W1.1 Introduction
W1.2 Basic Concepts

① Simultaneous Move Game
n players, each has a set of strategies S_i. $s=(s_1,\dots,s_n)=(s_i,s_{-i})$ is the vector of strategies. $S=\prod_i S_i$ is the strategy vector space. Denote $u_i(s)$ the utility of player i under $s \in S$.
② A strategy $s_i \in S_i$ is a dominant strategy for player i, if for all $s' \in S$, we have that $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$.
③ A strategy vector $s \in S$ is a dominant strategy equilibrium, if for each player i, and each alternate strategy vector $s' \in S$, we have that $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$.
④ A strategy vector $s \in S$ is said to be a Nash equilibrium if for all players i and each alternate strategy $s'_i \in S_i$, we have that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
⑤ Each player i picks a probability distribution p_i over his set of possible strategies S_i, such a choice is called a mixed strategy.
⑥ Risk-neutral: players act to maximize the expected payoff.

W2.1 Dominate Strategy and Truthfulness

① We say that a change from strategy s_i to s'_i is an improving response for player i if $u_i(s'_i, s_{-i}) \geq u_i(s)$ and best response if s'_i maximizes the players' utility $\max\limits_{s'_i \in S_i} u_i(s'_i, s_{-i})$
② An auction is truthful if reporting valuation truthfully is a dominant strategy for all participants/buyers.

W2.2 Mechanism Design (MD)

① A mechanism $(f, p_1, \dots, p_n)$ is called truthful (incentive compatible) if for every player i, every $v_1 \in V_1, \dots, v_n \in V_n$ and every $v_i \in V_i$, we have $v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v'_i, v_{-i})$ where $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$.
② A mechanism is truthful if and only if for every i and every v_{-i}:
- The payment p_i does not depend on v_i. $\forall v_{-i}, \exists p_a \in \mathbb{R}, \forall a \in A, s.t. \forall f(v_i, v_{-i}) = a, p(v_i, v_{-i}) = p_a$.
- The mechanism optimizes for each player. $\forall v_i, f(v_i, v_{-i}) \in argmax_a(v_i(a) - p_a)$, where $a \in f(\cdot, v_{-i})$.
③ Social welfare: $\sum_i v_i(a)$.
A social choice function f is efficient if it maximizes social welfare for all valuation reports. F is the set of all feasible social choice functions:
$\forall v \in V, f \in argmax_{f' \in F} \sum_i v_i(f'(v))$
④ Utility is quasi-linear. A mechanism is individually rational if for every player i, every $v \in V$, we have $u_i(f, p_1, \dots, p_n, v, v_i)$ $= v_i(f(v)) - p_i(v) \geq 0$. $(v \to v'?)$

W3.1 VCG (Vickrey-Clarke-Groves)

① f maximizes the social welfare: $f(v_1, \dots, v_n) \in argmax_{a \in A} \sum_i v_i(a)$;
For some functions $h_1, \dots, h_n$, where $h_i: V_{-i} \to \mathbb{R}$ (i.e. h_i does not depend on v_i), we have that for all $v_1 \in V_1, \dots, v_n \in V_n$: $p_i(v_1, \dots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \dots, v_n))$.
② $h_i(\cdot) = 0$ the maximum social welfare without i's participation: $h_i(v_{-i}) = \sum_{j \neq i} v_j(f(v_{-i}))$.

W3.2 Profit Maximization in MD

① Virtual Valuation $\Phi_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$.
Reserve Price = $\Phi_i^{-1}(0)$
② e.g. $b_1, b_2 \in [0,1]$ uniformly random. $\Pr[\min(b_1, b_2) > x] = (1-x)^2$. $F(x) = 1-(1-x)^2, f(x) = F'(x) = 2 - 2x$. $\mathrm{E}[\min(b_1, b_2)] = \int_0^1 x f(x) dx = 1/3$.
③ Myerson's Optimal Auction maximizes the seller's profit.

W4 MD Powered by Social Interactions
Incentive Diffusion Mechanism (IDM)

① i is j's diffusion critical node if all the information diffusion paths started from the seller s to j have to pass i.
② If a buyer or one of her "diffusion critical children" gets the item, then the buyer pays the highest bid of the others (without the buyer's participation).
③ In diffusion critical node path, if the payment of the next node is greater than the bid of the current node, passes it to the next node and receives the payment from the next node, otherwise, keep the item.
④ IC+IR+Seller revenue improved.

W5.1.1 Redistribution

① Budget balanced: The amount of extracted wealth that cannot be redistributed among the agents is 0.
② Myerson-Satterthwaite Theorem
No mechanism is capable of achieving IC, IR, efficiency and budget balance at the same time.
③ Asymptotically budget balanced: As the number of participants goes to infinity, the amount of extracted wealth that cannot be redistributed among the agents goes to 0.
④ **Cavallo's Method**:
$$r_i = \begin{cases} v'_3/n & for\ \ i = a_1, a_2 \\ v'_2/n & for\ \ i = a_3, \dots, a_n \end{cases}$$
IC, IR, efficient and asymptotically budget balanced.

W5.1.2 Sponsored Search Auction

① n bidders with pairs of keywords and bids, also a total budget. $m(<n)$ ad slots, Click Through Rate α_{ij}. i-th slot, bidder j.
② Generalized Second Price (GSP): score=$b_j w_j$. A bidder pays per click the lowest bid necessary to retain his position.
Rank by bid($w_j = 1$), revenue($w_j = \alpha_{1j}$)
Maximize social welfare:
$$\begin{aligned} \max \quad & \sum_{i=1}^{k}\sum_{j=1}^{n} \alpha_{ij} v_j x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall i = 1, \dots, k \\ & \sum_{i=1}^{k} x_{ij} \leq 1 \quad \forall j = 1, \dots, n \\ & x_{ij} \geq 0 \quad \forall i = 1, \dots, k,\ \forall j = 1, \dots, n \end{aligned}$$
where $x_{ij} = 1$ if bidder j is assigned to slot i and zero otherwise.

W5.2 Double Auctions/Exchanges

① In bilateral trading/double auctions, there does not exist a mechanism that is truthful, efficient, individually rational without outside subsidies (budget balanced).
② VCG: Deflicit; McAfee's: Remove one buyer-seller pair to set the payments.
③ Carsharing: VCG: High deficit; Fixed-price Mechanisms: Very inefficient; VCG with Reserve Prices. Dual-Role Exchanges: McAfee's Trade Reduction: not efficient, not IC; Balanced Trade Reduction: not efficient, IC, deficit control, requires k "backups".

W6.1 Social Choice (Voting)

① Let L be the set of all linear orders on A. Each voter i has a preference $\succ_i \in L$. A function $f: L^n \to A$ is called a social choice function. A function $F: L^n \to L$ is called a social welfare function.
② Majority vote: among two candidates, selects the candidate which has a majority vote, that is, more than half of the votes.
Condorcet's Paradox: $a \succ b \succ c \succ a$.
Plurality: the candidate that was placed first by the largest number of voters wins.
Borda count: each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i, and the candidate with most points wins.
③ A social choice function f can be strategically manipulated by voter i if for some $\prec_1, \dots, \prec_n \in L$ and some $\prec'_i \in L$ we have that $a \prec_i a'$ where $a = f(\prec_1, \dots, \prec_i, \dots, \prec_n)$ and $a' = f(\prec_1, \dots, \prec'_i, \dots, \prec_n)$; Voter i is a dictator in social choice function f if for all $\prec_1, \dots, \prec_n \in L$, $\forall$b $\neq$ a, a $\succ_i b \Rightarrow f(\prec_1, \dots, \prec_n) = a$. f is called a dictatorship if some i is a dictator in it.
Gibbard-Satterthwaite: Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.
④ Arrow's Theorem: Every social welfare function over a set of more than 2 candidates ($|A| \geq 3$) that satisfies unanimity and independence of irrelevant alternatives is a dictatorship. F satisfies unanimity if for every $\succ \in L, F(\succ, \dots, \succ) = \succ$. F satisfies independence of irrelevant alternatives if for every $a, b \in A$, every $\succ_1, \dots, \succ_n, \succ'_1, \dots, \succ'_n \in L$, if $\succ = F(\succ_1, \dots, \succ_n)$ and $\succ' = F(\succ'_1, \dots, \succ'_n)$, then $a \succ_i b \Leftrightarrow a \succ'_i b$ for all i implies $a \succ b \Leftrightarrow a \succ' b$.

W6.2 Social Choice

① Single-Peaked Preference: Each player i has a single-peaked preference $\succcurlyeq_i$ over A, i.e. there exists a point $p_i \in A$ s.t. $\forall x \in A \backslash \{p_i\}$ and $\forall \lambda \in [0,1), (\lambda x + (1-\lambda) p_i) \succ_i x$. Let $\mathcal{R}$ denote the class of single-peaked preferences.
② The Median Voter Rule: add n-1 fixed points, choose the median of the 2n-1 points.
③ Facility Location Games: minimize social cost (SC) / minimize maximum cost (MC).
=1= Locate the facility at the location of the median agent. Truthful and gives the optimal (minimum) social cost.
=2= Locate the facility at the location that minimizes the maximum cost.
=3= Locate the facility at the location of the first agent. Truthful and gives 2-approximation for the maximum cost.
④ Obnoxious Facility Location Games: Given a reported locations x' on $[0,2]$. Let n_1 be the number of agents located on $[0, l]$ and n_2 be the number of agents located on $[1,2]$. If $n_1 \geq n_2$, return $f(x') = 2$ and

otherwise return $f(x') = 0$.
Truthful and has an approximation ratio of 3 for the obnoxious facility game.

W7.1 Matching

① One-sided Matching: House Allocation. Without Initial Allocation: (Randomized) Serial dictatorship: Predefine a ranking of agents (could be randomized). Let each agent from the top of the ranking to choose her preferred house first.
② Pareto optimality is a state of allocation of resources from which it is impossible to reallocate so as to make any one individual or preference criterion better off without making at least one individual or preference criterion worse off.
③ A blocking coalition can, by trading / exchange among themselves, receive homes that each strictly prefers (or is equivalent) to the home she receives under current allocation, with at least one agent being strictly better off.
④ With Initial Allocation: Top Trading Cycle Mechanism: Each agent points to most preferred house (allow self-edge); Trade on cycles, agents and houses leave market; Each remaining agent points to its most preferred, remaining house. Repeat until no agents left. Truthful+ pareto optimal.
⑤ Two-sided Matching: A stable matching is a matching with no blocking pair, a blocking pair is two agents who prefer to match with each other.
⑥ Boy-Proposing Deferred Acceptance: Each man proposes to his top-ranked choice; Each woman who has received at least two proposals keeps (tentatively) her top-ranked proposal and rejects the rest; Each man who has been rejected proposes to his top-ranked choice among the women who have not rejected him; Each woman who has at least two proposals (including ones from previous rounds) keeps her top-ranked proposal and rejects the rest. It generates a stable matching.

W8.2 Cooperative Games & Cost Sharing

① Coalitional/Cooperative Game:
- A set of agents N.
- Each subset of agents (coalition) $S \subseteq N$ cooperate together can generate some value $v(S) \in \mathbb{R}$. Assume $v(\emptyset) = 0$. N is called grand coalition. $v: 2^N \to \mathbb{R}$ is called the characteristic function of the game. v is often assumed to be monotonic: $S \subseteq T \Rightarrow v(S) \leq v(T)$.
- The possible outcomes of the game is defined by $V(s) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$.

② For the grand coalition N, the allocation vector $x \in \mathbb{R}^N$ satisfy:
Efficiency if $\sum_{i \in N} x_i = v(N)$.
Individual Rationality if $\forall i \in N, x_i \geq v(\{i\})$.
The core of the coalitional game (N, v) is a set of vectors $x \in \mathbb{R}^N$ such that x is efficient and $\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S)$.
③ Given a coalitional game (N, v), the Shapley value of each player i is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S))$$

④ Properties of Shapley Value
Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$.
Symmetry: If i and j are two players who are equivalent in the sense that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N$ s.t. $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.
Linearity: $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.
Zero player (null player): $\phi_i(v) = 0$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$.
Shapley value is not necessarily in the core.
⑤ Cost Sharing: A cost function $c: 2^N \to \mathbb{R}^+$ and assume $c(\emptyset) = 0$.
⑥ A vector $\alpha \in \mathbb{R}^N$ is in the core of a cost sharing game (N, c) if $\sum_{i \in N} \alpha_i = c(N)$ and $\forall S \subseteq N, \sum_{j \in S} \alpha_j \leq c(S)$.

W9.1 Cost Sharing and Public Goods

① Public Goods: a commonly known cost c to build the public good. The public good is built iff $\sum_{i \in N} v_i > c$. a mechanism is budget balanced if: build the good and total payment $\geq c$, or not build and total payment ≥ 0.
② When there are only two agents, the only mechanisms that are IC, IR and budget balanced are fixed-price mechanisms. (It does not hold for more than three agents settings)
③ Cost Sharing of Excludable Good Production: The good can be shared by a subset of agents.
④ Mechanism: Find the largest k such that the highest k agents' valuation reports are at least c/k. Charge these k agents c/k and reject all others, i.e. the good is only shared by the k agents. IC+IR+Budget Balanced.

W9.2 Cake Cutting

① Proportionality: Each player receives a piece that he values as at least $1/n$ of the value of the entire cake.
Envy-freeness: Each player receives a piece that he values at least as much as every other piece. Envy-freeness implies proportionality.
② Divide and Choose: Two person share one cake. One person (the cutter) cuts the cake into two pieces. The other person chooses one (the chooser). Envy-free.
③ Last Diminisher: The players being ranged A, B, C, ... N; A cuts from the cake an arbitrary part; B has now the right, but is not obliged, to diminish the slice cut off; Whatever B does, C has the right (without obligation) to diminish still the already diminished (or not diminished) slice, and so on up to N; The rule obliges the "last diminisher" to take as his part the slice he was the last to touch. Proportional, not envy-free.
④ Moving-knife Protocol: The cake: interval [0,1]. n players $1, 2, \ldots, n$ and a referee. Referee starts a knife at 0 and moves the knife to the right; Repeat: When the piece to the left of the knife is worth $1/n$ to a player, the player shouts "stop", receives the piece, and exits; When only one player remains, she gets the remaining piece.
Complexity of moving-knife protocol: $\Theta(n^2)$ Proportional, not envy-free.
⑤ Even Paz: A piece of cake $[x, y]$. n agents. (Assume $n = 2k$ for simplicity).
- If $n = 1$, give $[x, y]$ to the single agent. Otherwise:
- Each agent mark a point z such that $v([x, z]) = v([z, y])$.
- Let z^* be the $(n/2)$-th mark from the left.
- Recurse on $[x, z^*]$ with the left $n/2$ agents, and on $[z^*, y]$ with the right $n/2$ agents. Uses a divide-and-conquer strategy, it is possible to achieve a division in time $O(n \log n)$. Proportional, not envy-free.
⑥ Any protocol returning a proportional allocation needs $\Omega(n \log n)$ queries.
Any protocol for finding an envy-free allocation requires $\Omega(n^2)$ queries.
Envy-free: $n = 2$: 2 queries; $n = 3$: 14 queries; $n = 4$: 171 queries;

W10 Crowdsourcing

① The Setting for Crowdsourcing Platforms
There is one requester with budget B, and n workers arrive sequentially (in a random order); Each worker a_i has cost $c_i \geq 0$ for each task and is willing to perform at most $t_i \geq 0$ tasks. The worker's utility per task if paid price p_i is $u_i = p_i - c_i$. (Assume all tasks are of the same kind. e.g., labeling.); In every round, a new worker arrives and reports c_i and t_i. The requester decides how many tasks to allocate and the price per task.
② Pay $p = c_i$ per task: Not truthful! Pay a fixed price p^* per task: May be too high (wasting budget) or too low (no enough workers)! Target: Maximize the number of tasks that are completed without exceeding the budget and be strategy-proof (truthful).
③ Dynamic Pricing:
If the requester can know next m workers' type profile: (assuming $c_1 \leq c_2 \leq \cdots \leq c_m$.):
- Let x_i be the number of tasks allocated to i
- When worker 1 comes, it is always worth improving price to $p = c_1$ to allocate $x_i = t_i$.
- When worker i comes, it is worth improving price to $p = c_i$ if we can further allocate at least one more task to i. We shall check whether $c_i\left(1 + \sum_{j<i} x_j\right) \leq B$ (we can afford the payments for first i agents when the price is c_i).
- Allocate as much work as possible to worker i: $x_i = \min\{t_i, \lfloor B/p \rfloor - \sum_{j<i} x_j\}$.
④ Use the past m workers' reports $\{(c_1, t_1), \ldots, (c_m, t_m)\}$ to predict future:
-1- Sort reports such that $c_1 \leq c_2 \leq \cdots \leq c_m$.
-2- Set $i = 1$.
-3- While $c_i \leq B/(1 + \sum_{j<i} x_j)$ do:
  -1- $p = c_i$
  -2- $x_i = \min\{t_i, \lfloor B/p \rfloor - \sum_{j<i} x_j\}$
  -3- $i = i + 1$
-4- Output p for the next m workers.
⑤ The Whole Pricing Mechanism
-1- First set $p_0 = \epsilon$ for the first worker.
-2- Then we have the history of the first worker, set $m = 1$ and we get a price p_1 for the next worker.
-…-
-3- Then we have the history of the first 2^k workers, set $m = 2^k$ and we get a price p_{k+1} for the next 2^k workers.
-4- For each bucket of workers, set $B_k = (2^{k-1}/n)B$.
The online pricing mechanism is budget feasible and strategy-proof.
⑥ If an agent pretends to be multiple agents to get more rewards, it is called Sybil attack.