Random Stuff 3

28 August 2024 19:01

Q) A number n in base 10 when written in base b is 503 and when written in $b+2$ is 305. Find the product of digits of n.

Ans:—

$$n = 3\times(b+2)^2 + 0\times(b+2)^1 + 5\times(b+2)^0$$

$$n = 5\times b^2 + 0\times b^1 + 3\times b^0$$

$$\Rightarrow 3(b+2)^2 + 5 = 5b^2 + 3$$

$$\Rightarrow 3b^2 + 12b + 12 + 5 = 5b^2 + 3 \Rightarrow 2b^2 - 12b - 14 = 0$$

$$\Rightarrow b^2 - 6b - 7 = 0$$

$$\Rightarrow (b-7)(b+1) = 0 \Rightarrow b = 7$$

$n = 248 \Rightarrow$ product of digits is 64

Q) $x > 0$ and $[x] + \left[\frac{1}{x}\right] = 2$. Find range of x.

Ans:— $x + \frac{1}{x} - \{x\} - \left\{\frac{1}{x}\right\} = 2$

as $\{x\} + \left\{\frac{1}{x}\right\} < 2$

$x + \frac{1}{x} \geq 4$

$\Rightarrow x^2 + 1 \geq 4x$

$\Rightarrow x^2 - 4x + 1 \geq 0$

$\Rightarrow x \geq 2+\sqrt{3},\ x \leq 2-\sqrt{3}$

$\Rightarrow$ this part will not be in solution

$\frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

So, $\Rightarrow x \in (2-\sqrt{3}, 2+\sqrt{3})$

Case 1:— $x = 0 + \varepsilon,\quad 0 < \varepsilon < 1$

$\Rightarrow [x] + \left[\frac{1}{x}\right] = 2 \Rightarrow 0 + \left[\frac{1}{\varepsilon}\right] = 2 \Rightarrow 2 \leq \frac{1}{\varepsilon} < 3$

$\Rightarrow \frac{1}{2} \geq \varepsilon > \frac{1}{3}$

$$\Rightarrow \boxed{x \in \left(\frac{1}{3}, \frac{1}{2}\right]}$$

Case 2:—

$x = 1 + \varepsilon,$

$[x] + \left[\frac{1}{x}\right] = 1 + \left[\frac{1}{1+\varepsilon}\right] = 2 \Rightarrow \left[\frac{1}{1+\varepsilon}\right] = 1 \Rightarrow 1 \leq \frac{1}{1+\varepsilon} < 2$

$\Rightarrow 1 \geq 1+\varepsilon > \frac{1}{2}$

$\Rightarrow$ for $\varepsilon = 0$

$\lfloor x \rfloor + \lfloor x \rfloor \quad \lfloor 1+\varepsilon \rfloor \qquad \Rightarrow 1 \leq 1+\varepsilon < 2$

$\boxed{x \in \{1\}}$ $\rightarrow$ for $\varepsilon = 0$ $\quad \Leftarrow \quad 1 > \varepsilon \geq 0$

Case 3:- $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \varepsilon < 1$

$x = 2 + \varepsilon$,

$$[x] + \left[\frac{1}{x}\right] = 2 + \left[\frac{1}{2+\varepsilon}\right] = 2 \Rightarrow \left[\frac{1}{2+\varepsilon}\right] = 0 \Rightarrow 2+\varepsilon > 1$$

$\Rightarrow x \in [2, 3)$

Case 4:- $\quad x = 3 + \varepsilon, \quad [x] + \left[\frac{1}{x}\right] = 3 + 0 = 3 \neq 2$ ✗

So we get the final range as $x \in \left(\frac{1}{3}, \frac{1}{2}\right] \cup \{1\} \cup [2, 3)$

Q> Let x, y, z be positive real numbers. Show that

$$x^4 + y^4 + z^2 \geq \sqrt{8}\, xyz$$

Ans:- $$\frac{x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2}}{4} \geq \sqrt[4]{\frac{x^4 y^4 z^4}{4}} \Rightarrow x^4 + y^4 + z^2 \geq \frac{4}{\sqrt{2}} xyz = \sqrt{8}\, xyz$$

Q> For any real number $x, y > 1$ prove that $\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8$.

Ans:- $$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 2 \frac{xy}{\sqrt{(x-1)(y-1)}} \Rightarrow 2 \frac{x}{\sqrt{x-1}} \frac{y}{\sqrt{y-1}} \geq 2 \cdot 2 \cdot 2 \geq 8$$

$=$ holds when $x = y = 2$

$(x-2)^2 = x^2 - 4x + 4 \geq 0$

$\Rightarrow x^2 \geq 4(x-1)$

$\Rightarrow x \geq 2\sqrt{x-1}$

$\Rightarrow \frac{x}{\sqrt{x-1}} \geq 2$

Similarly $\frac{y}{\sqrt{y-1}} \geq 2$

Q> Let $a, b \in \mathbb{R}$, $a \neq 0$. Show that, $a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq \sqrt{3}$