Signals and Systems Homework 3

Due Time: 21:59 March 30, 2018
Submitted in-class on Thu (Thu),
or to the box in front of SIST 1C 403E (the instructors office).

The process of solving a problem is a must. You can't score by giving only the result.

1. (15') Consider a continuous-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq 100 \\ 0, & |\omega| > 100 \end{cases}$$

When the input to this filter is a signal x(t) with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \xrightarrow{S} y(t) = x(t)$$

For what values of k is it guaranteed that $a_k = 0$?

Solution: $\omega_0 = 2\pi/T = 12$, and

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{S} y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0 t) e^{jk\omega_0 t}$$

since $y(t) = x(t)$, there must be:

$$\forall a_k \neq 0, k \in Z, |k\omega_0| \leq 100$$

This implies that $|k| \leq 8$. Therefore, for $|k| > 8$, a_k is guaranted to be 0.

2. (20') Consider a causal continuous-time LTI system whose frequency response is $H(j\omega) = \frac{1}{j\omega+4}$. Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

   (a) $x(t) = \cos 2\pi t$

   (b) $x(t) = \sin 4\pi t + \cos(6\pi t + \pi/4)$

   Solution:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0 t) e^{jk\omega_0 t}$$

   (a) Here $\omega_0 = 2\pi$, and the nonzero F.S. coefficiences of $x(t)$ is $a_1 = a_{-1}^* = 1/2$. Therefore, the nonzero F.S. coefficiences of $y(t)$ are:

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(4+2\pi j)}, b_{-1} = a_{-1} H(-j2\pi) = \frac{1}{4 - j2\pi}$$

   (b) Let $\omega_0 = 2\pi$, then the nonzero F.S. coefficiences of $x(t)$ is

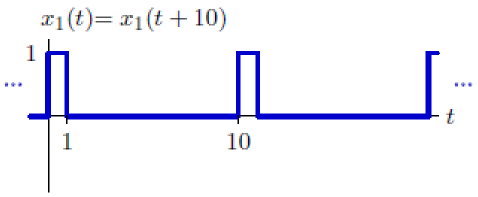
$$a_2 = a_{-2}^* = 1/2j, a_3 = a_{-3}^* = \frac{1}{2} e^{j\pi/4}$$

   therefore, the nonzero F.S. coefficiences of $y(t)$ are:

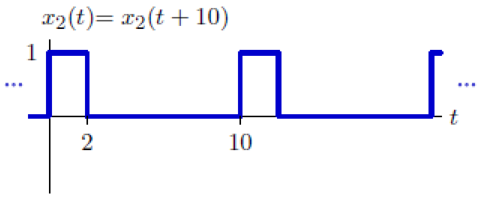
$$b_2 = a_2 H(j4\pi) = \frac{1}{2j(4+j4\pi)}, \qquad b_{-2} = b_2^* = -\frac{1}{2j(4-j4\pi)}$$

$$b_3 = a_3 H(j6\pi) = \frac{e^{j\pi/4}}{2(4+j6\pi)} \qquad b_{-3} = b_3^* = \frac{e^{-j\pi/4}}{2(4-j6\pi)}$$

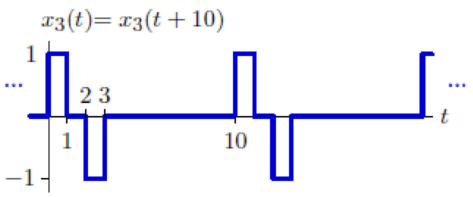
3. (a) (6') Determine the Fourier series coefficients a_k for $x_1(t)$ shown below.



(b) (6') Determine the Fourier series coefficients b_k for $x_2(t)$ shown below.



(c) (6') Determine the Fourier series coefficients c_k for $x_3(t)$ shown below.



Solution:
(a)

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{10}\int_0^1 1e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\left.\frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k}\right|_0^1 = \frac{1}{j2\pi k}\left(1 - e^{-j\pi k/5}\right)$$

Notice that this expression is badly formed at $k = 0$. We could use l'Hôpital's rule to evaluate this expression, but an easier method (which is also more robust against errors) is to simply evaluate the average value of $x_1(t)$ to find that $a_0 = 1/10$.

This solution could also be written in terms of sinusoids as

$$a_k = \begin{cases} \frac{1}{10} & k = 0 \\ \frac{1}{\pi k}e^{-j\pi k/10}\sin(\pi k/10) & k \neq 0 \end{cases}.$$

(b)

$$b_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{10}\int_0^2 1e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\left.\frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k}\right|_0^2 = \frac{1}{j2\pi k}\left(1 - e^{-j2\pi k/5}\right)$$

As with the previous part, this expression is badly formed for $k = 0$. We therefore obtain $b_0 = 1/5$ by calculating the average value of $x_2(t)$.

This solution could also be written in terms of sinusoids as

$$b_k = \begin{cases} \frac{1}{5} & k = 0 \\ \frac{1}{\pi k}e^{-j\pi k/5}\sin(\pi k/5) & k \neq 0 \end{cases}.$$

(c)

$$x_3(t) = x_1(t) - x_1(t-2)$$

$$\int_T x_1(t-2)e^{-j\frac{2\pi}{T}kt}dt = \int_T x_1(t)e^{-j\frac{2\pi}{T}k(t+2)}dt = e^{-j\frac{2\pi}{T}k2}\int_T x_1(t)e^{-j\frac{2\pi}{T}kt}dt = e^{-j\frac{2\pi}{T}k2}a_k$$

$$c_k = a_k - e^{-j\frac{2\pi}{T}k2}a_k = \left(1 - e^{-j2\pi k/5}\right)\frac{1}{j2\pi k}\left(1 - e^{-j\pi k/5}\right)$$

The average value of $x_3(t)$ is zero, so $c_0 = 0$.

This solution could also be written in terms of sinusoids as

$$c_k = \begin{cases} 0 & k = 0 \\ \frac{j2}{\pi k}e^{-j3\pi k/10}\sin(\pi k/5)\sin(\pi k/10) & k \neq 0 \end{cases}.$$

4. (15') Determine the CT signals with the following Fourier series coefficients. Assume that the signals are periodic in $T = 4$. Give an expression that is valid for $0 \leq t < 4$ (other values can be found by periodic extension).

$$a_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$$

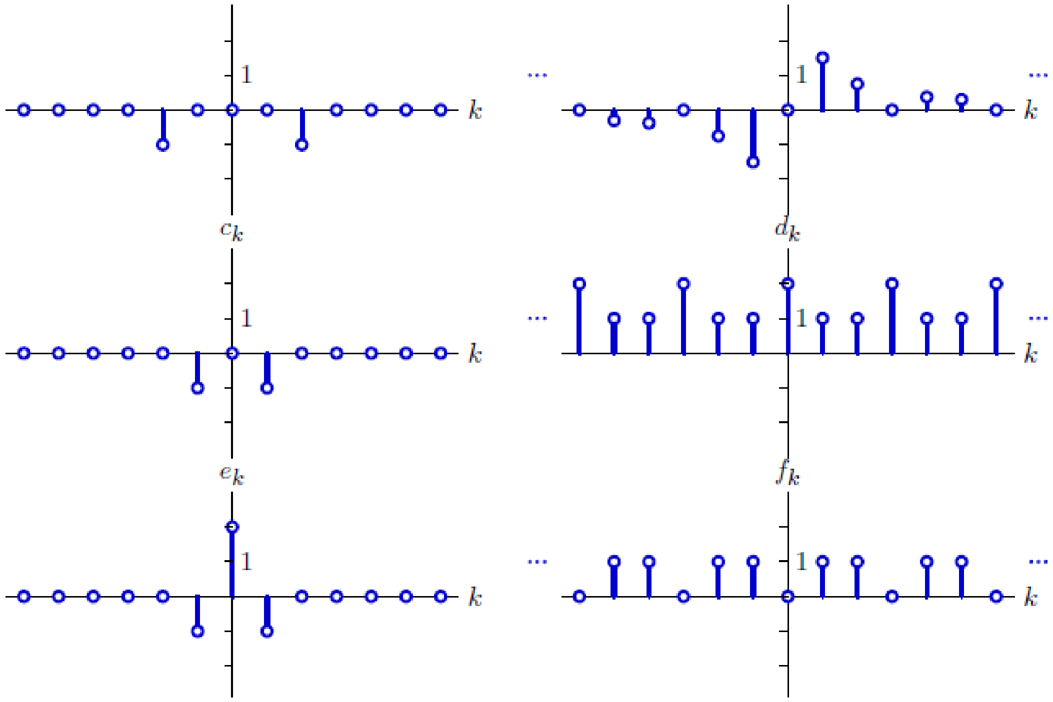
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} = -2je^{-j\frac{2\pi}{4}2t} - je^{-j\frac{2\pi}{4}t} + je^{j\frac{2\pi}{4}t} + 2je^{j\frac{2\pi}{4}2t}$$
$$= -2\sin(\pi t/2) - 4\sin(\pi t)$$

5. (20') Matching problem. You must explain why.

Consider the following Fourier series coefficients.

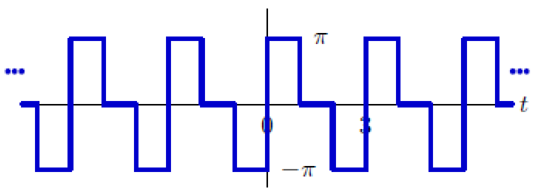
$$b_k = \begin{cases} \frac{3}{j2k} & k = \pm1, \pm2, \pm4, \pm5, \pm7, \ldots \\ 0 & k = 0, \pm3, \pm6, \cdots \end{cases}$$



(a) Which coefficients (if any) corresponds to the following periodic signals?

$$x_1(t) = 2 - 2\cos\left(\frac{2\pi}{3}t\right)$$

(b) Which (if any) set corresponds to the following periodic signal with period $T = 3$?



Solution:

(a)

From the constant 2, it is clear that the zero coefficient is 2. Since $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$, the coefficients for $k = \pm1$ are -1. Therefore the answer is e_k.

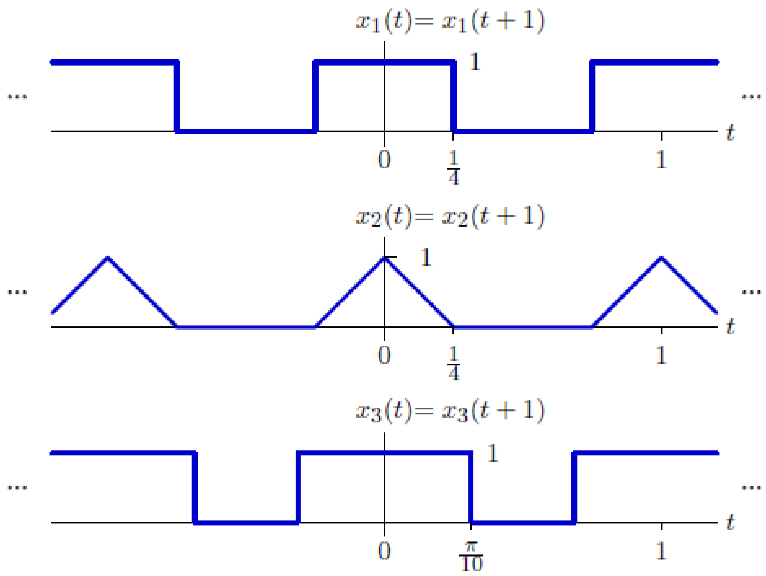
(b)

The signal is real and odd, so its FS coefficients must be purely imaginary and odd. Thus the only candidate is b_k. Solving

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x_3(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{3}\int_{-1}^{0} -\pi e^{-j\frac{2\pi}{3}kt}dt + \frac{1}{3}\int_{0}^{1} -\pi e^{-j\frac{2\pi}{3}kt}dt \\ &= \frac{1}{j2k}\left(e^{-j2\pi kt/3}\Big|_{-1}^{0} - e^{-j2\pi kt/3}\Big|_{0}^{1}\right) = \frac{1}{j2k}(2 - e^{j2\pi k/3} - e^{-j2\pi k/3}) \\ &= \frac{1}{jk}(1-\cos(2\pi k/3)) = \begin{cases} 0 & \text{if } k \text{ is evenly divisible by } 3 \\ 3/j2k & \text{otherwise} \end{cases} \end{aligned}$$

So the answer is b_k.

6. (12') Input/Output pairs The following signals are periodic with period $T = 1$.



Determine if the following systems could or could not be LTI.

$x_1(t) \longrightarrow$ System A $\longrightarrow x_2(t)$

$x_1(t) \longrightarrow$ System B $\longrightarrow x_3(t)$

$x_2(t) \longrightarrow$ System C $\longrightarrow x_1(t)$

$x_2(t) \longrightarrow$ System D $\longrightarrow x_3(t)$

$x_3(t) \longrightarrow$ System E $\longrightarrow x_1(t)$

$x_3(t) \longrightarrow$ System F $\longrightarrow x_2(t)$

Give a list of the systems that could *NOT* be LTI and *EXPLAIN WHY*, if your list is empty, write 'None'.

Hint: We can use the 'filter' idea as follows. First calculate the Fourier series coefficients. Then ask if each Fourier series coefficient in the output is a scaled version of the corresponding coefficient in the input.

$$x_2(t) \leftrightarrow b_k = \frac{4\sin^2(\pi k/4)}{\pi^2 k^2} = \begin{cases} 1/4, & k = 0 \\ \frac{2}{\pi^2 k^2}, & |k| = 1, 3, 5, 7, 9, 11, 13... \\ \frac{4}{\pi^2 k^2}, & |k| = 2, 6, 10, 14, ... \\ 0, & |k| = 4, 8, 12, 16 \end{cases}$$

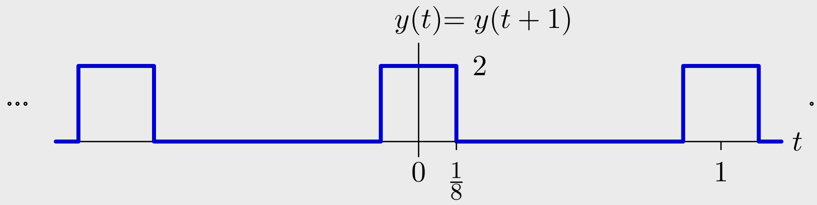
Solution:

We can use the "filter" idea as follows. First calculate the Fourier series coefficients. Then ask if each Fourier series coefficient in the output is a scaled version of the corresponding coefficient in the input.

$$x_1(t) \leftrightarrow a_k = \frac{1}{1}\int_{\frac{-1}{4}}^{\frac{1}{4}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{\pi k}{2}}{\pi k} = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{\pi k} & |k| = 1, 5, 9, 13, \ldots \\ -\frac{1}{\pi k} & |k| = 3, 7, 11, 15, \ldots \\ 0 & |k| = 2, 4, 6, 8, \ldots \end{cases}$$

$$x_2(t) = y(t) \underset{T}{*} y(t)$$

where $y(t)$ is the following signal:



$$y(t) \leftrightarrow d_k = \frac{1}{1}\int_{\frac{-1}{8}}^{\frac{1}{8}} 2e^{-j\frac{2\pi}{1}kt}dt = \frac{2\sin\frac{\pi k}{4}}{\pi k}$$

$$x_2(t) \leftrightarrow b_k = Td_k^2 = 1 \times \frac{4\sin^2\frac{\pi k}{4}}{\pi^2 k^2} = \begin{cases} \frac{1}{4} & k = 0 \\ \frac{2}{\pi^2 k^2} & |k| = 1, 3, 5, 7, 9, 11, 13, \ldots \\ \frac{4}{\pi^2 k^2} & |k| = 2, 6, 10, 14, \ldots \\ 0 & |k| = 4, 8, 12, 16, \ldots \end{cases}$$

$$x_3(t) \leftrightarrow c_k = \frac{1}{1}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{2\pi^2 k}{10}}{\pi k}$$

$$x_1(t) \longrightarrow \boxed{\text{System A}} \longrightarrow x_2(t)$$

The Fourier series coefficients at $k = 2, 6, 10, \ldots$ are zero in x_1 but these are not zero in x_2. Therefore the system could not be LTI.

$$x_1(t) \longrightarrow \boxed{\text{System B}} \longrightarrow x_3(t)$$

The Fourier series coefficients at $k = 2, 4, 6, 8, 10, \ldots$ are zero in x_1 but these are not zero in x_3. Therefore the system could not be LTI.

$$x_2(t) \longrightarrow \boxed{\text{System C}} \longrightarrow x_1(t)$$

All of the nonzero Fourier coefficients in x_1 are also present in x_2. Therefore the system could be LTI.

$$x_2(t) \longrightarrow \boxed{\text{System D}} \longrightarrow x_3(t)$$

The Fourier series coefficients at $k = 4, 8, 12, 16, \ldots$ are zero in x_2 but these are not zero in x_3. Therefore the system could not be LTI.

$$x_3(t) \longrightarrow \boxed{\text{System E}} \longrightarrow x_1(t)$$

All of the nonzero Fourier coefficients in x_1 are also present in x_3. Therefore the system could be LTI.

$$x_3(t) \longrightarrow \boxed{\text{System F}} \longrightarrow x_2(t)$$

All of the nonzero Fourier coefficients in x_2 are also present in x_3. Therefore the system could be LTI.

answer = $\boxed{\text{A, B, D}}$