EE150 Signals and Systems – Part 2: Linear Time-Invariant (LTI) System

Outline of LTI System

- Linearity & Time Invariance
- Convolution Sum & Convolution Integral
- Representation of Signals in terms of Impulses
- Convolution operator

Linear Systems

A system is linear if the following condition holds for any two inputs $x_1(t)$ and $x_2(t)$:

If

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t), \qquad x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t),$$

then

$$ax_1(t) + bx_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow ay_1(t) + by_2(t)$$

Recall: Linear Algebra

A is a matrix:

If $Ax_1 = y_1$, $Ax_2 = y_2$, then $A(ax_1 + bx_2) = ay_1 + by_2$

Thus matrix multiplication is linear

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Time (shift) Invariance

A system is time-invariant if the following holds for $x(t)$:

If

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$$

then

$$x(t-t_0) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-t_0)$$

Time (shift) Invariance

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1. A shift in the input produces the same shift in the output
2. The system has no internal way to keep time

Time invariance of a matrix

Think of an $\infty \times \infty$ matrix, $A = [a(i,j)]$, $-\infty < i,j < \infty$

- Consider any input $x = [x(j)]$, $-\infty < j < \infty$
- Then output $y = Ax = [y(i)]$, where

$$y(i) = \sum_{j=-\infty}^{\infty} a(i,j)x(j)$$

- When is this system shift invariant?

Time invariance of a matrix cont.

- A shift of input by k: $x_1 = [x_1(j)]$ where $x_1(j) = x(j-k)$, the output must be $y_1 = [y_1(i)]$ such that $y_1(i) = y(i-k)$.
- Hence for all x,

$$y(i-k) = \sum_{j=-\infty}^{\infty} a(i,j)x(j-k)$$

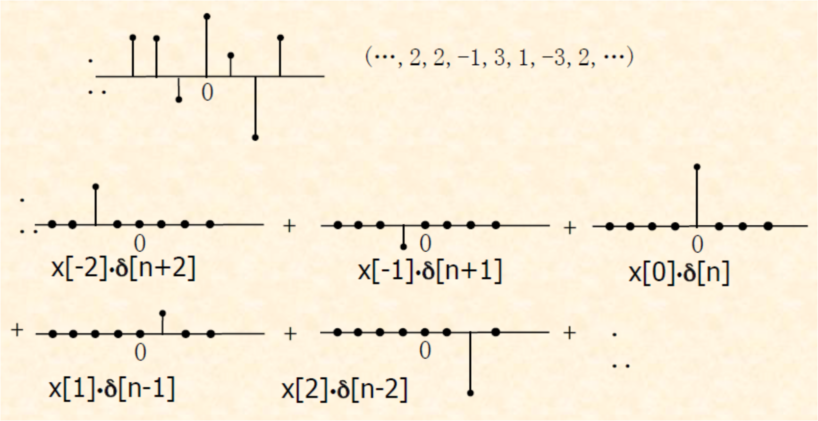
- By changing of variables ($r = i-k$, $s = j-k$),

$$y(r) = \sum_{s=-\infty}^{\infty} a(r+k, s+k)x(s)$$

Time invariance of a matrix cont.

- We must have
$$a(i,j) = a(i+k, j+k)$$
One may also try $x = e$
- Equivalently
$$a(i,j) = a(i-j, 0)$$
The matrix has only one distinct column
All other columns are shifts of this column
-
$$y(i) = \sum_{j=-\infty}^{\infty} a(i-j, 0)x(j)$$

Representation of Signal in terms of Impulses



Representation of Signal in terms of Impulses cont.

$$\delta[n-k]x[k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-k]x[k]$$

Goal: compute output of LTI system

Use linearity and time invariance as before

- Let output of $\delta[n]$ be $h[n]$,
  therefore output of $\delta[n-k]$ is $h[n-k]$ (time-invariance)
- We know that
  $x[n] = \sum x[k]\delta[n-k]$ (linear comb. of delta sequences)
  therefore $y[n] = \sum x[k]h[n-k]$ (linearity)

This operation is called *convolution*
Observe the similarity to matrices!

Impulse response

- $\delta[n]$ or $\delta(t)$ is called the impulse function
- When input is $\delta[n]$ (or $\delta(t)$), the output of a system, $h[n]$, is called impulse response
- Impulse response characterizes an LTI system.

  As we have seen

$$y[n] = \sum_k x[k]h[n-k]$$

Continuous time LTI systems

- As seen before

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

- Let $h(t)$ be the output when input is $\delta(t)$
- Then

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

The convolution operation

- Convolution of two signals $x(t)$ and $h(t)$, denoted by $x(t) * h(t)$, is defined by

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- For discrete-time

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Properties of convolution

1. Commutative: $x(t) * h(t) = h(t) * x(t)$
2. Bi-linear: $(ax_1(t) + bx_2(t)) * h(t) = a(x_1 * h) + b(x_2 * h)$, $x * (ah_1 + bh_2) = a(x * h_1) + b(x * h_2)$
3. Shift: $x(t - \tau) * h(t) = x(t) * h(t - \tau)$
4. Identity: $\delta(t)$ is the identity signal,
$$x * \delta = x = \delta * x$$
Identity is unique: $i(t) = i(t) * \delta(t) = \delta(t)$
5. Associative: $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$ (proof?)

Convolving $\delta(t)$ with itself

$\delta(t) * \delta(t)$

- Answer 1: It is $\delta(t)$, since $\delta(t)$ is identity
- But: the proof of identity works only for smooth $x(t)$

- Better answer: It is $\delta(t)$ since
  Let $y(t) = \delta(t) * \delta(t)$, then check that
  $y(0) = 0,\ t \neq 0,$
  $\int_{-\infty}^{\infty} y(t) dt = 1$

Convolving $\delta(t)$ with itself

- Third reason, if $x(t)$ is smooth

  $$x(t) * (\delta(t) * \delta(t)) = (x(t) * \delta(t)) * \delta(t) = x(t) * \delta(t) = x(t)$$

  Therefore $\delta(t) * \delta(t)$ is also an identity function.
  Since identity is unique $\delta(t) * \delta(t) = \delta(t)$

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Derivatives

- If $y(t) = x(t) * h(t)$ then note that (use linearity and time invariance)

$$\frac{y(t+\epsilon) - y(t)}{\epsilon} = \frac{x(t+\epsilon) - x(t)}{\epsilon} * h(t)$$

Hence

$$y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

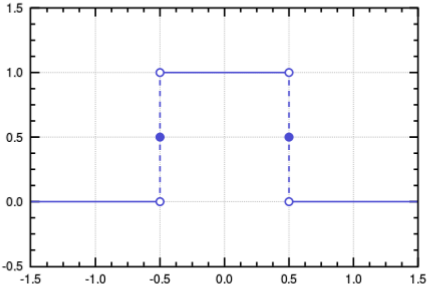
Convolution is smoothing

- If $x(t)$ has k derivatives and $h(t)$ has r derivatives, then $y(t)$ has at least $k + r$ derivatives
- $y^{(1)}(t) = x^{(1)}(t) * h(t) = x(t) * h^{(1)}(t)$
- $y^{(2)}(t) = x^{(2)}(t) * h(t) = x^{(1)}(t) * h^{(1)}(t) = x(t) * h^{(2)}(t)$
- ......
- $y^{(k+r)}(t) = x^{(k)}(t) * h^{(r)}(t)$

Examples on convolution

Rectangular function: $rect(t)$

$$rect(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2}) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}.$$



$$rect * rect = ?$$

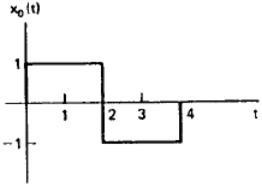
Examples on convolution cont.

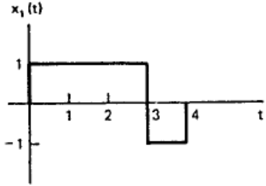
- $rect * rect = ?$: sliding window
- $rect * rect = \int \frac{d}{dt}(rect * rect)dt = ?$
- $rect * \int rect(t)dt = ?$

$rect * y = ?$, where y is

$$y(t) = \begin{cases} 0, & |t| > 1 \\ 1 - |t|, & -1 \leq t \leq 1 \end{cases}$$

Examples on convolution cont.





$x_0 * x_1 = ?$

Output of LTI systems

- Given an LTI system with impulse response $h(t)$
- The output $y(t)$ for an input $x(t)$ is given by

$$y(t) = x(t) * h(t)$$

- If the system is causal then $h(t) = 0$ for $t < 0$ (why?) (Hint: until $t = 0$, the system does not know if it was the 0 input or $\delta(t)$. 0 input has 0 output.)

An eigenvector basis

- A signal $a(t)$ is called an eigenvector if

$$a(t) * h(t) = \lambda a(t)$$

- $\delta(t)$ is not an eigenvector (unless $h(t) = \lambda \delta(t)$)
- What about $a(t) = e^{j2\pi ft}$

An eigenvector basis cont.

- What about $a(t) = e^{j2\pi ft}$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} a(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)}h(\tau)d\tau \\
&= e^{j2\pi ft}\int_{-\infty}^{\infty} e^{-j2\pi f\tau}h(\tau)d\tau \\
&= H(f)e^{j2\pi ft}
\end{aligned}
$$

Fourier basis

- Thus $\{e^{j2\pi ft}\}$ are eigenvectors for every LTI system!
- This is called the Fourier basis

- The function $H(f)$ is the Fourier transform of $h(t)$

$$H(f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} h(\tau) d\tau$$

Other eigenvector basis

- e^{st}: Laplace basis

$$e^{st} * h(t) = H(s) * e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$