EE150 Signals and Systems
– Part 7: z-transform (ZT)

$\downarrow$ Week 12, Tue, 20180515

z-transform

Remember the eigen-function for D-T LTI System:

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

z-transform (ZT):

$$x[n] \xleftrightarrow{ZT} X(z) \equiv \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{(Bilateral)}$$

ZT

In general: $z = r \cdot e^{j\omega}$ (polar form)

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] r^{-n} e^{-j\omega n}$$

$$\Longrightarrow X(z) = FT\{x[n]r^{-n}\}$$

$$X(z)|_{z=e^{j\omega}} = FT\{x[n]\}$$

ZT is a generalization of DTFT

ZT

Note: For different r value, $X(z)$ may or may not converge.

ROC: The set of z such that $\sum_{n=-\infty}^{\infty} |x[n]z^n|$ converges

ZT Example

$x[n] = 7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]$

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= 7 \sum_{n=-\infty}^{\infty} (\frac{1}{3})^n u[n] z^{-n} - 6 \sum_{n=-\infty}^{\infty} (\frac{1}{2})^n u[n] z^{-n} \\
&= 7 \sum_{n=0}^{\infty} (\frac{1}{3} z^{-1})^n - 6 \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n \\
&= \frac{7}{1 - \frac{1}{3} z^{-1}} - \frac{6}{1 - \frac{1}{2} z^{-1}} \qquad (*) \\
&= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}
\end{aligned}$$

ZT Example

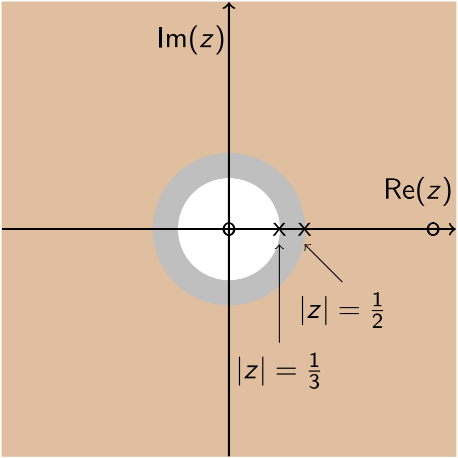
ROC: Both summations in $(*)$ have to converge

$$\Rightarrow \left|\frac{1}{3}z^{-1}\right| < 1 \quad \& \quad \left|\frac{1}{2}z^{-1}\right| < 1$$

$$\Rightarrow |z| > \frac{1}{3} \quad \& \quad |z| > \frac{1}{2}$$

$$\Rightarrow |z| > \frac{1}{2}$$

ZT Example



o: Zero
$z = 0, \quad z = \frac{3}{2}$

x: Pole
$z = \frac{1}{3}, \quad z = \frac{1}{2}$

ROC: $|z| > \frac{1}{2}$

ZT

Different $x[n]$ may have the same ZT

$$u[n] \xleftrightarrow{ZT} = \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| > 1$$

$$\begin{aligned} -u[-n-1] &\xleftrightarrow{ZT} -\sum u[-n-1]z^{-n} \\ &= -\sum_{-\infty}^{-1} z^{-n} = -\sum_{1}^{\infty} z^{n} \\ &= -\frac{z}{1-z} = \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| < 1 \end{aligned}$$

ZT

LTI system $x[n] * h[n] \xleftrightarrow{ZT} X(z) \cdot H(z)$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

A general method for solving difference equations: e.g.

$$
\begin{aligned}
& y[n] - ay[n-1] = \delta[n], \quad y[n] \text{ right-sided} \\
\Longrightarrow\ & Y(z) - az^{-1}Y(z) = 1 \\
\Longrightarrow\ & Y(z) = \frac{1}{1 - az^{-1}} \\
\Longrightarrow\ & Y(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots \\
\Longrightarrow\ & y[n] = a^n u[n]
\end{aligned}
$$

Inverse ZT

Can we use F^{-1} to obtain Z^{-1}? Consider:

$$
\begin{aligned}
X(z) &= X(re^{j\omega}) = F\{x[n]r^{-n}\} \\
\implies x[n]r^{-n} &= F^{-1}\{X(re^{j\omega})\} \\
\implies x[n] &= r^n F^{-1}\{X(re^{j\omega})\} \\
&= r^n \cdot \frac{1}{2\pi}\int_0^{2\pi} X(re^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega
\end{aligned}
$$

Inverse ZT

Note that $X(re^{j\omega}) \cdot (re^{j\omega})^n$ is a function of both "r" & "ω".

However, the integration is only respect to ω: an integration along a circle contour $z = re^{j\omega}$ in ROC, with a fixed r, and ω varying over a 2π interval.

By changing of variable, $dz = jre^{j\omega} d\omega$ or $d\omega = (\frac{1}{j}) z^{-1} dz$:

$$x[n] = \frac{1}{2\pi j} \oint_{|z|=r} X(z) z^{n-1} dz$$

Inverse ZT

$\oint$ integration around a counter-clockwise (CCW) closed circular contour centered at the origin with radius r

Remark: The formal inverse z-transform equation requires contour integration in complex plane

Alternative: Try to use partial-fraction expansion & table:
Express $X(z) = X_1(z) + X_2(z) + \dots$
in which X_1, X_2, $\dots$ have known ZT pairs

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Some Common ZT Pairs

- Right-sided signal, ROC is $|z| > a$
  e.g. $x[n] = u[n]$

- Left-sided signal, ROC is $|z| < b$
  e.g. $x[n] = u[-n-1]$

Some Common ZT Pairs

Table 10.2

signal	z-transform	ROC
(1) $\delta[n]$	1	all z
(2) $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
(3) $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
(4) $\delta[n-m]$	z^{-m}	$z \neq 0$ (for $m > 0$) $z \neq \infty$ (for $m < 0$)
(5) $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
(6) $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Some Common ZT Pairs

signal	z-transform	ROC
(7) $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
(8) $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
(9) $\cos(\omega_0 n) \cdot u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
(10) $\sin(\omega_0 n) \cdot u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
(11) $r^n \cos(\omega_0 n) \cdot u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
(12) $r^n \sin(\omega_0 n) \cdot u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Partial Fraction Expansion

A rational ZT can be expressed as

$$X(z) = \text{polynomial}(z^{-1}) + \sum_{i=1}^{I} \sum_{k=1}^{p_i} \frac{C_{i,k}}{(1 - a_i z^{-1})^k}$$

Partial Fraction Expansion

Example:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \qquad \frac{1}{4} < |z| < \frac{1}{3}$$

$$\implies X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$\implies X_1(Z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad \frac{1}{4} < |z|$$

$$X_2(Z) = \frac{2}{1 - \frac{1}{3}z^{-1}}, \qquad |z| < \frac{1}{3}$$

Partial Fraction Expansion

$$\implies x_1[n] = (\frac{1}{4})^n u[n]$$

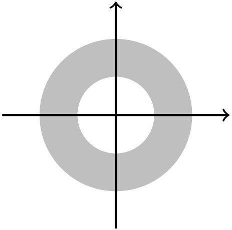
$$x_2[n] = -2(\frac{1}{3})^n u[-n-1]$$

$$\implies x_[n] = (\frac{1}{4})^n u[n] - 2(\frac{1}{3})^n u[-n-1]$$

Q:
ROC: $|z| < 1/4$
ROC: $|z| > 1/3$

Properties of ROC

1. ROC is a ring in the z-plane centered about origin.
   i.e. ROC is independent of ω



2. ROC does not contain any pole

Properties of ROC

3. If $x[n]$ has finite duration,

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

ROC is all z except possibly 0 or ∞:
If $X(z)$ contains negative power of z, then $X(0) = \infty$
If $X(z)$ contains positive power of z, then $X(\infty) = \infty$

For other values of z, always converge

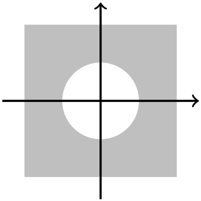
Properties of ROC

4. If $x[n]$ is right-sided,

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n}$$

then ROC takes the form: $|z| > c$

Or if $|z| = r_0$ is in ROC, then $|z| > r_0$ in ROC



Properties of ROC

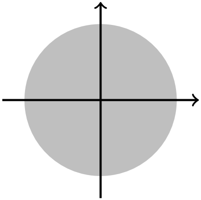
5. If $x[n]$ is left-sided,

$$X(z) = \sum_{n=-\infty}^{N_2} x[n] z^{-n}$$

then ROC takes the form: $0 < |z| < c$

Or if $|z| = r_0$ is in ROC, then $0 < |z| < r_0$ in ROC

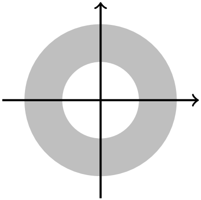
ROC will include $z = 0$, if $N_2 \leq 0$



Properties of ROC

6. If $x[n]$ is two-sided,
   then ROC takes the form: $c_1 < |z| < c_2$

   Or if $|z| = r_0$ is in ROC, then ROC is a RING that includes $|z| = r_0$



Properties of ZT

1. Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{ZT} aX_1(z) + bX_2(z)$$

   ROC <u>at least</u> $R_1 \cap R_2$

   ROC equals $R_1 \cap R_2$ if there is no pole-zero cancellation

2. Time-shifting

$$x[n - n_0] \xleftrightarrow{ZT} z^{-n_0}X(z)$$

   ROC: R possibly add or delete zero

Properties of ZT

3. Scaling in z-domain

$$z_0^n x[n] \xleftrightarrow{ZT} X(\frac{z}{z_0}), \qquad \text{ROC: } |z_0|R$$

$$\text{e.g. if } X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{ROC: } |z| > |a|$$

$$\text{then } z_0^n x[n] \xleftrightarrow{ZT} X(\frac{z}{z_0}) = \frac{1}{1 - a(\frac{z}{z_0})^{-1}} = \frac{1}{1 - az_0 z^{-1}}$$

ROC is $|z| > |az_0|$

4. Time-reversal

$$x[-n] \xleftrightarrow{ZT} X(\frac{1}{z}), \qquad \text{ROC: } \frac{1}{R}$$

Properties of ZT

5. Time-expansion

$$x_{(k)}[n] := \begin{cases} x[n/k], & \text{if } n \text{ is multiple of } k \\ 0, & \text{otherwise} \end{cases}$$

$$x_{(k)}[n] \xleftrightarrow{ZT} X(z^k), \qquad \text{ROC: } R^{1/k}$$

Proof:

$$\begin{aligned} X_{(k)}(z) &= \sum_{n=km,\ m=-\infty}^{\infty} x[n/k]z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m](z^k)^{-m} \\ &= X(z^k) \end{aligned}$$

$\square$

Properties of ZT

6. Conjugation

$$x^*[n] \xleftrightarrow{ZT} X^*(z^*), \qquad \text{ROC: } R$$

7. Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{ZT} X_1(z) \cdot X_2(z), \quad \text{ROC } \underline{\text{at least}} \ R_1 \cap R_2$$

Example: $w[n] = u[n] * x[n]$

$$w[n] \xleftrightarrow{ZT} U(z) \cdot X(z) = \frac{1}{1 - z^{-1}} \cdot X(z)$$

ROC contains intersection of R and $|z| > 1$

Properties of ZT

8. Differentiation in the z-domain

$$nx[n] \xleftrightarrow{ZT} -z\frac{d}{dz}X(z), \qquad \text{ROC: } R$$

9. Initial-Value Theorem

If $x[n] = 0,\ n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$, whereas for $n = 0$, $z^{-n} = 1$. $\square$

LTI System

LTI system with $h[n]$, the input & output are related by

$$Y(z) = X(z)H(z)$$

$H(z)$ is called the system function or transfer function of the system

Remember
(1) Eigen-function $x[n] = z^n \rightarrow y[n] = H(z)z^n$
(2) $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ frequency response of the system

LTI System

Causality:

An LTI system is causal iff $h[n] = 0,\ \forall n < 0$

$\implies$ $h[n]$ is right-sided and $H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$

$\implies$ ROC is the exterior of a circle, including ∞

Example:

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}, \quad \rightarrow \text{non-causal}$$

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2, \quad \rightarrow \text{causal}$$

LTI System

Stability:
An LTI system is stable iff $h[n]$ absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

or iff ROC of $H(z)$ includes unit circle $|z| = 1$

LTI System

Example:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

(1) ROC: $|z| > 2 \rightarrow$ causal, non-stable
(2) ROC: $\frac{1}{2} < |z| < 2 \rightarrow$ non-causal, stable
(3) ROC: $\frac{1}{2} > |z| \rightarrow$ non causal, non-stable

Inference:
A causal LTI system with rational $H(z)$ is stable iff all poles are within unit circle.

$\uparrow$ Week 12, Thu, 20180517

$\downarrow$ Week 13, Tue, 20180522

LCC Difference Eqn

General form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\xleftrightarrow{ZT} \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Note: Only $H(z)$ is not enough to find $h[n]$. Need extra information (like causality, stability) to find ROC and then $h[n]$

LCC Difference Eqn

Example: Given

(1) input $x_1[n] = (\frac{1}{6})^n u[n]$, and output:

$$y_1[n] = \left(a(\frac{1}{2})^n + 10(\frac{1}{3})^n \right) u[n]$$

(2) input $x_2[n] = (-1)^n$, and output

$$y_2[n] = \frac{7}{4}(-1)^n$$

Find the system LCC difference equation

LCC Difference Eqn

Answer:
From (1), $X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}$, $|z| > \frac{1}{6}$

$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}} = \frac{(a+10)-(5+\frac{a}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, |z| > 1/2$$

From (2),

$$H(-1) = \frac{7}{4} = \frac{Y_1(-1)}{X_1(-1)}$$

$$\Longrightarrow \frac{7}{4} = H(-1) = \frac{(a+10+5+\frac{a}{3})\cdot\frac{7}{6}}{\frac{3}{2}\cdot\frac{4}{3}}$$

$$\Longrightarrow a = -9$$

LCC Difference Eqn

$$\implies H(z) = \frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{z^2-\frac{13}{6}z+\frac{1}{3}}{z^2-\frac{5}{6}z+\frac{1}{6}}$$

Possible ROCs for $H(z)$: $|z| > \frac{1}{2}$, $\frac{1}{3} < |z| < \frac{1}{2}$, $|z| < \frac{1}{3}$

Since ROC of $Y_1(z)$ includes ROC of $X_1(z) \cap H(z)$,
$\implies$ ROC of $H(z)$ is $|z| > \frac{1}{2}$

$\implies$ the system is stable (includes $|z| = 1$) and casual (rational and exterior to the rightmost pole)

LCC Difference Eqn

The system can be characterized by:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{13}{6}x[n-1] + \frac{1}{3}x[n-2]$$

Block Diagram

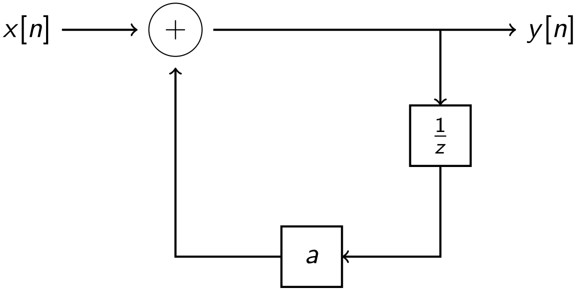
Block diagram for LTI characterized by LCC Difference Eqn

- Time-shifting $x[n] \rightarrow x[n-1]$

$$\longrightarrow \boxed{\frac{1}{z}} \longrightarrow$$

direct, parallel, series forms

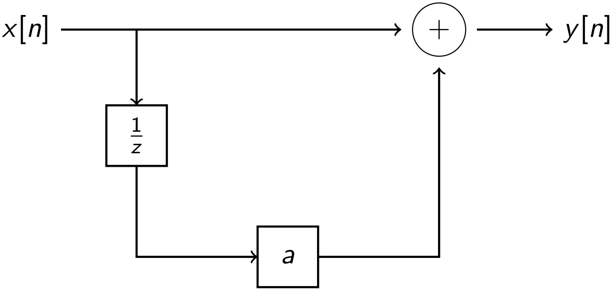
Block Diagram



$$y[n] = x[n] + a \cdot y[n-1]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

Block Diagram



$$y[n] = x[n] + a \cdot x[n-1]$$
$$H(z) = 1 + az^{-1}$$

Example 10.27

System: LTI + Causal + Stable + Rational $H(z)$.
$H(z)$ contains a pole $z = \frac{1}{2}$, a zero somewhere on unit circle, other poles and zeros are unknown.

The followings are true, false, or insufficient to determine?

(a) $F\{(\frac{1}{2})^n h[n]\}$ converges
(b) $H(e^{j\omega}) = 0$ for some ω
(c) $h[n]$ has finite duration
(d) $h[n]$ is real
(e) $g[n] = n \cdot (h[n] * h[n])$ is the impulse response of a stable system

Example 10.27

Answer:

(a) $F\{(\frac{1}{2})^n h[n]\}$ converges?

$$F\{(\frac{1}{2})^n h[n]\} = \sum_n (\frac{1}{2})^n h[n] e^{-j\omega n} = \sum_n h[n](2e^{j\omega})^{-n}$$

equivalent to ROC contains $|z| = 2$.
True since ROC contains the area exterior to the unit circle for LTI + stable + causal

Example 10.27

(b) $H(e^{j\omega}) = 0$ for some ω?

True: Since there is a zero on unit circle, implies $H(z) = 0$ for some $z = e^{j\omega}$

(c) $h[n]$ has finite duration?

False. If true, ROC includes $|z| \in (0, \infty)$, whereas $z = \frac{1}{2}$ is a pole.

Example 10.27

(d) $h[n]$ is real?

If true, $H(z) = H(z^*)^*$. Information is not sufficient

(e) system $g[n] = n \cdot (h[n] * h[n])$ is stable?

$G(z) = -z\frac{d}{dz}(H(z) \cdot H(z))$, ROC is at least R_H (actually equals, why?), includes unit circle, thus true

Causality Revisited

LTI + Causality:
An LTI system is causal iff $h[n] = 0, \ \forall n < 0$
$\Longleftrightarrow$ $h[n]$ is right-sided and $h[n] = 0$ for $n < 0$
$\Longleftrightarrow$ ROC is the exterior of a circle, including ∞

LTI + Rational + Causality:
A rational LTI system is causal
$\Longleftrightarrow$ ROC is the exterior of a circle outside the outermost pole, including ∞
$\Longleftrightarrow$ ROC is the exterior of a circle outside the outermost pole, the order of numerator $\leq$ the order of denominator in rational $H(z)$ expressed as ratio of polynomial $P(z)/Q(z)$

Causality Revisited

Example:

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}, \quad \rightarrow \text{non-causal}$$

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2, \quad \rightarrow \text{causal}$$

Summary

- ZT and inverse ZT (using partial fraction expansion)
- ROC
- properties of ZT:
  linearity, time shifting, scaling in the z-domain, time reverse and expansion, conjugation, convolution, differentiation in z-domain, the initial-value theorem.
- common ZT pairs
- analysis and characterization of LTI system using ZT:
  causality, stability, LTI system characterized by LCC difference equations (to find $h[n]$ or $H(z)$)