Tutorial 4

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Agenda

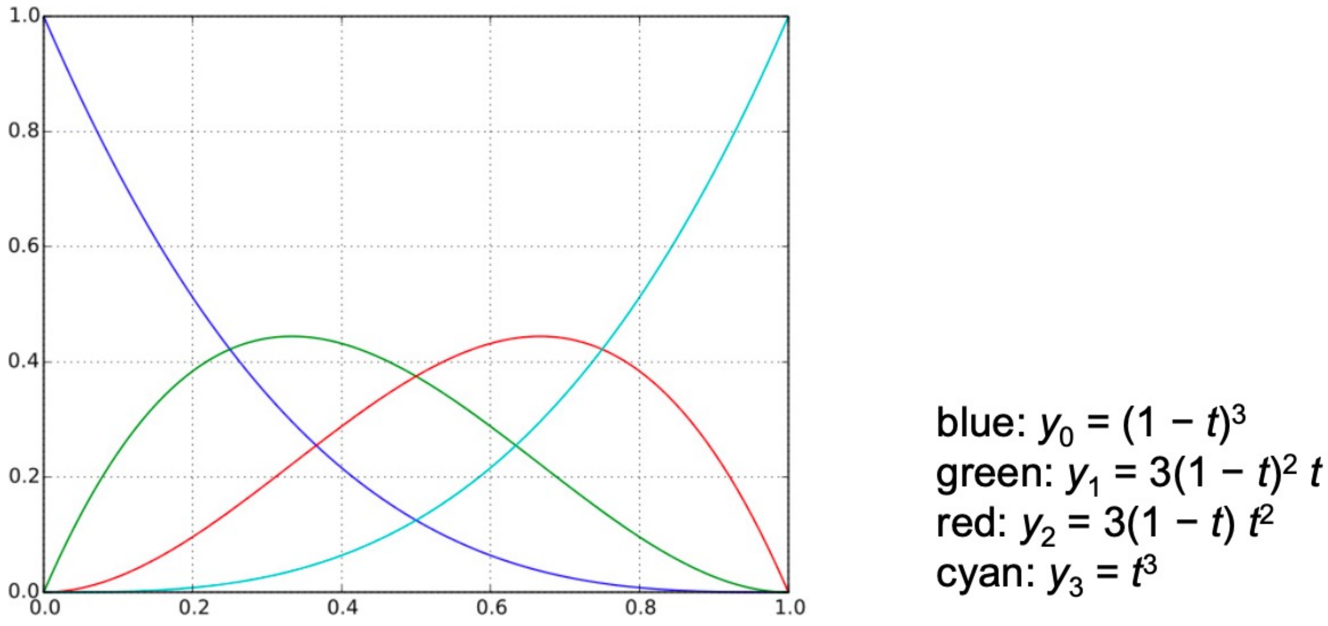
- **B-Spline Curve**
  - Formulation
  - Point evaluation
  - Tanget evaluation

- **B-Spline Surface**
  - Formulation
  - Point evaluation based on curve formulation
  - Normal evaluation

B-spline Curve

Motivation

- **Drawback of Bézier curve**
  - Editing one single point will effect the evaluation of the whole curve
  - Basis functions are global over the definition domain



Basis functions of Cubic Bézier Curves for t in [0, 1]

B-spline Curve

- *n*-order B-spline curve with *p+1* control points

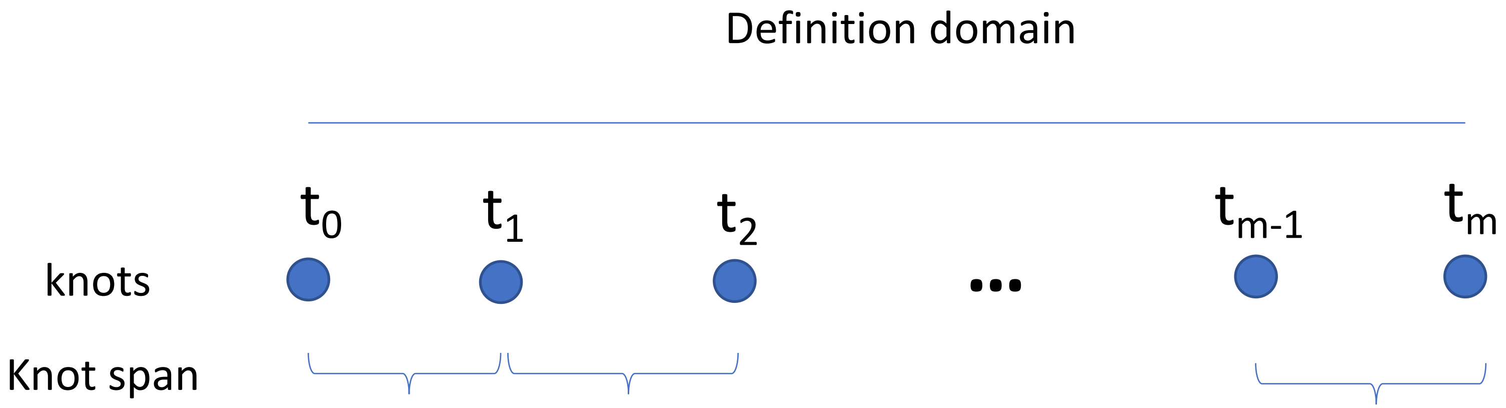
Control Points

$$S(t) = \sum_{j=0}^{p} P_j b_{j,n}(t), t \in [0, 1]$$

Local Basis Function

Basis Function of B-Spline Curve

- *m + 1* Knots

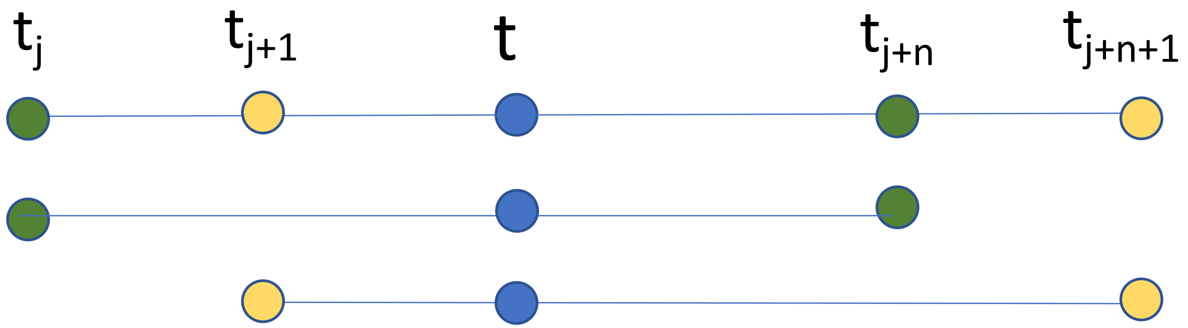


Basis Function of B-Spline Curve

- Recursive definition of $b_{j,n}(t)$

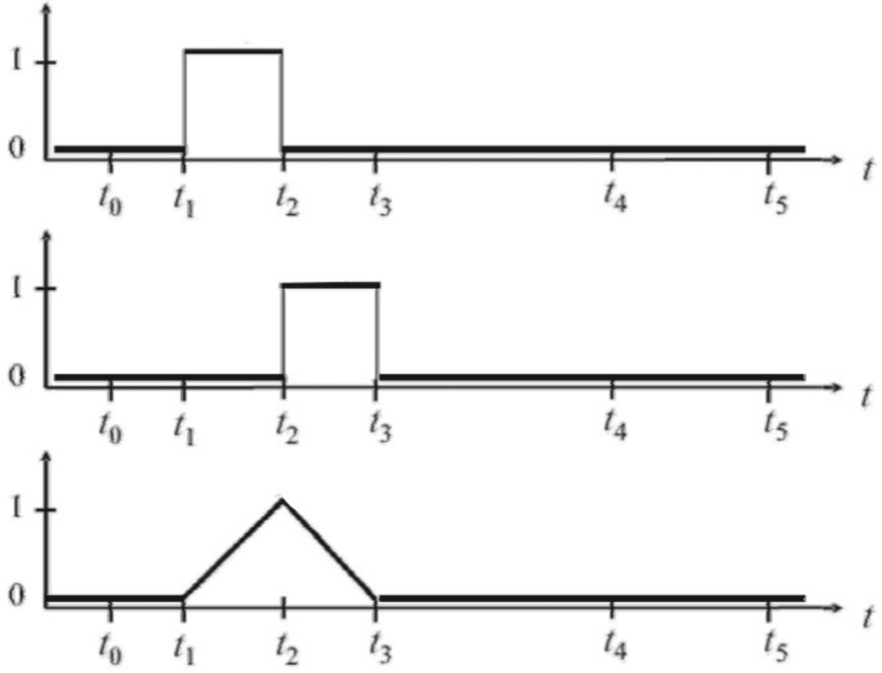
$$b_{j,0}(t) := \begin{cases} 1 & t_j < t < t_{j+1} \\ 0 & \dots \end{cases}$$

$$b_{j,n}(t) := \boxed{\frac{t - t_j}{t_{j+n} - t_j}} b_{j,n-1}(t) + \boxed{\frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}}} b_{j+1,n-1}(t).$$



Basis Function of B-Spline Curve

- **A simple example**
  - $b_{1,\,0}(t)$ is non-zero in [t_1, t_2], and $b_{2,\,0}(t)$ is non-zero in [t_2, t_3]
  - $b_{1,\,1}(t)$ is non-zero in [t_1, t_3]



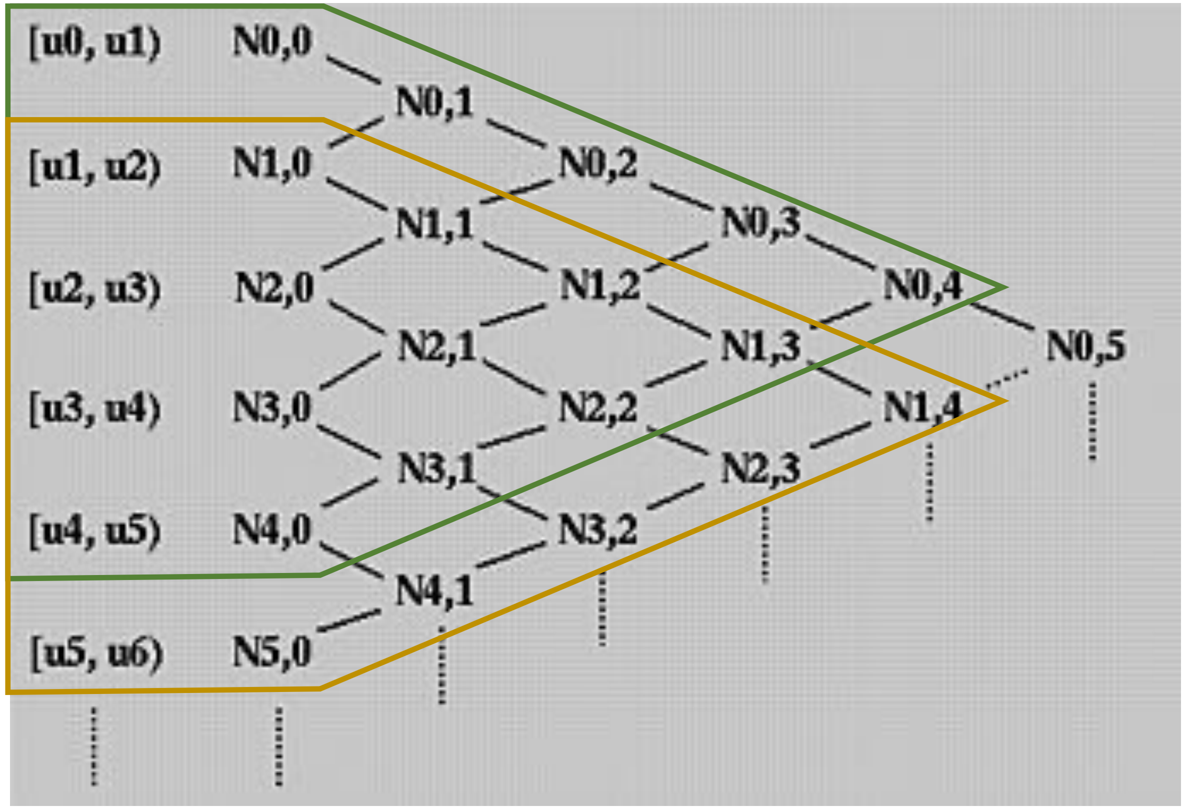
$b_{1,\,0}(t)$

$b_{2,\,0}(t)$

$b_{1,\,1}(t)$

Basis Function of B-Spline Curve

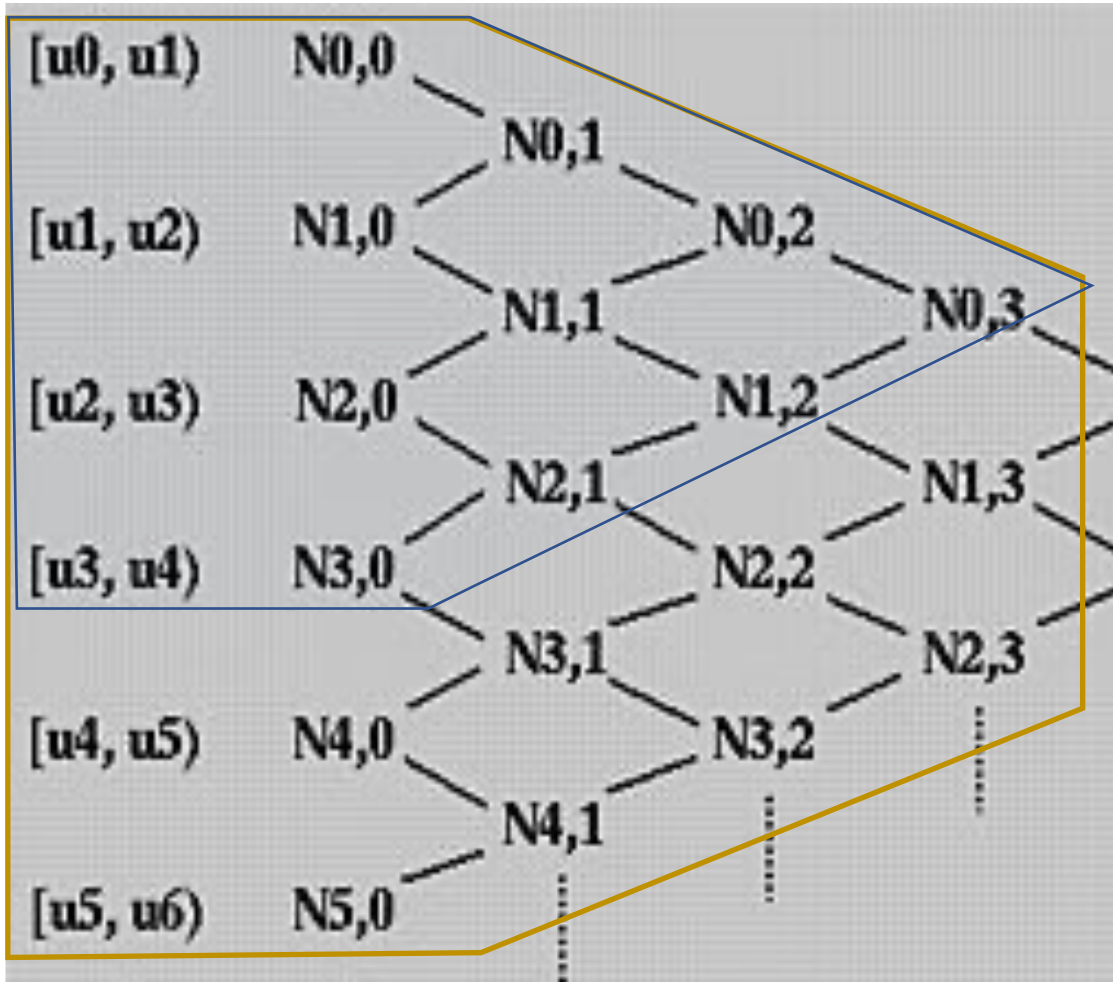
- An illustration on how to compute $b_{0,5}(t)$



- $b_{0,5}(t) = \frac{t-t_0}{t_5-t_0} b_{0,4}(t) + \frac{t_6-t}{t_6-t_1} b_{1,4}(t)$
- Its support is $[t_0, t_6]$

Relationship between *p+1, m+1, n*

- p+1: number of control points
- n: order of basis function
- m+1: number of knots
- Relationship: m = p + n + 1



Derivative of a B-spline Curve

- Formulation

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i$$

$$\frac{\mathrm{d}}{\mathrm{d}u} N_{i,p}(u) = N'_{i,p}(u) = \frac{p}{u_{i+p} - u_i} N_{i,p-1}(u) - \frac{p}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

$$\frac{\mathrm{d}}{\mathrm{d}u} \mathbf{C}(u) = \mathbf{C}'(u) = \sum_{i=0}^{n-1} N_{i+1,p-1}(u)\mathbf{Q}_i$$

$$\mathbf{Q}_i = \frac{p}{u_{i+p+1} - u_{i+1}} (\mathbf{P}_{i+1} - \mathbf{P}_i)$$

The derivative of a B-spline curve is another B-spline curve of degree p - 1 on the original knot vector with a new set of n control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, ..., $\mathbf{Q}_{n-1}$

https://pages.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/bspline-derv.html

Derivative of a B-spline Curve

- Finite difference method

$$\boldsymbol{C}'(u) = \frac{\boldsymbol{C}(u + \delta u) - \boldsymbol{C}(u)}{\delta u}$$

B-spline Surface

B-spline Surfaces: Construction

- **What we needs**
  - a set of m+1 rows and n+1 control points $\mathbf{p}_{i,j}$
    - where 0 <= i <= m and 0 <= j <= n;
  - a knot vector of $h + 1$ knots in the u-direction, $U = \{ u_0, u_1, ..., u_h \}$;
  - a knot vector of $k + 1$ knots in the v-direction, $V = \{ v_0, v_1, ..., v_k \}$;
  - the degree p in the u-direction;
  - the degree q in the v-direction;
- **We also have**
  - $h = m + p + 1$
  - $k = n + q + 1$

Evaluation position

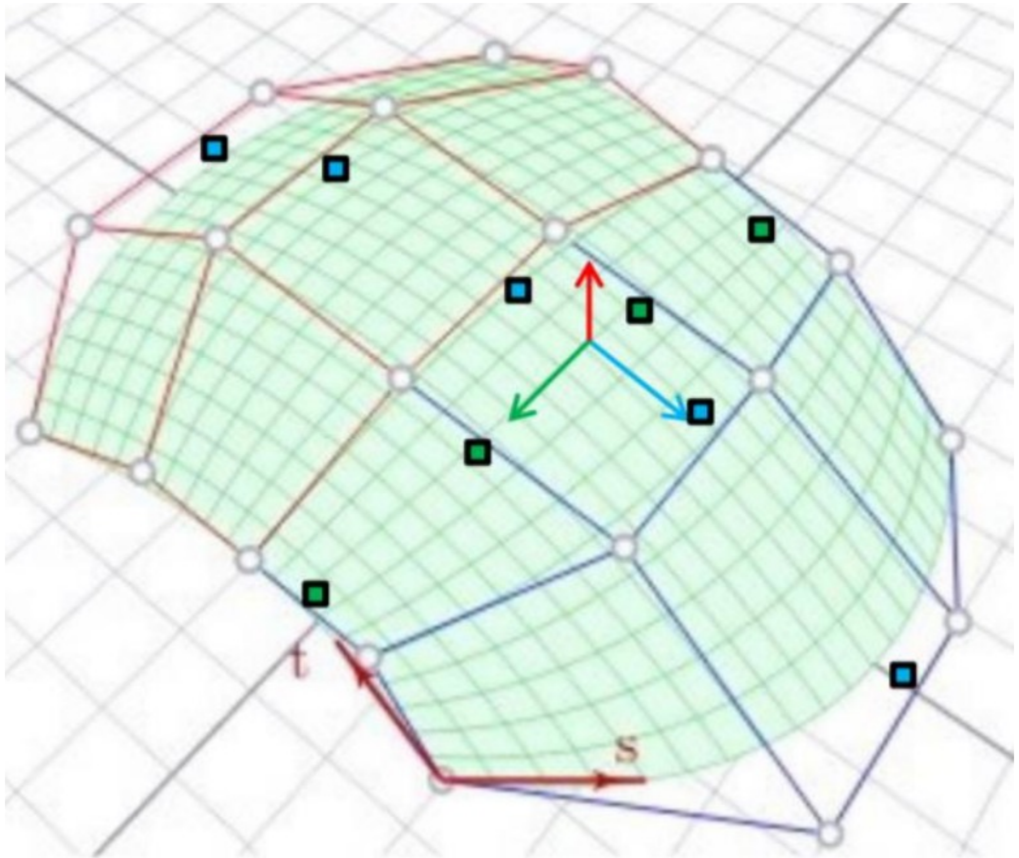
- Evaluate points in B-spline surface

$$\mathbf{p}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} N_{i,p}(u) N_{j,q}(v) \mathbf{k}_{i,j}$$

- First evaluate along u direction,
- Then evaluate along v direction

Normal Evaluation

- Similar to the normal evaluation of Bézier surface



Thanks