Geometric quantification of multiparty entanglement through orthogonality of vectors

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Abstract. The wedge product of vectors has been shown (Quantum Inf. Process. 16(5): 118,2017) to yield the generalised entanglement measure I-concurrence, wherein the separability of the multiparty qubit system arises from the parallelism of vectors in the underlying Hilbert space of the subsystems. Here, we demonstrate that the orthogonality and equality of the post-measurement vectors maximize the entanglement corresponding to the bi-partitions and can yield non-identical set of maximally entangled states. The Bell states for the two qubit case, GHZ and GHZ like states with superposition of four constituents for three qubits, naturally arise as the maximally entangled states. Interestingly, the later two have identical I-concurrence but different entanglement distribution. While the GHZ states can teleport a single qubit deterministically, the second set achieves the same for a more general state. The geometric conditions for maximally entangled two qudit systems are derived, leading to the generalised Bell states, where the reduced density matrices are maximally mixed. We further show that the reduced density matrix for an arbitrary finite dimensional subsystem of a general qudit state can be constructed from the overlap of the post-measurement vectors. Our geometric approach shows the complimentary character of intrinsic coherence and entanglement.

1. Introduction

Entanglement [1] is one of the most distinctive features of quantum mechanics which manifests as nonlocal correlations [2] in quantum systems [3, 4]. The maximally entangled Bell states [5] have played a significant role in illustrating the nature of quantum correlations, as well as their usefulness in quantum tasks [6]. They have found applications in quantum cryptography [7], quantum teleportation [8], super dense coding [9], to mention a few well known examples. The three particle Greenberger–Horne–Zeilinger (GHZ) state [10], its generalisations and W-states are known to show stronger non-locality and different entanglement properties [11, 12]. These states have found use in quantum information splitting [13–15], quantum teleportation [11, 16–18] and revealed subtle phase structure in atomic systems [4, 19]. It has been shown that a pure EPR pair [16] and a more general form of two qubit [17] can be teleported deterministically and faithfully, using GHZ like states as a channel and performing projective measurements over the GHZ like basis states. In contrast, a GHZ state and an SLOCC equivalent W-class state

[11] can only teleport an arbitrary single qubit state. Moreover, quantum secret sharing via the GHZ like states has been shown to be more efficient as compared to GHZ states [14]. From the application perspective, the GHZ like states are more robust and efficient as compared to GHZ and W-class of states. Significantly, the implementation of quantum tasks has revealed the physical nature of the quantum non-local correlations [20]. The non maximally entangled states fail to teleport deterministically [21] and the three qubit states get divided into two distinct classes [22]; the GHZ class shows deterministic teleportation and W class fails to do so [23]. The entangled states of higher dimensions are under intense investigation [24] as they provide advantage over qubit states and show better quantum correlations by strongly violating the Bell's inequality. Entanglement has also thrown considerable light on correlations in spin chains [25, 26] and quantum many body systems [27]. Therefore, understanding of the physical nature of entanglement has attracted considerable attention in the current literature [20, 24, 28].

Significant amount of work has been carried out to quantify entanglement, which for the two particle case has been well understood [3]. While there exists several entanglement measures for pure and mixed two qubit systems such as, concurrence [29], 3-tangle [30], entanglement of formation [31], Schmidt number [32] etc., there is no straightforward and unique way to characterize entanglement in multipartite setting. Among all these measures, concurrence is one of the most widely used, as it provides necessary and sufficient conditions for separability for a general pair of qubits. For arbitrary dimensions, the concurrence for multiparticle pure state was generalised by Rungta et al. [33] and is known as "I-Concurrence". Mintert et al. [34] carried out the general characterization of concurrence for multipartite quantum systems, showing that it is qualitatively different from other quantum correlations.

There have been various geometry based approaches to characterize and quantify entanglement [35–39]. Recently, a geometric perspective of I-concurrence has been given for multiparty entangled system of qudits, based on wedge product [40] and Lagrange's identity, a generalisation of the Brahmagupta–Fibonacci identity [41]. This framework was further explored in [39], where geometric representation of three way distributive entanglement, known as 3-tangle [30] in two dimensions is given, with a measure to quantify the distributive n-party entanglement. In [37], the minimum distance between a set of bi-partite n qudit density matrices with positive partial trace and maximally mixed states was calculated using a measure based on Euclidean distance between Hermitian matrices. Here, we use this wedge product formalism [38] to find geometrical configurations corresponding to the separable and maximally entangled states of finite dimensional multiparty systems. The concurrence corresponding to the bi-partitions of the multiparty system is expressed in terms of the wedge product of the post measurement vectors. Parallelism of post measurement vectors result in separability across the bi-partitions, whereas the orthogonality and equality of these vectors yield maximally entangled states. These constraints are employed to find maximally entangled states in case of two qubit and three qubit systems. For the two qubit case, one obtains the general expression of the maximally entangled states, which after local unitary transformations lead to the Bell states. In the three qubit case, starting with the Schmidt decomposed canonical forms [42, 43], we naturally obtain GHZ and GHZ like states [44], as maximally entangled states. We then characterize and compare these states with the W-state using the polynomial invariant 3-tangle [30], defined in the wedge product formalism [39]. The general conditions for maximally entangled states of a two qudit system is established leading to the generalised Bell states. We further describe a procedure to construct density matrices in terms of inner product of post measurement vectors. In addition to being efficient, this approach provides insight to the geometric origin of absolute maximal entanglement [45] and intrinsic coherence [46]. The intrinsic degree of coherence is written in terms of post-measurement vectors and shown that, the geometric conditions for extremum of intrinsic coherence are opposite to those of the entanglement.

The paper is organized as follows: In Sec. 2, we describe the general conditions for maximizing the global entanglement of a finite dimensional multiparty system and use these conditions to find maximally entangled states of two and three qubit systems. In Sec. 3, the geometric conditions for maximally entangled states in a general two-qudit system are obtained, which lead to the generalised Bell states. Sec. 4 is devoted to establish the connection of the reduced density matrix and post measurement vectors belonging to the Hilbert space of the subsystems. We conclude in Sec. 5 with a summary of the work and directions for future research.

2. Geometric conditions for maximally entangled states

Bhaskara and Panigrahi [38] have defined a global measure of entanglement as the sum of generalised concurrence corresponding to all the bi-partitions for a system using Lagrange's identity and showed that the concurrence of each bi-partition for a pure state can be quantified using wedge product of post-measurement vectors in the Hilbert space of the subsystems. This measure was shown to match with the I-concurrence [33], an extension of concurrence for multiparticle pure states that provides a faithful quantification of entanglement across any bi-partition.

The wedge product formalism provides a geometric representation of concurrence, leading to the understanding of entanglement in terms of configuration of vectors in the Hilbert space of the subsystems. It yields a set of different constraints, for maximally entangled and separable states. The separability across a bi-partition arises as a consequence of the parallelism of the post-measurement state vectors. In the following, the consequence of orthogonality and equality of the post-measurement state vectors will be shown to yield the maximally entangled states.

We consider a multiparty system $\mathcal{S}$, described by a pure state $|\Psi\rangle$ and a bi-partition $\mathcal{A}|\mathcal{B}$, of this system, with Hilbert space corresponding to $\mathcal{A}$ and $\mathcal{B}$ given by $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ respectively. If $\rho = |\Psi\rangle\langle\Psi|$ is the density matrix of the system, the I-concurrence, corresponding to the partition $\mathcal{A}|\mathcal{B}$ is given by:

$$C^2_{\mathcal{A}|\mathcal{B}} = 2(1 - \mathrm{Tr}(\rho^2_\mathcal{A})) = 2(1 - \mathrm{Tr}(\rho^2_\mathcal{B})), \tag{1}$$

where, $\rho_\mathcal{A} = \mathrm{Tr}_\mathcal{B}(\rho)$ and $\rho_\mathcal{B} = \mathrm{Tr}_\mathcal{A}(\rho)$ are the reduced density matrices. Without loss of generality, one can take $d_\mathcal{A} = dim(\mathcal{H}_\mathcal{A}) \leq dim(\mathcal{H}_\mathcal{B}) = d_\mathcal{B}$. If $\{|\phi_i\rangle \;\; i = 1, 2, ., d_\mathcal{A}\}$ is an orthonormal basis of $\mathcal{H}_\mathcal{A}$, it follows that,

$$|\Psi\rangle = \sum_{i=0}^{d_\mathcal{A}} |\phi_i\rangle\langle\phi_i|\Psi\rangle. \tag{2}$$

The set containing $\langle\phi_i|\Psi\rangle$ has $d_\mathcal{A}$ vectors (referred henceforth as post measurement vectors) corresponding to the respective $|\phi_i\rangle$'s in the Hilbert space $\mathcal{H}_\mathcal{B}$. Concurrence of the bi-partition $\mathcal{A}|\mathcal{B}$, defined in terms of the wedge products of post-measurement vectors, is then [38] :

$$C^2_{\mathcal{A}|\mathcal{B}} = 4\sum_{i<j} |\langle\phi_i|\Psi\rangle \wedge \langle\phi_j|\Psi\rangle|^2, \tag{3}$$

where i and j take values from 0 to $d_\mathcal{A}$. The condition for separability across this bi-partition is $C^2_{\mathcal{A}|\mathcal{B}} = 0$. For this, the vectors $\langle\phi_j|\Psi\rangle$ must be parallel to each other, $\forall\ i$ in the Hilbert space $\mathcal{H}_\mathcal{B}$, showing that the separability across bi-partitions can be viewed in terms of parallelism of vectors. For the maximally entangled state $|\Psi\rangle$, the I-concurrence $C_{\mathcal{A}|\mathcal{B}}$ must take maximum value for all the possible bi-partitions $\mathcal{A}|\mathcal{B}$. For the purpose of illustration of the geometric scenario corresponding to the maximally entangled state, we briefly discuss the wedge product of two vectors.

Consider an n dimensional space where $\{e_i\}$, with, $i = 1, 2, ..n$, is an orthonormal basis. The linear superposition of two vectors $\vec{a} = \sum_i a_i e_i$ and $\vec{b} = \sum_j b_j e_j$ forms a subspace, which naturally spans a complex plane containing the origin and these vectors. The wedge product of $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \wedge \vec{b} = \sum_{i<j} (a_i b_j - a_j b_i)\, e_i \wedge e_j. \tag{4}$$

The bivector $\vec{a} \wedge \vec{b}$ represents an oriented parallelogram in the plane, with adjacent sides as $\vec{a}$ and $\vec{b}$. Maximizing the magnitude of the wedge product corresponds to maximizing the area of the parallelogram. The area is maximum when $\vec{a}$ and $\vec{b}$ are orthogonal, $(\vec{a} \cdot \vec{b} = 0)$ and equal in length, if $|\vec{a}|^2 + |\vec{b}|^2 = c$, where c is a constant. This geometrically corresponds to the vectors $\vec{a}$ and $\vec{b}$, as the sides of a square in the underlying subspace.

Maximal $C_{\mathcal{A}|\mathcal{B}}$ for a particular bi-partition $\mathcal{A}|\mathcal{B}$ corresponds to the following conditions:

$$\langle \phi_i | \Psi \rangle^\dagger \langle \phi_j | \Psi \rangle = 0, \quad \forall \quad i \neq j, \tag{5}$$

$$|\langle \phi_i | \Psi \rangle| = |\langle \phi_j | \Psi \rangle|, \quad \forall \quad i, j. \tag{6}$$

Therefore, all the post measurement vectors are orthogonal to each other and have equal length in the Hilbert space $\mathcal{H}_{\mathcal{B}}$. Geometrically, it represents a $d_{\mathcal{A}}$ dimensional cube in the Hilbert space $\mathcal{H}_{\mathcal{B}}$. Now, a maximally entangled state $|\Psi\rangle$ must have maximal concurrence corresponding to all the possible bi-partitions, which occurs when the post measurement vectors for every bi-partition $\mathcal{A}|\mathcal{B}$ form a $d_{\mathcal{A}}$-cube in the Hilbert space $\mathcal{H}_{\mathcal{B}}$, where $d_{\mathcal{A}} \leq d_{\mathcal{B}}$.

One thus obtains a set of constraints for separability and maximal entanglement, in terms of the relations between post-measurement vectors corresponding to bi-partitions of the system. The parallelism of these vectors corresponds to separability across the bi-partition, and orthogonality and equality of these vectors lead to maximal entanglement.

In the following subsections, we obtain the maximally entangled states for two qubit system and two distinct maximally entangled states for three qubit system.

2.1. ***The two qubit entanglement***

For a two qubit system, the general state $|\psi\rangle$ in the computational basis is given by,

$$|\psi_{AB}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \tag{7}$$

where, $|ij\rangle = |i_A\rangle \otimes |j_B\rangle$ and $a, b, c, d \in \mathbf{C}$, satisfying the normalisation condition:

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \tag{8}$$

Here, we have only bi-partition $A|B$ and, both $\mathcal{H}_A$ and $\mathcal{H}_B$ are two dimensional Hilbert spaces. The post measurement state corresponding to $|0\rangle$ and $|1\rangle$ for particle A are, $\langle 0_A|\psi\rangle = a|0_B\rangle + b|1_B\rangle$ and $\langle 1_A|\psi\rangle = c\,|0_B\rangle + b|1_B\rangle$. These are two dimensional vectors in the Hilbert state $\mathcal{H}_{\mathcal{B}}$ (shown in Fig. 1). The generalized concurrence measure in terms of wedge product for $|\psi\rangle_{AB}$ has been obtained earlier as [38]:

$$E = 2\,|\langle 0_A \mid \psi\rangle \wedge \langle 1_A|\psi\rangle|\,. \tag{9}$$

The separability condition is the parallelism of vectors $\langle 0_A|\psi_{AB}\rangle$ and $\langle 0_A|\psi_{AB}\rangle$ in the Hilbert space $\mathcal{H}_B$, which corresponds to $\frac{a}{c} = \frac{b}{d}$. Maximizing E, geometrically corresponds to maximizing

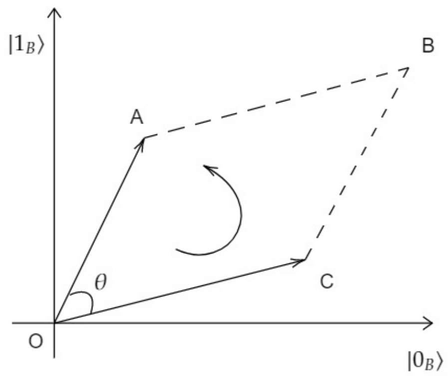

Figure 1. Bivector $\langle 0_A|\psi\rangle \wedge \langle 1_A|\psi\rangle$ as an oriented parallelogram in the Hilbert space $\mathcal{H}_B$, with the vectors $\vec{OC}$ and $\vec{OA}$ representing $\langle 0_A|\psi\rangle$ and $\langle 1_A|\psi\rangle$, respectively.

the area of the parallelogram formed by vectors $\vec{OC} = \langle 0_A|\psi_{AB}\rangle$ and $\vec{OA} = \langle 1_A|\psi_{AB}\rangle$, as two adjacent sides in $\mathcal{H}_B$. Maximal area corresponds to the following two conditions:

$$\langle 0_A|\psi\rangle^+ \langle 1_A|\psi\rangle = 0 \quad \text{and} \quad |\langle 0_A|\psi_{AB}\rangle|^2 = |\langle 1_A|\psi_{AB}\rangle|^2. \tag{10}$$

These two conditions lead to the general form of maximally entangled states for a two qubit system:

$$\bar{a}c + \bar{b}d = 0 \tag{11}$$
$$|a|^2 + |b|^2 = |c|^2 + |d|^2 \tag{12}$$

We have $\bar{a} = kd$ and $\bar{b} = -kc$, where $k \in \mathbf{C}$, and $\bar{a}$, $\bar{b}$ are complex conjugate of a, b respectively. From the second condition, we obtain $|k|^2 = 1$, hence $k = e^{i\theta}$. The general maximally entangled state takes the form:

$$|\psi\rangle_{AB} = a\,|00\rangle + b\,|01\rangle - \bar{b}e^{-i\theta}\,|10\rangle + \bar{a}e^{-i\theta}\,|11\rangle\,, \tag{13}$$

as the reduced density matrices ρ_A and ρ_B are maximally mixed.

$$\rho_A = Tr_B(\rho) = \sum_{i=0,1} \langle i_B|\rho|i_B\rangle = \frac{I}{2}. \tag{14}$$

The concurrence is found to be, $E = 2||a|^2 + |b|^2| = 1$, as expected. Bell states are obtained by taking appropriate values of the parameters a, b and θ, which are also obtained by performing local unitary transformations. For instance, for $a, b \in \mathcal{R}$, the unitary transformation $I \otimes U|\psi\rangle_{AB}$ results in the Bell states up to a phase factor:

$$|\psi_\pm\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}} \quad \text{for} \quad U = \sqrt{2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{15}$$

$$|\phi_\pm\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}} \quad \text{for} \quad U = \sqrt{2}\begin{pmatrix} -b & a \\ a & b \end{pmatrix}. \tag{16}$$

2.2. *The three qubit entanglement*

For a three qubit system described by state $|\psi_{ABC}\rangle$, we have three independent bi-partitions. Therefore, the global measure of entanglement E is given by sum of concurrence corresponding to all the three bi-partitions [38] :

$$E = E_{A|BC} + E_{B|AC} + E_{C|AB}. \tag{17}$$

In the wedge product formalism, the concurrence is given by:

$$E = 2 \sum_{i=A,B,C} |\langle 0_i|\psi_{ABC}\rangle \wedge \langle 1_i|\psi_{ABC}\rangle|. \tag{18}$$

As before, the separability condition across any bipartition is the parallelism of the post-measurement vectors, since the wedge product is zero in that case. Interestingly, for the three qubit case, separability across any two bipartition is sufficient for tripartite separability. It follows that, $E_{A|BC} = E_{B|AC} = 0$, is a sufficient condition for the state to be totally separable. $|\Psi_{ABC}\rangle$ can be written in the following form:

$$|\Psi_{ABC}\rangle = |0_A\rangle\langle 0_A|\Psi_{ABC}\rangle + |1_A\rangle\langle 1_A|\Psi_{ABC}\rangle. \tag{19}$$

Assuming the separability across bipartition $A|BC$, leads to $E_{A|BC} = 0$, and $\langle 1_A|\Psi_{ABC}\rangle = \alpha\langle 0_A|\Psi_{ABC}\rangle$, where $\alpha \in C$. One can separate out A from BC as:

$$|\Psi_{ABC}\rangle = (|0_A\rangle + \alpha|1_A\rangle) \otimes |\psi_{BC}\rangle \quad \text{, where} \quad |\psi_{BC}\rangle = \langle 0_A|\Psi_{ABC}\rangle. \tag{20}$$

Similarly, from separability across bipartition $B|AC$, $E_{B|AC} = 0$ leading to,

$$\langle 1_B|\Psi_{ABC}\rangle = \beta\langle 0_B|\Psi_{ABC}\rangle \quad \text{and hence,} \quad \langle 1_B|\psi_{BC}\rangle = \beta\langle 0_B|\psi_{BC}\rangle \tag{21}$$

where, $\beta \in C$. The state is then tripartite separable:

$$|\Psi_{ABC}\rangle = (|0_A\rangle + \alpha|1_A\rangle) \otimes (|0_B\rangle + \beta|1_B\rangle) \otimes |\psi_C\rangle \tag{22}$$

where, $|\psi_C\rangle = \langle 0_A 0_B|\Psi\rangle$. From the separated form of $|\Psi_{ABC}\rangle$, it is seen that: $\langle 1_c|\Psi_{ABC}\rangle = \gamma\langle 0_c|\Psi_{ABC}\rangle$, where $\gamma \in C$ and it follows that $E_{AB|C} = 0$. Hence, vanishing of the concurrence across any two bi-partition implies vanishing of concurrence across the third bi-partition. It is worth observing that, if the global entanglement E is greater than two, the state cannot be separated across any bi-partition.

We now discuss the conditions to obtain the maximally entangled state for three qubit systems, for which, a general state is given by :

$$\begin{aligned} |\psi_{ABC}\rangle = \lambda_0|000\rangle + \lambda_1|001\rangle + \lambda_2|010\rangle + \lambda_3|011\rangle + \lambda_4|100\rangle \\ + \lambda_5|101\rangle + \lambda_6|110\rangle + \lambda_7|111\rangle \end{aligned} \tag{23}$$

where, $\lambda_i \in C$, $i = 0 - 7$ satisfy the normalisation condition, $\sum_{i=0}^{7} |\lambda_i|^2 = 1$. To obtain the maximally entangled states, one needs to maximize $|\langle 0_i|\psi_{ABC}\rangle \wedge \langle 1_i|\psi_{ABC}\rangle|$ for all $i = A, B, C$. The conditions for maximally entangled states as discussed previously are : $\langle 0_i|\psi_{ABC}\rangle$ must be orthogonal to $\langle 1_i|\psi_{ABC}\rangle$ and $|\langle 0_i|\psi_{ABC}\rangle| = |\langle 1_i|\psi_{ABC}\rangle|$ for each $i = A, B, C$, leading to the following conditions,

$$\begin{aligned} \lambda_0\bar{\lambda}_4 + \lambda_1\bar{\lambda}_5 + \lambda_2\bar{\lambda}_6 + \lambda_3\bar{\lambda}_7 = 0, \\ \lambda_0\bar{\lambda}_2 + \lambda_1\bar{\lambda}_3 + \lambda_4\bar{\lambda}_6 + \lambda_5\bar{\lambda}_7 = 0, \\ \lambda_0\bar{\lambda}_1 + \lambda_2\bar{\lambda}_3 + \lambda_4\bar{\lambda}_5 + \lambda_6\bar{\lambda}_7 = 0; \end{aligned} \tag{24}$$

and

$$\begin{aligned}|\lambda_0|^2+|\lambda_1|^2+|\lambda_2|^2+|\lambda_3|^2&=|\lambda_4|^2+|\lambda_5|^2+|\lambda_6|^2+|\lambda_7|^2,\\|\lambda_0|^2+|\lambda_1|^2+|\lambda_2|^2+|\lambda_3|^2&=|\lambda_4|^2+|\lambda_5|^2+|\lambda_6|^2+|\lambda_7|^2,\\|\lambda_0|^2+|\lambda_1|^2+|\lambda_2|^2+|\lambda_3|^2&=|\lambda_4|^2+|\lambda_5|^2+|\lambda_6|^2+|\lambda_7|^2.\end{aligned}\tag{25}$$

These constraints may not lead to physically acceptable solutions. However, in the three qubit case, one finds that states with maximal entanglement measure does exist. We start with a canonical form of the three qubits and apply these constraint equations. Acín et al. [42] have shown that any pure three qubit can be reduced to a canonical form, having five amplitudes and one relative phase using the generalised Schmidt decomposition [43]. Any generic three qubit state can be reduced to several classes of canonical forms using five local base product states (LBPS), while preserving nonlocal properties of the state. Owing to different degrees of orthogonality, there exists three inequivalent classes of five LBPS. We begin with the general state built using the canonical forms from these LBPS classes. Once a maximally entangled state is obtained, we perform local unitary (LU) transformations to find other states that form a complete basis in the Hilbert space of three qubit system, as LU operations do not affect entanglement of the state [47].

The three qubit canonical form built from symmetric LBPS class, with qubits A, B and C is given by :

$$|\psi\rangle_{ABC}=k_0e^{i\theta}\,|000\rangle+k_1\,|001\rangle+k_2\,|010\rangle+k_3\,|100\rangle+k_4\,|111\rangle\,,\tag{26}$$

where k_i, $i=0,2,.,4$ are positive numbers and $0\leq\theta\leq\pi$.

From the orthogonality condition of the constraints :

$$k_0k_1=k_0k_2=k_0k_3=0.\tag{27}$$

The equality condition leads to :

$$|k_1|=|k_2|=|k_3|\quad\text{and}\quad|k_0|^2+|k_3|^2=|k_4|^2.\tag{28}$$

Two set of equations satisfy the above conditions :

(i) When $k_0\neq0,k_1=k_2=k_3=0$,

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle).\tag{29}$$

This maximally entangled state is the well known GHZ State.

(ii) When $k_0=0$,

$$|k_1|=|k_2|=|k_3|=|k_4|=\frac{1}{2}\tag{30}$$

Since the k_i's are positive numbers, the maximally entangled state is:

$$|\xi_1\rangle=\frac{1}{2}(|001\rangle+|010\rangle+|100\rangle+|111\rangle)\tag{31}$$

known in the literature as GHZ like state . It has maximal entanglement measure and satisfies the condition $\rho_A=\rho_B=\rho_C=I/2$.

As the entanglement measure remains invariant under unitary transformation, one can find other states that are also maximally entangled. For example, through a local unitary transformation $\sigma_z\otimes I\otimes I$:

$$|\psi_1\rangle=\frac{1}{\sqrt{2}}(|000\rangle-|111\rangle)\tag{32}$$

The other maximally entangled states obtained by performing local unitary transformations on $|\xi_1\rangle$ are:

$$U_1|\xi_1\rangle = |\xi_2\rangle = \frac{1}{2}(-|001\rangle - |010\rangle + |100\rangle + |111\rangle), \tag{33}$$

$$U_2|\xi_1\rangle = |\xi_3\rangle = \frac{1}{2}(|001\rangle - |010\rangle - |100\rangle + |111\rangle), \tag{34}$$

$$U_3|\xi_1\rangle = |\xi_4\rangle = \frac{1}{2}(-|001\rangle + |010\rangle - |100\rangle + |111\rangle), \tag{35}$$

where $U_1 = -\sigma_z \otimes I \otimes I$, $U_2 = -I \otimes I \otimes \sigma_z$ and, $U_3 = -I \otimes \sigma_z \otimes I$.

Different LBPS classes yield various maximally entangled states from the GHZ class (listed in Table 2.2). It can be seen that states obtained from the first two symmetric LBPS class are mutually orthogonal GHZ like states, that form a complete orthogonal basis on the eight dimensional Hilbert space.

Table 2.2 :Maximally entangled states belonging to different LBPS class

Classes of LBPS	Maximally entangled states
$\|000\rangle\,, \|011\rangle\,, \|110\rangle\,, \|101\rangle\,, \|111\rangle$	$\frac{1}{2}(\|000\rangle + \|110\rangle + \|101\rangle + \|011\rangle)$
	$\frac{1}{2}(\|000\rangle + \|110\rangle - \|101\rangle - \|011\rangle)$
	$\frac{1}{2}(\|000\rangle - \|110\rangle + \|101\rangle - \|011\rangle)$
	$\frac{1}{2}(\|000\rangle - \|110\rangle - \|101\rangle + \|011\rangle)$
$\|000\rangle\,, \|001\rangle\,, \|010\rangle\,, \|100\rangle\,, \|111\rangle$	$\frac{1}{2}(\|111\rangle + \|001\rangle + \|010\rangle + \|100\rangle)$
	$\frac{1}{2}(\|111\rangle + \|001\rangle - \|010\rangle - \|100\rangle)$
	$\frac{1}{2}(\|111\rangle - \|001\rangle + \|010\rangle - \|100\rangle)$
	$\frac{1}{2}(\|111\rangle - \|001\rangle - \|010\rangle + \|100\rangle)$
$\|000\rangle$,$\|010\rangle$,$\|100\rangle$ $\|101\rangle$,$\|111\rangle$	$\frac{1}{\sqrt{2}}(\|010\rangle \pm \|101\rangle)$
$\|000\rangle$,$\|001\rangle$,$\|110\rangle$ $\|101\rangle$,$\|111\rangle$	$\frac{1}{\sqrt{2}}(\|001\rangle \pm \|110\rangle)$
$\|000\rangle$,$\|010\rangle$,$\|100\rangle$ $\|101\rangle$,$\|111\rangle$	$\frac{1}{\sqrt{2}}(\|100\rangle \pm \|011\rangle)$

Using the entanglement measure of [38], we have been able to find states corresponding to the maximally entangled states as the geometry corresponding to the extrema (separable and maximally entangled states) are well defined. For other entangled states, the corresponding geometry is not unique, so one cannot find states that do not correspond to maximum value of the measure, the highly entangled W-state is one such example,

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \tag{36}$$

W-state satisfies all the orthogonality criteria for the maximally entangled state, but does not satisfy the equality of sides criterion, and hence its configuration does not correspond to that of maximal entanglement. In terms of persistence of entanglement, the GHZ like states are more like W- state, as both show robustness of entanglement under loss of any single qubit. This interesting property of GHZ like states differentiates it from GHZ states.

In order to characterise the genuine three party distributive entanglement in three qubit pure states $|\psi_{ABC}\rangle$, one can use the 3-tangle, which is a polynomial invariant quantity under permutation of qubits, follows from the Coffman–Kundu–Wooter's inequality [30] :

$$C^2_{A|BC} \geq C^2_{A|B} + C^2_{A|C}. \tag{37}$$

Here $C^2_{A|BC}$ corresponds to the squared concurrence corresponding to the bi-partition $A|BC$, and $C^2_{A|B(C)}$ is the square concurrence between A and B(C).

If the qubits A, B and C are entangled, the residual entanglement or 3-tangle is given by :

$$\tau = C^2_{A|BC} - C^2_{A|B} - C^2_{A|C} \tag{38}$$

In terms of wedge product [39], the 3-tangle takes the form:

$$\begin{aligned}\tau = 4|\langle 0_A|\psi_{ABC}\rangle \wedge \langle 1_A|\psi_{ABC}\rangle|^2 - 4|\sum_{i=0,1}\langle 0_A|\psi_{AB}\rangle_i \wedge \langle 1_A|\psi_{AB}\rangle_i|^2 \\ -4|\sum_{i=0,1}\langle 0_A|\psi_{AC}\rangle_i \wedge \langle 1_A|\psi_{AC}\rangle_i|^2,\end{aligned} \tag{39}$$

where, $|\psi_{AC}\rangle_i = \langle i_B|\psi_{ABC}\rangle$ and $|\psi_{AB}\rangle_i = \langle i_C|\psi_{ABC}\rangle$ are the post measurement vectors corresponding to B and C respectively. For the GHZ and GHZ like states, the 3-tangle obtained is one and vanishes for the W-state. The global entanglement for W state is also not maximal and has value $2\sqrt{2}$ as compared to 3 for GHZ and GHZ like states. Therefore, the amount of entanglement in the GHZ and GHZ like states is different from the W-state. These states belong to two inequivalent classes under stochastic local operations and classical communications (SLOCC) [22].

3. Entanglement measure for two qudit system

Consider the general form of two particle qudit state,

$$|\psi_{AB}\rangle = \sum_{i,j} a_{ij}|i_A\rangle \otimes |j_B\rangle, \tag{40}$$

where $i, j = 0, ..., d-1$ for a d -level computational basis. In terms of d post-measurement state vectors $\langle i_A|\psi_{AB}\rangle$ in the d-dimensional Hilbert space $\mathcal{H}_B$, measure of entanglement of A with B in terms of wedge product is defined as:

$$E^2 = 4\sum_{i<j}|\langle i_A|\psi_{AB}\rangle \wedge \langle j_A|\psi_{AB}\rangle|^2 \tag{41}$$

As has been done earlier, for maximally entangled state, one needs to maximize the magnitude of wedge product between all the post-measurement state vectors; this leads to the following conditions:

$$\langle i_A|\psi_{AB}\rangle^\dagger \langle j_A|\psi_{AB}\rangle = 0 \quad i \neq j, \tag{42}$$

$$|\langle i_A|\psi_{AB}\rangle| = |\langle j_A|\psi_{AB}\rangle| \quad \forall \quad i, j. \tag{43}$$

Therefore, the vectors $\langle i_A|\psi_{AB}\rangle$ are the sides of a d-cube in the Hilbert space $\mathcal{H}_B$. Alternatively, one can arrive at the same geometric conditions by maximizing the magnitude of wedge product of the d post-measurement vectors in $\mathcal{H}_B$. The wedge product $|\langle 0_A|\psi_{AB}\rangle \wedge ... \wedge \langle d_A|\psi_{AB}\rangle|$ corresponds to the volume of d-parallelepiped, whose sides are given by $\langle i_A|\psi_{AB}\rangle$, its volume is maximum for the case of d-cube. The entanglement corresponding to maximally entangled states is obtained as :

$$E^2 = \frac{4\ {}^d\mathbf{C}_2}{d^2}, \tag{44}$$

after using the normalisation condition $\sum_{i,j} |a_{ij}|^2 = 1$. Interestingly, for the maximally entangled case, as the vectors $\langle i_A|\psi_{AB}\rangle$ are the sides of a d-cube in $\mathcal{H}_B$, one can consider these vectors as a set of new computational basis and write $|\psi_{AB}\rangle$ in a Schmidt decomposed form. Taking $|i_B^{'}\rangle = \sqrt{d}\langle i_A|\psi_{AB}\rangle$ as the new orthonormal basis, we get the state vector in the following form,

$$|\psi_{AB}\rangle = \sum_i a_i^{'} |i_A\rangle \otimes |i_B\rangle, \tag{45}$$

with $a_i^{'} = \frac{1}{\sqrt{d}}$. Hence, for a two-qudit system, the maximally entangled state takes the form,

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle. \tag{46}$$

This is a generalised form of the Bell states, as is evident from the fact that the partial trace corresponding to subsystem A or B, leads to:

$$\rho_A = \mathrm{Tr}_B\, \rho = \frac{1}{d} \sum_{i=0}^{d-1} |i_A\rangle\langle i_A| = \frac{I}{d}, \tag{47}$$

where, $\rho = |\psi_{AB}\rangle\langle\psi_{AB}|$ is the density matrix of the system and I is the identity matrix. As an example, for a two qutrit state, the maximally entangled state is obtained as:

$$|\psi_{++}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{48}$$

The complete orthogonal basis of maximally entangled states can be found using local unitary transformations $U_1 \otimes U_2$ on $|\psi_{++}\rangle$.

For an n-qudit system, total number of bi-partitions are $2^{n-1} - 1$. A maximally entangled state must satisfy the constraints of orthogonality and equality of vectors for each of these bi-partitions. However, it is not guaranteed that such a state necessarily exists. In fact, maximally entangled state does not exist for a four qubit system. It is seen that the GHZ state for four qubits:

$$|GHZ\rangle_4 = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \tag{49}$$

does not satisfy the equality criterion for the post measurement vectors for the bi-partitions, where each subsystem has two qubits and hence, does not achieve the maximum value for global entanglement. This scenario also occurs for the $|GHZ\rangle_n = \frac{1}{\sqrt{2}}(|00...0_n\rangle + |11...1_n\rangle)$, describing the n-qubit system with $n \geq 4$.

4. Reduced density matrix in terms of post-measurement vectors

In this section, a procedure is described to obtain the reduced density matrix corresponding to any subsystem, in terms of the inner product between the post-measurement vectors. Suppose, $|\Psi\rangle$ describes a multiparty system $\mathcal{S}$ with the Hilbert space $\mathcal{H}$. We consider an arbitrary partition $\mathcal{A}|\mathcal{B}$ of $\mathcal{S}$, such that, $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, with $d_{\mathcal{A}} = dim(\mathcal{H}_{\mathcal{A}})$ and $d_{\mathcal{B}} = dim(\mathcal{H}_{\mathcal{B}})$, where subsystem $\mathcal{A}$ and $\mathcal{B}$ can be any finite collection of qudits. Suppose, $(|\phi_i\rangle_{\mathcal{A}})_{i=1}^{d_{\mathcal{A}}}$ to be an orthonormal basis of the Hilbert space $\mathcal{H}_{\mathcal{A}}$, then the post measurement vectors corresponding to subsystem $\mathcal{A}$ can $|\psi_i\rangle_{\mathcal{B}} = \langle\phi_i|\Psi\rangle$ where $i = 1, .., d_{\mathcal{A}}$. The post-measurement vectors $|\psi_i\rangle_{\mathcal{B}}$ lie in the Hilbert space $\mathcal{H}_{\mathcal{B}}$ and $\rho^{\mathcal{A}}$ (the reduced density matrix of the subsystem $\mathcal{A}$) can be written in terms of inner product of these vectors. The state vector of the system is :

$$|\Psi\rangle = \sum_{i=1}^{d_{\mathcal{A}}} |\phi_i\rangle_{\mathcal{A}} \otimes |\psi_i\rangle_{\mathcal{B}}. \tag{50}$$

Reduced density matrix of subsystem $\mathcal{A}$ is given by $\rho^{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}\,\rho$, where $\rho = |\Psi\rangle\langle\Psi|$ and,

$$\rho^{\mathcal{A}}_{ij} = \langle\phi_i|\rho^{\mathcal{A}}|\phi_j\rangle = \sum_{m=1}^{d_{\mathcal{B}}} \langle\xi_m|\psi_i\rangle\langle\psi_j|\xi_m\rangle, \tag{51}$$

where $(|\xi_i\rangle_{\mathcal{B}})_{i=1}^{d_{\mathcal{B}}}$ of is an orthonormal basis of $\mathcal{H}_{\mathcal{B}}$. Using completeness of the basis, it follows that $\rho^{\mathcal{A}}$ can be constructed in the form of a $d_{\mathcal{A}} \times d_{\mathcal{A}}$ matrix with the elements given by $\rho^{\mathcal{A}}_{ij} = \langle\psi_j|\psi_i\rangle$. Hence, the reduced density matrix can be constructed from the overlap of post-measurement vectors:

$$\rho^{\mathcal{A}} = \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \dots & \langle\psi_{d_{\mathcal{A}}}|\psi_1\rangle \\ . & .. & . \\ . & & . \\ \langle\psi_1|\psi_{d_{\mathcal{A}}}\rangle & .. & \langle\psi_{d_{\mathcal{A}}}|\psi_{d_{\mathcal{A}}}\rangle \end{pmatrix}. \tag{52}$$

This procedure to find reduced density matrix is not only computationally simple, it also gives one a perspective into why the geometrical constraints of orthogonality and equality of sides are obtained in the case of maximally entangled states. Following the discussion of the section 2, when $d_{\mathcal{A}} \leq d_{\mathcal{B}}$ for the bipartion $\mathcal{A}|\mathcal{B}$ of a system, the orthogonality constraint makes all the non-diagonal elements of reduced density matrix $\rho^{\mathcal{A}}$ zero, and the equality of the post measurement vectors forces all diagonal terms to be equal and hence leads to the maximally mixed density matrix, $\rho^{\mathcal{A}} = I/d_{\mathcal{A}}$.

Therefore, constraints for the maximally entangled state gives maximally mixed reduced density matrix for the smaller of the two bi-partition for every bi-partition of the system. States satisfying these conditions are also know as absolute maximally entangled states [45]. We have shown that for two qubit system, the Bell states and for three qubit system, the GHZ and GHZ like states satisfy all the constraints, and hence are absolute maximally entangled states. It has been shown that for four qubits and for n-qubit systems ($n \geq 7$), absolutely maximally entangled states do not exist [45], hence for such systems, no state exists which satisfies all the constraints. It would be interesting to explore further the absolute maximally entangled states of higher dimensions using the present approach.

The calculation of reduced density matrix provides geometric insights about the elements of matrix in terms of post measurement vectors. An alternate way to understand separability and entanglement for a two qudit system is through a recently defined measure of intrinsic coherence of the reduced density matrix. Following the definition in [46], the intrinsic coherence of subsystem $\mathcal{A}$ in terms of post measurement vectors is:

$$P_{\mathcal{A}}^{2}=\frac{(d\sum_{i,j=1}^{d}|\langle\psi_j|\psi_i\rangle|^2-1)}{(d-1)} \tag{53}$$

For the extreme cases (maximal coherence and zero coherence), we can see that the conditions are similar but opposite to that of maximally entangled states and separable states. When all the post measurement vectors, corresponding to the subsystem $\mathcal{A}$ are orthogonal and equal, $P_{\mathcal{A}}$ vanishes, and when they are parallel, $P_{\mathcal{A}}$ leads to maximum value one. Hence, intrinsic degree of coherence can be used to infer about separability and entanglement of the state describing any two qudit system, where separable states and maximally entangled state will correspond to maximum and maximum intrinsic coherence for the subsystem respectively.

It has been observed [48] that for composite bi-partite and tri-partite pure states, the quantum correlation measure I-concurrence, completes the complementary relation between the local properties of the subsystem (such as, coherence and predictability) and the global entanglement of the composite system. It would be interesting to further analyse the complementary characters of coherence and entanglement for the mixed states using the present approach. It would also be interesting to study the same for qutrit and higher dimensional systems, as it has strong implications in the multi-slit interferometry [49].

5. Conclusion

In conclusion, the wedge product formalism of I-concurrence naturally leads to the geometrical configurations corresponding to the separable and maximally entangled states. The geometrical condition for maximal entanglement is shown to be orthogonality and equality of post-measurement state vectors corresponding to bi-partitions of the multiparty system. These conditions lead to the general form of the maximally entangled two qubit state, GHZ and GHZ like states. In the case of a three qubit system, the vanishing of concurrence corresponding to any two bi-partition is shown to be sufficient for tripartite separability. We derived the general conditions for maximally entangled states of two qudits, which lead to the generalised Bell states. A method is presented to construct the density matrices in terms of inner product of post measurement vectors for an arbitrary finite dimensional system. This approach provides insight on the geometrical conditions for absolute maximal entanglement and intrinsic coherence, relating the intrinsic degree of coherence with separability and entanglement of states for a two qudit system. The present formalism can be extended to the intrapartite entanglement, where one deals with entanglement between various degrees of freedom for a single particle. However, the post-measurement vectors in such case need careful consideration, as they belong to different degrees of freedom of the same particle and independent unitary transformation needs careful study for the individual degrees of freedom.

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