Problem 1

(15 points) Compute the Fourier transform of each of the following signals:

(a)

$$x(t) = [e^{-\alpha t} \cos{(\omega_0 t)}]u(t), \alpha > 0$$

(b)

$$x(t) = e^{-3|t|} sin(2t)$$

(c)

$$x(t) = \begin{cases} 1 + \cos \pi t, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

.

Solution

(a) (5 points)

$$x(t) = e^{-\alpha t} \cos{(\omega_0 t)}\, u(t) = \frac{1}{2} e^{-\alpha t} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-\alpha t} e^{-j\omega_0 t} u(t)$$

$$X(j\omega) = \frac{1}{2\,(\alpha - j\omega_0 + j\omega)} + \frac{1}{2\,(\alpha + j\omega_0 + j\omega)}$$

(b) (5 points)

$$x(t) = e^{-3t} \sin(2t) u(t) + e^{3t} \sin(2t) u(-t)$$

$$x_1(t) = e^{-3t} \sin(2t) u(t) \overset{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}$$

$$x_2(t) = e^{3t} \sin(2t) u(-t) = -x_1(-t) \overset{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{1/2j}{3 - j2 - j\omega} - \frac{1/2j}{3 + j2 - j\omega}$$

$$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}$$

(c) (5 points)

$$X(j\omega) = \frac{2 \sin \omega}{\omega} + \int_{-1}^{1} \cos(\pi t) e^{-j\omega t} dt = \frac{2 \sin \omega}{\omega} + \frac{\sin \omega}{\pi - \omega} - \frac{\sin \omega}{\pi + \omega}$$

Problem 2

(15 points) Frequency response of a Linear Time-Invariant system is shown below:

$$H(\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

(a) Write out the differential equation that associates system input x(t) with output y(t).

(b) Determine the impulse response $h(t)$ of the system.

(c) Determine the output of the system with input $x(t) = e^{-4t}u(t)$.

Solution

(a) (5 points)

$\frac{dy^2(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$

(b) (5 points)

$H(w) = \frac{2}{j\omega+2} + \frac{-1}{j\omega+3}$

$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$

(c) (5 points)

$X(\omega) = \frac{1}{j\omega+4}$

$Y(\omega) = H(\omega)X(\omega) = \frac{1}{6-\omega^2+5j\omega} = \frac{1}{j\omega+2} - \frac{1}{j\omega+3}$

$y(t) = e^{-2t}u(t) - e^{-3t}u(t)$

Problem 3

(20 points) Ideal low pass filter frequency response is shown. Draw the spectrum of the output signal when input is the following function.

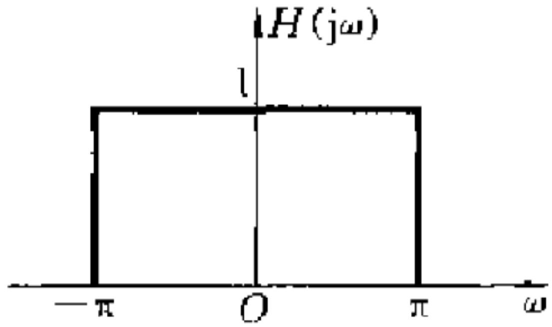


Figure 1: Ideal Low Pass Filter

(a)

$$f(t) = \frac{\sin(\pi t)}{\pi t}$$

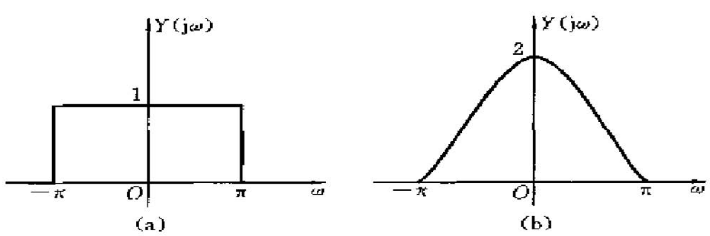
(b)

$$f(t) = \begin{cases} 1, |t| \leq 1 \\ 0, |t| > 1 \end{cases}$$

Solution

(a) (10 Points)
If $f(t) = \frac{\sin(\pi t)}{\pi t}$, $F(j\omega) = H(j\omega)$ so the spectrum of output signal $Y(j\omega) = H(j\omega)H(j\omega) = H(j\omega)$

(b) (10 Points)
$F(j\omega) = \frac{2Sin(\omega)}{\omega}$, so $Y(j\omega) = \frac{2Sin(\omega)}{\omega}H(j\omega)$. Results shown in figure below



Problem 4

(20 points) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{+\infty} x(\tau)z(t-\tau)d\tau - x(t)$$

where $z(t) = e^{-t}u(t) + 3\delta(t)$

(a) Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system.

(b) Determine the impulse response of the system.

Solution

(a) (10 Points)

According to the given differential equation, $j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega)$.

Therefore, $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega)-1}{j\omega+10} = \frac{2j\omega+3}{(j\omega+10)(j\omega+1)} = \frac{1}{9(j\omega+1)} + \frac{17}{9(j\omega+10)}$

(b) (10 Points)

$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t)$

Problem 5

(30 points) Let $x(t)$ and $y(t)$ be two real signals. Then the cross-correlation function of $x(t)$ and $y(t)$ is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)d\tau$$

Similarly, we can define $\phi_{yx}(t), \phi_{xx}(t)$, and $\phi_{yy}(t)$. The last two of these are called the autocorrelation functions of the signals $x(t)$ and $y(t)$, respectively. Let $\Phi_{xy}(j\omega)$, $\Phi_{yx}(j\omega)$, $\Phi_{xx}(j\omega)$, and $\Phi_{yy}(j\omega)$ denote the Fourier transforms of $\phi_{xy}(t), \phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$, respectively.

(a) Determine the relationship between $\Phi_{xy}(j\omega)$ and $\Phi_{yx}(j\omega)$.
Hint: You may need to prove $\phi_{yx}(t) = \phi_{xy}(-t)$ firstly.

(b) Find an expression for $\Phi_{xy}(j\omega)$ in terms of $X(j\omega)$ and $Y(j\omega)$.

(c) Show that $\Phi_{xx}(j\omega)$ is real and nonnegative for every ω.

(d) Suppose now that $x(t)$ is the input to an LTI system with a real-valued impulse response and with frequency response $H(j\omega)$ and that $y(t)$ is the output. Find expressions for $\Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ in terms of $\Phi_{xx}(j\omega)$ and $H(j\omega)$.

(e) Let $x(t)$ be as is illustrated in Figure 2, and let the LTI system impulse response be $h(t) = e^{-at}u(t), a > 0$. Compute $\Phi_{xx}(j\omega), \Phi_{xy}(j\omega)$, and $\Phi_{yy}(j\omega)$ using the results of parts (a)-(d).

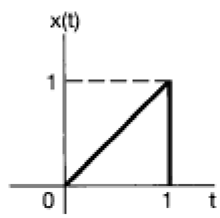


Figure 2: $x(t)$ in 5(e)

Solution

(a) (5 Points)

$$\phi_{xy}(t) = \phi_{yx}(-t)$$

$$\Phi_{xy}(j\omega) = \Phi_{yx}(-j\omega) \text{ or } \Phi_{xy}(j\omega) = \Phi_{yx}^{*}(j\omega)$$

(b) (5 Points)

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau = x(t) * y(-t)$$

$$\Phi_{xy}(j\omega) = X(j\omega)Y(-j\omega), \text{or } \Phi_{xy}(j\omega) = X(j\omega)Y^{*}(j\omega)$$

(c) (5 Points)

$$y(t) = x(t)$$

$$\Phi_{xx}(j\omega) = X(j\omega)X^{*}(j\omega) = |X(j\omega)|^2 \geq 0$$

(d) (5 Points)

$$\begin{aligned}\Phi_{xy}(j\omega) &= X(j\omega)Y^*(j\omega)\\ &= X(j\omega)[H(j\omega)X(j\omega)]^*\\ &= \Phi_{xx}(j\omega)H^*(j\omega)\end{aligned}$$

$$\begin{aligned}\Phi_{yy}(j\omega) &= Y(j\omega)Y^*(j\omega)\\ &= [H(j\omega)X(j\omega)][H(j\omega)X(j\omega)]^*\\ &= \Phi_{xx}(j\omega)|H(j\omega)|^2\end{aligned}$$

(e) (10 Points)

$$X(j\omega) = \frac{e^{-j\omega}-1}{\omega^2} - \frac{e^{-j\omega}}{j\omega}$$

$$H(j\omega) = \frac{1}{a+j\omega}$$

$$\begin{aligned}&\Phi_{xx}(j\omega) = |X(j\omega)|^2 = \tfrac{2-2\cos\omega}{\omega^4} - \tfrac{2\sin\omega}{\omega^3} + \tfrac{1}{\omega^2}\\ &\Phi_{xy}(j\omega) = \Phi_{xx}(j\omega)H^*(j\omega) = \left[\tfrac{2-2\cos\omega}{\omega^4} - \tfrac{2\sin\omega}{\omega^3} + \tfrac{1}{\omega^2}\right]\left[\tfrac{1}{a-j\omega}\right]\\ &\Phi_{yy}(j\omega) = \Phi_{xx}(j\omega)|H(j\omega)|^2 = \left[\tfrac{2-2\cos\omega}{\omega^4} - \tfrac{2\sin\omega}{\omega^3} + \tfrac{1}{\omega^2}\right]\left[\tfrac{1}{a^2+\omega^2}\right]\end{aligned}$$