Lecture 7
- Phasor

A beginning of AC circuits

AC = alternating current;
Circuits driven by sinusoidal current or voltage sources are *AC circuits*

Outline

- Sinusoidal signals

- Phasor

Sinusoidal Signal (Current or Voltage)

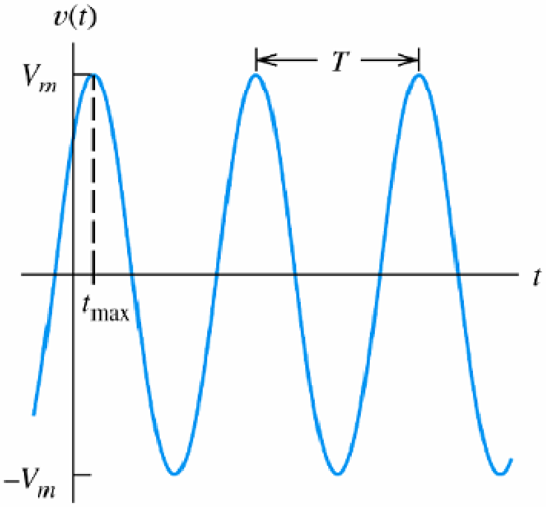
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

$(\omega t + \theta)$ is the **phase angle**

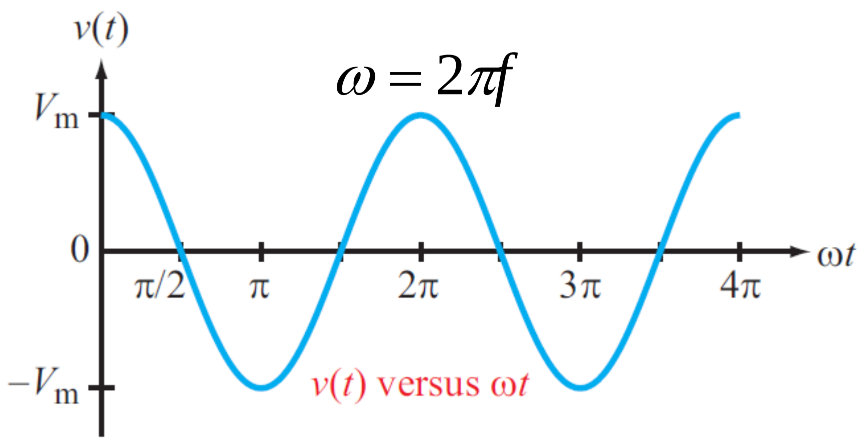
T is the **period**

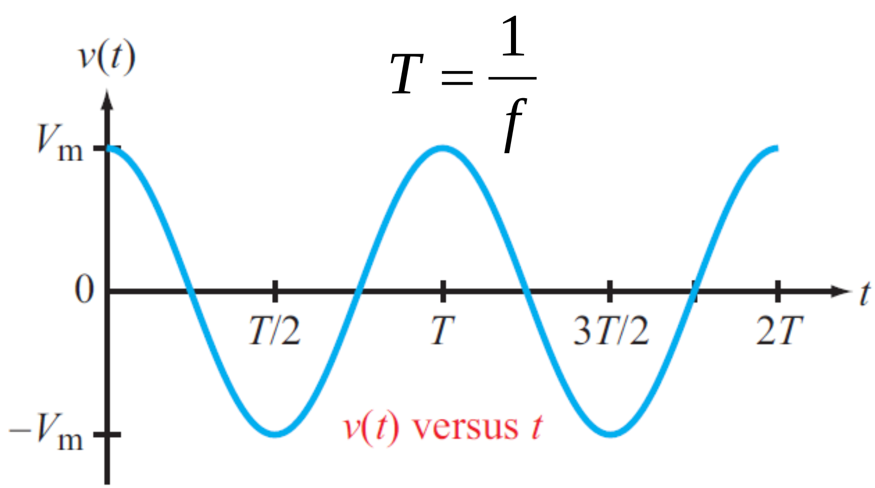


$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \theta)$$





Useful relations

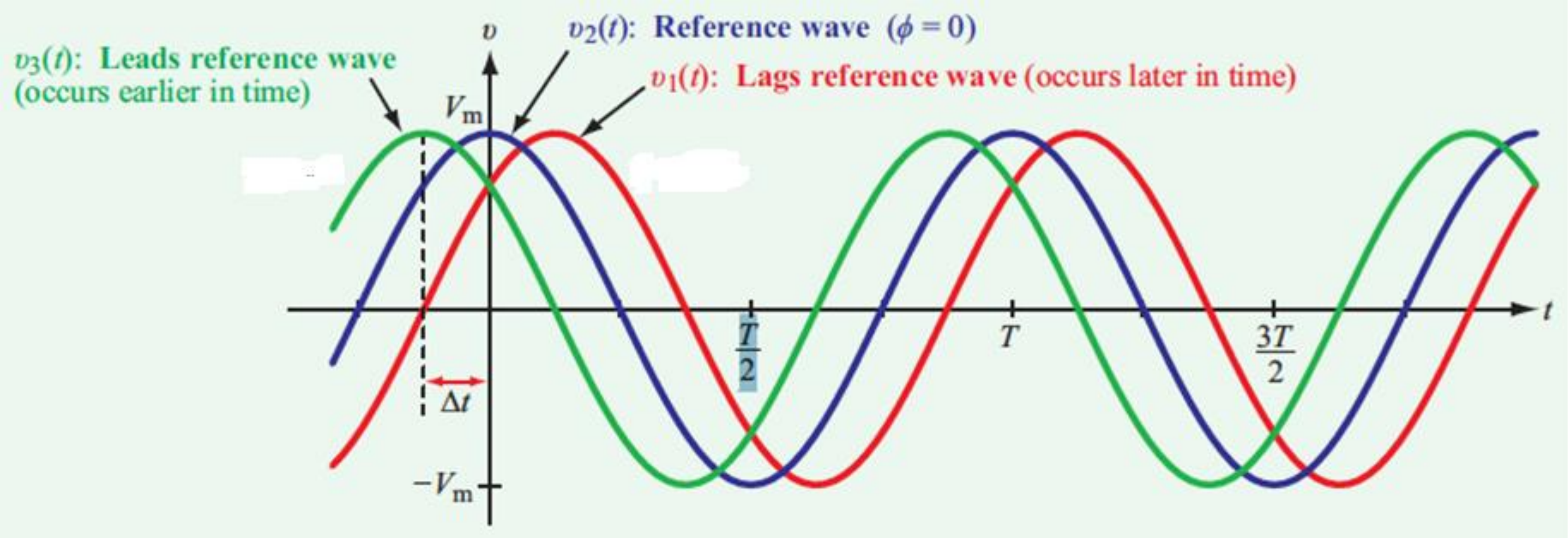
$$\sin x = \pm\cos(x \mp 90^\circ)$$
$$\cos x = \pm\sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

Phase Lead/Lag

$$V_m \cos\frac{2\pi t}{T} \qquad V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right) \qquad V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right)$$



[Source: Berkeley]

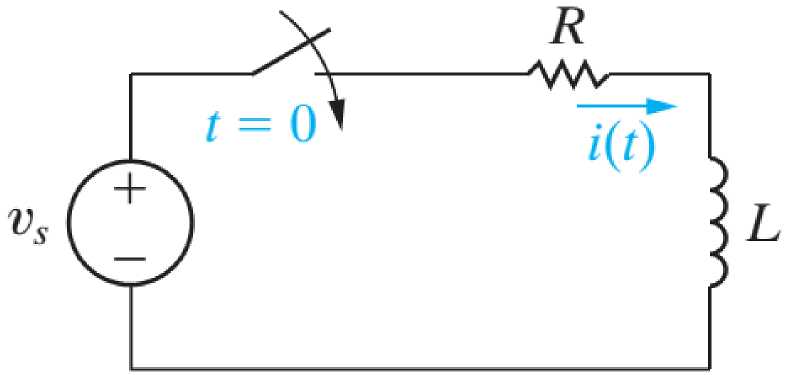
Why Sinusoidal signals?

- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface
- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.
- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.
- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.

[Source: Berkeley]

The Sinusoidal Response

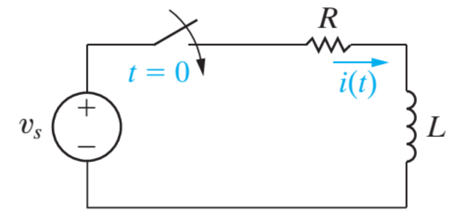
$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.
Find $i(t)$, $t \geq 0$.



$$\boxed{L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)}$$

Ordinary differential equation

Sinusoidal Steady-State Response



$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

Transient response

Steady-state response

- Steady-state solution is sinusoidal
- Response frequency = source frequency
- **Magnitude & phase** (<u>initial phase angle</u>) of S.S. response differs from that of source

Outline

- Sinusoidal signals

- Phasor

Phasor

- The idea of phasor representation is based on Euler's formula:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as

$$\cos\phi = \mathrm{Re}\left(e^{j\phi}\right)$$

$$\sin\phi = \mathrm{Im}\left(e^{j\phi}\right)$$

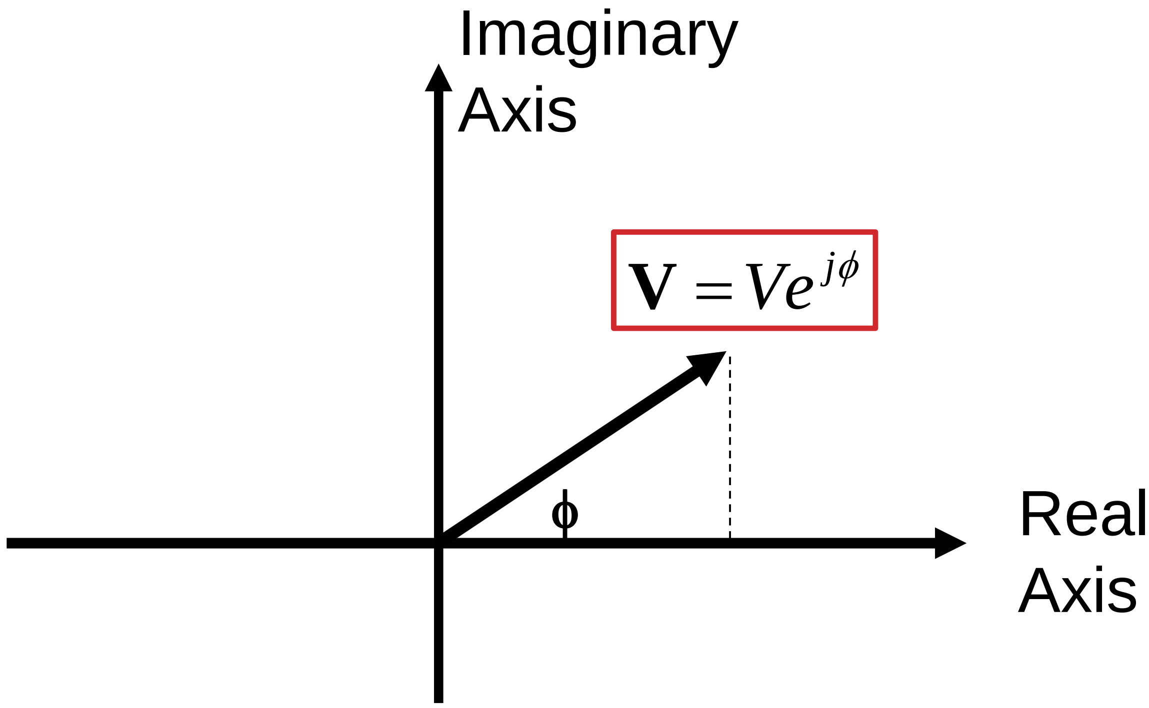
Similarly,

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{j\omega t}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \qquad \boxed{\mathbf{V} = Ve^{j\phi}}$$

Phasor:

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\{Ve^{j\phi}e^{j\omega t}\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \quad \boxed{\mathbf{V} = Ve^{j\phi}}$$

Complex representation of the magnitude and phase of a **sinusoid**



Phasor : Complex Numbers

- A powerful method for representing sinusoids is the phasor, a complex expression as well.
- A complex number z can be represented in *rectangular form* as:

$$z = x + jy \qquad \begin{aligned} \mathrm{Re}(z) &= x \\ \mathrm{Im}(z) &= y \end{aligned}$$

  - It can also be written in *polar* or *exponential* form as:

$$z = r\angle\phi = re^{j\phi}$$

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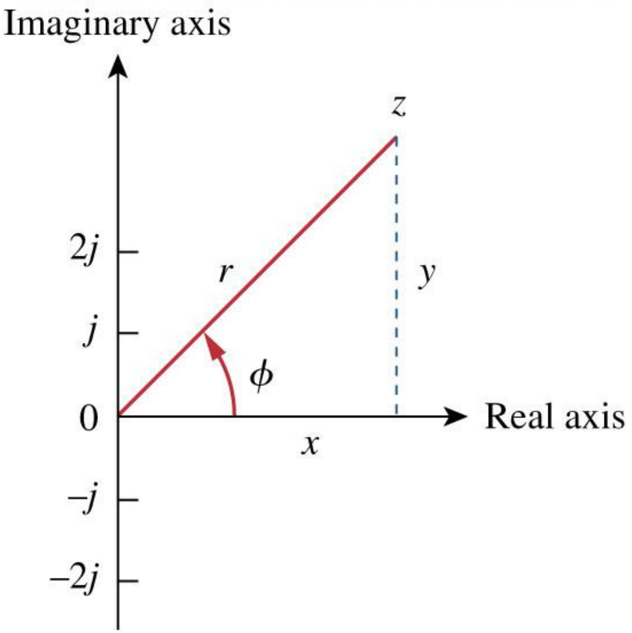

Arithmetic With Complex Numbers

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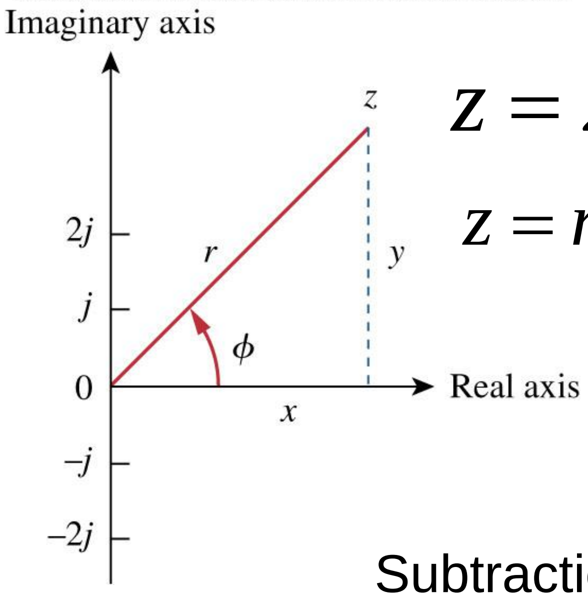

$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

Square Root

$$\boxed{\mathbf{z}^{1/2} = \pm|\mathbf{z}|^{1/2} e^{j\theta/2}}$$

Complex Conjugate

$$z^{*} = x - jy = r\angle -\phi = re^{-j\phi}$$

Relations for Complex Numbers

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\mathbf{z} = x + jy = |\mathbf{z}|e^{j\theta} \qquad \mathbf{z}^* = x - jy = |\mathbf{z}|e^{-j\theta}$$

$$x = \mathfrak{Re}(\mathbf{z}) = |\mathbf{z}|\cos\theta \qquad |\mathbf{z}| = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$$

$$y = \mathfrak{Im}(\mathbf{z}) = |\mathbf{z}|\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

$$\mathbf{z}^n = |\mathbf{z}|^n e^{jn\theta} \qquad \mathbf{z}^{1/2} = \pm|\mathbf{z}|^{1/2} e^{j\theta/2}$$

$$\mathbf{z}_1 = x_1 + jy_1 \qquad \mathbf{z}_2 = x_2 + jy_2$$

$$\mathbf{z}_1 = \mathbf{z}_2 \text{ iff } x_1 = x_2 \text{ and } y_1 = y_2 \qquad \mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$\mathbf{z}_1\mathbf{z}_2 = |\mathbf{z}_1||\mathbf{z}_2|e^{j(\theta_1+\theta_2)} \qquad \frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{|\mathbf{z}_1|}{|\mathbf{z}_2|}\, e^{j(\theta_1-\theta_2)}$$

$$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm 180^\circ$$

$$j = e^{j\pi/2} = 1\angle 90^\circ \qquad -j = e^{-j\pi/2} = 1\angle -90^\circ$$

$$\sqrt{j} = \pm e^{j\pi/4} = \pm\frac{(1+j)}{\sqrt{2}} \qquad \sqrt{-j} = \pm e^{-j\pi/4} = \pm\frac{(1-j)}{\sqrt{2}}$$

[Source: Berkeley]

Example

- Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Exercise

- Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Phasors

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

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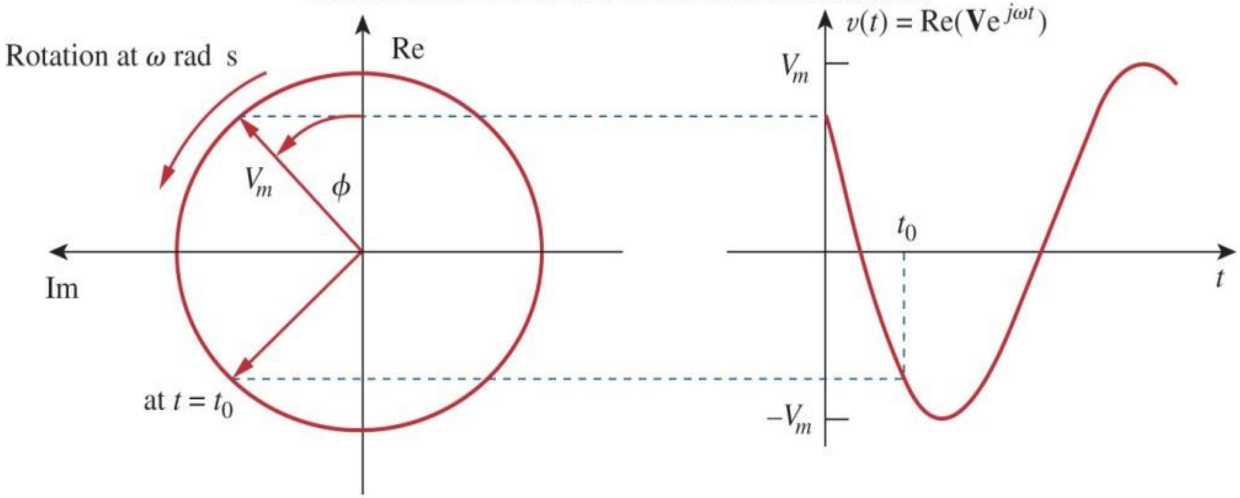

(a) (b)

$$V_m e^{j(\omega t+\phi)}$$

Example

- Transform these sinusoids to phasors

(a) $i = 6\cos(50t - 40^\circ)$ A

(b) $v = -4\sin(30t + 50^\circ)$ V

Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4\ \text{A}$$

$$\mathbf{V} = j8e^{-j20^\circ}\ \text{V}$$

$$\mathbf{V} = -25\angle 40^\circ\ \text{V}$$

$$\mathbf{I} = j(12 - j5)\ \text{A}$$

Phasor Relationships for Resistors

- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = R I_m \cos(\omega t + \phi) \qquad \mathbf{V} = R I_m \angle \phi$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$

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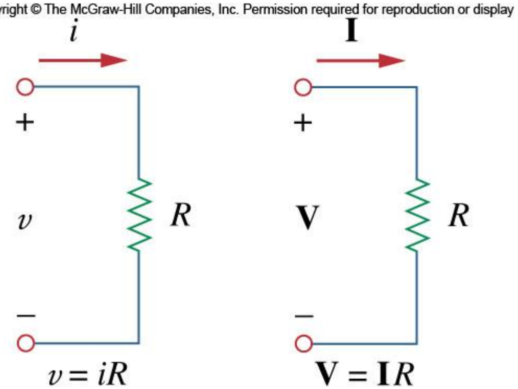

(a) (b)

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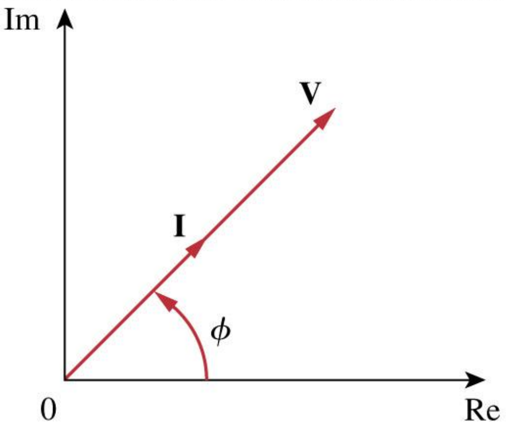
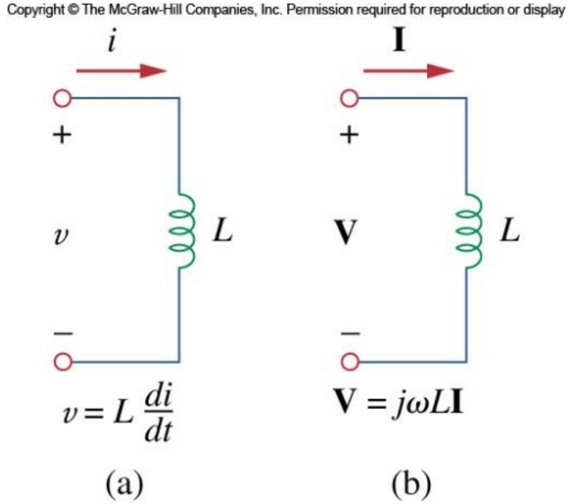


Phasor Relationships for Inductors



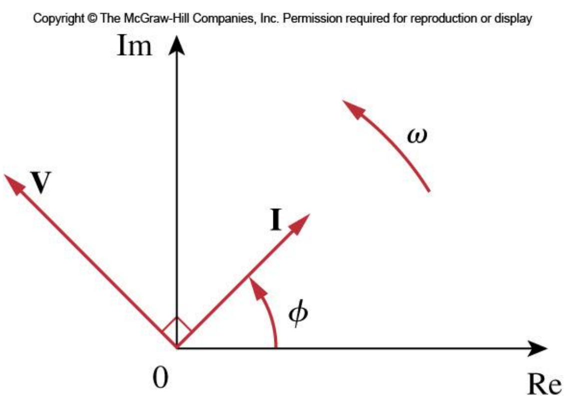
$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$
$$= \omega L I_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{V} = \omega L I_m \angle \phi + 90° = \omega L I_m e^{j(\phi + 90°)} = \omega L e^{j90°} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\boxed{\mathbf{V} = j\omega L \mathbf{I}}$$

- The voltage leads the current by 90° (phase shift = 90°)

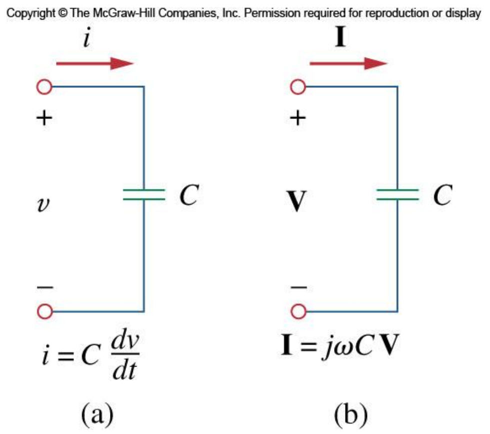


Phasor Relationships for Capacitors

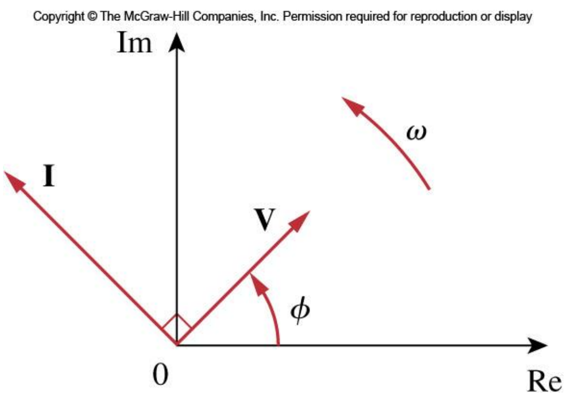
$$v = V_m \cos(\omega t + \phi) \qquad \mathbf{V} = V_m \angle \phi$$

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$
$$= \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90° = \cdots = j\omega C \cdot \mathbf{V}$$



$$\boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$



- The voltage lags the current by 90°.

Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\boldsymbol{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - Impedance is not a phasor! But it is (often) a complex number.
  - Impedance depends on the frequency ω.

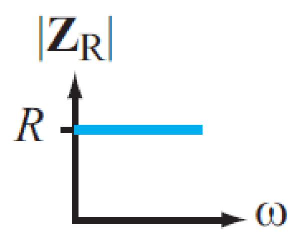
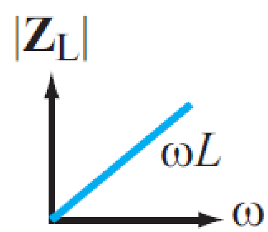
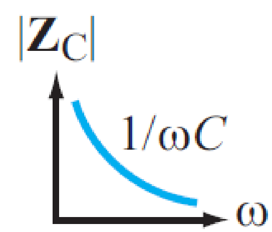
Admittance

Admittance is simply the inverse of impedance, unit: Simens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Summary of *R*, *L*, *C*

Property	R	L	C
v–i	$v = Ri$	$v = L\dfrac{di}{dt}$	$i = C\dfrac{dv}{dt}$
$\mathbf{V}$–$\mathbf{I}$	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
$\mathbf{Z}$	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

[Source: Berkeley]