Quiz Solutions

June 7, 2020

Lecture 27

$$g\left(\theta x_1 + (1-\theta)x_2\right) = \sup_{y\in A} f\left(\theta x_1 + (1-\theta)x_2, y\right).$$

Since $f(x,y)$ is convex x, we have

$$\begin{aligned}
\sup_{y\in A} f\left(\theta x_1 + (1-\theta)x_2, y\right) &\le \sup_{y\in A} \theta f\left(x_1, y\right) + \sup_{y\in A}(1-\theta) f\left(x_2, y\right) \\
&\le \theta \sup_{y\in A} f\left(x_1, y\right) + (1-\theta) \sup_{y\in A} f\left(x_2, y\right) \\
&= \theta g\left(x_1\right) + (1-\theta) g\left(x_2\right)
\end{aligned}$$

Therefore,

$$g\left(\theta x_1 + (1-\theta)x_2\right) \le \theta g\left(x_1\right) + (1-\theta) g\left(x_2\right),$$

namely, $g(x)$ is convex.

Lecture 28

Here, we consider the following standard Gaussian distribution, i.e., $\mu = 0, \sigma = 1$,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Recall that f is log-concave if and only if $f''(x)f(x) \le f'(x)^2$ for all x. We first calculate $f''(x)$ and $f'(x)$,

$$\begin{aligned}
f'(x) &= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} x = -f(x)x \\
f''(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^2 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = f(x)x^2 - f(x).
\end{aligned}$$

Clearly,

$$f''(x)f(x) = f(x)^2(x^2 - 1) \leq f(x)^2 x^2 = f'(x)^2,$$

which implies $f(x)$ is log-concave. The result can be readily generalized for any μ and σ.