Signals and Systems Homework 9
Solutions

1. (10')Let
$$g(t) = x(t) + \alpha x(-t)$$
where
$$x(t) = \beta e^{-t} u(t)$$
and the Laplace transform of $g(t)$ is
$$G(s) = \frac{s}{s^2 - 1}, \qquad -1 < Re\{s\} < 1$$
Determine the values of the constants α and β.

Solution: We have
$$X(S) = \frac{\beta}{s+1}, \quad Re\{s\} > -1$$
Also,
$$G(s) = X(s) + \alpha X(-s), \quad -1 < Res < 1$$
Therefore,
$$G(s) = \beta[\frac{1 - s + \alpha s + \alpha}{1 - s^2}].$$
Comparing with the given equation for $G(s)$,
$$\alpha = -1, \beta = \frac{1}{2}.$$

2. (10') Consider two right-sided signals $x(t)$ and $y(t)$ related through the differential equations
$$\frac{dx(t)}{dt} = -2y(t) + \delta(t)$$
and
$$\frac{dy(t)}{dt} = 2x(t)$$
Determine $Y(s)$ and $X(s)$, along with their regions of convergence.

Solution: Taking the Laplace transform of both sides of the two differential equations, we have
$$sX(s) = -2Y(s) + 1 \quad \text{and} \quad sY(s) = 2X(s).$$
Solving for $X(s)$ and $Y(s)$, we obtain
$$X(s) = \frac{s}{s^2 + 4} \quad \text{and} \quad Y(s) = \frac{2}{s^2 + 4}.$$
The region of convergence for both $X(s)$ and $Y(s)$ is $Re\{s\} > 0$ because both are right-hand signals.

3. (10') Consider an LTI system for which the system function $H(s)$ is rational and has the pole-zero pattern shown in the following figure.

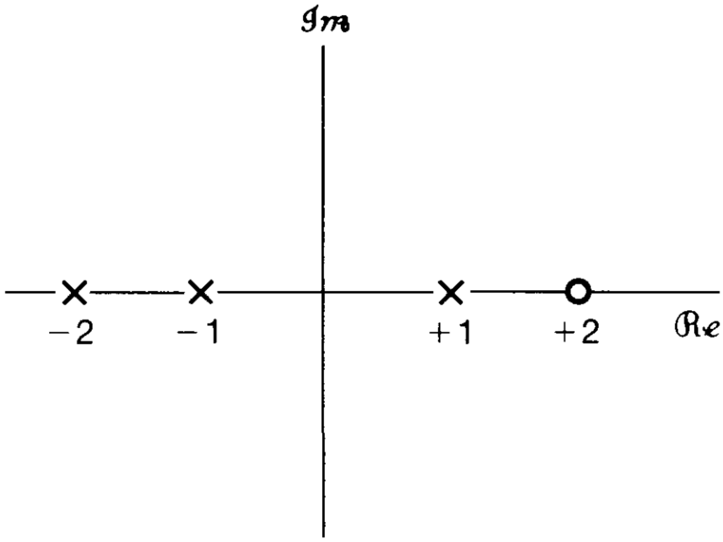

Figure 1:

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.

(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

Solution:

(a) The possible ROCs are

(1) $Re\{s\} < -2$.

(2) $-2 < Re\{s\} < -1$.

(3) $-1 < Re\{s\} < 1$.

(4) $Re\{s\} > 1$.

(b) (1) Unstable and anticausal.

(2) Unstable and non causal.

(3) Stable and non causal.

(4) Unstable and causal.

4. (20') Consider an LTI system with input $x(t) = e^{-t}u(t)$ and impulse response $h(t) = e^{-2t}u(t)$.

(a) Determine the Laplace transforms of $x(t)$ and $h(t)$.

(b) Using the convolution property, determine the Laplace transform $Y(s)$ of the output $y(t)$.

(c) From the Laplace transform of $y(t)$ as obtained in part (b), determine $y(t)$.

(d) Verify your result in part (c) by explicitly convolving $x(t)$ and $h(t)$.

Solution:

(a) Using Table 9.2, we obtain

$$X(s) = \frac{1}{s+1}, \quad Re\{s\} > -1.$$

and

$$H(s) = \frac{1}{s+2}, \quad Re\{s\} > -2.$$

(b) Since $y(t) = x(t) * h(t)$, we may use the convolution property to obtain

$$Y(s) = H(s)X(s) = \frac{1}{(s+1)(s+2)}.$$

The ROC of $Y(s)$ is $Re\{s\} > -1$.

(c) Performing partial fraction expansion on $Y(s)$, we obtain

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform, we get

$$y(t) = e^{-t}u(t) - e^{-2t}u(t).$$

(d) Explicit convolution of $x(t)$ and $h(t)$ gives us

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-2\tau}e^{-(t-\tau)}u(t-\tau)d\tau \\ &= e^{-t}\int_{0}^{t} e^{-\tau}d\tau \quad \text{for} \quad t > 0 \\ &= [e^{-t} - e^{-2t}]u(t). \end{aligned}$$

5. (10') Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system impulse response.

(a) Determine $H(s)$ as a ratio of two polynomials in s. Sketch the pole-zero pattern of $H(s)$.

(b) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal.

Solution:

(a) Taking the Laplace transform of both sides of the given differential equation and simplifying, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2}.$$

The pole-zero plot for $H(s)$ is as shown in the following figure:

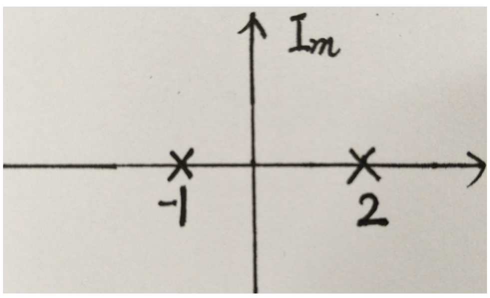

Figure 2:

(b) The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}.$$

1. If the system is stable, the ROC for $H(s)$ has to be $-1 < Re\{s\} < 2$. Therefore

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t).$$

2. If the system is causal, the ROC for $H(s)$ has to be $Re\{s\} > 2$. Therefore

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t).$$

3. If the system is neither stable nor causal, the ROC for $H(s)$ has to be $Res < -1$. Therefore

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t).$$

6. (20') Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

1. $H(1) = 0.2$
2. When the input is $u(t)$, the output is absolutely integrable.
3. When the input is $tu(t)$, the output is not absolutely integrable.
4. The signal $d^2h(t)/dt^2 + 2dh(t)/dt + 2h(t)$ is of finite duration.
5. $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

Solution: We know that

$$x_1(t) = u(t) \longleftrightarrow X_1(s) = \frac{1}{s}, \quad Re\{s\} > 0$$

Therefore, $X(s)$ has a pole at $s = 0$. Now, the Laplace transform of the output $y_1(t)$ of the system with $x_1(t)$ as the input is

$$Y_1(S) = H(s)X_1(s)$$

Since in clue 2, $Y_1(s)$ is given to be absolutely integrable, $H(s)$ must have a zero at $s = 0$ which cancels out the pole of $X_1(s)$ at $s = 0$.
We also know that

$$x_2(t) = tu(t) \longleftrightarrow X_2(s) = \frac{1}{s^2}, \quad Re\{s\} > 0$$

Therefore, $x_2(s)$ has two poles at $s = 0$. Now, the Laplace transform of the output $y_2(t)$ of the system with $x_2(t)$ as the input is

$$Y_2(s) = H(s)X_2(s)$$

Since in clue 3, $Y_2(s)$ is given to be not absolutely integrable, $H(s)$ does not have two zeros at $s = 0$. Therefore, we conclude that $H(s)$ has exactly one zero at $s = 0$. From clue 4 we know that the signal

$$p(t) = \frac{d^2h(t)}{dt^2} + 2\frac{dh(t)}{dt} + 2h(t)$$

is finite duration. Taking the Laplace transform of both sides of the above equation, we get

$$P(s) = s^2H(s) + 2sH(s) + 2H(s).$$

Therefore,

$$H(s) = \frac{P(s)}{s^2 + 2s + 2}$$

Since $p(t)$ is of finite duration, we know that $P(s)$ will have no poles in the finite s-plane. Therefore, $H(s)$ is of the form

$$H(s) = \frac{A\prod_{i=1}^{N}(s - z_i)}{s^2 + 2s + 2},$$

where $z_i, i = 1, 2, \ldots N$ represent the zeros of $P(s)$. Here, A is some constant.
From clue 5 we know that the denominator polynomial of $H(s)$ has to have a degree which is exactly one greater than the degree of the numerator polynomial. Therefore,

$$H(s) = \frac{A(s - s_1)}{s^2 + 2s + 2}$$

Since we already know that $H(s)$ has a zero at $s = 0$, we may rewrite this as $H(s) = \frac{As}{s^2+2s+2}$.
From clue 1 we know that $H(1)$ is 0.2. From this, we may easily show that $A = 1$. Therefore,

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

Since the pole of $H(s)$ are at $-1 \pm j$ and since $h(t)$ is causal and stable, the ROC of $H(s)$ is $Re\{s\} > -1$.

7. (20') Consider a stable and causal system with a real impulse response $h(t)$ and system function $H(s)$. It is known that $H(s)$ is rational, one of its poles is at $-1 + j$, one of its zeros is at $3 + j$, and it has exactly two zeros at infinity. For each of the following statements, determine whether it is true, whether it is false, or whether there is insufficient information to determine the statement's truth.

   (a) $h(t)e^{-3t}$ is absolutely integrable.
   (b) The ROC for $H(s)$ is $Re(s) > -1$.
   (c) The differential equation relating inputs $x(t)$ and outputs $y(t)$ for S may be written in a form having only real coefficients.
   (d) $\lim_{s\to\infty} H(s) = 1$.
   (e) $H(s)$ has no fewer than four poles.
   (f) $H(jw) = 0$ for at least one finite value of w.
   (g) If the input to S is $e^{3t}\sin t$, the output is $e^{3t}cost$.

   Solution:

   (a) True. Consider
   $$g(t) = h(t)e^{-3t} \longleftrightarrow G(s) = H(s + 3)$$
   The ROC of $G(s)$ will be the ROC of $H(s)$ shifted by 3 to the left. Clearly this ROC will still include the jw-axis. Therefore , $g(t)$ has to be stable.
   (b) Insufficient information. As mentioned earlier, $H(s)$ has some unknown poles. So we do not know which the rightmost pole is in $H(s)$. Therefore, we cannot determine what its exact ROC is.
   (c) True, Since $H(s)$ is rational, $H(s)$ may be expressed as a ratio of two polynomials in s. Furthermore, since $h(t)$ is real, then according to conjugation property of Laplace transform(If $x(t)$ is real and if $X(s)$ has a pole or zero at $s = s_0$, then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$), the coefficients of these polynomials will be real. Now, $H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}$.
   Here, $P(s)$ and $Q(s)$ are polynomials in s. The differential equation relating $x(t)$ and $y(t)$ is obtained by taking the inverse Laplace transform of $Y(s)Q(s) = X(s)P(s)$. Clearly, this differential equation has to have only real coefficients.
   (d) False. We are given that $H(s)$ has 2 zeros at $s = \infty$. Therefore, $lim_{s\to\infty}H(s) = 0$.
   (e) True. See the reasoning at the beginning of the problem.
   (f) Insufficient information. $H(s)$ may have other zeros. See reasoning at the beginning of the problem.

(g) False. We know that $e^{3t}sin(t) = (1/2j)e^{(3+j)t} - (1/2j)e^{(3-j)t}$. Both $e^{(3+j)t}$ and $e^{(3-j)t}$ are eigen functions of the LTI system. Therefore, the response of the system to these exponentials is $H(s+j)e^{(3+j)t}$ and $H(s-j)e^{(3-j)t}$, respectively. Since $H(s)$ has zeros at $3 \pm j$, we know that the output of the system to the two exponentials has to be zero. Hence, the response of the system to $e^{3t}sin(t)$ has to be zero.