Signals and Systems Solution 7

Due Time: 21:59 May 4, 2018
Submitted in-class on Thu (May 4),
or to the box in front of SIST 1C 403E (the instructor's office).

1. (20 points) The following are discrete-time signals and Fourier transforms. Determine the signal/FT for each one.

   (a) $x_1[n] = (\frac{1}{2})^{|n-1|}$

   (b) $\sin(\frac{\pi}{3}n+\frac{\pi}{4})$ (Determine the Fourier transform for $-\pi \leq \omega < \pi$. Hint: It's the Fourier transform for periodic signals).

   (c) $X_1(jw) = \frac{e^{-jw}-\frac{1}{5}}{1-\frac{1}{5}e^{-jw}}$

   (d) $X_2(jw) = \sum_{k=-\infty}^{+\infty}(-1)^k\delta(\omega - \frac{\pi}{2}k)$

   Solution

   (a) Using the Fourier transform equation, we can have that

$$
\begin{aligned}
X(j\omega) &= \sum_{n=-\infty}^{\infty} x_1[n]e^{-jwn} = \sum_{n=-\infty}^{0}(\frac{1}{2})^{-(n-1)}e^{-jwn} + \sum_{n=1}^{\infty}(\frac{1}{2})^{(n-1)}e^{-jwn} \\
&= \sum_{n=0}^{\infty}(\frac{1}{2})^{(n+1)}e^{jwn} + \sum_{n=0}^{\infty}(\frac{1}{2})^{n}e^{-jw(n+1)} \\
&= \frac{1}{2[1-\frac{1}{2}e^{j\omega}]} + \frac{e^{-j\omega}}{[1-\frac{1}{2}e^{-j\omega}]}
\end{aligned}
$$

   (b) Consider the signal $x_2[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$, the fundamental period is $N = 6$.
   The signal may be written as

$$
\begin{aligned}
x_2[n] &= \frac{1}{2j}e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - \frac{1}{2j}e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} \\
&= \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}
\end{aligned}
$$

   From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-2 \leq k \leq 3$ is

$$
a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}} \qquad\qquad a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}
$$

   Therefore, in the range $-\pi \leq \omega < \pi$, we have

$$
\begin{aligned}
X(j\omega) &= 2\pi a_1\sigma(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\sigma(\omega + \frac{2\pi}{6}) \\
&= \frac{\pi}{j}[e^{j\frac{\pi}{4}}\sigma(\omega - \frac{\pi}{3}) - e^{-j\frac{\pi}{4}}\sigma(\omega + \frac{\pi}{3})]
\end{aligned}
$$

   (c) The given Fourier transform may be written as

$$
\begin{aligned}
X(j\omega) &= \frac{e^{-jw}}{1-\frac{1}{5}e^{-jw}} - \frac{\frac{1}{5}}{1-\frac{1}{5}e^{-jw}} \\
&= e^{-jw}\sum_{n=0}^{\infty}(\frac{1}{5})^{n}e^{-jwn} - \frac{1}{5}\sum_{n=0}^{\infty}(\frac{1}{5})^{n}e^{-jwn}
\end{aligned}
$$

   Then we can have

$$
x[n] = (\frac{1}{5})^{n-1}u[n-1] - (\frac{1}{5})^{n+1}u[n]
$$

(d) $X_2(j\omega)$ is the Fourier transform of a periodic signal, from the expression we can have

$$\omega = \frac{\pi}{2} \qquad\qquad N = 4 \qquad\qquad a_k = \frac{(-1)^k}{2\pi}$$

Therefore, the signal is given by

$$x[n] = \frac{1}{2\pi}\sum_{k=0}^{3}(-1)^k e^{jk(\frac{\pi}{2})n} = \frac{1}{2\pi}[1 - e^{j\frac{\pi}{2}n} + e^{j\pi n} - e^{j\frac{3\pi}{2}n}]$$

2. (15 points) Given that $x[n]$ has Fourier transform $X(jw)$, express the Fourier transforms of the following signals in the terms of $X(jw)$.

(a) $x_1[n] = x[1-n] + x[-1-n]$.

(b) $x_2[n] = \frac{x^*[-n]+x[n]}{2}$.

(c) $x_3[n] = (n-1)^2 x[n]$

Solution

(a) Using the time reversal property, we have

$$x[-n] \longleftrightarrow X(-j\omega)$$

Using the time shift property on this, we have

$$x[-n+1] \longleftrightarrow e^{-j\omega}X(-j\omega) \qquad and \qquad x[-n-1] \longleftrightarrow e^{j\omega}X(-j\omega)$$

Therefore

$$x_1[n] \longleftrightarrow e^{-j\omega}X(-j\omega) + e^{j\omega}X(-j\omega) = 2\cos\omega X(-jw)$$

(b) Using the same conjugation property, we have

$$x^*[-n] \longleftrightarrow X^*(-j\omega)$$

Therefore

$$x_2[n] \longleftrightarrow \frac{1}{2}[X(j\omega) + X^*(j\omega)] \longleftrightarrow \Re e\{X(j\omega)\}$$

(c) Using the differentiation frequency property, we have

$$nx[n] \longleftrightarrow j\frac{dX(j\omega)}{d\omega} \qquad and \qquad n^2x[n] \longleftrightarrow -\frac{d^2X(j\omega)}{d\omega^2}$$

Therefore

$$x_3[n] \longleftrightarrow -\frac{d^2X(j\omega)}{d\omega^2} - 2j\frac{dX(j\omega)}{d\omega} + X(j\omega)$$

3. (15 points) Let

$$y[n] = (\frac{\sin\frac{\pi}{4}n}{\pi n})^2 * (\frac{\sin\omega_c n}{\pi n})$$

where $*$ denotes convolution and $|\omega_c n| \leq \pi$. Determine a stricter constraint on $\omega_c n$, which ensures that

$$y[n] = (\frac{\sin\frac{\pi}{4}n}{\pi n})^2$$

Solution
Consider the signal

$$x_1[n] = (\frac{\sin \frac{\pi}{4} n}{\pi n})$$

The Fourier transform of $x_1[n]$ is

$$X_1(j\omega) = \begin{cases} 1 & 0 \leq |\ \omega\ | \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\ \omega\ | \leq \pi \end{cases}$$

Now consider the signal $x_2[n] = (x_1[n])^2$. Using the multiplication property, we obtain the Fourier transform of $x_2[n]$

$$X_2(j\omega) = \frac{1}{2\pi}[X_1(j\omega) * X_1(j\omega)]$$

It's clear that $X_2(j\omega)$ is zero for $\frac{\pi}{2} \leq |\ \omega\ | \leq \pi$. Meanwhile, $FT\{\frac{\sin \omega_c n}{\pi n}\}$ is zero for $\omega_c \leq |\ \omega\ | \leq \pi$. Hence, ω_c must be satisfied that

$$\frac{\pi}{2} \leq |\ \omega_c\ | \leq \pi$$

4. (15 points) Let $x_1[n]$ be the discrete-time signal whose Fourier transform $X_1(jw)$ is depicted in Figure 1. Consider the signal $x_2[n]$ with Fourier transform $X_2(jw)$, as illustrated in Figure 2. Please express $x_2[n]$ in terms of $x_1[n]$.

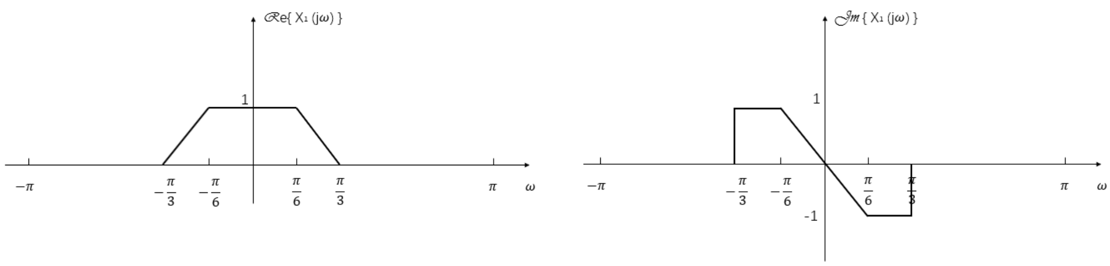

Figure 1: The real and imaginary parts of the Fourier transform $X_1(jw)$

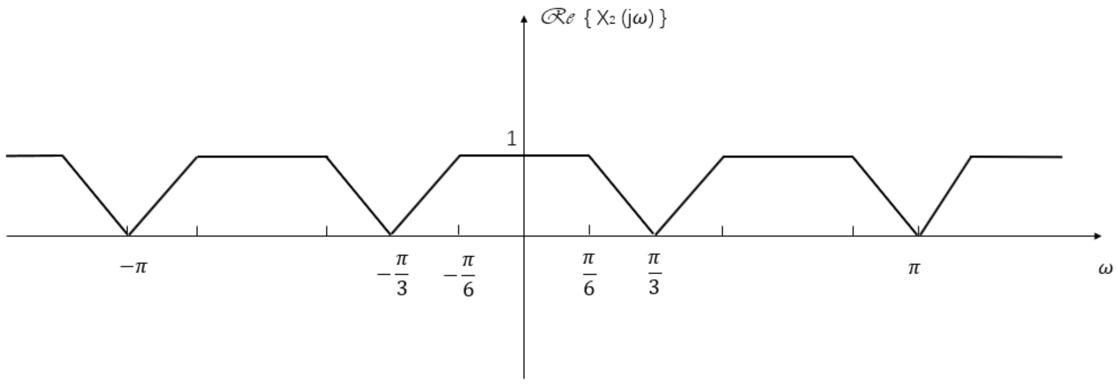

Figure 2: the Fourier transform $X_2(jw)$

Solution
From the Figure, we may express $X_2(jw)$ as

$$X_2(jw) = \Re e\{X_1(jw)\} + \Re e\{X_1(j(w - \frac{2\pi}{3}))\} + \Re e\{X_1(j(w + \frac{2\pi}{3}))\}$$

Using the conjugate symmetry property on this, we have

$$\Re e\{X_1(jw)\} \longleftrightarrow \varepsilon\nu\{x_1[n]\}$$

Using the frequency shift property on this, then

$$x_2[n] = \varepsilon\nu\{x_1[n]\}[1 + e^{j\frac{2\pi}{3}} + e^{-j\frac{2\pi}{3}}]$$

5. (15 points) Let $x[n] = e^{jwn}$ for $0 \le n < N$ and let $X[k]$ be the DFT of $x[n]$.

   (a) Calculate a simplified expression for $X[k]$ that is correct for any value of ω.

   (b) Calculate a simplified expression for $X[k]$ when $\omega = 2\pi m/N$ where m is an integer. And sketch a plot of $\mid X[k] \mid$

Solution

(a)

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} e^{jwn} e^{-j\frac{2\pi k}{N}n} \\ &= \frac{1 - e^{j(wN - 2\pi k)}}{1 - e^{j(w - \frac{2\pi k}{N})}} \qquad \text{since} e^{-j2\pi k} = 1 \text{ for } k \in Z \\ &= \frac{1 - e^{jwN}}{1 - e^{j(w - \frac{2\pi k}{N})}} \end{aligned}$$

(b) since $\omega = \frac{2\pi m}{N}$, for $m \in Z$
then we may write $x[n]$ as

$$x[n] = e^{j\frac{2\pi m}{N}n} = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi m}{N}n}$$

$$\Longrightarrow \qquad X[k] = \sigma[k - m]$$

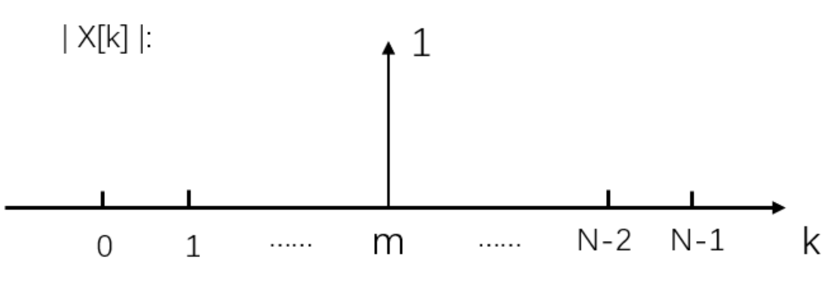

Figure 3:

6. (20 points) Let $x[n]$ be a signal of finite duration, that is, there is an integer N so that

$$x[n] = 0 \qquad \text{outside the interval } 0 \le n \le N-1$$

The DFT of $x[n]$ is denoted by $X[k]$, and $X(j\omega)$ denote the Fourier transform of $x[n]$.

(a) Show that

$$X[k] = \frac{1}{N} X(j(2\pi k/N))$$

(b) Let us consider samples of $X(j\omega)$ taken every $\frac{2\pi}{M}$, where $M < N$. These samples correspond to more than one sequence of duration N. To illustrate this, consider the two signals $x_1[n]$ and $x_2[n]$ depicted in Figure 3. Show that if we choose $M = 4$, we have

$$X_1(2\pi k/4) = X_2(j(2\pi k/4))$$

for all values of k.

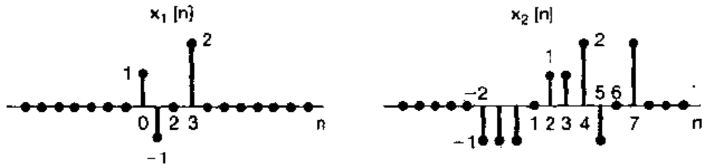

Figure 4: $x_1[n]$ and $x_2[n]$

Solution

(a) The analysis equation of the Fourier transform is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Comparing with the analysis equation of DFT,

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n}$$

then we have

$$X[k] = \frac{1}{N} X(j(2\pi k/N))$$

(b) From the figures we obtain

$$X_1(j\omega) = 1 - e^{-j\omega} + 2e^{-3j\omega}$$

and

$$X_2(j\omega) = -e^{2j\omega} - e^{j\omega} - 1 + e^{-2j\omega} + e^{-3j\omega} + 2e^{-4j\omega} - e^{-5j\omega} + 2e^{-7j\omega}$$

Now

$$X_1(j\frac{2\pi k}{4}) = 1 - e^{-j\frac{\pi k}{2}} + 2e^{-j\frac{3\pi k}{2}}$$

$$X_2(j\frac{2\pi k}{4}) = 1 - e^{-j\frac{\pi k}{2}} + 2e^{-j\frac{3\pi k}{2}} = X_1(j\frac{2\pi k}{4})$$