1 Nash Equilibrium

Given a two player game where the action space of both players is $\{A, B\}$. Consider whether the following statements are true or false. If true, give the proof, otherwise give a counterexample.

1.1 (1pt)

Suppose (A, A) is the unique pure strategy Nash equilibrium, then action A is a dominant strategy for at least one of the players.

The statement is True.
Since (B, B) is not a Nash equilibrium, then one of the following equations must be satisfied:

$$u_1(A, B) \geq u_1(B, B) \tag{1}$$
$$u_2(B, A) \geq u_2(B, B) \tag{2}$$

Suppose equation (1) is satisfied w.l.o.g.. Since (A, A) is the Nash equilibrium, then

$$u_1(A, A) \geq u_1(B, A) \tag{3}$$

Together with equation (1) and (3), A is the dominant strategy for player 1.

1.2 (1pt)

Suppose (A, A) is the unique Nash equilibrium, then action A is a dominant strategy for both players.

The statement is False.
The counterexample is not unique. A counterexample should satisfy the following requirements:

1. (A, A) is the *unique* Nash equilibrium;
2. A is not the dominant strategy for one of the players.

For example

1/2	A	B
A	5,5	3,4
B	1,3	2,6

2 Myerson's Mechanism

Suppose there are n agents who bid for one single item. Their probability density functions of their valuation distributions are of the Pareto's form and same (i.i.d.):

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \qquad x \geq 1$$

2.1 (1pt)

If $\alpha = 2$ and there are five bidders $\{A, B, C, D, E\}$ with bids $v_A = 20$, $v_B = 18$, $v_C = 16$, $v_D = 14$ and $v_E = 12$. Compute the allocation and payment of Myerson's mechanism.

$$F(x) = 1 - \frac{1}{x^\alpha}$$

When $\alpha = 2$, $\phi = v/2$. The allocation and payment will be same as VCG. Then A is the winner and she should pay 18.

2.2 (1pt)

If $\alpha = 1/2$, will the Myerson's Mechanism be truthful? Prove your statement.

$\phi = -v$, which is not monotone increasing. Hence the mechanism is not truthful.

3 Expected Revenue

Consider an auction for a single indivisible item where there are n buyers. Suppose all bidders have the same probability distribution of their valuations independently (i.i.d.) as uniform distribution on $[0, 1]$.

3.1 (1pt)

If the seller uses second price auction and $n = 3$, compute the expected revenue of the seller.

$$\mathbb{E}[rev] = 3 \times 2 \times \int_0^1 v^2(1-v)\mathrm{d}v = \frac{1}{2}$$

3.2 (1pt)

If the seller uses Myerson's mechanism and $n = k$ ($k > 0$), compute the expected revenue of the seller.

$$\mathbb{E}[rev] = k \cdot \left[\left(\frac{1}{2}\right)^k \cdot \frac{1}{2} + (k-1)\int_{\frac{1}{2}}^1 v(1-v)v^{k-2}\mathrm{d}v\right] = \frac{2^{-k}+k-1}{k+1}$$

3.3 (2pt)

If the seller uses second price auction and $n = k + 1$ ($k > 0$), compute the expected revenue of the seller. Comparing the result with that in 3.2, what can you observe?

$$\mathbb{E}[rev] = (k+1) \cdot k \cdot \int_0^1 v(1-v)v^{k-1}\mathrm{d}v = \frac{k}{k+2}$$

Note that for all $k > 0$, we have

$$\frac{k}{k+2} \geq \frac{2^{-k}+k-1}{k+1}$$

i.e., involving one more buyer with second price auction will achieve higher revenue than applying Myerson's mechanism.