Signals and Systems Homework 8
Solutions

1. (15') Consider the signal
$$x(t) = e^{-5t}u(t-1)$$
and denote its Laplace transform by $X(s)$.

   (a) (5') Using $X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$, evaluate $X(s)$ and specify its region of convergence.

   (b) (10') Determine the values of the finite numbers A and t_0 such that the Laplace transform $G(s)$ of
$$g(t) = Ae^{-5t}u(-t-t_0)$$
has the same algebraic form as $X(s)$. What is the region of convergence corresponding to $G(s)$?

   Solution:

   (a)
$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-5t}u(t-1)e^{-st}dt \\ &= \int_{1}^{\infty} e^{-(5+s)t}dt \\ &= \frac{e^{-(5+s)}}{s+5} \end{aligned}$$
The ROC will be $\mathscr{R}\{s\} > -5$.

   (b) We can easily show that $g(t) = Ae^{-5t}u(-t-t_0)$ has the Laplace transform
$$G(s) = -\frac{Ae^{(s+5)t_0}}{s+5}$$
The ROC is specified as $\mathscr{R}\{s\} < -5$. Therefore, $A = -1$ and $t_0 = -1$.

2. (15') For the Laplace transform of
$$x(t) = \begin{cases} e^t \sin 2t, & t \leq 0 \\ 0, & t > 0 \end{cases}$$
indicate the location of its poles and its region of convergence.
Solution:

We know that
$$x_1(t) = -e^{-t}\sin(2t)u(t) \xleftrightarrow{\mathscr{L}} X_1(s) = -\frac{2}{(s+1)^2+2^2}, \quad \mathscr{R}\{s\} > -1$$
We also know that
$$x(t) = x_1(-t) \xleftrightarrow{\mathscr{L}} X(s) = X_1(-s)$$
The ROC of $X(s)$ is such that if s_0 was in the ROC of $X_1(s)$, then $-s_0$ will be in the ROC of $X(s)$. Putting the two above equations together ,we have
$$x(t) = x_1(-t) = e^t \sin(2t)u(-t) \xleftrightarrow{\mathscr{L}} X(s) = X_1(-s) = -\frac{2}{(s-1)^2+2^2}, \quad \mathscr{R}\{s\} < 1.$$
The denominator of the form $s^2 - 2s + 5$. Therefore, the poles of $X(s)$ are $1+2j$ and $1-2j$.

3. (15') How many signals have a Laplace transform that may be expressed as

$$\frac{(s-1)}{(s+2)(s+3)(s^2+s+1)}$$

in its region of convergence? Please write down their region of convergence.
Solution:

We may find different signal with the given Laplace transform by choosing different regions of convergence. The poles of the given Laplace transform are

$$s_0 = -2, \quad s_1 = -3, \quad s_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j, \quad s_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$$

Based on the locations of the locations of these poles , we my choose form the following regions of convergence:
(1) $\mathscr{R}\{s\} > -\frac{1}{2}$
(2) $-2 < \mathscr{R}\{s\} < -\frac{1}{2}$
(3) $-3 < \mathscr{R}\{s\} < -2$
(4) $\mathscr{R}\{s\} < -3$
Therefore, we may find four different signals the given Laplace transform.

4. (10') Given that

$$e^{-at}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s+a}, \quad \mathscr{R}\{s\} > \mathscr{R}\{-a\}$$

determine the inverse Laplace transform of

$$X(s) = \frac{2(s+2)}{s^2+7s+12} \quad \mathscr{R}\{s\} > -3$$

Solution:

Using partial fraction expansion

$$X(s) = \frac{4}{s+4} - \frac{2}{s+3}.$$

Taking the inverse Laplace transform,

$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t).$$

5. (20') A causal LTI system S with impulse response $h(t)$ has its input $x(t)$ and output $y(t)$ related through a linear constant-coefficient differential equation of the form

$$\frac{d^3y(t)}{dt^3} + (1+\alpha)\frac{d^2y(t)}{dt^2} + \alpha(\alpha+1)\frac{dy(t)}{dt} + \alpha^2 y(t) = x(t).$$

(a) (10') If

$$g(t) = \frac{dh(t)}{dt} + h(t).$$

how many poles does $G(s)$ have?

(b) (10') For what real values of the parameter α is S guaranteed to be stable?

Solution:
Taking the Laplace transform of both sides of the given differential equations, we obtain

$$Y(s)[s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2] = X(s).$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2}.$$

(a) Taking the Laplace transform of both sides of the given equation, we have

$$G(s) = sH(s) + H(s).$$

Substituting for $H(s)$ from above,

$$G(s) = \frac{s+1}{s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2} = \frac{1}{s^2 + \alpha s + \alpha^2}.$$

Therefore, $G(s)$ has 2 poles.

(b) We know that

$$H(s) = \frac{1}{(s+1)(s^2 + \alpha s + \alpha^2)}.$$

Therefore, $H(s)$ has poles at -1, $\alpha(-\frac{1}{2} + j\frac{\sqrt(3)}{2})$, and $\alpha(-\frac{1}{2} - j\frac{\sqrt(3)}{2})$. If the system has to be stable, then the real part of the poles has to be less than zero. For this to be true, we require that $-\alpha/2 < 0$, i.e., $\alpha > 0$.

6. (10') Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = e^{-3t}u(t)$$

Given that

$$e^{-at}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathscr{R}\{s\} > -a,$$

use properties of the Laplace transform to determine the Laplace transform $Y(s)$ of $y(t)$.
Solution:

We have

$$x_1(t) = e^{-2t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} X_1(s) = \frac{1}{s+2}, \quad \mathscr{R}\{s\} > -2,$$

and

$$x_2(t) = e^{-3t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} X_2(s) = \frac{1}{s+3}, \quad \mathscr{R}\{s\} > -3,$$

Using the time-shifting time-scaling properties, we obtain

$$x_1(t-2) \stackrel{\mathscr{L}}{\longleftrightarrow} e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, \quad \mathscr{R}\{s\} > -2,$$

and

$$x_2(-t+3) \stackrel{\mathscr{L}}{\longleftrightarrow} e^{-3s}X_2(-s) = \frac{e^{-3s}}{3-s}, \quad \mathscr{R}\{s\} > -3,$$

Therefore, using the convolution property we obtain

$$y(t) = x_1(t-2) * x_2(-t+3) \stackrel{\mathscr{L}}{\longleftrightarrow} Y(s) = \frac{e^{-5s}}{(s+2)(3-s)}$$

7. (15') The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Determine and sketch the response $y(t)$ when the input is

$$x(t) = e^{-|t|}, \quad -\infty < t < \infty.$$

Solution:

Since $x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$,

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1}, \quad -1 < \mathscr{R}\{s\} < 1$$

We are also given that

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Since the poles of $H(s)$ are at $-1 \pm j$, and since $h(t)$ is causal, we may conclude that the ROC of $H(s)$ is $\mathscr{R}\{s\} > -1$. Now,

$$Y(s) = H(s)X(s) = \frac{-2}{(s^2+2s+2)(s-1)}.$$

The ROC of $Y(s)$ will be the intersection of the ROCs of $X(s)$ and $H(s)$. This is $-1 < \mathscr{R}\{s\} < 1$. We may obtain the following partial fraction expansion for $Y(s)$:

$$Y(s) = -\frac{2/5}{s-1} + \frac{2s/5 + 6/5}{s^2+2s+2}.$$

We may rewrite this as

$$Y(s) = -\frac{2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2+1}\right].$$

Nothing that the ROC of $Y(s)$ is $-1 < \mathscr{R}\{s\} < 1$, we obtain

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos tu(t) + \frac{4}{5}e^{-t}\sin tu(t).$$