1 HEAD or TAIL

Alice and Bob are playing a game. They both have a coin. The game requires Alice and Bob to choose one side (HEAD or TAIL) of the coin. Then they show their choices at the same time, and the result is decided according to the following rules.

a) If they show different sides, Bob wins \$3 from Alice.

b) If they both show HEAD, Alice wins \$1 from Bob. If they both show TAIL, Alice wins \$5 from Bob.

1.1 Mixed Strategies

Suppose Alice's strategy is showing HEAD with probability p and Bob's strategy is showing HEAD with probability q. Calculate Bob's expected payoff in term of p and q.

The expected payoff of Bob is

$$-pq + 3p(1-q) + 3(1-p)q - 5(1-p)(1-q) = 8(p+q) - 12pq - 5$$

1.2 Best Response

If Alice choose her side uniformly randomly ($p = 0.5$), then which strategy is Bob's best response?

When $p = 0.5$, Bob has expected payoff $2q - 1$, which is maximized when $q = 1$. Hence, Bob's best response is to always choose HEAD.

1.3 Nash Equilibrium

Calculate the Nash equilibrium of this game. What are the expected payoffs of Alice and Bob in the equilibrium? Do you think the game is fair?
Suppose the Nash equilibrium is (p, q), then we have

$$\begin{cases} q - 3(1-q) & = -3q + 5(1-q) \\ -p + 3(1-p) & = 3p - 5(1-p) \end{cases}$$

from which we derive that $p = q = 2/3$. Hence, the Nash equilibrium of the game is $(2/3, 2/3)$. In the equilibrium, the expected payoffs of Alice and Bob is $-1/3$ and $1/3$ respectively. The game is actually unfair since when Bob choose $q = 2/3$, she cannot prevent lose in expectation.

2 Aladdin Lamp

Tom and Jerry have found Aladdin Lamp which can give them power but the lamp only has 10 units of power. Thus the lamp has made a rule. They should show how much power they wish to have at the same time. If the sum of their wishes exceeds 10, the lamp will only satisfy the lower power (with random tie-breaking). If the sum of their wishes is within 10, the lamp will satisfy both of them.

2.1 Action Spaces

Write down the action spaces of Tom and Jerry.

Both Tom and Jerry's action spaces are $\mathbb{R}$ (Alternatively, $\mathbb{Z}$ is also correct where each unit of power cannot be divided. Limiting the space in non-negative number or less than 10 is also okay for this problem).

2.2 Nash Equilibrium

Find a Nash equilibrium for this game.

$(5, 5)$ is a Nash equilibrium for this game.

2.3 A New Rule

Suppose the lamp has made an alternative rule. If the sum of wishes exceeds 10, the lamp will satisfy the lower power first and give the rest of the power to the other. If the sum of their wishes is within 10, the lamp will still satisfy both of them. Now, in this new rule, is the strategy you have found in 2.2 still a Nash equilibrium? If yes, is it the unique Nash equilibrium? If it is no longer a Nash equilibrium, find a Nash equilibrium for the new rule.

$(5, 5)$ is still a Nash equilibrium for the new rule. Moreover, it is also the unique Nash equilibrium (If in the first question, $\mathbb{Z}$ is used as the action space, then $(5, 5)$ is not the unique Nash equilibrium. e.g., $(5, 6)$ is also a Nash equilibrium).

3 Second Price Auction with Budget

Consider a second price auction for a single indivisible item. Suppose each bidder i has a value $v_i > 0$ and a budget $c_i > 0$. If a bidder wins the object and has to pay higher than the budget, the bidder will simply drop out from the auction but is charged with a small penalty $\epsilon > 0$. Compute a bid in the auction for each player i which will be a weakly dominant strategy for the player.

If the bidder i bids a value larger than v_i or c_i, she may get a negative utility; if the bidder i bids a value smaller than v_i and c_i, she may lose the auction. Therefore, the weakly dominant strategy for the player i is $\min\{v_i, c_i\}$.

4 Vickrey-Clarke-Groves Mechanism

A mechanism $(f, p_1, \ldots, p_n)$ is called Vickrey-Clark-Groves (VCG) mechanism if

- $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_i v_i(a)$; that is, f maximizes the social welfare, and
- for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \to \mathbb{R}$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$, $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$.

Prove that VCG mechanism is incentive compatible.

Fix i, v_{-i}, v_i, and v'_i. We need to show that for player i with valuation v_i, the utility when declaring v_i is not less than the utility when declaring v'_i. Denote $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$. The utility of i, when declaring v_i, is $v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i})$, but when declaring v'_i is $v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i})$. But since $a = f(v_i, v_{-i})$ maximizes social welfare over all alternatives, $v_i(a) + \sum_{j \neq i} v_j(a) > v_i(a') + \sum_{j \neq i} v_j(a')$ and thus the same inequality holds when subtracting the same term $h_i(v_{-i})$ from both sides.