Random Stuff 3

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Q) A number n in base 10 when written in base b is 503 and when written in $b+2$ is 305. Find the product of digits of n.

Ans:-

$$n = 3\times(b+2)^2 + 0\times(b+2)^1 + 5\times(b+2)^0$$

$$n = 5\times b^2 + 0\times b^1 + 3\times b^0$$

$$\Rightarrow 3(b+2)^2 + 5 = 5b^2 + 3$$

$$\Rightarrow 3b^2 + 12b + 12 + 5 = 5b^2 + 3 \Rightarrow 2b^2 - 12b - 14 = 0$$

$$\Rightarrow b^2 - 6b - 7 = 0$$

$$\Rightarrow (b-7)(b+1) = 0 \Rightarrow b = 7$$

$n = 248 \Rightarrow$ product of digits is 64

Q) $x > 0$ and $[x] + \left[\frac{1}{x}\right] = 2$. Find range of x.

Ans:- $x + \frac{1}{x} - \{x\} - \left\{\frac{1}{x}\right\} = 2$

as $\{x\} + \left\{\frac{1}{x}\right\} < 2$

$$x + \frac{1}{x} \geq 4$$

$$\Rightarrow x^2 + 1 \geq 4x$$

$$\Rightarrow x^2 - 4x + 1 \geq 0$$

$$\Rightarrow x \geq 2+\sqrt{3},\ x \leq 2-\sqrt{3}$$

$\Rightarrow$ this part will not be in solution

$$\frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

So, $\Rightarrow x \in (2-\sqrt{3}, 2+\sqrt{3})$

Case 1:-

$x = 0 + \varepsilon$, $\quad 0 < \varepsilon < 1$

$$\Rightarrow [x] + \left[\frac{1}{x}\right] = 2 \Rightarrow 0 + \left[\frac{1}{\varepsilon}\right] = 2 \Rightarrow 2 \leq \frac{1}{\varepsilon} < 3$$

$$\Rightarrow \frac{1}{2} \geq \varepsilon > \frac{1}{3}$$

$$\Rightarrow \boxed{x \in \left(\frac{1}{3}, \frac{1}{2}\right]}$$

Case 2:-

$x = 1 + \varepsilon$,

$$[x] + \left[\frac{1}{x}\right] = 1 + \left[\frac{1}{1+\varepsilon}\right] = 2 \Rightarrow \left[\frac{1}{1+\varepsilon}\right] = 1 \Rightarrow 1 \leq \frac{1}{1+\varepsilon} < 2$$

$$\Rightarrow 1 \geq 1+\varepsilon > \frac{1}{2}$$

$\Rightarrow$ for $\varepsilon = 0$

$\boxed{x \in \{1\}}$

$\boxed{x \in \{1\}}$ $\xrightarrow{\text{for } \varepsilon = 0}$ $\Leftarrow \quad 1 > \varepsilon \geq 0$

Case 3:-

$0 \leq \varepsilon < 1$

$x = 2 + \varepsilon$,

$[x] + \left[\frac{1}{x}\right] = 2 + \left[\frac{1}{2+\varepsilon}\right] = 2 \Rightarrow \left[\frac{1}{2+\varepsilon}\right] = 0 \Rightarrow 2 + \varepsilon > 1$

$\Rightarrow x \in [2, 3)$

Case 4:- $x = 3 + \varepsilon$, $[x] + \left[\frac{1}{x}\right] = 3 + 0 = 3 \neq 2$ ×

So we get the final range as $x \in \left(\frac{1}{3}, \frac{1}{2}\right] \cup \{1\} \cup [2, 3)$

Q) Let x, y, z be positive real numbers. Show that

$$x^4 + y^4 + z^2 \geq \sqrt{8}\, xyz$$

Ans:- $\frac{x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2}}{4} \geq \sqrt[4]{\frac{x^4 y^4 z^4}{4}} \Rightarrow x^4 + y^4 + z^2 \geq \frac{4}{\sqrt{2}} xyz = \sqrt{8}\, xyz$

Q) For any real number $x, y > 1$ prove that $\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8$.

Ans:- $\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 2 \frac{xy}{\sqrt{(x-1)(y-1)}} \Rightarrow 2 \frac{x}{\sqrt{x-1}} \frac{y}{\sqrt{y-1}} \geq 2 \cdot 2 \cdot 2 \geq 8$

= holds when $x = y = 2$

$(x-2)^2 = x^2 - 4x + 4 \geq 0$

$\Rightarrow x^2 \geq 4(x-1)$

$\Rightarrow x \geq 2\sqrt{x-1}$

$\Rightarrow \frac{x}{\sqrt{x-1}} \geq 2$

Similarly

$\frac{y}{\sqrt{y-1}} \geq 2$

Q) Let $a, b \in \mathbb{R}$, $a \neq 0$. Show that, $a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq \sqrt{3}$

Ans:- $a^2 + \left(b + \frac{1}{2a}\right)^2 + \frac{3}{4a^2}$

$\left(b + \frac{1}{\ }\right)^2 + a^2 + \frac{3}{\ } = \left(b + \frac{1}{\ }\right)^2 + \left(a - \sqrt{\frac{3}{4}}\frac{1}{a}\right)^2 + 2\sqrt{\frac{3}{4}}\frac{1}{a}(a)$

Ans:-

$$a^2 + \left(b + \frac{1}{2a}\right) + \frac{\;}{4a^2}$$

$$= \left(b + \frac{1}{2a}\right)^2 + a^2 + \frac{3}{4a^2} = \left(b + \frac{1}{2a}\right)^2 + \left(a - \left(\sqrt{\frac{3}{4}}\right)\frac{1}{a}\right)^2 + 2\left(\sqrt{\frac{3}{4}}\right)\frac{1}{a}(a)$$

$$= \underbrace{\left(b + \frac{1}{2a}\right)^2}_{\to 0} + \underbrace{\left(a - \sqrt{\frac{3}{4a}}\right)^2}_{\to 0} + \sqrt{3} \geq \sqrt{3}$$

minimum