Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time

The Sampling Theorem

- ❑ What is sampling?

  Converting continuous-time signals to discrete-time signals
- ❑ Why sampling?

  To use the well-developed digital technology
- ❑ But, a signal could not always be uniquely specified by equally-spaced samples

$$x_1(kT) = x_2(kT) = x_3(kT)$$

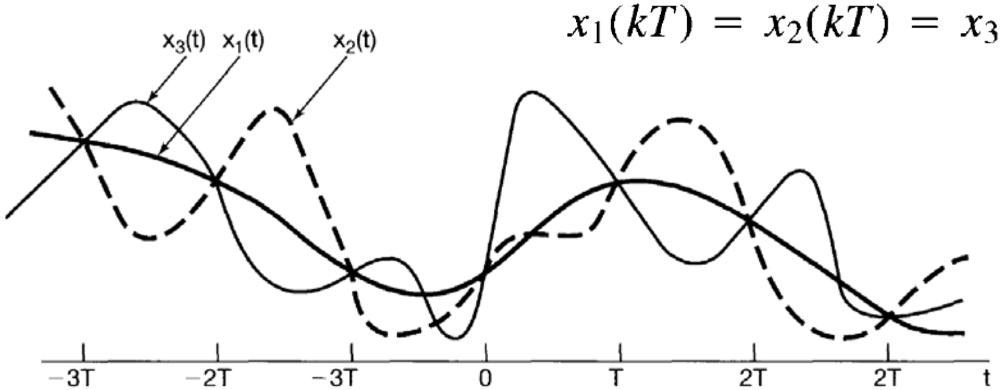
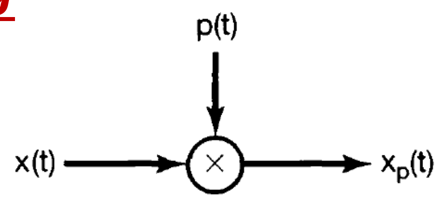


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

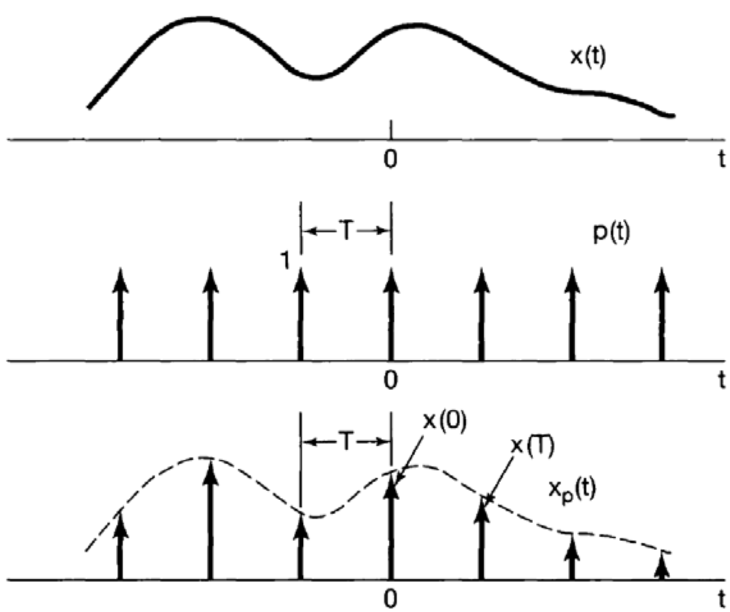
- ❑ The sampling theorem should be satisfied

Impulse-Train Sampling



$$x_p(t) = x(t) \cdot p(t)$$

- Time domain

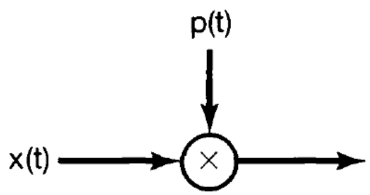


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

The Sampling Theorem

Impulse-Train Sampling



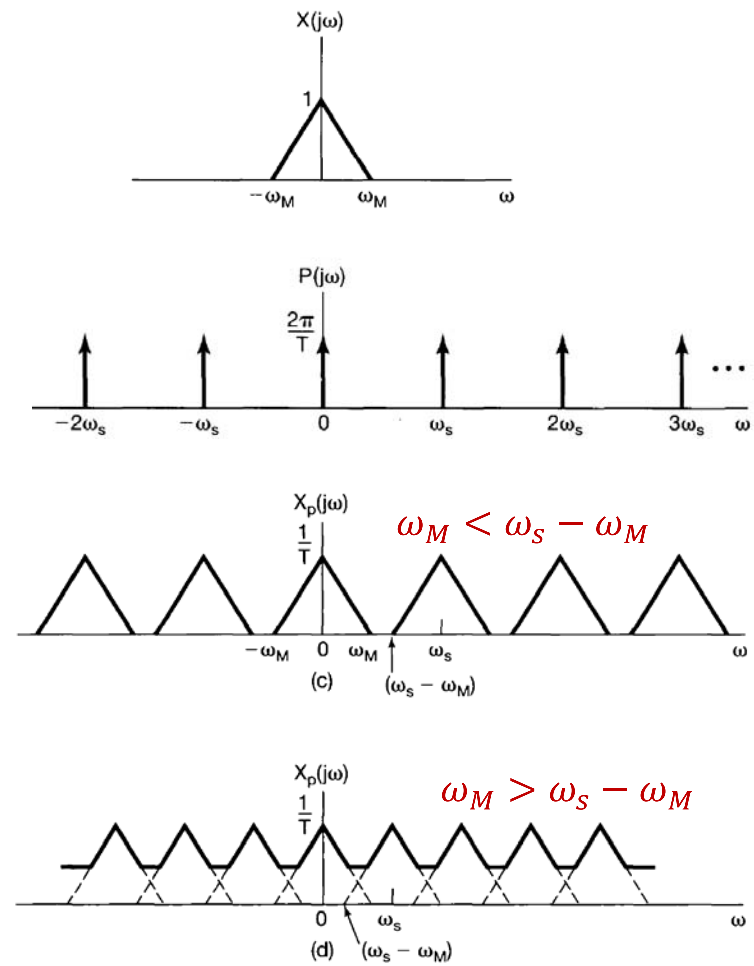
$$x_p(t) = x(t) \cdot p(t)$$

- Frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\big(j(\omega - k \cdot \omega_s)\big)$$



The Sampling Theorem

Sampling Theorem

Sampling Theorem:
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm1, \pm2, \ldots$, if

$$\omega_s > 2\omega_M,$$

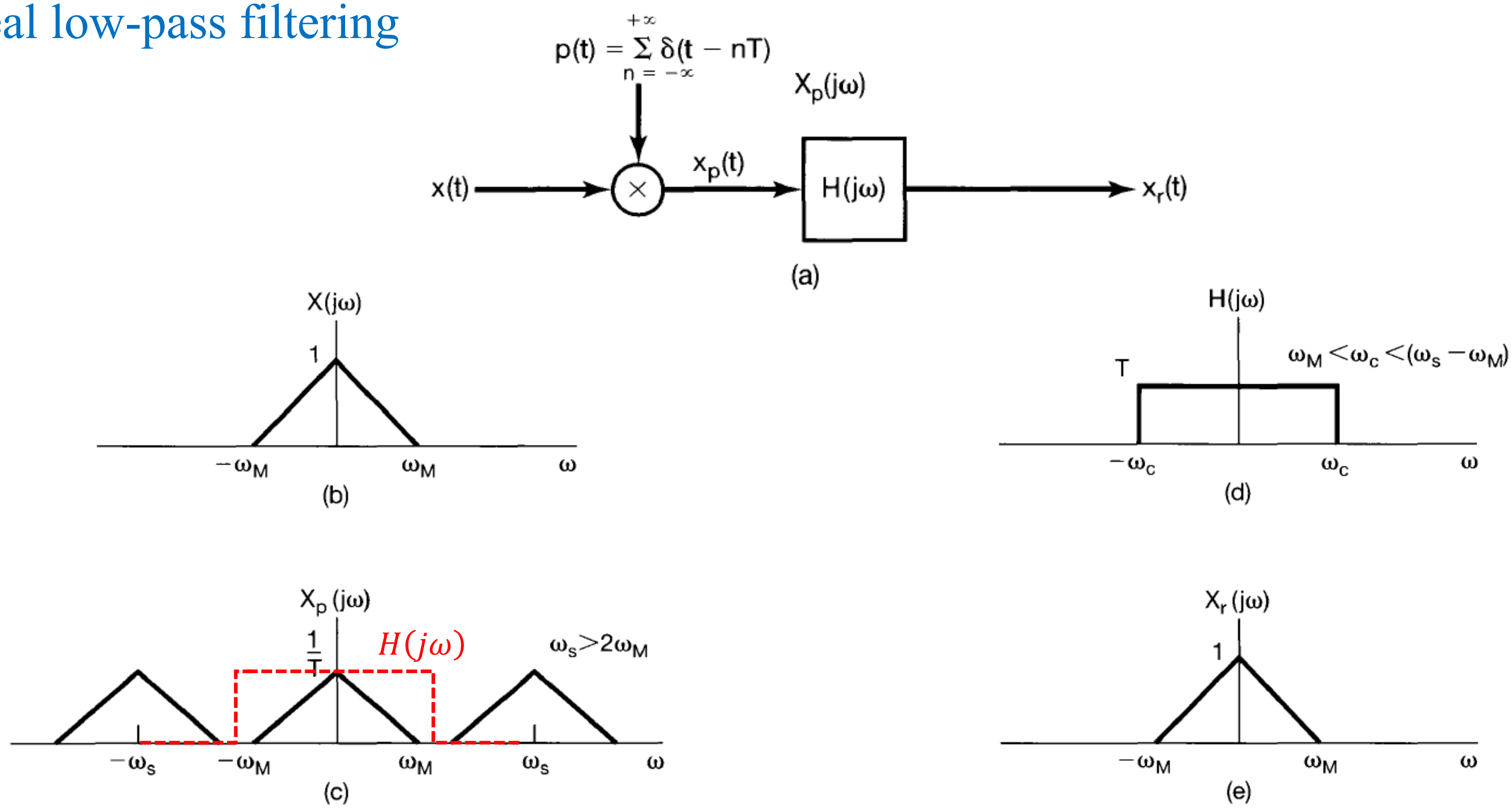
where

$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

Recovery of the CT signal

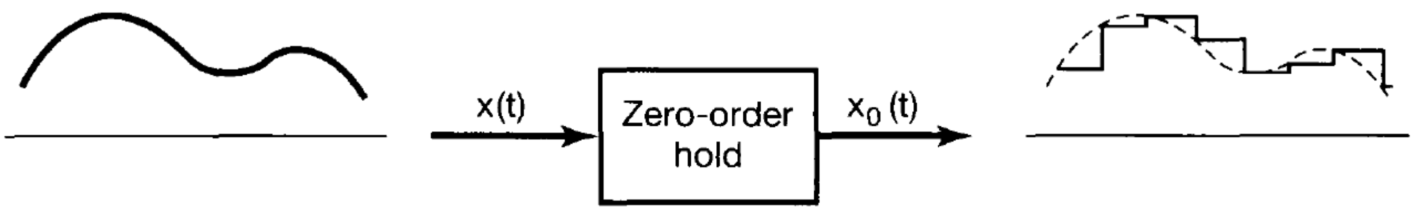
- Ideal low-pass filtering



The Sampling Theorem

Sampling with a Zero-order Hold

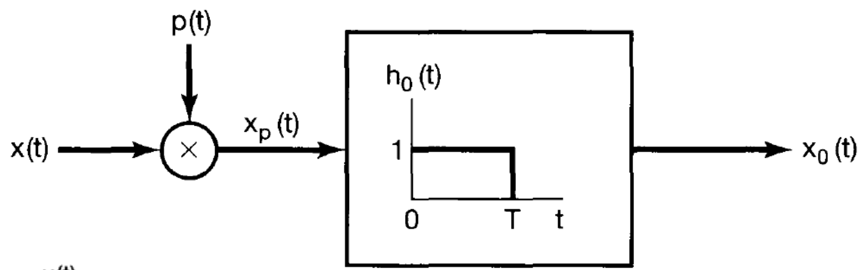
- Why: Impulse-train is difficult to generate
- Principle: Samples $x(t)$ at a given instant and holds that value until the next instant

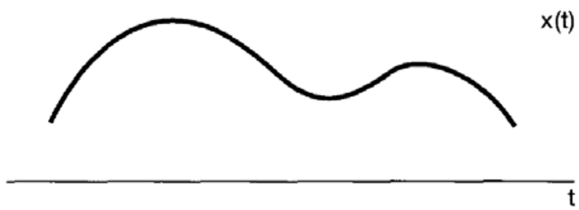


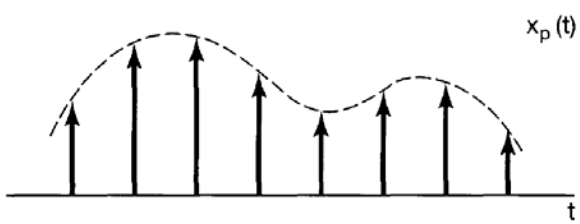
The Sampling Theorem

Sampling with a Zero-order Hold

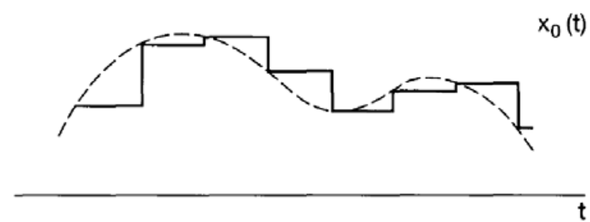
- Equivalent: Impulse-train sampling + an LTI system with a rectangular impulse response





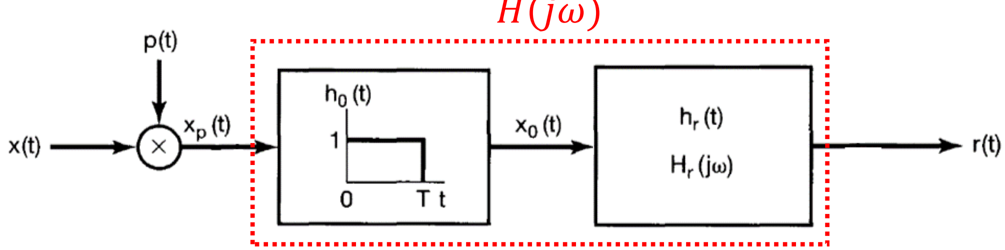


$$x_0(t) = x_p(t) * h(t) \Longrightarrow$$

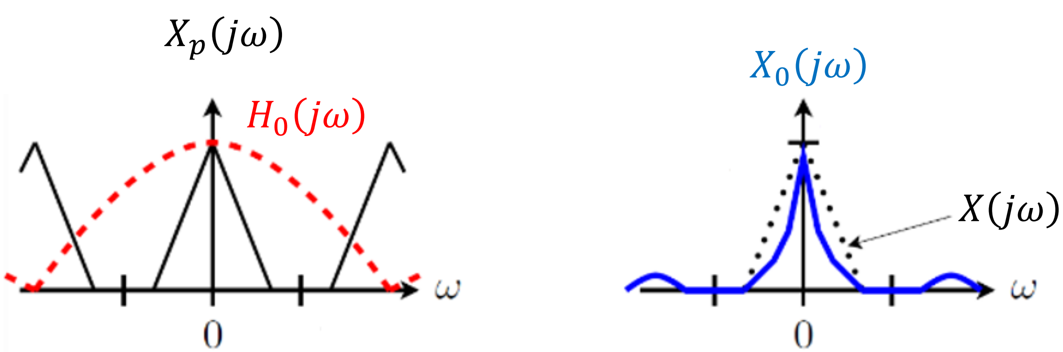


Sampling with a Zero-order Hold

❑ Compensation filter

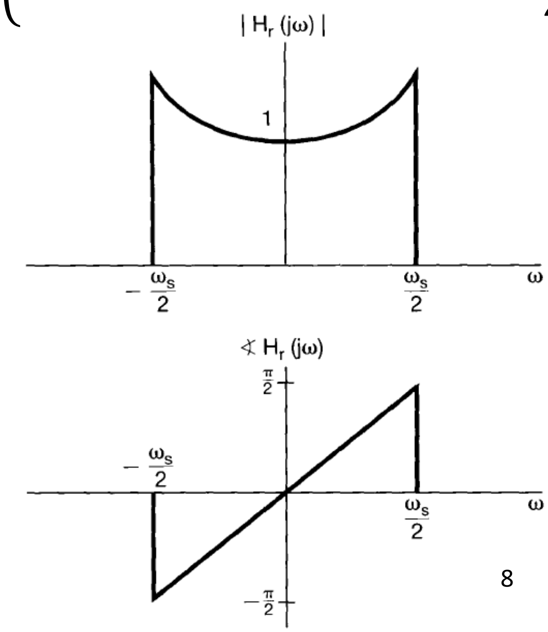


$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin\omega T}{\omega}\right]$$



Let $H_0(j\omega)H_r(j\omega) = H(j\omega)$

$$H_r(j\omega) = \begin{cases} e^{-j\omega T/2} \Big/ \left[\frac{2\sin\omega T}{\omega}\right], & |\omega| \le \frac{\omega_s}{2} \\ 0, & |\omega| > \frac{\omega_s}{2} \end{cases}$$



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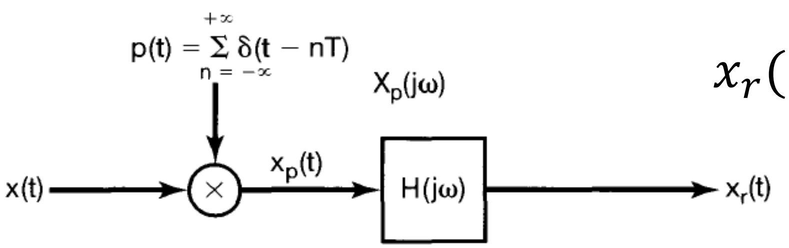
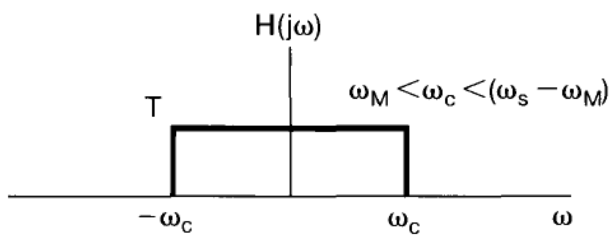
Reconstruction of a Signal Using Interpolation

Band-limited interpolation: (ideal low-pass filter)





$$h(t) = \frac{T\omega_c}{\pi}\frac{\sin \omega_c t}{\omega_c t}$$

$$x_r(t) = x_p(t) * h(t) = \left[\underbrace{\sum_{n=-\infty}^{\infty} x(nT)\cdot\delta(t-nT)}_{x_p(t)}\right] * h(t)$$

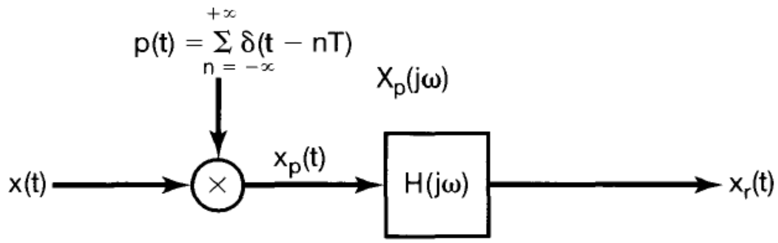
$$= \sum_{n=-\infty}^{\infty} x(nT)[\delta(t-nT) * h(t)\,]$$

$$= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$$

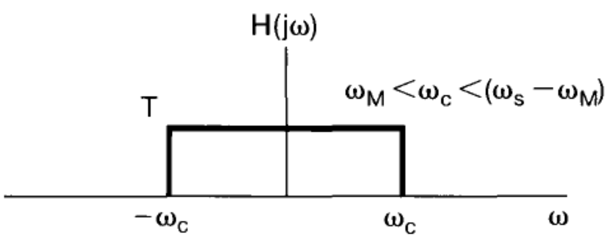
$$= \sum_{n=-\infty}^{\infty} x(nT)\frac{T\omega_c}{\pi}\frac{\sin \omega_c(t-nT)}{\omega_c(t-nT)}$$

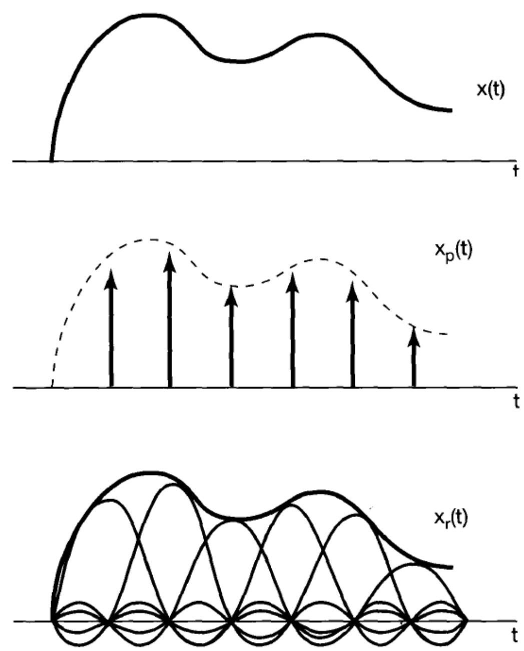
Reconstruction of a Signal Using Interpolation

Band-limited interpolation: (ideal low-pass filter)



$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{T\omega_c}{\pi}\,\frac{\sin\omega_c(t-nT)}{\omega_c(t-nT)}$$

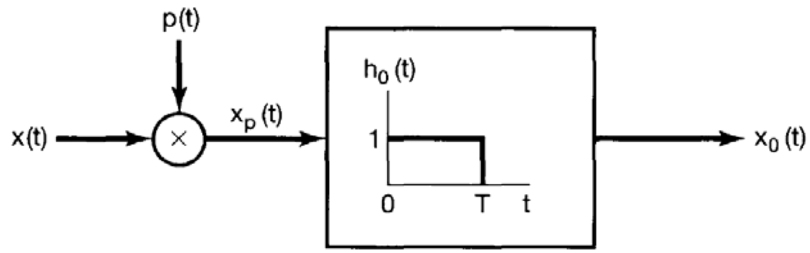




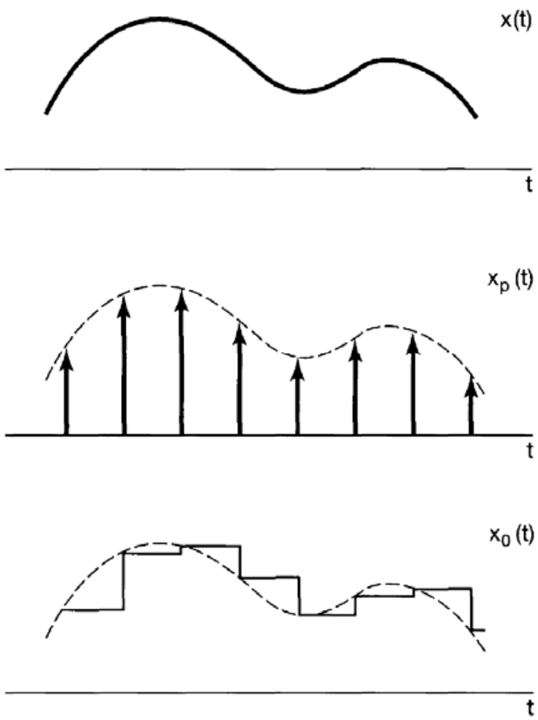
$$h(t) = \frac{T\omega_c}{\pi}\,\frac{\sin\omega_c t}{\omega_c t}$$

Reconstruction of a Signal Using Interpolation

Zero-order hold

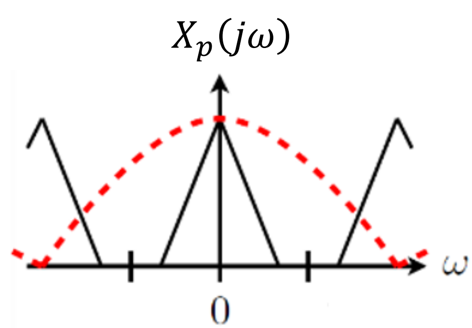


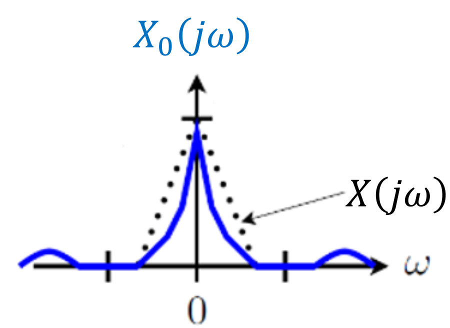
- Time domain



- Frequency domain

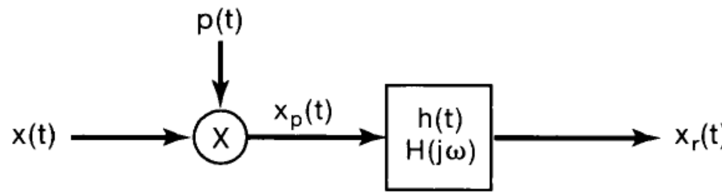
$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin \omega T}{\omega}\right]$$



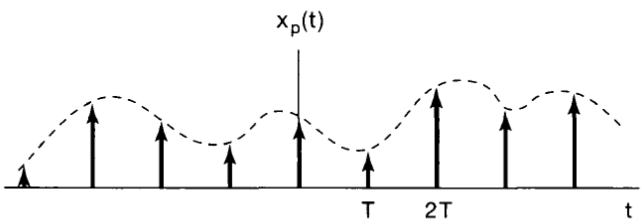


Reconstruction of a Signal Using Interpolation

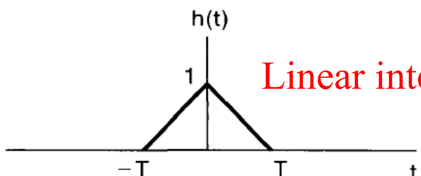
First-order hold: Impulse-train sampling + an LTI system with a tri angular impulse response



- Time domain

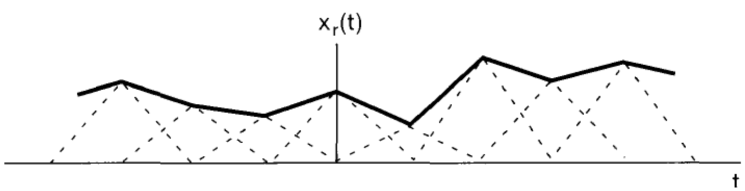


(b)



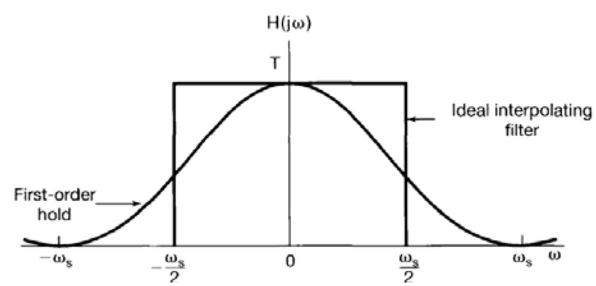
(c)

Linear interpolation



- Frequency domain

$$H_0(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$



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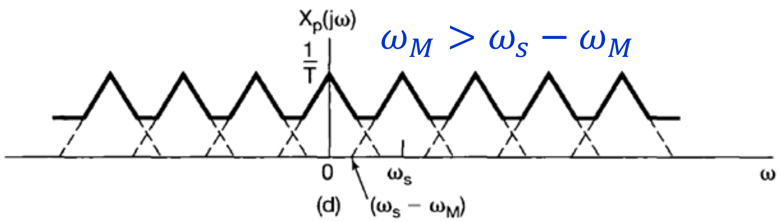
Aliasing

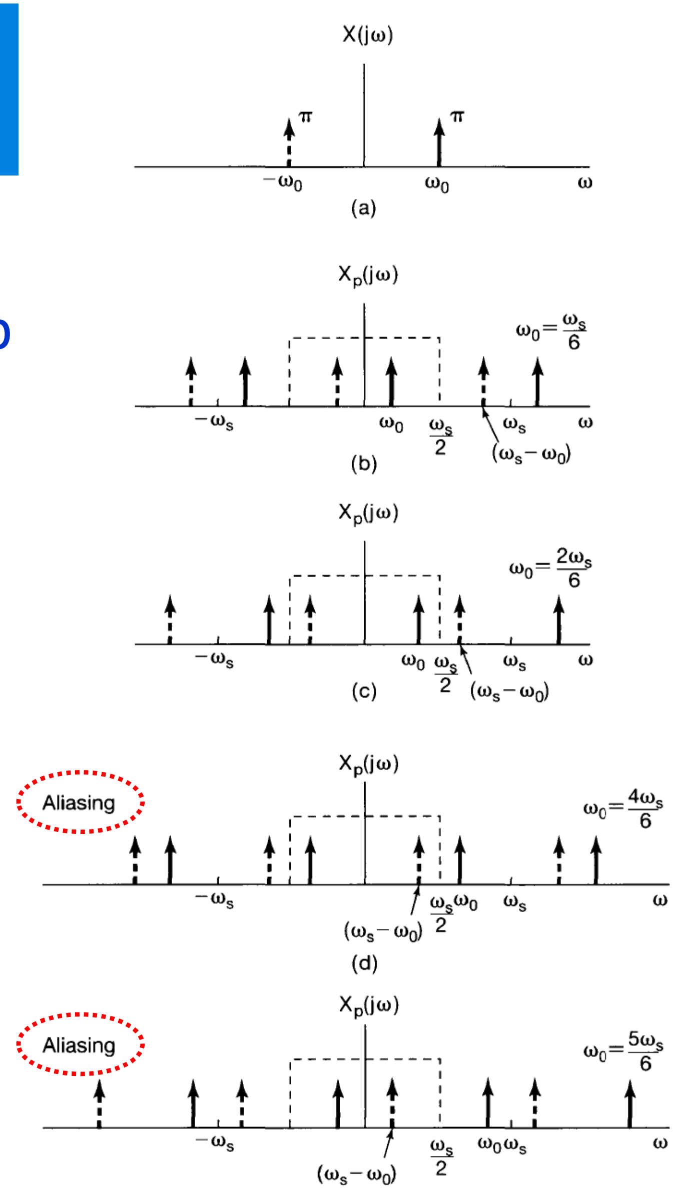
Aliasing

- When $\omega_s < 2\omega_M$, the individual spectrums overlap



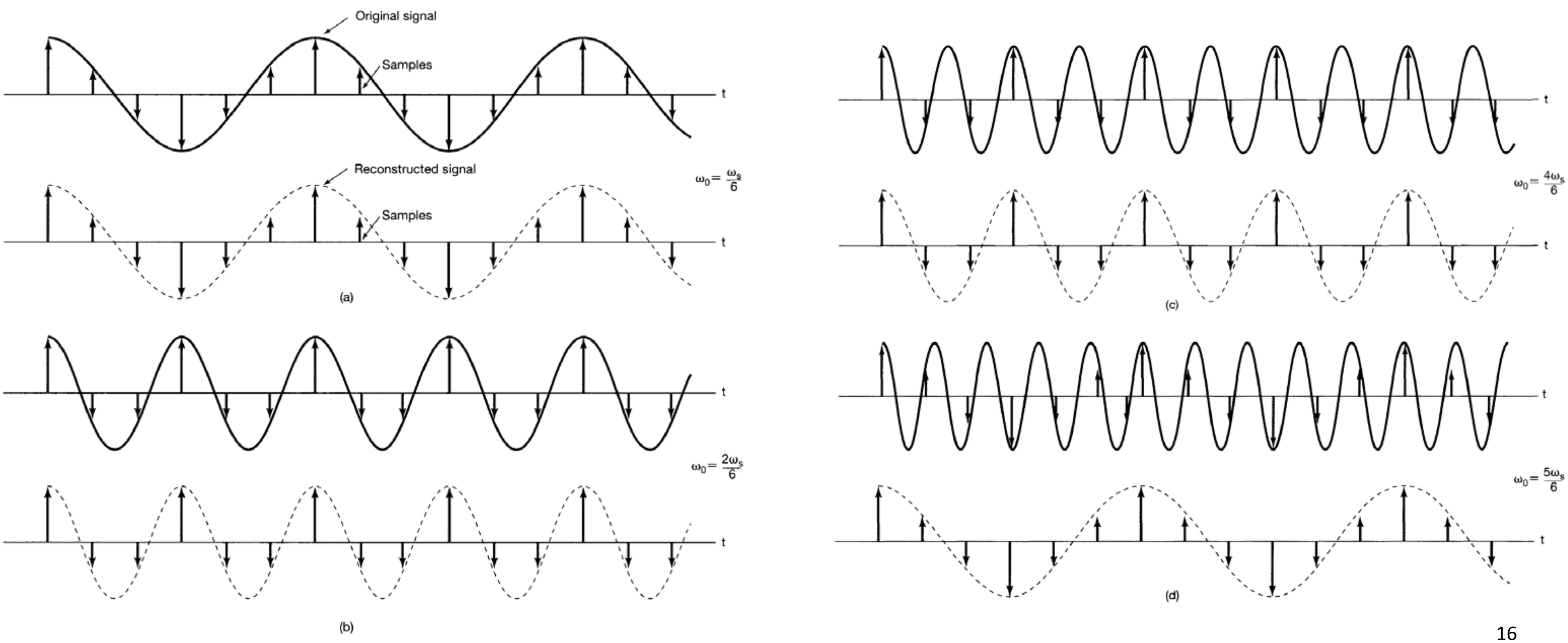
- Consider original signal is $x(t) = \cos\omega_0 t$, with different ω_0 but sampled at same ω_s

  - When aliasing occurs, the original frequency ω_0 takes on the identity of lower frequency $(\omega_s - \omega_0)$.



Aliasing

$x(t) = \cos \omega_0 t$ Time domain



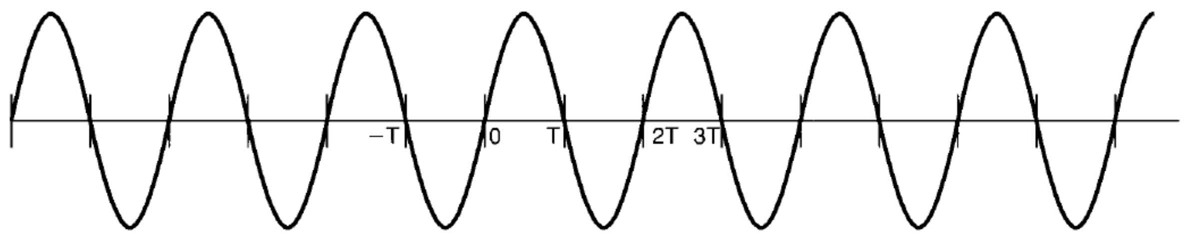
Aliasing

Aliasing

- $\omega_s = 2\omega_M$ is not sufficient to avoid aliasing
  - Consider a signal $x(t) = \cos(\omega_0 t + \emptyset)$ is sampled using impulse sampling with $\omega_s = 2\omega_0$
  - The reconstructed signal using ideal low-pass filter is

$$x_r(t) = \cos(\emptyset)\cos(\omega_0 t) \;\; = x(t) \text{ only if } \emptyset = 2k\pi$$

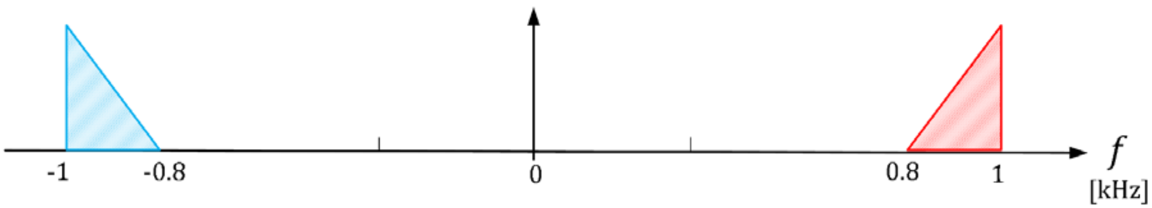
  - Particularly, if $\emptyset = -\pi/2$, then $x(t) = \sin \omega_0 t$ and $x_r(t) = 0$



Aliasing

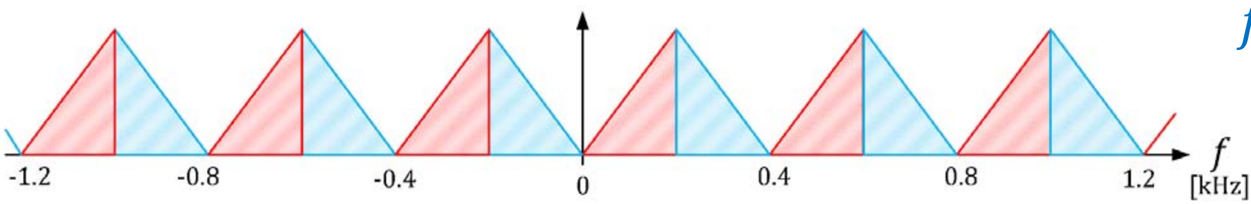
Aliasing

- For signal with $f_c > B/2$, where $f_c = (f_h + f_l)/2$ and $B = f_h - f_l$



$f_l = 800$ Hz, $f_h = 1000$ Hz

Determine the lowest f_s with no aliasing

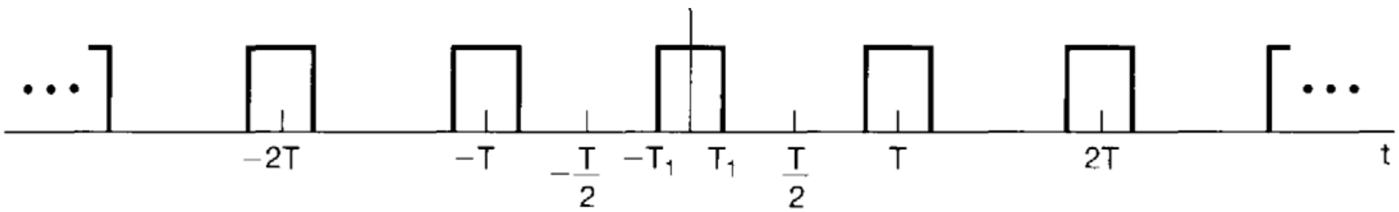


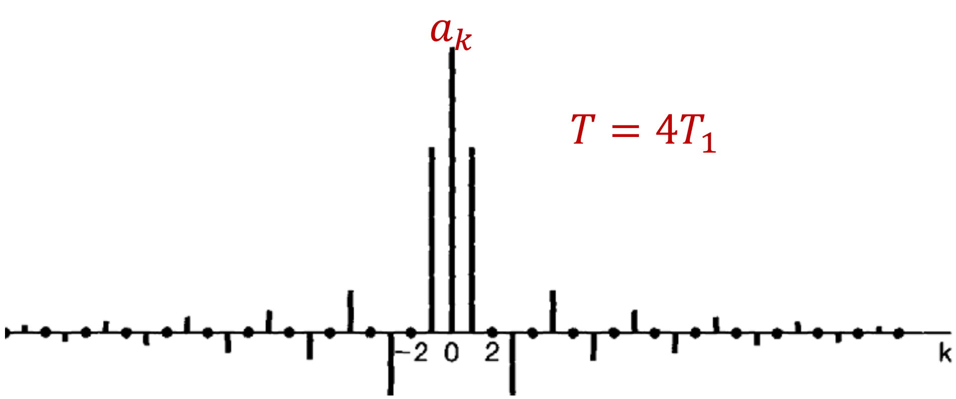
$f_s = 400$ Hz

Q: What about $f_l = 850$ Hz?

Aliasing

- ❑ For harmonic related signal, e.g., a square wave





- $\omega_s > 2K\omega_0$, with K the kth harmonics you want to include
- Low-pass filtering before sampling