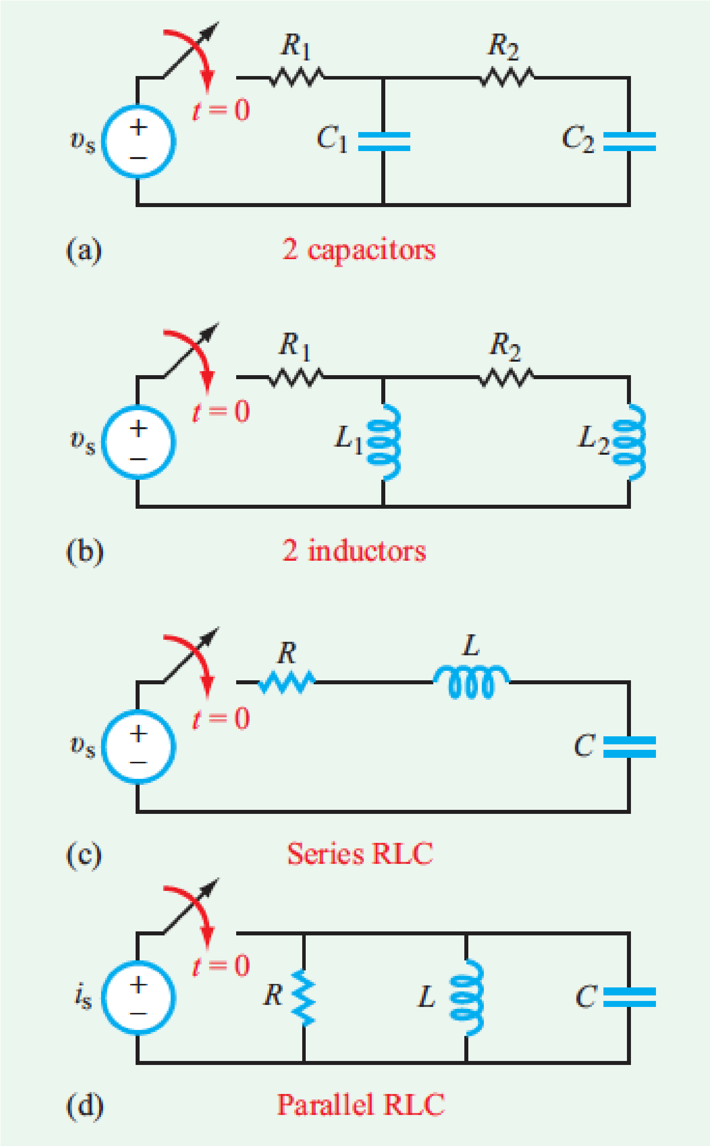
Lecture 6

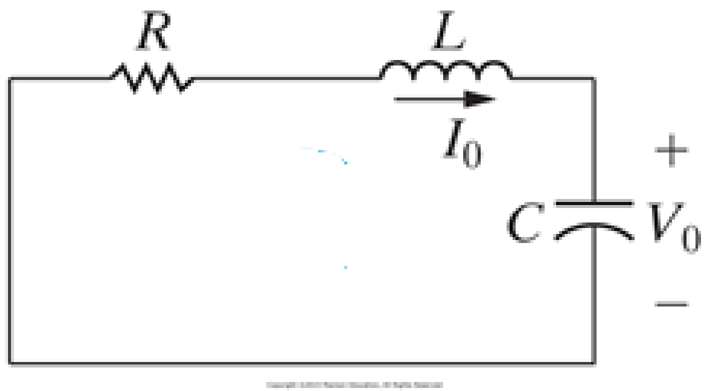
- Second-Order Circuits

Second-Order Circuits

- Two energy storage elements
- Analysis: Determine voltage or current as a function of time
- A second order circuit is characterized by a second order differential equation.
- Initial/final values of voltage/current, *and their derivatives are* needed



Source-Free Series RLC

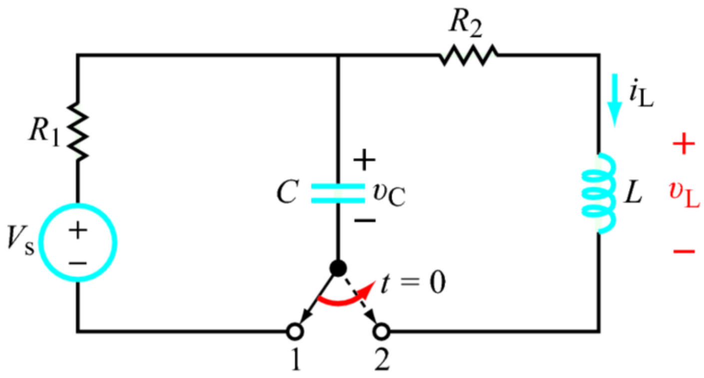


$$\frac{d^2v}{d^2t}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0$$

$$\boxed{s^2+\frac{R}{L}s+\frac{1}{LC}=0}$$

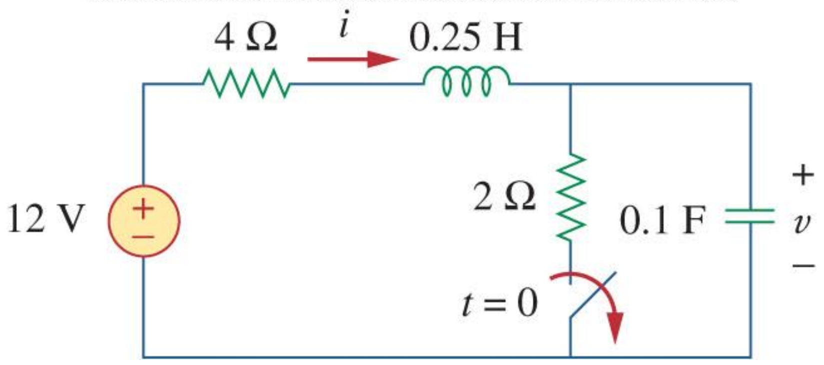
$$s=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

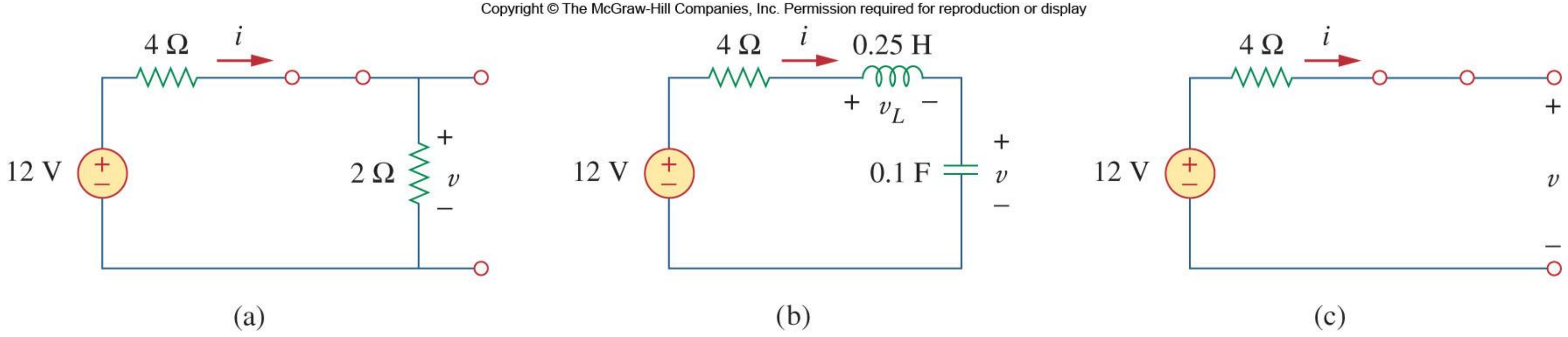
Initial and Final Conditions



Example

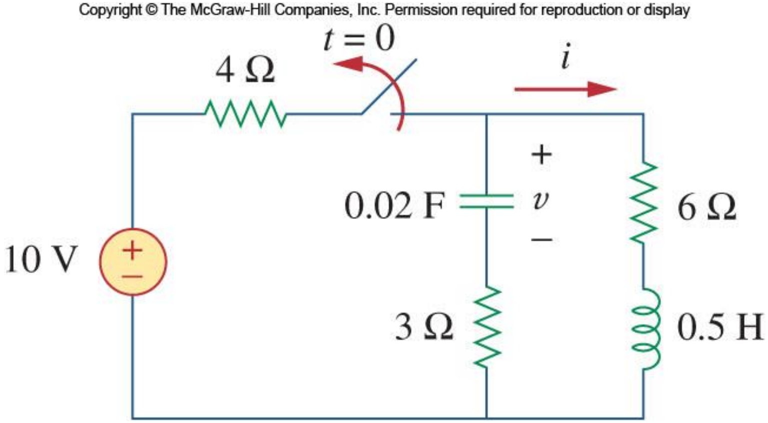
- The switch has been closed for a long time. It is open at $t = 0$. Find
  - $i(0^+)$, $v(0^+)$
  - $di(0^+)/dt$, $dv(0^+)/dt$
  - $i(\infty)$, $v(\infty)$



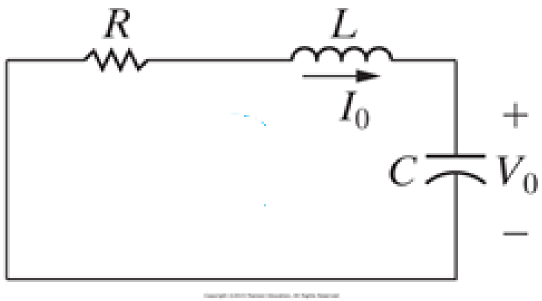


Exercise

- Assume the circuit has reached steady state at $t = 0^{-}$. Find
  - $i(0^{+})$, $v(0^{+})$
  - $di(0^{+})/dt$, $dv(0^{+})/dt$
  - $i(\infty)$, $v(\infty)$



Source-Free Series RLC



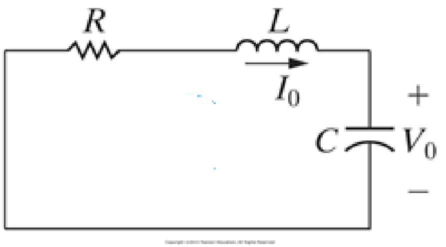
$$\frac{d^2v}{d^2t} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}}v = 0$$

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

[Source: Berkeley]

Case 1: Overdamped ($\alpha > \omega_0$)



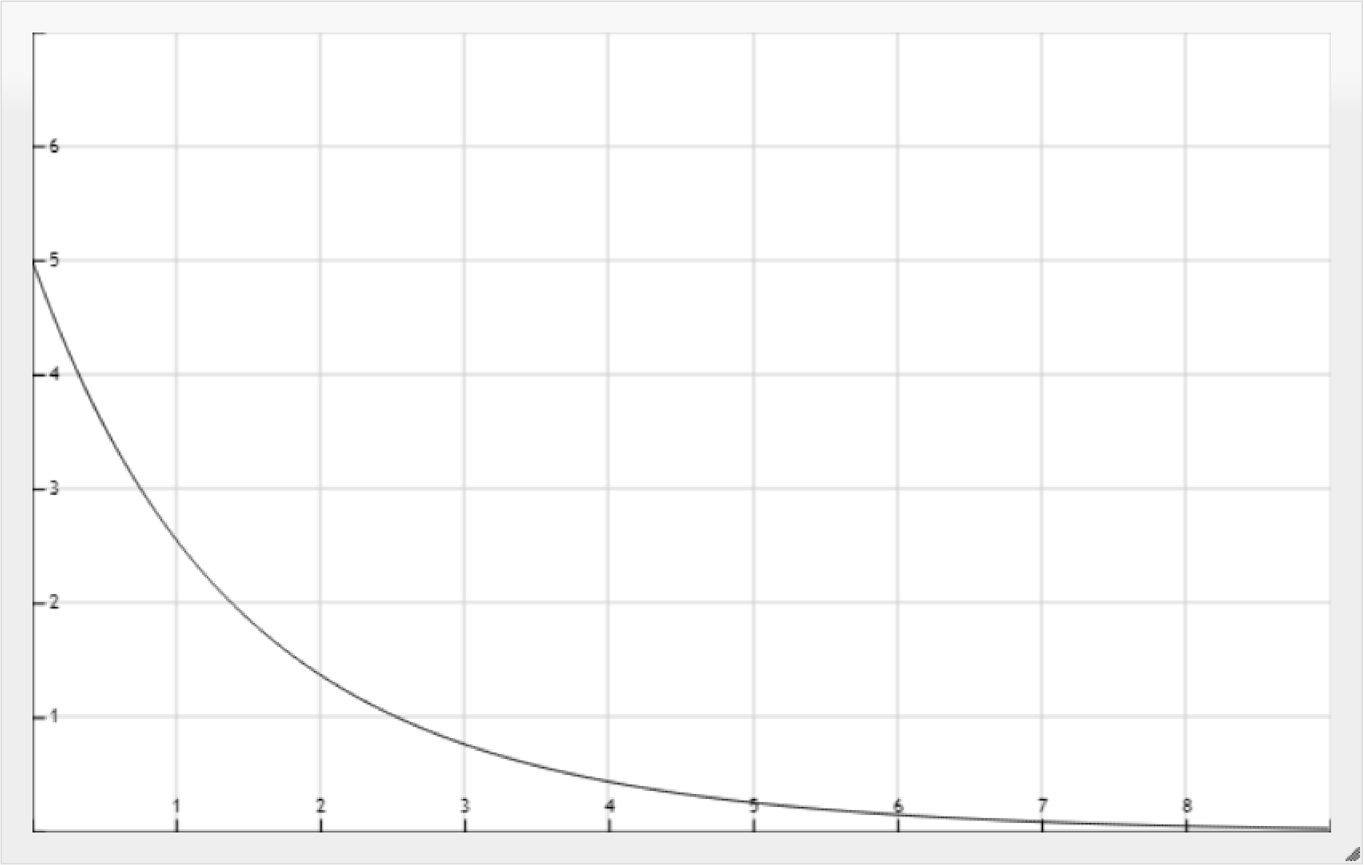
$$\frac{d^2 v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}} v = 0$$

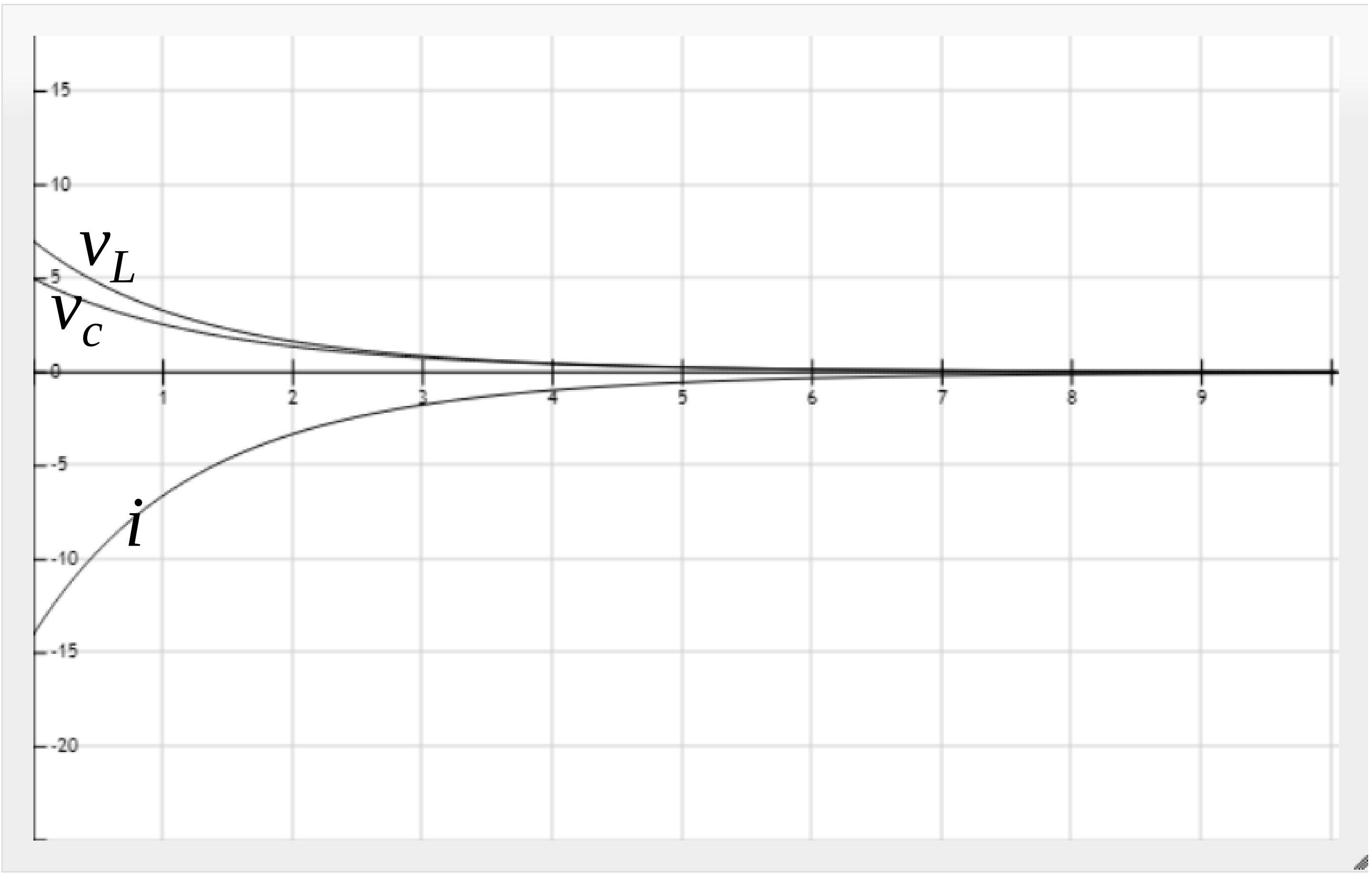
$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

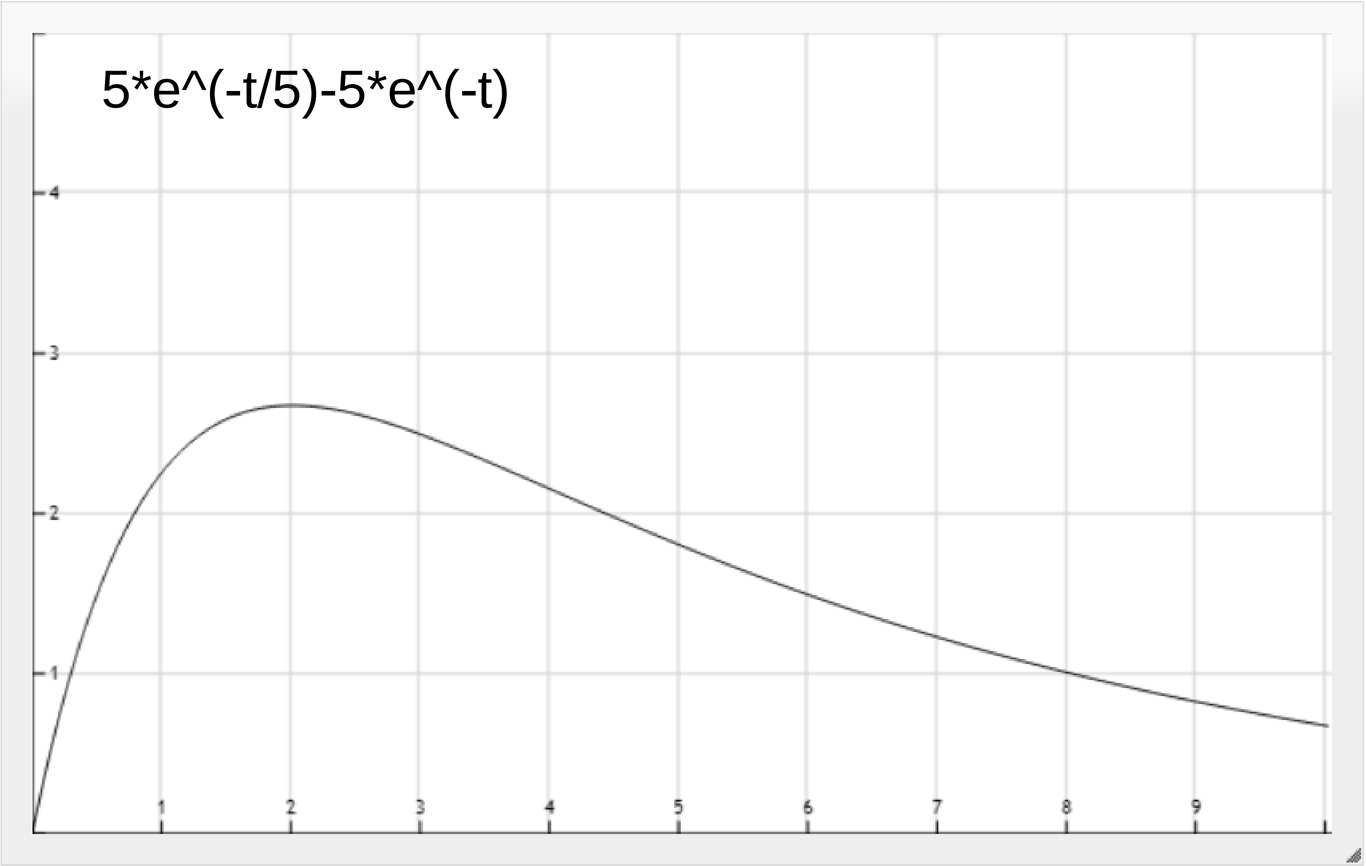
An example

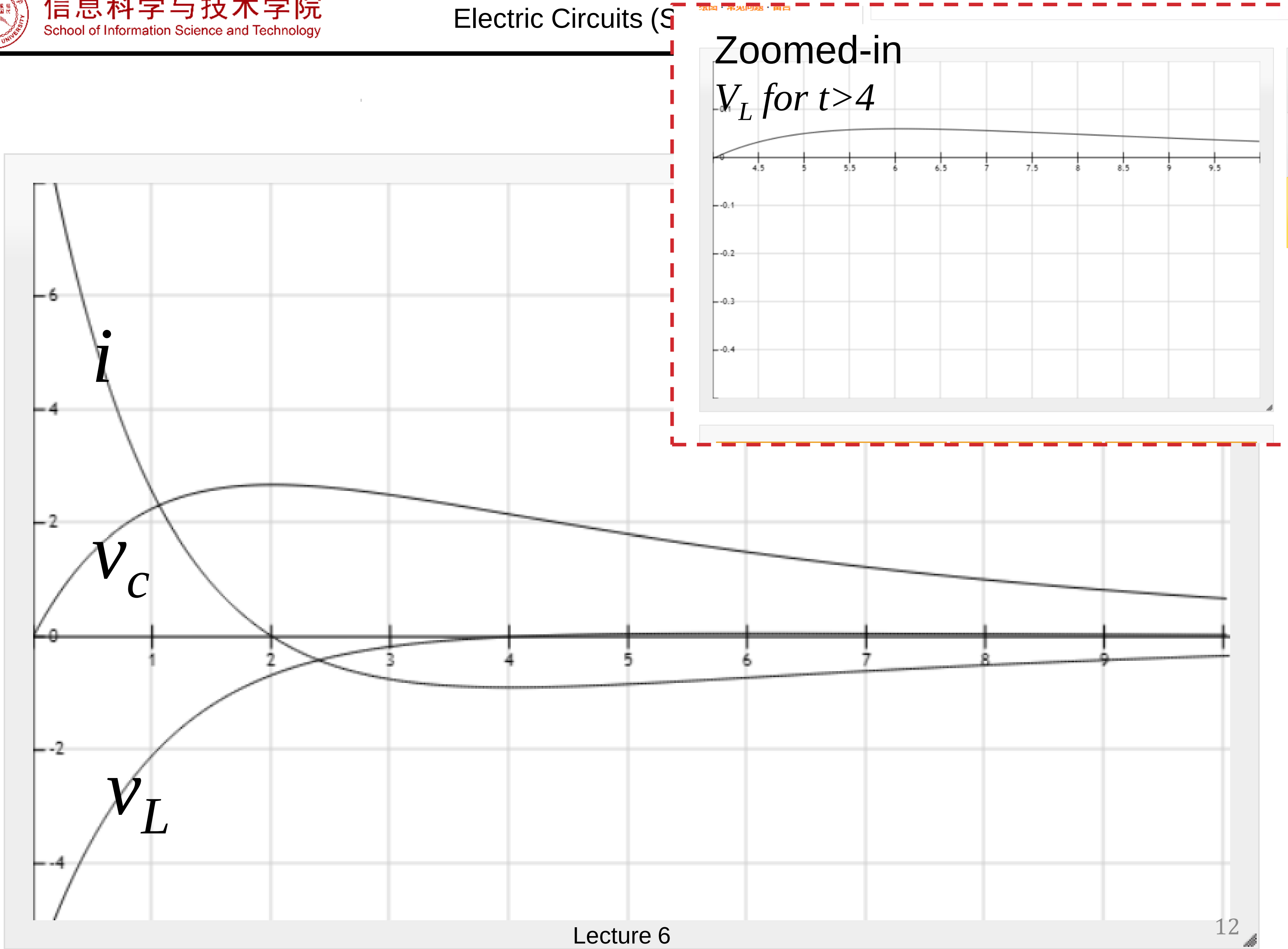
$V_c=$ 2e^(-t)+3e^(-t/2)




15
10
5
0
-5
-10
-15
-20
1
2
3
4
5
6
7
8
9
v_L
v_C
i

Another example

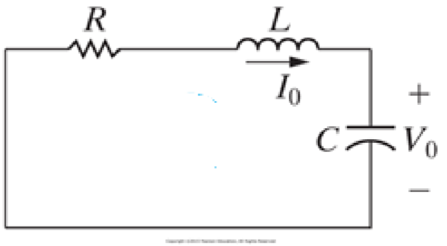




Zoomed-in

V_L for $t>4$

Case 2: Critically Damped ($\alpha=\omega_0$)



$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \qquad v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Case 2: Critically Damped ($\alpha=\omega_0$)

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$

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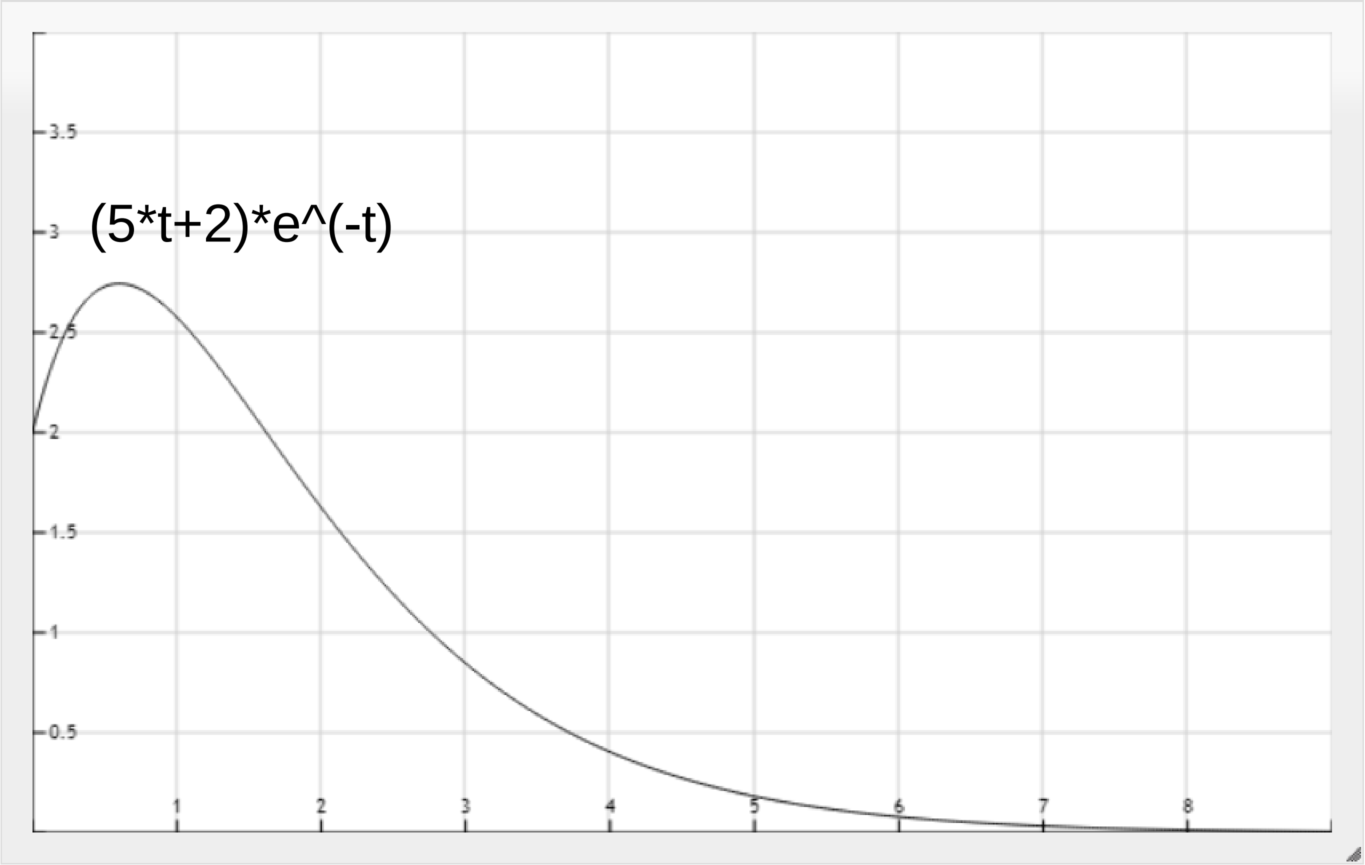
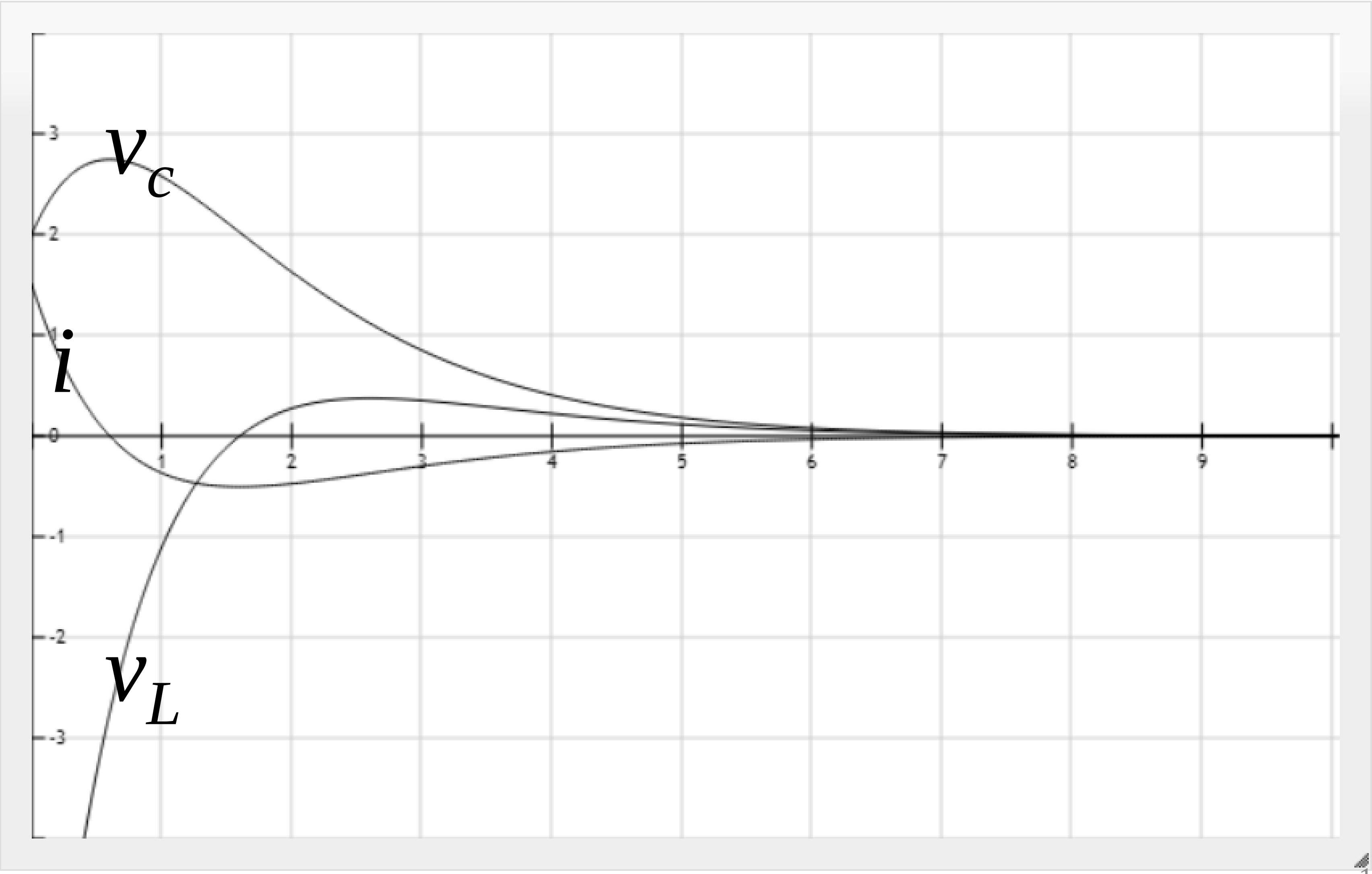



v_c
i
v_L
3
2
1
0
-1
-2
-3
1
2
3
4
5
6
7
8
9

Case 3: Underdamped ($\alpha<\omega_0$)

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

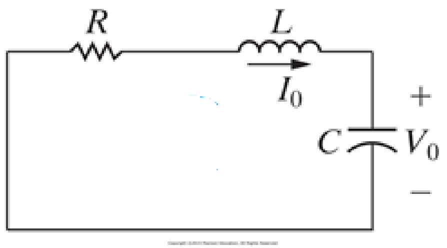
- ω_0 is often called the resonant frequency;
- ω_d is called the damping frequency.

The natural response

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

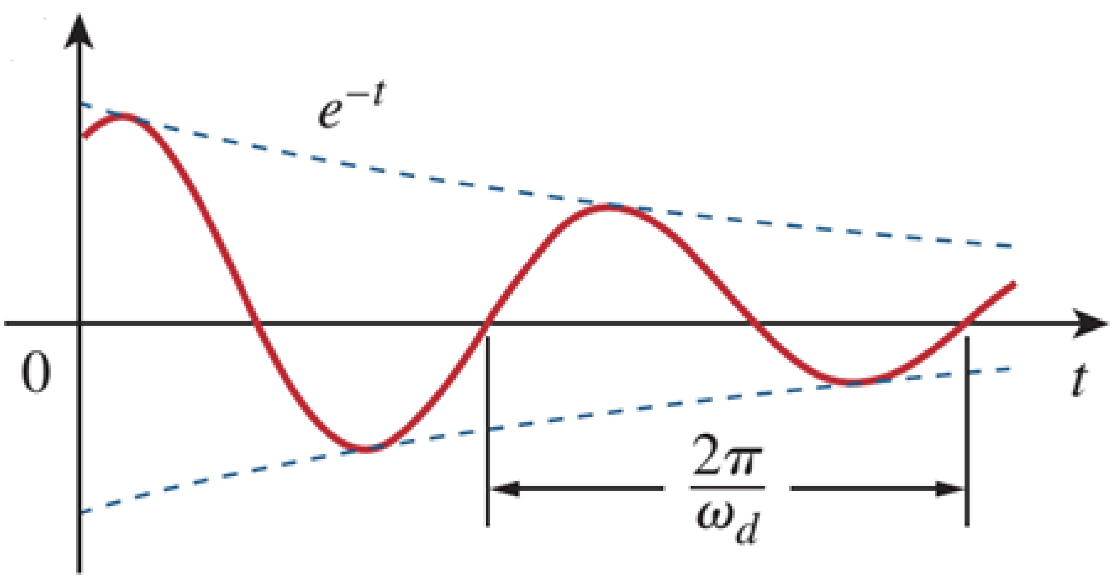
becomes

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

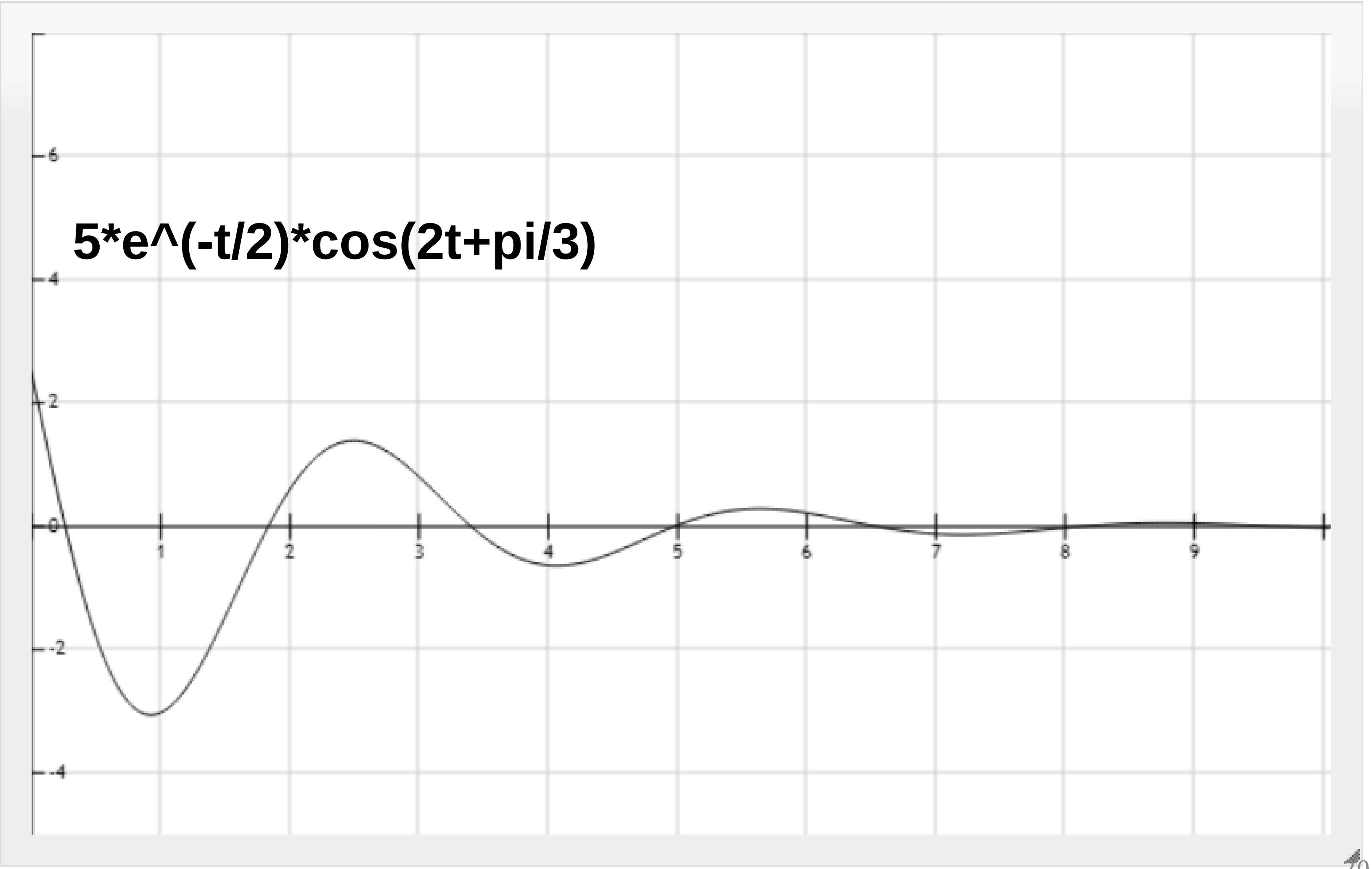


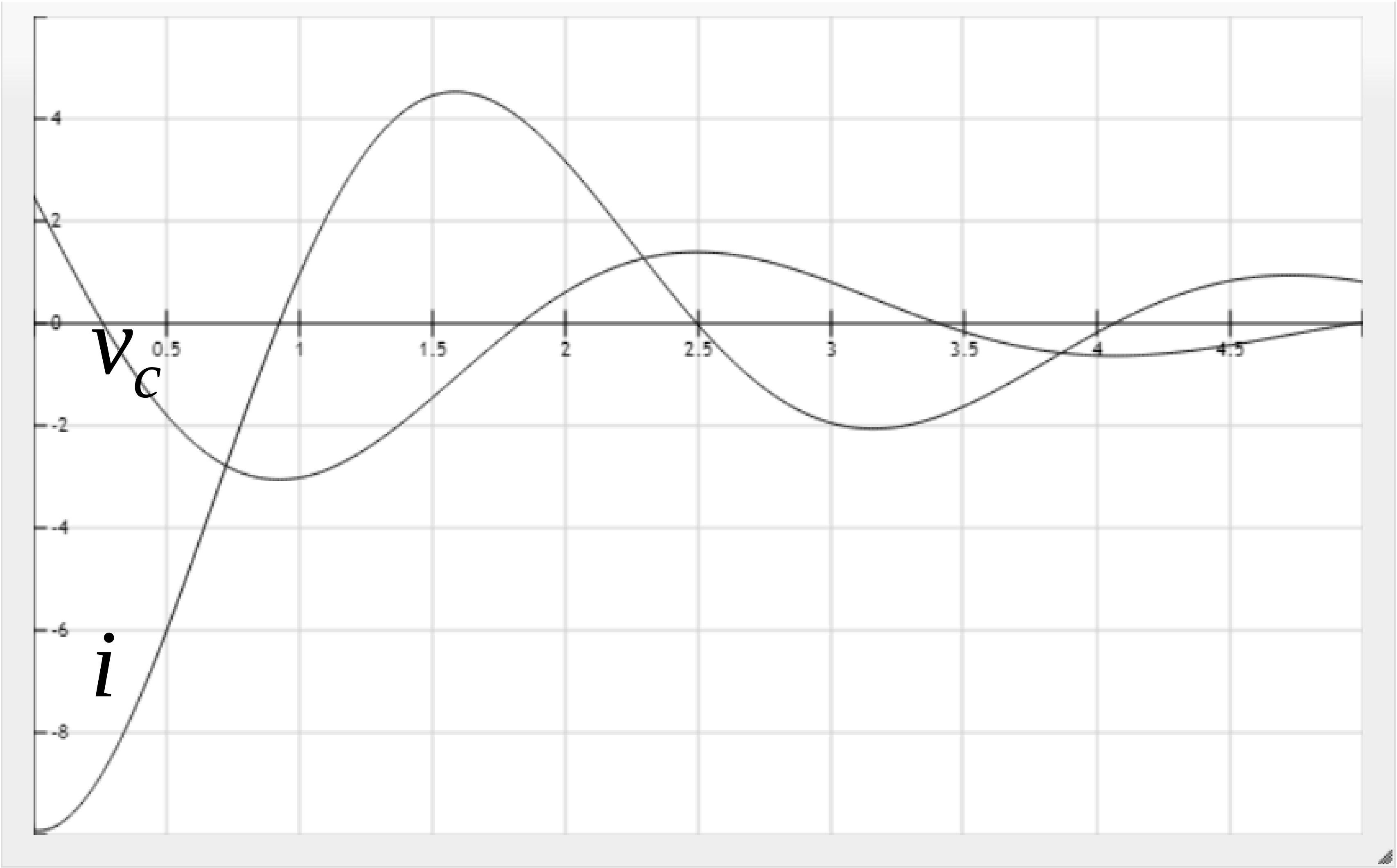
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

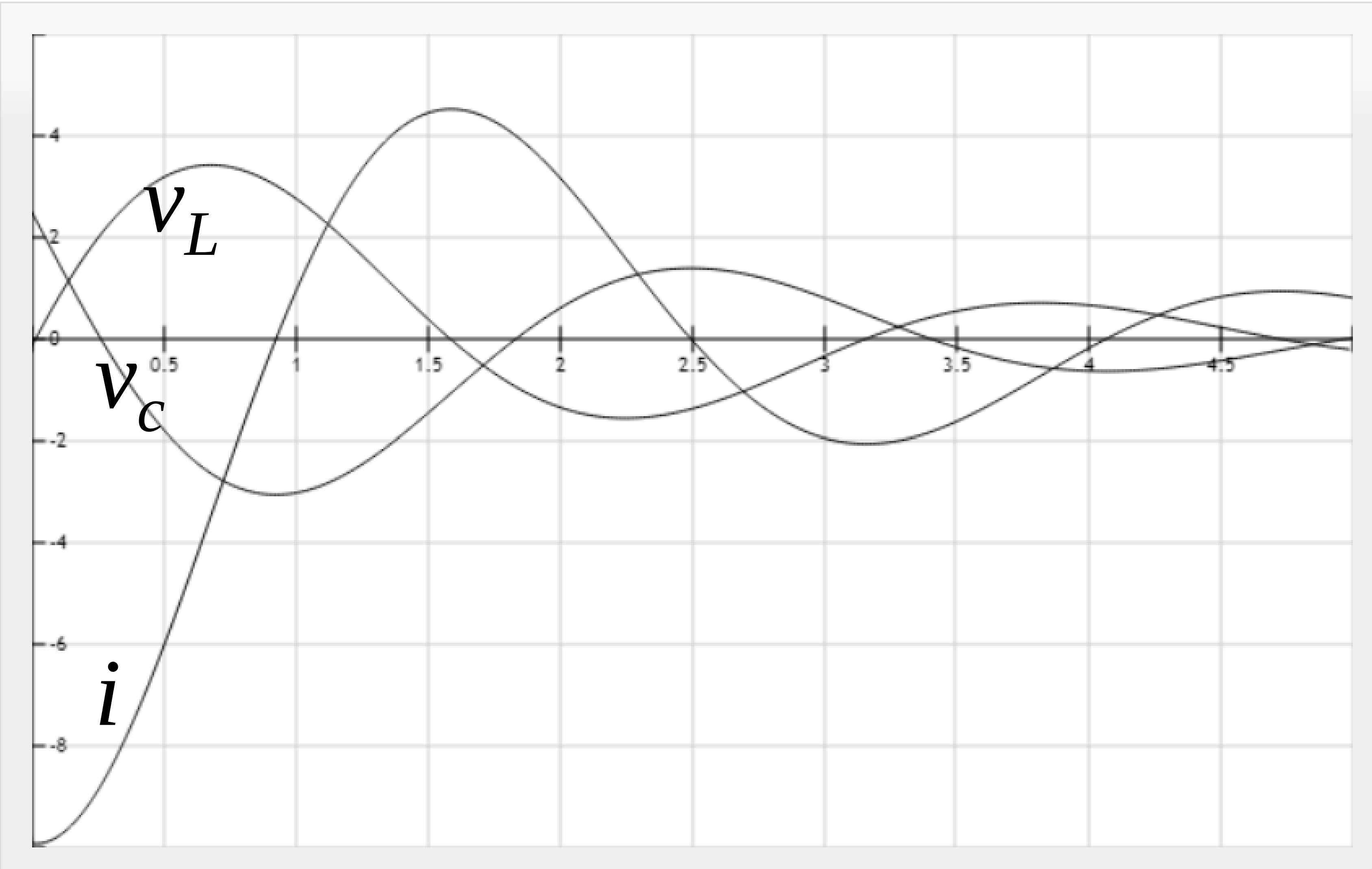
- Exponential $e^{-\alpha t}$ * Sine/Cosine term
  - Exponentially damped, time constant = $1/\alpha$
  - Oscillatory, period $T = \frac{2\pi}{\omega_d}$



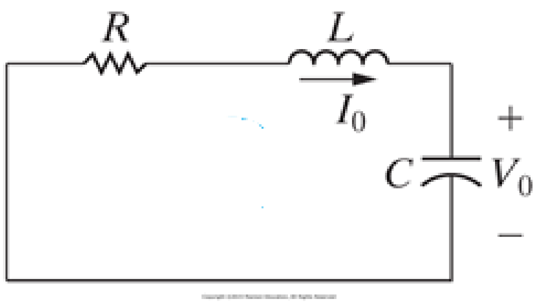
T=?




4
2
0
-2
-4
-6
-8
0.5
1
1.5
2
2.5
3
3.5
4
4.5
v_C
i


v_L
v_C
i
4
2
0
-2
-4
-6
-8
0.5
1
1.5
2
2.5
3
3.5
4
4.5

Properties of Series RLC Network



- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - α determines the rate of damping
    - $\alpha > \omega_0$ (i.e., $R > 2\sqrt{\frac{L}{C}}$), overdamped

      $$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

    - $\alpha = \omega_0$ (i.e., $R = 2\sqrt{\frac{L}{C}}$), critically damped

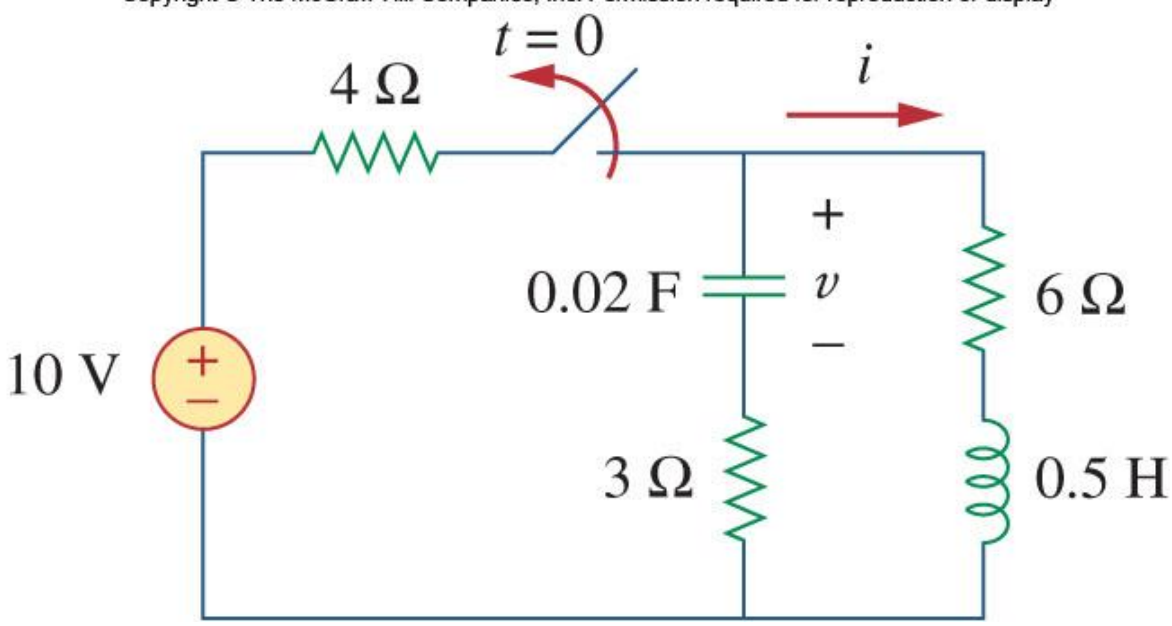
      $$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

    - $\alpha < \omega_0$ (i.e., $R < 2\sqrt{\frac{L}{C}}$), underdamped

      $$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

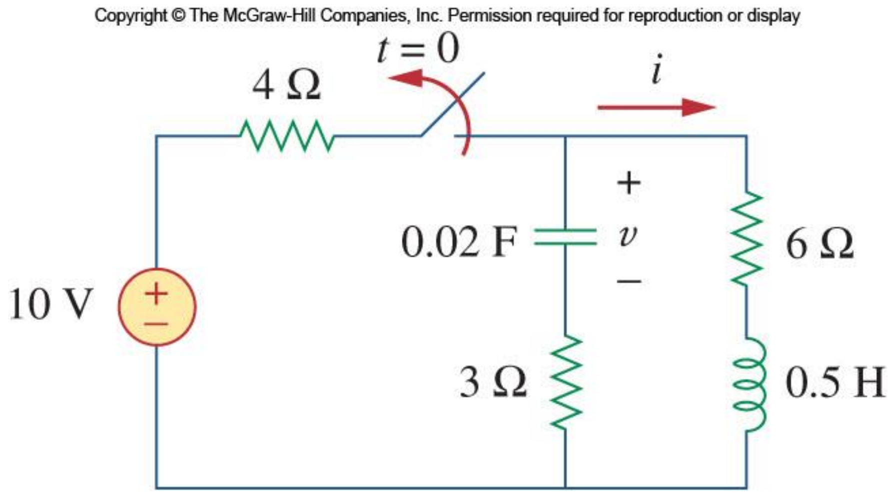
Example

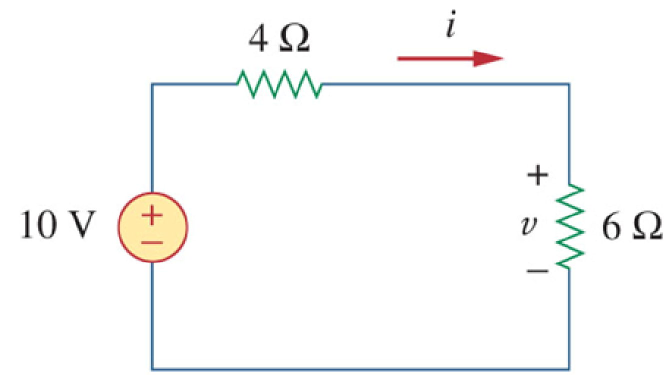
- Find $v(t)$ &*i(t)* in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.

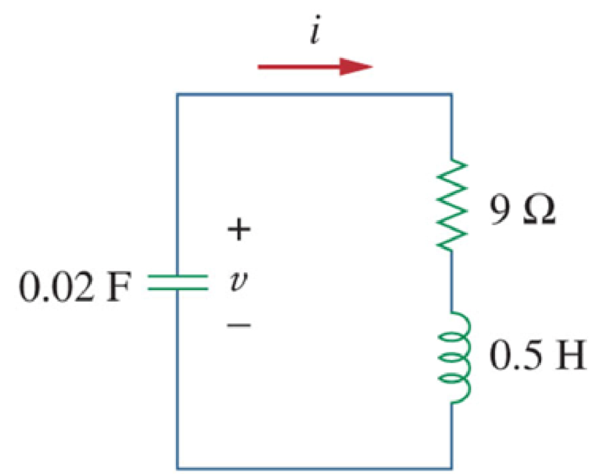


Example

- Find $v(t)$ &*i(t)* in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.



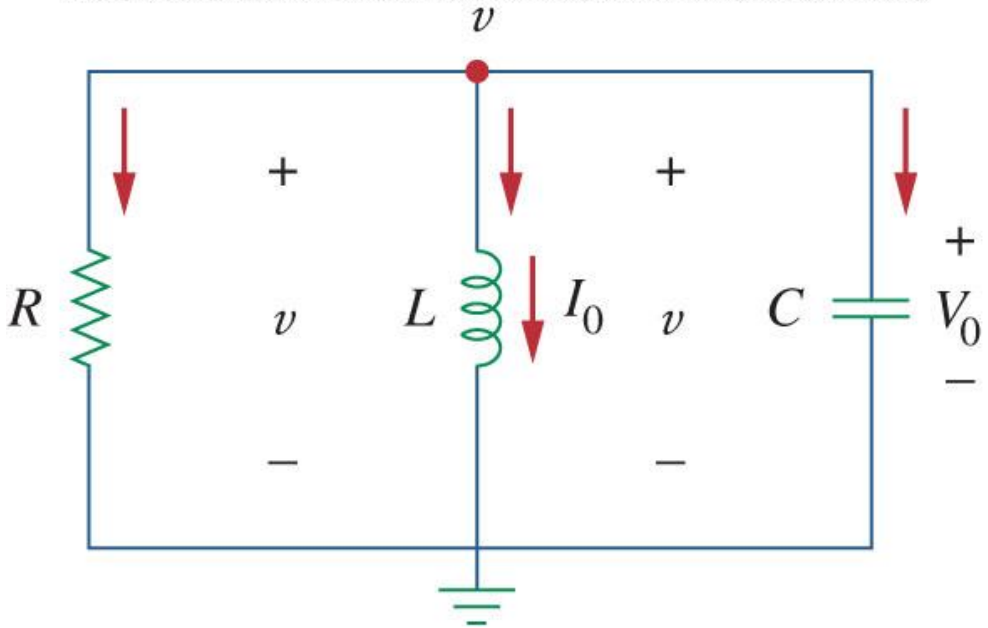




$$\alpha = \frac{R}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

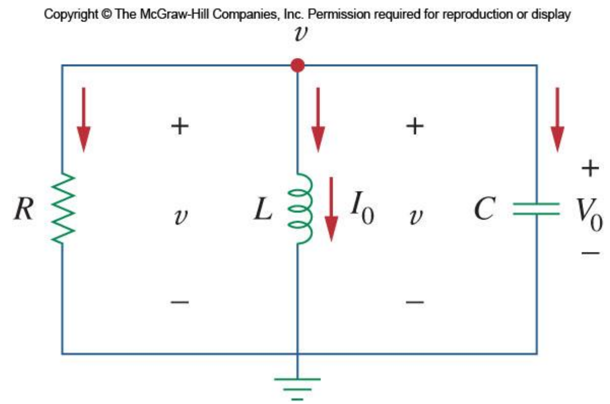
$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = -9 \mp j4.359$$

Source-Free Parallel RLC Network



Source-Free Parallel RLC Network

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$



- The characteristic equation is:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

From this, we can find the roots of the characteristic equation to be:

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

As in last time, there are three scenarios to consider.

Three Damping Cases

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\quad \omega_0=\frac{1}{\sqrt{LC}}$$

- For the overdamped case, the roots are real and negative,

$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

- For critically damped, the roots are real and equal

$$v(t)=(A_2+A_1t)e^{-\alpha t}$$

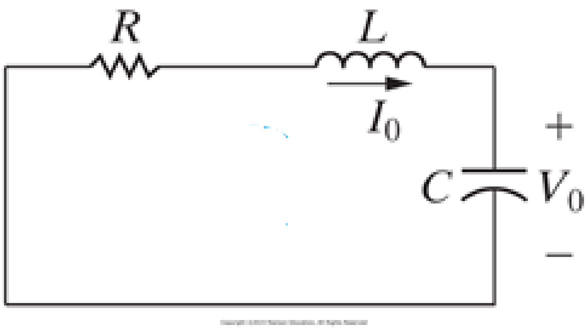
- In the underdamped case, the roots are complex

$$v(t)=e^{-\alpha t}(A_1\cos\omega_dt+A_2\sin\omega_dt)$$

Series vs. Parallel (Source-Free RLC Network)

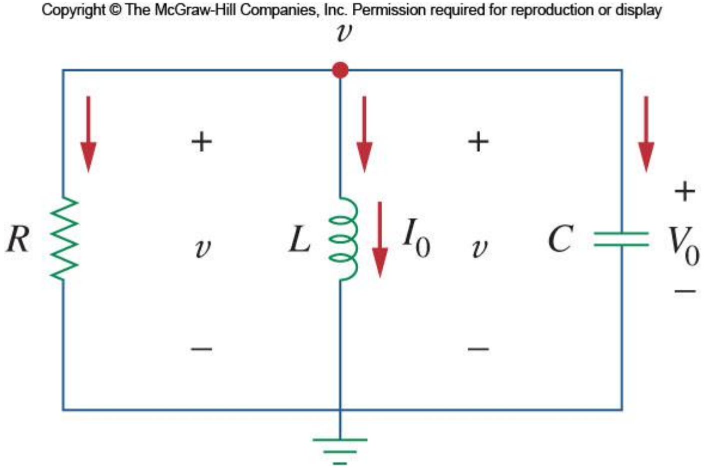
- Series

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

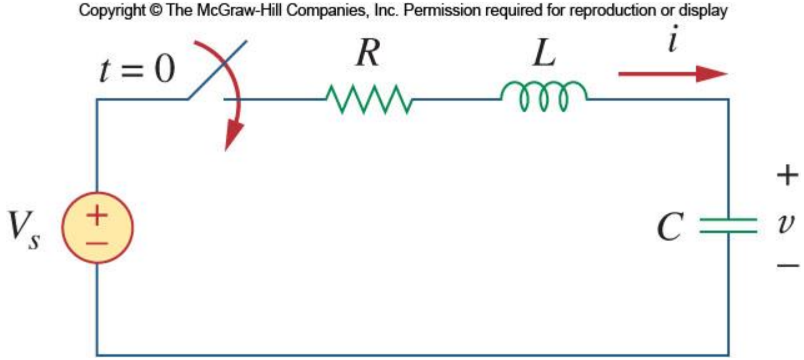


- Parallel

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v(t) = (A_2 + A_1 t) e^{-\alpha t}$$
$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$



Step Response of a Series RLC Circuit



$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

- The solution

$$v(t)=v_t(t)+v_{ss}(t)$$

- The complete solutions for the three conditions of damping are:

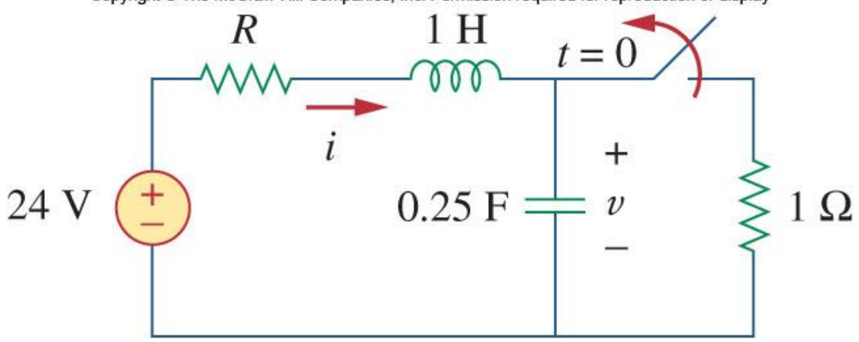
$$v(t)=V_s+(A_1e^{s_1t}+A_2e^{s_2t})\ \text{(Overdamped)}$$

$$v(t)=V_s+(A_1+A_2t)e^{-\alpha t}\ \text{(Critically Damped)}$$

$$v(t)=V_s+(A_1\cos\omega_dt+A_2\sin\omega_dt)e^{-\alpha t}\ \text{(Underdamped)}$$

Example

- Find $v(t)$ and $i(t)$ for $t > 0$.
  Consider three cases:
  - $R = 5\Omega$
  - $R = 4\Omega$
  - $R = 1\Omega$



When $R = 5\Omega$,

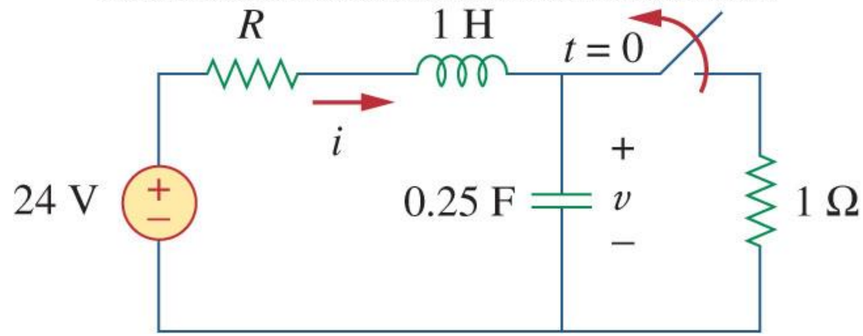
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4A = C\frac{dv(0)}{dt},\ \ v(0) = 4V,\ \ \frac{dv(0)}{dt} = 16$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2.5,\ \ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ \ s_{1,2} = -1, -4$$ Overdamped.

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$



When $R = 4\Omega$,

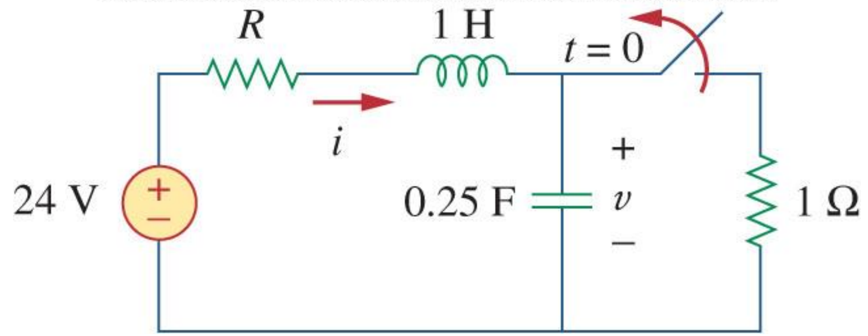
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4.8A = C\frac{dv(0)}{dt},\ \ v(0) = 4.8V,\ \ \frac{dv(0)}{dt} = 19.2$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2,\ \ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ \ s_{1,2} = -2$$ Critically damped

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$



When $R = 1\Omega$,

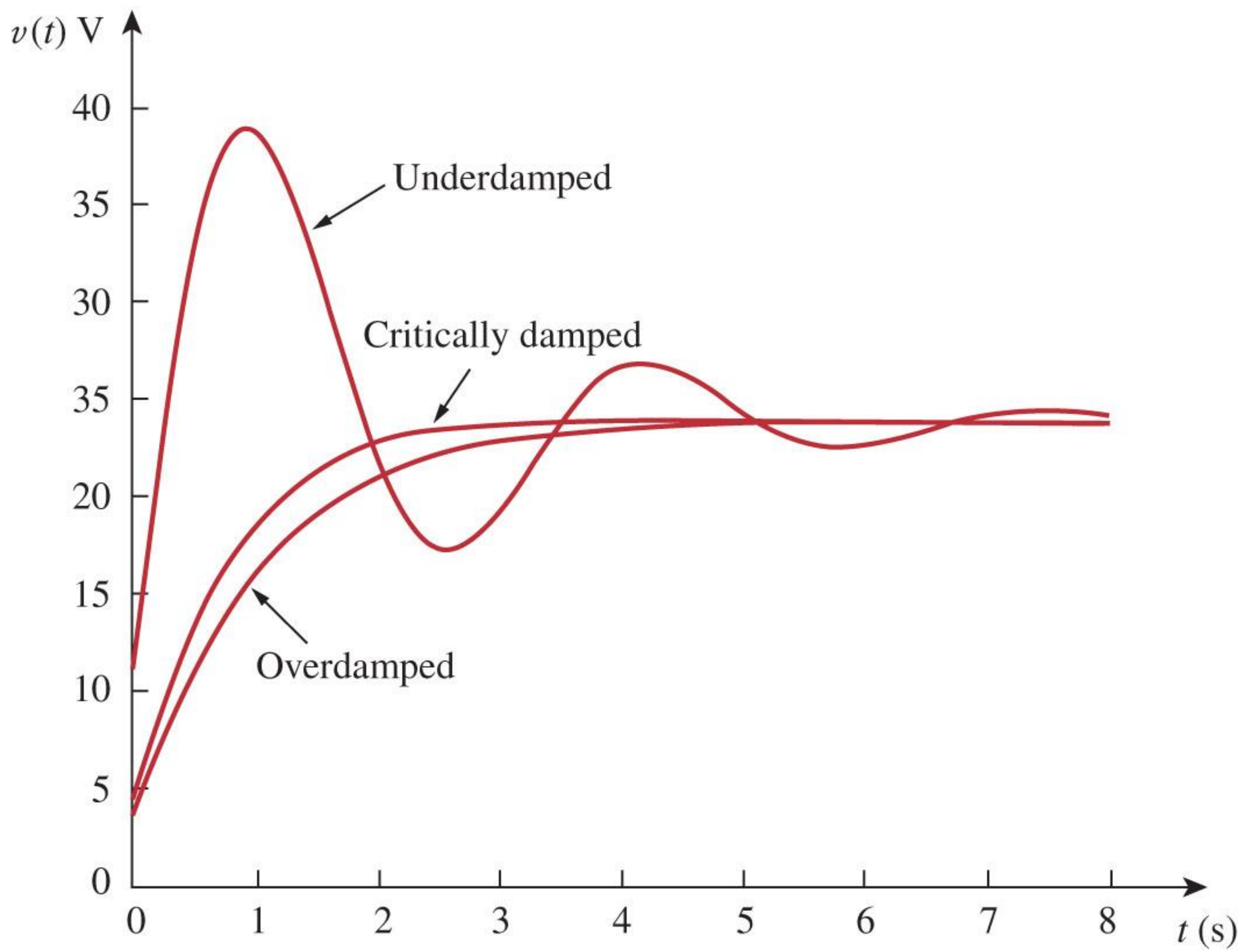
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 12A = C\frac{dv(0)}{dt},\ \ v(0) = 12V,\ \ \frac{dv(0)}{dt} = 48$$

- For $t > 0$, switch open, a series RLC network

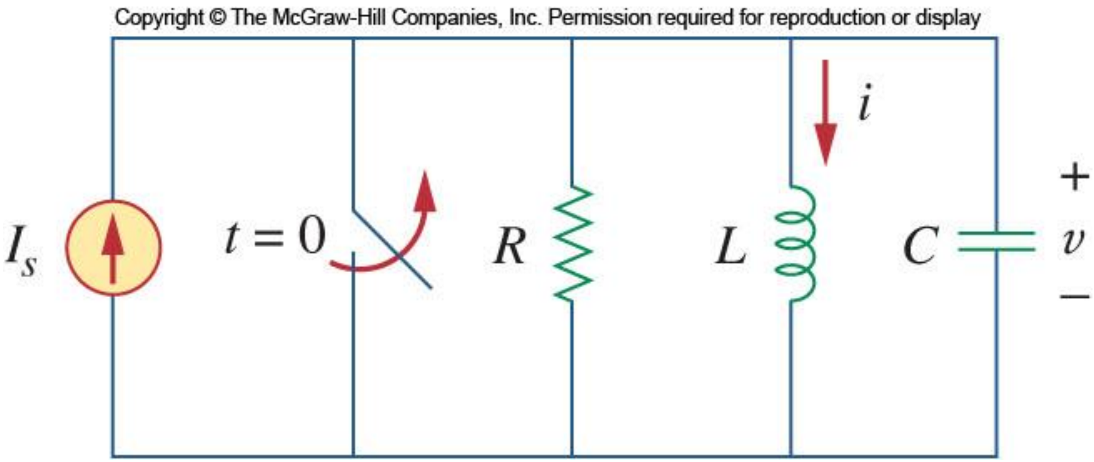
$$\alpha = \frac{R}{2L} = 0.5,\ \ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ \ s_{1,2} = -0.5 \pm j1.936$$ Underdamped

$$v(t) = v_{ss} + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t}$$



v(t) V
40
35
30
35
20
15
10
5
0
Underdamped
Critically damped
Overdamped
0
1
2
3
4
5
6
7
8
t (s)

Step Response of a Parallel RLC Circuit



Apply KCL,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

But

$$v = L\frac{di}{dt}$$

So we get

$$\frac{d^2 i}{d^2 t} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

Step Response of a Parallel RLC Circuit

$$\frac{d^2 i}{d^2 t} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

- As in the series RLC case, the response is a combination of transient and steady state responses:

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

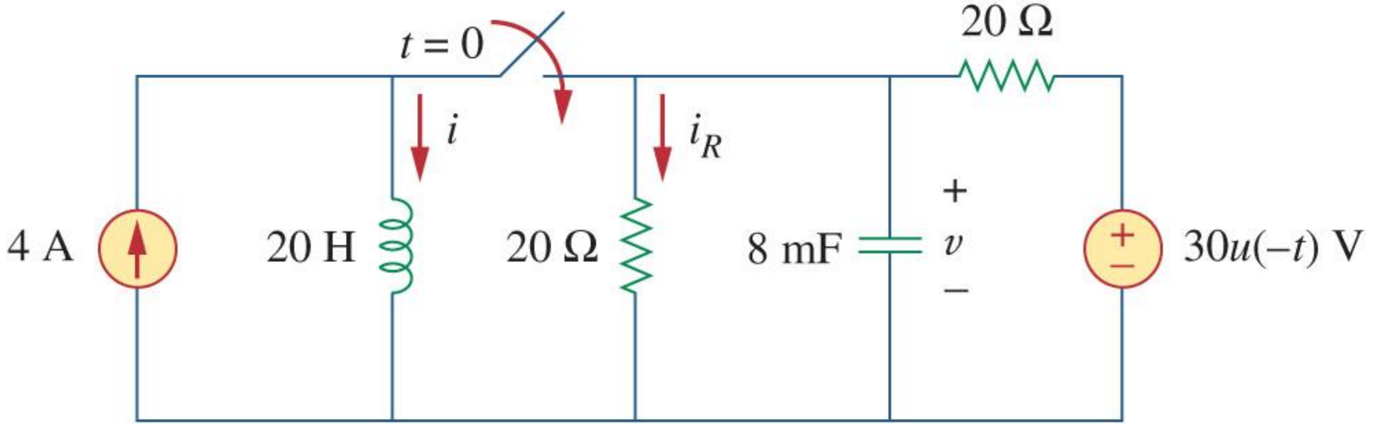
$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critally Damped)}$$

$$i(t) = I_s + (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}$$

Here the variables A_1 and A_2 are obtained from the initial conditions, *i(0)* and *di(0)/dt*.

Example

- Find $i(t)$ and $i_R(t)$ for $t > 0$.



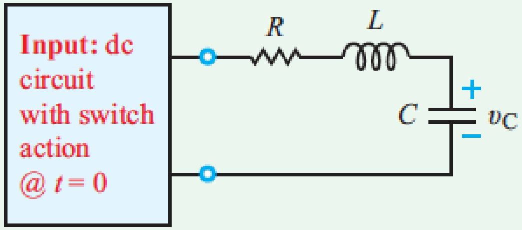
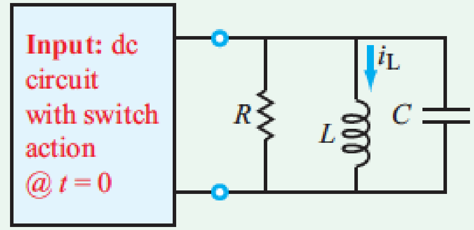
Initial values ($t < 0$): $i(0) = 4A$, $v(0) = \frac{20}{20+20} \times 30V = 15V = L\frac{di(0)}{dt}$

For $t > 0$, $\alpha = \frac{1}{2RC} = \frac{1}{2\times10\times8\times10^{-3}} = 6.25$, $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times8\times10^{-3}}} = 2.5$

$$s_{1,2} = -6.25 \pm 5.7282$$

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Step response of RLC circuits for $t \geq 0$.

Series RLC	Parallel RLC
	
Total Response	**Total Response**
Overdamped $(\alpha > \omega_0)$ $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty)$ $A_1 = \dfrac{\frac{1}{C}\, i_C(0) - s_2[v_C(0) - v_C(\infty)]}{s_1 - s_2}$ $A_2 = \left[\dfrac{\frac{1}{C}\, i_C(0) - s_1[v_C(0) - v_C(\infty)]}{s_2 - s_1}\right]$	**Overdamped** $(\alpha > \omega_0)$ $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + i_L(\infty)$ $A_1 = \dfrac{\frac{1}{L}\, v_L(0) - s_2[i_L(0) - i_L(\infty)]}{s_1 - s_2}$ $A_2 = \left[\dfrac{\frac{1}{L}\, v_L(0) - s_1[i_L(0) - i_L(\infty)]}{s_2 - s_1}\right]$
Critically Damped $(\alpha = \omega_0)$ $v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$ $B_1 = v_C(0) - v_C(\infty)$ $B_2 = \frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]$	**Critically Damped** $(\alpha = \omega_0)$ $i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$ $B_1 = i_L(0) - i_L(\infty)$ $B_2 = \frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]$
Underdamped $(\alpha < \omega_0)$ $v_C(t) = e^{-\alpha t}(D_1 \cos\omega_d t + D_2 \sin\omega_d t) + v_C(\infty)$ $D_1 = v_C(0) - v_C(\infty)$ $D_2 = \dfrac{\frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d}$	**Underdamped** $(\alpha < \omega_0)$ $i_L(t) = e^{-\alpha t}(D_1 \cos\omega_d t + D_2 \sin\omega_d t) + i_L(\infty)$ $D_1 = i_L(0) - i_L(\infty)$ $D_2 = \dfrac{\frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]}{\omega_d}$

Auxiliary Relations

$$\alpha = \begin{cases} \dfrac{R}{2L} & \text{Series RLC} \\[2ex] \dfrac{1}{2RC} & \text{Parallel RLC} \end{cases} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

General Second-Order Circuits

- The principles of the approach to solve the series and parallel forms of RLC circuits can **be applied** to general second-order circuits, by taking the following six steps:

  1. First determine the initial conditions, $x(0)$ and $dx(0)/dt$.

  2. Applying KVL and KCL, to find the general second-order differential equation to describe $x(t)$. 3.**Depending on the roots of C.E. , the form of the general solution (3 cases) of homogeneous equation can be determined.**

  4. We obtain the **particular solution** by observation/calculation, **specially** for a DC/step response

$$x_{p.s.}(t)=x(\infty)$$

  5. The total response = general solution + particular solution.

$$X(t)=x_{p.s.}(t)+ x_{g.s.}(t)$$

  6. Using the initial conditions to determine the constants of $X(t)$.

General solution for second-order circuits for $t \geq 0$.

$x(t)$ = unknown variable (voltage or current)

Differential equation: $x'' + ax' + bx = c$

Initial conditions: $x(0)$ and $x'(0)$

Final condition: $x(\infty) = \dfrac{c}{b}$

$$\alpha = \frac{a}{2} \qquad \omega_0 = \sqrt{b}$$

Overdamped Response $\alpha > \omega_0$

$$x(t) = [A_1 e^{s_1 t} + A_2 e^{s_2 t} + x(\infty)]$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$A_1 = \frac{x'(0) - s_2[x(0) - x(\infty)]}{s_1 - s_2} \qquad A_2 = -\left[\frac{x'(0) - s_1[x(0) - x(\infty)]}{s_1 - s_2}\right]$$

Critically Damped $\alpha = \omega_0$

$$x(t) = [(B_1 + B_2 t)e^{-\alpha t} + x(\infty)]$$

$$B_1 = x(0) - x(\infty) \qquad B_2 = x'(0) + \alpha[x(0) - x(\infty)]$$

Underdamped $\alpha < \omega_0$

$$x(t) = [D_1 \cos \omega_d t + D_2 \sin \omega_d t]e^{-\alpha t} + x(\infty)$$

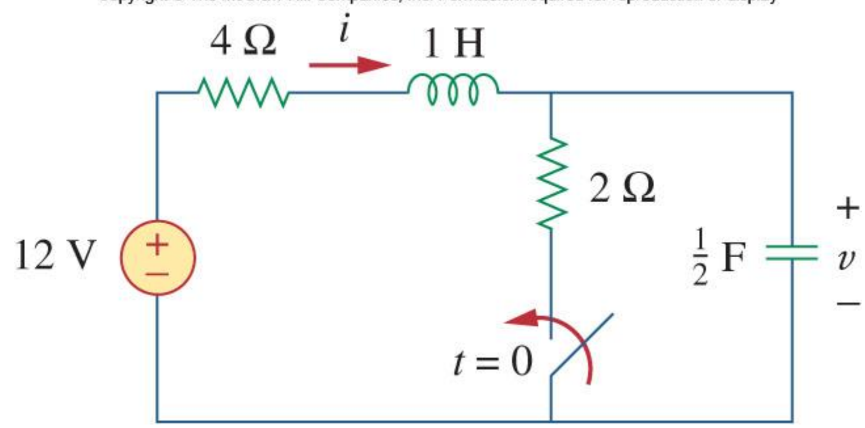
$$D_1 = x(0) - x(\infty) \qquad D_2 = \frac{x'(0) + \alpha[x(0) - x(\infty)]}{\omega_d}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$x(t)$ = unknown variable (voltage or current)	
Differential equation:	$x'' + ax' + bx = c$
Initial conditions:	$x(0)$ and $x'(0)$
Final condition:	$x(\infty) = \dfrac{c}{b}$

[Important]
1.This table works well when c is a constant, as $x(\infty)$ is actually a particular solution (特解) of the equation.
2. When c is a function of time (t), such as $c= 5t$; $c= t^2+3$; you should also be able to solve the equation (Requirement of the course).

General RLC Circuits

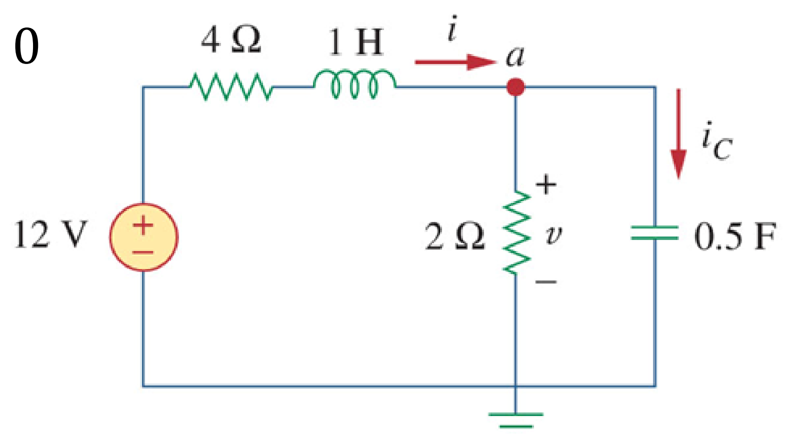


- Find the complete response v for $t > 0$ in the circuit.

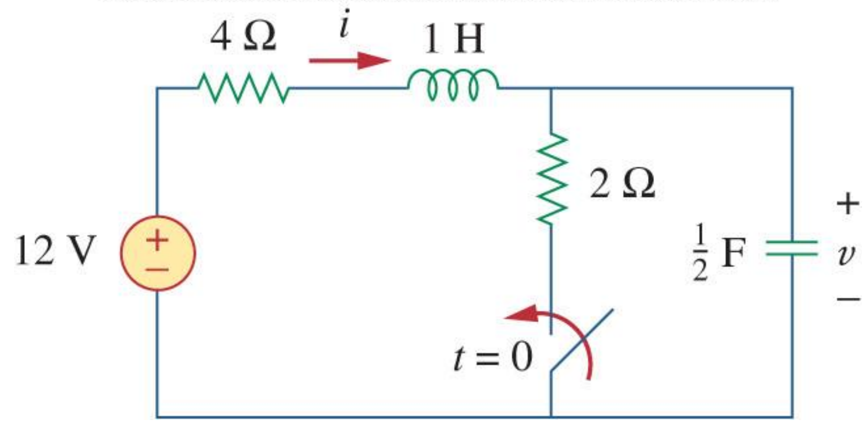
1. Initial conditions

$$v(0^+) = v(0^-) = 12V,\ i(0^+) = i(0^-) = 0$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-12/2}{0.5} = -12\ V/s$$



General RLC Circuits



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2. KCL at node *a*: $i = \frac{v}{2} + 0.5\frac{dv}{dt}$

KVL on left mesh: $4i + 1\frac{di}{dt} + v = 12$

$\Rightarrow$ $\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 24$ $\Rightarrow$ General Solution $v_t(t) = A_1e^{-2t} + A_2e^{-3t}$

3. Particular Solution : Steady-state response $v_{ss}(t) = 4V$

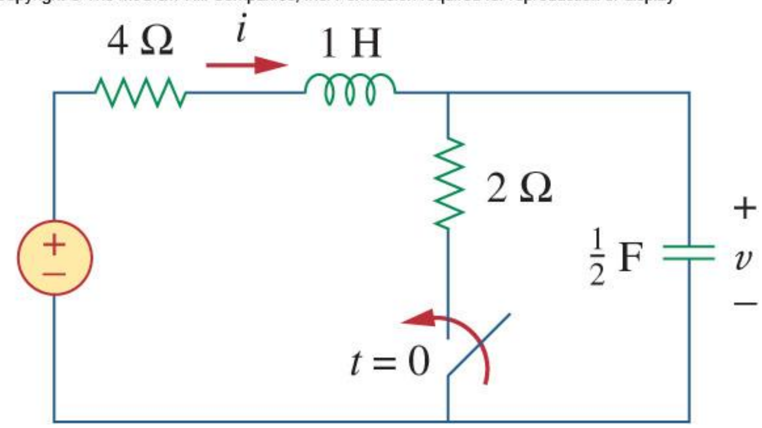
4. Put together : $v(t) = 4 + \mathrm{A}_1e^{-2t} - \mathrm{A}_2e^{-3t}$

5. Using initial conditions to determine A_1 , A_2

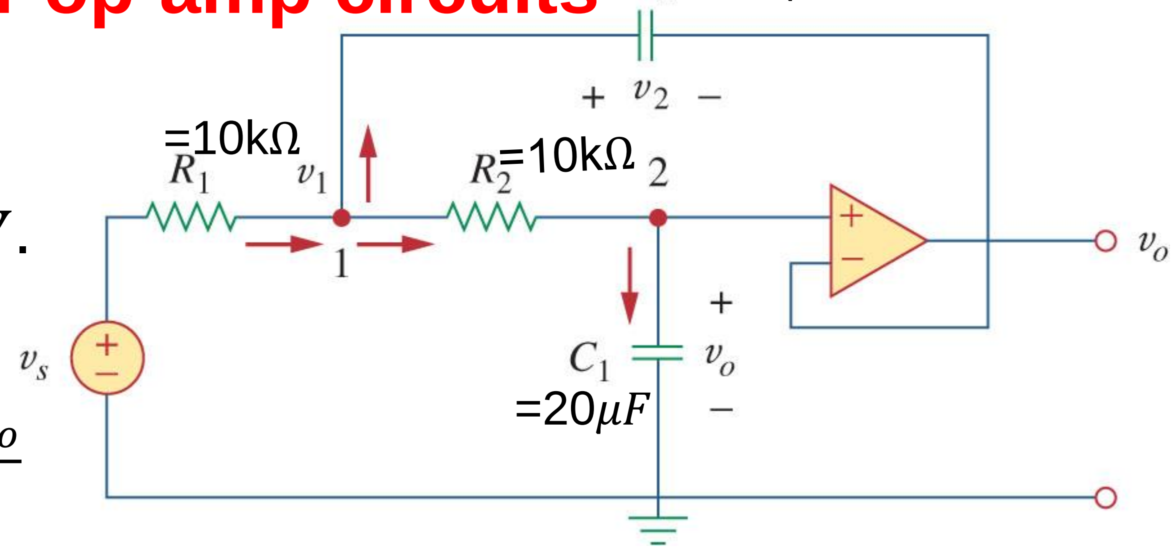
Self-test-General RLC Circuit

- Find the complete response v for $0 < t < 1$ in the circuit.

V=*12t*



Example of 2nd-order op-amp circuits



- Find v_o for $t > 0$ when $v_s = 10u(t)mV$.

KCL at node 1:

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2}$$

KCL at node 2:

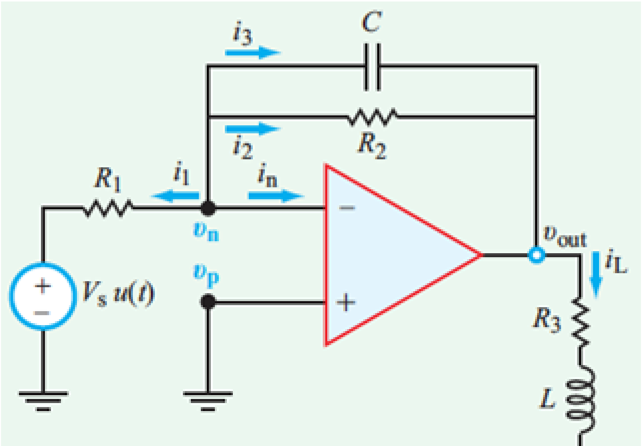
$$C_1\frac{dv_o}{dt} = \frac{v_1 - v_o}{R_2}$$

$and\ we\ have\ v_1 - v_2 = v_o$

$$\Rightarrow \frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_o}{dt} + \frac{v_o}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2}$$

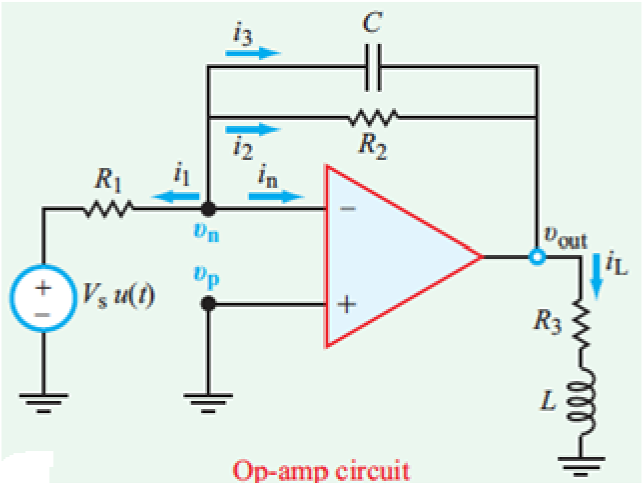
Initial conditions: $v_o(0^+) = 0,\ C_1\frac{dv_o(0^+)}{dt} = \frac{v_1(0^+)-v_o(0^+)}{R_2} = \frac{v_2(0^+)}{R_2} = 0$

Example---求Vsu(t) 状态下I_L



Op-amp circuit

Example



$$i_L(0) = i_L(0^-) = 0, \quad i_L'(0) = \frac{1}{L}\,v_L(0) = 0.$$

$$\frac{R_3}{R_2}\,i_L + \left(\frac{L}{R_2} + R_3C\right)\frac{di_L}{dt} + LC\,\frac{d^2 i_L}{dt^2} = -\frac{V_s}{R_1}$$