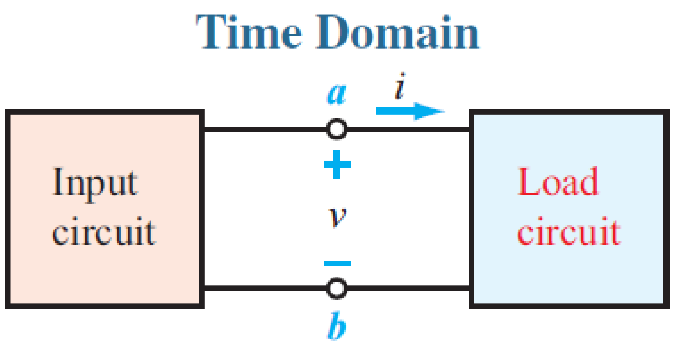
Lecture 9

- AC Power Calculation

Outline

- Instantaneous power
- Average power
- Apparent power
- Power Factor
- Complex power

AC Power in Time Domain: Instantaneous



$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

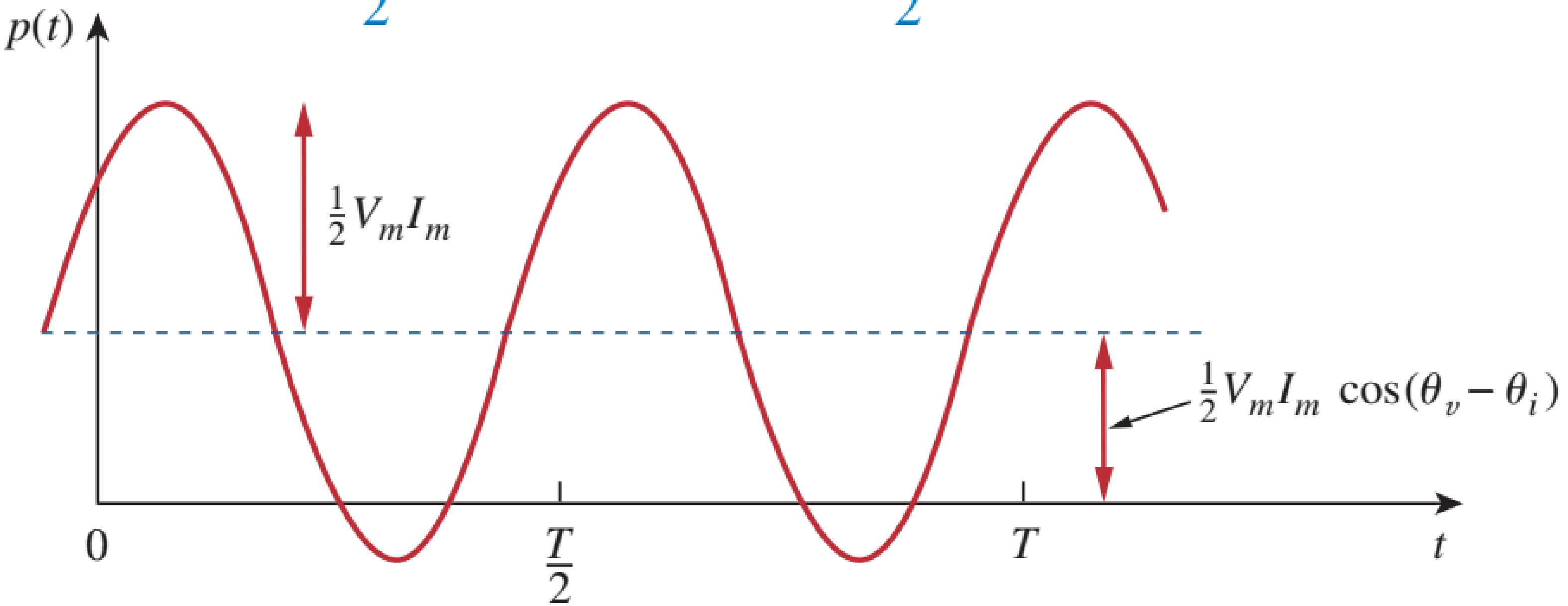
Instantaneous power:
power at any instant of time.

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

AC Power in Time Domain: Instantaneous

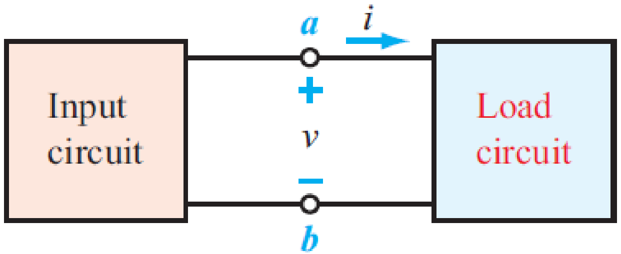
$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



Average Power *P* (*Capitalized*)

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

$$\boxed{p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)}$$



<u>Average (or real)</u> power (unit: watts)

The **average power**, in watts, is the average of the instantaneous power over one period.

$$P = \frac{1}{T} \int_0^T p(t)\, dt$$

Average Power *P (time domain)*

$$
\begin{aligned}
P &= \frac{1}{T}\int_0^T \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)\, dt \\
&\quad + \frac{1}{T}\int_0^T \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)\, dt \\
&= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T}\int_0^T dt \\
&\quad + \frac{1}{2} V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt
\end{aligned}
$$

$$
\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}
$$

Average Power *P (phasor domain)*

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

$$\mathbf{V} = V_m \angle \theta_v \text{ and } \mathbf{I} = I_m \angle \theta_i,$$

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle \theta_v - \theta_i$$

$$= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

Two special cases for average power *P*

- For a purely resistive load R：

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R \quad \text{where } |\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$$

- For a purely reactive load:

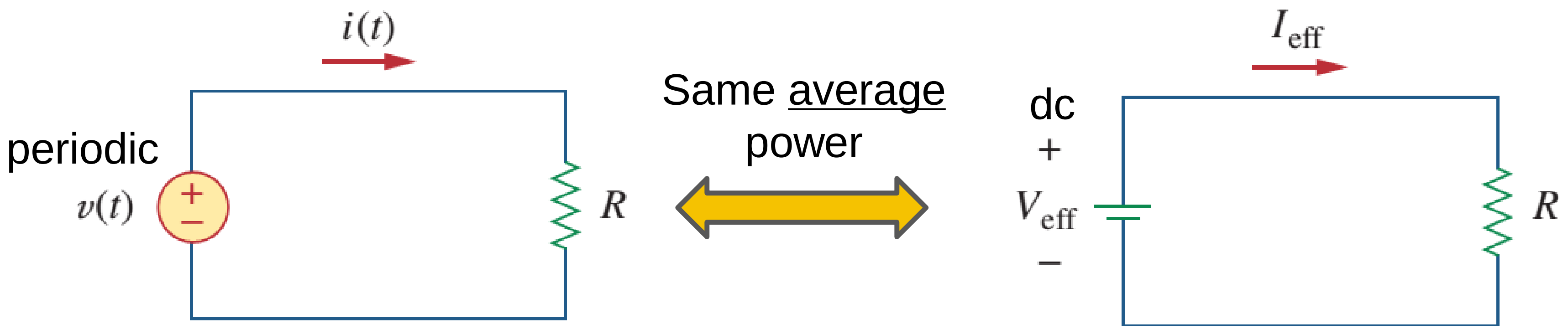
$$P = \frac{1}{2}V_m I_m \cos 90^\circ = 0$$

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Effective Value (RMS)

- For any periodic function $x(t)$ in general, its rms value is

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt}$$



$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \qquad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$

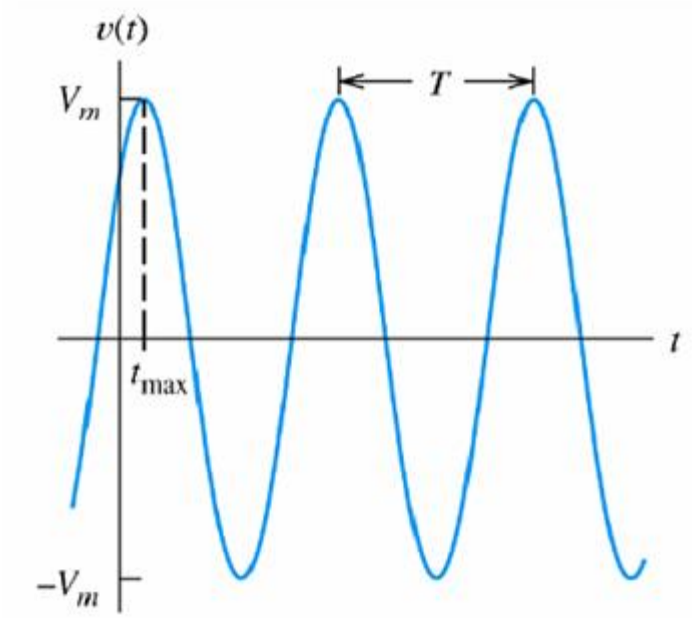
Example: RMS of a Sinusoidal

- The RMS value of $v(t) = V_m \cos(\omega t + \phi)$ is

$$V_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt}$$

$$= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cdot \cos^2(\omega t + \phi)dt}$$

$$= \frac{V_m}{\sqrt{2}}$$



Average Power

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$

$$= V_{\mathrm{rms}} I_{\mathrm{rms}} \cos(\theta_v - \theta_i)$$

Apparent Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= \boxed{V_{\text{rms}} I_{\text{rms}}} \cos(\theta_v - \theta_i)$$

$$\boxed{S = V_{rms} I_{rms}}$$

Unit: volt-amp (VA)

It seems <u>apparent</u> that the power should be the voltage-current product, by analogy with dc resistive circuits.

Power Factor

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \boxed{\cos(\theta_v - \theta_i)}$$

- The power factor

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)}$$

  - $(\theta_v - \theta_i)$ is called <u>power factor angle</u>.
    - >0 means a *lagging* pf (current lags voltage)
    - <0 means a *leading* pf (current leads voltage)
  - pf ranges from 0 to 1.

Power Factor-2

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \boxed{\cos(\theta_v - \theta_i)}$$

- The power factor

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

  - $(\theta_v - \theta_i)$ is called power factor angle.
  - $(\theta_v - \theta_i)$ is equal to the angle of the load impedance

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i$$

Also $$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i$$

Outline

- Instantaneous power
- Average power
- Apparent power
- Power Factor
- **Complex power**

Complex Power

$$v(t) = V_m \cos(\omega t + \theta_v) \Longrightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Longrightarrow \mathbf{I} = I_m \angle \theta_i$$

We observe that:

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i)$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + j\frac{1}{2} V_m I_m \sin(\theta_v - \theta_i)$$

- Define a *single* power metric

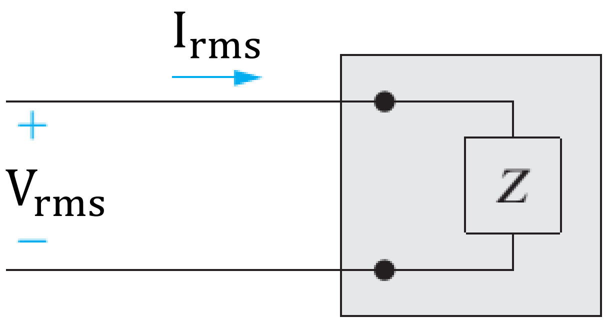
$$\boxed{\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\mathrm{rms}}\mathbf{I}_{\mathrm{rms}}^* = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = P + jQ}$$

Unit: volt-amp (VA)

Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

Another Way to Calculate Complex Power

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*}$$



$$\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}} Z$$

$$\begin{aligned}\mathbf{S} &= \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* \\ &= \mathbf{V}_{\text{rms}} \left(\frac{\mathbf{V}_{\text{rms}}}{Z}\right)^* \\ &= \frac{|\mathbf{V}_{\text{rms}}|^2}{Z^*}\end{aligned}$$

$$\begin{aligned}\mathbf{S} &= \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* \\ &= \mathbf{I}_{\text{rms}} Z \mathbf{I}_{\text{rms}}^* \\ &= |\mathbf{I}_{\text{rms}}|^2 Z \\ &= |\mathbf{I}_{\text{rms}}|^2 (R + jX) \\ &= |\mathbf{I}_{\text{rms}}|^2 R + j|\mathbf{I}_{\text{rms}}|^2 X \\ &= I_{rms}^2 R + j I_{rms}^2 X\end{aligned}$$

$$\boxed{\begin{aligned}P &= \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \\ Q &= \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X\end{aligned}}$$

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\mathrm{rms}}\mathbf{I}_{\mathrm{rms}}^* = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = P + jQ$$

$$P = \mathrm{Re}(\mathbf{S}) = I_{\mathrm{rms}}^2 R$$

$$Q = \mathrm{Im}(\mathbf{S}) = I_{\mathrm{rms}}^2 X$$

- Average (or real) power

$$P = \mathrm{Re}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: Watts

- Reactive power

$$Q = \mathrm{Im}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: Volt Amperes Reactive (VARs)

- Apparent power

$$s = |\mathbf{S}| = \frac{1}{2}V_m I_m = V_{rms}I_{rms}$$

Unit: volt-amp (VA)

$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$

$$= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\underline{/\theta_v - \theta_i}$$

$$\text{Apparent Power} = S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$

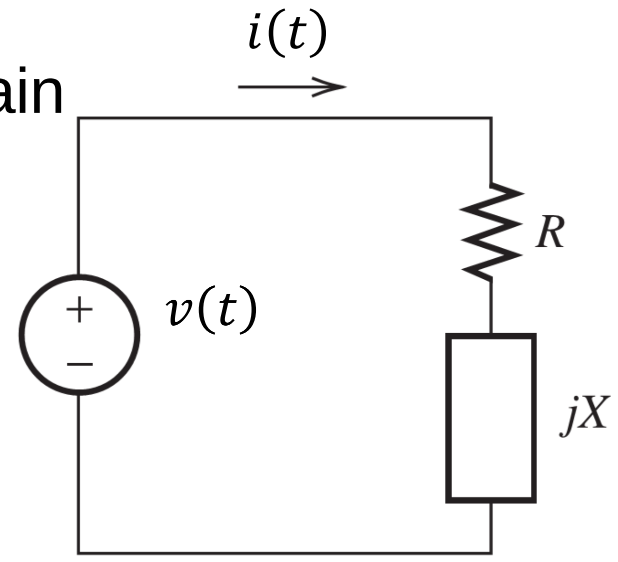
$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$

$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

Reactive Power *Q*

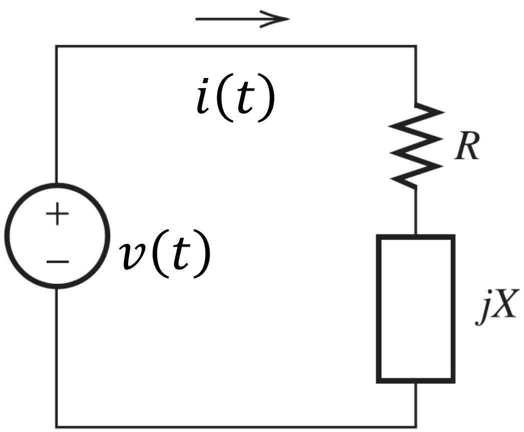
Let us look at Instantaneous power again



$$p(t) = v(t)i(t)$$
$$p(t) = pR(t)+p_X(t)$$
$$p_R(t) =$$
$$p_X(t) =$$

Reactive Power *Q*: Peak Exchanged Power

- Definition: The peak instantaneous power associated with the energy storage elements contained in a general load.



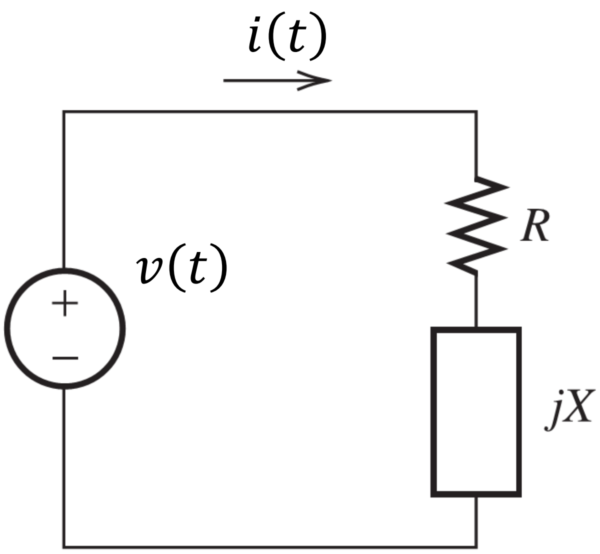
$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

$$Q = \begin{cases} 0 & \text{for a resistive } (\theta_v - \theta_i = 0^\circ) \\ \frac{1}{2} V_m I_m & \text{for inductive } (\theta_v - \theta_i = 90^\circ) \\ -\frac{1}{2} V_m I_m & \text{for capacitive } (\theta_v - \theta_i = -90^\circ) \end{cases}$$

- Reactive power is still of concern to power-system engineers
  - Transmission lines/transformers/fuses et al. must be capable of withstanding the current associated with reactive power.

Example

- Find the average power and reactive power absorbed by an impedance $Z = 30 - j70\Omega$, when a voltage $\mathbf{V} = 120\angle 0^{\circ}$ is applied across it.



$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{120\angle 0^{\circ}}{30 - j70} = \frac{120\angle 0^{\circ}}{76.16\angle -66.8^{\circ}}$$
$$= 1.576\angle 66.8^{\circ}\ \text{A}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = 37.24\text{W}$$

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = -86.91\text{VAR}$$

Power Triangle

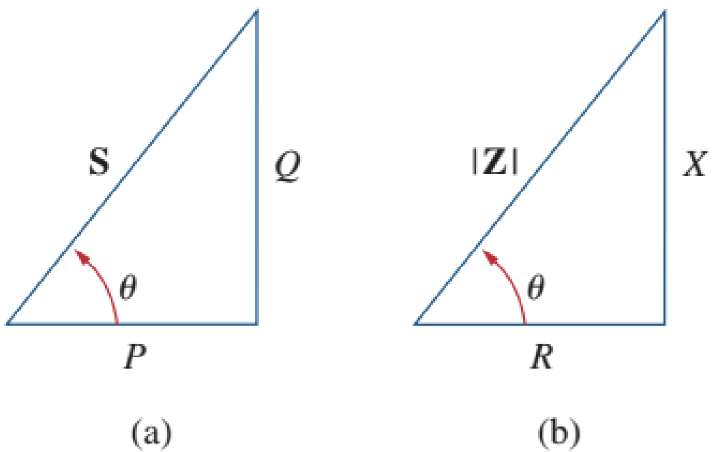


Figure 11.21
(a) Power triangle, (b) impedance triangle.

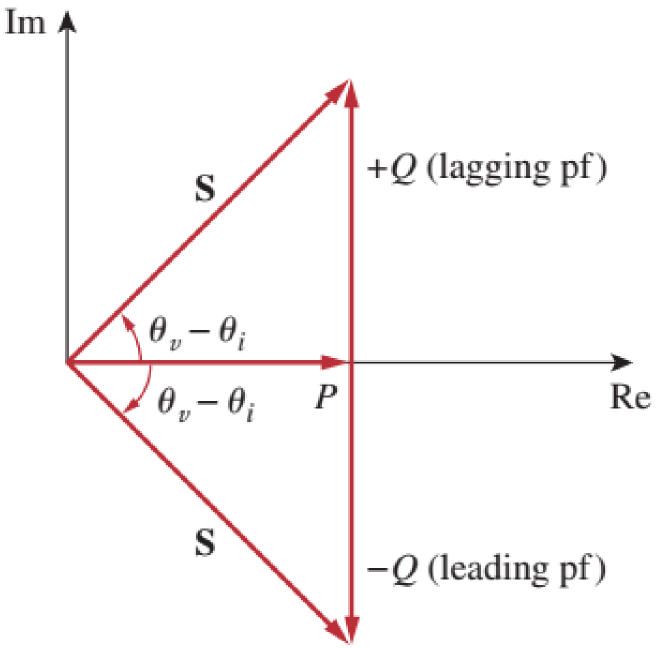
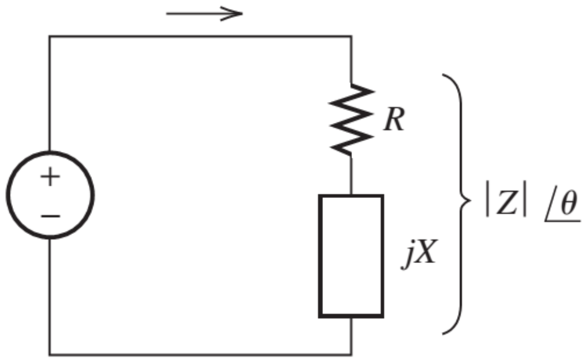


Figure 11.22
Power triangle.

Quantity	Units
Complex power	volt-amps
Average power	watts
Reactive power	var

Power Factor



Power factor leading and lagging relationships for a load $\mathbf{Z} = R + jX$.

Load Type	$\phi_z = \phi_v - \phi_i$	I-V Relationship	*pf*
Purely Resistive ($X = 0$)	$\phi_z = 0$	**I** in-phase with **V**	1
Inductive ($X > 0$)	$0 < \phi_z \le 90°$	**I** lags **V**	lagging
Purely Inductive ($X > 0$ and $R = 0$)	$\phi_z = 90°$	**I** lags **V** by 90°	lagging
Capacitive ($X < 0$)	$-90° \le \phi_z < 0$	**I** leads **V**	leading
Purely Capacitive ($X < 0$ and $R = 0$)	$\phi_z = -90°$	**I** leads **V** by 90°	leading

Example

- A series-connected load draws a current

$$i(t) = 4\cos(100\pi t + 10°)\text{A}$$

when the applied voltage is

$$v(t) = 120\cos(100\pi t - 20°)\text{V}$$

  - Find the apparent power and the power factor of the load.
  - Determine the values that form the series-connected load.

$$V_{\text{rms}} I_{\text{rms}} = 240 \text{ VA}$$

$$\text{pf} = \cos(\theta_v - \theta_i) = 0.866 \qquad \text{(leading)}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 25.98 - j15\ \Omega$$

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Exercise

- The voltage across a load is $v(t) = 60\cos(\omega t - 10°)\mathrm{V}$, and the current through the load is $i(t) = 1.5\cos(\omega t + 50°)$. Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = 45\underline{/-60°}\ \text{VA}$$

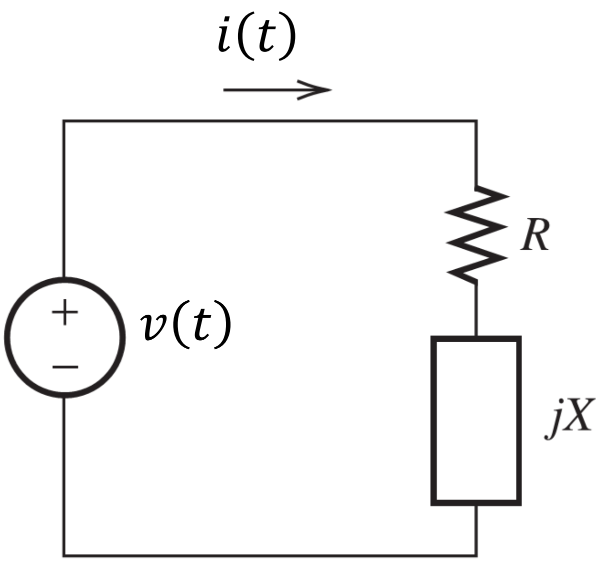
$$\text{pf} = 0.5\ \text{(leading)}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 40\underline{/-60°}\ \Omega$$

Quick Summary – Power Calculation

$$v(t) = V_m \cos(\omega t + \theta_v) \Longrightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Longrightarrow \mathbf{I} = I_m \angle \theta_i$$



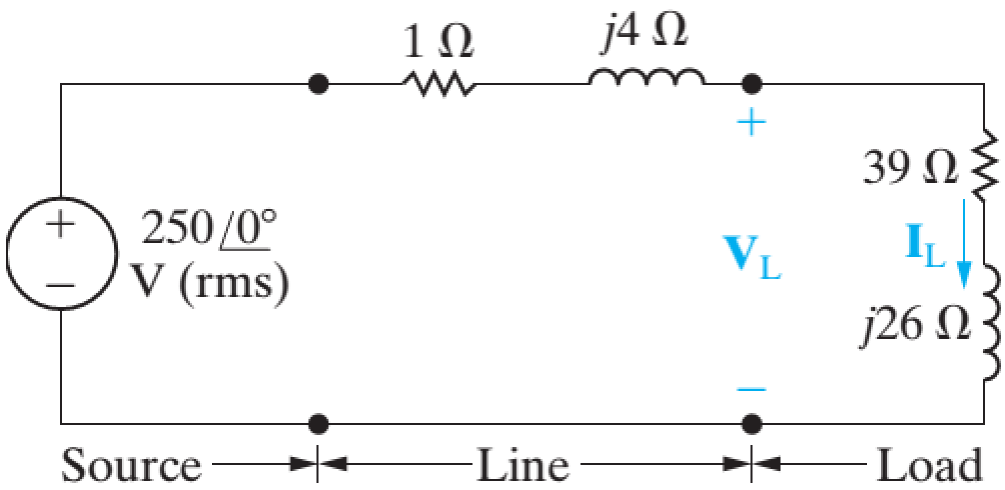
$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

$$S = \frac{1}{2} V_m I_m = V_{rms} I_{rms}$$

$$\boxed{\mathbf{S} = S \angle (\theta_v - \theta_i) = P + jQ}$$

Example



- Find $\mathbf{V}_L$ and $\mathbf{I}_L$.
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

$$\mathbf{I}_L = \frac{250\angle 0^\circ}{40+j30} = 4 - j3$$
$$= 5\angle -36.87^\circ \text{ (rms)}$$

$$\mathbf{V}_L = \mathbf{I}_L(39 + j26)$$
$$= 234 - j13$$
$$= 234.36\angle -3.18^\circ$$

Load:

$$\mathbf{V}_L\mathbf{I}_L^* = 975 + j650 \text{ VA}$$

Line:

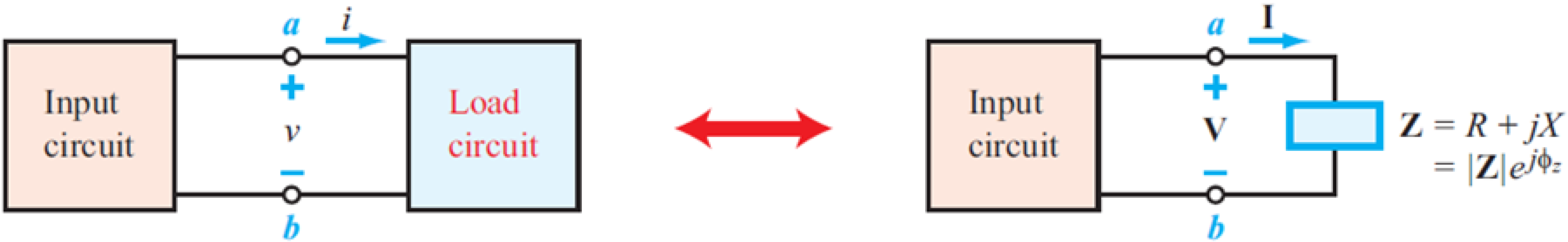
$$P = (5)^2(1) = 25 \text{ W}$$
$$Q = (5)^2(4) = 100 \text{ VAR}$$

Source:

$$250\angle 0^\circ \mathbf{I}_L^* = 1000 + j750 \text{ VA}$$

Complex Power

Time Domain | **Phasor Domain**



Time Domain		Phasor Domain
$v(t) = V_m \cos(\omega t + \phi_v)$	⟷	$\mathbf{V} = V_m e^{j\phi_v}$
$i(t) = I_m \cos(\omega t + \phi_i)$	⟷	$\mathbf{I} = I_m e^{j\phi_i}$
$V_{rms} = V_m/\sqrt{2}$	⟷	$\mathbf{V}_{rms} = V_{rms} e^{j\phi_v}$
$I_{rms} = I_m/\sqrt{2}$	⟷	$\mathbf{I}_{rms} = I_{rms} e^{j\phi_i}$

Complex Power

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{rms}\mathbf{I}_{rms}^* = P + jQ$$

Real Average Power

$$\begin{aligned} P &= \Re\mathfrak{e}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \cos(\phi_v - \phi_i) \\ &= I_{rms}^2 R \end{aligned}$$

Reactive Power

$$\begin{aligned} Q &= \Im\mathfrak{m}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \sin(\phi_v - \phi_i) \\ &= I_{rms}^2 X \end{aligned}$$

Apparent Power

$$\begin{aligned} S = |\mathbf{S}| &= \sqrt{P^2 + Q^2} \\ &= V_{rms} I_{rms} \\ &= I_{rms}^2 |\mathbf{Z}| \end{aligned}$$

$$\mathbf{S} = S e^{j\phi_s}$$

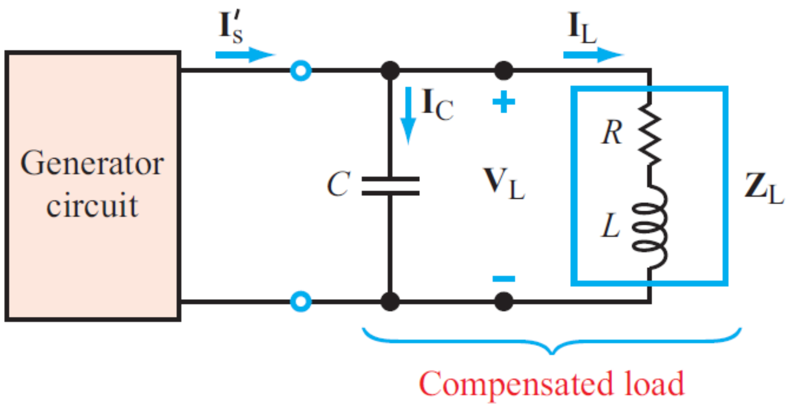
$$\phi_s = \phi_v - \phi_i = \phi_z$$

Power Factor

$$\begin{aligned} pf &= \frac{P}{S} \\ &= \cos(\phi_v - \phi_i) \\ &= \cos\phi_z \end{aligned}$$

Content for the Discussion Session

- Power factor correction



- Maximum power transfer