上海科技大学 ShanghaiTech University

CS240 Algorithm Design and Analysis

Lecture 16

Local Search

Fall 2021

2021.11.15

Last Time – What you need to know

- Coping with NP-Completeness
- **Parameterized complexity.** FPT (fixed parameter tractable) with respect to some parameter k is the class of problems solvable in time $f(k) \cdot \mathrm{poly}(n)$
  - Finding Small Vertex Covers
  - Independent Set on Trees: Greedy Algorithm
  - Weighted Independent Set on Trees: DP Algorithm
  - Circular Arc Coloring: Dynamic Programming Algorithm

Local Search

- Gradient descent
- Metropolis algorithm
- Maximum Cut
- Nash Equilibria

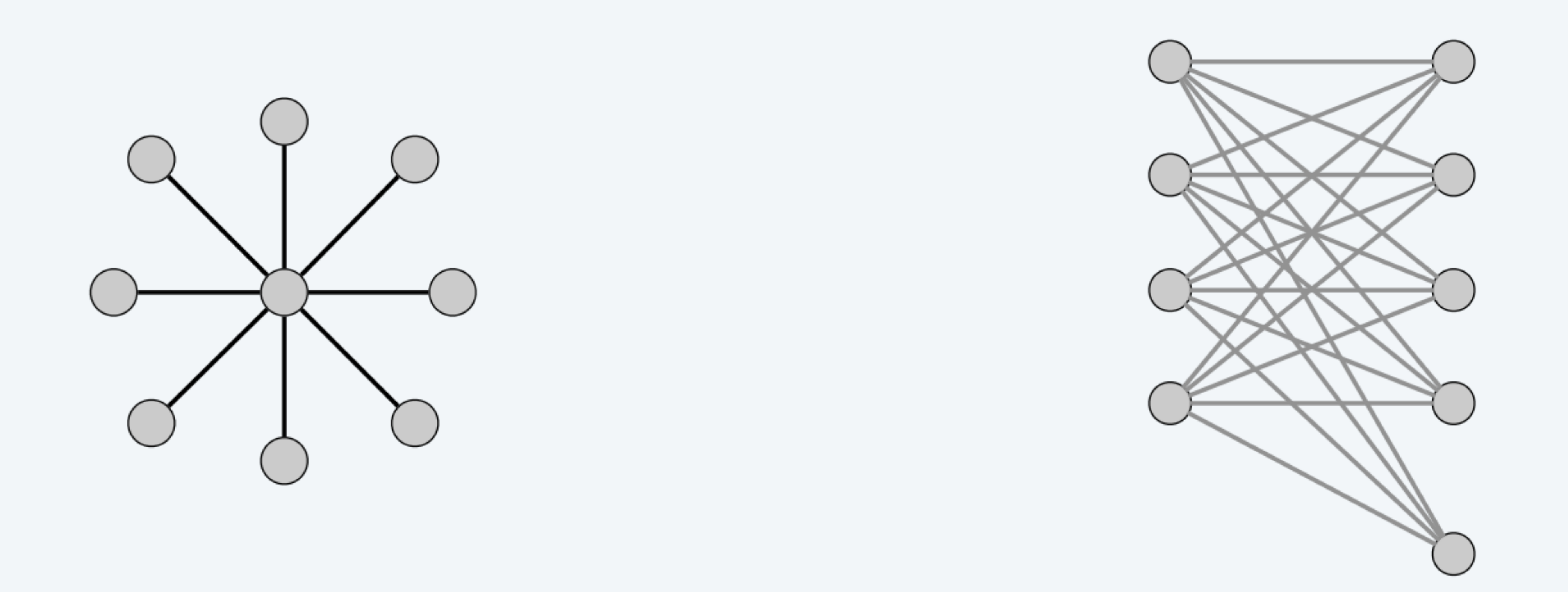
Local Search: Gradient Descent

Gradient descent: vertex cover

- **Vertex cover.** Given a graph $G = (V, E)$, find a subset of nodes S of minimal cardinality such that for each $(u, v) \in E$, either u or v (or both) are in S
- **Neighbor relation.** $S \sim S'$ if S' can be obtained from S by adding or deleting a single node. Each vertex cover S has at most n neighbors
- **Gradient descent.** Start with $S = V$. If there is a neighbor S' that is a vertex cover and has lower cardinality, replace S with S'
- **Remark.** Algorithm terminates after at most n steps since each update decreases the size of the cover by one

Gradient descent: vertex cover

- **Local optimum.** No neighbor is strictly better



Optimum = center node only
Local optimum = all other nodes

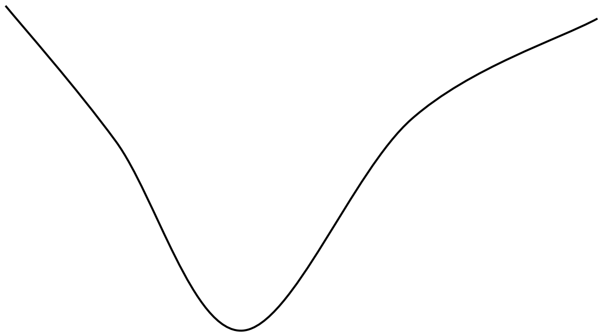
Optimum = all nodes on left side
Local optimum = all nodes on right side

Local Search

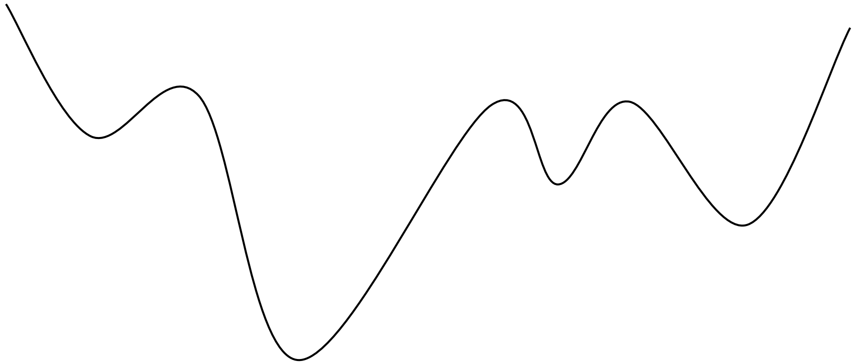
Local search. Algorithm that explores the space of possible solutions in sequential fashion, moving from a current solution to a "nearby" one.

Neighbor relation. Let $S \sim S'$ be a neighbor relation for the problem.

Gradient descent. Let S denote current solution. If there is a neighbor S' of S with strictly lower cost, replace S with the neighbor whose cost is as small as possible. Otherwise, terminate the algorithm.



A funnel



A jagged funnel

Local Search: Metropolis Algorithm

Metropolis Algorithm

Metropolis algorithm. [Metropolis, Rosenbluth, Rosenbluth, Teller, Teller 1953]

- Simulate behavior of a physical system according to principles of statistical mechanics.
- Globally biased toward "downhill" steps, but occasionally makes "uphill" steps to break out of local minima.

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 21, NUMBER 6 JUNE, 1953

Equation of State Calculations by Fast Computing Machines

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(Received March 6, 1953)

A general method, suitable for fast computing machines, for investigating such properties as equations of state for substances consisting of interacting individual molecules is described. The method consists of a modified Monte Carlo integration over configuration space. Results for the two-dimensional rigid-sphere system have been obtained on the Los Alamos MANIAC and are presented here. These results are compared to the free volume equation of state and to a four-term virial coefficient expansion.

Gibbs-Boltzmann Function

- **Gibbs-Boltzmann function.** The probability of finding a physical system in a state with energy E is proportional to $e^{-E/(kT)}$, where T > 0 is temperature and k is a constant
  - For any temperature T > 0, function is monotone decreasing function of energy E
  - System more likely to be in a lower energy state than higher one
    - T larger: high and low energy states have roughly same probability
    - T small: low energy states are much more probable

Metropolis Algorithm

- **The algorithm.**
  - Given a fixed temperature T, maintain current state S.
  - Randomly perturb current state S to new state S' $\in$ N(S).
  - If E(S') $\leq$ E(S), update current state to S'
    Otherwise, update current state to S' with probability $e^{-\Delta E / (kT)}$, where ΔE = E(S') - E(S) > 0.

- **Theorem.** Let $f_S(t)$ be fraction of first t steps in which simulation is in state S. Then, assuming some technical conditions, with probability 1:

$$\lim_{t\to\infty} f_S(t) = \frac{1}{Z} e^{-E(S)/(kT)}$$

where $Z = \sum_{S\in N(S)} e^{-E(S)/(kT)}$

- **Intuition.** Simulation spends roughly the right amount of time in each state, according to Gibbs-Boltzmann equation

Simulated Annealing

Simulated annealing

- T large $\Rightarrow$ probability of accepting an uphill move is large.
- T small $\Rightarrow$ uphill moves are almost never accepted.
- Idea: turn knob to control T.
- Cooling schedule: $T = T(i)$ at iteration i.

Physical analog

- Take solid and raise it to high temperature, we do not expect it to maintain a nice crystal structure
- Take a molten solid and freeze it very abruptly, we do not expect to get a perfect crystal either
- Annealing: cool material gradually from high temperature, allowing it to reach equilibrium at succession of intermediate lower temperatures

Local Search: maximum cut

Maximum Cut

Maximum cut. Given an undirected graph $G = (V, E)$ with positive integer edge weights w_e, find a node partition (A, B) such that the total weight of edges crossing the cut is maximized.

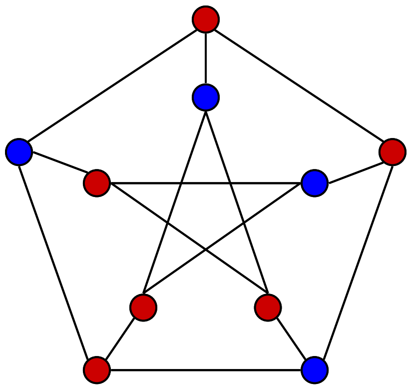
$$w(A,B) := \sum_{u \in A,\, v \in B} w_{uv}$$

Note:

- The Min-Cut problem can be solved in poly time.
- The Max-Cut problem is NP-hard.

Toy application

- n activities, m people
- Each person wants to participate in two of the activities
- Schedule each activity in the morning or afternoon to maximize number of people that can enjoy both activities



- **Real applications.** Circuit layout, statistical physics

Maximum Cut

Single-flip neighborhood. Given a partition (A, B), move one node from A to B, or one from B to A if it improves the solution.

Local search algorithm.

```
Max-Cut-Local (G, w) {
   Pick a random node partition (A, B)

   while (∃ improving node v) {
      if (v is in A) move v to B
      else           move v to A
   }

   return (A, B)
}
```

Maximum Cut: Local Search Analysis

Theorem. Let (A, B) be a locally optimal partition and let (A*, B*) be the optimal partition. Then $w(A, B) \geq \frac{1}{2} \sum_e w_e \geq \frac{1}{2} w(A^*, B^*)$.

Weights are nonnegative

Pf.

- Local optimality implies that for any $u \in A$: $\sum_{v \in A} w_{uv} \leq \sum_{v \in B} w_{uv}$
  Adding up all these inequalities yields:

$$2 \sum_{\{u,v\} \subseteq A} w_{uv} \leq \sum_{u \in A,\, v \in B} w_{uv} = w(A, B)$$

- Similarly

$$2 \sum_{\{u,v\} \subseteq B} w_{uv} \leq \sum_{u \in A,\, v \in B} w_{uv} = w(A, B)$$

- Now,

each edge counted once

$$\sum_{e \in E} w_e = \underbrace{\sum_{\{u,v\} \subseteq A} w_{uv}}_{\leq \frac{1}{2} w(A, B)} + \underbrace{\sum_{u \in A,\, v \in B} w_{uv}}_{w(A, B)} + \underbrace{\sum_{\{u,v\} \subseteq B} w_{uv}}_{\leq \frac{1}{2} w(A, B)} \leq 2 w(A, B)$$

▪

Maximum Cut: Big Improvement Flips

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Local search. Within a factor of 2 for MAX-CUT, but not poly-time!

Big-improvement-flip algorithm. Only choose a node which, when flipped, increases the cut value by at least $\frac{2\varepsilon}{n} w(A, B)$

Claim. Upon termination, big-improvement-flip algorithm returns a cut (A, B) with $(2 + \varepsilon)\ w(A, B) \geq w(A^*, B^*)$.

Pf idea. Add $\frac{2\varepsilon}{n} w(A, B)$ to each inequality in original proof.

Claim. Big-improvement-flip algorithm terminates after $O(\varepsilon^{-1}\ n \log W)$ flips, where $W = \Sigma_e\ w_e$.

- Each flip improves cut value by at least a factor of $(1 + \varepsilon/n)$.
- After n/ε iterations the cut value improves by a factor of 2
- Cut value can be doubled at most $\log_2 W$ times.

if $x >= 1$, $(1+1/x)^x >= 2$

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Maximum Cut: the State of Art

Theorem. [Sahni-Gonzales 1976] There exists a ½-approximation algorithm for MAX-CUT

Theorem. [Goemans-Williamson 1995] There exists an 0.878567-approximation algorithm for MAX-CUT.

$$\uparrow$$

$$\min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1-\cos\theta}$$

Theorem. [Håstad 1997] Unless P = NP, no 16/17 approximation algorithm for MAX-CUT.

$$\uparrow$$

0.941176

Neighbor Relations for Max Cut

- **1-flip neighborhood.** Cuts (A, B) and (A′, B′) differ in exactly one node
- **K-flip neighborhood.** Cuts (A, B) and (A′, B′) differ in at most k nodes
- **KL-neighborhood.** [Kernighan-Lin 1970]
  - To form neighborhood of (A, B):
    - Iteration 1: flip node from (A, B) that results in best cut value (A_1, B_1), and mark that node
    - Iteration k: flip node from (A_{i-1}, B_{i-1}) that results in best cut value (A_i, B_i) among all nodes not yet marked

Cut value of (A_1, B_1) may be worse than (A, B)

  - Neighborhood of (A, B) = $(A_1, B_1), \ldots, (A_{n-1}, B_{n-1})$
  - Neighborhood includes some very long sequences of flips, but without the computational overhead of a k-flip neighborhood
  - Practice: powerful and useful framework
  - Theory: explain and understand its success in practice

Nash Equilibria

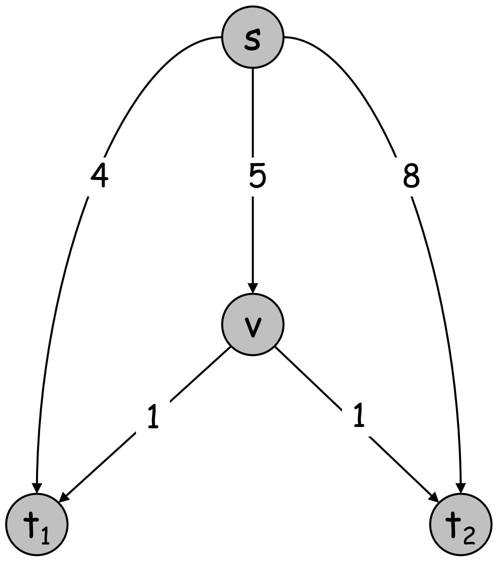
Multicast Routing with Fair Cost Sharing

Multicast routing. Given a directed graph $G = (V, E)$ with edge costs $c_e \geq 0$, a source node s, and k agents located at terminal nodes $t_1, \ldots, t_k$.

Agent j must construct a path P_j from node s to its terminal t_j.

Fair share. If x agents use edge e, they each pay c_e / x.

1	2	1 pays	2 pays
outer	outer	4	8
outer	middle	4	5 + 1
middle	outer	5 + 1	8
middle	middle	5/2 + 1	5/2 + 1



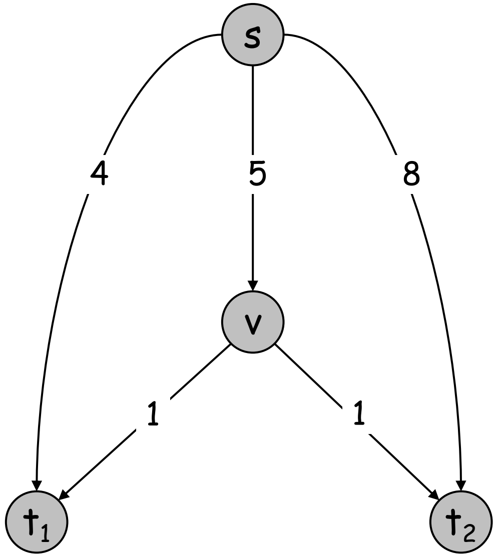
Nash Equilibrium

Best response dynamics. Each agent is continually prepared to improve its solution in response to changes made by other agents.

Nash equilibrium. Solution where no agent has an incentive to switch.

Ex:

- Two agents start with outer paths.
- Agent 1 has no incentive to switch paths (since $4 < 5 + 1$), but agent 2 does (since $8 > 5 + 1$).
- Once this happens, agent 1 prefers middle path (since $4 > 5/2 + 1$).
- Both agents using middle path is a Nash equilibrium.



Note. Best response dynamics may not terminate since no single objective function is being optimized.

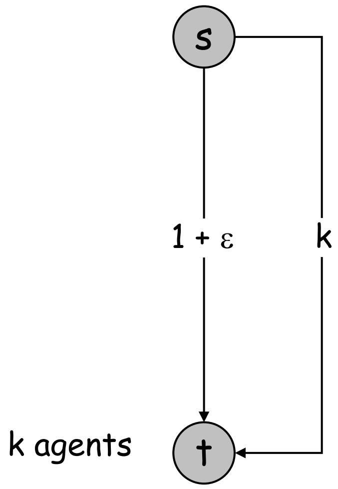
Nash Equilibrium and Local Search

- **Local search algorithm.** Each agent is continually prepared to improve its solution in response to changes made by other agents
- **Analogies**
  - Nash equilibrium: local search
  - Best response dynamics: local search algorithm
  - Unilateral move by single agent: local neighborhood
- **Contrast.** Best-response dynamics need not terminate since no single objective function is being optimized

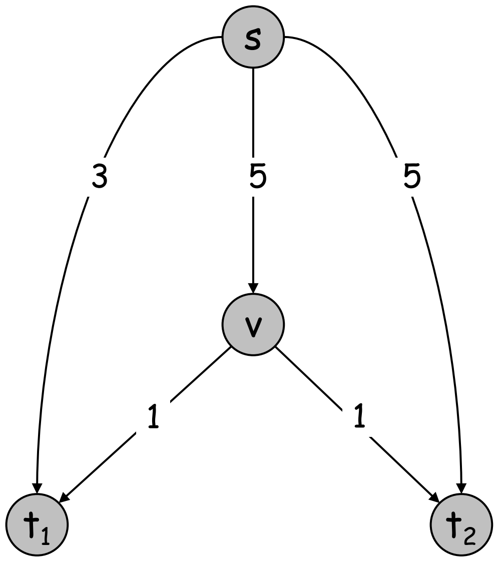
Social Optimum

Social optimum. Minimizes total cost to all agent.

Observation. In general, there can be many Nash equilibria. Even when it's unique, it does not necessarily equal the social optimum.



Social optimum = 1 + ε
Nash equilibrium A = 1 + ε
Nash equilibrium B = k



Social optimum = 7
Unique Nash equilibrium = 8

Price of Stability

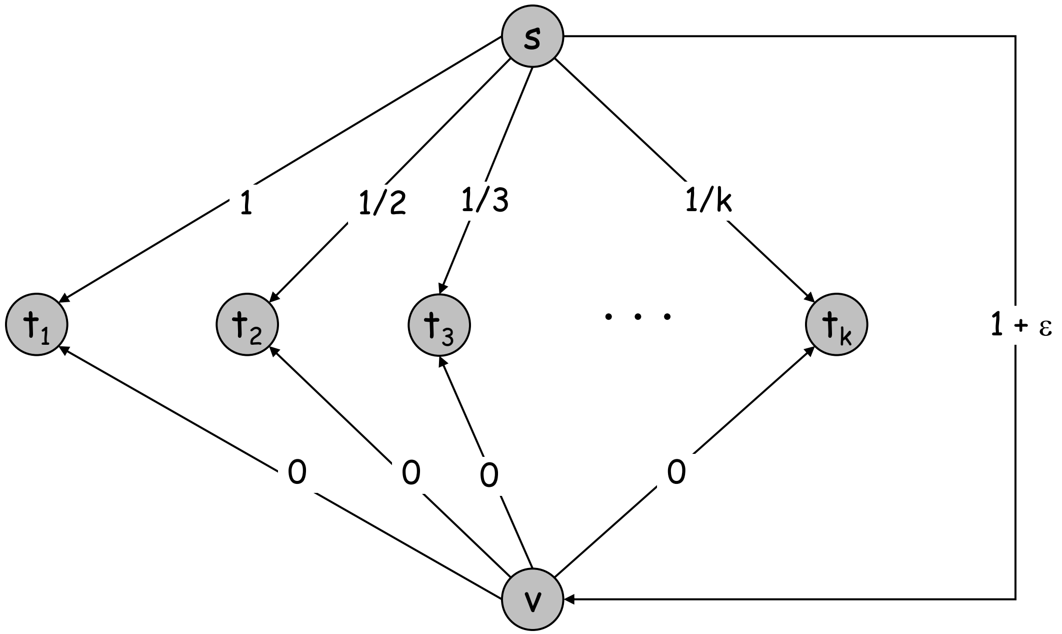
Price of stability. Ratio of best Nash equilibrium to social optimum.

Fundamental question. What is price of stability?

Ex: Price of stability = $\Theta(\log k)$.

- **Social optimum:** Everyone takes bottom paths.
- **Unique Nash equilibrium:** Everyone takes top paths.
- **Price of stability:** $H(k) / (1 + \varepsilon)$.

$\underbrace{H(k)}_{1 + 1/2 + \dots + 1/k}$



Finding a Nash Equilibrium

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Theorem. The following algorithm terminates with a Nash equilibrium (but its running time may be exponential).

```
Best-Response-Dynamics(G, c) {
   Pick a path for each agent
   while (not a Nash equilibrium) {
      Pick an agent i who can improve by switching paths
      Switch path of agent i
   }
}
```

$H(0) = 0$
$H(k)=\sum_{i=1}^{k} \frac{1}{i}$

Pf. Consider a set of paths $P_1, \ldots, P_k$.

- Let x_e denote the number of paths that use edge e.
- Let $\Phi(P_1, \ldots, P_k) = \sum_{e \in E} c_e \cdot H(x_e)$ be a potential function.
- Since there are only finitely many sets of paths, it suffices to show that Φ strictly decreases in each step.

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Finding a Nash Equilibrium

Pf. (continued)

- Consider agent j switching from path P_j to path P_j'.
- Agent j switches because

$$\underbrace{\sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}}_{\text{newly incurred cost}} < \underbrace{\sum_{e \in P_j - P_j'} \frac{c_e}{x_e}}_{\text{cost saved}}$$

- Φ increases by

$$\sum_{f \in P_j' - P_j} c_f \left[H(x_f + 1) - H(x_f) \right] = \sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}$$

- Φ decreases by

$$\sum_{e \in P_j - P_j'} c_e \left[H(x_e) - H(x_e - 1) \right] = \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}$$

- Thus, net change in Φ is negative. ▪

Bounding the price of stability

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Claim. Let $C(P_1, \ldots, P_k)$ denote the total cost of selecting paths $P_1, \ldots, P_k$.

For any set of paths $P_1, \ldots, P_k$, we have

$$C(P_1, \ldots, P_k) \;\le\; \Phi(P_1, \ldots, P_k) \;\le\; H(k) \cdot C(P_1, \ldots, P_k)$$

Pf. Let x_e denote the number of paths containing edge e.

- Let E^+ denote set of edges that belong to at least one of the paths.

$$C(P_1, \ldots, P_k) \;=\; \sum_{e \in E^+} c_e \;\le\; \underbrace{\sum_{e \in E^+} c_e\, H(x_e)}_{\Phi(P_1, \ldots, P_k)} \;\le\; \sum_{e \in E^+} c_e\, H(k) \;=\; H(k)\; C(P_1, \ldots, P_k)$$

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Bounding the price of stability

Theorem. There is a Nash equilibrium for which the total cost to all agents exceeds that of the social optimum by at most a factor of H(k).

Pf.

- Let (P_1^*, ..., P_k^*) denote set of socially optimal paths.
- Run best-response dynamics algorithm starting from P*.
- Since Φ is monotone decreasing $\Phi(P_1, \ldots, P_k) \leq \Phi(P_1^*, \ldots, P_k^*)$.

$$C(P_1, \ldots, P_k) \underset{\substack{\uparrow \\ \text{previous claim} \\ \text{applied to } P}}{\leq} \Phi(P_1, \ldots, P_k) \leq \Phi(P_1^*, \ldots, P_k^*) \underset{\substack{\uparrow \\ \text{previous claim} \\ \text{applied to } P^*}}{\leq} H(k) \cdot C(P_1^*, \ldots, P_k^*)$$

Summary

Existence. (Pure) Nash equilibria always exist for k-agent multicast routing with fair sharing.

Price of stability. Best Nash equilibrium is never more than a factor of H(k) worse than the social optimum.

Big open problem. Find any Nash equilibrium in poly-time, even for 2 players. Known to be PPAD-complete for a general game [Chen and Deng, 2005].

Next Time:
Lower Bounds