## 'Complementarity' in paraxial and non-paraxial optical beams

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Compiled April 30, 2021

Using the correspondence of a two dimensional paraxial and three dimensional non-paraxial optical beams with the qubit and qutrit systems respectively, we derive a complementarity relation between Hilbert-Schmidt coherence, generalized predictability and linear entropy. The linear entropy, a measure of mixedness is shown to saturate the complimentarity relation for mixed bi-partite states, which for the pure two qubit and qutrit systems quantifies the global entanglement and reduces the complimentarity relation to the triality relation between coherence, predictability and entanglement.

Entanglement has played a significant role in advancing our understanding of multiparty quantum systems [1, 2]. This non-local correlation is a quantum mechanical property, which is absent in classical systems. However, the analogous mathematical structure of vector spaces in classical optics [3, 4] enables the realization of local entanglement [5] in a single system arising out of the inseparability between its various degrees of freedom. This type of intra-system entanglement has been referred to as classical or non-quantum entanglement [6–8]. This manifests in optical beams, when the polarization state and spatial degrees of freedom are inseparably entangled.

Traditionally, the complimentarity has been explored in the context of wave-particle duality of quantum objects [9–11]. Recent investigations have revealed the complimentarity relations in optical system having different non-separable degrees of freedom [12–14]. One such complimentarity exists between the degree of polarization and the entanglement measure, concurrence [15]. The degree of polarization is related to the coherence and predictability via Polarization Coherence Theorem [16]. This leads to the triality relation [17–19],  $P^2 + C^2 + \mathcal{E}^2 = 1$ , connecting coherence (C), predictability (P) and entanglement ( $\mathcal{E}$ ). These relations hold for paraxial electromagnetic field having planar wavefront, that are described by a 2 × 2 polarization coherence matrix.

In this letter, we investigate the complementarity for 3-D non paraxial electromagnetic field, where the wavefronts are not necessarily planar, and described by a  $3\times 3$  polarization-coherence matrix. The extension of polarization-coherence matrix for this case requires a generalization of Stokes parameter,

which from being four parameters in the two dimensional case will now have nine parameters [20-22]. Also, the expansion of the general state in terms of the Hermitian matrix requires the Gell-Mann matrices [23] and the description of the nonseparability of different degrees of freedom of optical beams requires the extension of the concurrence to multipartite qudit systems, namely I-concurrence [24, 25]. We use the correspondance of a paraxial and non-paraxial electromagnetic field having two independent degrees of freedom with the qubit and qutrit systems respectively. We then derive a complimentarity relation involving Hilbert-Schmidt coherence, generalized predictability and a measure of mixedness, linear entropy which saturates for mixed states. For pure bi-partite systems, the linear entropy takes the form of entanglement measure, concurrence and the complimentarity relation reduces to the usual triality relation [17].

Electric field of a non-uniformly polarized beam of light with planar wavefront in the x-y plane, can be described in general by,

$$\mathbf{E}(\mathbf{r}) = a\mathbf{e}_{x}\psi(\mathbf{r}) + b\mathbf{e}_{x}\phi(\mathbf{r}) + c\mathbf{e}_{y}\psi(\mathbf{r}) + d\mathbf{e}_{y}\phi(\mathbf{r}), \tag{1}$$

where, a,b,c,d are complex coefficients and,  $\psi(\mathbf{r})$  and  $\phi(\mathbf{r})$  are orthonormal functions. Depending on the coefficients, the polarization and spatial degrees of freedom (DOF) can be separable or entangled. To meaningfully characterize the amount of inseparability, one can find a correspondence of this form of non-uniformly polarized beam with four dimensional Hilbert space,  $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$ , where,  $\mathcal{H}_p = \mathrm{Span}\{\mathbf{e}_x, \mathbf{e}_y\}$  and  $\mathcal{H}_s = \mathrm{Span}\{\psi(\mathbf{r}), \phi(\mathbf{r})\}$ , and following the prescription,  $|0\rangle_p = \mathbf{e}_x$ ,  $|1\rangle_p = \mathbf{e}_y$ ,  $|0\rangle_s = \psi(\mathbf{r})$ ,  $|1\rangle_s = \phi(\mathbf{r})$ , the electric field in (1) can be identified as [8],

$$|\mathbf{E}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$
 (2)

where,  $|ij\rangle = |i\rangle_p \otimes |j\rangle_s$ , and  $|\mathbf{E}\rangle$  is a vector in the Hilbert space  $\mathcal{H}$ . With this identification, one can use the standard techniques from quantum information to analyse the entanglement. For Instance, from configuration of the vectors,  $\langle 0_p | \mathbf{E} \rangle = a | 0 \rangle_s + b | 1 \rangle_s$  and  $\langle 1_p | \mathbf{E} \rangle = c | 0 \rangle_s + d | 1 \rangle_s$  one can find whether the polarization and spatial degree of freedom are separable or maximally entangled. The separability of polarization and spatial DOF corresponds to the  $\langle 0_p | \mathbf{E} \rangle$  and  $\langle 1_p | \mathbf{E} \rangle$  being parallel in  $\mathcal{H}_s$  [25], whereas the orthogonality and equality of these vectors correspond to the maximal entanglement. The reduced density matrix corresponding to the polarization degree of freedom

is known as polarization-coherence matrix, which for the pure state  $|\mathbf{E}\rangle$  is obtained as,

$$\Phi = \begin{bmatrix} \langle 0_p | \mathbf{E} \rangle^{\dagger} \langle 0_p | \mathbf{E} \rangle & \langle 1_p | \mathbf{E} \rangle^{\dagger} \langle 0_p | \mathbf{E} \rangle \\ \langle 0_p | \mathbf{E} \rangle^{\dagger} \langle 1_p | \mathbf{E} \rangle & \langle 1_p | \mathbf{E} \rangle^{\dagger} \langle 1_p | \mathbf{E} \rangle \end{bmatrix}. \tag{3}$$

It should be noted that  $\Phi$  is a Hermitian matrix and,  $\text{Tr}(\Phi)$  corresponds to the intensity  $\langle E|E\rangle$  of the beam. Moreover, since the above is obtained for a pure state,  $\det(\Phi)=0$ . In general, polarization-coherence matrix for a beam with planar wavefront is a  $2\times 2$  Hermitian, positive semi-definite matrix,

$$\Phi = \frac{1}{2}(I + \sum_{i=1}^{3} S_{i}\sigma_{i}), \tag{4}$$

where,  $S_i = \langle \sigma_i \rangle = \operatorname{Tr}(\Phi \sigma_i)$  are the Stokes parameters and  $\{\sigma_1, \sigma_2, \sigma_3\}$  are the Pauli matrices. As a consequence of positive semi-definiteness of  $\Phi$ , one obtains  $S_1^2 + S_2^2 + S_3^2 \leq 1$ . Therefore, set of all polarization-coherence matrix of a beam with planar wavefront can be represented by a solid ball of unit radius (Poincare sphere) with the center representing the unpolarized beam (maximally mixed state) and the points on the surface representing the polarized beams (pure states).

We now define the fundamental quantities that characterize the properties of an optical beam.

The generalized predictability is defined as,

$$\mathcal{P}^2 = \frac{2(n-1)}{n} \left[ \sum_{i=1}^n (\Phi_{ii})^2 - \frac{2}{n-1} \sum_{i < j} \Phi_{ii} \Phi_{jj} \right]$$
 (5)

where, n is the dimension of the polarization-coherence matrix  $\Phi$ . It is apparent that the predictability depends only on the diagonal terms of the matrix, i.e., on the fraction of energy across the orthogonal components of the field. The above definition of predictability is motivated by the following reasons. Firstly, it satisfies the boundary conditions i.e., zero when fraction of energy across the orthogonal component of the field are equal and maximum when only one of the diagonal elements is non-zero. Secondly, with this predictability, the entanglement measure that appears in the complementary relations is the well established measure concurrence [15], and its extension I-concurrence [24, 25], for 2-D and 3-D pure bipartite fields respectively. Furthermore, this definition along with the Hilbert-Schmidt coherence extends the well known polarization-coherence theorem [16] for the 3-D case, which leads to the important result that, degree of polarization is given by predictability only when  $\Phi$  is diagonal [26].

For paraxial beams given by (4), the predictability obtained is the difference in intensity corresponding to  $|0\rangle_p$  and  $|1\rangle_p$ ,

$$\mathcal{P}^2 = (\Phi_{11} - \Phi_{22})^2 = S_3^2, \tag{6}$$

equivalently, it is the square of expectation value of the observable  $\sigma_z$ . Hilbert-Schmidt coherence is defined in terms of the off-diagonal terms of the density matrix in the basis  $\{|i\rangle_p\}$ ,

$$C^2 = 2\sum_{i \neq i} |\Phi_{ij}|^2 \tag{7}$$

where i, j = 1, 2. In terms of the observable, it is obtained from the expectation values of  $\sigma_x$  and  $\sigma_y$  as,

$$C^2 = \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2.$$
 (8)

For the pure states, predictability and coherence obey the well known duality relation [27],

$$\mathcal{P}^2 + \mathcal{C}^2 = \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2$$
  
=  $S_1^2 + S_2^2 + S_3^2 = 1$ , (9)

which reflects the complementary character of these properties. When  $\mathcal{P}$  is maximum, i.e., when the beam is in the state  $|0\rangle_p$  or  $|1\rangle_p$ , the coherence is zero, whereas when the predictability is zero i.e., when  $|a|^2 + |b|^2 = |c|^2 + |d|^2$  for the beam (1), the coherence obtained is maximum.

When one deals with a bipartite system, with each party representing a two level system, state of each party is described by the reduced density matrix thus obtained after tracing out the other. In general, the reduced density matrix obtained is not pure, i.e., cannot be represented by a particular state vector, rather an ensemble. In classical optics, the reduced density matrix corresponding to the polarization DOF represents the polarization-coherence matrix,  $P^2 = \sum_i S_i^2$  is defined to be degree of polarization of the beam. This realization trivially leads to the polarization-coherence theorem,  $P^2 = \mathcal{P}^2 + \mathcal{C}^2$  [16]. Since the polarization coherence matrix is mixed in general, the duality relation becomes,  $\mathcal{P}^2 + \mathcal{C}^2 \leq 1$ .

Pure bipartite systems are described by a state vector of the form (2). For fields given by (2), if the polarization and spatial DOF is separable, the resulting polarization-coherence matrix represents a polarized state and the duality relation saturates. However, if both DOF are inseparable, the resulting polarization coherence matrix represents a partially polarized state. It has been shown that the amount of entanglement quantified by the concurrence [15], which for the state (2) is given by,  $\mathcal{E}=2|ad-bc|$  obeys a triality relation with predictability and coherence,  $\mathcal{E}^2+\mathcal{P}^2+\mathcal{C}^2=1$  [17]. Therefore, the amount of inseparability of polarization and spatial DOF bounds the degree of polarization of the beam [12, 14]. If the beam itself is mixed the above triality relation becomes an inequality [28].

When the bipartite system itself is mixed, concurrence does not remain a bonafide measure of mixedness of the reduced density matrices, therefore fails to saturate the complementary relation for the subsystems. Concurrence provides a necessary and sufficient condition for the separability of the density matrix, that is, whether the density matrix can be obtained from an ensemble containing only pure states. Therefore, even when the concurrence is zero, the reduced density matrix can be mixed, and hence  $\mathcal{P}^2 + \mathcal{C}^2 < 1$ . Now, we show that the linear entropy of mixedness saturates the complementary relation in general, and for the pure two qubit state, this exactly quantifies the global entanglement concurrence.

The linear entropy is given by,

$$\mathcal{M}^2(\rho) = \frac{d}{d-1}(1 - \text{Tr}(\rho^2)),$$
 (10)

where, d is the dimension of the density matrix  $\rho$ . This quantity ranges from 0 to 1, the former value is for the pure states and the latter value for the maximally mixed state. For two level system given by the reduced density matrix  $\rho_A$ , it becomes,  $\mathcal{M}^2(\rho_A) = 2(1 - \text{Tr}(\rho_A^2))$ . Suppose the reduced density matrix thus obtained is of the form  $\rho = (I + \mathbf{S}_A \cdot \sigma)/2$ , one obtains,

$$\mathcal{M}_A^2 + \mathcal{C}_A^2 + \mathcal{P}_A^2 = 2\left[1 - \text{Tr}\left(\frac{I(1+|\mathbf{S}_A|^2)}{4}\right)\right] + |\mathbf{S}_A|^2 = 1,$$
(11)

where,  $\mathcal{M}_A = \mathcal{M}(\rho_A)$ . Similarly, for the other subsystem the complementary relation obtained is,

$$\mathcal{M}_B^2 + \mathcal{C}_B^2 + \mathcal{P}_B^2 = 1.$$
 (12)

We observe that the amount of mixedness in a two level system bounds the total amount of local information it can possess in the form of coherence and predictability. It is worth noting that the linear entropy of mixedness for a two level system reduces to,  $\mathcal{M}^2(\rho_A)=2(1-\text{Tr}((\rho_A)^2))=4\det(\rho_A)$ , which for a pure bi-partite state exactly quantifies the entanglement between the subsystem, however, such is not the case for the mixed bipartite state. For the pure bipartite systems, the mixedness of the individual subsystem arises only due to the entanglement between two party. Therefore, the triality relation involving mixedness reduces to the triality relation involving the concurrence for the pure-bipartite system.

Electromagnetic waves with arbitrary wavefront is given by a  $3 \times 3$  Hermitian, positive semi-definite polarization-coherence matrix, therefore isomorphic to a density matrix of a three level (qutrit) system [22]. The general form of polarization-coherence matrix requires the use of Gell-Mann matrices  $\{\lambda_i\}$  given by [23],

$$\lambda_{1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_{2} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{4} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_{5} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \lambda_{6} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_{7} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad \lambda_{8} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In terms of the identity I and  $\{\lambda_i\}$ , an arbitrary polarization-coherence matrix in 3-D takes the form,

$$\Phi_3 = \frac{1}{3} \left( I + \sqrt{3} \sum_{i=1}^8 S_i \lambda_i \right), \tag{13}$$

where,  $S_i$  are the generalized Stokes parameter obtained as  $S_i = \sqrt{3} \operatorname{Tr}(\lambda_i \Phi_3)/2$ . The generalized Stokes parameters are real valued as the polarization-coherence matrix and the Gell-Mann matrices are Hermitian. For  $\Phi_3$  to be associated with a Jones vector, one must have  $\Phi_3^2 = \Phi_3$ ,

$$\Phi_{3}^{2} = \frac{1}{9} \left( I + 2\sqrt{3} \sum_{i} S_{i} \lambda_{i} + 3 \sum_{i,j} S_{i} S_{j} \lambda_{i} \lambda_{j} \right) 
= \frac{1}{9} \left( I + 2 \sum_{i} S_{i}^{2} I + 2\sqrt{3} \sum_{i} S_{i} \lambda_{i} + 3 \sum_{ijk} d_{ijk} S_{i} S_{j} \lambda_{k} \right),$$
(14)

where we have used,  $\lambda_i \lambda_j = \frac{2}{3} \delta_{ij} I + \sum_k (d_{ijk} + i f_{ijk}) \lambda_k$  and the antisymmetry of the structure constant  $f_{ijk}$ . For a pure state,  $\Phi_3^2 = \Phi_3$ , which implies  $\sum_i S_i^2 = 1$  and  $\sqrt{3} \sum_{ij} d_{ijk} S_i S_j = S_k$ . Therefore, fully polarized beams (pure states) lie on the surface

of the eight dimensional sphere embedded in the nine dimensional Euclidean space. It is worth noting that, due to the additional condition on the Stokes vectors, only a subset of points on the surface represents the polarization-coherence matrix.

Hilbert-Schmidt coherence for the polarization-coherence matrix  $\Phi_3$ , in terms of the generalized Stokes parameters is obtained as,

$$\begin{split} C_{HS}^2 &= 2 \sum_{i \neq j} |(\Phi_3)_{ij}|^2 \\ &= \frac{4}{3} \left[ S_1^2 + S_2^2 + S_4^2 + S_5^2 + S_6^2 + S_7^2 \right]. \end{split} \tag{15}$$

The complementary quantity predictability, using (5) is obtained as,

$$\mathcal{P}^2 = \frac{4}{3} \left[ \sum_{i=1}^3 ((\Phi_3)_{ii})^2 - \sum_{i < j} (\Phi_3)_{ii} (\Phi_3)_{jj} \right]$$

$$= \frac{4}{3} \left[ S_3^2 + S_8^2 \right].$$
(16)

To check explicitly that predictability defined above satisfy the boundary conditions, consider a pure state for a three level system  $|\psi\rangle=a|0\rangle+b|1\rangle+c|2\rangle$ , where  $a,b,c\in C$  and  $|a|^2+|b|^2+|c|^2=1$ . Predictability for this pure state is obtained as,

$$\mathcal{P}^2 = \frac{4}{3} \left( |a|^4 + |b|^4 + |c|^4 - |ab|^2 - |ac|^2 - |bc|^2 \right) \tag{17}$$

When the probability for finding all the outcomes are equal, i.e.,  $|a|=|b|=|c|=\frac{1}{\sqrt{3}}$ , the predictability is zero and, the coherence is maximum for such states. When the probability of finding one of the outcome is one and others zero, for example, |a|=1, |b|=|c|=0, the predictability obtained is maximum and coherence for such states is zero. It is worth noting that if  $(\alpha,\beta,\gamma)$  represent the probability of the three outcomes respectively, under any permutation of the individual probabilities, the predictability remains unchanged. For pure state, (15) and (16) leads to the duality relation,

$$\mathcal{P}^2 + C_{HS}^2 = \frac{4}{3}. ag{18}$$

Complementary character of predictability and coherence for the fields with arbitrary wavefront is evident from the above duality relation. We note in passing, that the degree of polarization for 3-D fields is obtained completely in terms of fraction of energy in the orthogonal components, when the polarizationcoherence matrix is diagonal i.e., when coherence is zero. Similar result for the paraxial beam was obtained in Ref. [26].

Polarization basis for a non-paraxial beam is three dimensional, with  $\mathcal{H}_p = \mathrm{Span}\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ . Along with spatial DOF, it yields an analogous vector space structure to that of a bipartite qutrit system. Therefore, complementary relation for the subsystem of a bipartite qutrit system will result in the complementary relation for 3D beam given by  $\Phi_3$ . Next, we derive complementary relation for pure and mixed bipartite qutrit systems.

Consider a pure two qutrit state given by,  $|\psi\rangle_{AB}$ , with A and B representing the qutrits. For such systems, the reduced density matrix for the subsystems A and B will be mixed in general, and the duality relation becomes an inequality,  $\mathcal{P}^2 + C_{HS}^2 \leq \frac{4}{3}$ , where the predictability and coherence are obtained for the respective subsystems and, it saturates when the density matrix

representing the subsystem is pure. For a pure two qutrit system represented the vectors corresponding to the subsystem A will be  $|\phi_i\rangle=\langle i_A|\psi\rangle$ , where i=0,1,2. The global entanglement, I-concurrence in terms of the wedge product is given by [25],

 $\mathcal{E}_{AB}^2 = 4\sum_{i < j} ||\phi_i\rangle \wedge |\phi_j\rangle|^2.$  (19)

For maximally entangled bipartite qutrit systems,  $\mathcal{E}_{AB}^2=4/3$ . We observe that for such cases, all the local properties of system are lost, in the sense that coherence and predictability both vanishes. The reduced density matrix of the subsystem A is obtained as,  $(\rho_A)_{ij}=\langle\phi_j|\phi_i\rangle$ . Therefore, the Hilbert-Schmidt coherence for A is obtained as,

$$C_{HS}^2 = 2\sum_{i \neq j} |\langle \phi_j | \phi_i \rangle|^2 = 4\sum_{i < j} |\langle \phi_j | \phi_i \rangle|^2.$$
 (20)

From this form it is evident that when the vectors are orthogonal, as in the case of maximally entangled state, the coherence of the subsystem is zero. Since the probability of individual outcomes for A are  $\langle \phi_i | \phi_i \rangle$ , from eq. (16) predictability for this subsystem becomes,

$$\mathcal{P}^2 = rac{4}{3} \left( \sum_i |\langle \phi_i | \phi_i 
angle|^2 - \sum_{i < j} \langle \phi_i | \phi_i 
angle \langle \phi_j | \phi_j 
angle 
ight),$$
 (21)

It is evident from the above expression that when all the outcomes are equally probable (which is the case for maximally entangled states), the predictability is zero. Therefore, we observe that for the maximally entangled state both the coherence and predictability are zero. Using (19), (20), and (21), one obtains,

$$\begin{split} \mathcal{E}_{AB}^2 + C_{HS}^2 + \mathcal{P}^2 &= 4 \sum_{i < j} ||\phi_i\rangle \wedge |\phi_j\rangle|^2 + 4 \sum_{i < j} |\langle \phi_j | \phi_i\rangle|^2 \\ &\quad + \frac{4}{3} \left( \sum_i |\langle \phi_i | \phi_i\rangle|^2 - \sum_{i < j} \langle \phi_i | \phi_i\rangle \langle \phi_j | \phi_j\rangle \right) \\ &= \frac{4}{3} \left( \sum_i |\langle \phi_i | \phi_i\rangle|^2 + 2 \sum_{i < j} \langle \phi_i | \phi_i\rangle \langle \phi_j | \phi_j\rangle \right) \\ &= \frac{4}{3} \left( \sum_i \langle \phi_i | \phi_i\rangle \right)^2 = \frac{4}{3} \end{split}$$

where in the second step, Lagrange-Brahmagupta identity [29],  $|a \wedge b|^2 = |a|^2|b|^2 - |a \cdot \bar{b}|^2$  and in the last step, normalisation condition is used. Therefore, the global entanglement, predictability and the coherence obeys a tight triality relation. When the two qutrit state is separable, entanglement is zero and one obtains the usual duality relation between coherence and predictability. When the state is inseparable, the global entanglement imposes an upper bound on the amount of coherence and predictability the single qutrit can possess. Therefore, local characteristics of the system reduces when the entanglement is non-zero.

When the two qutrit system itself it mixed, as in the case of two qubit case, entanglement is replaced by the the linear entropy of mixedness. For a qutrit with density matrix  $\rho_A$ , the linear entropy of mixedness is given by,

$$\mathcal{M}^2(\rho_A) = \frac{3}{2}(1 - \text{Tr}(\rho_A^2)).$$
 (23)

Using the expression of  $\rho_A^2$  obtained as in Eq. (14), one gets,

$$\frac{4}{3}\mathcal{M}^2(\rho_A) + \mathcal{P}^2 + C_{HS}^2 = \frac{4}{3}.$$
 (24)

Therefore, in general mixedness of a qutrit bounds the amount of coherence and predictability. Interestingly, for the pure bipartite qutrit system,  $\mathcal{E}_{AB}^2 = \frac{4}{3}\mathcal{M}^2(\rho_A)$ , and triality relation (24) reduces to one with entanglement, coherence and predictability (22).

In future, we would like to investigate the role of local entanglement between polarization and field envelope in observed correlated motion of particle in an optical trap [30] and the role of spin-orbit interaction in near field optics [31]. We expect that the present approach in deriving complimentarity relations will stimulate further research on classical entanglement.

**Funding.** Department of Science and Technology (DST), India (DST/ICPS/QuEST/Theme-1/2019/6).

**Acknowledgments.** AKR and NKC thank Department of Science and Technology (DST), Govt. of India for Inspire Scholarship.

**Disclosures.** The authors declare no conflicts of interest.

## **REFERENCES**

- E. Schrödinger, Math. Proc. Camb. Philos. Soc. 32, 446 (1936).
- 2. M. Erhard, M. Krenn, and A. Zeilinger, Nat. Rev. Phys. 2, 365 (2020).
- M. Born and E. Wolf, Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light (Cambridge University Press, 1999).
- E. Wolf, Introduction to the Theory of Coherence and Polarization of Light (Cambridge University Press, 2007).
- S. Azzini, S. Mazzucchi, V. Moretti, D. Pastorello, and L. Pavesi, Adv. Quantum Technol. 3, 2000014 (2020).
- B. N. Simon, S. Simon, F. Gori, M. Santarsiero, R. Borghi, N. Mukunda, and R. Simon, Phys. Rev. Lett. 104, 023901 (2010).
- J. H. Eberly, X.-F. Qian, A. A. Qasimi, H. Ali, M. A. Alonso, R. Gutiérrez-Cuevas, B. J. Little, J. C. Howell, T. Malhotra, and A. N. Vamivakas, Phys. Scripta 91, 063003 (2016).
- F. Töppel, A. Aiello, C. Marquardt, E. Giacobino, and G. Leuchs, New J. Phys. 16, 073019 (2014).
- 9. N. Bohr, Nature 121, 580 (1928).
- 10. W. K. Wootters and W. H. Zurek, Phys. Rev. D 19, 473 (1979).
- 11. D. M. Greenberger and A. Yasin, Phys. Lett. A 128, 391 (1988).
- X.-F. Qian, T. Malhotra, A. N. Vamivakas, and J. H. Eberly, Phys. Rev. Lett. 117, 153901 (2016).
- X.-F. Qian, K. Konthasinghe, S. K. Manikandan, D. Spiecker, A. N. Vamivakas, and J. H. Eberly, Phys. Rev. Res. 2, 012016 (2020).
- 14. F. D. Zela, Opt. Lett. 43, 2603 (2018).
- 15. W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
- 16. J. H. Eberly, X.-F. Qian, and A. N. Vamivakas, Optica 4, 1113 (2017).
- 17. M. Jakob and J. A. Bergou, Opt. Commun. 283, 827 (2010).
- M. L. W. Basso and J. Maziero, J. Phys. A: Math. Theor. 53, 465301 (2020).
- 19. T. Qureshi, Opt. Lett. 46, 492 (2021).
- T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, Phys. Rev. E 66, 016615 (2002).
- J. Ellis, A. Dogariu, S. Ponomarenko, and E. Wolf, Opt. Commun. 248, 333 (2005).
- 22. C. J. R. Sheppard, Phys. Rev. A 90, 023809 (2014).
- 23. M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
- P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A 64, 042315 (2001).
- V. S. Bhaskara and P. K. Panigrahi, Quantum Inf. Process. 16, 1 (2017).
- A. Al-Qasimi, O. Korotkova, D. James, and E. Wolf, Opt. Lett. 32, 1015 (2007).

 M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Phys. Rev. A 92, 012118 (2015).

- 28. A. Al-Qasimi, J. Opt. Soc. Am. A 37, 1526 (2020).
- 29. J. Stillwell, Mathematics and Its History (Springer, 2002).
- 30. B. Roy, N. Ghosh, S. Dutta Gupta, P. K. Panigrahi, S. Roy, and A. Banerjee, Phys. Rev. A 87, 043823 (2013).
- V. V. Kotlyar, A. G. Nalimov, A. A. Kovalev, A. P. Porfirev, and S. S. Stafeev, Phys. Rev. A 102, 033502 (2020).