CS 243: Homework #3

Due on January 1, 2017

Dengji ZHAO

ShanghaiTech

Contents

Problem 1: Deferred Acceptance

(a)

Given the preference lists below, show the stable matchings given by the Deferred Acceptance Algorithm with men-proposing and women-proposing respectively.

$$
\begin{array}{cc}
Men\{u, v, w, x, y, z\} & Women\{a, b, c, d, e, f\} \\
u : a \succ_u b \succ_u d \succ_u c \succ_u f \succ_u e & a : z \succ_a x \succ_a y \succ_a u \succ_a v \succ_a w \\
v : a \succ_v b \succ_v c \succ_v f \succ_v e \succ_v d & b : y \succ_b z \succ_b w \succ_b x \succ_b v \succ_b u \\
w : c \succ_w b \succ_w d \succ_w a \succ_w f \succ_w e & c : v \succ_c x \succ_c w \succ_c y \succ_c u \succ_c z \\
x : c \succ_x a \succ_x d \succ_x b \succ_x e \succ_x f & d : w \succ_d y \succ_d u \succ_d x \succ_d z \succ_d v \\
y : c \succ_y d \succ_y a \succ_y b \succ_y f \succ_y e & e : u \succ_e v \succ_e x \succ_e w \succ_e y \succ_e z \\
z : d \succ_z e \succ_z f \succ_z c \succ_z b \succ_z a & f : u \succ_f w \succ_f x \succ_f v \succ_f z \succ_f y
\end{array}
$$

Solution

Men proposing:

$a : \cancel{u} \quad \cancel{v} \quad x \qquad b : \cancel{v} \quad w \quad \cancel{u} \qquad c : \cancel{w} \quad \cancel{x} \quad \cancel{y} \quad v \quad \cancel{u} \qquad d : \cancel{z} \quad y \quad \cancel{u} \qquad e : z \qquad f : u$

So the matching is $a - x \quad b - w \quad c - v \quad d - y \quad e - z \quad f - u$

Women proposing:

$u : \cancel{c} \quad f \qquad v : c \quad \cancel{e} \qquad w : d \qquad x : e \qquad y : b \qquad z : a$

So the matching is $u - f \quad v - c \quad w - d \quad x - e \quad y - b \quad z - a$

(b)

In a matching problem with n men and n women, each man/woman assigns $n - i$ points to the i-th person in his or her preference list. Let the weight of a pair to be the sum of the points assigned by the two person to each other. Construct an example showing that a maximum weighted matching is not a stable matching.

Solution

$$
\begin{array}{cc}
Men\{x, y, z\} & Women\{a, b, c\} \\
x : a \succ_x c \succ_x b & a : y \succ_a x \succ_a z \\
y : a \succ_y b \succ_y c & b : y \succ_b z \succ_b x \\
z : c \succ_z a \succ_z b & c : z \succ_c x \succ_c y
\end{array}
$$

maximum weight: $1 + 2 + 2 + 1 + 2 + 2 = 10$ ($a - x \quad b - y \quad c - z$).
stable: $2 + 2 + 0 + 0 + 2 + 2 = 8$ ($a - y \quad b - x \quad c - z$).
the answer is not exclusive

Problem 2: TTC

Consider an instance of TTC (top-trading cycles) problem: a candy allocation problem with four people $\{1,2,3,4\}$ and four candy items $\{A,B,C,D\}$.

Person	Preference	Initial Allocation
1	$C \succ_1 A \succ_1 D \succ_1 B$	A
2	$A \succ_2 C \succ_2 D \succ_2 B$	B
3	$D \succ_3 C \succ_3 A \succ_3 B$	C
4	$A \succ_4 D \succ_4 C \succ_4 B$	D

(a)

Execute TTC on the allocation problem. What is the final allocation?

Solution At first, the link status is $A \rightarrow C \rightarrow D \rightarrow A$, $B \rightarrow A$, so A, C, D will trade together. The final allocation is $1-C$, $2-B$, $3-D$, $4-A$.

(b)

Is there any blocking coalition in the allocation given by TTC, why?

No. According to the final allocation and preferences, A, C and D all get their most wanted items. Then, if there exists a blocking coalition, it must include at least one of A, C and D. But any one of A, C and D who gives out its current item will be definitely worse off.

Problem 3: School Choice

Consider a Public School Choice problem with five students $\{s_1, ..., s_5\}$ and four schools $\{c_1, ..., c_4\}$. Only c_2 has two slots, and every other school has only one slot. All schools have the same priority list: $\{s_1\} \succ \{s_2\} \succ \{s_3\} \succ \{s_4\} \succ \{s_5\}$, since all of the schools like students with higher scores. Students' preferences are given as follows:

- $s_1 : \{c_1\} \succ_{s_1} \{c_2\} \succ_{s_1} \{c_3\} \succ_{s_1} \{c_4\}$,
- $s_2 : \{c_1\} \succ_{s_2} \{c_2\} \succ_{s_2} \{c_3\} \succ_{s_2} \{c_4\}$,
- $s_3 : \{c_1\} \succ_{s_3} \{c_3\} \succ_{s_3} \{c_2\} \succ_{s_3} \{c_4\}$,
- $s_4 : \{c_2\} \succ_{s_4} \{c_1\} \succ_{s_4} \{c_3\} \succ_{s_4} \{c_4\}$,
- $s_5 : \{c_3\} \succ_{s_5} \{c_1\} \succ_{s_5} \{c_2\} \succ_{s_5} \{c_4\}$,

(a) Boston Mechanism

The Boston mechanism (used in Boston high schools until 2005) is defined as follows:

Boston Mechanism:

- In step one, each student proposes to her first choice school, and students are matched with a school in order of school priority while there remains capacity.

- In each subsequent step $k > 1$: each un-matched student proposes to her k-th most preferred school, and students are matched with a school in order of school priority while there remains capacity. The mechanism terminates when all students are matched.

a.1 What is the matching produced by Boston mechanism? Please show the steps and the results.

Solution

s1,s2,s3 will request c1
so c1 will choose s1 and s2,s3 are rejected
s4 will request c2
so c2 will choose s4
s5 will request c3
c3 choose s5
s2,s3 remain rejected
S2 will chose c2,
S3 will chose c3
C2 will take s2
c3 is already full so will reject s3
So s3 remain rejected
S3 will go for c2
Since c2 is already full with so will reject s3
So s3 will go for c4
C4 will accept s3 as it is vacant
So matching are as follow
C1 s1
C2 s2,s4
C3 s5
C4 s3

a.2 Is there an agent who has an incentive to misreport her preferences (assuming other agents truth-telling)?

Solution
c3 can decline s5 to get s3
when s3 will apply for c3
yes

(b) Shanghai Mechanism

The Shanghai mechanism was first implemented as a high school admissions mechanism in Shanghai. In 2008, variants of the mechanism were implemented in nine provinces as the parallel college admissions mechanisms to replace the sequential mechanisms, which corresponds to the Boston mechanism with categories. We now describe a stylized version of the parallel mechanism, adapted for the school choice context.

1. Each student applies to his/her first ranked school.
2. If a school receives more applications than its capacity, then it retains the students with the highest priority up to its capacity and rejects the remaining students (throughout the allocation process, a school can hold no more applications than its capacity).

3. Whenever a student is rejected by a school, his/her application is sent to his/her next highest ranked school.

4. Whenever a school receives new applications, these applications are considered together with the retained applications for that school. Among the retained and new applications, the ones with the highest priority up to its capacity are retained.

5. The allocation is finalized at every e steps. That is, in steps e, $2e$ and $3e$ etc. (in this case, let e=2), each student is assigned a school that holds his or her application in that step. These students and their assignments are removed from the system.

6. The allocation process terminates when no more applications can be rejected.

What is the matching produced by Shanghai Mechanism? Please show the steps and the results. Explain if there is any student who has an incentive to misreport her preference (assuming other students are truth-telling) in the above setting. Further show whether it is always the case, if not, give an example.

Solution
round 1
step 1 :c1(s2,s3, choose s1), c2(choose s4), c3(choose s5)
step 2:c1(s2,s3, choose s1), c2(choose s2,s4), c3(s5 choose s3)
round 2 c4(s5)
match: c1(s1), c2(s2s4),c3(s3),c4(s5)
In this case, they can not misreport.
No. If initially s5$c_1 > c_2 > c_3 > c_4$, s5$c3 > c_2 > c_1 > c_4$ is better for s5.