Problem 1

(15 points)

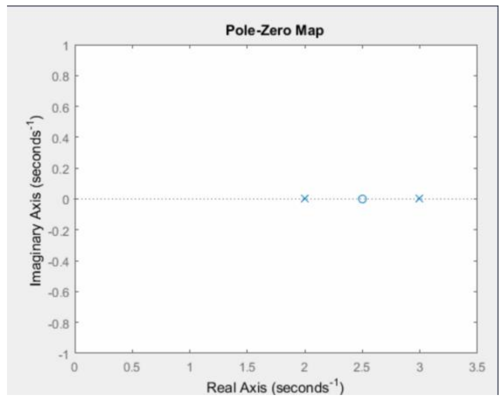
Determine the Laplace transform by definition and the associated ROC and pole-zero plot for each of the following functions of time.

(a) $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$

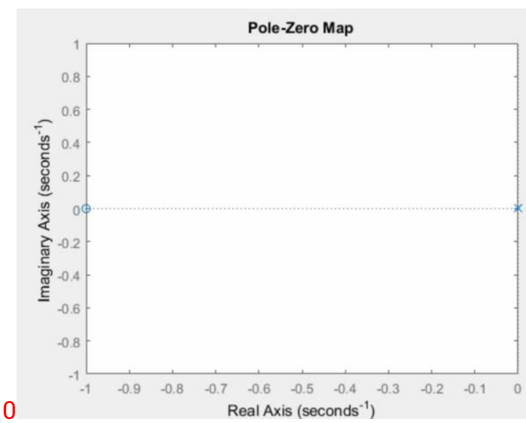
(b) $x(t) = \delta(3t) + u(3t)$

(c) $x(t) = |t|e^{-2|t|}$

Solution:

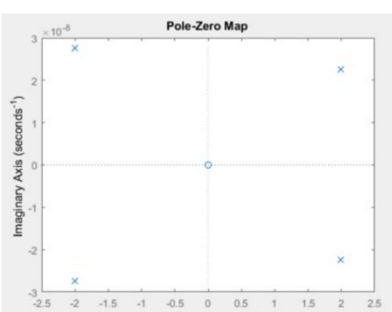


(a) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \frac{5-2s}{s^2+6-5s}$ ROC: $Re\{Z\} < 2$



(b) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \frac{1}{3} + \frac{1}{s}$ ROC: $Re\{Z\} > 0$

(c) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} (|t|e^{-2|t|})e^{-st}dt = -\int_{-\infty}^{0} te^{(2-s)t}dt + \int_{0}^{\infty} te^{(-2-s)t}dt = \frac{1}{(s-2)^2} + \frac{1}{(s+2)^2}$ ROC: $-2 < Re\{Z\} < 2$



Problem 2

(15 points)
A causal LTI system with impulse response h(t) has the following properties:
1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{1}{6}e^{2t}$ for all t.
2. The impulse response h(t) satisfies the differential equation:

$$\frac{dh(t)}{dt} + 2h(t) = e^{-4t}u(t) + bu(t),$$

where b is an unknown constant.
Determine the system function H (s) of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in the answer.

Solution:

From properties 1: $x(t) = e^{2t}$ produces $y(t) = \frac{1}{6}e^{2t}$ for all t, then we can get $H(2) = \frac{1}{6}$.

From properties 2: apply the Laplace transform to the both side of the equation, we can get the

$$H(s) = \frac{s + b(s+4)}{s(s+2)(s+4)}$$

Since $H(2) = \frac{1}{6}$, we can deduce that $b = 1$

Therefore,

$$H(s) = \frac{2}{s(s+4)}, \ Re\{s\} > 0$$

Problem 3

(20 points)
We are given the following five facts about a real signal x(t) with Laplace transform X(s):
1. X(s) has exactly two poles.

2. X(s) has no zeros in the finite s-plane.

3. X(s) has a pole at s = -1 + j.

4. $e^{2t}x(t)$ is not absolutely integrable.

5. X(0) = 8.

Determine X(s) and specify its region of convergence.

Solution:

From fact 1 and 2: we can know that X(s) is the form of

$$X(s) = \frac{A}{(s+a)(s+b)}$$

From fact 3: we get that one of the poles is X(s) = -1+j. Furthermore, since x(t) is real, so the two poles must be conjugate pairs, so the poles are: -1+j and -1-j. Therefor,

$$X(s) = \frac{A}{(s+1-j)(s+1+j)}$$

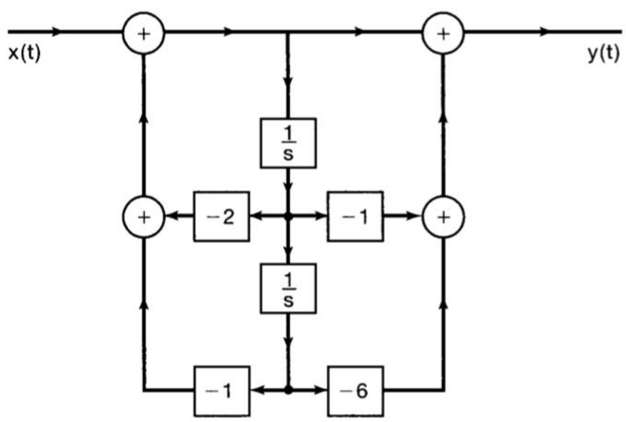
From fact 5: we get that X(0)=8, then A=16,

$$X(s) = \frac{16}{s^2+2s+2}$$

From fact 4: since $y(t) = e^{2t}x(t) \overset{L}{\leftrightarrow} Y(s) = X(s-2)$, so X(s-2) is not absolutely integrable. So the ROC of X(s-2) should not contain jw-axis. The ROC of X(s-2) is just the ROC of X(s) shifted by 2 to the right. Since X(s) has two poles s1=-1+j and s2= -1-j. So ROC of X(s) should be Re{z} > -1.

Problem 4

(25 points)
The input x(t) and output y(t) of a causal LTI system are related through the block- diagram representation shown below.



(a) Determine a differential equation relating y(t) and x(t).

(b) Is this system stable?

Solution:

(a) From diagram, we get that:

$$Y(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1} X(s)$$

Therefore, we can get that:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t)$$

(不一定要按照这个思路)
(b) since that,

$$H(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1}$$

So pole is s=-1.Since causal LTI system, right-hand-side signal, ROC must be the right of s=-1. Therefore, ROS must contain the jw-axis, so stable.

Problem 5

(25 points)
Consider the system S characterized by the differential equation

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = x(t)$$

(a) Determine the zero-state response of this system for the input $x(t) = e^{-4t}u(t)$.

(b) Determine the zero-input response of the system for $t > 0^-$, given that

$$y(0^-) = 1, \quad \lim_{t=0^-}\frac{dy(t)}{dt} = -1, \quad \lim_{t=0^-}\frac{d^2y(t)}{dt^2} = 1$$

(c) Determine the output of S when the input is $x(t) = e^{-4t}u(t)$ and the initial conditions are the same as those specified in part (b).

Solution:

Apply the inverse unilateral transform of both sides of the given equation, we can get that:

$$s^3Y(s) - s^2Y(0^-) - sY'(0^-) - Y''(0^-) + 6s^2Y(s) - 6sY(0^-) - 6Y'(0^-) + 11sY(s) - 11Y(0^-) + 6Y(s) = X(s)$$

(a) Zero-state, so assume all initial conditions are zero. So we get that:

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = X(s)$$

Since $x(t) = e^{-4t}u(t)$, we can get that: $X(s) = \frac{1}{s+4}$, ROC: Re{s} > -4

So $Y(s) = \frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{\frac{1}{6}}{s+1} + \frac{-\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+3} + \frac{-\frac{1}{6}}{s+4}$, Re{s} >-1

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t)$$

(b) zero-input, Use the given initial and input = 0, so we get that:

$$s^3Y(s) - s^2 + s - 1 + 6s^2Y(s) - 6s + 6 + 11sY(s) - 11 + 6Y(s) = 0$$

We can get that:

$$Y(s) = \frac{1}{s+1}$$

Thus(unilateral),

$$y(t) = e^{-t}u(t)$$

(c) just the total sum of (a) and (b), the result is:

$$y(t) = \frac{7}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t)$$