MA 556 PRESENTATION

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Sectional Curvature

Definition of Sectional Curvature

Let M be a Riemann manifold. Let σ be a two dimensional subspace of T_pM Let (v, w) be a basis of σ. Then the scalar curvature is defined as

$$K(v, w) = \frac{\langle R(v, w)w, v) \rangle}{|v \wedge w|^2} \tag{1}$$

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Note that the above definition of the sectional curvature is independent of the choice of basis (v, w) and is thus a property of the plane σ alone. This will be proven in the slides to come.

Intuitive Version of Sectional Curvature

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Figure for Intuitive Version of Sectional Curvature

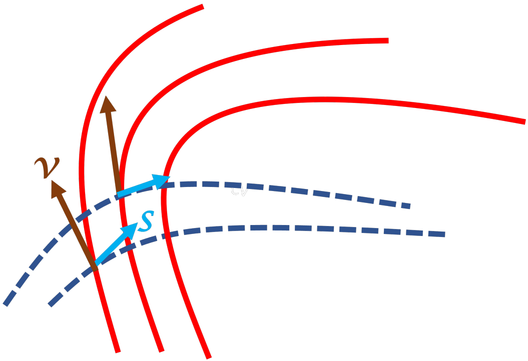

Figure: Geodesics on some arbitrary surface. Note that v indicates the tangent vector to the geodesic s is the orthogonal vector that measures the distance between two geodesics.

Getting into the Math behind Curvature

We will start with some basic assumptions. Let (M, g) be our Riemann Manifold.

- We choose ∇ to be the connection whose torsion is 0.
- We will let the lie brackets of the vector fields V and S.

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Recall that in multi-variable calculus, we often used the second derivative to check curvature. This works because the second derivative measures the change of change of function. Change of function can occur even without curvature since the change can be uniform (linear). We will apply an analogous concept here by replacing the double derivative with double covariant derivative of the vector field V.

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$$\nabla_{\vec{v}}\vec{v} = 0 \tag{2}$$

Math behind Curvature

Continuing,

$$\nabla_{\vec{v}}\vec{v} = 0 \tag{3}$$

Taking covariant derivative with respect to $\vec{s}$ on both sides,

$$\nabla_{\vec{s}}\nabla_{\vec{v}}\vec{v} = 0 \tag{4}$$

$$\Rightarrow \nabla_{\vec{s}}\nabla_{\vec{v}}\vec{v} - \nabla_{\vec{v}}\nabla_{\vec{s}}\vec{v} + \nabla_{\vec{v}}\nabla_{\vec{s}}\vec{v} = 0 \tag{5}$$

Using the definition of Riemann Curvature, the above equation reduces to

$$R(\vec{s},\vec{v})\vec{v} + \nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} = 0 \tag{6}$$

More Figures

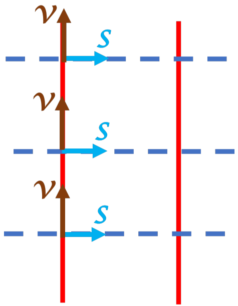

Figure: Case I: We have straight geodesics

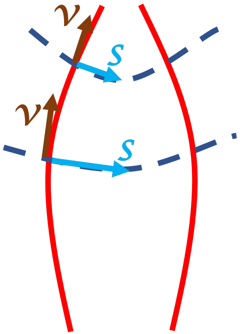

Figure: Case II: We have converging geodesics (positive curvature)

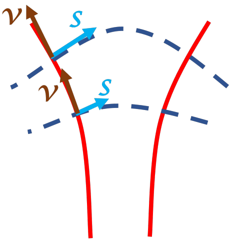

Figure: Case III: We have diverging geodesics (negative curvature)

Figures Explained

Note that $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s}$ indicates the rate of change of how fast the $\vec{s}$ vector changes with respect to travelling on the geodesic. First we consider the sign. In case I, $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} = 0$ (since $\vec{s}$ is uniform throughout the geodesic) . In case II, we have $\vec{s}$ decreasing in magnitude and this decrease decreases continuously. Thus, $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} = 0$ is opposite to $\vec{s}$. This yields $\langle \nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s}, \vec{s} \rangle < 0$. Using equation 6, we get $\langle R(\vec{s}, \vec{v})\vec{v}, \vec{s} \rangle > 0$. Likewise, for case III $\langle R(\vec{s}, \vec{v})\vec{v}, \vec{s} \rangle < 0$. This is the intuition behind curvature.

Is This It?

We have intuitively concluded the form of curvature as $K(\vec{s}, \vec{v}) = \langle R(\vec{s}, \vec{v})\vec{v}, \vec{s}\rangle$, which holds true upto a constant. Note that curvature must be independent of the choice of basis. This can be done by suitable normalization.

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Consider a new set of basis $(\tilde{v}, \tilde{w}) = (av + bw, cv + dw)$ for $a, b, c, d \in \mathbb{R}$. The curvature should not change in this new basis. We will use the fact that

$$||\tilde{v} \wedge \tilde{w}|| = ||av + bw \wedge cv + dw|| \tag{7}$$

$$= |ad - bc|\, ||v \wedge w|| \tag{8}$$

Normalization: Part I

Now,

$$
\begin{aligned}
R(\tilde{v}, \tilde{w}) &= R(av + bw, cv + dw) && (9)\\
&= R(av, cv) + R(av, dw) + R(bw, cv) + R(bw, dw) && (10)\\
&= acR(v, v) + adR(v, w) + bcR(w, v) + bdR(w, w) && (11)\\
&= (ad - bc)R(v, w) && (12)
\end{aligned}
$$

where equations 10 and 11 use the multi linearity of tensors and equation 12 uses the fact that $R(v, w) = -R(w, v)$.

Normalization: Part II

Using $K(v,w) = \langle R(v,w)w, v\rangle$ from the previous slides,

$$\begin{aligned} K(\tilde{v},\tilde{w}) &= \langle R(\tilde{v},\tilde{w})\tilde{w},\tilde{v}\rangle && (13)\\ &= (ad-bc)\langle R(v,w)\tilde{w},\tilde{v}\rangle && (14)\\ &= (ad-bc)\langle R(v,w)av+bw, cv+dw\rangle && (15)\\ &= (ad-bc)^2\langle R(v,w)v, w\rangle && (16)\end{aligned}$$

where equation 14 follows from equation 12 and equation 16 follows from the axioms of inner product of vectors.

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\end{aligned}$$

where equation 14 follows from equation 12 and equation 16 follows from the axioms of inner product of vectors.
Equation 16 shows that we need to scale down the curvature by $(ad-bc)^2$ in order to normalize it.

Final Result

Thus, we can define curvature as

$$K(v, w) = \frac{\langle R(v, w)w, v\rangle}{||v \wedge w||^2} \tag{17}$$

We can show that the above quantity is indeed a tensor and invariant under a change of basis.

Problem 8 of Notes Chapter 5

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Show that the sectional curvature of the sphere at all points is 1.

Problem 8: Motivation

We will derive the sectional curvature of a spherical surface of radius R. Note that while the calculations are straight-forward using the formulae for Christoffel symbols and Riemann tensor, the calculations are rather cumbersome and non-trivial. We aim to provide a detailed calculation of the various parameters, most of which may be skipped over due to want of time. The slides are however there to stay and will contain the detailed derivations.

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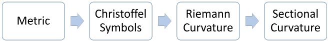

Figure: Flowchart of our calculations

Problem 8: Setting Up

Some assumptions that we will make:

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- We will use the formula for Christoffel symbols from metric.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) \tag{18}$$

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  where α is a dummy summation index.
- We will use the formula for Riemann curvature from Christoffel symbols.

$$R_{\lambda\mu\nu\sigma} = g_{\lambda\alpha}(\partial_\sigma \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\sigma} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\sigma}) \tag{19}$$

  where α and γ are dummy summation indices.

Problem 8: Specific Case of Sphere

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The distance metric is

$$g = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \tag{20}$$

i.e., $g_{\theta\theta} = R^2, g_{\theta\phi} = g_{\phi\theta} = 0, g_{\phi\phi} = R^2 \sin^2\theta$.

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Calculating the inverse

$$g^{-1} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2\theta} \end{pmatrix} \tag{21}$$

i.e., $g^{\theta\theta} = \frac{1}{R^2}, g^{\theta\phi} = g^{\phi\theta} = 0, g^{\phi\phi} = \frac{1}{R^2 \sin^2\theta}$.

Problem 8: Computing the Christoffel Symbols

We will use equation 18 to calculate the Christoffel symbols. We will also use the fact that the cross terms in the metric are zero.

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}) = 0 \tag{22}$$

$$\Gamma^{\theta}_{\theta\phi} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\theta\phi,\theta}) = 0 \tag{23}$$

$$\Gamma^{\theta}_{\phi\theta} = \frac{1}{2}g^{\theta\theta}(g_{\theta\phi,\theta} + g_{\theta\theta,\phi} - g_{\phi\theta,\theta}) = 0 \tag{24}$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}g^{\theta\theta}(g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}) = -\frac{1}{2}g^{\theta\theta}g_{\phi\phi,\theta} = -\sin\theta\cos\theta \tag{25}$$

Problem 8: Continuing the Computation of the Christoffel Symbols

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}g^{\phi\phi}(g_{\phi\theta,\theta} + g_{\theta\phi,\theta} - g_{\theta\theta,\phi}) = 0 \tag{26}$$

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$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi}) = 0 \tag{29}$$

Problem 8: Riemann Curvature

We shall calculate only $R_{\theta\phi\theta\phi}$ for calculating sectional curvature. It is straight-forwards but tedious to show that only the permutations of this tensor are non zero. We will use equation 19.

$$R_{\theta\phi\theta\phi} = g_{\theta\theta}(\partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\theta\phi}\Gamma^{\phi}_{\phi\phi} - \Gamma^{\theta}_{\phi\theta}\Gamma^{\theta}_{\phi\theta} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}) \tag{30}$$

$$= R^2(\sin^2\theta - \cos^2\theta - 0 + 0 + 0 - 0 - (-\sin\theta\cos\theta)(\frac{\cos\theta}{\sin\theta})) \tag{31}$$

$$= R^2 \sin^2\theta \tag{32}$$

Problem 8: Sectional Curvature

So far we have

$$R_{\theta\phi\theta\phi} = R^2 \sin^2\theta \tag{33}$$

Also, we have the formula for sectional curvature

$$K(v,w) = \frac{\langle R(v,w)w, v)\rangle}{|v \wedge w|^2} = \frac{R_{vwvw}}{\det(g)} \tag{34}$$

Hence, we get the sectional curvature of the sphere to be

$$K(\theta,\phi) = \frac{R_{\theta\phi\theta\phi}}{\det(g)} = \frac{R^2\sin^2\theta}{R^4\sin^2\theta} = \frac{1}{R^2} \tag{35}$$

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Thus, for a unit sphere, curvature is constant $(= 1)$ at all points, irrespective of the values of θ, ϕ.

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Bonus

What surface has constant negative curvature (say, -1)?

Riemann Curvature from Sectional Curvature

Riemann Curvature from Sectional Curvature

We can deduce the Riemann curvature at a point in the manifold given the sectional curvature of the surface.

Motivation

We will define some basic assumptions that we will use in out proof.

- Define

$$R(x,y,z,w) = \langle R(x,y)z, w \rangle \tag{36}$$

- Skew symmetry

$$R(x,y,z,w) = -R(y,x,z,w) \tag{37}$$

$$R(x,y,z,w) = -R(x,y,w,z) \tag{38}$$

- Symmetry

$$R(x,y,z,w) = R(z,w,x,y) \tag{39}$$

- Bianchi Identity

$$\sum_{\pi(x,y,z)} R(x,y,z,w) = 0 \tag{40}$$

Setting Up

Now, define a polynomial

$$f(t) = R(x + tw, y + tz, y + tz, x + tw) \tag{42}$$
$$- t^2(R(x, z, z, x) + R(w, y, y, w)) \tag{43}$$

Since all terms are symmetric, $f(t)$ can be expressed in terms of the sectional curvature and norm of wedge products. Thus, the RHS of equation 42 is a polynomial in t whose coefficients are the sectional curvatures.

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Consider the coefficient of t^2 in equation 42. Using the multi linearity property of R tensor, we can write it as

$$f''(0) = R(x,y,z,w) + R(x,z,y,w) + R(w,z,y,x) + R(w,y,z,x) \tag{44}$$

Some Simple Manipulations

Using equations 37, 38, 39, we obtain

$$f''(0) = 2R(x, y, z, w) - 2R(z, x, y, w) \tag{45}$$

Now, exchange x and y and define a new polynomial

$$g(t) = R(y + tw, x + tz, x + tz, y + tw) \tag{46}$$
$$- t^2(R(y, z, z, y) + R(w, x, x, w)) \tag{47}$$

Since all terms are symmetric, $g(t)$ can again be expressed in terms of the sectional curvature and norm of wedge products. Following a similar procedure, we obtain

$$g''(0) = -2R(x, y, z, w) + 2R(y, z, x, w) \tag{48}$$

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Now, 45-48 gives

$$f''(0) - g''(0) = 4R(x,y,z,w) - 2R(z,x,y,w) - 2R(y,z,x,w) \tag{49}$$

Completion of Proof

Using 40,

$$R(z,x,y,w)+R(y,z,x,w)=-R(x,y,z,w) \tag{50}$$

Substituting 50 in 49,

$$f''(0)-g''(0)=6R(x,y,z,w) \tag{51}$$

$$\Rightarrow R(x,y,z,w)=\frac{f''(0)-g''(0)}{6}=\frac{(f-g)''(0)}{6} \tag{52}$$

Final Result

We need the polynomials f and g in terms of sectional curvature. Observe the form of f in 42. Using the formula for sectional curvature,

$$\begin{aligned} f(t) =& K(x+tw, y+tz)\,||(x+tw)\wedge(y+tz)||^2 && (53)\\ &- t^2(K(x,z)\,||x\wedge z||^2 + K(w,y)\,||w\wedge y||^2) && (54)\end{aligned}$$

Likewise

$$\begin{aligned} g(t) =& K(y+tw, x+tz)\,||(y+tw)\wedge(x+tz)||^2 && (55)\\ &- t^2(K(y,z)\,||y\wedge z||^2 + K(w,x)\,||w\wedge x||^2) && (56)\end{aligned}$$

Halloween Special: A Scary Expression

Complete evaluation of equations 51, 53, and 55 gives an explicit formula for $R(x, y, z, w)$.

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$$\begin{aligned}
6\langle R(x,y)z,w\rangle =&K(x+w,y+z)\,||(x+w)\wedge(y+z)||^2\\
&-K(y+w,x+z)\,||(y+w)\wedge(x+z)||^2\\
&-K(x,y+z)\,||x\wedge(y+z)||^2-K(y,x+w)\,||y\wedge(x+w)||^2\\
&-K(z,x+w)\,||z\wedge(x+w)||^2-K(w,y+z)\,||w\wedge(y+z)||^2\\
&+K(x,y+w)\,||x\wedge(y+w)||^2+K(y,z+w)\,||y\wedge(z+w)||^2\\
&+K(z,y+w)\,||z\wedge(y+w)||^2+K(w,x+z)\,||w\wedge(x+z)||^2\\
&+K(x,z)\,||x\wedge z||^2+K(y,w)\,||y\wedge w||^2\\
&-K(x,w)\,||x\wedge w||^2-K(y,z)\,||y\wedge z||^2
\end{aligned}$$

Quite scary, isn't it?

THANK YOU!

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