Discussion 6
Second-Order Circuits

Overview

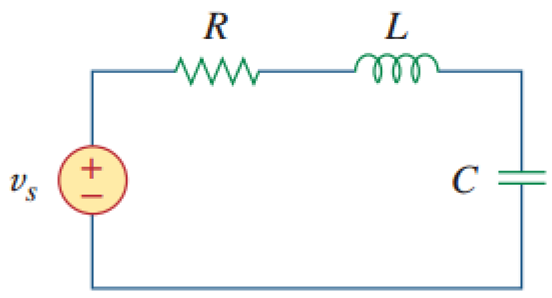
Second-Order Circuits

- Initial and final values
- Source-Free RLC Circuits
  - Series
  - Parallel
- Step Response of RLC Circuits
  - Series
  - Parallel
- Second-Order Op Amp Circuits

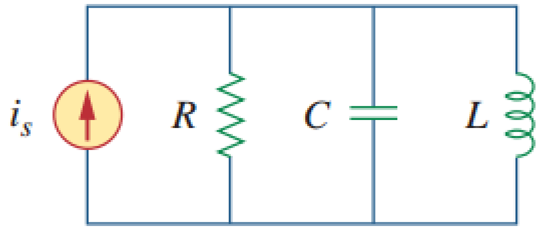
Application

- automobile ignition circuit

Second Order Circuits

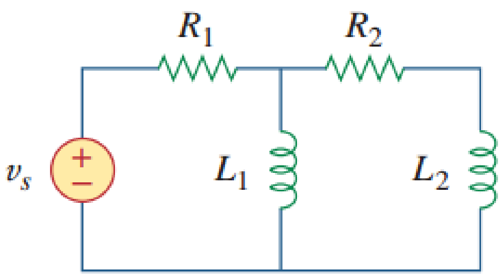


(a)

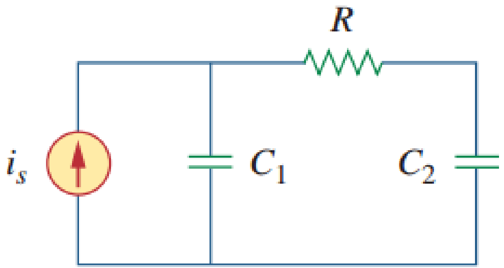


(b)

Different storage element



(c)



(d)

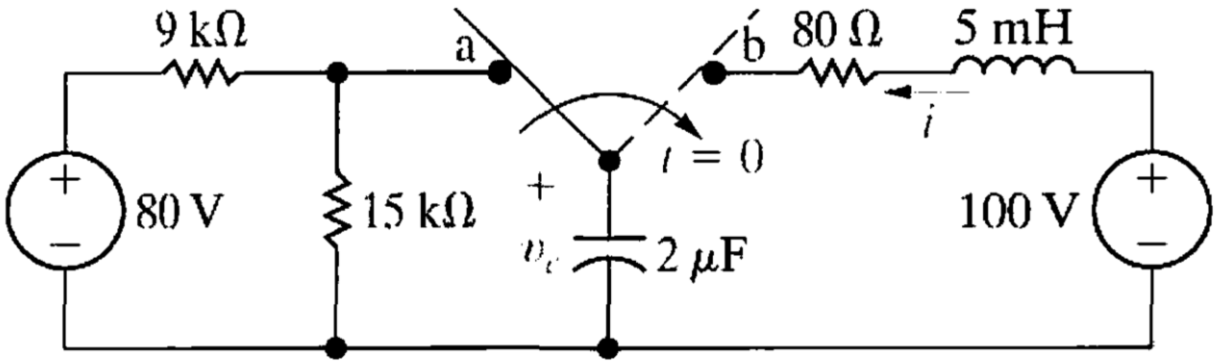
Same storage element

Second Order Circuits

- This time, though we will only consider **DC independent** sources
- Steps to find a solution $x(t)$:
  - **Step one:** Find initial and final values.
  - **Step two:** Extract the second-order differential equation out of the circuit.
  - **Step three:** Solve the second-order ODE and get the general solution of homogeneous equation -- $x_t(t)$.
  - **Step four:** $x(t) = x_{ss} + x_t(t)$ where x_{ss} is the final state value.
  - **Step five:** Substitute the initial values in and confirm the coefficients.

Initial and final values

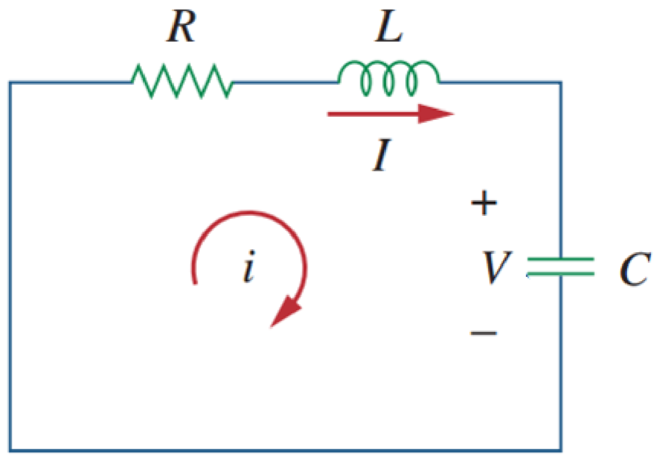
- Use methods of analysis you have learned before to find the **capacitor voltage** and the **inductor current** on **initial** and **final** states
- Regard the capacitor as **open circuit** and inductor as **short circuit**.
- **Example:**

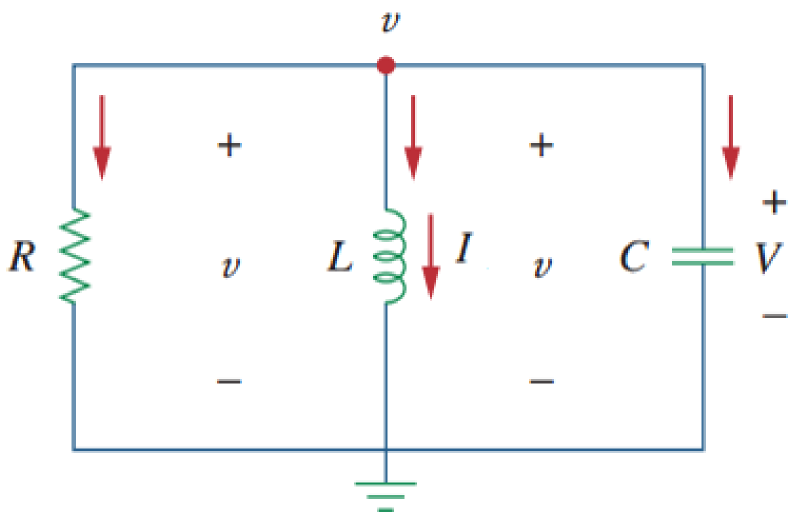


Initial state: $i_l(0^+) = i_l(0^-) = 0$
$v_c(0^+) = v(0^-) = 50\text{V}$

Final state: $i_l(\infty) = 0$
$v_c(\infty) = 100\text{V}$

Source Free Second Order Circuits





KVL: $Ri + L\frac{di}{dt} + v = 0$

But i= $C\frac{dv}{dt}$

So: $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$

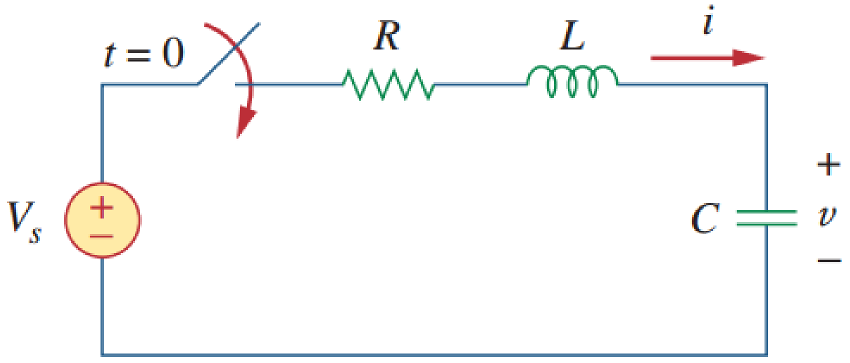
KCL: $\frac{v}{R} + C\frac{dv}{dt} + i = 0$

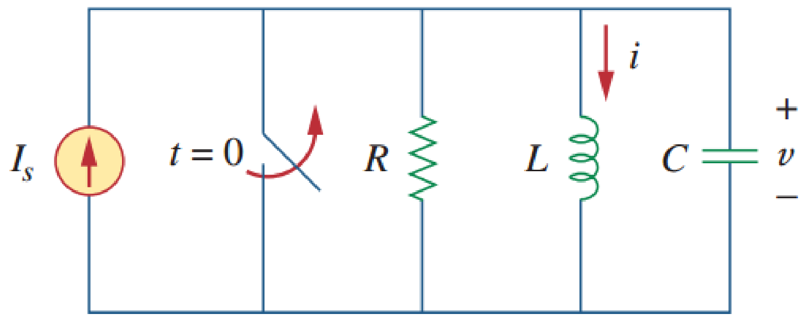
But v= $L\frac{di}{dt}$

So: $\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} v = 0$

Step Response of RLC Circuits

- When there is a DC source, the ODE becomes Inhomogeneous:





KVL: $Ri + L\frac{di}{dt} + v = V_s$

But $i = L\frac{dv}{dt}$

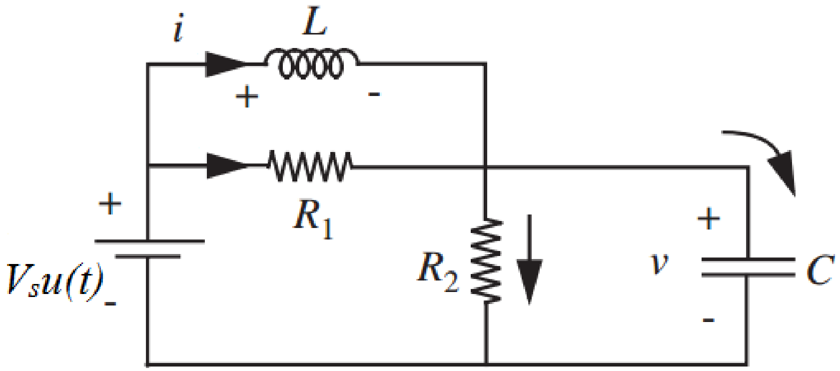
So: $\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{V_s}{LC}$

KCL: $\frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} vdt$

But $v = C\frac{di}{dt}$

So: $\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{I_s}{LC}$

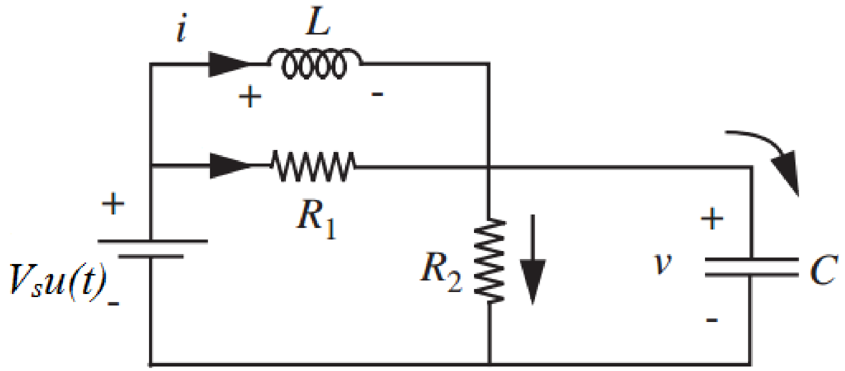
Example



Initial state: $i(0^+) = i(0^-) = 0$ $v(0^+) = v(0^-) = 0$

Final state: $i(\infty) = \frac{V_s}{R_2}$ $v(\infty) = V_s$

Example



Since we got a circuit with voltage source, so generate the differential equation in terms of **capacitor voltage *v***. Apply KVL we obtain:

$$v + L\frac{di}{dt} = V_s \quad ①$$

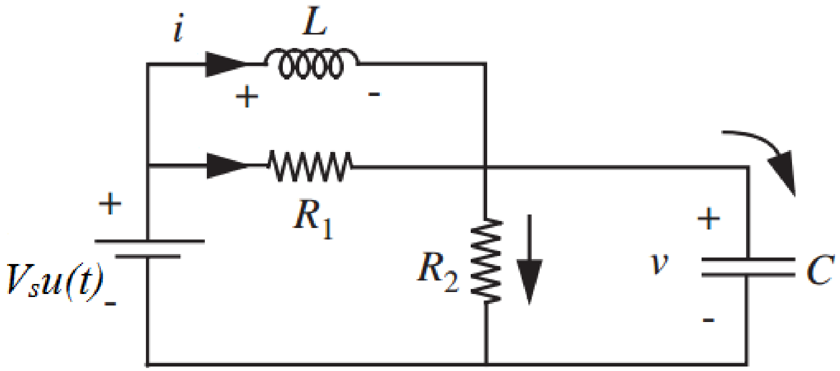
Then we rewrite i in terms of v:

$$i = C\frac{dv}{dt} + \frac{v}{R_2} - \frac{(V_s - v)}{R_1} \quad ②$$

Substitute ② into ① then

$$\frac{d^2v}{dt^2} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{1}{LC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

Example



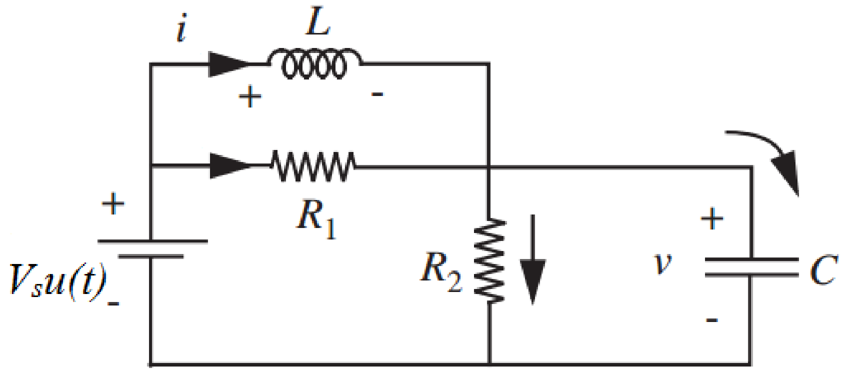
$$\frac{d^2v}{dt^2} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{1}{C}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

Firstly we solve the homogeneous equation ③ to find the general solution.

$$\frac{d^2v}{dt^2} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{1}{C}\frac{dv}{dt} + \frac{1}{LC}v = 0 \quad ③$$

Is the circuit **overdamped, critically damped or underdamped**?

Example



It depends on the coefficients $\boldsymbol{\alpha} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{1}{C}$ and $\boldsymbol{\omega_0} = \frac{1}{\sqrt{LC}}$

$$\boldsymbol{s_{1,2}} = -\boldsymbol{\alpha} \pm \sqrt{\boldsymbol{\alpha}^2 - \boldsymbol{\omega}_0^2}$$

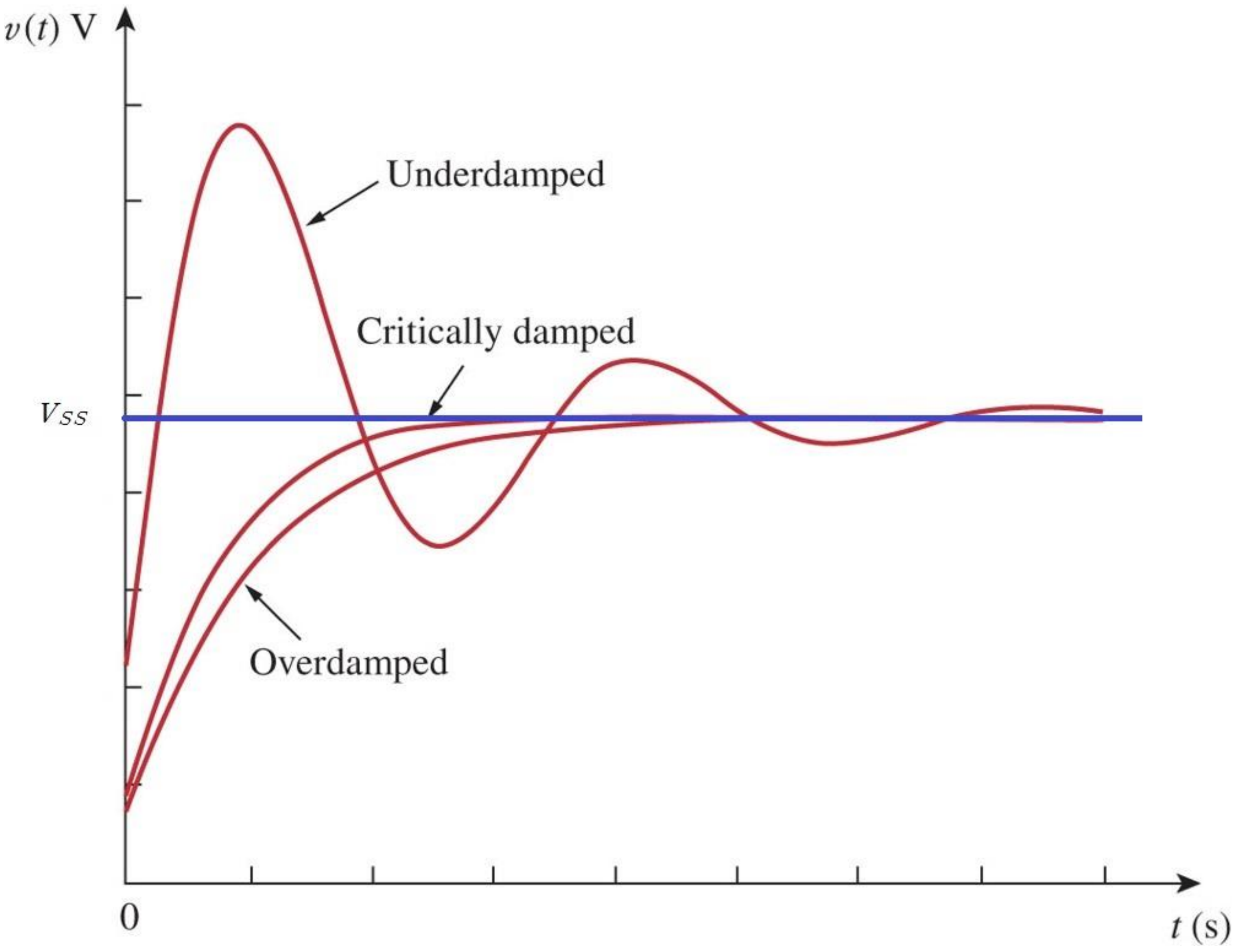
$\boldsymbol{\alpha} > \boldsymbol{\omega_0} \Rightarrow$ two distinct real solutions $\Rightarrow v_t(t) = Ae^{s_1 t} + Be^{s_2 t}$ overdamped

$\boldsymbol{\alpha} = \boldsymbol{\omega_0} \Rightarrow$ only one real root r $\Rightarrow v_t(t) = Ae^{s_1 t} + Bte^{s_2 t}$ critically damped

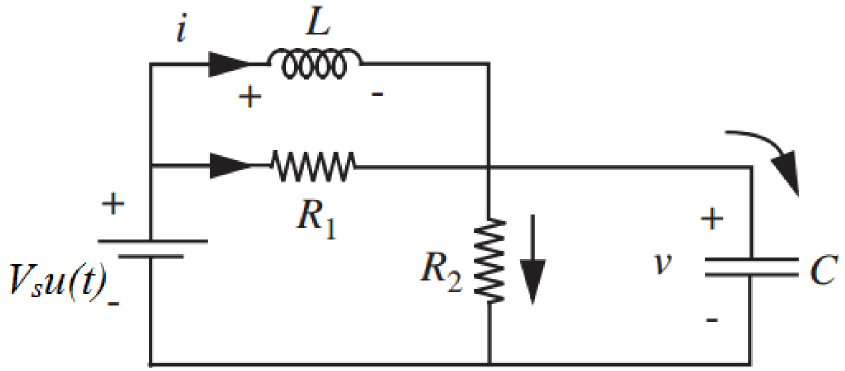
$\boldsymbol{\alpha} < \boldsymbol{\omega_0} \Rightarrow$ two complex conjugate roots $\Rightarrow v_t(t) = Ae^{s_1 t} + Be^{s_2 t}$ underdamped

$$\Rightarrow v_t(t) = Ae^{-\alpha t}\cos(\omega_d t) + Be^{-\alpha t}\sin(\omega_d t) \quad \text{where } \omega_d = \sqrt{\boldsymbol{\omega}_0^2 - \boldsymbol{\alpha}^2}$$

Example



Example



We know that $v(\infty) = V_s$

So

$$v(t) = v_s + v_t(t)$$

Then substitute the initial values in and confirm the coefficients A and B.

$$i(0^+) = i(0^-) = 0 \qquad v(0^+) = v(0^-) = 0$$

Second Order Circuits

- Please think: what if the voltage source is an AC source?
  - Is there still a steady state?
  - How to solve the ODE?

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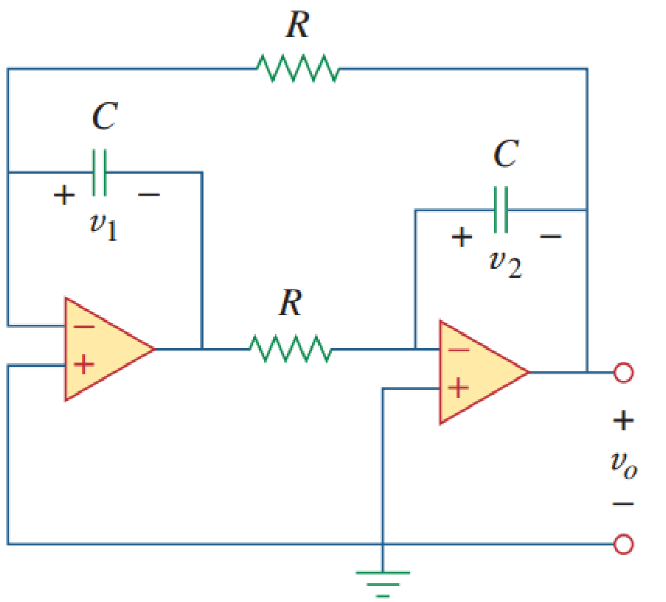
Second-Order Op Amp Circuits

Determine the differential equation in terms of v_o.



For the first op amp:

$$\frac{v_0 - 0}{R} = C\frac{dv_1}{dt}$$

For the second op amp:

$$\frac{-v_1 - 0}{R} = C\frac{dv_2}{dt}$$

But

$$v_0 = -v_2$$

So

$$\frac{d^2 v_o}{dt^2} - \frac{1}{R^2C^2}v_o = 0$$

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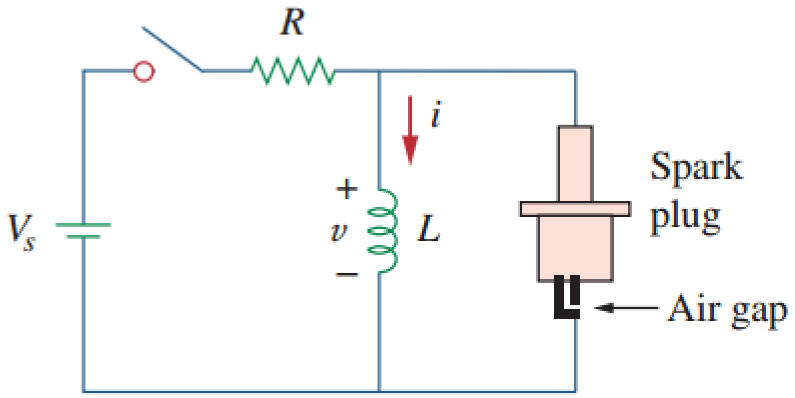
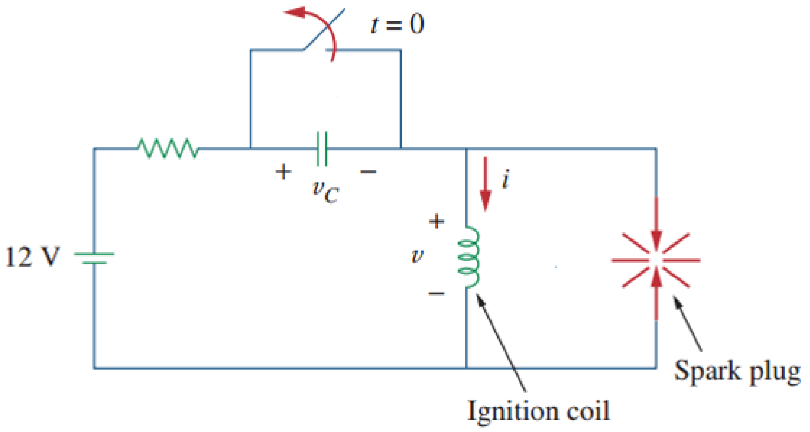
Application: Automobile Ignition System

- Mechanism： generate a voltage much higher than the voltage source through the transient response of the inductor.
- Please think:
  - What’s different between (1) and (2) ?
  - Which one is better?
  - Is the circuit (2) overdamped, critically damped or underdamped when we could get a much higher voltage?
  - How to choose the value of R, C and L?



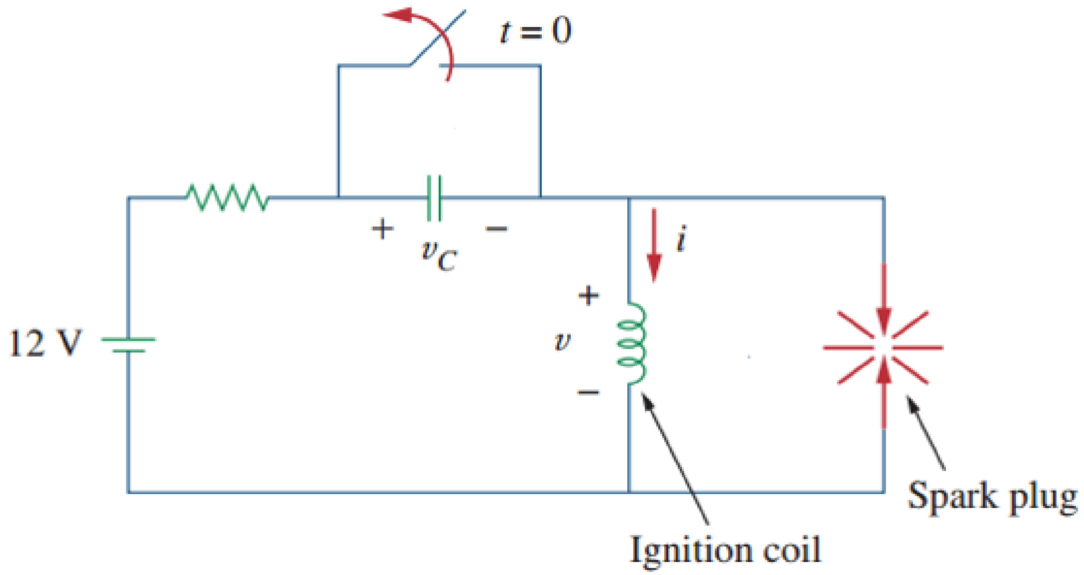
(1)



(2)

Application: Automobile Ignition System

- Regard the spark plug as open circuit. It becomes a series RLC circuit, in which:
- $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$
  - $\alpha = \frac{R}{2L}$
  - $\omega_0 = \frac{1}{\sqrt{LC}}$
  - $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
  - $i_{ss} = 0$
- $i(t) = Ae^{-\alpha t}\cos(\omega_d t) + Be^{-\alpha t}\sin(\omega_d t)$
- $v(t) = L\frac{di(t)}{dt}$



(2)

End