Randomized algorithms 4
Linear programming

CS240 Spring 2021

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Linear programming

- A farmer has the following problem.
  - He has 100 acres of land, on which he can plant wheat or barley or both.
  - He has 420 kg of fertilizer, and 160 kg of pesticide.
  - Each acre of barley requires 5 kg of fertilizer and 1 kg of pesticide.
  - Each acre of wheat requires 3 kg of fertilizer and 2 kg of pesticide.
  - Wheat sells for $3 per acre, barley sells for $4 per acre.
    - Actually, he could probably make $300 for wheat and $400 for barley. Choose $3 and $4 for simplicity.
- How many acres of wheat and barley should the farmer plant his field to maximize his income?
- Let w, b be acres of wheat and barley farmer plants.
- He wants to maximize 3w+4b, subject to the land, pesticide and fertilizer constraints.

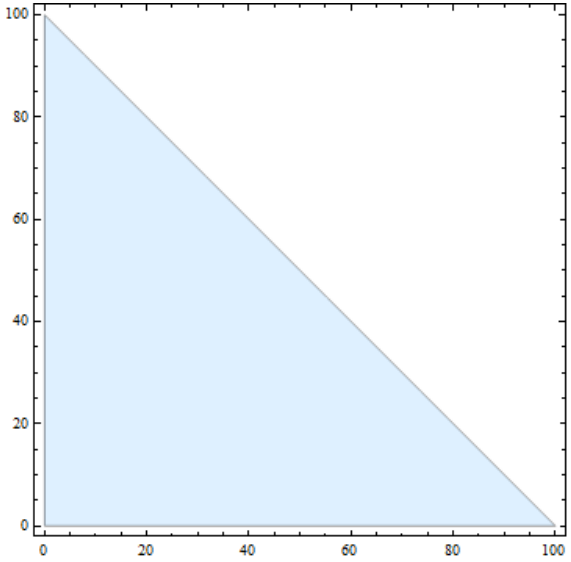
Linear programming

- **maximize** 3w+4b **subject to**

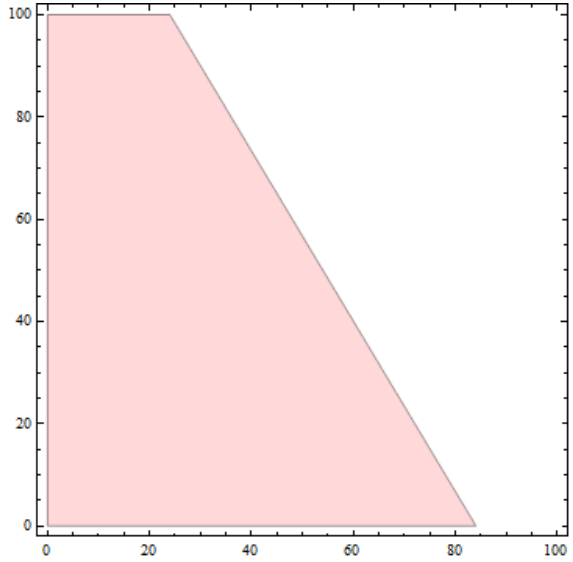
  $w+b \leq 100$ (land)

  $3w+5b \leq 420$ (fertilizer)

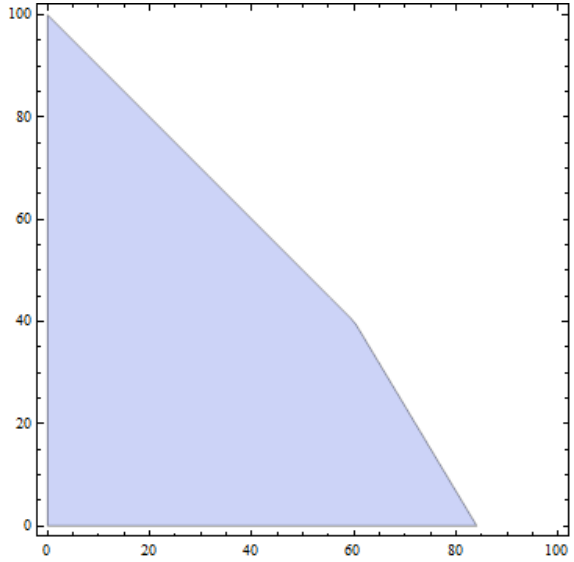
  $2w+b \leq 160$ (pesticide)



land constraint



fertilizer constraint



land + fertilizer constraints

Linear programming

- **maximize** 3w+4b **subject to**

  $w+b \leq 100$ (land)

  $3w+5b \leq 420$ (fertilizer)

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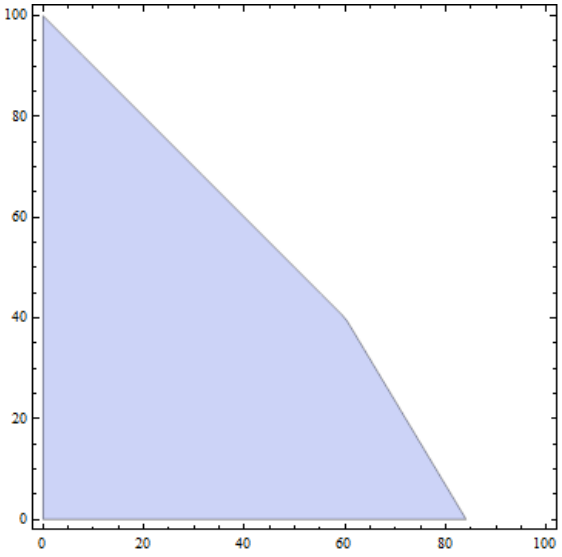
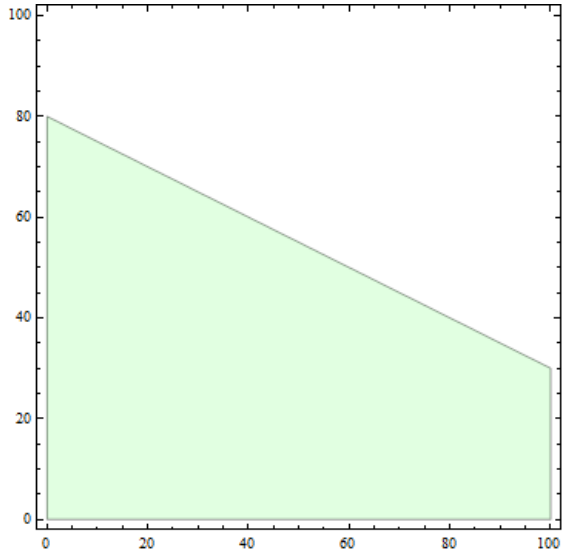
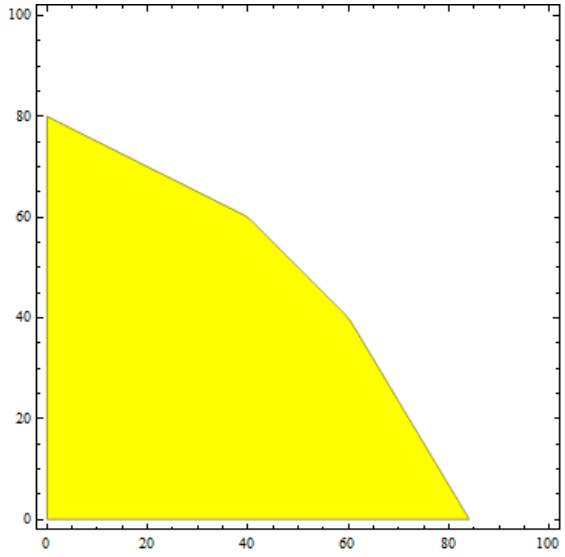


land + fertilizer constraints



pesticide constraints



all three constraints

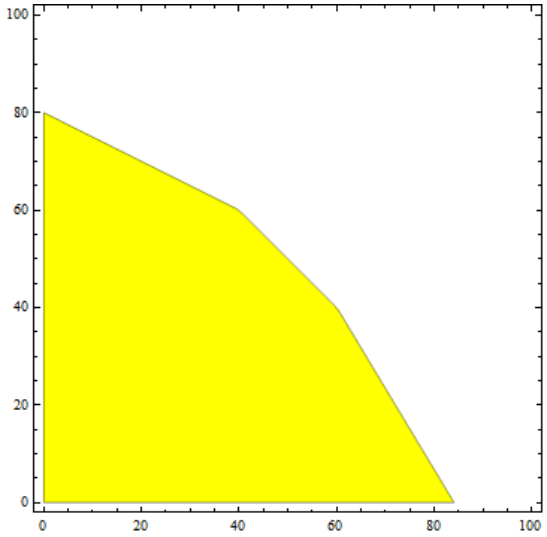
Linear programming

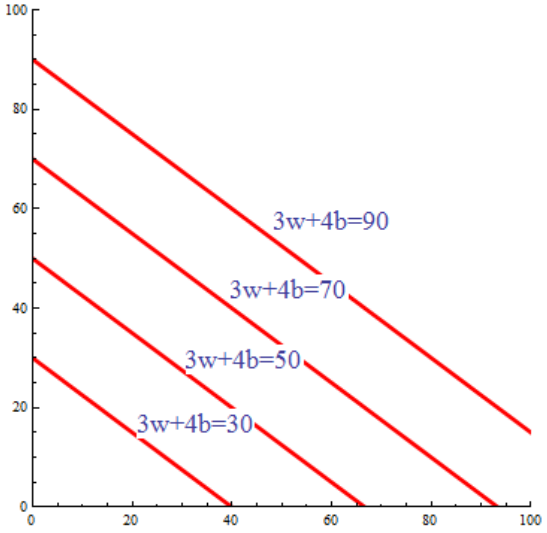
- **maximize** 3w+4b **subject to**

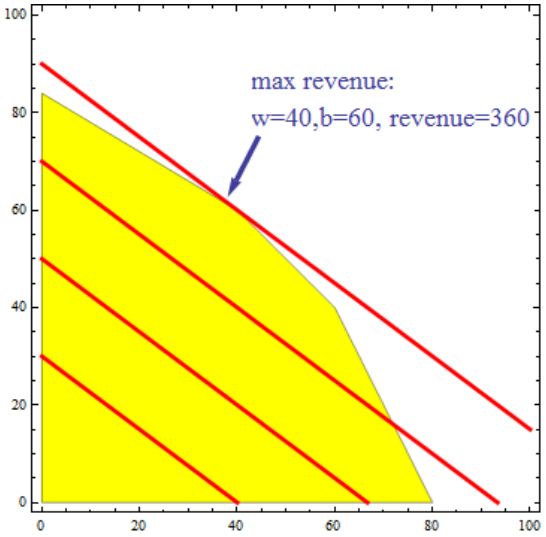
  w+b ≤ 100 (land)

  3w+5b ≤ 420 (fertilizer)

  2w+b ≤ 160 (pesticide)

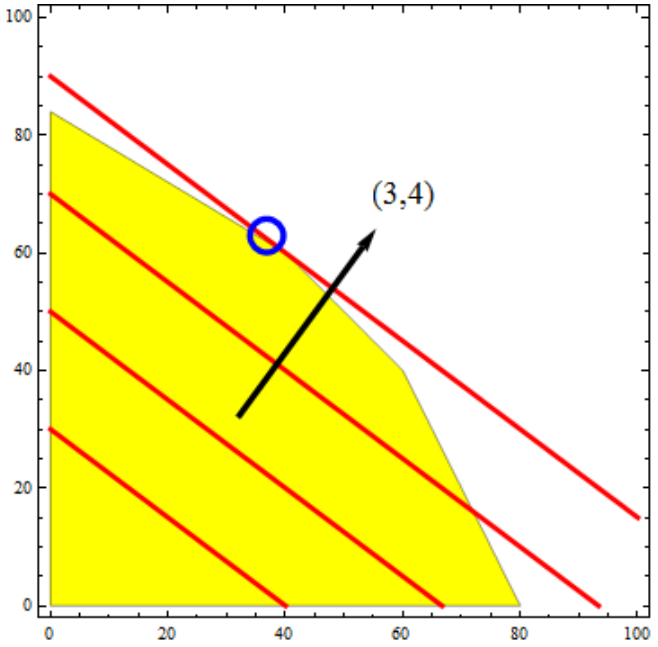






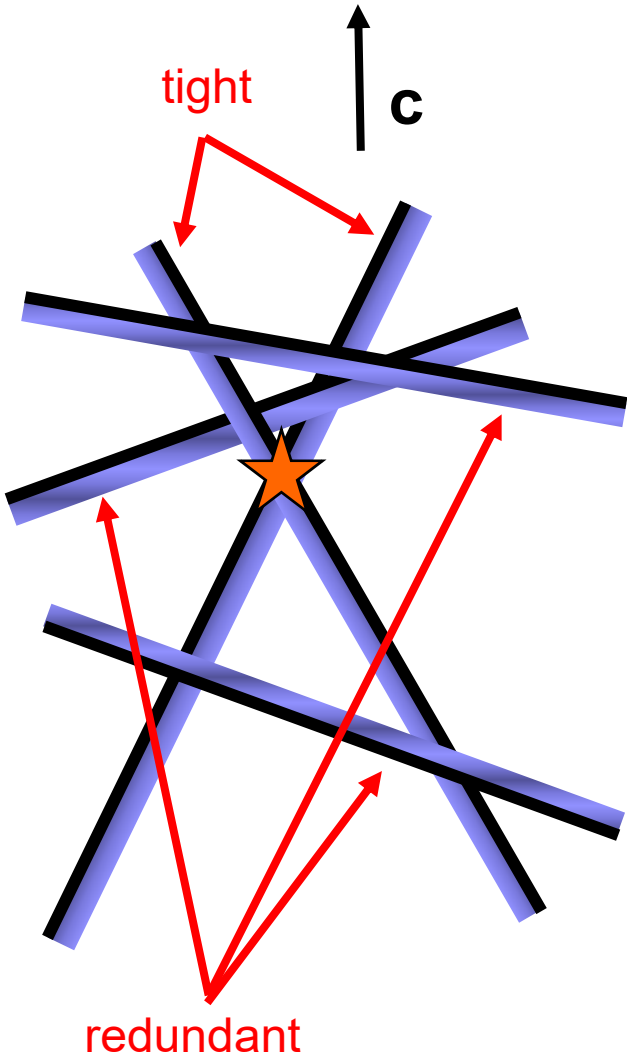
Linear programming

- The feasible region is the area corresponding in which all the constraints are satisfied.
- Key Fact The optimum lies at an extreme point (corner).
- Find optimum by taking a line perpendicular to the direction pointed by the objective function, and shifting the line till when it will stop touching the feasible region.
- The optimum lies at the intersection of two constraints.
  - Call these the basis of the optimum.
  - For simplicity, assume constraints are general position, i.e. no 3 intersect at a point.



Randomized LP in 2D

- Since the optimum is defined by two constraints, the other constraints are redundant!
- A constraint is tight if it the optimum lies on its defining line.
- Let H be set of n constraints. If pick random constraint, there's only 2/n probability it's tight.
- If constraint's not tight, we can discard it without changing optimum.
- How do we tell if it's tight?
  - For any constraint set G, let B(G) denote optimum.
- $h \in H$ is redundant iff $B(H)=B(H-\{h\})$.
  - I.e. the optimum is the same with or without h. So opt doesn't lie on h.

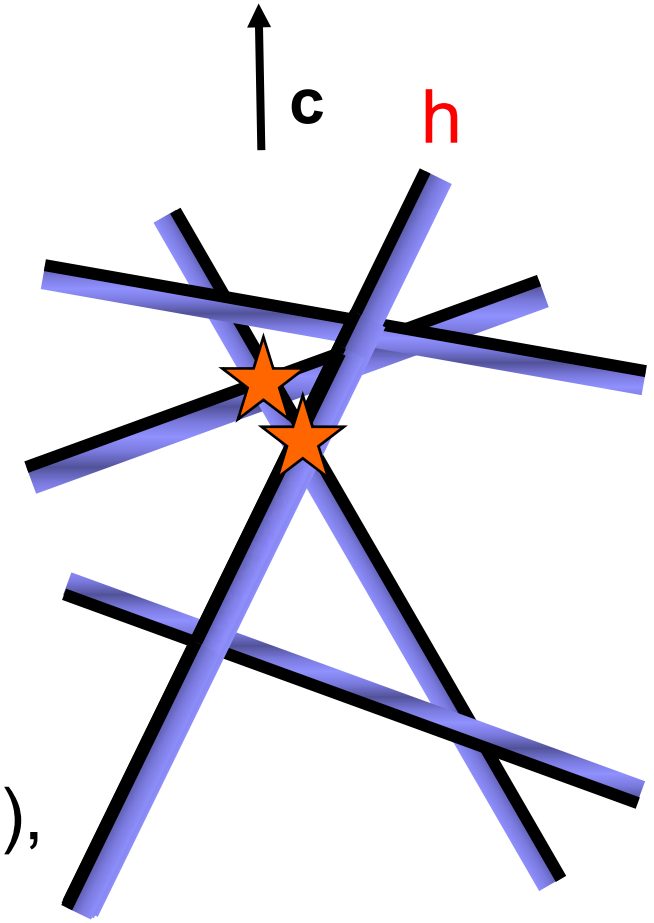


2D LP algorithm

- If |H|=2, output intersection of the 2 halfplanes.
- Pick random constraint $h \in H$.
- Recursively find opt=B(H-{h}).
- If opt doesn't violate h, output opt.
  - opt violates h if opt lies outside h.
- Else project H-{h} onto h's boundary to obtain a 1D LP.
- Output the opt of the 1D LP.

Projection

- Given constraint h, let ∂(h) be its boundary, i.e. the line defining h.
- Suppose B(H-{h}) violates h.
  - Then B(H) must lie on the boundary of h.
- Project a halfplane onto ∂(h), reducing it to a line segment bounded on one or two sides.
- After projecting all H-{h} onto ∂(h), we're left with a segment representing feasible region to 1D LP.
- Optimizing this is easy. The opt is one of the endpoints.

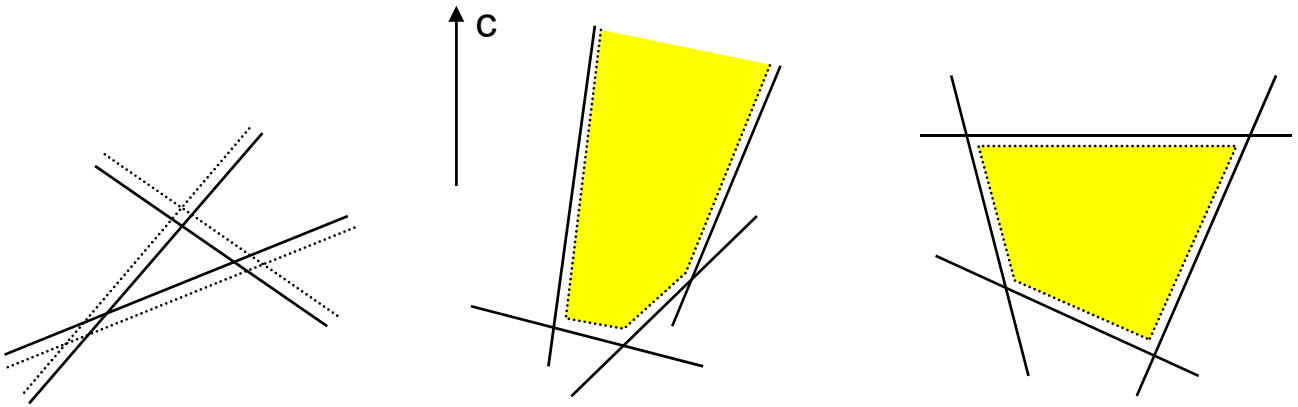


Analysis

- Let T(n) be expected time to solve 2D LP with n constraints.
- $T(n) \leq T(n-1)+O(1)+2/n(O(n)+O(1))$.
- T(n-1) time recursively find opt=B(H-{h}).
- First O(1) is time to check whether opt violates h.
- There's 2/n probability opt violates h, in which case we project all constraints onto ∂(h).
  - O(n) to project H-{h} onto ∂(h).
  - Final O(1) to solve 1D LP.
- T(n) solves to O(n).
  - So we can solve 2D LP with n constraints in expected linear time.

Corner cases

- For simplicity, we ignored several corner cases.
  - Infeasible means no points satisfy all the constraints.
    - Ex Constraints $x_1>1$ and $x_1<0$.
  - Unbounded means the optimum is infinite.
    - Ex Maximize x_2 s.t. $x_1+x_2>0$.
  - Non-unique optimum means an infinite number of points maximize the objective.
    - Ex Maximize x_2 s.t. $x_2 \leq 0$.
- Preprocess input to check for corner cases.

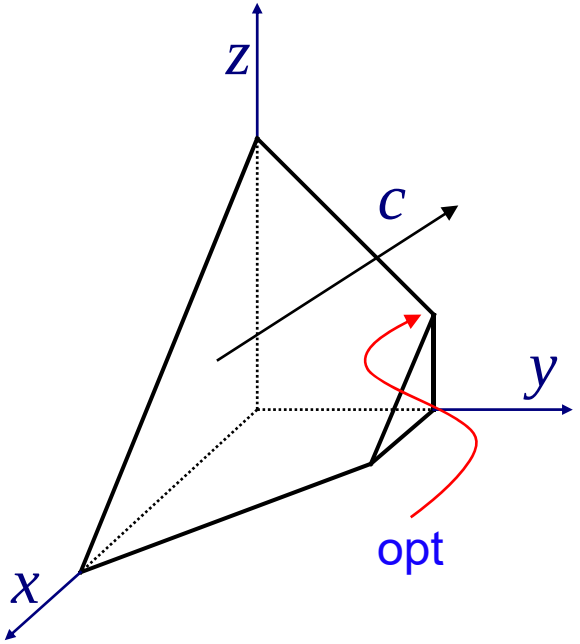


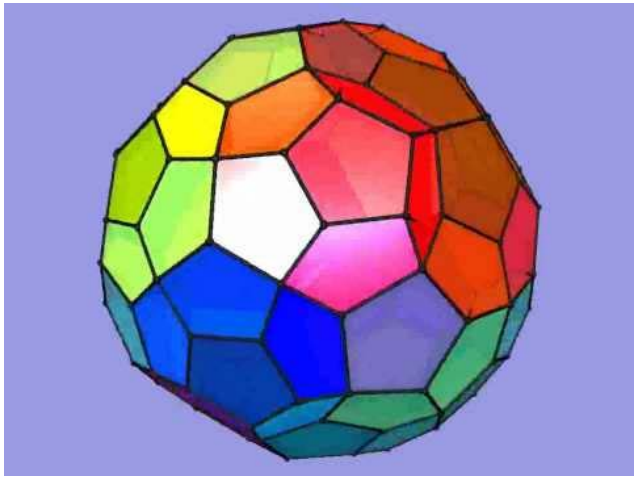
Infeasible Unbounded Non-unique optimum

http://www.cse.yorku.ca/~andy/courses/6114/Slides/CG5-LP.pptx

Higher dimensions

- In d>2 dimensions, lines become planes and each constraint corresponds to the space to one side of a plane, called a halfspace.
- The intersection of the halfspaces defines the feasible region.
  - This is a convex region called a polytope.
- Each extreme point (corner) of the polytope is the intersection of d halfspaces.
- The objective function defines a direction. Take a plane perpendicular to this direction and shift it till it stops touching feasible region.
- Hence optimum again lies at intersection of d halfspaces.
- Polytopes can be very complicated.





d-Dimensional LP algorithm

- If |H|=d, output their intersection.
- Pick random constraint $h \in H$.
- Recursively find opt=B(H-{h}).
- If opt doesn't violate h, output opt.
- Else project H-{h} onto h's boundary to obtain a d-1 dimensional LP.
- Recursively solve the d-1 dim LP.

Analysis

- Let T(n,d) be expected time to solve d-dim LP with n constraints.
- $T(n,d) \leq T(n-1,d)+O(d)+d/n(O(dn)+T(n-1,d-1))$.
- T(n-1,d) time recursively find opt=B(H-{h}).
- O(d) time to check whether opt violates h.
- There's d/n probability opt violates h.
  - Because opt is defined by d of the n halfspaces.
  - In this case we project all constraints onto ∂(h).
  - O(dn) to project H-{h} onto ∂(h).
  - We obtain a d-1 dim LP with n-1 constraints.
  - T(n-1,d-1) time to solve this.
- T(n,d) solves to O(d! n)
  - Linear in number of constraints.
  - Exponential in dimensions.

Matrix formulation

- **maximize** 3w+4b **subject to**

  $w+b \leq 100$

  $3w+5b \leq 420$

  $2w+b \leq 160$

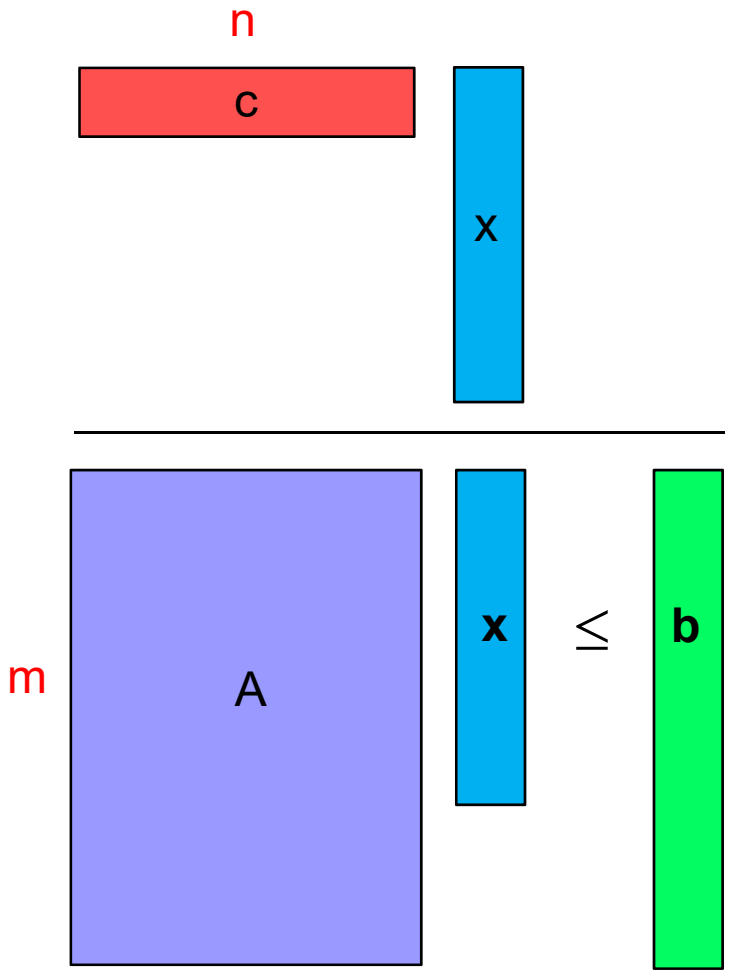
- $\text{maximize} \begin{bmatrix} 3,4 \end{bmatrix} \cdot \begin{bmatrix} w \\ b \end{bmatrix} \text{s.t.}$

  $$\begin{bmatrix} 1 & 1 \\ 3 & 5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} w \\ b \end{bmatrix} \leq \begin{bmatrix} 100 \\ 420 \\ 160 \end{bmatrix}$$

- Let $x \in \Re^{n \times 1}, c \in \Re^{1 \times n}, A \in \Re^{m \times n}, b \in \Re^{m \times 1}$.

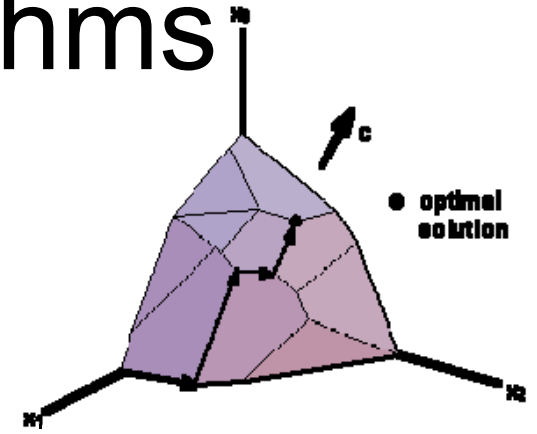
  $\text{maximize}\, c \cdot x \,\text{s.t.}$

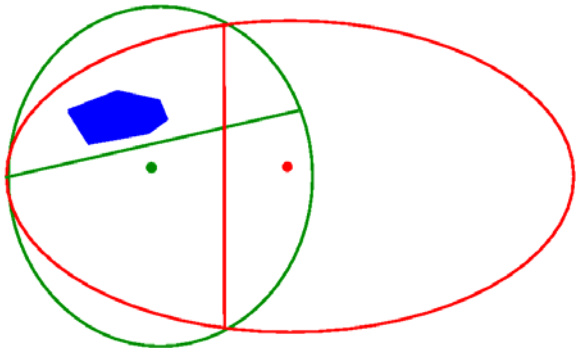
  $A \cdot x \leq b$

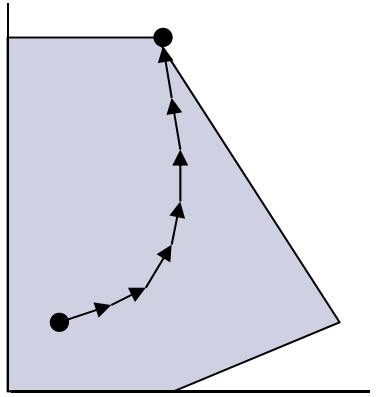


Deterministic LP algorithms

- Simplex method (Dantzig 1947)
  - Optimum lies at the farthest corner in direction of c.
  - Walk along boundary of polytope in directions that improve c.
  - Number of steps can be exponential in worst case.
  - But works fast in practice.
- Ellipsoid method (Khachiyan 1979)
  - First polynomial solution.
  - Interesting mostly in theory.
- Interior point method (Karmarker 1984)
  - Practical polynomial method.

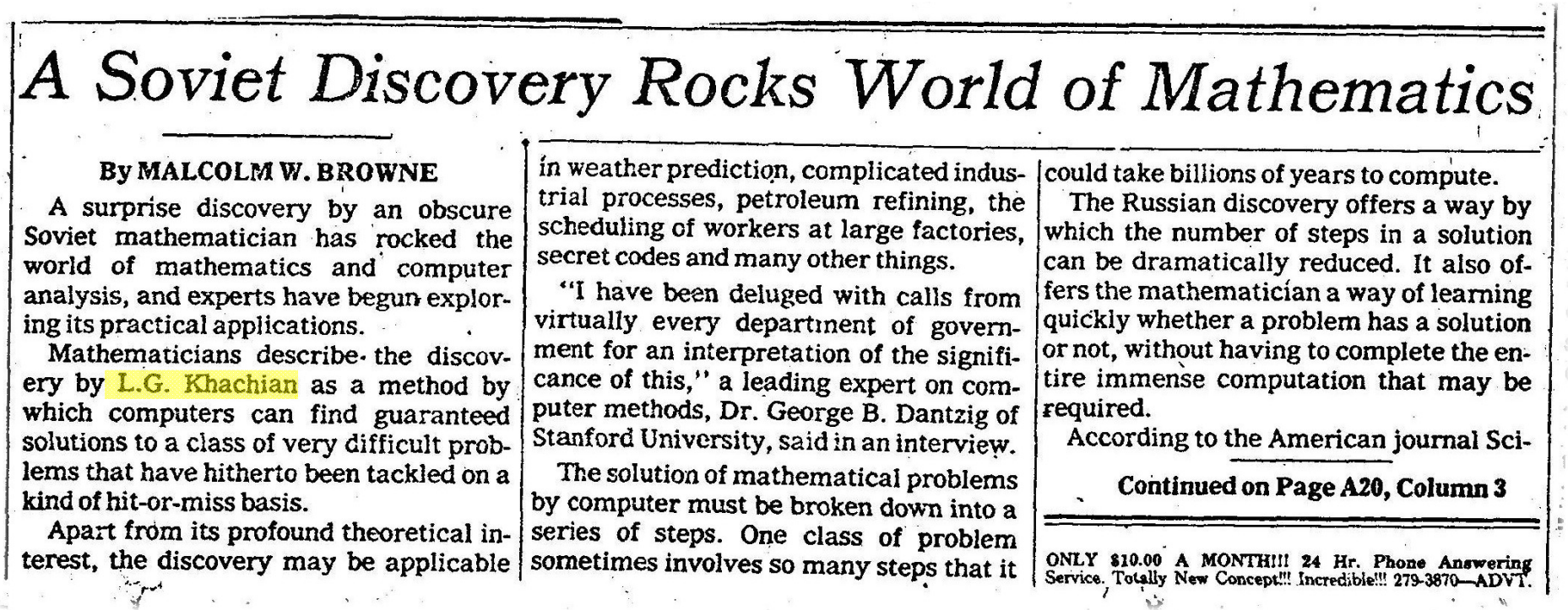






Applications of LP

- Linear programming is one of the most important and widely used algorithms in the world.
- NY Times, Nov. 27, 1979



A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

Mathematicians describe the discovery by L.G. Khachian as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit-or-miss basis.

Apart from its profound theoretical interest, the discovery may be applicable in weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers at large factories, secret codes and many other things.

"I have been deluged with calls from virtually every department of government for an interpretation of the significance of this," a leading expert on computer methods, Dr. George B. Dantzig of Stanford University, said in an interview.

The solution of mathematical problems by computer must be broken down into a series of steps. One class of problem sometimes involves so many steps that it could take billions of years to compute.

The Russian discovery offers a way by which the number of steps in a solution can be dramatically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire immense computation that may be required.

According to the American journal Sci-

Continued on Page A20, Column 3

Applications of LP: Network flow

$$\max \sum_{v \in V} f(s, v)$$

total flow

$$f(u, v) \leq c(u, v) \quad \forall u, v \in V$$

capacity constraints

$$f(u, v) = -f(v, u) \quad \forall u, v \in V$$

flow symmetry

$$\sum_{v \in V} f(u, v) = 0 \quad \forall u \in V - \{s, t\}$$

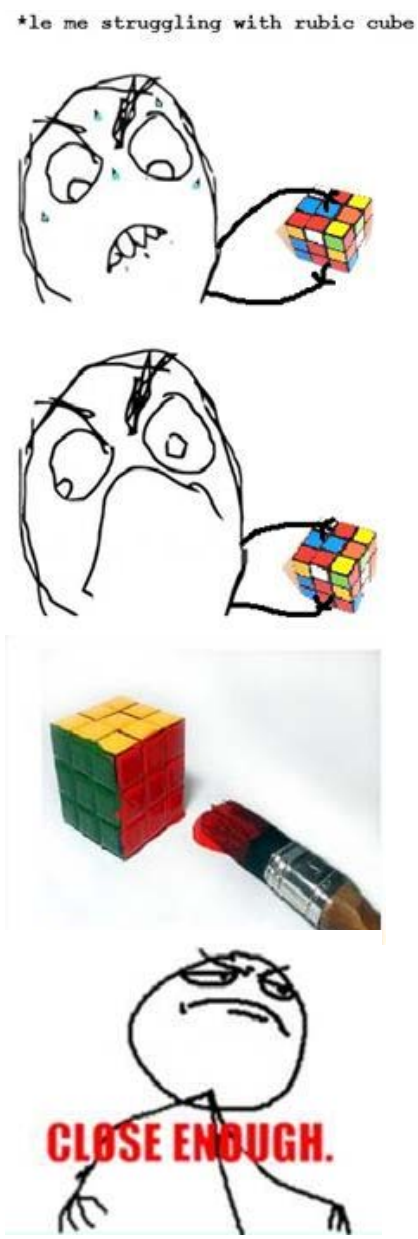
conservation of flow at u

- However, it's more efficient to solve max flow using e.g. Dinic's algorithm than more general LP algorithms.

Approximation algorithms 1
Set cover

Approximation algorithms

- Up to now, most of our algorithms have been exact. I.e. they find an optimal solution.
- But there are many problems for which we don't know how to find an optimal solution.
  - A key example is NP-complete problems. We don't know efficient algorithms for any NPC problem.
- Many such problems are important in practice. What do we do?
- If we can't get find the best answer, let's try for good enough.
- Approximation algorithms find an approximately optimal answer.

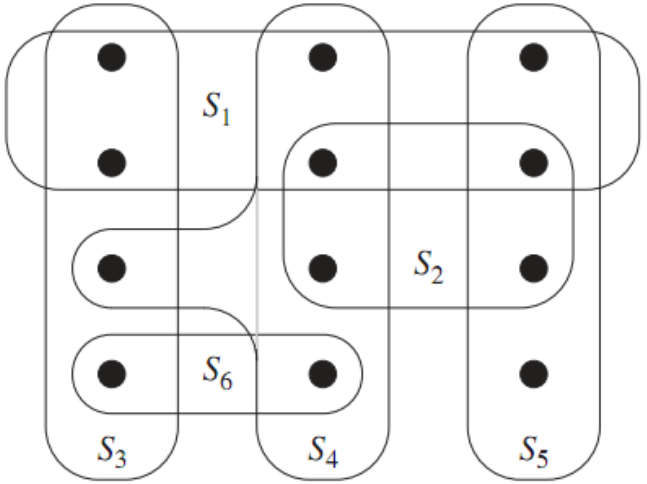


Approximation ratio

- Let X be a maximization problem. Let A be an algorithm for X.
- Let α>1 be a constant.
- A is an α-approximation algorithm for X if A always returns an answer that's at least $1/\alpha$ times the optimal.
  - Ex If X is max-flow, A is a 2-approx algorithm if it always returns a flow that's at least ½ the optimal.
  - The closer α is to 1, the better the approximation.
- If X is a minimization problem, A is an α-approximation algorithm for X if it always returns an answer that's at most α times larger than the optimal.
  - Ex If X is min-cut, A is a 2-approx algorithm if it always returns a cut that's at most 2 times the size of the optimal.
  - Again, the closer α is to 1, the better the approximation.

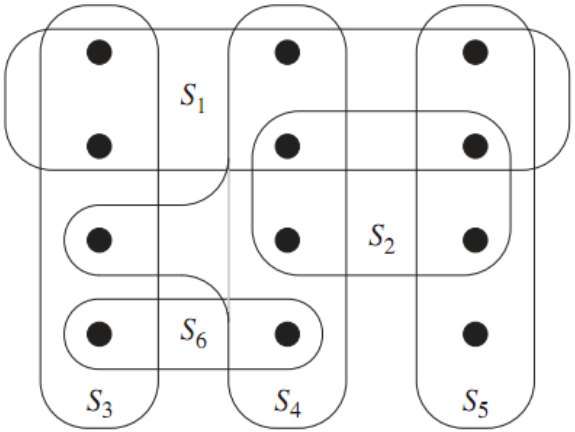
Coverings

- Suppose there's a set of teachers, and each can teach a certain set of classes.
  - Let S_i be the set of classes teach i can teach.
- The entire set of classes is X.
- We want to pick the minimum set of teachers to teach all the classes.
  - Let T be set of teachers we pick.
  - We want $\bigcup_{i \in T} S_i = X$, and T to be the smallest possible.



Set covering

- Input A collection F of sets. Each set has a cost. The union of all the sets is X.
- Output A subset G of F, whose union is X.
- Goal Minimize the total cost of the sets in G.



If all sets have same cost, S_3, S_4 and S_5 is a min cost set cover of X.

- Minimum cost set cover is NP-complete.
- We'll see a ln(n)-approximation algorithm, where n=|X|.

A greedy approximation alg

- A natural greedy heuristic is to choose sets which cover points most cheaply.
  - For each set, let c be its cost, and m be the number of points it covers.
  - We want to use the set with the smallest c/m value, because this is the cheapest way to cover some new points.
- After we pick this set, remove all the points it covers. Then we consider the per unit cost of the remaining sets and again choose the cheapest.

A greedy approximation alg

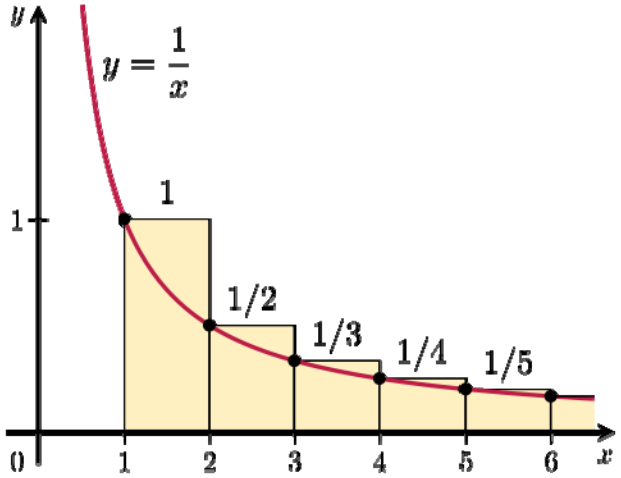
- F is the entire collection of sets. The union of these sets is X.
- Each set S in F has a cost cost(S).
- U is the set of elements of X we haven't covered yet.
- C is the set cover we eventually output.

- U = X
- C = ∅
- while U≠∅
  - choose S∈F-C with min |cost(S)|/|S∩U|
  - C = C∪{S}
  - U = U – S
- output C

Per unit cost to cover new elements.

Proof of correctness

- We always output a set cover, because the while loop continues till X is covered.
- We'll prove the approximation ratio is at most 1+1/2+1/3+...+1/n ≈ ln(n).
  - If the min cost of a set cover is V, our set cover costs at most ln(n)*V.
- The basic plan is to bound the cost of the set cover the algorithm outputs using the "average cost" per element.



Proof of correctness

- Order the sets in C by when they're added to C, earliest set first.
  - Let the order be $S_1, S_2,\ldots,S_m$.
- Cost of the set cover is $L=\sum_i \text{cost}(S_i)$.
- Order the elements in X by when they're added, earliest element first.
  - Let the order be $e_1, e_2,\ldots,e_n$.
  - So, the first few e's are added by S_1, the next few added by S_2, etc.
  - Every element in X is in the list, because C covers X.

Proof of correctness

- Let n_i be the number of new elements S_i covers.
  - So, n_i is the number of elements in S_i, but not in $S_1,...,S_{i-1}$.
- Divide the cost of S_i evenly among the new elements it covers.
  - If e is newly covered by S_i, then cost(e)= cost(S_i)/n_i.
- Σ_k cost(e_k) = Σ_i n_i *cost(S_i)/n_i = Σ_i cost(S_i) = L.
  - Every element is covered by some S_i, and S_i covers n_i new elements.
- We'll prove cost(e_k) $\leq$ OPT/(n-k+1), for any k.
- Suppose this is true, then

$$L = \Sigma_k \text{cost}(e_k) \leq \Sigma_k \text{OPT}/(n-k+1) \approx \ln(n)*\text{OPT}$$

The per element cost

- Let's focus on some element e_k, and let S_j be the set which covers e_k for the first time.
- Let $C_1,\ldots,C_m$ be the sets in an optimal cover, each of which covers some elements of $U=\{e_k,e_{k+1},e_{k+2},\ldots,e_n\}$.
  - Let $n'_1,\ldots,n'_m$ be the number of elements of U which $C_1,\ldots,C_m$ cover.
- Obs 1 $\sum_i n'_i \geq n-k+1$.
  - Because $C_1,\ldots,C_m$ cover U.
- Obs 2 $\sum_i \text{cost}(C_i) \leq \text{OPT}$.
  - Because $C_1,\ldots,C_m$ are a subset of an optimal cover, which has cost OPT.

The per element cost

- Obs 3 None of $C_1,...,C_m$ are among $S_1,...,S_{j-1}$.
  - If some C_i is among $S_1,...,S_{j-1}$, then since C_i covers some e in U, e would be covered by $\{S_1,...,S_{j-1}\}$. So, e would be among the first k-1 elements covered. Contradiction.
- Obs 4 There exists some C_i among $C_1,...,C_m$ with $cost(C_i)/n'_i \leq OPT/(n-k+1)$.
  - If every C_i in $C_1,...,C_m$ has $cost(C_i)/n'_i > OPT/(n-k+1)$, then

$OPT \geq \Sigma_i\, cost(C_i) = \Sigma_i\, n'_i * cost(C_i)/n'_i >$

$\Sigma_i\, n'_i * OPT/(n-k+1) \geq OPT/(n-k+1)\ \Sigma_i\, n'_i \geq$

$OPT/(n-k+1)*(n-k+1) = OPT.$

Contradiction.

Proof of approximation ratio

- Lemma $\text{cost}(S_j)/n_j \le \text{OPT}/(n-k+1)$.
- Proof When choosing S_j, the only sets the algorithm is not allowed to choose are $S_1,\ldots,S_{j-1}$.
  - By obs 3, $C_1,\ldots,C_m$ aren't in $S_1,\ldots,S_{j-1}$.
  - By obs 4, there's some C_i in $C_1,\ldots,C_m$, with $\text{cost}(C_i)/n'_i \le \text{OPT}/(n-k+1)$.
  - S_j was chosen so that $\text{cost}(S_j)/n_j$ is min among all sets not in $S_1,\ldots,S_{j-1}$.
  - So $\text{cost}(S_j)/n_j \le \text{cost}(C_i)/n'_i \le \text{OPT}/(n-k+1)$.
- Since $\text{cost}(S_j)/n_j = \text{cost}(e_k)$, we have $\text{cost}(e_k) \le \text{OPT}/(n-k+1)$.
- The approx ratio follows because

$$L = \sum_k \text{cost}(e_k) = \sum_k \text{OPT}/(n-k+1) \approx \ln(n)*\text{OPT}$$