs: Lemma 2.25 follows immediately. It says that a smooth 1-form on M is a $C(M)$-linear functional on the space of smooth vector fields. We recommend that you now take a look at Lemma 2.26 and its proof. Also see how you can finish off Lemma 2.29.

Remark 3.29. Let E and F be any smooth vector bundles. In the literature, a smooth section of $E \otimes F$ is called a *tensor*. A map $\Gamma(E) \to \Gamma(F)$ of $C(M)$-modules is also called a tensor. This makes sense because such a map is a smooth section of $E^* \otimes F$.

3.6. Dual of the category of smooth vector bundles

We discuss the category $\mathsf{Bundle}^{\text{sop}}$. It is the dual of the category of smooth vector bundles. However, it is not simply the opposite category. The term "sop" stands for "strange opposite".

The consideration of this category is important. It allows us to properly understand the functoriality of the dual bundle. This enables us to interpret the construction of the cotangent bundle as a functor on manifolds. It also allows us to do base change for the functor of smooth sections.

3.6.1. Dual category.

Definition 3.30. Let $\mathsf{Bundle}^{\mathrm{sop}}$ denote the following category. Objects are smooth vector bundles. A morphism from $E \to M$ to $E' \to M'$ is a smooth map $f : M \to M'$ and a linear map $E'_{f(p)} \to E_p$ for each point p of M such that the dual of these linear maps defines a smooth bundle map $(E')^* \to E^*$, where $(E')^*$ and E^* are the dual bundles.

Better to say this using the pullback bundle: A smooth map $f : M \to M'$ and a morphism $f^*(E') \to E$ over M. The above definition can then become a lemma.

What role is the last condition playing? We explain by some examples.

Suppose M' is a point. So the bundle E' just consists of a vector space, say V. What is a morphism from $E \to M$ to $E' \to M'$ in this case. It is a family of linear maps $V \to E_p$, one for each $p \in M$. But this is not all. We need this family to be smooth. For instance, the image of a vector $v \in V$ should sweep out a smooth section of M.

Let $\mathsf{Bundle}^{\mathrm{sop}}_M$ denote the subcategory of bundles over M in which we allow only those morphisms for which $f = \mathrm{id}$. The smoothness condition assures that

$$\mathsf{Bundle}^{\mathrm{sop}}_M = \mathsf{Bundle}^{\mathrm{op}}_M,$$

the latter being the opposite category of Bundle_M, which was used in the discussion on smooth functors.

If E is a vector bundle, then so is E^*. Proposition 3.24 shows that this assigment is functorial over a fixed base. However taking the dual does not require working with a fixed base. What happens to functoriality then? Duality defines an equivalence of categories

$$\mathsf{Bundle} \to \mathsf{Bundle}^{\mathrm{sop}}, \qquad E \mapsto E^*.$$

Equivalent means that we can dispense with one of the two categories. However, in practice, it is better to keep both, since one is usually more convenient than the other depending on the context.

Note the dual of TM is T^*M, so by composing functors, we obtain

$$\begin{tikzcd} & \mathsf{Bundle} \arrow[dr] & \\ \mathsf{Manifold} \arrow[ur, "T"] \arrow[rr, "T^*"'] & & \mathsf{Bundle}^{\mathrm{sop}}. \end{tikzcd}$$

Similar to T^*, the bundle of differential forms is also a functor

$$\mathsf{Manifold} \to \mathsf{Bundle}^{\mathrm{sop}}.$$

3.6.2. Base change for the functor of smooth sections. Let AlgMod denote the category of algebra-modules: An object is a pair (R, M), where R is an algebra and M is a R-module. A morphism $(R, M) \to (S, N)$ consists of an algebra morphism $\varphi : R \to S$ and a morphism $f : M \to N$ of abelian groups which is a map of R-modules (viewing N as a R-module via φ:

$$f(rm) = \varphi(r)f(m)$$

for all $r \in R$ and $m \in M$.

Remark 3.31. One can consider the category of monoid-modules in any monoidal category. Consider the monoidal category of vector spaces, with monoidal structure being the tensor product. Then a monoid-module in this category is precisely an algebra-module. More precisely, AlgMod is the category of monoid-modules in the monoidal category of abelian groups.

There is a contravariant functor

$$\mathsf{Bundle}^{\mathrm{sop}} \to \mathsf{AlgMod}, \qquad (E, \pi, M) \to (C(M), \Gamma(E)).$$

If we take a usual bundle map $E \to E'$, then there is no map relating $\Gamma(E)$ and $\Gamma(E')$ in either direction. This is where "sop" comes in. If we instead take a sop bundle map $E \to E'$, then we have maps $E'_{f(p)} \to E_p$ for each p, which induces a linear map $\Gamma(E') \to \Gamma(E)$. We also have the algebra morphism $C(M') \to C(M)$. It is straightforward to verify that this defines a morphism of algebra-modules.

We provide a small illustration on how this construction can be useful. By precomposing with the cotangent bundle functor, we obtain a contravariant functor

$$\mathsf{Manifold} \to \mathsf{AlgMod}, \qquad M \to (C(M), \Gamma(T^*M)).$$

Recall that $\Gamma(T^*M) = \Omega^1(M)$ is the $C(M)$-module of 1-forms on M. Functoriality says that a smooth map $M \to N$ induces compatible maps

$$C(N) \to C(M) \quad \text{and} \quad \Omega^1(N) \to \Omega^1(M).$$

Post-composing with the functor $\mathsf{AlgMod} \to \mathsf{Alg}$, which projects on the first coordinate, we obtain the functor of smooth funtions C.

In the above discussion, we can replace the cotangent bundle functor by the differential forms bundle functor. This has the effect of replacing $\Omega^1(M)$ by $\Omega(M)$.

Punchline: We think of vector fields and 1-forms as dual notions. But this demands care. We never pushforward vector fields but we always pullback 1-forms.

Problems

(1) Recall that for V finite-dimensional, $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Suppose $f : V \to W$ is a linear map. How would you express it as an element of $V^* \otimes W$?
(2) Show that the tensor product functor $\mathsf{Vec}^2 \to \mathsf{Vec}$ which sends (V, W) to $V \otimes W$ is smooth.
(3) Show that:
   - The Mobius bundle is not a trivial bundle.
   - The canonical line bundle over $\mathbb{R}P^1$ is indeed the Mobius bundle. (First note that $\mathbb{R}P^1 = S^1$.)
   - The pullback of the Mobius bundle under the map $f : S^1 \to S^1$ defined by $f(z) := z^2$ (in complex notation) is the trivial line bundle $S^1 \times \mathbb{R}$. What happens if we replace z^2 by z^n for a positive integer n?
   - The tensor product of the Mobius bundle E over S^1 with itself is isomorphic to the trivial line bundle over S^1. What does it imply about the $C(S^1)$-module $\Gamma(E)$?
(4) The normal bundle to S^n in $\mathbb{R}^{n+1}$ is a line bundle over S^n defined as follows. The total space E consists of pairs (p, x) such that $p \in S^n$ and $x \in \mathbb{R}^{n+1}$ is a scalar multiple of p. The canonical projection $\pi : E \to S^n$ is $(p, x) \mapsto p$. For $p \in S^n$, a local trivialization $\varphi_U : E_U \to U \times \mathbb{R}$ (for U small enough) sends

(q, y) to $(q, \langle y, e \rangle)$, where e is the unit vector pointing at p. Check that this indeed defines a vector bundle. Is this bundle trivial?

In general, how would you define the normal bundle of a codimension-one submanifold of $\mathbb{R}^{n+1}$? When would this vector bundle be trivial?

(5) What is the initial object, terminal object, product, coproduct in Bundle? What about Bundle_M?
(6) Suppose E is a vector bundle over M, and U is an open set containing $p \in M$. Let s be a smooth section of E_U. Show that there exists a smooth section t of E whose restriction to U agrees with s is some neighborhood of p.
(7) Show that a smooth section of the pullback bundle f^*E is a smooth map from the bundle $N \times \mathrm{triv} \to N$ to the bundle $E \to M$.
(8) Suppose E and F are vector bundles over M and $\varphi : \Gamma(E) \to \Gamma(F)$ is a map of $C(M)$-modules. If s is a smooth section of E which vanishes at p, then show that $\varphi(s)$ also vanishes at p. (In the literature, this is referred to as the local property of a tensor.) Use this to answer the "Why?" in Lemma 2.25.
(9) Suppose E and E' are vector bundles over M, and F and F' are vector bundles over N. Show that bundle maps $E \to F$ and $E' \to F'$ induce bundle maps $E \oplus E' \to F \oplus F'$ and $E \otimes E' \to F \otimes F'$.
(10) For a vector bundle E over M, to show that $\Gamma(E)$ is a projective $C(M)$-module, it suffices to construct k smooth sections $s_1, \ldots, s_k$ which span E. Why? The number k will be strictly bigger than the rank of E unless E is trivial.
(11) Let M be a smooth manifold. Show that the tensor product over $C(M)$ of two projective $C(M)$-modules is again a projective $C(M)$-module.
(12) More generally, show that the tensor product of projective modules over any commutative ring R is again projective.
(13) Is the functor $\mathsf{Bundle}^{\mathrm{sop}} \to \mathsf{AlgMod}$ full and faithful?

CHAPTER 4

Connections and curvature

4.1. Covariant derivatives wrt vector fields

Recall that one can differentiate a smooth function f wrt a vector field X on a manifold M. It yields the smooth function that we have denoted $X(f)$. In local coordinates, this is the directional derivative. Now a smooth function f on M is the same as a smooth section of the trivial line bundle on M. So a natural question arises: Can one differentiate a smooth section of any vector bundle E wrt a vector field X on M? The answer in general is no. To make this idea work, we need to postulate it as an extra requirement on E. This is the notion of a covariant derivative. It was introduced by Koszul around 1950.

Definition 4.1. Let $\pi : E \to M$ be a smooth vector bundle. A *covariant derivative* on E is a family of maps

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$

one for each smooth vector field X on M, such that

$$\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s), \tag{4.1a}$$

$$\nabla_{fX}(s) = f\nabla_X(s), \tag{4.1b}$$

$$\nabla_X(s_1 + s_2) = \nabla_X(s_1) + \nabla_X(s_2), \tag{4.1c}$$

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s. \tag{4.1d}$$

We call $\nabla_X(s)$ the *covariant derivative* of the section s wrt the vector field X.

Note very carefully that ∇_X is not a map of $C(M)$-modules due to the extra term $X(f)s$ in (4.1d). However, it is $\mathbb{R}$-linear.

Remark 4.2. Let (A, M) be an object of the category AlgMod. Explicitly, A is an algebra and M is an A-module. A *derivation* of (A, M) is a pair (D, D') such that $D : A \to A$ is a derivation of the algebra A, and $D' : M \to M$ is a linear map satisfying

$$D'(am) = aD'(m) + D(a)m.$$

In our context, for a manifold M, the pair (X, ∇_X) is a derivation of $(C(M), \Gamma(E))$.

Example 4.3 (Trivial covariant derivative). Consider the trivial line bundle $E = M \times \mathbb{R}$. Recall that $\Gamma(E) = C(M)$. Define a covariant derivative on E by

$$\nabla_X(f) := X(f). \tag{4.2}$$

Condition (4.1d) takes the form

$$\nabla_X(fg) = f\nabla_X(g) + \nabla_X(f)g.$$

This is the familiar derivation property of vector fields. In this case, the conditions for a covariant derivative reflect the fact that the space of vector fields is the space of derivations of the algebra of smooth functions.

More generally, consider the trivial bundle $E = M \times \mathbb{R}^d$. In this case, $\Gamma(E)$ is the free rank d module over $C(M)$. Thus, a smooth section of this vector bundle is a d-tuple $(f_1, \dots, f_d)$ of smooth functions on M. Define a covariant derivative on E by

$$\nabla_X(f_1, \dots, f_d) := (Xf_1, \dots, Xf_d). \tag{4.3}$$

The conditions of a covariant derivative clearly hold. We call this the *trivial covariant derivative.*

Example 4.4 (Tangent bundle of euclidean space). Recall that the tangent bundle of $\mathbb{R}^n$ is trivial. More precisely, $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ be the canonical frame vector field. By the previous example, we have the trivial covariant derivative (∇_X) on $T\mathbb{R}^n$. Explicitly,

$$\nabla_X\big(f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}\big) = X(f_1)\frac{\partial}{\partial x_1} + \cdots + X(f_n)\frac{\partial}{\partial x_n}. \tag{4.4}$$

This is an instance of (4.3).

Example 4.5 (Tangent bundle of submanifolds of euclidean space). Let M be a submanifold of $\mathbb{R}^n$. Then we can define a covariant derivative on the tangent bundle TM as follows. Suppose X and Y are vector fields on M. Extend them to vector fields on $\mathbb{R}^n$, and continue to denote them by X and Y. Then $\nabla_X(Y)$ is a vector field on $\mathbb{R}^n$ (as in the previous example), but it will not be a vector field on M in general. So decompose this vector field into its tangential and normal components wrt M. Taking the tangential component of $\nabla_X(Y)$ defines a covariant derivative on M. One needs to check that this does not depend on the choice of the extensions.

Covariant derivative is a local notion:

Lemma 4.6. *Suppose (∇_X) is a covariant derivative on E. Then if for some open set U either $X|_U = 0$ or $s|_U = 0$, then*

$$(\nabla_X(s))(p) = 0$$

for all $p \in U$.

In other words, $\nabla_X(s)(p)$ only depends on the values of X and s in a neighborhood of p. Compare with Lemma 2.12.

Proof. Suppose $s|_U = 0$. Choose a smooth function f such that $f(p) = 1$ and f is zero outside U. Then $fs \equiv 0$. So

$$0 = \nabla_X(fs) = f\nabla_X(s) + X(f)s$$

Now evaluate the rhs at p. Since $s(p) = 0$ and $f(p) = 1$, we obtain $(\nabla_X(s))(p) = 0$ as required.

The argument for $X|_U = 0$ is similar and left as an exercise for you. □

For an open set U of M, recall that E_U is the smooth vector bundle obtained by restricting E to U. A covariant derivative (∇_X) on E restricts to a covariant derivative on E_U, which we denote by (∇^U_X). Explicitly, for a vector field X on U and a section s of E_U,

$$\nabla^U_X(s)(p) := \nabla_{X'}(s')(p),$$

where X' is a vector field on M which extends X, and s' is a section of E which extends s. (Why do such extensions exist?) By Lemma 4.6, the rhs above is independent of the extensions chosen.

Also note that as U varies, the (∇_X^U) satisfy the following compatibility. Whenever $V \subseteq U$, the restriction of (∇_X) first to U, and then to V, is the same as its restriction to V.

Lemma 4.7. *The value of $\nabla_X(s)$ at a point p depends only on X_p (and not on the values of X in a neighborhood of p).*

Proof. By Lemma 4.6, to calculate $\nabla_X(s)$ we can work in a chart U containing the point p. Let $(X_1, \dots, X_n)$ be a tuple of linearly independent vector fields on U (where n is the dimension of M). Write $X = f_1 X_1 + \dots + f_n X_n$. Then, by (4.1a) and (4.1b),

$$\nabla_X(s) = f_1 \nabla_{X_1}(s) + \dots + f_n \nabla_{X_n}(s).$$

So its value at p only depends on $f_1(p), \dots, f_n(p)$ and the fixed vector fields X_i. (The above formula is notationally incorrect, but I have prefer it over making a notational mess.)

Alternatively, in view of (4.1a), we need to show that if $X_p = 0$, then $\nabla_X(s)(p) = 0$. For this, we use Lemma 3.16 applied to $E = TM$, and write $X = f_1 X_1 + \dots + f_n X_n$, where the f_i vanish at p. (Now n may no longer be the dimension of M.) Now use the above formula, which is notationally correct. □

For the trivial covariant derivative on the trivial line bundle, Lemma 4.7 says that $X(f)(p)$ only depends on X_p. This is clear since $X(f)(p) = X_p(f)$.

Intuitively, one may think of $\nabla_{X_p}(s)$ as a measure of the way the section s is changing at p in the direction specified by X_p.

4.2. Connections on vector bundles

We saw how a covariant derivative allows us to differentiate a section of a vector bundle wrt a vector field. We now provide a more abstract way of understanding a covariant derivative by using the concept of a bundle-valued 1-form. This is the notion of a connection.

4.2.1. Connection on a vector bundle.

Recall that T^*M is the cotangent bundle of M, and a section of this bundle is a 1-form on M. Also recall that one can take the tensor product of two vector bundles. The discussion below makes use of the bundle $T^*M \otimes E$, where E is an arbitrary but fixed vector bundle. A section of this bundle is called an E-valued 1-form on M.

Definition 4.8. Let $\pi : E \to M$ be a smooth vector bundle. A *connection* on E is a $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that

$$\nabla(fs) = f\nabla(s) + df \otimes s \tag{4.5}$$

for any smooth section s of E and smooth function f on M.

The $\mathbb{R}$-linearity says that

$$\nabla(s_1 + s_2) = \nabla(s_1) + \nabla(s_2) \quad \text{and} \quad \nabla(cs) = c\nabla(s).$$

The second condition of scalar multiplication also follows from (4.5) by taking f to be the constant function c. Note very carefully that ∇ is not a map of $C(M)$-modules due to the extra term $df \otimes s$ in (4.5).

Warning. Some authors use the letter D to denote a connection.

By Theorem 3.26, the functor of smooth sections Γ is a strong monoidal functor wrt tensor products. In particular,

$$\Gamma(T^*M \otimes E) \;\cong\; \Gamma(T^*M) \otimes_{C(M)} \Gamma(E).$$

Hence, a section of $T^*M \otimes E$ can be written as a sum of tensor products of a section of T^*M and a section of E. The action of $C(M)$ can be used on either term of the tensor product. That is, if the section is $\omega \otimes s$, then the action of f can be written as $f\omega \otimes s$ or $\omega \otimes fs$.

Warning. We point out that

$$\Gamma(T^*M) \otimes_{C(M)} \Gamma(E) \neq \Gamma(T^*M) \otimes_{\mathbb{R}} \Gamma(E).$$

Many books while writing the tensor product do not explicitly specify whether the tensor product is over $C(M)$ or over $\mathbb{R}$.

Remark 4.9. Some books formulate a connection as a linear map

$$\nabla : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$$

satisfying appropriate conditions. Note that the tensor product is over $\mathbb{R}$ and not over $C(M)$, which makes this formulation awkward. Why cannot the tensor product be over $C(M)$?

Is there a category of vector bundles with connections? The sop bundle seems relevant here.

4.2.2. Equivalence between connections and covariant derivatives.

Proposition 4.10. *For E a smooth vector bundle, a connection on E is the same as a covariant derivative on E.*

Proof. Suppose ∇ is a connection on E. For any vector field X, there is a map

$$\nabla_X : \Gamma(E) \to \Gamma(E), \qquad \nabla_X(s) := \nabla(s)(X),$$

where $\nabla(s)(X)$ is the contraction of X with the 1-form appearing in $\nabla(s)$. To illustrate, if $\nabla(s) = \omega \otimes t$, then $\nabla_X(s) = \omega(X)t$. We claim that this defines a covariant derivative: (4.1c) follows since ∇ is $\mathbb{R}$-linear, (4.1d) follows from (4.5). If ω is a 1-form, then

$$\omega(X + Y) = \omega(X) + \omega(Y) \quad \text{and} \quad \omega(fX) = f\omega(X),$$

so (4.1a) and (4.1b) automatically follow.

Thus, a connection leads to a covariant derivative. Conversely, the connection can be reconstructed from the covariant derivative as follows.

Let $(X_1, \dots, X_n)$ be a tuple of linearly independent vector fields defined on an open set U of M (where n is the dimension of M), that is, a frame vector field on U. Define

$$\nabla^U : \Gamma(E_U) \to \Gamma(T^*U \otimes E_U), \qquad s \mapsto \sum_{i=1}^n X_i^* \otimes \nabla_{X_i}(s),$$

where the X_i^* denote the basis of 1-forms dual to the X_i. By (4.1c) and (4.1d) (applied to constant functions), this is a $\mathbb{R}$-linear map. Further, the ∇^U patch together, that is, ∇^U and ∇^V agree on $U \cap V$. This means that if $(Y_1, \dots, Y_n)$ is another tuple of linearly independent vector fields on U, then

$$\sum_{i=1}^n X_i^* \otimes \nabla_{X_i}(s) = \sum_{j=1}^n Y_j^* \otimes \nabla_{Y_j}(s).$$

To check this, we evaluate both sides on Y_j. Writing $Y_j = \sum_i f_i X_i$, the evaluation of the lhs is

$$\sum_{i=1}^n f_i \nabla_{X_i}(s) = \nabla_{\sum_i f_i X_i}(s) = \nabla_{Y_j}(s),$$

which is also the evaluation of the rhs. Note that we used (4.1a) and (4.1b). This yields the connection ∇, with (4.1d) translating to (4.5). $\square$

Example 4.11 (Trivial connection). We now connect to Example 4.3. Consider the trivial line bundle $E = M \times \mathbb{R}$. We saw that $\nabla_X(f) := Xf$ is a covariant derivative on E. Since E is the trivial line bundle, $\Gamma(E) = C(M)$ and $\Gamma(T^*M \otimes E) = \Gamma(T^*M)$, which is the space of 1-forms. Observe that the associated connection on E is

$$\nabla(f) = df. \tag{4.6}$$

In other words, ∇ coincides with the exterior derivative d from 0-forms to 1-forms in (2.20). We call this the *trivial connection* on E.

This generalizes as follows. Consider the trivial bundle $E = M \times \mathbb{R}^d$. Let us write $(e_1, \dots, e_d)$ for the canonical frame field. That is, $e_1(p) = (p, (1, 0, \dots, 0))$, and so on. We saw that

$$\nabla_X\big(\sum_{j=1}^d f_j e_j\big) := \sum_{j=1}^d X(f_j)\, e_j$$

is a covariant derivative on E. This is the same formula as (4.3). The associated connection can be written as

$$\nabla\big(\sum_{j=1}^d f_j e_j\big) = \sum_{j=1}^d df_j \otimes e_j. \tag{4.7}$$

We call this the *trivial connection* on E. The special case (4.4) of the tangent bundle on $\mathbb{R}^n$ can be written as

$$\nabla\big(f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n}\big) = d(f_1) \otimes \frac{\partial}{\partial x_1} + \dots + d(f_n) \otimes \frac{\partial}{\partial x_n}. \tag{4.8}$$

4.3. Connections and matrix-valued one-forms

Recall that a connection ∇ on a vector bundle E specifies a map from sections of E to E-valued one-forms. Using a local frame field of E, we will now see how a connection is determined locally by a matrix-valued one-form. This is called the connection form. It can be equivalently expressed using a family of locally defined smooth functions which are called Christoffel symbols of the connection. For a trivial bundle, the connection form and Christoffel symbols are defined globally.

4.3.1. Connection forms. Let ∇ be a connection on a rank d smooth vector bundle E over a smooth n-manifold M. Choose a local frame field $(s_1, \ldots, s_d)$ on an open set U of M which trivializes E. Write

$$\nabla^U(s_j) = \sum_{k=1}^{d} \theta_k^j \otimes s_k, \tag{4.9}$$

where θ_k^j are smooth 1-forms on U. The $d \times d$ matrix (θ_k^j) of 1-forms is called the *connection form*. It determines ∇^U completely, since by (4.5), for $s = \sum_j f_j s_j$,

$$\nabla^U(s) = \sum_{j=1}^{d} f_j \nabla^U(s_j) + df_j \otimes s_j.$$

It is customary to write this as $\theta(s) + d(s)$, but it is important to note that θ and d are not defined globally. If the bundle is trivial, then they are, see Example 4.13 below.

4.3.2. Christoffel symbols. Even more explicitly, we may assume that $(x_1, \ldots, x_n)$ is a coordinate system on U, and write

$$\theta_k^j = \sum_{i=1}^{n} \Gamma_{ij}^k dx_i, \tag{4.10}$$

where Γ_{ij}^k are smooth functions on U. The Γ_{ij}^k are called the *Christoffel symbols* of the connection. (They first appeared in 1869.) Here i varies between 1 and n, while j and k vary between 1 and d. In terms of covariant derivatives,

$$\nabla_{\frac{\partial}{\partial x_i}}(s_j) = \sum_{k=1}^{d} \Gamma_{ij}^k s_k. \tag{4.11}$$

It is convenient to keep U implicit and drop it from the notation.

4.3.3. Change of frame fields. Let $(s_1, \ldots, s_d)$ and $(s_1', \ldots, s_d')$ be two local frame fields on U with connection forms θ_k^j and $(\theta')_k^j$, respectively. We want to understand the relation between θ and θ'. Write

$$s_i' = \sum_j P_j^i s_j \quad \text{and} \quad s_i = \sum_j (P^{-1})_j^i s_j'. \tag{4.12}$$

Here P_j^i are smooth functions on U. They define a $d \times d$ matrix $P = (P_j^i)$. It is invertible at all points in U and $(P^{-1})_j^i$ are the entries of the inverse matrix P^{-1}. We calculate:

$$\nabla(s_i') = \nabla(\sum_j P_j^i s_j) = \sum_j \nabla(P_j^i s_j)$$

$$
\begin{aligned}
&= \sum_j P^i_j \nabla(s_j) + d(P^i_j) \otimes s_j \\
&= \sum_j P^i_j \sum_k \theta^j_k \otimes s_k + \sum_j d(P^i_j) \otimes s_j \\
&= \sum_k \sum_j P^i_j \theta^j_k \otimes s_k + \sum_k d(P^i_k) \otimes s_k \\
&= \sum_k \big(\sum_j P^i_j \theta^j_k + d(P^i_k) \big) \otimes s_k \\
&= \sum_k \big(\sum_j P^i_j \theta^j_k + d(P^i_k) \big) \otimes \big(\sum_l (P^{-1})^k_l s'_l \big) \\
&= \sum_k \sum_l \big(\sum_j P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l \big) \otimes s'_l \\
&= \sum_l \big(\sum_j \sum_k P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l \big) \otimes s'_l .
\end{aligned}
$$

Note carefully how we used linearity of the connection and property (4.5) and properties of the tensor product (which is over $C(M)$). This yields

$$(\theta')^i_l = \sum_j \sum_k P^i_j \theta^j_k (P^{-1})^k_l + d(P^i_k)(P^{-1})^k_l .$$

This formula can be written compactly in matrix notation as

$$\theta' = P\theta P^{-1} + (dP)P^{-1}. \tag{4.13}$$

This is an identity of $d \times d$ matrices whose entries are 1-forms on U.

4.3.4. Space of all connections. Recall that the space of sections $\Gamma(E)$ of any vector bundle E over M is a module over $C(M)$. The term $f\nabla(s)$ is the action of f on the section $\nabla(s)$. Due to the additional term $df \otimes s$ in (4.5), a connection is *not* a map of $C(M)$-modules. However, the difference of two connections is:

If ∇_1 and ∇_2 are two connections on E, then $\nabla_1 - \nabla_2$ is a map of $C(M)$-modules:

$$
\begin{aligned}
(\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs) \\
&= (f\nabla_1(s) + df \otimes s) - (f\nabla_2(s) + df \otimes s) \\
&= f\nabla_1(s) - f\nabla_2(s) \\
&= f(\nabla_1(s) - \nabla_2(s)).
\end{aligned}
$$

Hence by Corollary 3.28, $\nabla_1 - \nabla_2$ can be viewed as a 1-form on M with values in the endomorphism bundle $\mathrm{End}(E) = E \otimes E^*$:

$$(\nabla_1 - \nabla_2) \in \Gamma(T^*M \otimes \mathrm{End}(E)).$$

A connection *cannot* be viewed in this manner but the difference of two connections can be.

Conversely, if ∇ is a connection on E and θ is a 1-form on M with values in $\mathrm{End}(E)$, then $\nabla + \theta$ is a connection on E:

$$
\begin{aligned}
(\nabla + \theta)(fs) &= \nabla(fs) + \theta(fs) \\
&= f\nabla(s) + df \otimes s + f\theta(s) \\
&= f(\nabla + \theta)(s) + df \otimes s.
\end{aligned}
$$

This yields:

Proposition 4.12. *Suppose ∇^0 is a connection on a smooth vector bundle E. Then every connection ∇ on E can be uniquely written in the form $\nabla = \nabla^0 + \theta$, where θ is an* $\mathrm{End}(E)$*-valued* 1*-form on M.*

In other words, the space of connections on E is an affine space for $\Omega^1(\mathrm{End}(E))$. It is also common to use the symbol A instead of θ. We point out that $\theta = 0$ is permissible but not $\nabla = 0$ (unless the bundle itself is 0), that is, the zero map does not define a connection.

Example 4.13 (Connections on the trivial bundle). Recall from Example 4.11 that when E is a trivial bundle, it carries a canonical connection which is called the trivial connection. Thus, we can take ∇^0 to be the trivial connection, and then by Proposition 4.12, every connection is given by ∇^0 plus a matrix-valued 1-form on M. Let us spell this out.

What are all connections on the trivial line bundle $E = M \times \mathbb{R}$? We claim that a connection on E can be uniquely written as

$$\nabla(f) = f\omega + df \tag{4.14}$$

for a smooth 1-form ω. To see this, suppose θ is a map $C(M) \to \Gamma(T^*M)$ of $C(M)$-modules. Then θ is determined by its value on the constant function 1, and this value can be arbitrary. Thus $\theta(f) = f\omega$, where $\omega := \theta(1)$.

The Christoffel symbols of the connection are present in ω. In local coordinates, if we write $\omega = \sum_{i=1}^n g_i dx_i$, then $\Gamma^1_{i1} = g_i$. In particular, the Christoffel symbols of the trivial connection are zero.

More generally, let us consider the trivial bundle $E = M \times \mathbb{R}^d$. Let us write $(e_1, \dots, e_d)$ for the canonical frame field. Any connection on E is uniquely given by

$$\nabla(s) = \sum_{j=1}^d f_j \nabla(e_j) + df_j \otimes e_j, \quad \text{with} \quad \nabla(e_j) = \sum_{k=1}^d \theta^j_k \otimes e_k, \tag{4.15}$$

where θ^j_k are arbitrary smooth 1-forms on M. In this case, θ and d are globally defined. Recall that θ is a $(E^* \otimes E)$-valued 1-form. In the above notation,

$$\theta = \sum_{j,k} \theta^j_k \otimes e^*_j \otimes e_k.$$

4.4. Operations on connections

In Section 3.4, we discussed some important operations on smooth vector bundles such as direct sum, tensor product and taking duals. We now look at similar operations on connections.

4.4.1. Direct sum and tensor product. Suppose ∇ is a connection on E, and ∇' is a connection on E', both smooth vector bundles over M. Then we get a connection ∇'' on $E \oplus E'$ by setting

$$\nabla''_X(s, s') := (\nabla_X(s), \nabla'_X(s')). \tag{4.16}$$

Similarly, we get a connection ∇'' on $E \otimes E'$ by setting

$$\nabla''_X(s \otimes s') := \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s'). \tag{4.17}$$

The above formulas are written in terms of covariant derivatives. For writing them in terms of connections, simply remove the subscripts X on ∇; we are now equating 1-form valued sections.

4.4.2. Duals. Suppose ∇ is a connection on E. Then we get a connection ∇'' on E^* by setting

$$\nabla''_X(\alpha)(s) := X(\alpha(s)) - \alpha(\nabla_X(s)). \tag{4.18}$$

Here α is a section of E^* and s is a section of E. Since $E^* = \mathrm{Hom}(E, \mathrm{triv})$, a section of E^* is the same as a map $\alpha : \Gamma(E) \to C(M)$ of $C(M)$-modules. Equivalently,

$$\nabla''(\alpha)(s) := d(\alpha(s)) - \alpha(\nabla(s)). \tag{4.19}$$

This is an identity of 1-forms. Why do we need the minus sign? This is because $\nabla''_X(\alpha)$ must be a $C(M)$-functional on $\Gamma(E)$, that is,

$$\nabla''_X(\alpha)(gs) = g\nabla''_X(\alpha)(s).$$

Try to expand the lhs using the definition, and the derivation properties of X and ∇. Note that the term $X(g)\alpha(s)$ shows up twice and cancels due to the minus sign.

Remark 4.14. Equation (4.18) can be rewritten

$$\nabla_X(\alpha(s)) = \alpha(\nabla_X(s)) + \nabla_X(\alpha)(s),$$

where all covariant derivatives are denoted by ∇_X. If we call $\alpha(s)$ the contraction of α and s, then the above equation says that ∇_X commutes with contractions.

The above discussion shows that a connection on E induces a connection on the various tensor bundles associated to E. In particular, a connection on the tangent bundle of a manifold induces a connection on the bundle of (r, s)-tensors.

Question 4.15. Suppose ∇_i is a connection on E_i, where $i = 1, \dots, n$. Suppose $\mathcal{F}$ is a smooth functor on n variables. Then when would we get an induced connection on $\mathcal{F}(E_1, \dots, E_n)$?

4.5. Differential forms on a vector bundle

We have studied differential forms on a manifold M in Section 2.7. We now tensor the bundle of differential forms on M with any vector bundle E over M. This gives the notion of an E-valued differential form on M. The connection now plays the role of the exterior derivative. We recover the classical setup by taking E to be the trivial line bundle with trivial connection.

4.5.1. Differential forms on a manifold. Let M be a manifold. Recall that $\wedge^k(T^*M)$ denotes the k-th exterior power of the cotangent bundle T^*M. A section of $\wedge^k(T^*M)$ is a k-form on M. The space of all k-forms on M is denoted $\Omega^k(M)$. Taking direct sum over k yields the algebra of differential forms $\Omega(M)$.

4.5.2. Differential forms on a vector bundle. Now suppose E is a vector bundle over M. We consider the $C(M)$-module

$$\Omega^k(E) := \Gamma(\wedge^k(T^*M) \otimes E).$$

An element of this space is called an E-valued k-form on M, or a k-form on M with values in E, or simply a k-form on E. Taking direct sum over k yields the space $\Omega(E)$.

We note that $\Omega(E)$ is a bimodule over the algebra $\Omega(M)$: The left and right actions (both denoted by $\wedge$) are

$$\omega \wedge (\omega' \otimes s) := (\omega \wedge \omega') \otimes s \quad \text{and} \quad (\omega \otimes s) \wedge \omega' := (\omega \wedge \omega') \otimes s. \tag{4.20}$$

Since we implicitly identify $E \otimes \mathrm{triv} \cong E$, many different looking notations specify the same element, for instance,

$$\omega \wedge (1 \otimes s) = \omega \wedge s = \omega \otimes s, \qquad f \wedge s = f \otimes s = fs.$$

Note that $\Omega(\mathrm{triv}) = \Omega(M)$ and it is a bimodule over itself with left and right actions given by left and right multiplication in $\Omega(M)$.

4.5.3. Gauge exterior derivative. In the language of k-forms on a vector bundle E, observe that a connection on E is a $\mathbb{R}$-linear map

$$\nabla : \Omega^0(E) \to \Omega^1(E)$$

satisfying (4.5).

Theorem 4.16. *Suppose ∇ is a connection on a smooth vector bundle E. Then there is a unique sequence of $\mathbb{R}$-linear maps*

$$\nabla : \Omega^k(E) \to \Omega^{k+1}(E), \qquad k = 0, 1, 2, \dots$$

such that for $k = 0$, ∇ is the given connection, and

$$\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^k \omega \wedge \nabla(s) \tag{4.21}$$

for any smooth k-form ω on M, and any smooth section s of E.

The term $\omega \wedge \nabla(s)$ refers to the left action of ω on $\nabla(s)$ as defined in (4.20), where ∇ is the connection that we started with. Note that for $k = 0$, (4.21) reduces to (4.5).

Proof. The uniqueness assertion is clear since terms of the form $\omega \otimes s$ linearly span $\Omega^k(E)$, and the rhs of (4.21) is known. One needs to check that (4.21) is well-defined, that is,

$$\begin{aligned} \nabla((\omega_1 + \omega_2) \otimes s) &= \nabla(\omega_1 \otimes s) + \nabla(\omega_2 \otimes s), \\ \nabla(\omega \otimes (s_1 + s_2)) &= \nabla(\omega \otimes s_1) + \nabla(\omega \otimes s_2), \\ \nabla(f\omega \otimes s) &= \nabla(\omega \otimes fs). \end{aligned}$$

The first two are clear; we check the last one.

$$\begin{aligned} \nabla(f\omega \otimes s) &= d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= (df \wedge \omega) \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s). \end{aligned}$$

$$\begin{aligned} \nabla(\omega \otimes fs) &= d\omega \otimes fs + (-1)^k \omega \wedge \nabla(fs) \\ &= f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s) + (df \wedge \omega) \otimes s. \end{aligned}$$

So both sides are equal as required. $\square$

The extension of ∇ is sometimes called the *exterior covariant derivative*. In some contexts, it is called the *gauge exterior derivative*.

Note that if E is the trivial line bundle $M \times \mathbb{R}$, and ∇ is the trivial connection given by the exterior derivative d, then the above extension of ∇ is the exterior derivative d on higher differential forms on M. Recall that the exterior derivative satisfies $d^2 = 0$. However, in general, $\nabla \circ \nabla \neq 0$; this is of significance as we will see later.

Lemma 4.17. *Suppose ∇ is a connection on a smooth vector bundle E. For any smooth k-form θ on E and smooth function f on M,*

$$\nabla(f\theta) = df \wedge \theta + f\nabla(\theta). \tag{4.22}$$

PROOF. It suffices to check for $\theta = \omega \otimes s$.

$$\begin{aligned}\nabla(f\omega \otimes s) &= d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= (df \wedge \omega) \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla(s) \\ &= df \wedge \theta + f\nabla(\theta).\end{aligned}$$

We made use of the identity in Theorem 2.27, item (3). □

More generally, by the same calculation:

Lemma 4.18. *Suppose ∇ is a connection on a smooth vector bundle E. For any smooth k-form ω on M and smooth m-form θ on E,*

$$\begin{aligned}\nabla(\omega \wedge \theta) &= d\omega \wedge \theta + (-1)^k \omega \wedge \nabla(\theta) \\ \nabla(\theta \wedge \omega) &= \nabla(\theta) \wedge \omega + (-1)^m \theta \wedge d\omega.\end{aligned} \tag{4.23}$$

There is one identity for the left action and one for the right action.

4.5.4. Forms as alternating maps. The following result generalizes Lemma 2.29.

Lemma 4.19. *For a smooth vector bundle E over M, there is a canonical linear isomorphism between $\Omega^k(E)$ and alternating $C(M)$-module maps from the k-fold tensor product of $\Gamma(TM)$ to $\Gamma(E)$.*

Explicitly, for 1-forms ω_i, a section s, and vector fields X_j,

$$(\omega_1 \wedge \cdots \wedge \omega_k \otimes s)(X_1 \otimes \cdots \otimes X_k) = \det(\omega_i(X_j))s.$$

The following result generalizes Lemma 2.30.

Lemma 4.20. *Suppose ∇ is a connection on a smooth vector bundle E. For any smooth k-form θ on E, and smooth vector fields $X_1, \ldots, X_{k+1}$,*

$$\begin{aligned}\nabla(\theta)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{X_i}(\theta(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),\end{aligned} \tag{4.24}$$

where the hat over X_i indicates that X_i is deleted from the sequence.

4.6. Curvature

We discuss curvature. This notion makes sense on any vector bundle with a connection. Recall that connection and covariant derivative are equivalent. Hence the notion of curvature can be formulated in either language. We explain both approaches independently.

4.6.1. Curvature of a connection.

Definition 4.21. Let ∇ be a connection on a smooth vector bundle $\pi : E \to M$. Its *curvature* is the map

$$R := \nabla \circ \nabla : \Omega^0(E) \longrightarrow \Omega^2(E).$$

The connection is called *flat* if $R = 0$, that is, $R(s) = 0$ for every smooth section s of E.

Recall that a connection is not a map of $C(M)$-modules. However:

Lemma 4.22. *The curvature R of a connection ∇ is a map of $C(M)$-modules. That is,*

$$R(fs) = fR(s) \quad \textit{and} \quad R(s_1 + s_2) = R(s_1) + R(s_2).$$

PROOF. The second identity is clear. For the first, we calculate using (4.5), (4.21) and (4.22):

$$\begin{aligned}\nabla(\nabla(fs)) &= \nabla(f\nabla(s) + df \otimes s)\\ &= (df \wedge \nabla(s) + f\nabla(\nabla(s))) + (d^2 f \otimes s - df \wedge \nabla(s))\\ &= f\nabla(\nabla(s)).\end{aligned}$$

Since $d^2 = 0$, the term $d^2 f \otimes s$ is zero. The minus sign in front of $df \wedge \nabla(s)$ occurs because df is a 1-form (and 1 is an odd integer). □

Thus, curvature is a map of $C(M)$-modules

$$\Gamma(E) \to \Gamma(\wedge^2(T^*M) \otimes E),$$

and by Corollary 3.28, it can be viewed as an element of

$$\Gamma(\wedge^2(T^*M) \otimes E^* \otimes E).$$

Thus, curvature is a 2-form on M with values in $\mathrm{End}(E)$. (Recall that we had a similar statement for the difference of two connections.)

4.6.2. Matrix of curvature 2-forms. Choose a local frame field $(s_1, \ldots, s_d)$ on an open set U of M which trivializes E. Write

$$R(s_j) = \sum_{k=1}^{d} R^j_k \otimes s_k, \tag{4.25}$$

or equivalently,

$$R = \sum_{j,k=1}^{d} R^j_k \otimes s^*_j \otimes s_k, \tag{4.26}$$

where R^j_k are smooth 2-forms on U. (These depend on the choice of the local frame field.) Let us express them in terms of the connection form (4.9):

$$\begin{aligned}\nabla \circ \nabla(s_j) &= \sum_{k=1}^{d} \nabla(\theta^j_k \otimes s_k) \\ &= \sum_{k=1}^{d} d\theta^j_k \otimes s_k - \sum_{k=1}^{d}\sum_{l=1}^{d} \theta^j_k \wedge \theta^k_l \otimes s_l \quad \text{using (4.21)} \\ &= \sum_{k=1}^{d} \Big(d\theta^j_k - \sum_{m=1}^{d} \theta^j_m \wedge \theta^m_k\Big) \otimes s_k \\ &= \sum_{k=1}^{d} R^j_k \otimes s_k.\end{aligned}$$

Thus,

$$R^j_k = d\theta^j_k - \sum_{m=1}^{d} \theta^j_m \wedge \theta^m_k. \tag{4.27}$$

This can be expressed in short by $R = d\theta - \theta \wedge \theta$. But it is important to know what this exactly means. Some books will write $R = d\theta + \theta \wedge \theta$. Note here that formula (4.27) can be rewritten as

$$R^j_k = d\theta^j_k + \sum_{m=1}^{d} \theta^m_k \wedge \theta^j_m. \tag{4.28}$$

Also some books will write θ^k_j for θ^j_k and R^k_j for R^j_k.

4.6.3. Change of frame fields. Let $(s_1, \ldots, s_d)$ and $(s'_1, \ldots, s'_d)$ be two local frame fields on U with curvature forms R^j_k and $(R')^j_k$, respectively. Then

$$R' = PRP^{-1}, \tag{4.29}$$

where P is the matrix for change of frame fields (4.12). This is a familiar formula from linear algebra on how a matrix transforms when we change basis. Exactly the same happens here because curvature is $C(M)$-linear.

Compare and contrast with (4.13). If we were to do a change of frame fields for a difference of two connections, then the formula would be as in (4.29).

Some books will say: R transforms as a tensor.

4.6.4. Flat connections on the trivial bundle. Recall that the trivial connection on the trivial line bundle $M \times \mathbb{R}$ is nothing but the exterior derivative d. Since $d^2 = 0$, the trivial connection is flat.

Consider the rank d trivial bundle E on M. Formula (4.28) shows that flat connections on E correspond to solutions of the equation

$$d\theta^j_k + \sum_{m=1}^{d} \theta^m_k \wedge \theta^j_m = 0. \tag{4.30}$$

This is called the *Maurer-Cartan equation*. We need to solve for the θ^j_k. Of course, the trivial connection is flat since $\theta^j_k \equiv 0$ in that case.

For $d = 1$, the Maurer-Cartan equation simply says that $d\theta = 0$. So flat connections on the trivial line bundle on M correspond to closed 1-forms on M.

4.6.5. Curvature of a covariant derivative. Let us now look at curvature from the perspective of covariant derivatives.

Definition 4.23. Let (∇_X) be a covariant derivative on a smooth vector bundle $\pi : E \to M$. Its *curvature* is the map which assigns to every pair of smooth vector fields (X, Y) on M the operator

$$R(X,Y) : \Gamma(E) \longrightarrow \Gamma(E)$$

defined by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{4.31}$$

We permit ourselves to also write $R_{X,Y}$ instead of $R(X,Y)$. Some formal consequences are discussed below.

Lemma 4.24. *For any smooth vector fields X and Y, the curvature operator $R(X,Y)$ in (4.31) is a map of $C(M)$-modules. That is,*

$$R(X,Y)(fs) = fR(X,Y)(s),$$
$$R(X,Y)(s_1 + s_2) = R(X,Y)(s_1) + R(X,Y)(s_2).$$

Proof. We check the first identity. Using (4.1d),

$$\begin{aligned}\nabla_X \nabla_Y (fs) &= \nabla_X (f \nabla_Y (s) + Y(f)s) \\ &= f \nabla_X \nabla_Y (s) + X(f) \nabla_Y (s) + Y(f) \nabla_X (s) + XY(f)s.\end{aligned}$$

It follows that

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fs) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(s) + [X,Y](f)s.$$

Thus, the commutator of ∇_X and ∇_Y fails to be a map of $C(M)$-modules. Observe how this defect is corrected by the term $\nabla_{[X,Y]}$. $\square$

Lemma 4.25. *The curvature operator $R(X,Y)$ in (4.31) is $C(M)$-linear in X and in Y. That is,*

$$R(fX, Y) = fR(X,Y),$$
$$R(X_1 + X_2, Y) = R(X_1, Y) + R(X_2, Y),$$
$$R(X, Y_1 + Y_2) = R(X, Y_1) + R(X, Y_2).$$

In addition,

$$R(X,Y) = -R(Y,X).$$

Proof. We explain the first identity, the rest are clear. Recall from (2.29) that the bracket of vector fields behaves as follows under scalar multiplication by smooth functions.

$$[fX, gY] = fX(g)Y - gY(f)X + fg[X,Y].$$

For symmetry purposes, we have multiplied in both coordinates.

Using this formula, and (4.1b),

$$R(fX, gY) = f\nabla_X (g\nabla_Y) - g\nabla_Y (f\nabla_X) - fX(g)\nabla_Y + gY(f)\nabla_X - fg\nabla_{[X,Y]}.$$

Now use (4.1d) and note how the unwanted terms $fX(g)\nabla_Y$ and $gY(f)\nabla_X$ cancel out to give

$$R(fX, gY) = fgR(X,Y).$$

□

Some books will say: R is tensorial wrt X, Y and s.

Lemma 4.26. *The value of $R(X,Y)(s)(p)$ only depends on X_p and Y_p (and not on the values of X and Y at other points of M).*

Proof. Due to the antisymmetry relation $R(X,Y) = -R(Y,X)$, it suffices to prove the result in one of the two coordinates. Accordingly, suppose $X_p = 0$. As in the proof of Lemma 4.7, use Lemma 3.16 applied to $E = TM$, and write $X = f_1X_1 + \cdots + f_nX_n$, where the f_i vanish at p. Now use Lemma 4.25 to deduce that $R(X,Y)(s)(p) = 0$. □

Lemma 4.27. *The value of $R(X,Y)(s)(p)$ only depends on $s(p)$ (and not on the values of s at other points of M).*

Proof. Suppose $s(p) = 0$. By Lemma 3.16, write $s = \sum_i f_i s_i$ with $f_i(p) = 0$. Now use Lemma 4.24 to deduce that $R(X,Y)(s)(p) = 0$. □

4.6.6. Equivalence of the two approaches. Suppose (θ^j_k) is the connection 1-form of ∇. We know from formula (4.28) that

$$R(s_j) = \sum_{k=1}^{d} \big(d\theta^j_k + \sum_{m=1}^{d} \theta^m_k \wedge \theta^j_m\big) \otimes s_k.$$

Hence,

$$R(X,Y)(s_j) = \sum_{k=1}^{d} \big(d\theta^j_k(X,Y) + \sum_{m=1}^{d} (\theta^m_k \wedge \theta^j_m)(X,Y)\big) s_k.$$

Now, by formula (2.28),

$$d\theta^j_k(X,Y) = X(\theta^j_k(Y)) - Y(\theta^j_k(X)) - \theta^j_k([X,Y]),$$

and by formula (2.26),

$$(\theta^m_k \wedge \theta^j_m)(X,Y) = \theta^m_k(X)\theta^j_m(Y) - \theta^m_k(Y)\theta^j_m(X).$$

Further,

$$\nabla_X(s_j) = \sum_k \theta^j_k(X) s_k,$$

from which we have

$$\nabla_X\nabla_Y(s_j) = \sum_{k=1}^{d} \big(X(\theta^j_k(Y)) + \sum_{m=1}^{d} \theta^m_k(X)\theta^j_m(Y)\big) s_k,$$

and a similar expression for $\nabla_Y\nabla_X(s_j)$. Putting all this information together, we deduce

$$R(X,Y)(s_j) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(s_j),$$

as required.

We could have done this calculation earlier and turned Definition 4.23 into a proposition. Lemma 4.25 would then immediately follow since we are contracting the pair of vector fields on a 2-form. Similarly, Lemma 4.24 would follow from Lemma 4.22.

4.7. Parallel transport

Why is it that one needs to postulate a covariant derivative? Given a vector bundle E, what is the difficulty in defining $\nabla_{X_p}(s)(p)$? A natural thing to do is to take a curve on M passing through p which is tangential to X_p, and see how the section s changes along this curve? However, the values of s at different points on the curve lie in different fibers, so how do we compare them? This is where the problem lies. You may think that we could pick a trivialization around p, and then we have an identification of the fibers. But this process is not canonical, a different trivialization would give a different identification.

The point is that a covariant derivative is precisely what allows us to identify fibers over points on a curve. This is called parallel transport. Details follow.

4.7.1. Sections along a curve. Consider the following setup.

- ∇ is a connection on a smooth vector bundle E over M.
- c is a curve on M, that is,
$$c : [a, b] \to M, \qquad t \mapsto c(t)$$
is a smooth function.
- X is a smooth vector field on M, which is tangential to c, that is, $X_{c(t)} = c_*(\frac{d}{dt})$.
- s is a smooth section of E.

By Lemma 4.7, the notation $\nabla_{X_p}(s)(p)$ makes sense, where X_p denotes the tangent vector at p given by the vector field X. We compute
$$\nabla_{X_{c(t)}}(s)(c(t)),$$
the covariant derivative of s wrt X along the curve c.

Let U be a open set on M with coordinates $(x_1, \dots, x_n)$. Write $c(t) = (c_1(t), \dots, c_n(t))$. Then
$$X_{c(t)} = \sum_i \dot{c}_i(t) \frac{\partial}{\partial x_i},$$
with the dot on top indicating derivative wrt t. Also, let $(s_1, \dots, s_d)$ be a frame field over U. Write $s = \sum_k h_k s_k$, where h_k are smooth functions on U.
$$\begin{aligned}
\nabla_{X_{c(t)}}(s)(c(t)) &= \sum_k \nabla_{X_{c(t)}}(h_k s_k)(c(t)) \\
&= \sum_k X_{c(t)}(h_k) s_k(c(t)) + h_k(c(t)) \nabla_{X_{c(t)}}(s_k)(c(t)) \\
&= \sum_k \dot{h_k}(c(t)) s_k(c(t)) + \sum_{k,i} h_k(c(t)) \dot{c}_i(t) \nabla_{\frac{\partial}{\partial x_i}}(s_k)(c(t)) \\
&= \sum_k \dot{h_k}(c(t)) s_k(c(t)) + \sum_{k,i,j} h_k(c(t)) \dot{c}_i(t) \Gamma^j_{ik}(c(t)) s_j(c(t)).
\end{aligned}$$

Lemma 4.28. *In the above setup, $\nabla_X(s)$ at any point on the curve c only depends on the values of s on the curve (and not on its values at other points of M).*

PROOF. This follows from the above calculation. The key step is when we replace $X_{c(t)}(h_k)$ by $\dot{h_k}(c(t))$. The general fact is that $X_p(f)$ only depends on the values of f along any curve passing through p whose tangent vector at p is X_p. $\square$

We say s is a *section along* c if it is defined only at points of c. Smoothness would mean that $s(c(t))$ is a smooth section of the pullback bundle $c^*(E)$.

Now consider the "differential" equation

$$\nabla_{X_{c(t)}}(s)(c(t)) = 0.$$

The variable is s (it is a section along c), with the remaining parameters ∇, X and c fixed. Also, suppose an initial condition is specified, that is, $s(c(a))$ is given.

Lemma 4.29. *In the above setup, there is a unique solution for s along c.*

PROOF. Locally, around $c(a)$, by the above calculation, we need to solve

$$\sum_k \dot{h_k}(c(t))s_k(c(t)) + \sum_{k,i,j} h_k(c(t))\dot{c_i}(t)\Gamma^j_{ik}(c(t))s_j(c(t)) = 0.$$

The variables are the functions $h_k(c(t))$. The lhs is a section. Since $(s_1, \dots, s_d)$ is a frame field, we can put the coefficient of each s_k to be zero. Thus we obtain

$$\dot{h_k}(c(t)) + \sum_{i,j} h_j(c(t))\dot{c_i}(t)\Gamma^k_{ij}(c(t)) = 0, \quad k = 1, \dots, d. \tag{4.32}$$

This is a linear system of ODE in the d functions h_k. So there is a unique solution satisfying the given initial condition.

Break the interval $[a, b]$ into finite number of intervals such that the above analysis works in each of those intervals. By concatenating the solutions, we get a solution along all of c. $\square$

In the trivial connection, the Christoffel symbols are zero, so we are left to solving $\dot{h_k}(c(t)) = 0$. Thus all the h_k (and hence s) are constant along c, as expected. This is how we parallel transport vectors in say $\mathbb{R}^n$ implicitly all the time.

Definition 4.30. A section s along c is called *parallel* if $\nabla_{\dot{c}}(s) = 0$.

Lemma 4.29 shows that a parallel section along c exists and is unique if its value is specified at one point of c.

One can call a section parallel if $\nabla_X(s) = 0$ for all vector fields X. However this system of equations is overdetermined, so such global parallel sections will not exist in general.

4.7.2. Parallel transport. A connection defines a linear isomorphism

$$P_c : E_{c(a)} \xrightarrow{\cong} E_{c(b)}. \tag{4.33}$$

We explain this below.

- Suppose $v \in E_{c(a)}$. Then by Lemma 4.29, we can uniquely parallel transport v to an element of $E_{c(b)}$. This is defined to be $P_c(v)$. Thus there is a unique parallel section along c going from v to $P_c(v)$.
- The map P_c is linear. This is because if s_1 works for v_1, and s_2 for v_2, then $s_1 + s_2$ works for $v_1 + v_2$.
- Let $\bar{c}$ be the opposite curve joining $c(b)$ to $c(a)$. The map

  $$P_{\bar{c}} : E_{c(b)} \xrightarrow{\cong} E_{c(a)},$$

  is inverse to P_c. So P_c is an isomorphism.

With a way to identify nearby fibers, covariant derivative assumes a form similar to the classical derivative:

Proposition 4.31. *Let ∇ be a connection, $c : [a,b] \to M$ be a curve and s be a section along c. For any $a \le t \le b$, let*

$$P_{c,t} : E_{c(a)} \xrightarrow{\cong} E_{c(t)}$$

be the parallel transport map. Then

$$\nabla_{\dot{c}(a)}(s)(c(a)) = \lim_{t\to a} \frac{P_{c,t}^{-1}(s(c(t))) - s(c(a))}{t}. \tag{4.34}$$

Proof. Pick a frame $s_1, \ldots, s_d$ of parallel sections along d. Write $s = \sum_k h_k s_k$, where h_k are smooth functions along c. The covariant derivative of all the s_k along c is zero. Hence, by (4.1d),

$$\nabla_{X_{c(a)}}(s)(c(a)) = \sum_k \dot{h_k}(c(a))s_k(c(a))$$

Since $P_{c,t}^{-1}(s_k(c(t))) = s_k(c(a))$, the above limit evaluates to the rhs. □

Consider the trivial bundle $E = M \times \mathbb{R}^k$. Here there is a canonical identification of fibers. Check that this parallel transport corresponds to the trivial connection on E. To be very concrete, take M to be euclidean space, and $E = TM$. Then a tangent vector at p can be moved to any other point q by keeping it parallel. This should make the terminology clear.

Note that if we parallel translate a vector v along a closed curve c, then in the above example, we will end up again at v, but in general, we will not. For instance, try such an experiment on S^2. Not returning at the starting point says that there is curvature in the connection. Thus, the trivial connection has no curvature.

4.7.3. Reversing the process. tentative discussion Now imagine that we do not have a connection, but instead are given the linear isomorphisms (4.33), one for each curve. What axioms must we impose on the P_c that will allow us to construct a connection whose parallel transport is precisely the P_c?

(1) If $c = c_2 \cdot c_1$, then $P_c = P_{c_2} \circ P_{c_1}$.
(2) $P_{\bar{c}} = P_c^{-1}$.
(3) P_c varies smoothly on c, that is, for any $v \in E_{c(a)}$, $P_{c,t}(v)$ is a smooth section along c.
(4) P_c varies smoothly with c: If

$$C : [0,1] \times [a,b] \to M, \qquad c(u,-) =: c_u : [a,b] \to M$$

is a smooth family of curves, and s is a smooth section along the curve $C(-,a)$, then $P_{c_u}(s(u,a))$ is a smooth section along the curve $C(-,b)$.

Given the P_c, the covariant derivative is defined using (4.34). It seems that we should also impose the condition that the limit does not depend on the particular choice of c.

This approach is due to Knebelman (1951).

4.7.4. Horizontal tangent space of a vector bundle. Suppose E is a vector bundle. Consider the tangent bundle TE. This is the tangent bundle not of M but of the vector bundle itself.

Let $v \in E_p$ be any point of E. Let us look at the tangent space T_vE. It has a distinguished "vertical" subspace which can be identified with E_p. But there is no distinguished complementary "horizontal" subspace (unless $v = 0$). In other words, we do not know what it means to move horizontally across fibers starting at v. You may again think that we can do this by picking a trivialization containing v. But this will not work.

Now suppose that E has a connection ∇. Take any tangent vector of M at p. Call it X. Parallel translate v along any curve on M passing through p whose tangent at p is X. This will yield a curve $v(t)$ in E. The vectors

$$\lim_{t\to a} v(t)$$

as X varies over all tangent vectors at p, will span a subspace in T_vE, which is complementary to E_p. Thus, a connection provides us with a notion of a horizontal. For more on these ideas, see [14, Section 12.4] on Ehresmann connections.

4.7.5. Curvature as a two-dimensional limit. Roughly speaking, the curvature at a point p measures the change in a vector v at p when it is parallel transported along a small closed curve starting and ending at p. The precise result is given below.

Proposition 4.32. *Suppose X_p and Y_p are linearly independent tangent vectors at p, and $v \in E_p$. Then*

$$R(X_p, Y_p)v = -\lim_{t,u\to 0} \frac{v_{t,u} - v}{tu},$$

where $v_{t,u}$ is obtained by parallel transporting v along a closed curve defined as follows: Choose local coordinates $(x_1, \dots, x_n)$ around p such that $X_p = \frac{\partial}{\partial x_1}$ and $Y_p = \frac{\partial}{\partial x_2}$. The (piecewise smooth) curve goes counterclockwise along the boundary of the rectangle with vertices $(0,0)$, $(t,0)$, $(0,u)$, and (t,u) (with remaining $n-2$ coordinates 0).

Proof. This is stated and proved in [14, Theorem 12.47]. Let A be the two-dimensional submanifold of M corresponding to the plane in $\mathbb{R}^n$, whose last $n-2$ coordinates are 0. Define a section s along A, whose value at (t,u) is defined by parallel transporting v first from $(0,0)$ to $(t,0)$, and then from $(t,0)$ to $(0,u)$. One needs to check that s is smooth. We have

$$[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}] = 0, \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x_2}}(s) = 0.$$

Therefore,

$$R(X_p, Y_p)v = \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}}(s) - \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}}(s) = -\nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}}(s)$$

Let P_u denote parallel translation along the y-axis from $(0,0)$ to $(0,u)$. Then

$$\nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}}(s) = \lim_{u\to 0} \frac{P_u^{-1}(\nabla_{\frac{\partial}{\partial x_1}}(s)(0,u)) - \nabla_{\frac{\partial}{\partial x_1}}(s)(0,0)}{u} = \lim_{u\to 0} \frac{P_u^{-1}(\nabla_{\frac{\partial}{\partial x_1}}(s)(0,u))}{u}$$

(The covariant derivative of s in the horizontal direction is not zero in general, but along the x-axis it is zero.) Now let $P_{t,u}$ denote parallel translation in the horizontal direction from $(0,u)$ to (t,u). Then

$$\nabla_{\frac{\partial}{\partial x_1}}(s)(0,u)) = \lim_{t\to 0}\frac{P_{t,u}^{-1}(s)(t,u) - s(0,u)}{t}$$

Substituting this back into the calculation, and noting that

$$P_u^{-1}P_{t,u}^{-1}(s)(t,u) = v_{t,u} \quad \text{and} \quad P_u^{-1}s(0,u) = v,$$

we obtain the desired result. □

Problems

(1) Give an example of a connection on the canonical line bundle.
(2) Consider the trivial rank 2 bundle $E = M\times\mathbb{R}^2$. Does $\nabla_X(f,g) := (Xg, Xf)$ define a covariant derivative on E?
(3) Let h be a nowhere zero smooth function on a manifold M. It induces a bundle isomorphism from the trivial line bundle $E = M\times\mathbb{R}$ to itself which sends (p,x) to $(p, xh(p))$. The trivial covariant derivative (4.2) induces a covariant derivative on E by conjugating by this isomorphism.
  - Write down a formula for it; also check (4.1d) explicitly.
  - Write the corresponding connection in the form (4.14).
  - Interpret the relation between this connection and the trivial one as an instance of (4.13).

  Generalize the above analysis to $E = M\times\mathbb{R}^k$.
(4) Let $S^2(r)$ denote the sphere of radius r centered at the origin in $\mathbb{R}^3$. The map $S^2(1)\to S^2(2)$ which multiplies by 2 is a diffeomorphism. The trivial connection on the tangent bundle of $\mathbb{R}^3$ induces connections on the tangent bundles of $S^2(1)$ and $S^2(2)$. Show that the scaling map does not respect connections.
(5) Let ∇_1 and ∇_2 be two connections on E. Show that $a\nabla_1 + b\nabla_2$ is a connection whenever $a+b=1$.
(6) Check that formulas (4.16), (4.17), (4.18) indeed define connections.
(7) Explain clearly how the first formula in (4.23) generalizes all of (4.5), (4.21), (4.22). Also deduce the second formula in (4.23) using the first.
(8) Check directly that for the trivial line bundle, the connection ∇ defined by (4.14) has curvature $R(f) = f\,d\omega$. (In this example, curvature is a map of $C(M)$-modules from 0-forms to 2-forms on M.) Deduce that R is flat iff $d\omega = 0$.
(9) How does the calculation in Section 4.6.6 change if we start with formula (4.27) instead of (4.28)?
(10) Suppose ∇ is a connection on E. In a local frame field, write $\nabla = d+\theta$. Show that the induced connection ∇ on E^* is given by

$$\nabla(s_k^*) = -\sum_{j=1}^{d}\theta_k^j\otimes s_j^*.$$

Observe that the induced connection ∇ on $\mathrm{End}(E) = E^*\otimes E$ is given by

$$\nabla(s_j^*\otimes s_k) = -\sum_m \theta_j^m\otimes s_m^*\otimes s_k + \sum_l \theta_l^k\otimes s_j^*\otimes s_l.$$

Now consider an arbitrary section

$$s = \sum_{j,k} f_k^j \otimes s_j^* \otimes s_k$$

of $\mathrm{End}(E)$. Show that

$$\nabla(s) = \sum_{j,k} \left(d(f_k^j) + \sum_m f_m^j \theta_k^m - \sum_l f_k^l \theta_l^j \right) \otimes s_j^* \otimes s_k.$$

A useful shorthand for this formula is

$$\nabla(s) = d(s) + [\theta, s].$$

(11) For any smooth vector bundle E, $\Omega(\mathrm{End}(E))$ carries the structure of an algebra and $\Omega(E)$ is a module over it. Explain this statement. Show that for any E-valued k-form θ,

$$\nabla(\nabla(\theta)) = R \wedge \theta,$$

where the rhs denotes the action of R on θ. Both sides are E-valued $(k+2)$-forms. For $k = 0$, the identity is just the definition of R.

(12) Suppose ∇ is a connection on E. Then it induces a connection ∇ on $\mathrm{End}(E)$, which further extends to forms on $\mathrm{End}(E)$. Show that ∇ of the curvature is zero. Recall that the latter is a 2-form on $\mathrm{End}(E)$. This is known as the *second Bianchi identity*.

(13) In local coordinates $(x_1, \dots, x_n)$ and frame field $(s_1, \dots, s_d)$, set

$$R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})(s_l) = \sum_k R_{lij}^k s_k.$$

Write down an expression for the functions R_{lij}^k in terms of the Christoffel symbols.

(14) Let ∇ be a connection on the tangent bundle TM. The *torsion* of ∇ is defined to be

$$T_{X,Y} := \nabla_X Y - \nabla_Y X - [X, Y].$$

Check that

$$\Gamma(TM) \otimes_{C(M)} \Gamma(TM) \to \Gamma(TM), \qquad X \otimes Y \mapsto T_{X,Y}$$

is a map of $C(M)$-modules. One says that torsion is a tensor.

(15) How does parallel translation work on the tangent bundle of $\mathbb{R}$ wrt an arbitrary connection?

CHAPTER 5

Metric aspects

5.1. Inner products on a vector space

We review the familiar notion of an inner product. An inner product on a vector spaces allows us to talk of lengths, areas, and similar metric notions which we normally associate with geometry.

5.1.1. Category of inner product spaces. Let V be a finite-dimensional vector space over $\mathbb{R}$. A bilinear form on V is a linear map

$$V \otimes V \to \mathbb{R}, \qquad v \otimes w \mapsto \langle v, w \rangle.$$

An *inner product* on V is a bilinear form on V which is

- symmetric: $\langle v, w \rangle = \langle w, v \rangle$.
- nondegenerate: if $\langle v, w \rangle = 0$ for all w, then $v = 0$.
- positive definite: $\langle v, v \rangle \geq 0$ for all v, and $\langle v, v \rangle = 0$ iff $v = 0$.

A vector space with an inner product is called an inner product space.

Suppose V and W are vector spaces with an inner product. A morphism from V to W is a linear map $f : V \to W$ such that its restriction $(\ker f)^\perp \to W$ preserves inner products. (For notation $U^\perp$, see below.) Thus every morphism factors canonically as a composite of a surjective morphism followed by an injective morphism.

$$\begin{tikzcd}
V \arrow[rr] \arrow[dr, two heads] & & W \\
& (\ker f)^\perp \arrow[ur, hook] &
\end{tikzcd}$$

This defines the category of vector spaces with inner product.

An isomorphism is this category is called an *isometry*. Explicitly, an isometry is a linear isomorphism $f : V \to W$ such that $\langle v, v' \rangle = \langle f(v), f(v') \rangle$.

The zero space has the zero inner product, it is the initial and terminal object in this category.

Similarly, if V and W have inner products, then so does $V \oplus W$. This is the product and coproduct in this category.

5.1.2. Metric notions. Suppose V has an inner product. Then

- V admits an orthonormal basis, that is, a basis $(e_1, \ldots, e_n)$ such that

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

- there is a canonical isomorphism

$$V \xrightarrow{\cong} V^*, \qquad v \mapsto \langle v, - \rangle.$$

- for any subspace U of V, one has $V = U \oplus U^{\perp}$, where
$$U^{\perp} := \{v \mid \langle u, v\rangle = 0 \text{ for all } u \in U\}.$$
It is called the *orthogonal complement* of U.
- one can talk of lengths of vectors, and angles between vectors. The length of a vector $\|v\|$ and the area of the parallelogram $|u \wedge v|$ defined by u and v are given by
$$\|v\| := \langle v, v\rangle^{1/2} \quad \text{and} \quad |u \wedge v| := |\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2|^{1/2}$$
If $u = (a, b)$ and $v = (c, d)$ in an orthonormal basis, then
$$|u \wedge v| = |ad - bc| = |(a^2 + b^2)^2(c^2 + d^2)^2 - (ac + bd)^2|^{1/2}.$$
- V is a metric space, that is, there is a notion of distance between two vectors.
$$d(u, v) := \|u - v\|.$$

Remark 5.1. In geometry, it is also important to consider more general bilinear forms where $\langle v, v\rangle$ can be negative. For instance, this happens in general relativity. A good discussion on such bilinear forms and semi-riemannian manifolds is given in [14, Section 7.6].

5.2. Vector bundles with metric

We discuss vector bundles with metric. These are vector bundles equipped with an inner product on each fiber which varies smoothly.

5.2.1. Category of metric bundles. Let E be a vector bundle over M. A *metric* on E is a choice of an inner product $\langle\ ,\ \rangle_p$ on E_p for each point p of M which varies smoothly with p: $\langle s_1(p), s_2(p)\rangle_p$ is a smooth function on M for any smooth sections s_1 and s_2.

Here are some alternative formulations:

- A metric on E is a morphism $\Gamma(E)\otimes\Gamma(E) \to C(M)$ of $C(M)$-modules, whose induced bilinear form $E_p \otimes E_p \to \mathbb{R}$ is an inner product for each p.
- A metric on E is a smooth bundle map $E \otimes E \to \text{triv}$, whose restriction to each fiber is an inner product.
- A metric on E is a smooth section h of $E^* \otimes E^*$ such that the bilinear form on E_p induced by $h(p)$ is an inner product.

In a local frame field $(s_1, \dots, s_d)$, the metric can be written as
$$h = \sum_{i,j=1}^{d} h_{ij} s_i^* \otimes s_j^*,$$
where h_{ij} are smooth functions. In the inner product notation, this translates to
$$\langle s_i, s_j\rangle = h_{ij}.$$

How do we compare vector bundles with a metric? We first consider the situation when the bundles have the same base: If E and F are two metric vector bundles over M, then a morphism from E to F is a bundle map over M such that at each point p of M, the linear map $E_p \to F_p$ is a morphism of inner product spaces.

This defines the category $\mathsf{mBundle}_M$ of metric vector bundles over M.

Proposition 5.2. *If $f : N \to M$ is a smooth map and $E \to M$ is a vector bundle with a metric, then so is the pullback bundle f^*E. The inner product of the fiber $(f^*E)_p$ is defined to be the inner product of $E_{f(p)}$.*

PROOF. We need to check that the above assignment is smooth. So suppose s_1 and s_2 are two sections of f^*E. By problem (7), these are bundle maps from the trivial bundle on N to E. So $\langle s_i, s_2\rangle$ is the composite

$$N \to \mathrm{triv}_N \otimes \mathrm{triv}_N \xrightarrow{(s_1,s_2)} E \otimes E \to \mathrm{triv}_M \to \mathbb{R}.$$

The notation triv_N denotes the trivial line bundle on N. The first map sends p to $(p, 1 \otimes 1)$. The last map sends (p, x) to x. Since all maps are smooth, so is their composite. □

Now we tackle the general case. Suppose $E \to M$ and $E' \to M'$ are two metric vector bundles. A morphism between them is a bundle map such that the induced map $E \to f^*E'$ of bundles over M is a morphism in $\mathsf{mBundle}_M$. We denote this category by $\mathsf{mBundle}$.

There are another possibility for defining a morphism, namely: A morphism is a strange bundle map such that the induced map $f^*E' \to E$ of bundles over M is a morphism in $\mathsf{mBundle}_M$. We denote this category by $\mathsf{mBundle}^{\mathrm{sop}}$.

An isomorphism is any of these categories is called an *isometry*.

5.2.2. Orthonormal frame fields. Let E be a vector bundle with a metric. A local frame field $(s_1, \ldots, s_d)$ on E is called *orthonormal* if at each point p, $(s_1(p), \ldots, s_d(p))$ is an orthonormal basis of E_p.

Lemma 5.3. *A vector bundle with a metric admits a local orthonormal frame field.*

PROOF. Start with any local frame field $(s_1, \ldots, s_d)$ and apply the Gram-Schmidt orthonormalization procedure in each fiber. All steps in this procedure are smooth, for instance,

$$s_2' = s_2 - \frac{\langle s_1, s_2\rangle}{\langle s_1, s_1\rangle} s_1,$$

so s_2' is a smooth section. Note how the smoothness of the metric gets used here. Hence, the procedure results in a local orthonormal frame field. □

5.2.3. Metric connections. Let E be a vector bundle with a metric. A *metric connection* on E is a connection ∇ on E which satisfies

$$d\langle s, s'\rangle = \langle \nabla(s), s'\rangle + \langle s, \nabla(s')\rangle \tag{5.1}$$

for any sections s and s'. This is an identity of 1-forms. In terms of covariant derivatives, the condition is

$$X\langle s, s'\rangle = \langle \nabla_X(s), s'\rangle + \langle s, \nabla_X(s')\rangle \tag{5.2}$$

for any sections s and s', and vector field X. This is an identity of functions.

Proposition 5.4. *Let ∇ be a connection on a vector bundle E with metric. Then ∇ is metric iff parallel transport by ∇ is an isometry on the fibers E_p.*

PROOF. Suppose ∇ is metric. Let c be any curve, and s (and s') be a section along c which is parallel. Hence $\nabla_{\dot c}(s) = 0$. Condition (5.2) now implies that

$$\frac{d}{dt}\langle s(c(t)), s'(c(t))\rangle = 0.$$

Thus the inner product of two vectors, as they are parallel transported along c, remains constant.

Conversely, suppose parallel transport is an isometry. We verify (5.2) at a point p. Pick any curve c passing through p whose tangent at p is X_p. Let $P_{c,t}$ denote the isometry between the fiber over $p = c(a)$ and the fiber over $c(t)$. Then

$$X_p\langle s, s'\rangle = \frac{d}{dt}\langle s(c(t)), s'(c(t))\rangle = \frac{d}{dt}\langle P_{c,t}^{-1}s(c(t)), P_{c,t}^{-1}s'(c(t))\rangle$$

Now all the action is happening in the fiber over p, with the derivative taken at $t = a$. So we can continue the calculation as follows.

$$\begin{aligned} &= \langle \frac{d}{dt}P_{c,t}^{-1}s(c(t)), P_{c,t}^{-1}s'(c(t))\rangle + \langle P_{c,t}^{-1}s(c(t)), \frac{d}{dt}P_{c,t}^{-1}s'(c(t))\rangle \\ &= \langle \nabla_{X_p}(s), s'\rangle + \langle s, \nabla_{X_p}(s')\rangle. \end{aligned}$$

The same argument in different garb is as follows. Take an orthonormal basis $s_1(p), \ldots, s_d(p)$ for E_p. Let $s_1(t), \ldots, s_d(t)$ be their parallel translates at $c(t)$. They continue to be orthonormal by hypothesis. Let $s = \sum_i f_i s_i$ and $s' = \sum_i f_i' s_i$. Then

$$\begin{aligned} X_p\langle s, s'\rangle = X_p\langle \sum_i f_i s_i, \sum_j f_j' s_j\rangle &= X_p(\sum_{i,j} f_i f_j' \langle s_i, s_j\rangle) \\ &= \sum_{i,j} X_p(f_i f_j') = \sum_{i,j} X_p(f_i)f_j' + f_i X_p(f_j'). \end{aligned}$$

The two terms in the rhs corresponds to the covariant derivatives of s and s':

$$\langle \nabla_{X_p}(s), s'\rangle = \langle \sum_i \nabla_{X_p}(f_i s_i), \sum_j f_j' s_j\rangle = \sum_{i,j} X_p(f_i)f_j'$$

since $\nabla_{X_p}(s_i) = 0$. $\square$

The connection 1-forms of a metric connection in an orthonormal frame field are skew-symmetric.

5.3. Riemannian manifolds

We discuss riemannian manifolds. They have origins in work of Riemann around 1855. The focus now shifts from abstract vector bundles to tangent bundles.

5.3.1. Riemannian manifolds.

Definition 5.5. A *riemannian manifold* is a smooth manifold equipped with a metric on its tangent bundle. A riemannian manifold is denoted (M, g), where M is the smooth manifold and g is the metric.

A morphism of riemannian manifolds is a smooth map $M \to N$ such that the induced map of bundles $TM \to TN$ is a morphism of inner product spaces on each fiber.

This defines the category of riemannian manifolds. An isomorphism of riemannian manifolds is called an *isometry*.

An *isometric immersion* is a morphism of riemannian manifolds in which the smooth map on manifolds is an immersion. An *isometric embedding* is defined similarly.

Lemma 5.6. *An isometric immersion from (M_1, g_1) to (M_2, g_2) is a smooth map $M_1 \to M_2$ such that the pullback of g_2 is g_1.*

Remark 5.7. It is conceivable that one may want to consider more general morphisms. For instance, one take take all morphisms of metric tangent bundles. This would be a full subcategory of $\mathsf{mBundle}$. Another possibility would be the full subcategory of $\mathsf{mBundle}^{\mathrm{sop}}$.

By Lemma 5.3, a riemannian manifold admits a local frame of orthonormal vector fields. However, it is important to note: In general, one cannot choose a local coordinate system $(x_1, \ldots, x_n)$ so that the orthonormal frame field is $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Think of this: The Lie bracket of the coordinate vector fields is always zero, but that will not be the case in general for a local frame of orthonormal vector fields.

The metric is usually denoted g and written in the form

$$g = \sum_{i,j=1}^{d} g_{ij} dx_i \otimes dx_j,$$

where g_{ij} are smooth functions.

In a riemannian manifold M, there are canonical isomorphisms

$$T_pM \xrightarrow{\cong} T_p^*M, \qquad v \mapsto \langle v, - \rangle$$

which then yields a canonical isomorphism between TM and T^*M. This further induces an isomorphism

$$\Gamma(TM) \xrightarrow{\cong} \Gamma(T^*M), \qquad X \mapsto \langle X, - \rangle$$

between the spaces of vector fields and 1-forms.

5.3.2. Levi-Civita connection. Let ∇ be a connection on the tangent bundle TM of a manifold (not necessarily riemannian). Recall from problem (14) in Chapter 3 that the *torsion* of ∇ is defined by

$$T_{X,Y} := \nabla_X Y - \nabla_Y X - [X, Y],$$

and the map

$$\Gamma(TM) \otimes_{C(M)} \Gamma(TM) \to \Gamma(TM), \qquad X \otimes Y \mapsto T_{X,Y}$$

is a map of $C(M)$-modules.

A connection on TM is called *torsion-free* or *symmetric* if its torsion is zero, that is,

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{5.3}$$

Theorem 5.8. *Suppose (M, g) is a riemannian manifold. Then there is a unique metric connection on M whose torsion is zero.*

Proof. Suppose ∇ is a metric connection on (M, g) whose torsion is zero. Let X, Y, Z and W be any vector fields on M. Then, by the metric property (5.2),

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{aligned}$$

Adding the first two and subtracting the third,

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\qquad = \langle \nabla_X Y + \nabla_Y X, Z\rangle + \langle \nabla_Y Z - \nabla_Z Y, X\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle. \end{aligned}$$

Since the torsion is zero, we can continue the calculation as follows.

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = \langle \nabla_X Y, Z\rangle + \langle [Y, X], Z\rangle + \langle [Y, Z], X\rangle + \langle [X, Z], Y\rangle.$$

So ∇ is necessarily unique, since we can write

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ - \langle [Y, X], Z\rangle - \langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle. \end{aligned} \tag{5.4}$$

Conversely, one can check that ∇ defined by the above equation is a connection which is both metric and torsion-free. This is left as an exercise. □

The connection defined by (5.4) is called the *Levi-Civita connection* of (M, g). It is the unique metric and torsion-free connection on (M, g). It first appeared in work of Levi-Civita in 1916.

Example 5.9. Consider the riemannian manifold $\mathbb{R}^n$ with metric

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle = \delta_{ij},$$

the Kronecker delta, or equivalently,

$$g = dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n.$$

In other words, (g_{ij}) is the identity matrix.

Consider the Levi-Civita connection on $(\mathbb{R}^n, g)$. Formula (5.4) shows that the Christoffel symbols are zero wrt the standard frame field $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. So the Levi-Civita connection is the trivial connection. This can also be seen as follows. The trivial connection is clearly torsion-free. Further, usual parallel transport is an isometry along fibers, so by Proposition 5.4, the trivial connection is metric. Theorem 5.8 then forces it to be the Levi-Civita connection.

Example 5.10. Let M be a submanifold of $\mathbb{R}^n$. Then T_pM is a subspace of $\mathbb{R}^n$, so it inherits an inner product from the standard inner product on $\mathbb{R}^n$. This turns M into a riemannian manifold. One can show that the Levi-Civita connection on M coincides with the connection on M induced from the trivial connection on $\mathbb{R}^n$ as mentioned in Example 4.5.

5.4. Riemann curvature

We briefly discuss the Riemann curvature associated to a riemannian manifold. We have already seen that curvature is a 2-dimensional concept in the same way that covariant derivative is a 1-dimensional concept. A equivalent approach is through sectional curvature which is obtained by taking 2-dimensional slices of the smooth manifold.

5.4.1. Riemann curvature. Let (M, g) be a riemannian manifold. The curvature R of the Levi-Civita connection on M is called the *Riemann curvature* of M. It is standard to think of R as a map

$$\begin{aligned} \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) &\longrightarrow C(M), \\ X \otimes Y \otimes Z \otimes W &\longmapsto g\langle R(X, Y)Z, W\rangle. \end{aligned}$$

This is a map of $C(M)$-modules, all tensor products are over $C(M)$.

As a shorthand, it is customary to write

$$(X, Y, Z, W) := g\langle R(X, Y)Z, W\rangle.$$

Here X, Y, Z and W are vector fields, and (X, Y, Z, W) is a function, that is, a scalar field.

Lemma 5.11. *The value of (X, Y, Z, W) at a point p only depends on X_p, Y_p, Z_p and W_p, that is, the notation (X_p, Y_p, Z_p, W_p) is well-defined.*

Proof. Follows from Lemma 3.16, part (2). Alternatively, one may use Lemmas 4.26 and 4.27. $\square$

Lemma 5.12. *The Riemann curvature satisfies*

$$\begin{aligned}(X, Y, Z, W) &= -(Y, X, Z, W)\\ (X, Y, Z, W) &= -(X, Y, W, Z)\\ (X, Y, Z, W) &= (Z, W, X, Y).\end{aligned}$$

Proof. The first identity is a restatement of $R(X, Y) = -R(Y, X)$ given in Lemma 4.25. The remaining two can be established by manipulating using the metric and symmetry of the Levi-Civita connection. Do this as an exercise, or see [8, Proposition 2.5]. $\square$

5.4.2. Sectional curvature.

Lemma 5.13. *Let M be a riemannian manifold. Let σ be a two-dimensional subspace of T_pM. Let (v, w) be a basis of σ. Then*

$$K(v, w) = \frac{\langle R(v, w)w, v\rangle}{|v \wedge w|^2}$$

is independent of the choice of v and w, so the notation $K(\sigma)$ is well-defined.

Proof. By Lemma 5.12, we observe that $K(v, w) = K(w, v)$, $K(v, w) = K(v + cw, w)$, and $K(v, w) = K(cv, w)$ if $c \neq 0$. Since one can go from one basis of σ to another by such operations, the claim follows. $\square$

The real number $K(\sigma)$ is called the *sectional curvature* of σ at p. There must be some nice way to motivate this definition. Perhaps the interpretation of curvature as a two-dimensional limit may be relevant.

Remark 5.14. Do Carmo [8] uses a different convention. He writes (v, w, v, w) instead of (v, w, w, v), but then his definition of $R(X, Y)$ also differs from ours by a sign.

Lemma 5.15. *The sectional curvature determines the Riemann curvature. That is, if we know the sectional curvature of every two-dimensional tangent plane at all points, then the Riemann curvature is uniquely determined.*

Proof. See [8, Lemma 3.3, Chapter 4] or [14, Proposition 13.27]. $\square$

5.5. Curvature of surfaces in euclidean space

By a surface in $\mathbb{R}^3$, we mean a two-dimensional submanifold of $\mathbb{R}^3$. It is a riemannian manifold with metric induced from the standard metric of $\mathbb{R}^3$. We derive an expression for the sectional curvature of a surface. This is simply a function on the surface. It is also called the Gaussian curvature.

5.5.1. Sectional curvature of a surface. Let M be any two-dimensional riemannian manifold, and (x,y) be local coordinates. Let us try to find an expression for the sectional curvature of M in terms of its Christoffel symbols:

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}(\frac{\partial}{\partial y}) &= \nabla_{\frac{\partial}{\partial x}}(\Gamma^1_{22}\frac{\partial}{\partial x}+\Gamma^2_{22}\frac{\partial}{\partial y})\\ &= \frac{\partial\Gamma^1_{22}}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\Gamma^2_{22}}{\partial x}\frac{\partial}{\partial y}+\Gamma^1_{22}\nabla_{\frac{\partial}{\partial x}}(\frac{\partial}{\partial x})+\Gamma^2_{22}\nabla_{\frac{\partial}{\partial x}}(\frac{\partial}{\partial y})\\ &= (\frac{\partial\Gamma^1_{22}}{\partial x}+\Gamma^1_{22}\Gamma^1_{11}+\Gamma^2_{22}\Gamma^1_{12})\frac{\partial}{\partial x}+(\frac{\partial\Gamma^2_{22}}{\partial x}+\Gamma^1_{22}\Gamma^2_{11}+\Gamma^2_{22}\Gamma^2_{12})\frac{\partial}{\partial y}\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}(\frac{\partial}{\partial y}) &= \nabla_{\frac{\partial}{\partial y}}(\Gamma^1_{12}\frac{\partial}{\partial x}+\Gamma^2_{12}\frac{\partial}{\partial y})\\ &= \frac{\partial\Gamma^1_{12}}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\Gamma^2_{12}}{\partial y}\frac{\partial}{\partial y}+\Gamma^1_{12}\nabla_{\frac{\partial}{\partial y}}(\frac{\partial}{\partial x})+\Gamma^2_{12}\nabla_{\frac{\partial}{\partial y}}(\frac{\partial}{\partial y})\\ &= (\frac{\partial\Gamma^1_{12}}{\partial y}+\Gamma^1_{12}\Gamma^1_{21}+\Gamma^2_{12}\Gamma^1_{22})\frac{\partial}{\partial x}+(\frac{\partial\Gamma^2_{12}}{\partial y}+\Gamma^1_{12}\Gamma^2_{21}+\Gamma^2_{12}\Gamma^2_{22})\frac{\partial}{\partial y}\end{aligned}$$

Putting these together,

$$\begin{aligned}&\langle R(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\frac{\partial}{\partial y},\frac{\partial}{\partial x}\rangle =\\ &\quad(\frac{\partial\Gamma^1_{22}}{\partial x}+\Gamma^1_{22}\Gamma^1_{11}+\Gamma^2_{22}\Gamma^1_{12}-\frac{\partial\Gamma^1_{12}}{\partial y}-\Gamma^1_{12}\Gamma^1_{21}-\Gamma^2_{12}\Gamma^1_{22})g_{11}\\ &\qquad+(\frac{\partial\Gamma^2_{22}}{\partial x}+\Gamma^1_{22}\Gamma^2_{11}+\Gamma^2_{22}\Gamma^2_{12}-\frac{\partial\Gamma^2_{12}}{\partial y}-\Gamma^1_{12}\Gamma^2_{21}-\Gamma^2_{12}\Gamma^2_{22})g_{21}\end{aligned} \tag{5.5}$$

The sectional curvature is the above formula divided by $\det(g_{ij})$. The Christoffel symbols can also be expressed in terms of the metric. This will give an expression for K solely in terms of g (which we are not going to write).

5.5.2. Graph of a function. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a smooth function of two variables. Let M be the surface in $\mathbb{R}^3$ defined as the graph of f:

$$M := \{(x,y,f(x,y))\}.$$

The smooth structure on M is given by a single coordinate chart

$$\mathbb{R}^2\to M, \qquad (x,y)\mapsto(x,y,f(x,y)).$$

Clearly, M is a submanifold of $\mathbb{R}^3$. The standard metric on $\mathbb{R}^3$ turns M into a riemannian manifold. Explicitly, in the above chart, the metric is given by

$$g=\begin{pmatrix}1+f_x^2 & f_xf_y\\ f_xf_y & 1+f_y^2\end{pmatrix}.$$

This is calculated by taking the dot products of

$$\frac{\partial}{\partial x}=(1,0,f_x) \quad\text{and}\quad \frac{\partial}{\partial y}=(0,1,f_y)$$

in all possible ways. Check that the inverse of the metric is

$$\frac{1}{1+f_x^2+f_y^2}\begin{pmatrix}1+f_y^2 & -f_xf_y\\ -f_xf_y & 1+f_x^2\end{pmatrix}.$$

Lemma 5.16. *The Christoffel symbols of* (M, g) *are*

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} = \Gamma^1_{21} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} = \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} f_x f_{xx} & f_x f_{xy} & f_x f_{yy} \\ f_y f_{xx} & f_y f_{xy} & f_y f_{yy} \end{pmatrix}$$

PROOF. There is an explicit formula for computing the Christoffel symbols in terms of the metric, see problem (3). This calculation is doable but painful.

Alternatively, we can directly compute the covariant derivatives. Let $N := (-f_x, -f_y, 1)$. This is a vector field orthogonal to the surface M. Note that

$$f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y} + N = (0, 0, 1 + f_x^2 + f_y^2). \tag{5.6}$$

To calculate $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$, we differentiate $(1, 0, f_x)$ wrt x to get $(0, 0, f_{xx})$ and, express it as a linear combination of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and N:

$$(0, 0, f_{xx}) = \Gamma^1_{11} \frac{\partial}{\partial x} + \Gamma^2_{11} \frac{\partial}{\partial y} + bN.$$

The formulas for Γ^1_{11} and Γ^2_{11} immediately follow from (5.6). The remaining formulas are obtained in a similar manner. (The scalar b is related to the second fundamental form of M.) □

Lemma 5.17. *The sectional curvature of* (M, g) *is given by*

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}. \tag{5.7}$$

PROOF. Calculate using Lemma 5.16 and formula (5.16). Also $\det(g_{ij}) = 1 + f_x^2 + f_y^2$.

The calculation is easy to do at points where $f_x = f_y = 0$, that is, the tangent plane is horizontal: Only the terms $\frac{\partial \Gamma^2_{22}}{\partial x}$ and $\frac{\partial \Gamma^1_{12}}{\partial y}$ contribute. The terms involving products of Christoffel symbols are automatically zero, since either f_x or f_y is a factor.

I leave it to you to check the general case. □

Sectional curvature at the point $(0, 0)$:

- of the paraboloid $z = x^2 + y^2$ is 4,
- of the hyperboloid $z = x^2 - y^2$ is -4,
- of the plane $z = 0$ is 0.

5.5.3. Gauss map of a surface. Suppose M is a (oriented) surface in $\mathbb{R}^3$. Let N denote the unit normal to M. This is a vector field of $\mathbb{R}^3$, but only defined on N. This defines a map

$$N : M \to S^2.$$

This is called the *Gauss map*.

The derivative of the Gauss map yields a linear map $T_pM \to T_pS^2$ at each point p. Since these two tangent planes are parallel, we may identify them, so in fact we have a linear map $T_pM \to T_pM$.

Theorem 5.18. *The determinant of the derivative of the Gauss map is precisely the sectional curvature* K *at* p.

Proof. It suffices to consider the case when the surface is the graph of a function f. The normal to the surface is $(-f_x, -f_y, 1)$. The Gauss map is

$$(x, y) \mapsto \frac{1}{(1 + f_x^2 + f_y^2)^{1/2}}(-f_x, -f_y, 1).$$

Differentiate wrt x to get a vector, say u, and differentiate wrt y to get a vector, say v. Then both u and v belong to the plane spanned by $(1, 0, f_x)$ and $(0, 1, f_y)$. The resulting matrix is

$$\frac{1}{(1 + f_x^2 + f_y^2)^{3/2}} \begin{pmatrix} -f_{xx}(1 + f_y^2) + f_x f_y f_{xy} & -f_{xy}(1 + f_x^2) + f_x f_y f_{xx} \\ -f_{xy}(1 + f_y^2) + f_x f_y f_{yy} & -f_{yy}(1 + f_x^2) + f_x f_y f_{xy} \end{pmatrix}$$

The determinant of this matrix is indeed the formula for sectional curvature given in Lemma 5.17: terms involving $f_{xx}f_{xy}$ and $f_{yy}f_{xy}$ cancel out, the coefficient of the terms $f_{xx}f_{yy}$ and $f_{xy}f_{xy}$ is $(1 + f_x^2 + f_y^2)$ up to sign, so it cancels one factor from $(1 + f_x^2 + f_y^2)^3$. □

Note that the approach to curvature of a surface via the Gauss map gives the impression that curvature depends on the embedding of the surface in $\mathbb{R}^3$. However, we know that it only depends on the metric of the surface, meaning that two isometric surfaces in $\mathbb{R}^3$ will have the same sectional curvatures. This is the Theorem Egregium of Gauss.

Problems

(1) Show that a linear map is an isometry iff it takes an orthonormal basis to another orthonormal basis.

(2) Suppose c and c' are curves in $\mathbb{R}^n$. Show that

$$\frac{d}{dt}\langle c(t), c'(t)\rangle = \langle \frac{d}{dt}c(t), c'(t)\rangle + \langle c(t), \frac{d}{dt}c'(t)\rangle,$$

where $\langle -, -\rangle$ is any inner product in $\mathbb{R}^n$. (This result was used in Proposition 5.4.)

(3) Suppose ∇ is a connection on TM. The Christoffel symbols of ∇ wrt local coordinates $(x_1, \ldots, x_n)$ are defined by

$$\sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k} := \nabla_{\frac{\partial}{\partial x_i}}(\frac{\partial}{\partial x_j})$$

Now let (M, g) be a riemannian manifold, and ∇ its Levi-Civita connection. Express the Christoffel symbols Γ_{ij}^k in terms of the g_{ij}.

(4) Show that a connection on TM is symmetric iff the Christoffel symbols satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$.

(5) Put $R_{klij} := \sum_m g_{km} R_{lij}^m$, with notation as in problem (13) in Chapter 3. Show that

$$R_{klij} = \langle R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\rangle.$$

(6) Let M be a surface in $\mathbb{R}^3$ with metric induced from the standard metric of $\mathbb{R}^3$. Let $c : [a.b] \to M$ be a curve on M, and Y a vector field on M along c. Then Y defines a map $[a.b] \to \mathbb{R}^3$. Show that Y is parallel along c iff $\frac{dY}{dt}$ is perpendicular to $T_{c(t)}M$ for all t.

(7) Show that the Riemann curvature satisfies
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$
This is called the Bianchi identity.

(8) Show that the sectional curvature of the sphere at all points is 1.

(9) Give the flat metric to the torus by viewing it as a quotient of $\mathbb{R}^2$. Show that the standard embedding of the torus in $\mathbb{R}^3$ is not an isometric immersion. Can you find an isometric immersion (in $\mathbb{R}^4$)?

What about the cylinder with the flat metric? Is its standard embedding an isometric immersion?

(10) Suppose E is a vector bundle with a connection ∇. Does there exist a metric on E which makes ∇ a metric connection?

(11) Is there some generalization of the Levi-Civita connection to arbitrary metric bundles?

APPENDIX A

Tensor, shuffle, symmetric, exterior algebras

We discuss some important algebras associated to a vector space listed below.

- Tensor algebra
- Shuffle algebra
- Symmetric algebra (Two versions)
- Exterior algebra (Two versions)

A.1. Tensor product of vector spaces

Fix a field $\Bbbk$. The category of vector spaces Vec has vector spaces over $\Bbbk$ as objects and $\Bbbk$-linear maps as morphisms. The zero space is the initial as well as the terminal object. The product and coproduct are both given by direct sum.

Let V and W be vector spaces over $\Bbbk$. An important construction is the *tensor product* $V \otimes W$. Elements of this vector space are formal sums

$$\sum_{i=1}^{n} a_i(v_i \otimes w_i), \qquad a_i \in \Bbbk, v_i \in V, w_i \in W$$

subject to the relations

$$\begin{aligned}(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w,\\ w \otimes (v_1 + v_2) &= w \otimes v_1 + w \otimes v_2,\\ a(v \otimes w) &= av \otimes w = v \otimes aw.\end{aligned}$$

If V has basis $\{e_1, \dots, e_m\}$ and W has basis $\{f_1, \dots, f_n\}$, then $V \otimes W$ has basis $\{e_i \otimes f_j\}_{ij}$.

The tensor product is functorial. If $f : V \to V'$ and $g : W \to W'$ are linear maps, then there is an induced linear map $f \otimes g : V \otimes W \to V' \otimes W'$.

Note that the tensor product is neither the product nor the coproduct in Vec, but it defines a monoidal structure on Vec. (Products and coproducts are examples of monoidal structures on any category.) Monoids wrt the tensor product are called algebras. More explicitly, an algebra is a vector space A equipped with linear maps

$$A \otimes A \to A \quad \text{and} \quad \Bbbk \to A$$

subject to associativity and unitality axioms.

A.2. Tensor algebra and shuffle algebra

Let V be a vector space over the field $\Bbbk$. The *tensor algebra* of V is

$$\mathcal{T}(V) := \bigoplus_{k \geq 0} V^{\otimes k},$$

where $V^{\otimes k}$ denotes the k-fold tensor product of V. The product is concatenation of tensors:

$$(u_1 \otimes \cdots \otimes u_m) \otimes (v_1 \otimes \cdots \otimes v_n) \mapsto u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n.$$

Note that $\mathcal{T}(V)$ is a graded algebra, the grading is given by the number of tensor factors. The product is *not* commutative.

If V is finite-dimensional with basis $(e_1, \dots, e_n)$, then elements of $\mathcal{T}(V)$ are noncommutative polynomials in the e_i, with product being the usual multiplication of polynomials. The standard notation for this is $\Bbbk\langle e_1, \dots, e_n\rangle$.

A related construction is that of the *shuffle algebra* of a vector space:

$$\mathcal{T}^\vee(V) := \bigoplus_{k \geq 0} V^{\otimes k}.$$

It has the same underlying vector space as $\mathcal{T}(V)$, but the product is different. It is the sum over all ways to shuffle the two tensors. For example:

$$(u \otimes v) \otimes w \mapsto u \otimes v \otimes w + u \otimes w \otimes v + w \otimes u \otimes v.$$

Note that $\mathcal{T}^\vee(V)$ is also a graded algebra, the grading is given by the number of tensor factors. Further, the product is commutative.

There is a signed version of this product, which keeps track of the parity of the shuffle. For example:

$$(u \otimes v) \otimes w \mapsto u \otimes v \otimes w - u \otimes w \otimes v + w \otimes u \otimes v.$$

This defines the *signed shuffle algebra*, which we denote by $\mathcal{T}^\vee_{-1}(V)$. The product is now graded commutative.

A.3. Symmetric algebra (Coinvariant and invariant versions)

Consider the *symmetrization*

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The sum is over all permutations on k letters. For example,

$$u \otimes v \mapsto u \otimes v + v \otimes u.$$

Adding over all $k \geq 0$, we obtain a map

$$\kappa : \mathcal{T}(V) \to \mathcal{T}^\vee(V). \tag{A.1}$$

This is called the *norm map*.

Proposition A.1. *The norm map is a morphism of graded algebras, that is, symmetrization relates concatenation to shuffling.*

Proof. Exercise. □

The norm map is far from being an isomorphism; this is of significance. Let $\mathcal{S}^\vee(V)$ denote its image and $\mathcal{S}(V)$ denote its coimage. This yields the following commutative diagram of graded algebras.

$$\begin{tikzcd} \mathcal{T}(V) \arrow[r, "\kappa"] \arrow[d, two heads] & \mathcal{T}^\vee(V) \\ \mathcal{S}(V) \arrow[r, "\cong"'] & \mathcal{S}^\vee(V) \arrow[u, hook] \end{tikzcd} \tag{A.2}$$

(This is a general principle; starting with any morphism of algebras one obtains a commutative diagram as above by looking at the image and coimage of the morphism.)

The algebras $\mathcal{S}(V)$ and $\mathcal{S}^\vee(V)$ are isomorphic. Either one is called the *symmetric algebra* on V. Elements of $\mathcal{S}(V)$ are tensors such as $u \otimes v$ subject to the relation

$$u \otimes v = v \otimes u$$

while elements of $\mathcal{S}^\vee(V)$ are symmetric tensors such as

$$u \otimes v + v \otimes u.$$

If V is finite-dimensional with basis $(e_1, \dots, e_n)$, then elements of $\mathcal{S}(V)$ are commutative polynomials in the e_i, with product being the usual multiplication of polynomials. The standard notation for this is $\Bbbk[e_1, \dots, e_n]$, and is commonly referred to as the polynomial algebra. The quotient map $\mathcal{T}(V) \twoheadrightarrow \mathcal{S}(V)$ is

$$\Bbbk\langle e_1, \dots, e_n\rangle \twoheadrightarrow \Bbbk[e_1, \dots, e_n], \qquad e_i \mapsto e_i.$$

There is another way of describing the symmetric algebras using invariants and coinvariants: Consider the left action of the symmetric group S_k on $V^{\otimes k}$ where the element $\sigma \in \mathrm{S}_k$ acts by

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The symmetric algebra of V is

$$\mathcal{S}(V) = \bigoplus_{k \geq 0} (V^{\otimes k})_{\mathrm{S}_k} \quad \text{and} \quad \mathcal{S}^\vee(V) = \bigoplus_{k \geq 0} (V^{\otimes k})^{\mathrm{S}_k},$$

where $(-)_{\mathrm{S}_k}$ denotes the space of S_k-coinvariants, and $(-)^{\mathrm{S}_k}$ denotes the space of S_k-invariants.

A.4. Exterior algebra (Coinvariant and invariant versions)

Consider the *antisymmetrization*

$$V^{\otimes k} \to V^{\otimes k}, \qquad v_1 \otimes \cdots \otimes v_k \mapsto \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)},$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ. For example,

$$u \otimes v \mapsto u \otimes v - v \otimes u.$$

Adding over all $k \geq 0$, we obtain a morphism of algebras

$$\kappa_{-1} : \mathcal{T}_{-1}(V) \to \mathcal{T}^\vee_{-1}(V). \tag{A.3}$$

This is the *signed norm map*. Here $\mathcal{T}_{-1}(V)$ is the tensor algebra $\mathcal{T}(V)$; we use the subscript -1 to ensure uniformity of notation. For a deeper reason, see [1, Section 2.6].

The signed norm map is also far from being an isomorphism. Let $\Lambda^\vee(V)$ denote its image and $\Lambda(V)$ denote its coimage. This yields the following commutative diagram of graded algebras.

$$\begin{tikzcd} \mathcal{T}_{-1}(V) \arrow[r, "\kappa_{-1}"] \arrow[d, two heads] & \mathcal{T}^\vee_{-1}(V) \\ \Lambda(V) \arrow[r, "\cong"'] & \Lambda^\vee(V) \arrow[u, hook] \end{tikzcd} \tag{A.4}$$

The algebras $\Lambda(V)$ and $\Lambda^\vee(V)$ are isomorphic. Either one is called the *exterior algebra* on V. The term *Grassmann algebra* is also used. Elements of $\Lambda(V)$ are wedges such as $u \wedge v$ subject to the relation

$$u \wedge v = -v \wedge u.$$

(It is standard to use wedge instead of tensor to distinguish this quotient from $\mathcal{S}(V)$.) Elements of $\Lambda^\vee(V)$ are antisymmetric tensors such as

$$u \otimes v - v \otimes u.$$

The isomorphism between the two exterior algebras antisymmetrizes the wedge.

$$\Lambda(V) \to \Lambda^\vee(V), \qquad u \wedge v \mapsto u \otimes v - v \otimes u.$$

If V is finite-dimensional with basis $(e_1, \dots, e_n)$, then $\Lambda(V)$ has a basis given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n}.$$

Thus $\Lambda(V)$ is a finite dimensional graded algebra:

$$\Lambda(V) = \bigoplus_{k=1}^{n} \Lambda^k(V).$$

The k-th component is called the k-th exterior power of V, its dimension is $\binom{n}{k}$.

The exterior algebras can also be described using invariants and coinvariants by making use of the signed action of the symmetric groups.

For any finite-dimensional vector space W, there is a canonical linear isomorphism

$$W \to \mathrm{Hom}(W^*, \Bbbk) = (W^*)^*, \quad w \mapsto (f \mapsto f(w)).$$

Now take $W = (\Lambda^\vee)^k(V)$. First note that $(\Lambda^\vee)^k(V)^* \cong \Lambda^k(V^*)$. Further, a linear map from $\Lambda^k(V^*)$ to $\Bbbk$ is the same as a linear map

$$V^* \otimes \cdots \otimes V^* \to \Bbbk$$

(with k tensor factors) which is alternating. Alternating means that the value of the map changes sign if the two adjacent tensor factors are switched. By precomposing by the isomorphism from $\Lambda^k(V)$ to $(\Lambda^\vee)^k(V)$:

Lemma A.2. *For a finite-dimensional vector space V, there is a canonical linear isomorphism between $\Lambda^k(V)$ and alternating linear maps from the k-fold tensor product of V^* to $\Bbbk$.*

Explicitly,

$$(v_1 \wedge \cdots \wedge v_k)(f_1 \otimes \cdots \otimes f_k) = \det(f_j(v_i)).$$

(We need to antisymmetrize the wedge and then evaluate.)

Warning. Many books use a normalization factor of $\frac{1}{k!}$ in front of the determinant.

Remark A.3. Let q be any scalar. One can define an algebra $\mathcal{T}_q^\vee(V)$, whose product is given by q-shuffling: The coefficient of the shuffle is q power the number of interchanges of the shuffle. This is the q-shuffle algebra. For $q = 1$, we recover the shuffle algebra, and for $q = -1$, we recover the signed shuffle algebra.

Next, one can then define the q-norm map $\kappa_q : \mathcal{T}(V) \to \mathcal{T}_q^\vee(V)$ by using q-symmetrization. This map is an isomorphism if q not a root of unity. Observe that, for $q = 0$, the map is in fact the identity.

Bibliography

[1] Marcelo Aguiar and Swapneel Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010. 52, 93

[2] Mark Anthony Armstrong, *Basic topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1983, Corrected reprint of the 1979 original. v, 4

[3] Marcel Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, 2003. v

[4] William M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, second ed., Pure and Applied Mathematics, vol. 120, Academic Press Inc., Orlando, FL, 1986. v, 6, 12, 14

[5] Theodor Bröcker and Klaus Jänich, *Introduction to differential topology*, Cambridge University Press, Cambridge, 1982, Translated from the German by C. B. Thomas and M. J. Thomas. v

[6] Isaac Chavel, *Riemannian geometry*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction. v

[7] Manfredo Perdigão do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976, Translated from the Portuguese. v

[8] ______, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. v, 9, 85

[9] ______, *Differential forms and applications*, Universitext, Springer-Verlag, Berlin, 1994, Translated from the 1971 Portuguese original. v

[10] Victor Guillemin and Alan Pollack, *Differential topology*, AMS Chelsea Publishing, Providence, RI, 2010, Reprint of the 1974 original. 14

[11] Dale Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994. 42, 43

[12] Klaus Jänich, *Topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1984, With a chapter by Theodor Bröcker, Translated from the German by Silvio Levy. v

[13] Jürgen Jost, *Riemannian geometry and geometric analysis*, sixth ed., Universitext, Springer, Heidelberg, 2011. v

[14] Jeffrey M. Lee, *Manifolds and differential geometry*, Graduate Studies in Mathematics, vol. 107, American Mathematical Society, Providence, RI, 2009. v, 43, 50, 52, 75, 80, 85

[15] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. v

[16] John W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original. 14

[17] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76. 18, 49

[18] Shigeyuki Morita, *Geometry of differential forms*, Translations of Mathematical Monographs, vol. 201, American Mathematical Society, Providence, RI, 2001, Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics. v, 14

[19] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. 3

[20] ______, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000, Second edition of [MR0464128]. v, 1, 5, 6

[21] Jet Nestruev, *Smooth manifolds and observables*, Graduate Texts in Mathematics, vol. 220, Springer-Verlag, New York, 2003, Joint work of A. M. Astashov, A. B. Bocharov, S. V.

Duzhin, A. B. Sossinsky, A. M. Vinogradov and M. M. Vinogradov, Translated from the 2000 Russian edition by Sossinsky, I. S. Krasil′schik and Duzhin. v, 18, 46, 52

[22] Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. v

[23] Andrew Pressley, *Elementary differential geometry*, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2010. v

[24] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I-V*, second ed., Publish or Perish Inc., Wilmington, Del., 1979. v

[25] John A. Thorpe, *Elementary topics in differential geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1994, Corrected reprint of the 1979 original. v

[26] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy. v

[27] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition. v

[28] Jeffrey R. Weeks, *The shape of space*, second ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 249, Marcel Dekker Inc., New York, 2002. 4