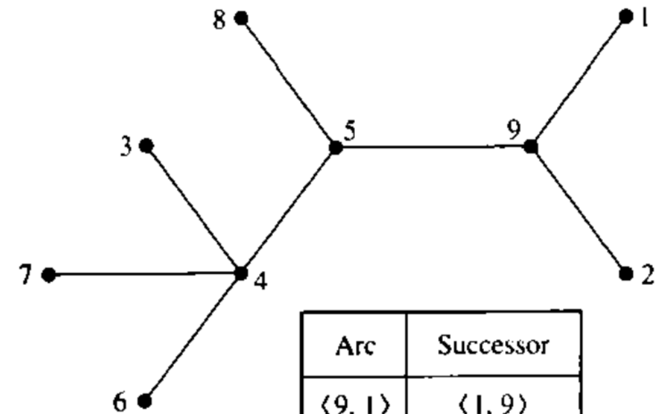
PRAM 3
Tree algorithms

CS121 Parallel Computing

Spring 2019

Euler tours

- An Euler tour of a graph is a cycle that goes through every edge of the graph.
  - It may go through a vertex multiple times.
- A connected, directed graph has an Euler tour if and only if the indegree and outdegree of every vertex are equal.
- Suppose we take an undirected graph, and for edge (u,v), create two directed edges (u,v) and (v,u).
  - Then every vertex has equal indegree and outdegree, and so has an Euler tour.
- Consider a tree where each edge has been doubled.
  - To find an Euler tour of the tree, first order the edges adjacent to each node arbitrarily.
  - Say the neighbors of a node v are ordered $u_0, \ldots, u_{d-1}$. Then set the successor of edge (u_i, v) on the tour to $\left(v, u_{(i+1) \bmod d}\right)$.
- The Euler tour of a tree can be computed in O(1) parallel time.



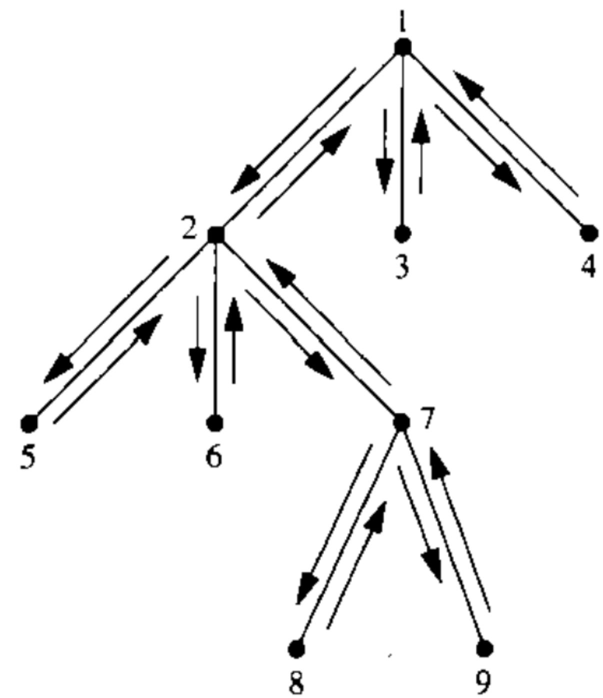
Arc	Successor
$\langle 9,1\rangle$	$\langle 1,9\rangle$
$\langle 9,2\rangle$	$\langle 2,9\rangle$
$\langle 4,3\rangle$	$\langle 3,4\rangle$
$\langle 5,4\rangle$	$\langle 4,3\rangle$
$\langle 3,4\rangle$	$\langle 4,7\rangle$
$\langle 7,4\rangle$	$\langle 4,6\rangle$
$\langle 6,4\rangle$	$\langle 4,5\rangle$
$\langle 8,5\rangle$	$\langle 5,4\rangle$
$\langle 4,5\rangle$	$\langle 5,9\rangle$
$\langle 9,5\rangle$	$\langle 5,8\rangle$
$\langle 4,6\rangle$	$\langle 6,4\rangle$
$\langle 4,7\rangle$	$\langle 7,4\rangle$
$\langle 5,8\rangle$	$\langle 8,5\rangle$
$\langle 5,9\rangle$	$\langle 9,2\rangle$
$\langle 2,9\rangle$	$\langle 9,1\rangle$
$\langle 1,9\rangle$	$\langle 9,5\rangle$

$\langle 9,1\rangle \to \langle 1,9\rangle \to \langle 9,5\rangle \to \langle 5,8\rangle \to \langle 8,5\rangle \to \langle 5,4\rangle \to$
$\langle 4,3\rangle \to \langle 3,4\rangle \to \langle 4,7\rangle \to \langle 7,4\rangle \to \langle 4,6\rangle \to \langle 6,4\rangle \to$
$\langle 4,5\rangle \to \langle 5,9\rangle \to \langle 9,2\rangle \to \langle 2,9\rangle \to \langle 9,1\rangle$

Source: Introduction to Parallel Algorithms, Jaja

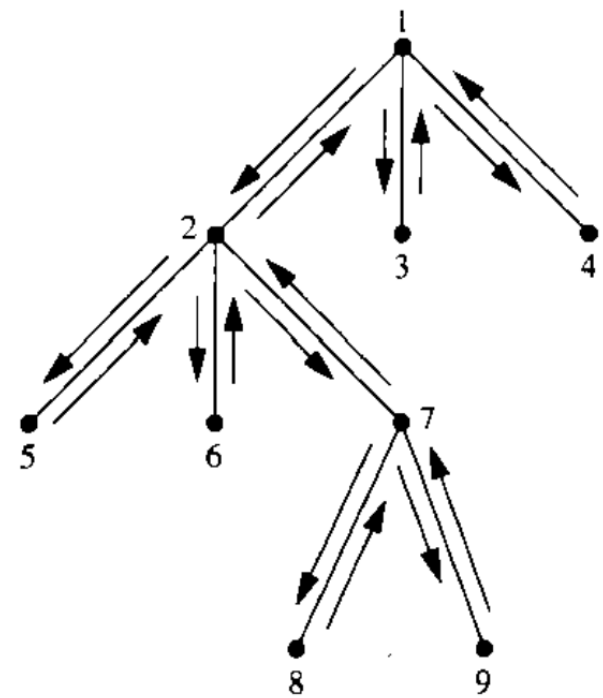
Parallel tree operations

- Many operations on trees can be done in parallel using Euler tours and prefix sum.
- These operations in turn are used in other parallel graph algorithms.
- We first root a tree in parallel.
  - I.e. set an arbitrary node r as the tree's root. Then each node v needs to compute p(v), its parent in the rooted tree.
  - To do this, assign a weight of 1 to each edge in an Euler tour of the tree.
  - Then compute the parallel prefix sum of the edges.
  - For each edge (u,v), set u=p(v) whenever the prefix sum of (u,v) is less than the prefix sum of (v,u).
  - Thus, we can root a tree with n nodes in O(log n) time and O(n) work.



Node depths

- For each node, compute its depth in a rooted tree.
  - For each node v, let p(v) be its parent.
  - Set the weight of edge (p(v), v) to 1, an the weight of edge (v, p(v)) to -1.
  - Compute a parallel prefix sum of the Euler tour starting at the root.
  - The depth of node v is the prefix sum of edge (p(v), v).
- For a tree with n nodes, this takes O(log n) time using O(n) work.

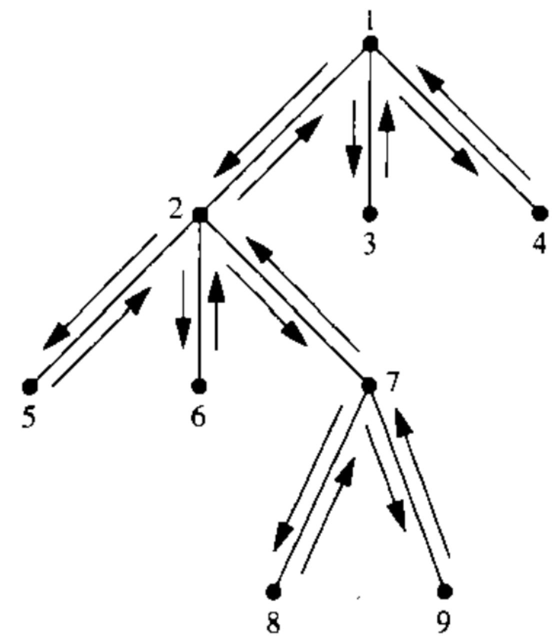


Postorder numbering

- Traverse a rooted tree in postorder, starting from the root r.
  - Start an Euler tour from r. For each node v, we want to visit v's children in the order of the tour, then visit v itself.
  - Ex

v	1	2	3	4	5	6	7	8	9
n(v)	9	6	7	8	1	2	5	3	4

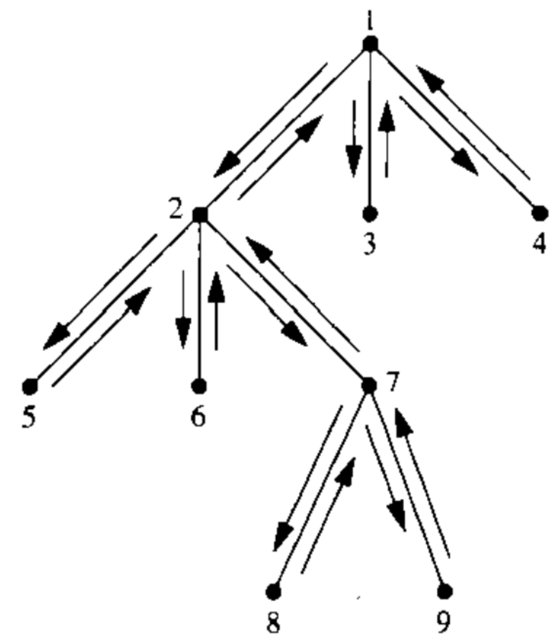
  - For each node v, set the weight of edge (v, p(v)) to 1, and the weight of (p(v), v) weight 0.
  - Compute a parallel prefix sum of the Euler tour.
  - For each $v \neq r$, set n(v) to the prefix sum of edge (v, p(v)). Set n(r)=n.
- In a tree with n nodes, we can compute the postorder numbering in O(log n) time and O(n) work.



Euler Path	Weight	Prefix Sums
⟨1, 2⟩	0	0
⟨2, 5⟩	0	0
⟨5, 2⟩	1	1
⟨2, 6⟩	0	1
⟨6, 2⟩	1	2
⟨2, 7⟩	0	2
⟨7, 8⟩	0	2
⟨8, 7⟩	1	3
⟨7, 9⟩	0	3
⟨9, 7⟩	1	4
⟨7, 2⟩	1	5
⟨2, 1⟩	1	6
⟨1, 3⟩	0	6
⟨3, 1⟩	1	7
⟨1, 4⟩	0	7
⟨4, 1⟩	1	8

Number of descendants

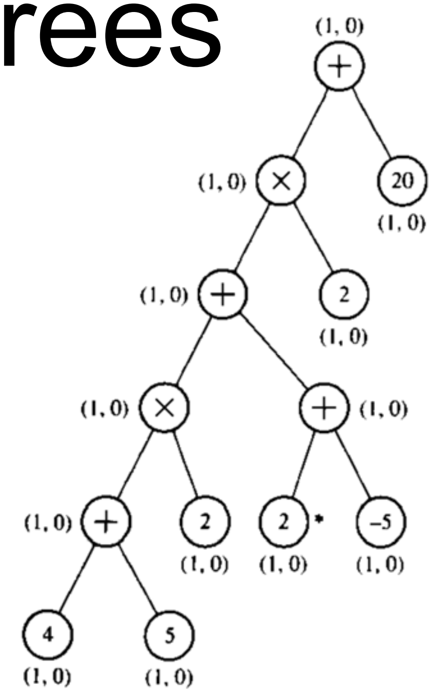
- For each node v in a rooted tree, compute the number of nodes in the subtree rooted at v.
- To do this, we compute prefix sums as in the postorder numbering.
- Then the number of descendants of a node v equals the prefix sum of (v, p(v)) minus the prefix sum of (p(v), v).
- In a tree with n nodes, we can compute the number of descendants in O(log n) time and O(n) work.

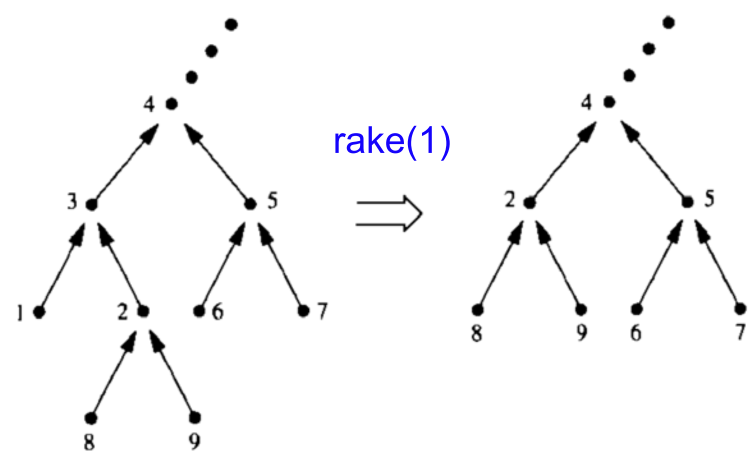


Euler Path	Weight	Prefix Sums
⟨1, 2⟩	0	0
⟨2, 5⟩	0	0
⟨5, 2⟩	1	1
⟨2, 6⟩	0	1
⟨6, 2⟩	1	2
⟨2, 7⟩	0	2
⟨7, 8⟩	0	2
⟨8, 7⟩	1	3
⟨7, 9⟩	0	3
⟨9, 7⟩	1	4
⟨7, 2⟩	1	5
⟨2, 1⟩	1	6
⟨1, 3⟩	0	6
⟨3, 1⟩	1	7
⟨1, 4⟩	0	7
⟨4, 1⟩	1	8

Evaluating expression trees

- An expression tree is a binary tree with values at the leaves and operators (+ or ×) in the interior nodes.
- We want to quickly evaluate an expression tree in parallel.
- The main tool is the rake operation.
  - Given a node u with sibling v, parent p and grandparent p', rake(u) removes u and p, and connects v with p'.
- We repeatedly rake an expression tree in parallel to contract it to 3 nodes with the same value as the original tree.





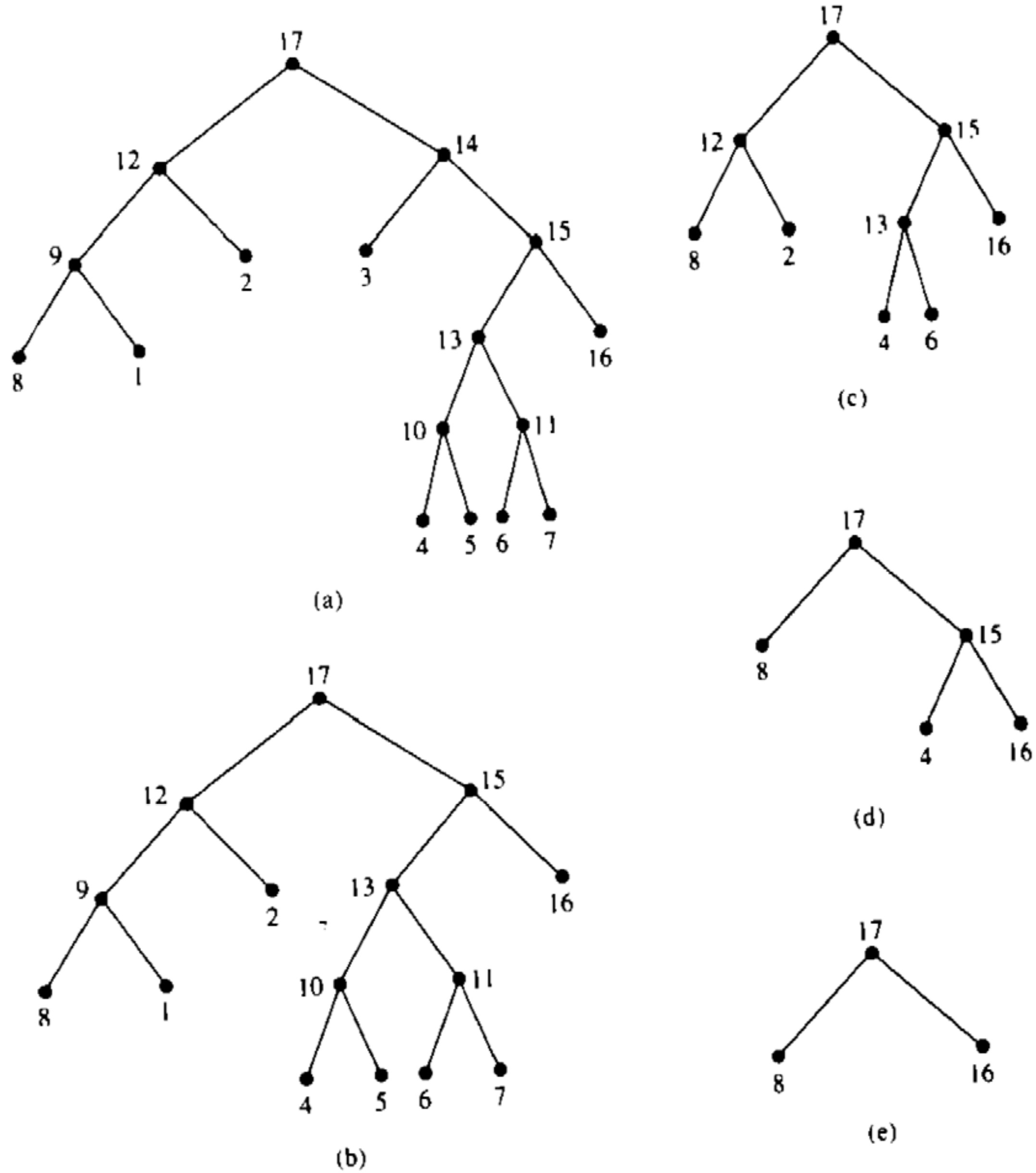
Raking in parallel

- When raking multiple nodes in parallel, we have to avoid concurrent changes to the same node.
- Given an expression tree, label the leaves from left to right (except for the first and last leaf), and call the set A.
  - Let A_{odd} and A_{even} be the subset of A with odd and even labels, resp.
- Repeat for $\lceil \log(n+1) \rceil$ rounds, where $n = |A|$.
  - Concurrently rake all the leaves in A_{odd} that are left children.
  - Concurrently rake the rest of the leaves in A_{odd}.
  - Set $A = A_{even}$.

Example

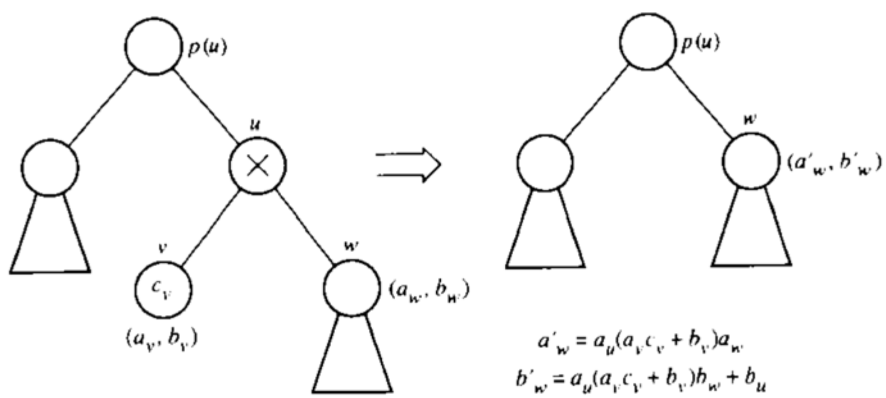
- First rake node 3.
- Then rake nodes 1, 5, 7.
- Then rake leaves 2, 6.
- Then rake leaf 4.

- Nodes raked concurrently don’t have the same parent. So the raking process works correctly.
- Each round reduces the number of leaves by a factor of 2.
- After $[\log(n + 1)]$ rounds, there will be two leaves.

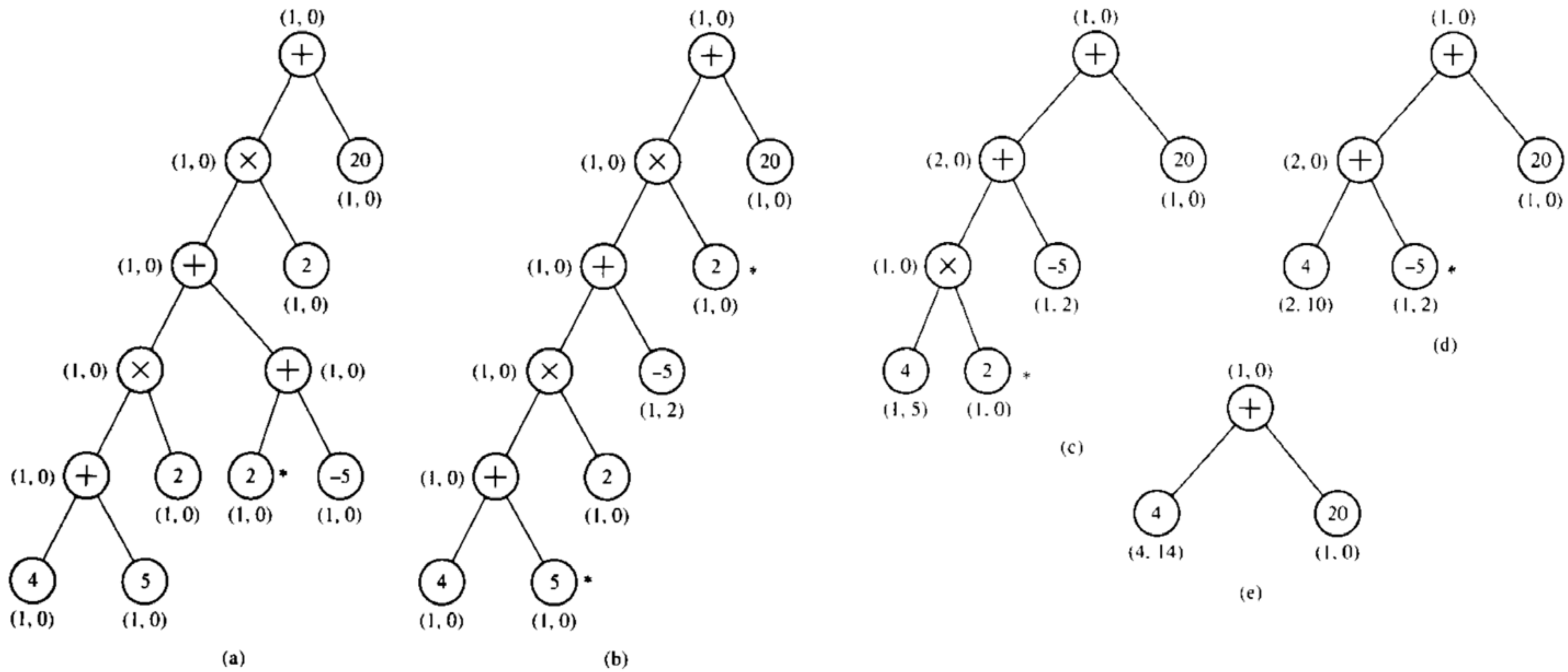


Parallel expression evaluation

- When raking an expression tree, we combine the raked node's value with its sibling's value.
- However, when we rake many nodes in parallel, some nodes may not have their values computed yet.
  - Imagine a chain graph with leaves hanging off the side. Nodes in the middle of the chain do not have values yet when they take part in a rake.
- Given a node v, we assign a label (a_v, b_v) to v.
  - If v is a leaf, v also has a numerical label x, and the numerical value of v is $a_v x + b_v$.
- When nodes are raked, we also update their labels.
  - Ex In the example below, suppose w has a still undermined value X. Then after raking v, w's value is $a_u(a_v c_v + b_v)(a_w X + b_w) + b_u = [a_u(a_v c_v + b_v)a_w]X + [a_u(a_v c_v + b_v)b_w + b_u]$.
    - Thus, the new label of w is $(a_u(a_v c_v + b_v)a_w, a_u(a_v c_v + b_v)b_w + b_u)$.
  - Leaves always have a numerical label, so their value can be immediately evaluated.
- Since raking in parallel reduces the tree to 3 nodes in $O(\log n)$ time, the tree's value can be evaluated in $O(\log n)$ time.



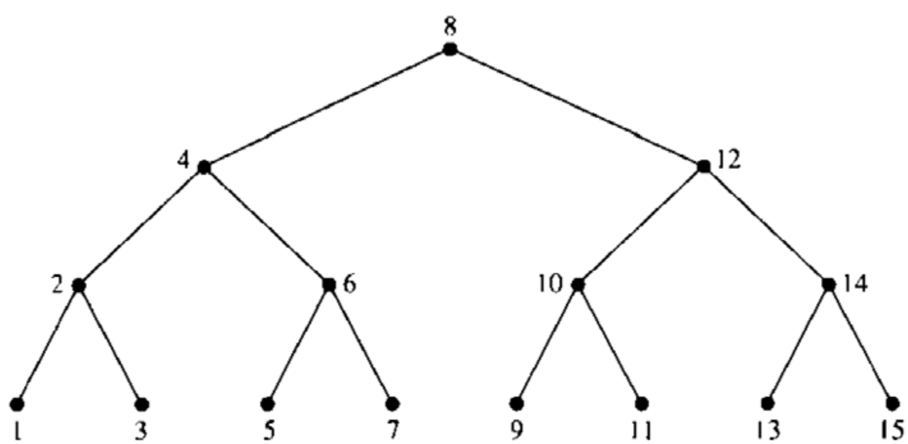
Example



Nodes marked by * are raked in parallel in each round.

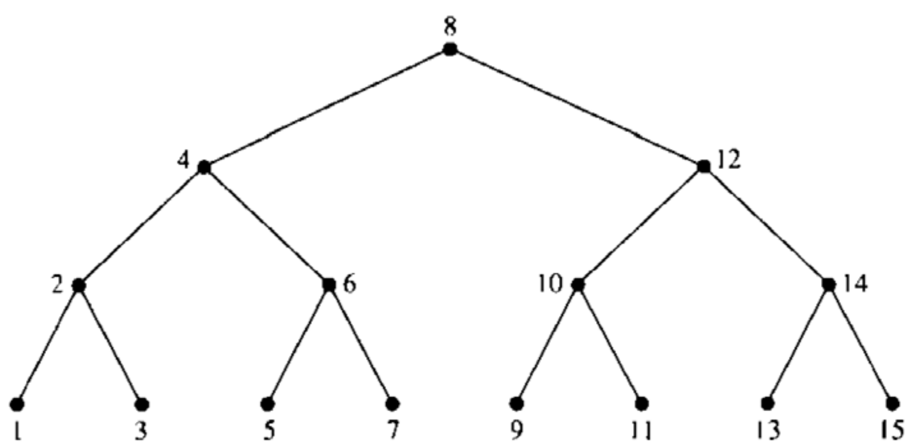
Lowest common ancestors

- Given two nodes u, v in a tree, LCA(u,v) is the lowest node in the tree with u and v as descendants.
  - Ex LCA(1,5) = 4, LCA(3,10) = 8.
- Given a tree, we want to preprocess it so that LCA queries can be answered in O(1) time.



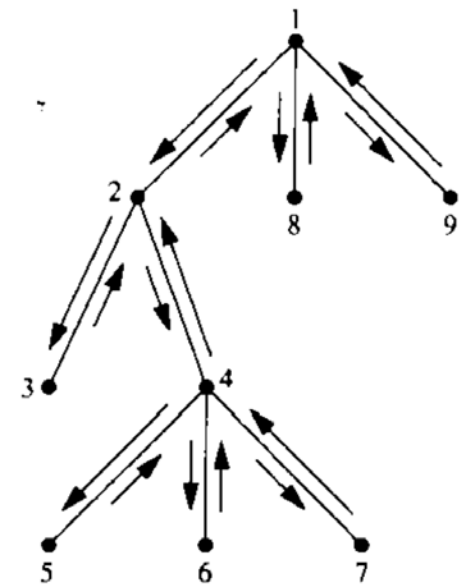
Simple cases

- For a path graph, the LCA of two nodes is whichever node is closer to the root, which can be computed in O(1) time.
- For a complete binary tree, suppose all nodes are labeled by their inorder number.
  - Given u and v, write their labels in binary. Suppose $u = (z_1 z_2 \dots)_2$.
  - Let i be the most significant bit on which u and v differ.
  - Then $LCA(u, v) = (z_1 z_2 \dots z_{i-1} 10 \dots 0)_2$.
  - Ex 9 = $(1001)_2$ and 13=$(1101)_2$. The MSB on which they differ is 2. So LCA(9,13) = $(1100)_2$ = 12.
  - Thus, the LCA can be computed in O(1) time.



LCA and Euler tours

- Consider a general tree with n nodes.
- First compute an Euler tour of the tree, in O(1) parallel time.
  - Label nodes by their order of appearance in the tour.
- Next compute the depths of all nodes, in O(log n) parallel time.
- The Euler tour can be described by a size 2n-1 array A listing the order of nodes visited.
- Create a corresponding array B giving the levels (i.e. depths) of the nodes in A.

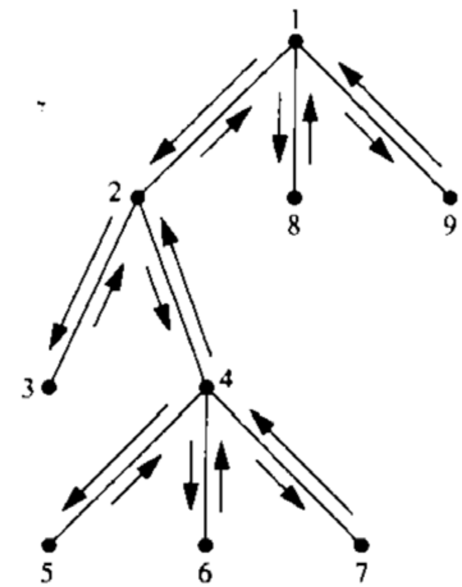


$A = (1, 2, 3, 2, 4, 5, 4, 6, 4, 7, 4, 2, 1, 8, 1, 9, 1)$

$B = (0, 1, 2, 1, 2, 3, 2, 3, 2, 3, 2, 1, 0, 1, 0, 1, 0)$

LCA and Euler tours

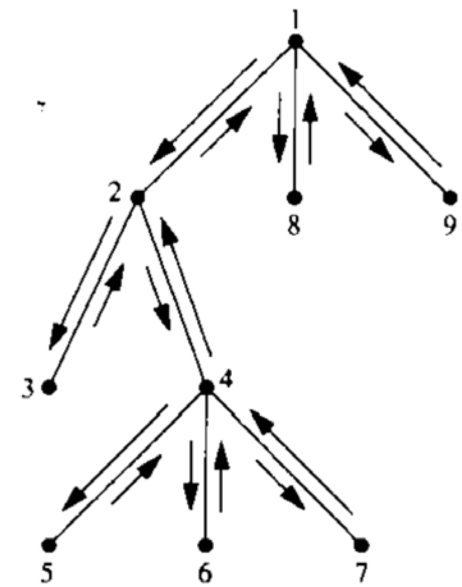
- Given a node v, define the following
  - level(v) is the level of v.
  - l(v), r(v) are the indices of the first, resp. last occurrence of v in A.
    - Ex l(4) = 4, r(4) = 10.
- Lem Given a node v, we have the following.
  - $l(v) = i$ iff $level(A[i-1]) = level(v) - 1$.
  - $r(v) = i$ iff $level(A[i+1]) = level(v) - 1$.
- Ex $l(4) = 4$, and $level(A[3]) = level(2) = 1 = level(4) - 1$.
- Ex $r(4) = 10$, and $level(A[11]) = level(2) = 1 = level(4) - 1$.
- Assume A and B are given. Then we can compute $l(v)$ and $r(v)$ for all nodes v in parallel O(1) time.
  - Use one processor for each node in A and check the conditions in the lemma.
  - If either condition holds for a node v, record l(v) or r(v).



$A = (1, 2, 3, 2, 4, 5, 4, 6, 4, 7, 4, 2, 1, 8, 1, 9, 1)$
$B = (0, 1, 2, 1, 2, 3, 2, 3, 2, 3, 2, 1, 0, 1, 0, 1, 0)$

LCA and Euler tours

- Lem Given nodes u and v in the tree, the following properties hold
  - u is an ancestor of v if and only if $l(u) < l(v) < r(u)$.
  - If $r(u) < l(v)$, then LCA(u,v) is the node with the minimum level in the interval $[r(u), l(v)]$ of A.
- Ex 2 is an ancestor of 4, and $l(2) = 1 < l(4) = 3 < r(2) = 11$.
- Ex $r(4) = 10 < l(8) = 13$, and LCA(4,8) is the node with min level in $A[10{:}13]$, namely node 1 with level 0.
- Using the lemma, we can find LCAs in O(1) time, if we can find the node with min level in second property of lemma in O(1) time.
- In particular, given an array X and any interval $[i, j]$, we want to find the min value in $X[i{:}j]$ in O(1) time.

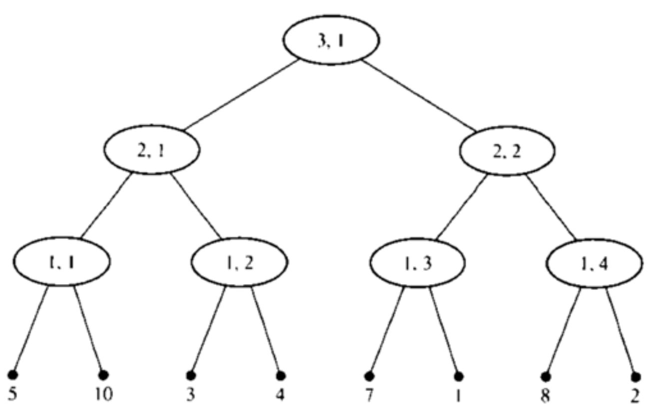


$A = (1, 2, 3, 2, 4, 5, 4, 6, 4, 7, 4, 2, 1, 8, 1, 9, 1)$

$B = (0, 1, 2, 1, 2, 3, 2, 3, 2, 3, 2, 1, 0, 1, 0, 1, 0)$

Range minima problem

- Given an array X of size n, preprocess X so that given any $1 \leq i,j \leq n$, we can find $Rmin[i,j] = \min(X_i, \dots, X_j)$ in O(1) time.
  - The preprocessing will take $O(\log n)$ parallel time and $O(n \log n)$ work.
- Assume for simplicity n is a power of 2.
- Form a complete binary tree with the values of X as the leaves.
  - Label each node in the tree by its height and index within a layer.
- For each node (h,j), create two arrays P(h,j) and S(h,j).
  - Let the values of the leaf nodes in the subtree rooted at (h,j) be $X_p, X_{p+1}, \dots, X_q$.
  - P(h,j) is the array of prefix minima of $(X_p, X_{p+1}, \dots, X_q)$, i.e. $P(h,j)[k] = \min(X_p, \dots, X_{p+k-1})$.
  - S(h,j) is the array of suffix minima of $(X_q, X_{q-1}, \dots, X_p)$, i.e. $S(h,j)[k] = \min(X_q, X_{q-1}, \dots, X_{q-k+1})$.



Ex Array $X = [5,10,3,4,7,1,8,2]$.

$P(2,1) = (5,5,3,3)$, $S(2,1) = (4,3,3,3)$

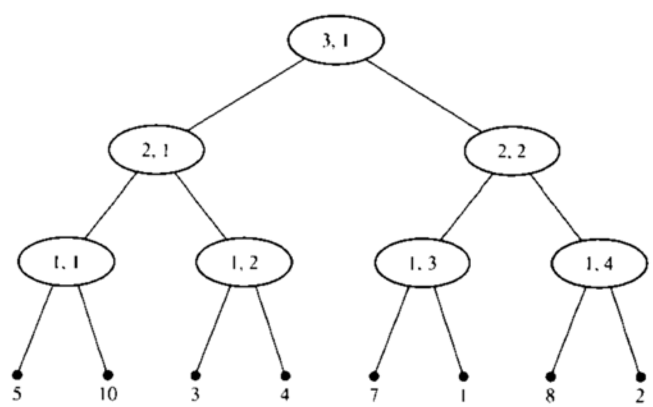
$P(2,2) = (7,1,1,1)$, $S(2,2) = (2,2,1,1)$

$P(3,1) = (5,5,3,3,3,1,1,1)$

$S(3,1) = (2,2,1,1,1,1,1,1)$

Computing range minima

- Given the P and S arrays, we can compute range minima in O(1) time.
- Given an interval [i,j], let $v = LCA(i,j)$.
  - Since X is represented by a complete binary tree, LCA can be computed in O(1) time.
  - Let u and w be the left, resp. right child of v.
  - Let X_i be the p'th node counting from the right in the subtree rooted at u.
  - Let X_j be the q'th node counting from the left in the subtree rooted at w.
  - Then $Rmin[i,j] = \min(S(u)[p], P(w)[q])$.
- Ex To find $Rmin[2,5]$, we have $LCA(2,5)$ is the root (3,1).
  - Then u = (2,1), w = (2,2). Also, $S(u) = (4,3,3,3)$ and $P(w) = (7,1,1,1)$.
  - Leaf 2 is the second child from the right in u's subtree, and leaf 5 is the second child from the left in w's subtree.
  - So $Rmin[2,5] = \min(10,3,4,7) = \min(S(u)[2], P(w)[2]) = \min(3,1) = 1$.



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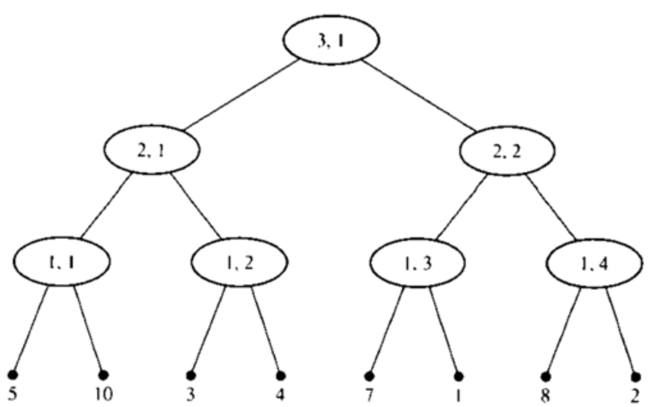
$P(3,1) = (5,5,3,3,3,1,1,1)$

$S(3,1) = (2,2,1,1,1,1,1,1)$

Computing P and S arrays

- We compute the P and S arrays from the bottom up.
- For a leaf with value X_i, the P and S arrays for it are just (X_i).
- Given a node v, let u and w be its left and right children, and suppose we've computed the P and S arrays for u and v.
  - Let x_u be the last value in $P(u)$, and x_w be the last value in $S(w)$.
  - Let $P'(w)$ be the elementwise min of $P(w)$ and x_u.
  - Let $S'(u)$ be the elementwise min of $S(u)$ and x_w.
- Then $P(v) = P(u) \circ P'(w)$, i.e. $P(u)$ followed by $P'(w)$. Also, $S(v) = S(w) \circ S'(u)$.



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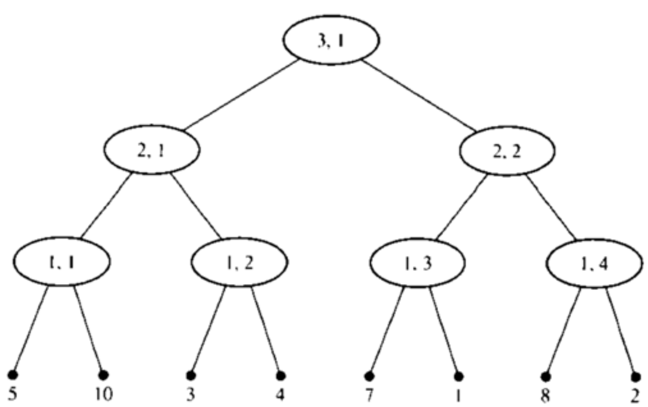
$S(3,1) = (2,2,1,1,1,1,1,1)$

Computing P and S arrays

- Ex Suppose we computed the P and S arrays for nodes $u = (2,1)$ and $w = (2,2)$, and want to compute it for node $v = (3,1)$.
  - $x_u = 3$ and $x_w = 1$, so $P'(w) = (3,1,1,1)$ and $S'(u) = (1,1,1,1)$.
  - Then $P(3,1)$ and $S(3,1)$ are as shown below.
- Thm Given a tree with n leaves, all the P and S arrays in the tree can be computed in O(log n) parallel time and O(n log n) work.
- Proof Given two P (resp. S) arrays of size k, computing the parent P (resp. S) array takes O(1) parallel time and O(k) work.
  - On each layer of the tree, the total sizes of all P and S arrays is O(n).
  - So can compute each layer in O(1) parallel time and O(n) work.
  - There are O(log n) layers. So the theorem follows.
- The work can be reduced to O(n) using accelerated cascading.



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