Lecture 7 – Image Reconstruction （图像重建）

This lecture will cover:

– Reconstruction modalities（重建模式）

– Reconstruction from projection（投影重建算法）

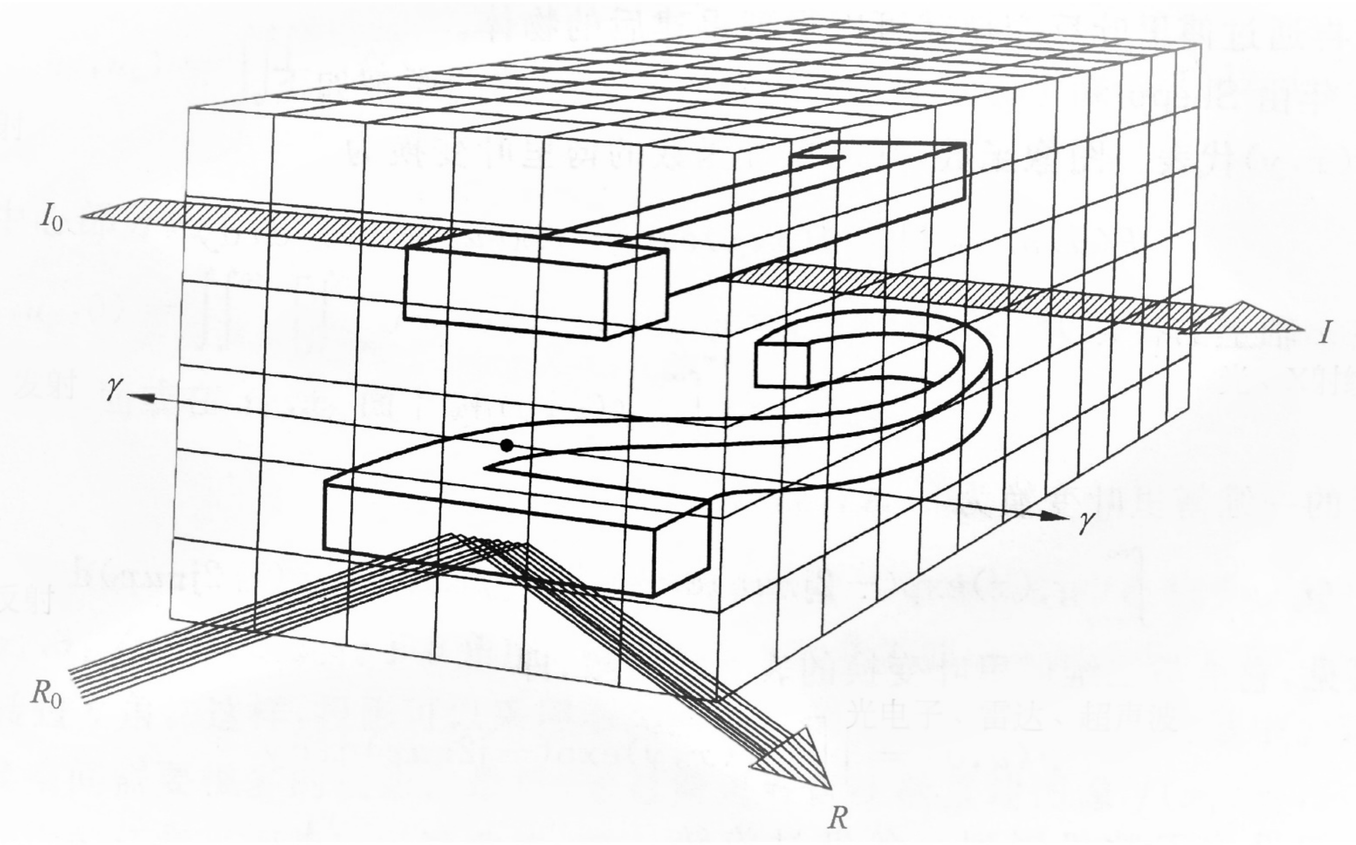
- Computed Tomography (计算机断层成像)
- Radon transform（雷登变换）
- The Fourier-Slice Theorem（傅里叶切片定理）
- Parallel-Beam Filtered Backprojections（平行射线束滤波反投影）
- Fan-Beam Filtered Backprojections（扇形射线束滤波反投影）

– Reflection imaging

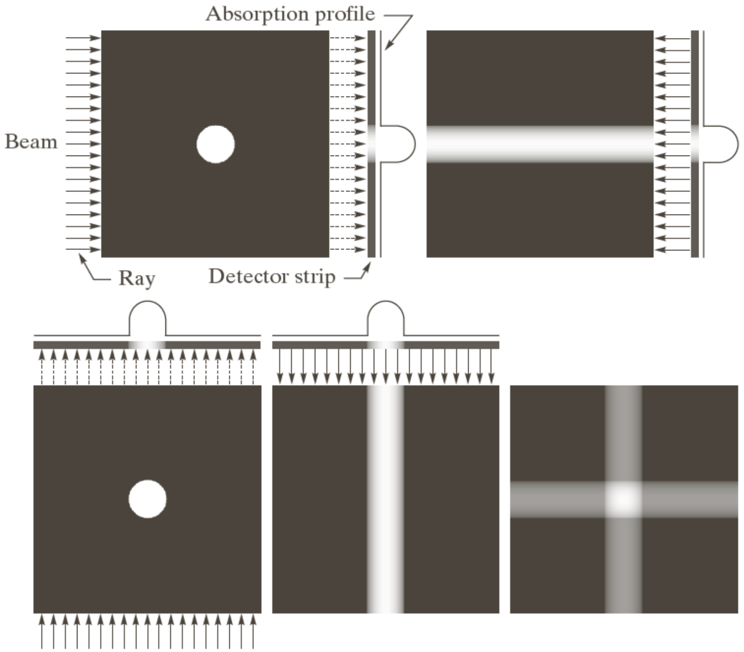
- Time of flight
- Born Approximation and Inverse theory（玻恩近似与反演理论）

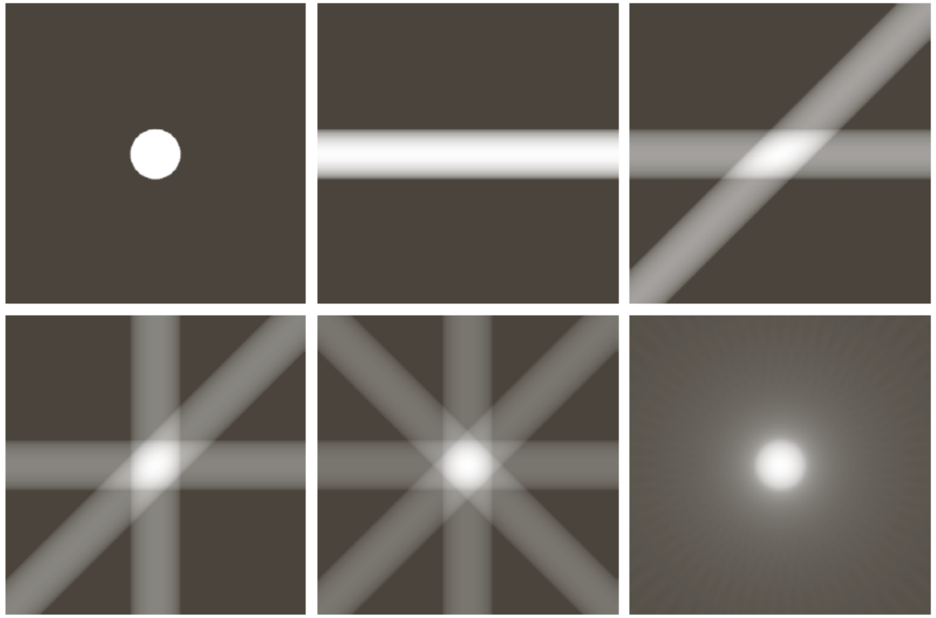
Reconstruction Modalities

- **Transmission**
- **Emission**
- **Reflection**

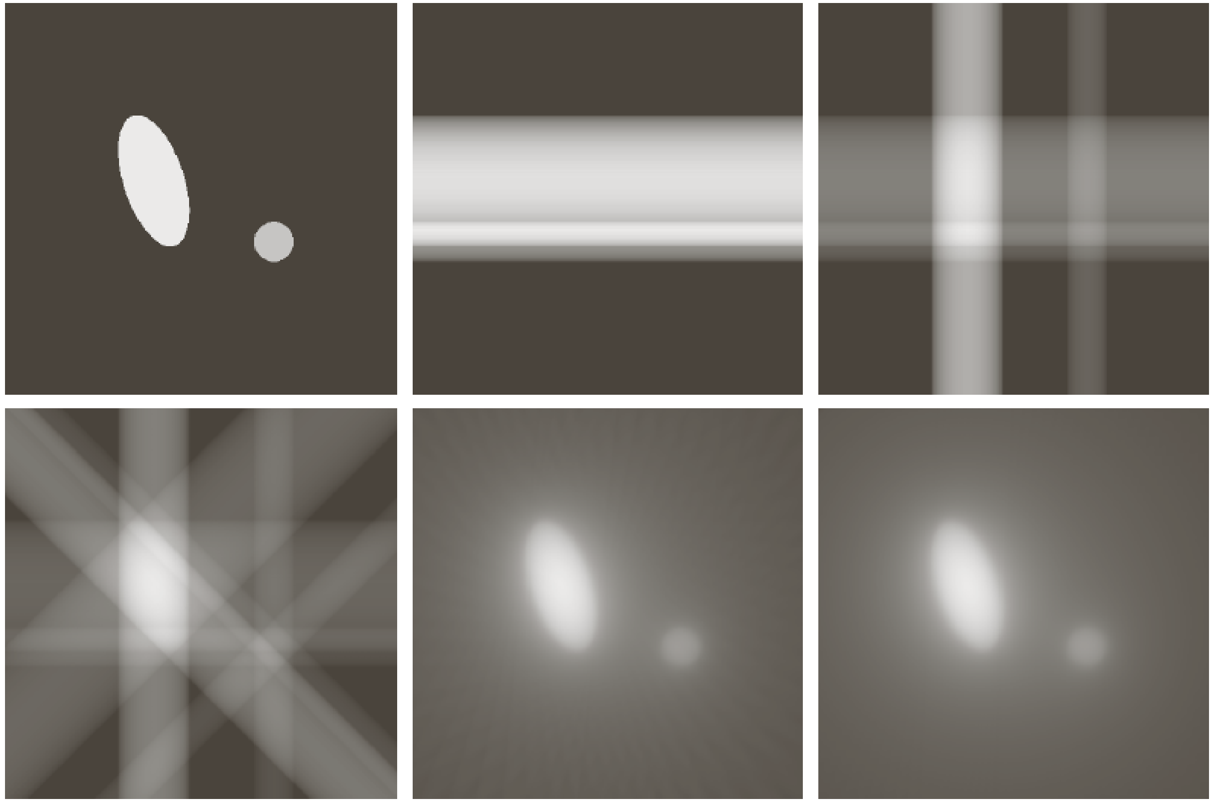


Back Projection

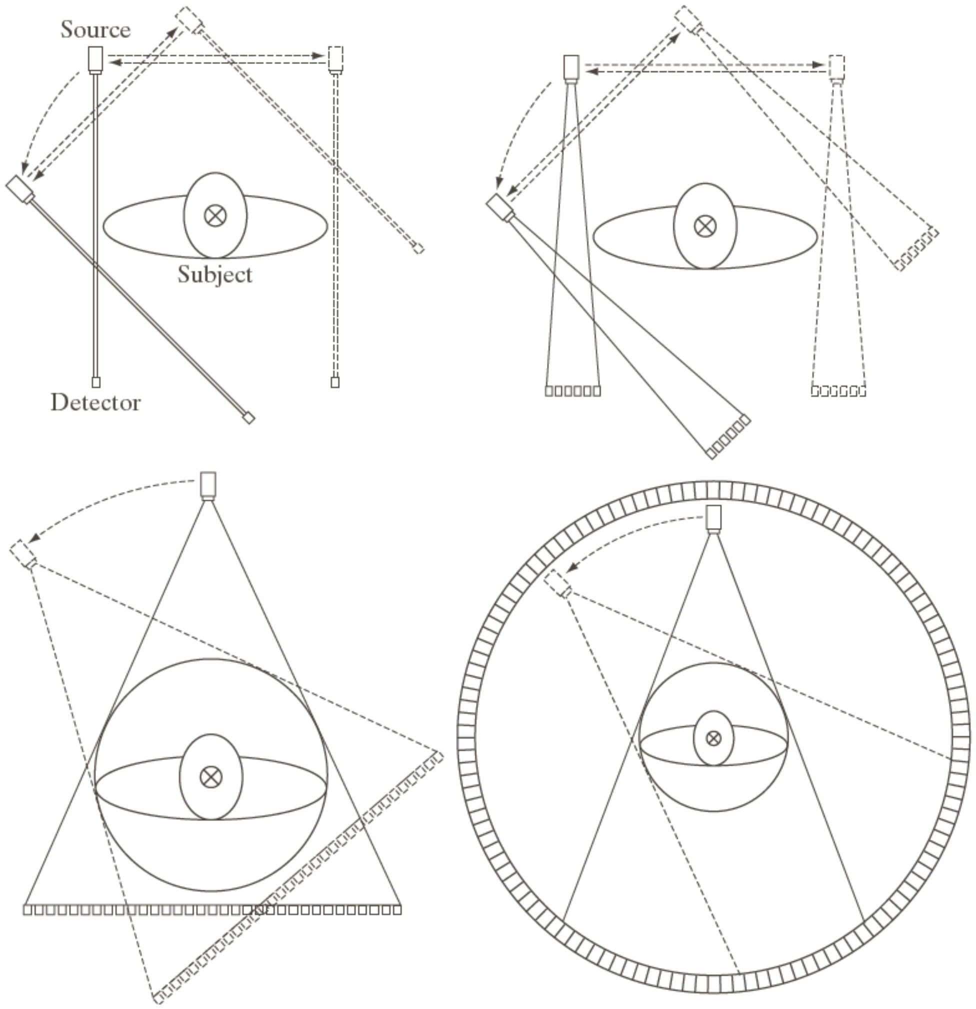




Back Projection



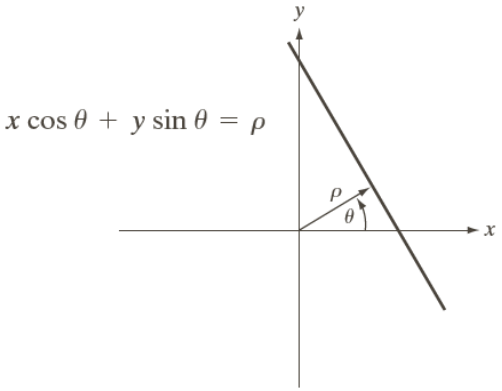
Computed Tomography



Radon Transform

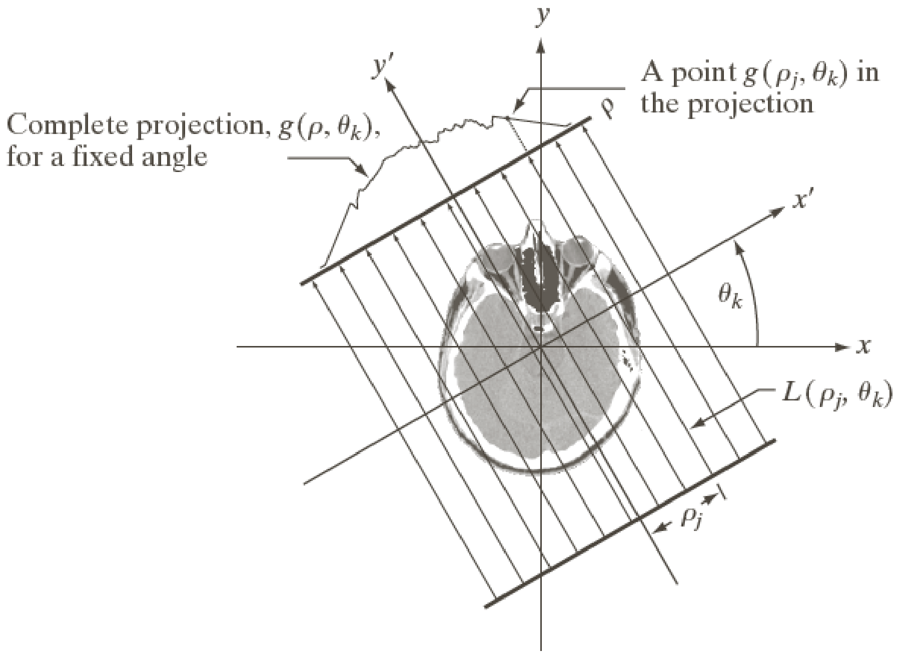
Normal representation for a line:

$$x \cos\theta + y \sin\theta = \rho$$

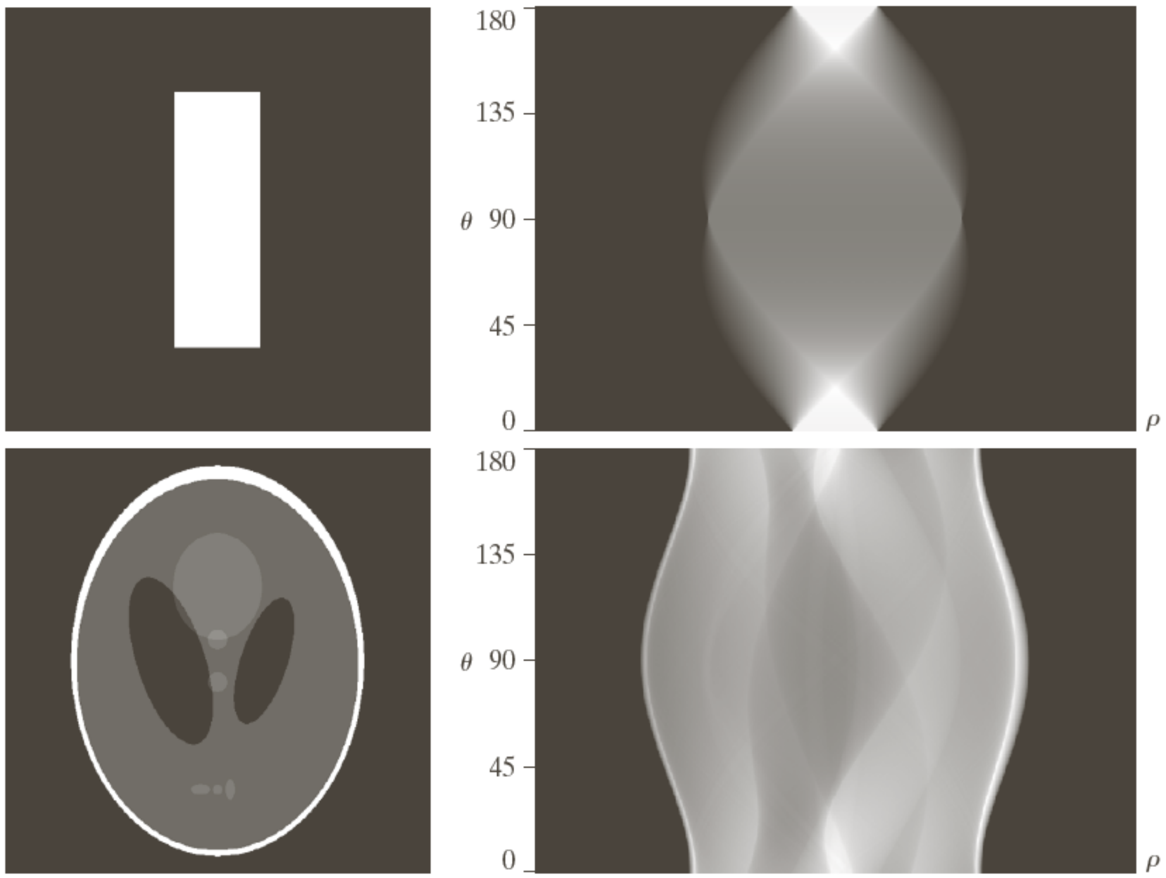


The projection of $f(x, y)$ along an arbitrary line in the xy-plane:

$$g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y-0}^{N-1} f(x, y)\delta(x \cos\theta + y \sin\theta - \rho)$$



Sinogram (正弦图)



Back Projection from Radon Transform

For a fixed value of rotation θ_k:

$$f_{\theta_k}(x, y) = g(\rho, \theta_k) = g(x\cos\theta_k + y\sin\theta_k, \theta_k)$$

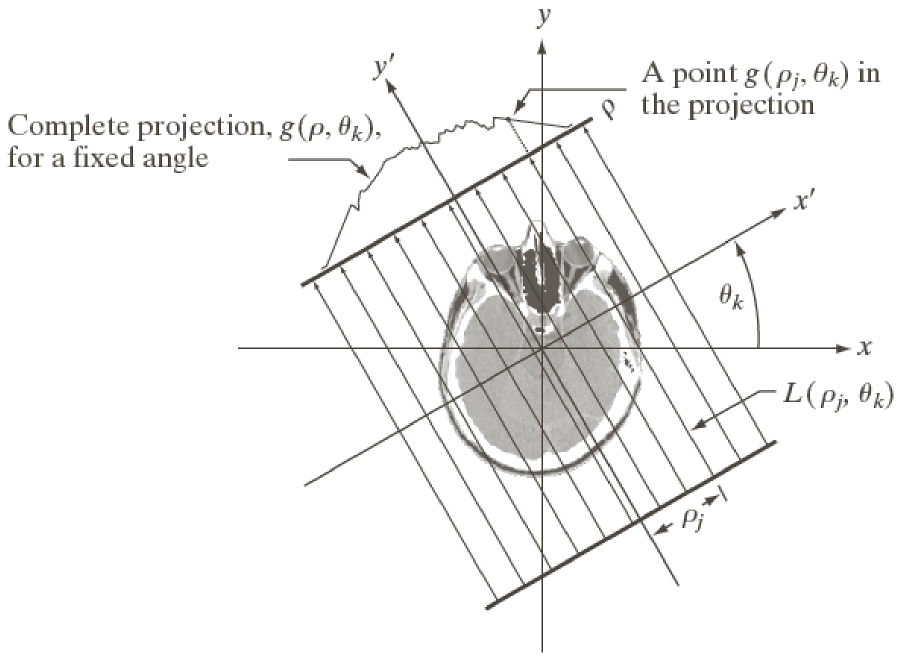
Then a single backprojection obtained at an angle θ :

$$f_\theta(x, y) = g(x\cos\theta + y\sin\theta, \theta)$$

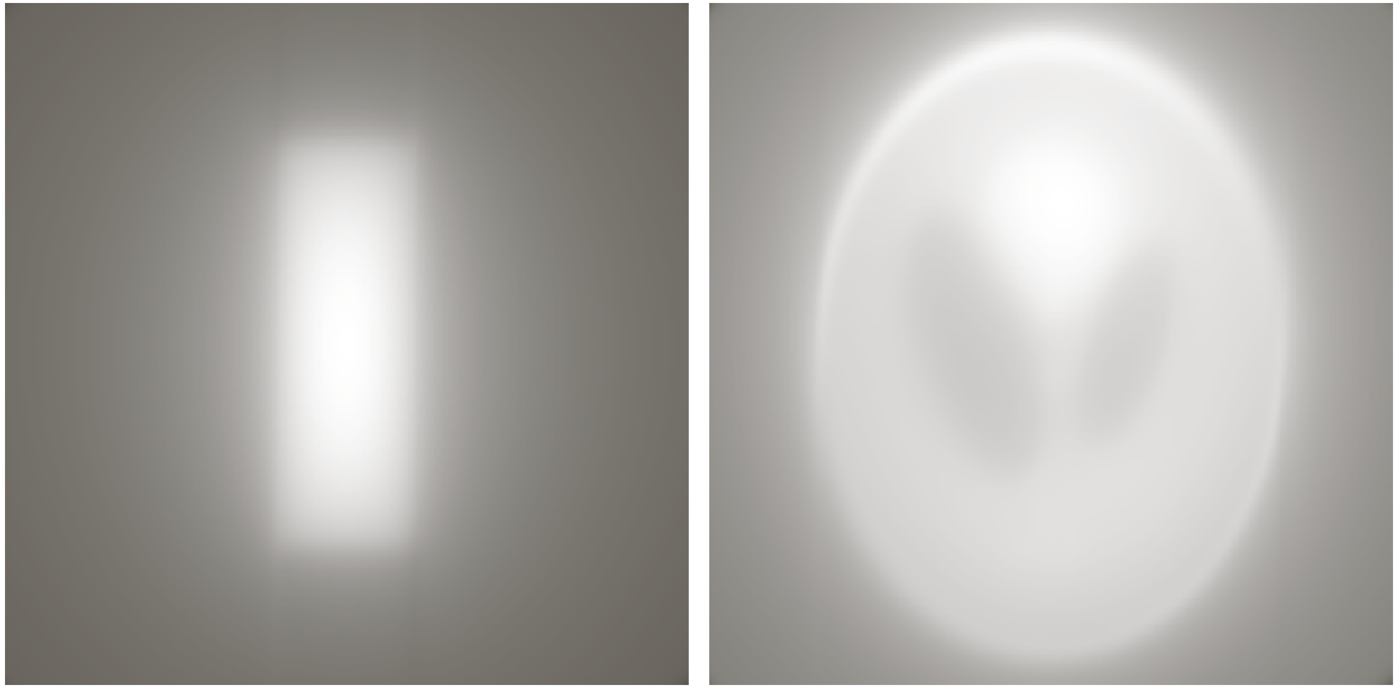
Where $g(\rho, \theta)$ is the projection value.

The final image by summing over all the back-projected images

$$f(x, y) = \sum_{\theta=0}^{\pi} f_\theta(x, y)$$



Back Projection from Radon Transform



The Fourier-Slice Theorem

The 1D FT of a projection with respect of ρ:

$$G(\omega,\theta)=\int_{-\infty}^{\infty} g(\rho,\theta)e^{-j2\pi\omega\rho}d\rho$$

where projection $g(\rho,\theta)$ is

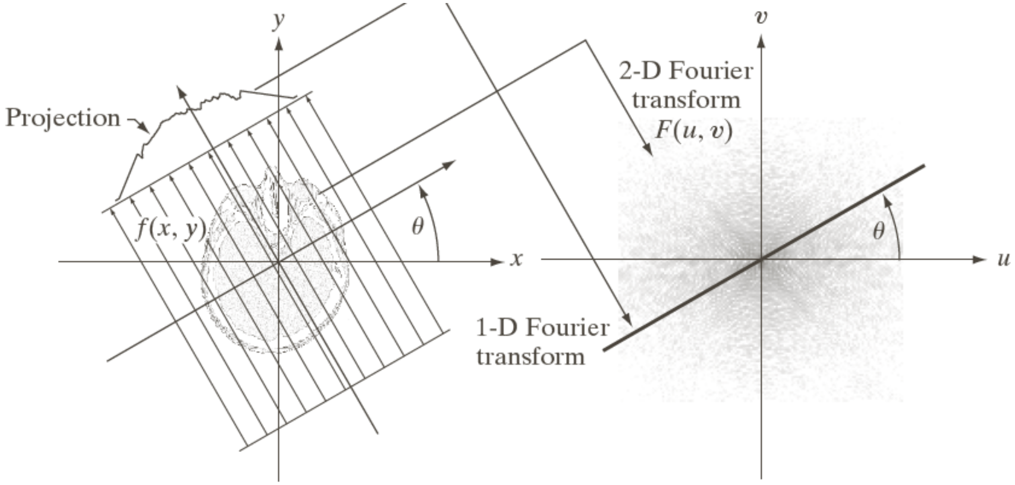
$$g(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)dxdy$$

then

$$G(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dxdy$$

$$=\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy\right]_{u=\omega\cos\theta;v=\omega\sin\theta}$$

Therefore $\quad G(\omega,\theta)=F(\omega\cos\theta,\omega\sin\theta)$



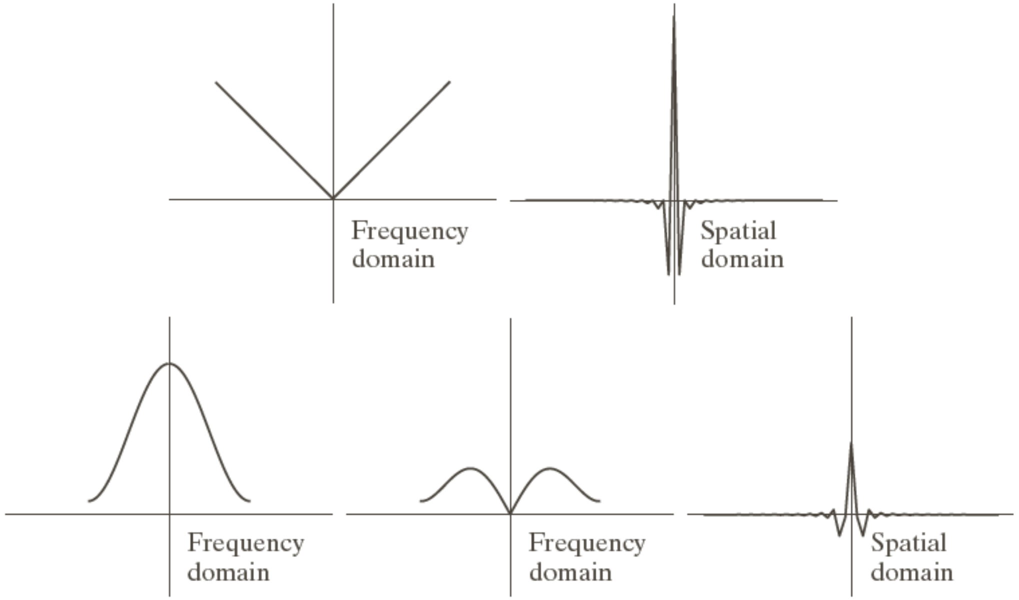
Parallel-Beam Filtered Backprojections

The 2D IFT of $F(u,v)$ with Fourier-slice theorem:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta$$

$$= \int_{0}^{\pi}\left[\int_{-\infty}^{\infty} |\omega| G(\omega,\theta)e^{j2\pi\omega\rho}d\omega\right]_{\rho=x\cos\theta+y\sin\theta} d\theta$$

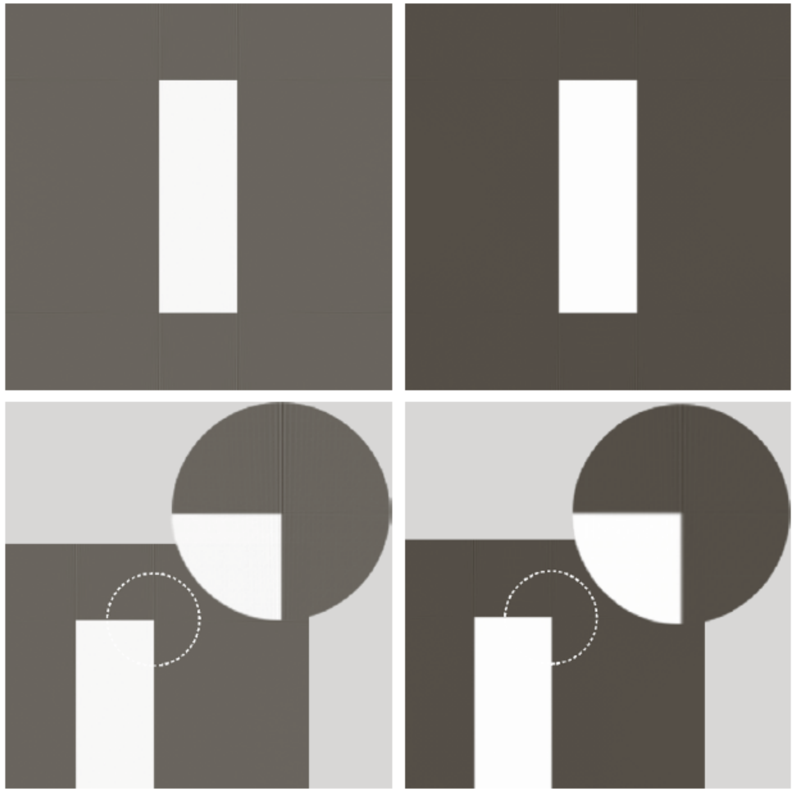


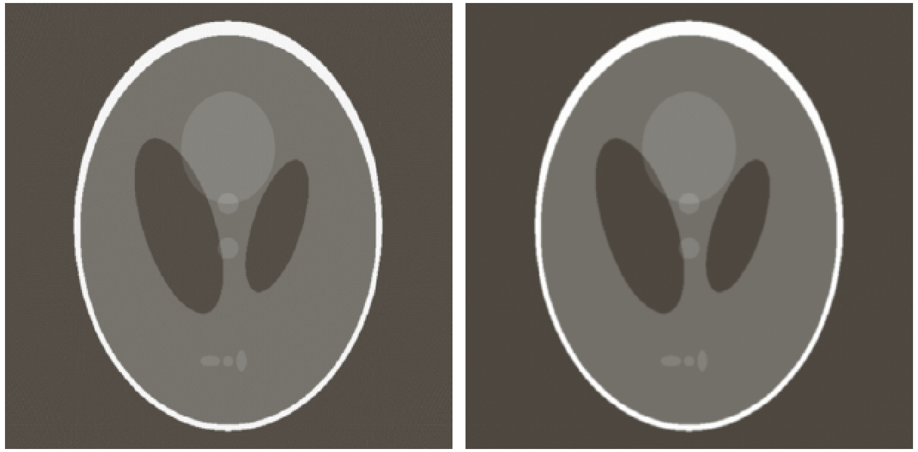
Convolution backprojection

$$f(x,y) = \int_{0}^{\pi} [s(\rho)\star g(\rho,\theta)]_{\rho=x\cos\theta+y\sin\theta} d\theta$$

Where $s(\rho) = \mathrm{IFT}(|\omega|)$, $g(\rho,\theta) = \mathrm{IFT}[G(\omega,\theta)]$

Parallel-Beam Filtered Backprojections



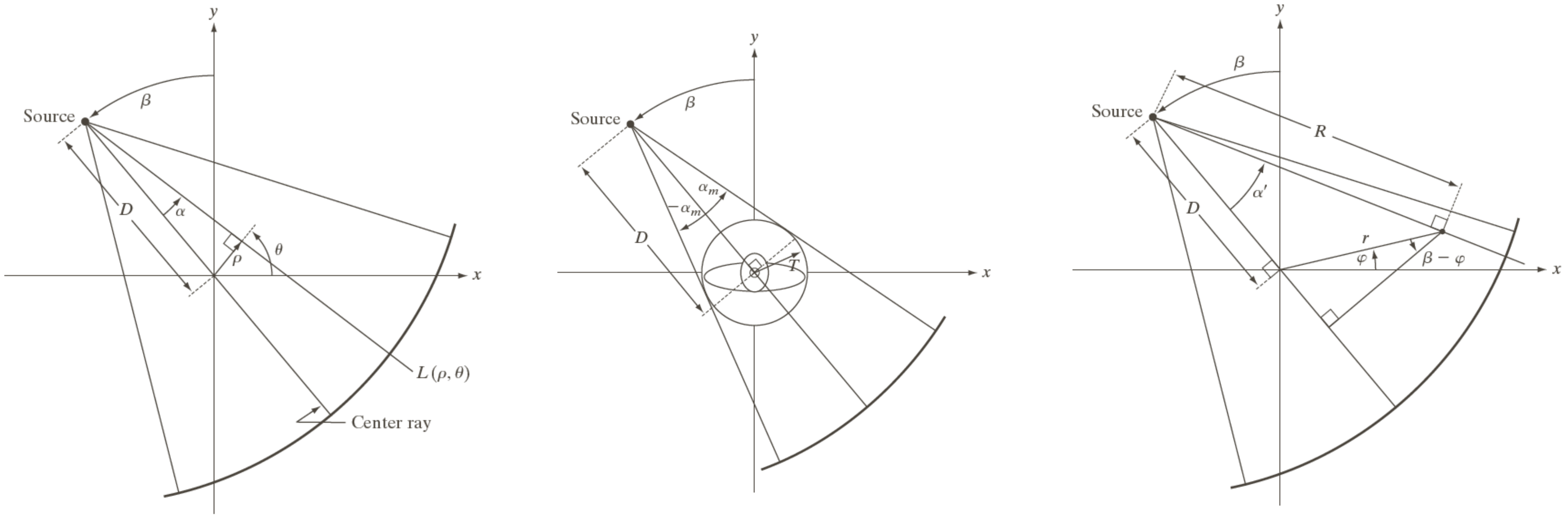


Fan-Beam Filtered Backprojections

Fundamental fan-beam reconstruction based on filtered backprojection:

$$f(r,\varphi)=\int_0^{2\pi}\frac{1}{R^2}\left[\int_{-\alpha_m}^{\alpha_m}q(\alpha,\beta)h(\alpha'-\alpha)\mathrm{d}\alpha\right]\mathrm{d}\beta$$

Where $h(\alpha)=\frac{1}{2}\left(\frac{\alpha}{\sin\alpha}\right)^2 s(\alpha)$, $\quad q(\alpha,\beta)=p(\alpha,\beta)D\cos\alpha$, $\quad p(\alpha,\beta)=g(\rho,\theta)=g(D\sin\alpha,\alpha+\beta)$



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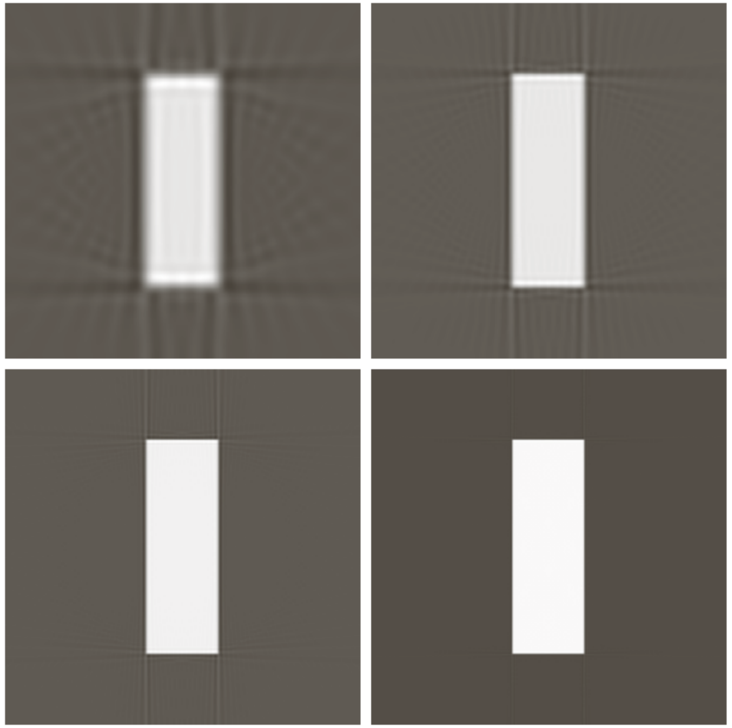
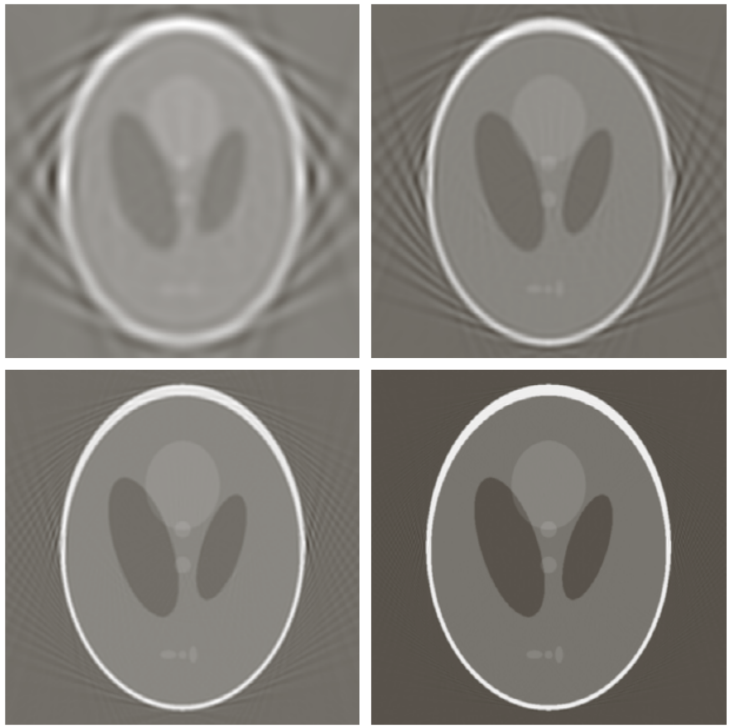
Fan-Beam Filtered Backprojections

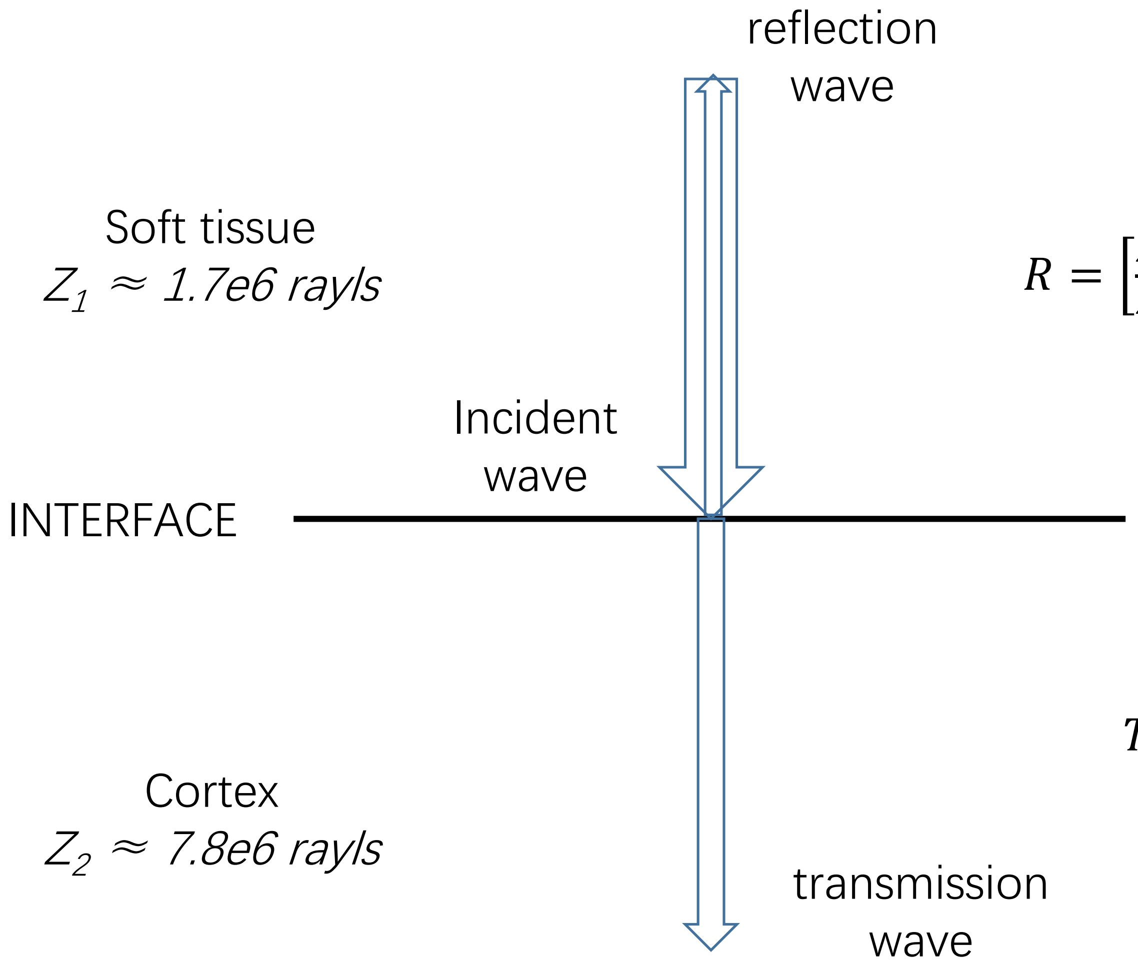




Ultrasound wave reflection

Soft tissue

$Z_1 \approx 1.7e6$ *rayls*



$$R = \left[\frac{Z_2 - Z_1}{Z_2 + Z_1}\right]^2 = 0.41$$

$$T = 1 - R$$

Cortex

$Z_2 \approx 7.8e6$ *rayls*

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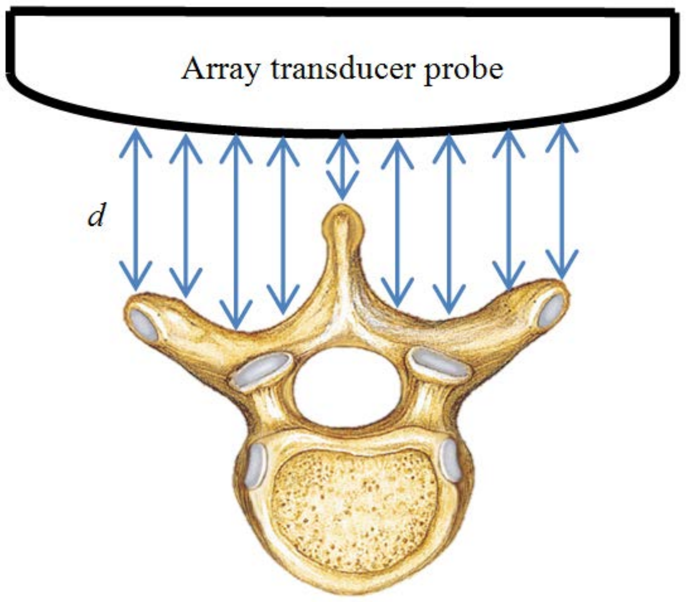
Ultrasound imaging

- Reflection from the interface between different tissues
- Using time of flight (*t*) to determine the distance (*d*) and locate the structures

$$t = \frac{2 * d}{V}$$

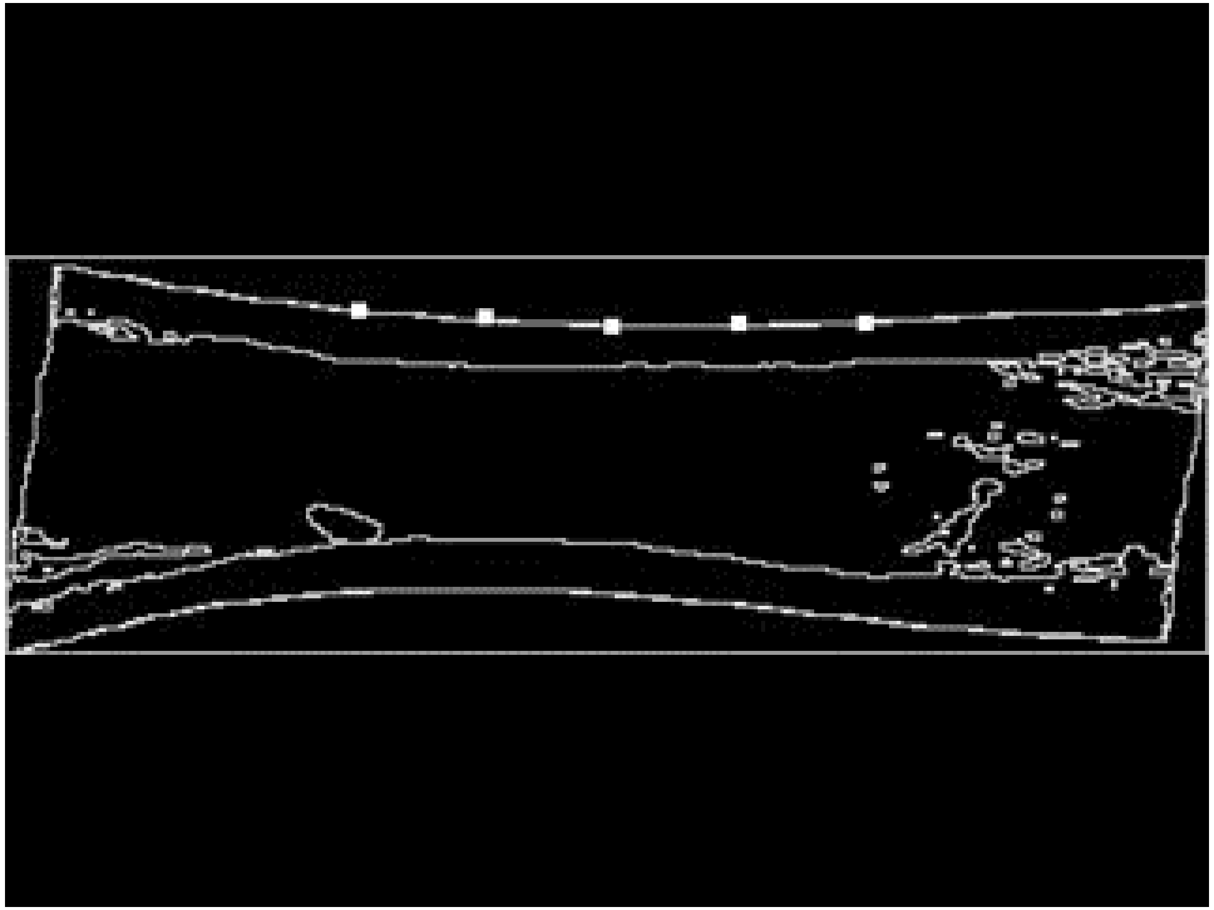
- Use gray level to indicate the amplitude (B mode)



The schematic of ultrasound reflection waves from the vertebra cortex / soft tissue interface.

Ultrasound image

Wave Propagation



Born Approximation

Based on the perturbation velocity profile assumption:

$$\frac{1}{c^2(\vec{x})} = \frac{1}{c_0{}^2(\vec{x})} + f(\vec{x})$$

An approximate solution for acoustic wave equation (Born Approximation) is

$$d(\vec{s},\vec{r},\omega) = \omega^2 \int d^3x G_0(\vec{x},\vec{r},\omega) f(\vec{x}) G_0(\vec{s},\vec{x},\omega)$$

***$G_0(x,y,\omega)$* : Green function**

$$G_0(\vec{x},\vec{y},\omega) = A(\vec{x},\vec{y}) e^{-j\omega\tau(\vec{x},\vec{y})}$$

(*Miller et al. 1987, Weglein, 1982; Nemeth et al, 1999*)

Operator & cost function

Rewrite in operator form: $D(\vec{s},\vec{r}) = L(\vec{s},\vec{r},\vec{x},\omega_i) * F(\vec{x})$

Where, the cost function ***J*** $J = \left\| D - D^{obs} \right\|_2^2$

Where

D : the theoretical data ***D**^obs^*: the observed data

Inversion is to minimize *J* to find the best solution of $F(\vec{x})$

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Operator & cost function

- Adjoint operator：$F(\vec{x}) = L^{*}(\vec{s},\vec{r},\vec{x},\omega) * D(\vec{s},\vec{r})$

- Minimum norm solution (LS)：$J = \left\| LF - D \right\|_2^2$

- Damped minimum norm solution (DLS)：$J = \left\| LF - D \right\|_2^2 + \mu \left\| F \right\|_2^2$

- Weighted minimum norm solution (WLS)：$J = \left\| LF - D \right\|_2^2 + \mu \left\| WF \right\|_2^2$

Conjugate Gradient Method

Consider a system of linear equations： **L**x=y

Let $\mathbf{x} = \mathbf{x}_0$, $\mathbf{p}_0 = \mathbf{r}_0 = \mathbf{y} - \mathbf{L}\mathbf{x}_0$, and k **=0**. The following steps will be repeated until number of iterations or the tolerance limit for convergence is reached.

$$\text{(a)} \quad \alpha_k = \frac{\mathbf{r}_k{}^T \mathbf{r}_k}{\mathbf{p}_k{}^T \mathbf{L} \mathbf{p}_k} \quad .$$

$$\text{(b)} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \quad .$$

$$\text{(c)} \quad \mathbf{r}_{k+1} = \mathbf{y} - \mathbf{L}\mathbf{x}_{k+1} \quad .$$

$$\text{(d)} \quad \beta_k = \frac{\mathbf{r}_{k+1}{}^T \mathbf{r}_{k+1}}{\mathbf{r}_k{}^T \mathbf{r}_k} \quad .$$

$$\text{(e)} \quad \mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k \quad .$$

$$\text{(f)} \quad k = k + 1 \quad .$$