Signals and Systems Homework 10

Due Time: 21:59 May 25, 2018
Submitted in-class on Thu (May 24),
or to the box in front of SIST 1C 403E (the instructor's office).

1. Using partial-fraction expansion and the fact that

$$a^n u[n] \longleftrightarrow \frac{1}{1 - az^{-1}}, |z| > |a|$$

find the inverse z-transform of

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})}, |z| > 2$$

Solution: Using partial-fraction expansion,

$$X(z) = \frac{2/9}{1 - z^{-1}} + \frac{7/9}{1 + 2z^{-1}}, \qquad |z| > 2$$

Taking the inverse z-transform,

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$

2. Consider a left-sided sequence $x[n]$ has z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

(a) Write $X(z)$ as a ratio of polynomials in z instead of z^{-1}.

(b) Using a partial-fraction expansion , express $X(z)$ as a sum of terms, where each term represents a pole from your answer in part (a).

(c) Determine $x[n]$.

Solution:

(a)

$$X(z) = \frac{z^2}{(z - \frac{1}{2})(z - 1)}$$

(b)

$$X(z) = \frac{-z}{z - \frac{1}{2}} + \frac{2z}{z - 1}$$

or

$$X(z) = 2z[\frac{-z}{z - \frac{1}{2}} + \frac{z}{z - 1}]$$

(c) Note that $x[n]$ is a left-side signal, then the ROC for this signal is $|z| < \frac{1}{2}$. Using the fact, we may find the inverse z-transform is:

$$x[n] = (\frac{1}{2})^n u[-n-1] - 2u[-n-1]$$

or

$$x[n] = 2(\frac{1}{2})^{n+1} u[-n-2] - 2u[-n-2]$$

3. Consider an even sequence $x[n](x[n] = x[-n])$ with rational z-transform $X(z)$.

   (a) From the definition of the z-transform, show that $X(z) = X(\frac{1}{z})$.

   (b) From your results in part (a), show that if a pole(zero) of $X(z)$ occurs at $z = z_0$, then a pole(zero) must also occurs at $z = \frac{1}{z_0}$.

   (c) Verify the results in part (b) for each of the following sequences:
   (i)$\sigma[n+1] + \sigma[n-1]$ (ii)$\sigma[n+1] - \frac{5}{2}\sigma[n] + \sigma[n-1]$

Solution:

(a) First let us determine the z-transform $X_1(z)$ of the sequence $x_1[n] = x_1[-n]$ in terms of $X(z)$.

$$\begin{aligned} X_1(z) &= \sum_{-\infty}^{\infty} x[-n] z^{-n} \\ &= \sum_{-\infty}^{\infty} x[n] z^{n} \\ &= X(\frac{1}{z}) \end{aligned}$$

Therefore, if $x[n] = x[-n]$, then $X(z) = X(\frac{1}{z})$.

(b) If z_0 is a pole, then $\frac{1}{X(z_0)} = 0$. From the result of part (a), we know that $X(z_0) = X(\frac{1}{z_0})$. Therefore, $\frac{1}{X(z_0)} = \frac{1}{X(\frac{1}{z_0})} = 0$. This implies that there is a pole at $z = \frac{1}{z_0}$.

If z_0 ia a zero, then $X(z_0) = 0$. From the result of part (a), we know that $X(z_0) = X(\frac{1}{z_0}) = 0$. This implies that there is a zero at $z = \frac{1}{z_0}$.

(c) (1) In this case,

$$X(z) = z + z^{-1} = \frac{1+z^2}{z}, |z| > 0$$

$X(z)$ has zero at $z_1 = j$ and $z_2 = -j$. Also, $X(z)$ has pole at $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = \frac{1}{z_1}$ and $p_2 = \frac{1}{p_1}$, it satisfies the conclusion in part (b).

(2) In this case,

$$X(z) = z - \frac{5}{2} + z^{-1} = \frac{1 - \frac{5}{2}z + z^2}{z}, |z| > 0$$

$X(z)$ has zero at $z_1 = \frac{1}{2}$ and $z_2 = 2$. Also, $X(z)$ has pole at $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = \frac{1}{z_1}$ and $p_2 = \frac{1}{p_1}$, it satisfies the conclusion in part (b).

4. The input $x(t)$ and output $y(t)$ of a causal LTI system are related through the block-diagram representation shown in the following figure.

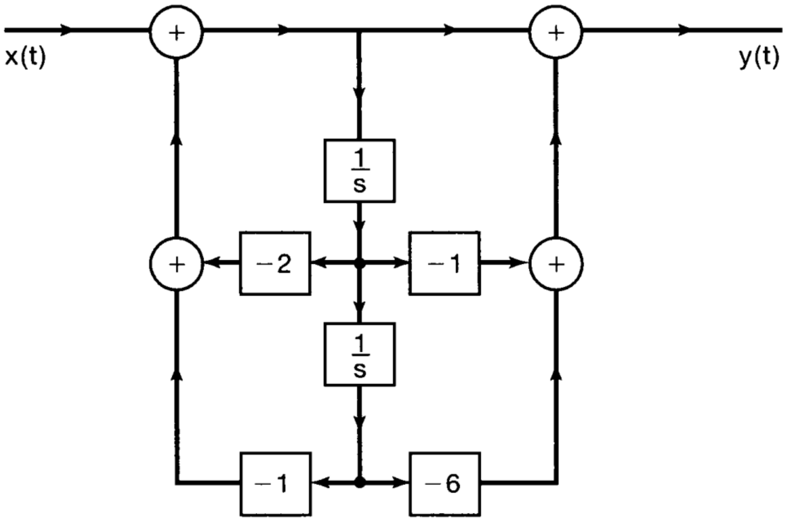

Figure 1:

(a) Determine a differential equation relating $y(t)$ and $x(t)$.

(b) Is this system stable?

Solution:

(a) From the figure ,it is clear that

$$\frac{F(s)}{s} = Y_1(s)$$

Therefore, $f(t) = \frac{dy_1(t)}{dt}$. Similarly, $e(t) = \frac{df(t)}{dt}$, therefore, $e(t) = \frac{d^2y_1(t)}{dt^2}$.
From the block diagram it is clear that

$$y(t) = e(t) - f(t) - 6y_1(t) = \frac{d^2y_1(t)}{dt^2} - \frac{dy_1(t)}{dt} - 6y_1(t)$$

Then

$$Y(s) = (s^2 - s - 6)Y_1(s)$$

Now ,let us determine the relationship between $y_1(t)$ and $x(t)$. This may be done by concentrating on the lower half of the above figure .
It is clear that $y_1(t)$ and $x(t)$ must be related by the following differential equation:

$$\frac{d^2y_1(t)}{dt^2} + 2\frac{dy_1(t)}{dt} + y_1(t) = x(t)$$

So

$$Y_1(s) = \frac{X(s)}{s^2 + 2s + 1}$$

Then we get

$$Y(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1}X(s)$$

Taking the inverse Laplace transform ,we obtain

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t).$$

(b) The two poles of the system are at -1.Since the system is causal; the ROC must be to the right of $s = -1$. Therefore, the ROC must include the $j\omega$-axis. Hence, the system is stable.

5. Consider a fourth-order causal LTI system S whose system function is specified as

$$H(s) = \frac{1}{(s^2 - s + 1)(s^2 + 2s + 1)}$$

   (a) Draw a block diagram representation for S as a *cascade* interconnection of two second-order system, each of which is represented in direct form. There should be no multiplications by nonreal coefficients in the resulting block diagrame.

   (b) Draw a block diagram representation for S as a *parallel* interconnection of two second-order system, each of which is represented in direct form. There should be no multiplications by nonreal coefficients in the resulting block diagrame.

   Solution:

   (a) We may write $H(s)$ as

$$H(s) = \frac{1}{s^2 + 2s + 1} \frac{1}{s^2 - s + 1} = H_1(s)H_2(s)$$

   (b) We may write $H(s)$ as

$$H(s) = \frac{1}{3} \frac{s + 2}{s^2 + 2s + 1} + \frac{1}{3} \frac{1 - s}{s^2 - s + 1} = H_3(s) + H_4(s)$$

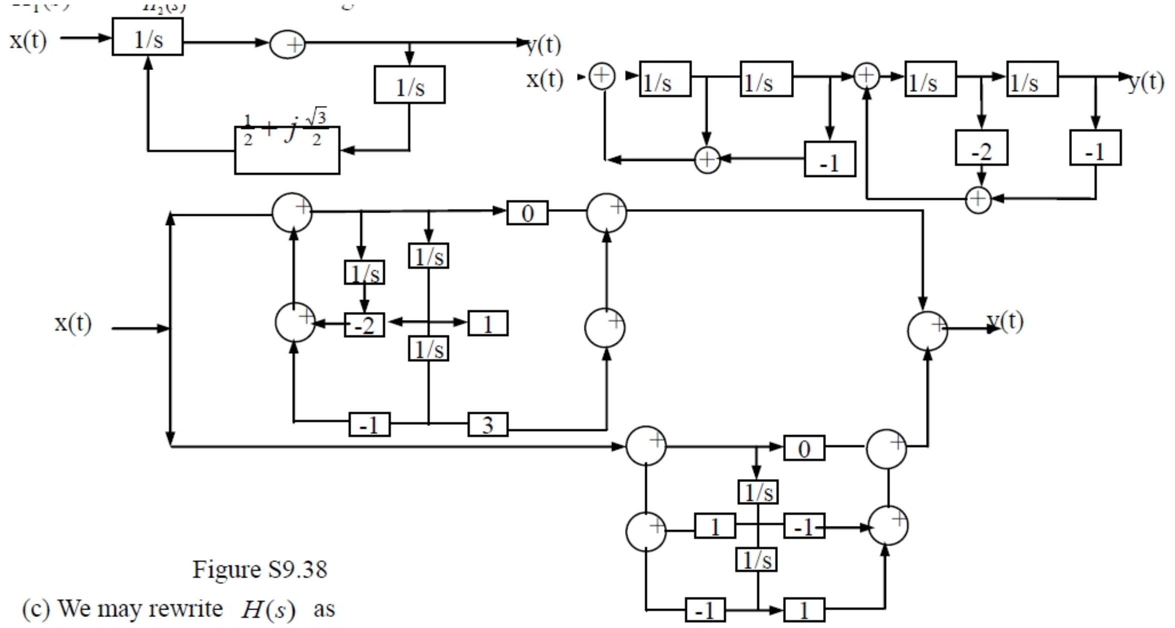

Figure S9.38

(c) We may rewrite $H(s)$ as