Inequality Extensions

SI252 Reinforcement Learning

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Cauchy-Schwarz

Theorem

For any r.v.s X and Y with finite variances,

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)\,\mathbb{E}(Y^2)}.$$

Example: Second Moment Method

Let X be a nonnegative random variable, then

$$\Pr(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}(X^2)}.$$

Example: Application of Second Moment Method

Assume $X = I_1 + \cdots + I_n$, where the I_j are uncorrelated indicator r.v.s. Let $p_j = \mathbb{E}(I_j)$. Upper bound of $\Pr(X = 0)$?

Example: Application of Second Moment Method

Suppose there are 14 people in a room. How likely is it that there are two people with the same birthday or birthdays one day apart?

Markov's Inequality & Chebyshev's Inequality

Theorem (Markov's Inequality)

For any r.v. X and constant $a > 0$,

$$P(|X| \geq a) \leq \frac{E|X|}{a}.$$

Theorem (Chebyshev's Inequality)

Let X have mean μ and variance σ^2. Then for any $a > 0$,

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

Example: Coin Flipping

Find bounds on the probability of having no more than $n/4$ heads or fewer than $3n/4$ heads in a sequence of n fair coin flips.

Chernoff's Technique

Theorem

For any r.v. X and constant a,

$$P(X \geq a) \leq \inf_{t>0} \frac{E\left(e^{tX}\right)}{e^{ta}},$$

$$P(X \leq a) \leq \inf_{t<0} \frac{E\left(e^{tX}\right)}{e^{ta}}.$$

Example: Sum of Independent Bernoulli R.V.s

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables such that $\Pr(X_i = 1) = p_i$, $\Pr(X_i = 0) = 1 - p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}(X)$. Then for $0 < \delta < 1$,

$$\Pr\left(|X - \mu| \geq \delta\mu\right) \leq 2e^{-\mu\delta^2/3}.$$

Example: Revisit Example of Coin Flipping

Find bounds on the probability of having no more than $n/4$ heads or fewer than $3n/4$ heads in a sequence of n fair coin flips.