CS101 Algorithms and Data Structures

Divide and Conquer

Textbook Ch 1.4

Outline

- Master Theorem
- Integer Multiplication
- Matrix Multiplication
- Convolution and FFT

Master Theorem

- Goal. Recipe for solving common divide-and-conquer recurrences:

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

with $T(0) = 0$ and $T(1) = \Theta(1)$.

- Terms.
  - $a \geq 1$ is the number of subproblems.
  - $b \geq 2$ is the factor by which the subproblem size decreases.
  - $f(n) \geq 0$ is the work to divide and combine subproblems.
- Recursion tree. [assuming n is a power of b]
  - a = branching factor.
  - a^k = number of subproblems at level k.
  - $1 + \log_b n$ levels.
  - n / b^k = size of subproblem at level k.

Master Theorem

- If $T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$ for constants $a > 0, b > 1, d \geq 0$, then:

$$T(n) = \begin{cases} O(n^d) & if\ d > log_b a \\ O(n^d log n) & if\ d = log_b a \\ O(n^{log_b a}) & if\ d < log_b a \end{cases}$$

Outline

- Master Theorem
- Integer Multiplication
- Matrix Multiplication
- Convolution and FFT

Integer addition and subtraction

- Addition. Given two n-bit integers x and y, compute $x + y$.
- Subtraction. Given two n-bit integers x and y, compute $x - y$.
- Grade-school algorithm. $\Theta(n)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Remark. Grade-school addition and subtraction algorithms are optimal.
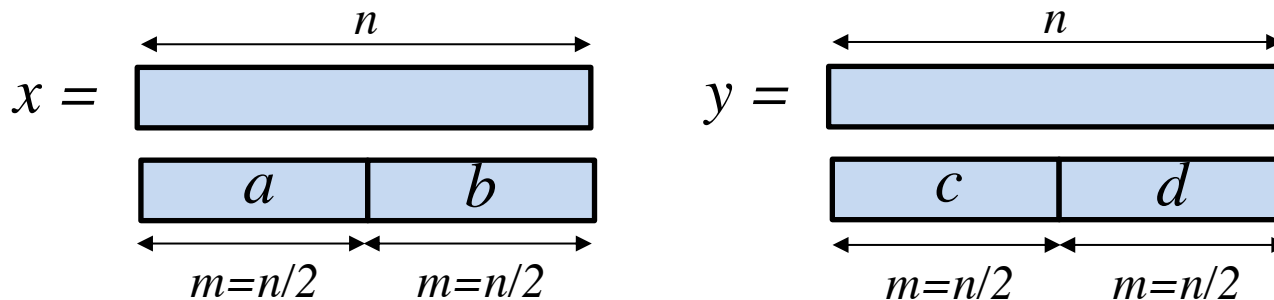
Integer multiplication

- Multiplication. Given two n-bit integers x and y, compute $x \times y$.
- Grade-school algorithm (long multiplication). $\Theta(n^2)$ bit operations.

		1	1	0	1	0	1	0	1							
×		0	1	1	1	1	1	0	1							
		1	1	0	1	0	1	0	1							
	0	0	0	0	0	0	0	0								
1	1	0	1	0	1	0	1									
1	1	0	1	0	1	0	1									
1	1	0	1	0	1	0	1									
1	1	0	1	0	1	0	1									
1	1	0	1	0	1	0	1									
0	0	0	0	0	0	0	0									
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Divide-and-conquer multiplication

- To multiply two n-bit integers x and y:
  - Divide x and y into low- and high-order bits.
  - Multiply four $\tfrac{1}{2}n$-bit integers, recursively.
  - Add and shift to obtain result.

Then

$$x\,y \;=\; (2^m\,a + b)\,(2^m\,c + d) \;=\; 2^{2m}\,\boxed{ac} \;+\; 2^m\,(\boxed{bc} + \boxed{ad}) \;+\; \boxed{bd}$$

Ex. $x = 1\,0\,0\,0\ 1\,1\,0\,1$ $\quad y = 1\,1\,1\,0\ 0\,0\,0\,1$

MULTIPLY(x, y, n)

IF ($n = 1$)

RETURN $x\,y$.

ELSE

$m \leftarrow \lceil n/2 \rceil$.

$a \leftarrow \lfloor x/2^m \rfloor; \quad b \leftarrow x \bmod 2^m$.

$c \leftarrow \lfloor y/2^m \rfloor; \quad d \leftarrow y \bmod 2^m$.

$\Theta(n)$

$e \leftarrow$ MULTIPLY(a, c, m).

$f \leftarrow$ MULTIPLY(b, d, m).

$4\,T((n/2))$

$g \leftarrow$ MULTIPLY(b, c, m).

$h \leftarrow$ MULTIPLY(a, d, m).

RETURN $2^{2m}\,e + 2^m\,(g + h) + f$. $\Theta(n)$

Question 1

- How many bit operations to multiply two n-bit integers using the divide-and-conquer multiplication algorithm?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ & \text{if } n > 1 \end{cases}$$

A. $T(n) = \Theta(n^{1/2})$.

B. $T(n) = \Theta(n \log n)$.

C. $T(n) = \Theta(n^{\log_2 3}) = O(n^{1.585})$.

D. $T(n) = \Theta(n^2)$.

Recursion Tree for Integer Multiplication

Level		Problem #	Problem size	Work
0		1	n	n
1		4	$(n/2)$	$4(n/2)$
2	4 4 4 4	4^2	$(n/4)$	$4^2(n/4)$
. . .	. . .	. . .	. . .	. . .
$k = \log_2 n$	. . . . . .	$4^k = 4^{\log_2 n}$	1	$4^k(n / 2^k) = 4^{\log_2 n}(n / 2^{\log_2 n})$

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

Total work $= \sum_1^k 4^k(n / 2^k) = \sum_1^k n2^k = n^2$

Karatsuba trick

- To multiply two n-bit integers x and y:

    - Divide x and y into low- and high-order bits.
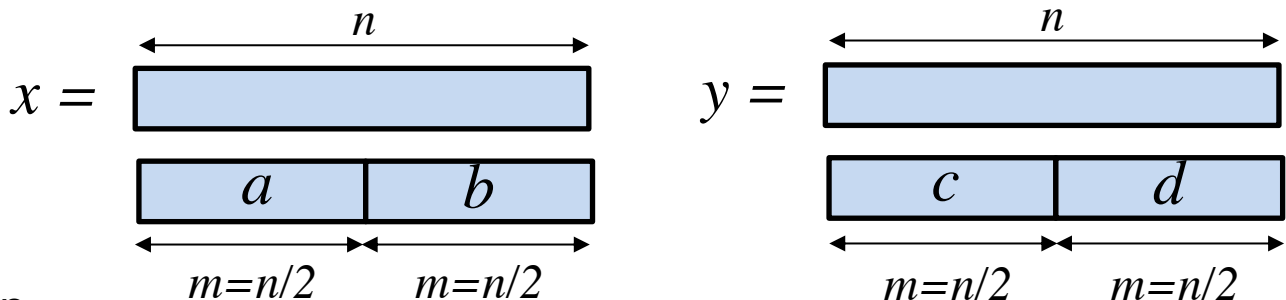   - To compute middle term $bc + ad$, use identity:

$$bc + ad = ac + bd - (a - b)(c - d)$$

- Multiply only three $\frac{1}{2}n$-bit integers, recursively.

Divide-and-conquer multiplication

- To multiply two n-bit integers x and y:
  - Divide x and y into low- and high-order bits.
  - Multiply four $\frac{1}{2}n$-bit integers, recursively.
  - Add and shift to obtain result.

Then

$$x\,y \;=\; (2^m\,a + b)\,(2^m\,c + d) \;=\; 2^{2m}\,\boxed{ac}\; +\; 2^m\,(\boxed{bc} + \boxed{ad})\; +\; \boxed{bd}$$

$$=\; 2^{2m}\,\boxed{ac}\; +\; 2^m\,(\boxed{ac} + \boxed{bd} - \boxed{(a-b)(c-d)})\; +\; \boxed{bd}$$

KARATSUBA-MULTIPLY(x, y, n)

IF $(n = 1)$

    RETURN $x\, y$.

ELSE

    $m \leftarrow \lceil n / 2 \rceil$.

    $a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$.

    $c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$. $\leftarrow \Theta(n)$

    $e \leftarrow$ KARATSUBA-MULTIPLY(a, c, m).

    $f \leftarrow$ KARATSUBA-MULTIPLY(b, d, m). $\leftarrow 3\, T((n / 2))$

    $g \leftarrow$ KARATSUBA-MULTIPLY$((a - b), (c - d), m)$.

    Flip sign of g if needed.

    RETURN $2^{2m}\, e + 2^m\, (e + f - g) + f$. $\leftarrow \Theta(n)$

Question 2

- How many bit operations to multiply two n-bit integers using the divide-and-conquer multiplication algorithm?

$$
T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 3\,T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}
$$

A. $T(n) = \Theta(n^{1/2})$.

B. $T(n) = \Theta(n \log n)$.

C. $T(n) = \Theta(n^{\log_2 3}) = O(n^{1.585})$.

D. $T(n) = \Theta(n^2)$.

Recursion Tree for Integer Multiplication

Level		Problem #	Problem size	Work
0	████████████████	1	n	n
1	████ ████ ████ ████	3	(n/2)	3(n/2)
2	██...██ ██...██ ██...██ ██...██	3^2	(n/4)	3^2(n/4)
	3 3 3 3			
. . .	. . .	. . .	. . .	. . .
$k=\log_2 n$	. . . . . .	$3^k = 3^{\log_2 n}$	1	$3^k(n / 2^k) = 3^{\log_2 n}(n / 2^{\log_2 n})$

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n)$$

Total work $= \sum_0^{k-1} 4^k(n / 2^k) = \sum_0^{k-1} n(3/2)^k = n^{\log_2 3} = n^{1.585}$

Outline

- Master Theorem
- Integer Multiplication
- Matrix Multiplication
- Convolution and FFT

Dot product

- Dot product. Given two length-n vectors a and b, compute $c = a \cdot b$.
- Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$

$$a = \begin{bmatrix} .70 & .20 & .10 \end{bmatrix}$$
$$b = \begin{bmatrix} .30 & .40 & .30 \end{bmatrix}$$
$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Matrix multiplication

- Matrix multiplication.

    Given two n-by-n matrices A and B, compute $C = AB$.

- $\Theta(n^3)$ arithmetic operations.

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Block matrix multiplication

$$C_{11}$$

$$A_{11} \quad A_{12} \quad B_{11}$$

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$B_{21}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$
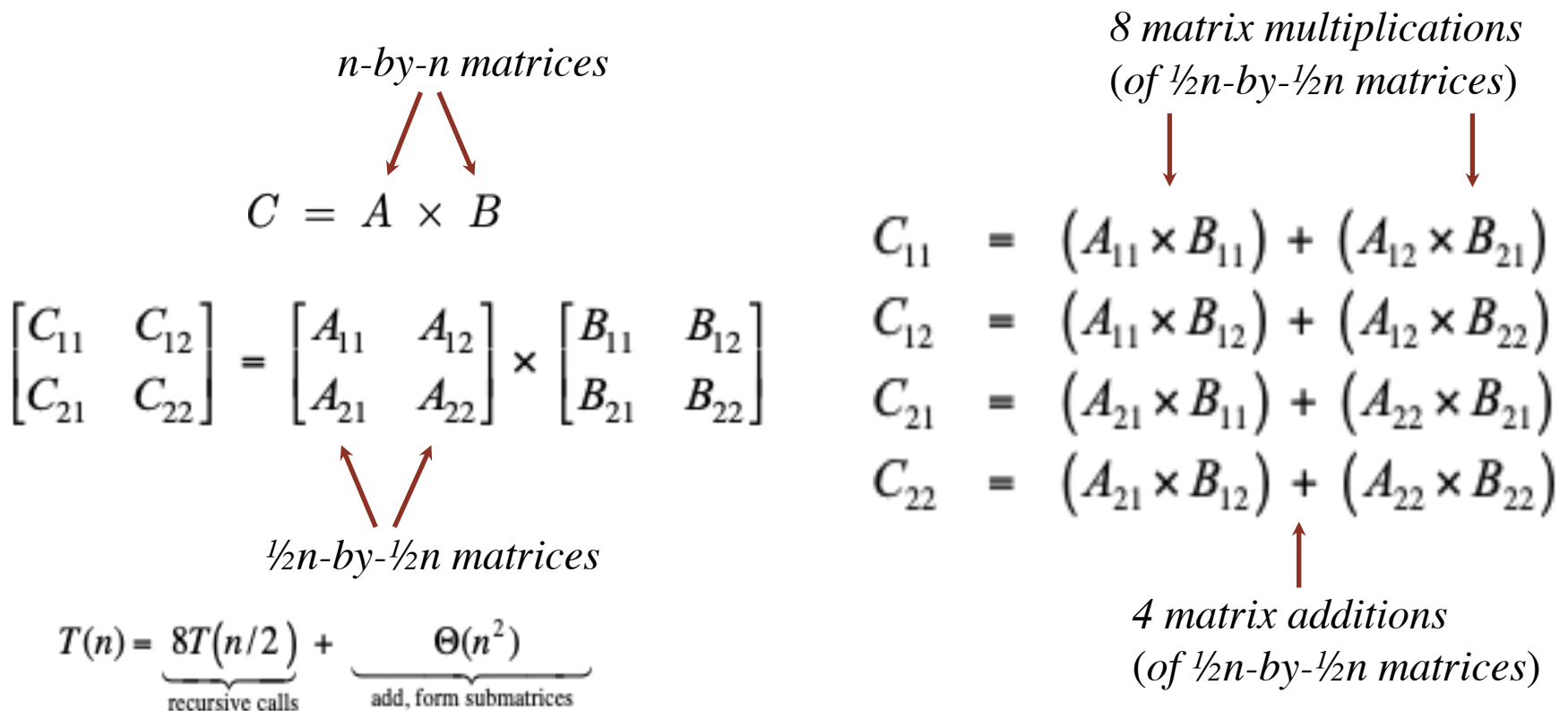
Block matrix multiplication

C_{12} A_{11} A_{12} B_{21}

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

B_{22}

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

$$C_{12} = \left(A_{11} \times B_{12} \right) + \left(A_{12} \times B_{22} \right)$$

Block matrix multiplication: warmup

- To multiply two n-by-n matrices A and B:
  - Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
  - Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
  - Combine: add appropriate products using 4 matrix additions.

n-by-n matrices

8 matrix multiplications
(of ½n-by-½n matrices)

$$C = A \times B$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$
\begin{aligned}
C_{11} &= \left(A_{11} \times B_{11} \right) + \left(A_{12} \times B_{21} \right) \\
C_{12} &= \left(A_{11} \times B_{12} \right) + \left(A_{12} \times B_{22} \right) \\
C_{21} &= \left(A_{21} \times B_{11} \right) + \left(A_{22} \times B_{21} \right) \\
C_{22} &= \left(A_{21} \times B_{12} \right) + \left(A_{22} \times B_{22} \right)
\end{aligned}
$$

½n-by-½n matrices

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}}$$

4 matrix additions
(of ½n-by-½n matrices)

Strassen's trick

- Key idea. Can multiply two 2-by-2 matrices via 7 scalar multiplications
  (plus 11 additions and 7 subtractions).

$$P_1 \leftarrow A_{11}\ (B_{12} - B_{22})$$

scalars

$$P_2 \leftarrow (A_{11} + A_{12})\ B_{22}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$P_3 \leftarrow (A_{21} + A_{22})\ B_{11}$$

$$P_4 \leftarrow A_{22}\ (B_{21} - B_{11})$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

$$P_5 \leftarrow (A_{11} + A_{22})\ (B_{11} + B_{22})$$

$$C_{12} = P_1 + P_2$$

$$P_6 \leftarrow (A_{12} - A_{22})\ (B_{21} + B_{22})$$

$$C_{21} = P_3 + P_4$$

$$P_7 \leftarrow (A_{11} - A_{21})\ (B_{11} + B_{12})$$

$$C_{22} = P_1 + P_5 - P_3 - P_7$$

7 scalar multiplications

assume n is a power of 2

STRASSEN(n, A, B)

IF $(n = 1)$ RETURN $A\,B$.

Partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.

$P_1 \leftarrow$ STRASSEN$(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN$(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN$(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{22}), (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN$(n / 2, (A_{12} - A_{22}), (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN$(n / 2, (A_{11} - A_{21}), (B_{11} + B_{12}))$.

$7\,T(n / 2) + \Theta(n^2)$

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

$\Theta(n^2)$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Analysis of Strassen's algorithm

Theorem. Strassen's algorithm requires $O(n^{2.81})$ arithmetic operations to multiply two n-by-n matrices.

Pf.

– When n is a power of 2, apply Case 1 of the master theorem:

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \implies T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

– When n is not a power of 2, pad matrices with zeros to be n'-by-n', where $n \le n' < 2n$ and n' is a power of 2.

$$
\begin{bmatrix}
1 & 2 & 3 & 0 \\
4 & 5 & 6 & 0 \\
7 & 8 & 9 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\times
\begin{bmatrix}
10 & 11 & 12 & 0 \\
13 & 14 & 15 & 0 \\
16 & 17 & 18 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
=
\begin{bmatrix}
84 & 90 & 96 & 0 \\
201 & 216 & 231 & 0 \\
318 & 342 & 366 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Strassen's algorithm: practice

- **Implementation issues.**
  - Sparsity.
  - Caching.
  - n not a power of 2.
  - Numerical stability.
  - Non-square matrices.
  - Storage for intermediate submatrices.
  - Crossover to classical algorithm when n is "small."
  - Parallelism for multi-core and many-core architectures.

- Common misperception. *"Strassen's algorithm is only a theoretical curiosity."*
  - Apple reports 8x speedup when $n \approx 2,048$.
  - Range of instances where it's useful is a subject of controversy.

Question 3

- **Suppose that you could multiply two 3-by-3 matrices with 21 scalar multiplications. How fast could you multiply two n-by-n matrices?**

  A. $\Theta(n^{\log_3 21})$
  B. $\Theta(n^{\log_2 21})$
  C. $\Theta(n^{\log_9 21})$
  D. $\Theta(n^2)$

Question 4

- **Is it possible to multiply two 3-by-3 matrices using only 21 scalar multiplications?**

  A. Yes.

  B. No.

  C. Unknown.

Fast matrix multiplication: theory

- Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

- A. Yes! [Strassen 1969]

$$\Theta(n^{\log_2 7}) = O(n^{2.81})$$

- Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

- A. Impossible. [Hopcroft–Kerr, Winograd 1971]

$$\Theta(n^{\log_2 6}) = O(n^{2.59})$$

- Begun, the decimal wars have. [Pan 1978, Bini et al., Schönhage, …]

  – Two 70-by-70 matrices with 143,640 scalar multiplications. $O(n^{2.7962})$

  – Two 48-by-48 matrices with 47,217 scalar multiplications. $O(n^{2.7801})$

  – A year later. $O(n^{2.7799})$

  – December 1979. $O(n^{2.521813})$

  – January 1980. $O(n^{2.521801})$

History of arithmetic complexity of matrix multiplication

| year	algorithm	arithmetic operations
1858	"grade school"	$O(n^3)$
1969	**Strassen**	$O(n^{2.808})$
1978	**Pan**	$O(n^{2.796})$
1979	**Bini**	$O(n^{2.780})$
1981	**Schönhage**	$O(n^{2.522})$
1982	**Romani**	$O(n^{2.517})$
1982	**Coppersmith–Winograd**	$O(n^{2.496})$
1986	**Strassen**	$O(n^{2.479})$
1989	**Coppersmith–Winograd**	$O(n^{2.3755})$
2010	**Strother**	$O(n^{2.3737})$
2011	**Williams**	$O(n^{2.372873})$
2014	**Le Gall**	$O(n^{2.372864})$
	???	$O(n^{2+\varepsilon})$

Galactic

algoithms

Number of arithmetic operations to multiply two n–by–n matrices

Outline

- Master Theorem
- Integer Multiplication
- Matrix Multiplication
- Convolution and FFT

Outline

- **Fourier theorem.** [Fourier, Dirichlet, Riemann] Any (sufficiently smooth) periodic function can be expressed as the sum of a series of sinusoids.
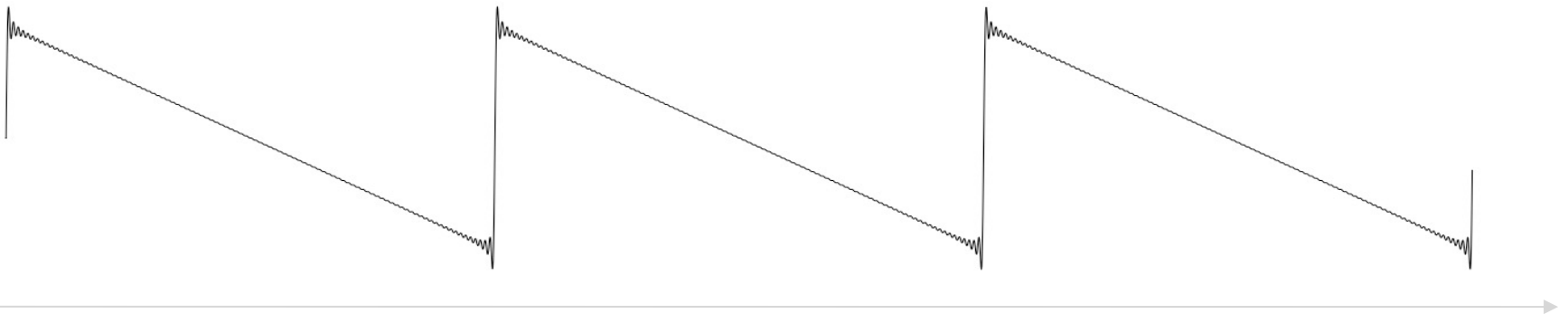
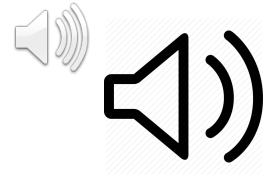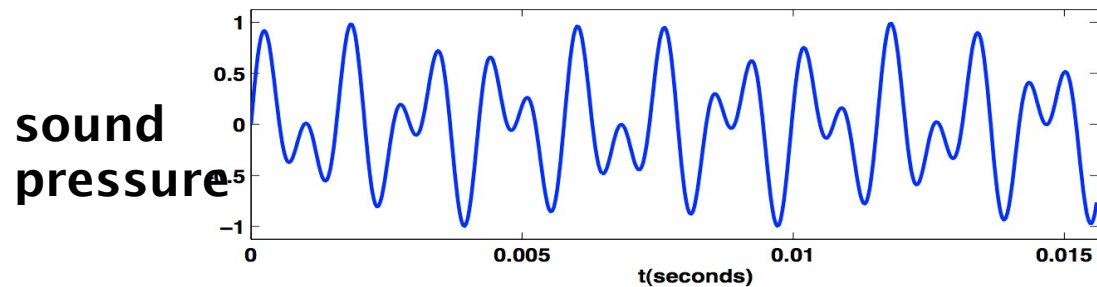$$y(t) = \frac{2}{\pi} \sum_{k=1}^{n} \frac{\sin kt}{k} \qquad n = 100$$

Time domain vs. frequency domain
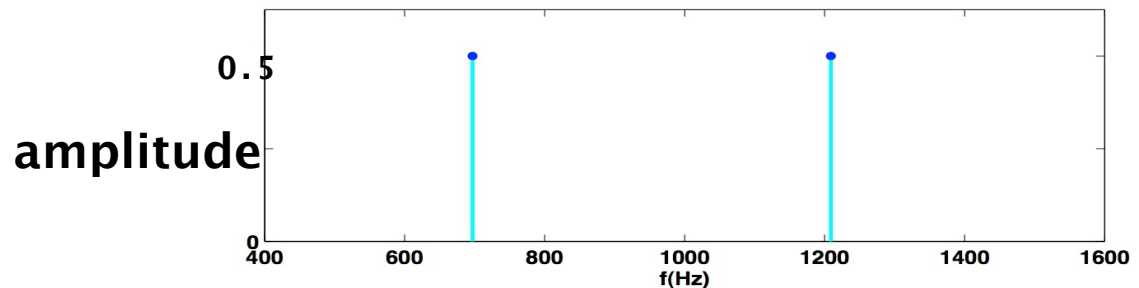
- Signal. [touch tone button 1]

$$y(t) \;=\; \tfrac{1}{2}\sin(2\pi \cdot 697\; t) \;+\; \tfrac{1}{2}\sin(2\pi \cdot 1209\; t)$$

- Time domain.

sound pressure

- Frequency domain.

amplitude

Reference: Cleve Moler, Numerical Computing with MATLAB

Fast Fourier Transform (FFT)

- **FFT.** Fast way to convert between time domain and frequency domain.

- **Alternate viewpoint.** Fast way to multiply and evaluate polynomials.

we take this approach

Fast Fourier transform: applications

- Applications.
    - Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry, …
    - Digital media. [DVD, JPEG, MP3, H.264]
    - Medical diagnostics. [MRI, CT, PET scans, ultrasound]
    - Numerical solutions to Poisson's equation.
    - Integer and polynomial multiplication.
    - Shor's quantum factoring algorithm.

" The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. "

— Charles van Loan

10

F · R · O · N · T · I · E · R · S
IN APPLIED MATHEMATICS
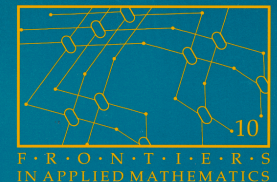
Computational Frameworks
for the Fast Fourier Transform

Charles Van Loan

siam

Fast Fourier transform: brief history

- Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

- Runge–König (1924). Laid theoretical groundwork.

- Danielson–Lanczos (1942). Efficient algorithm, x-ray crystallography.

- Cooley–Tukey (1965). Detect nuclear tests in Soviet Union and track submarines. Rediscovered and popularized FFT.

An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2.

Importance not fully realized until emergence of digital computers.

Polynomials: coefficient representation

- **Univariate polynomial.** [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$$
$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1}$$

- **Addition.** $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_{n-1} + b_{n-1})x^{n-1}$$

- **Evaluation.** $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \ldots + x(a_{n-2} + x(a_{n-1}))\ldots)))$$

- **Multiplication (linear convolution).** $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: point-value representation

- **Fundamental theorem of algebra.** A degree n univariate polynomial with complex coefficients has exactly n complex roots.

- **Corollary.** A degree $n - 1$ univariate polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x.
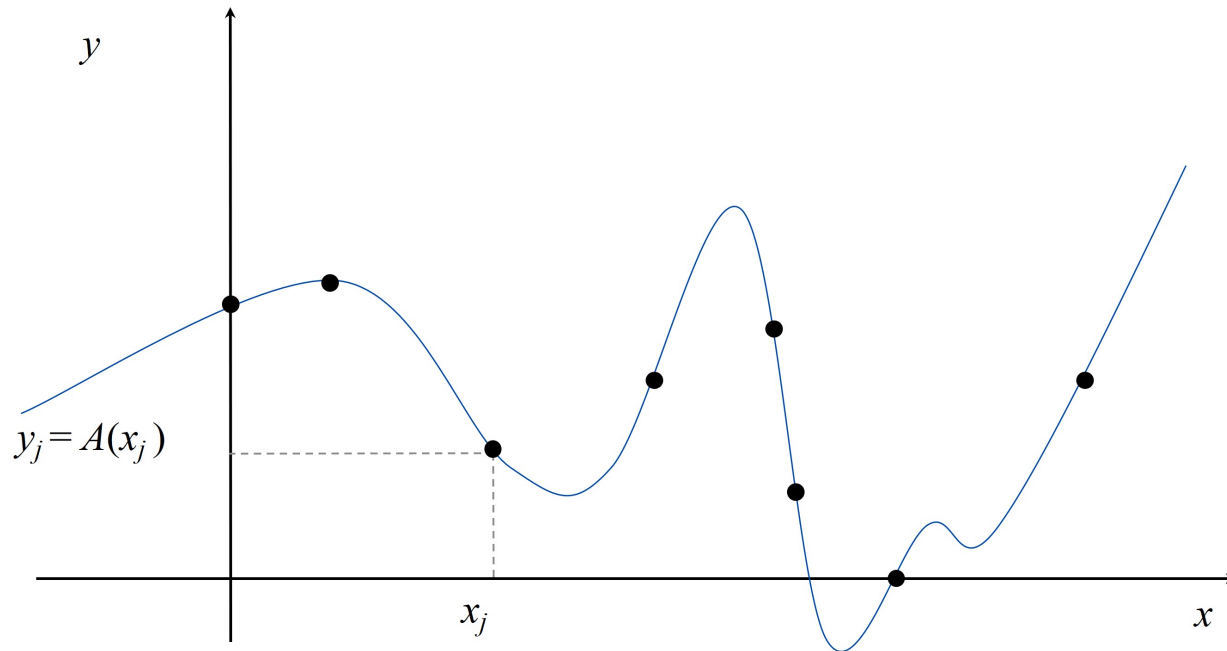
Polynomials: point-value representation

- **Univariate polynomial.** [point-value representation]

$$A(x): \ (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$
$$B(x): \ (x_0, z_0), \ldots, (x_{n-1}, z_{n-1})$$

- **Addition.** $O(n)$ arithmetic operations.

$$A(x) + B(x): \ (x_0, y_0 + z_0), \ldots, (x_{n-1}, y_{n-1} + z_{n-1})$$

- **Multiplication.** $O(n)$, but represent $A(x)$ and $B(x)$ using $2n$ points.

$$A(x) \, B(x): \ (x_0, y_0 \cdot z_0), \ldots, (x_{2n-1}, y_{2n-1} \cdot z_{2n-1})$$
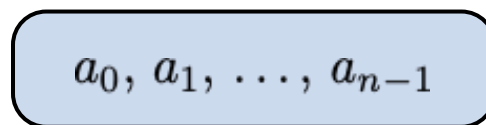
Converting between two representations

- **Tradeoff.** Either fast evaluation or fast multiplication. We want both!
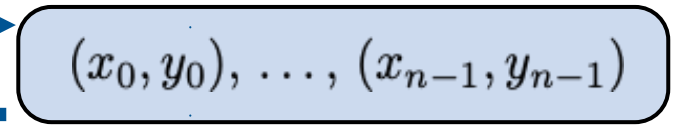
representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

- **Goal.** Efficient conversion between two representations $\Rightarrow$ all ops fast.

coefficient representation   **point-value representation**

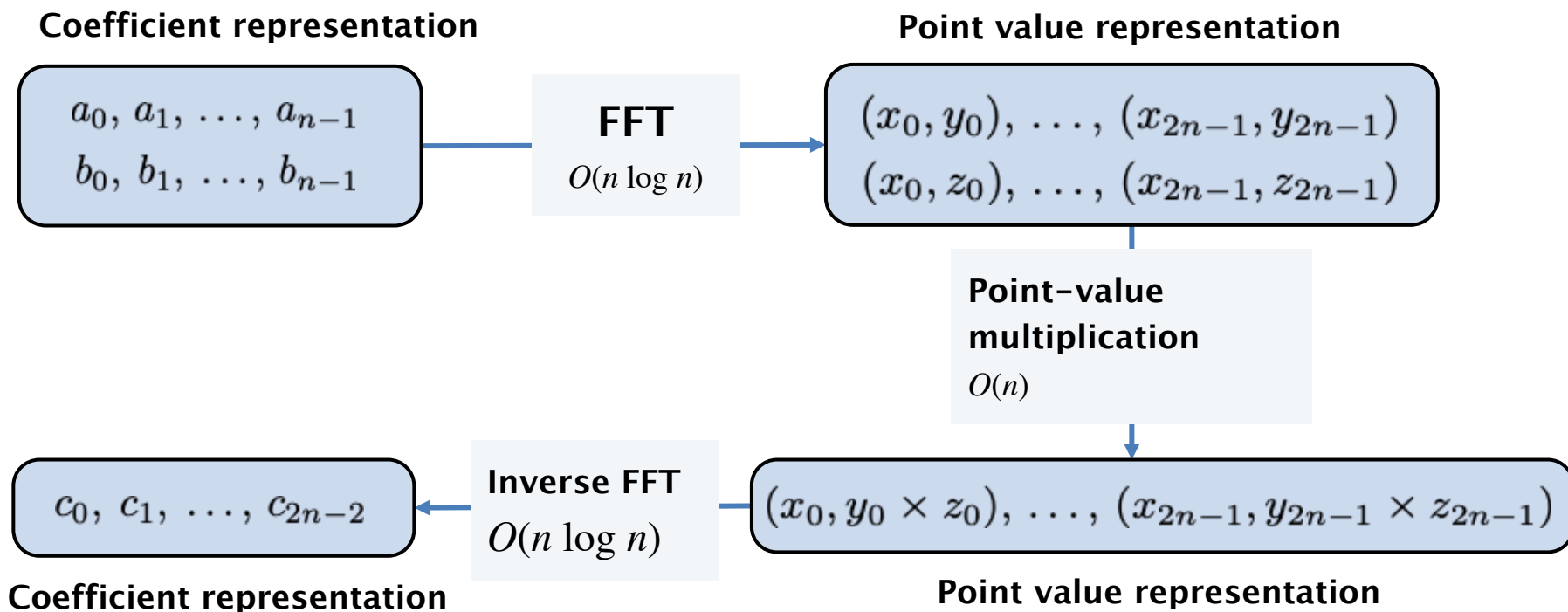$$a_0, a_1, \ldots, a_{n-1}$$ $$(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$

Converting between two representations

- Application. Polynomial multiplication (coefficient representation).

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} \boxed{x(n)} e^{-i\frac{2\pi k}{N}n} = \sum_{n=0}^{N-1} x(n) \boxed{W_N^{kn}} \qquad W_N^k = e^{-i\frac{2\pi k}{N}}$$

$$a_0, a_1, \cdots, a_{n-1} \qquad\qquad 1, x^1, x^2, \ldots, x^{n-1}$$

Coefficient representation

$$a_0, a_1, \ldots, a_{n-1}$$
$$b_0, b_1, \ldots, b_{n-1}$$

FFT
$O(n \log n)$

Point value representation

$$(x_0, y_0), \ldots, (x_{2n-1}, y_{2n-1})$$
$$(x_0, z_0), \ldots, (x_{2n-1}, z_{2n-1})$$

Point–value multiplication
$O(n)$

$$c_0, c_1, \ldots, c_{2n-2}$$

Inverse FFT
$O(n \log n)$

$$(x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Coefficient representation

Point value representation

Converting between two representations: brute force

- **Coefficient $\Rightarrow$ point-value.** Given a polynomial $A(x) = a_0 + a_1 x + ... + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, ..., x_{n-1}$.

$$
\begin{bmatrix}
y_0 \\
y_1 \\
y_2 \\
\vdots \\
y_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n-1}
\end{bmatrix}
$$

- Running time. $O(n^2)$ via matrix–vector multiply (or n Horner's).

Converting between two representations: brute force

- **Point-value** $\Rightarrow$ **coefficient.**   Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $A(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible if x_i distinct

- **Running time.** $O(n^3)$ via Gaussian elimination.

or $O(n^{2.38})$ via fast matrix multiplication

Question 5

- **Which divide-and-conquer approach to use to evaluate polynomials?**

  $$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$$

  A. Divide polynomial into low- and high-degree terms.

  $$A_{low}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

  $$A_{high}(x) = a_4 + a_5 x + a_6 x^2 + a_7 x^3.$$

  B. Divide polynomial into even- and odd-degree terms.

  $$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$$

  $$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$$

  C. Either A or B.
  D. Neither A nor B.

Divide-and-conquer

- Decimation in time. Divide into even- and odd- degree terms.
  - $A(x) \qquad = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
  - $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
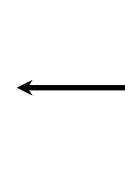 - $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
  - $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$

- Decimation in frequency. Divide into low- and high-degree terms.
  - $A(x) \qquad = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
  - $A_{low}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$
  - $A_{high}(x) = a_4 + a_5 x + a_6 x^2 + a_7 x^3.$
  - $A(x) = A_{low}(x) + x^4\, A_{high}(x).$

Coefficient to point-value representation: intuition

- **Coefficient** $\Rightarrow$ **point-value.** Given a polynomial $A(x) = a_0 + a_1 x + ... + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0 , ..., x_{n-1}$. $\longleftarrow$

We get to choose which ones!

- **Divide.** Break up polynomial into even- and odd-degree terms.
  - $A(x)$ $= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
  - $A_{even}(x)$ $= a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
  - $A_{odd}(x)$ $= a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
  - $A(x)$ $= A_{even}(x^2) + x A_{odd}(x^2).$
  - $A(-x)$ $= A_{even}(x^2) - x A_{odd}(x^2).$

- **Intuition.** Choose two points to be ± 1.
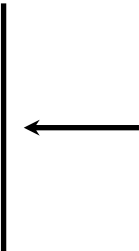 - $A(1)$ $= A_{even}(1) + 1 A_{odd}(1).$
  - $A(-1)$ $= A_{even}(1) - 1 A_{odd}(1).$

Can evaluate polynomial of degree n−1 at 2 points by evaluating two polynomials of degree ½n − 1 at only 1 point. $\longleftarrow$

Coefficient to point-value representation: intuition

- **Coefficient $\Rightarrow$ point-value.** Given a polynomial $A(x) = a_0 + a_1 x + ... + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, ..., x_{n-1}$.

  We get to choose which ones!

- **Divide.** Break up polynomial into even- and odd-degree terms.
  - $A(x)$ $\quad = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
  - $A_{even}(x)$ $\quad = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
  - $A_{odd}(x)$ $\quad = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
  - $A(x)$ $\quad = A_{even}(x^2) + x A_{odd}(x^2).$
  - $A(-x)$ $\quad = A_{even}(x^2) - x A_{odd}(x^2).$

- **Intuition.** Choose two points to be ± 1.
  - $A(1)$ $\quad = A_{even}(1) \ +1 A_{odd}(1).$
  - $A(-1)$ $\quad = A_{even}(1) \ -1 A_{odd}(1).$
  - $A(i)$ $\quad = A_{even}(-1) +i A_{odd}(-1).$
  - $A(-i)$ $\quad = A_{even}(-1) -i A_{odd}(-1).$

  Can evaluate polynomial of degree n−1 at 4 points by evaluating two polynomials of degree ½n − 1 at only 2 points.
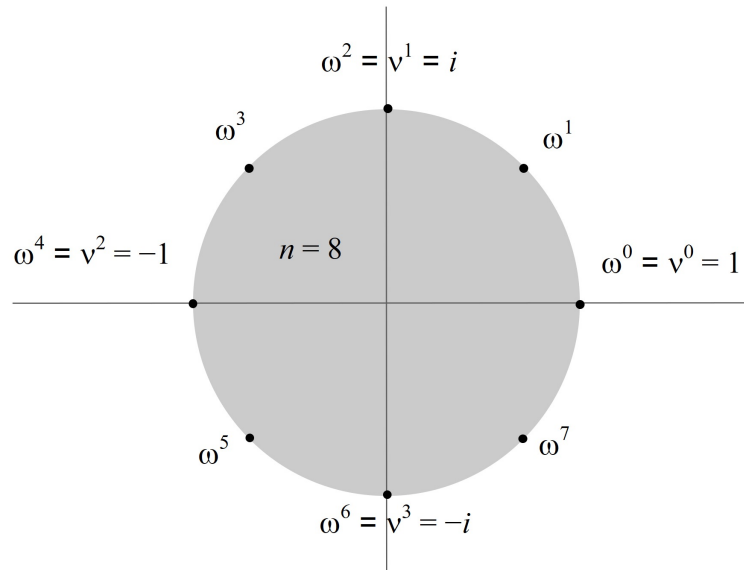
Euler's identity

- Euler's identity. $e^{ix} = \cos x + i \sin x$.

- Sinusoids. Sum of sine and cosines = sum of complex exponentials.

Roots of unity

- Def. An nth root of unity is a complex number x such that $x^n = 1$.

- Fact. The nth roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.
- Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

- Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $v^0, v^1, \ldots, v^{n/2-1}$ where $v = \omega^2 = e^{4\pi i / n}$.

$$\omega^k = -\omega^{k + \frac{1}{2}n}$$

Fast Fourier transform

- **Goal.** Evaluate a degree $n - 1$ polynomial $A(x) = a_0 + \ldots + a_{n-1}\, x^{n-1}$ at its nth roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

- **Divide.** Break up polynomial into even- and odd-degree terms.
  - $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2} x^{n/2-1}$.
  - $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1} x^{n/2-1}$.
  - $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
  - $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

- **Conquer.** Evaluate $A_{even}(x)$ and $A_{odd}(x)$ at the $\tfrac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

- **Combine.**

$$\boxed{\omega^k = \omega^{k+\tfrac{1}{2}n}}$$

  - $y_k = A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$.
  - $y_{k+\tfrac{1}{2}n} = A(\omega^{k+\tfrac{1}{2}n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$.

FFT: implementation

- **Goal.** Evaluate a degree $n-1$ polynomial $A(x) = a_0 + \ldots + a_{n-1}\, x^{n-1}$ at its nth roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

  - $y_k = A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k),\;\; 0 \le k < n/2.$
  - $y_{k+\frac{1}{2}n} = A(\omega^{k+\frac{1}{2}n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k),\;\; 0 \le k < n/2.$

$FFT(n, a_0, a_1, a_2, \ldots, a_{n-1})$

IF $(n = 1)$ RETURN a_0.

$(e_0, e_1, \ldots, e_{n/2-1}) \leftarrow FFT(n/2, a_0, a_2, a_4, \ldots, a_{n-2}).$

$(d_0, d_1, \ldots, d_{n/2-1}) \leftarrow FFT(n/2, a_1, a_3, a_5, \ldots, a_{n-1}).$

$2\,T(n/2)$

FOR $k = 0$ TO $n/2 - 1$.

$\omega^k \leftarrow e^{2\pi i\, k/n}.$

$y_k \leftarrow e_k + \omega^k d_k.$

$y_{k+n/2} \leftarrow e_k - \omega^k d_k.$

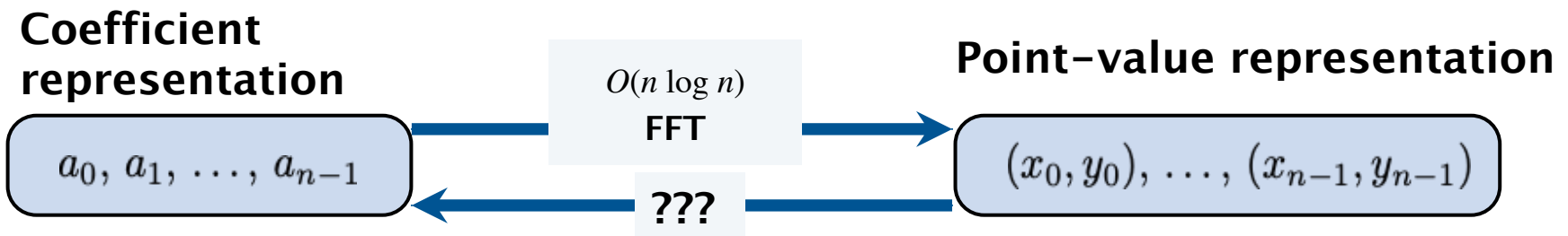$\Theta(n)$

RETURN $(y_0, y_1, y_2, \ldots, y_{n-1}).$
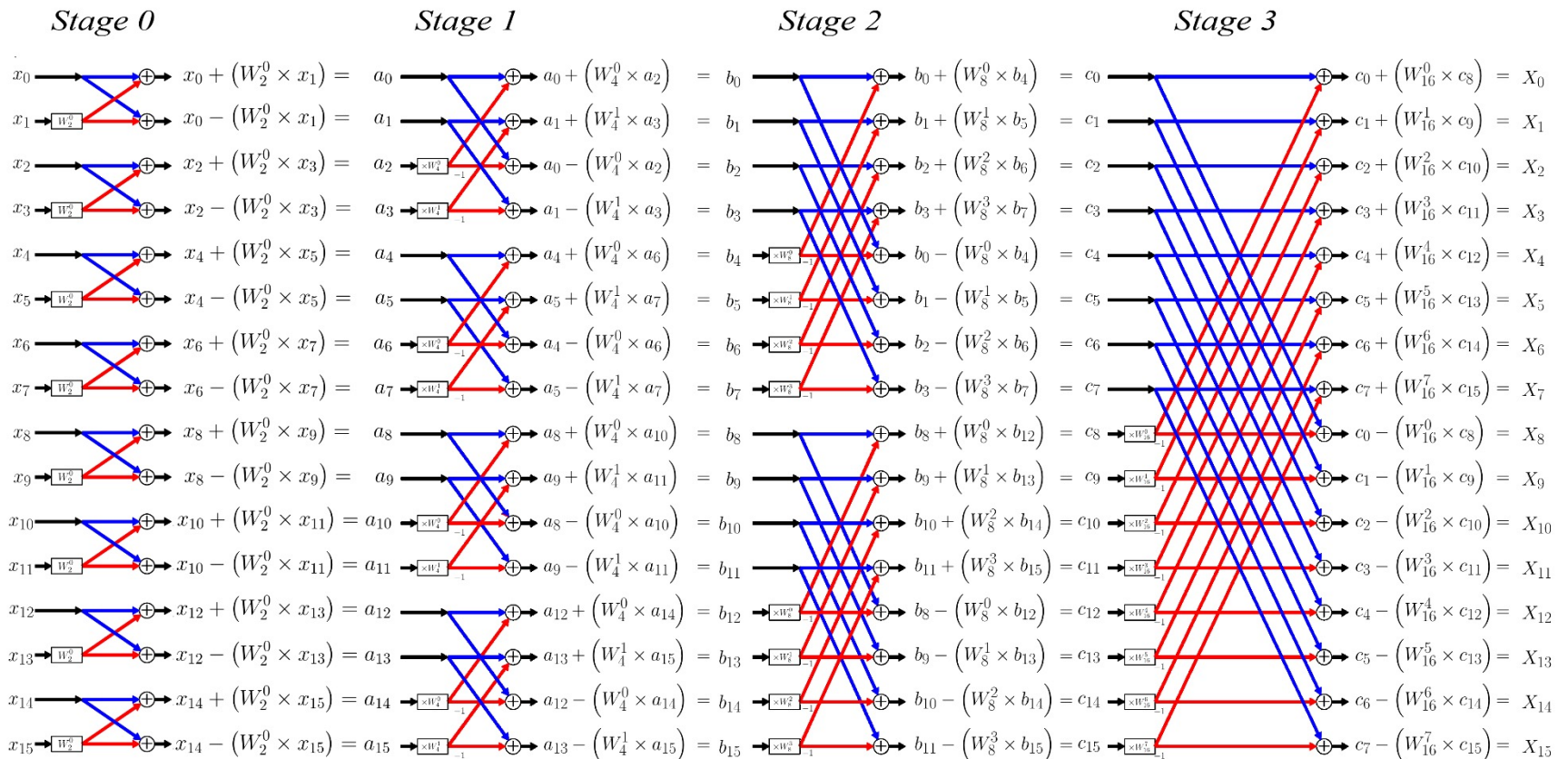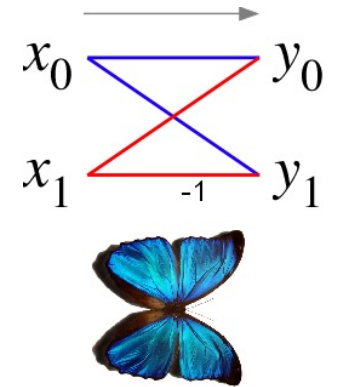
FFT: summary

- **Theorem.** The FFT algorithm evaluates a degree $n-1$ polynomial at each of the nth roots of unity in $O(n \log n)$ arithmetic operations and $O(n)$ extra space.

- **Pf.**

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Coefficient representation

$a_0, a_1, \ldots, a_{n-1}$

$O(n \log n)$
FFT

???

Point-value representation

$(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$

FFT Butterfly