Lab 3 Analysis of Periodic Signals in the Frequency Domain

Objective

- Explore the relationship between the time domain and the frequency domain.
- Understand the Fourier series of periodic signals.
- Master the expression of the signal spectrum.

Content

Fourier Series of Periodic Signal

Fourier analysis can be divided into Fourier series and Fourier transform. The former is used to represent a periodic signal by a discrete sum of complex exponentials, while the latter is used to represent an aperiodic signal by a continuous superposition or integral of complex exponentials.

According to the theory of Fourier analysis, periodic signals which satisfy the Dirichlet conditions can be decomposed into sinusoid signals with different magnitude and phase.
The trigonometric Fourier series is shown as follows:

$$\mathrm{f}(t) = a_{0+} \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$$a_0 = \frac{1}{T_1} \int_{t_0}^{t_0+T1} f(t)\, dt$$

$$a_n = \frac{2}{T_1} \int_{t_0}^{t_0+T1} f(t) \cos n\omega_1 t\, dt$$

$$b_n = \frac{2}{T_1} \int_{t_0}^{t_0+T1} f(t) \sin n\omega_1 t\, dt$$

Example: The periodic rectangular pulse is shown in Figure 1. Calculate the Fourier series and find out the relationship between the rectangular pulse and its harmonics.

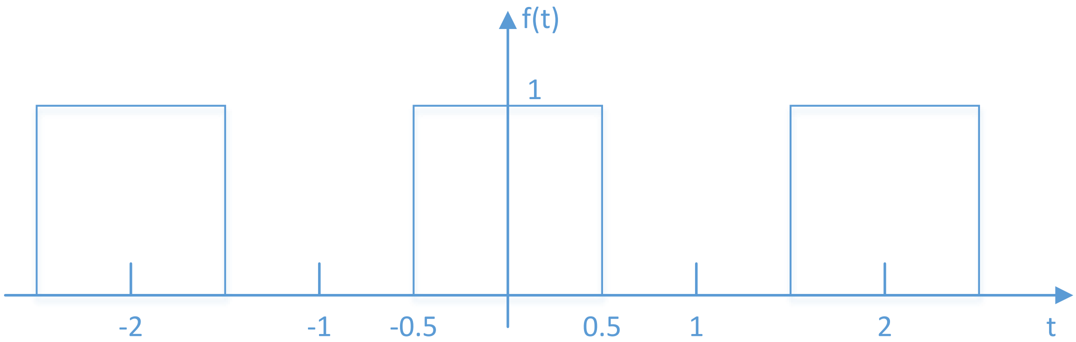

Figure 1 Periodic Rectangular Pulse

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{2}\int_{-0.5}^{0.5} 1dt = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t dt = \int_{-0.5}^{0.5} \cos n\pi t\, dt = \frac{1}{n\pi}\sin n\pi t \Big|_{-0.5}^{0.5} = \frac{2}{n\pi}\sin\frac{n\pi}{2} = Sa(\frac{n\pi}{2})$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t dt = \int_{-0.5}^{0.5} \sin n\pi t\, dt = -\frac{1}{n\pi}\cos n\pi t \Big|_{-0.5}^{0.5} = 0$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} Sa(\frac{n\pi}{2})\cos n\pi t\,, \omega = \frac{2\pi}{T} = \pi$$

Realize Fourier Series with Numeric Method

Fourier Series with Loop Structure:

```
% numeric method: loop function, trapz
clear;clf;
T = 2; f = 1/T; w1 = 2*pi*f;
dt = 0.01;
t = -2:dt:2;
tao = -0.5:dt:0.5;
a0 = trapz(tao,tao.^0)/T;
f = a0;
N = input('N=');
an = zeros(1,N);
bn = zeros(1,N);
for n = 1:N
    fcos = 1.*cos(n*w1*tao); an(n)=trapz(tao,fcos)*2/T;
    fsin = 1.*sin(n*w1*tao); bn(n)=trapz(tao,fsin)*2/T;
    f = f+ an(n)*cos(n*w1*t)+bn(n)*sin(n*w1*t);
end
```

```
plot(t,f);xlabel('t(s)');ylabel('ft'); grid on;
title(['Numeric Loop with N=' num2str(N)]);
```

Fourier Series with Matrix Operation:

Except loop structure, we can also calculate the Fourier Series by the advantage of matrix.

For $a_n = \frac{2}{T_1}\int_{t_0}^{t_0+T1} f(t)\cos n\omega_1 t\,dt$, let ω_n be $n\omega_1$, let t_m be $(0:\mathrm{m}-1) * \mathrm{dt}$.

Then we can change a_n to a matrix like this:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(t)\bullet\cos w_1 t \\ f(t)\bullet\cos w_2 t \\ \vdots \\ f(t)\bullet\cos w_n t \end{bmatrix} dt$$

$$= \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(t_1)\bullet\cos w_1 t_1 & f(t_2)\bullet\cos w_1 t_2 & \cdots & f(t_m)\bullet\cos w_1 t_m \\ f(t_1)\bullet\cos w_2 t_1 & f(t_2)\bullet\cos w_2 t_2 & \cdots & f(t_m)\bullet\cos w_2 t_m \\ \vdots & \vdots & \ddots & \vdots \\ f(t_1)\bullet\cos w_n t_1 & f(t_2)\bullet\cos w_n t_2 & \cdots & f(t_m)\bullet\cos w_n t_m \end{bmatrix} dt$$

$$= \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+\mathrm{T}_1} f(t)\cdot *\cos\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} * \begin{bmatrix} t_1 & t_2 & \cdots & t_m \end{bmatrix}\right) dt$$

When calculating $\sum_{n=1}^{\infty}(a_n cos w_n t)$, we can also change it to the matrix calculation, like:

$$\sum_{n=1}^{\infty}(a_n \cos w_n t)$$

$$= [a_1 \cos w_1 t + a_2 \cos w_2 t + \cdots a_n \cos w_n t]$$

$$= \begin{bmatrix} a_1 \cos w_1 t_1 + a_2 \cos w_2 t_1 + \cdots a_n \cos w_n t_1 \\ a_1 \cos w_1 t_2 + a_2 \cos w_2 t_2 + \cdots a_n \cos w_n t_2 \\ \vdots \\ a_1 \cos w_1 t_m + a_2 \cos w_2 t_m + \cdots a_n \cos w_n t_m \end{bmatrix}$$

$$= \begin{bmatrix} \cos w_1 t_1 & \cos w_2 t_1 & \cdots & \cos w_n t_1 \\ \cos w_1 t_2 & \cos w_2 t_2 & \cdots & \cos w_n t_2 \\ \vdots & \vdots & \ddots & \vdots \\ \cos w_1 t_m & \cos w_2 t_m & \cdots & \cos w_n t_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \cos\left(\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

```
% numeric method: matrix calculation
clear; clf;
T = 2; f = 1/T; w1 = 2*pi*f;
dt = 0.01;
t = -2:dt:2;
tao = -0.5:dt:0.5;
a0 = trapz(tao,tao.^0)/T;
N = input('N=');
n = 1:N;
fcos = 1.*cos(n'*w1*tao); an = trapz(tao,fcos,2)*2/T;
fsin = 1.*sin(n'*w1*tao); bn = trapz(tao,fsin,2)*2/T;
f = a0 + an'*cos(n'*w1*t) + bn'*sin(n'*w1*t);
plot(t,f); xlabel('t(s)');ylabel('ft'); grid on;
title(['Numeric Matrix with N=' num2str(N)]);
```

Realize the Fourier Series with Symbolic Method

Calculate the Fourier Series with MATLAB (symbolic method):

```
% Symbolic method: loop function, int
clear; clf;
N = input('N=');
syms t1 a0 an bn n
T1 = 2; f = 1/T1; w1 = 2*pi*f;
range = [-0.5,0.5];
a0 = 1/T1*int(t1^0,t1,range);
```

```
f = a0;
for n=1:N
    an = 2/T1*int(cos(n*w1*t1),t1,range);
    bn = 2/T1*int(sin(n*w1*t1),t1,range);
    f = f+an*cos(n*w1*t1)+bn*sin(n*w1*t1);
end
fplot(f); xlabel('t');ylabel('y(t)');
title(['Symbolic Loop with N=' num2str(N)]);
axis([-2,2,-0.2,1.2]); grid on;
```

1. Try to compare the three results by plotting them.
2. Change the value of N. What can you find?

Gibbs Phenomenon

Generally, a signal can be reconstructed with a small number of Fourier series if the original signal is smooth. For discontinuous signals, plenty of high-frequency components are required to reconstruct the signal accurately.

For signals with discontinuities, which are infinite in the frequency domain, only a portion of the Fourier series will be superimposed during reconstruction. As a result, there will a substantial overshoot near these discontinuities. As N increases, overshoot will occur at smaller and smaller intervals. However, the increase in N does not reduce the magnitude of the overshoot. This is called the Gibbs phenomenon, which is shown in figure 2.

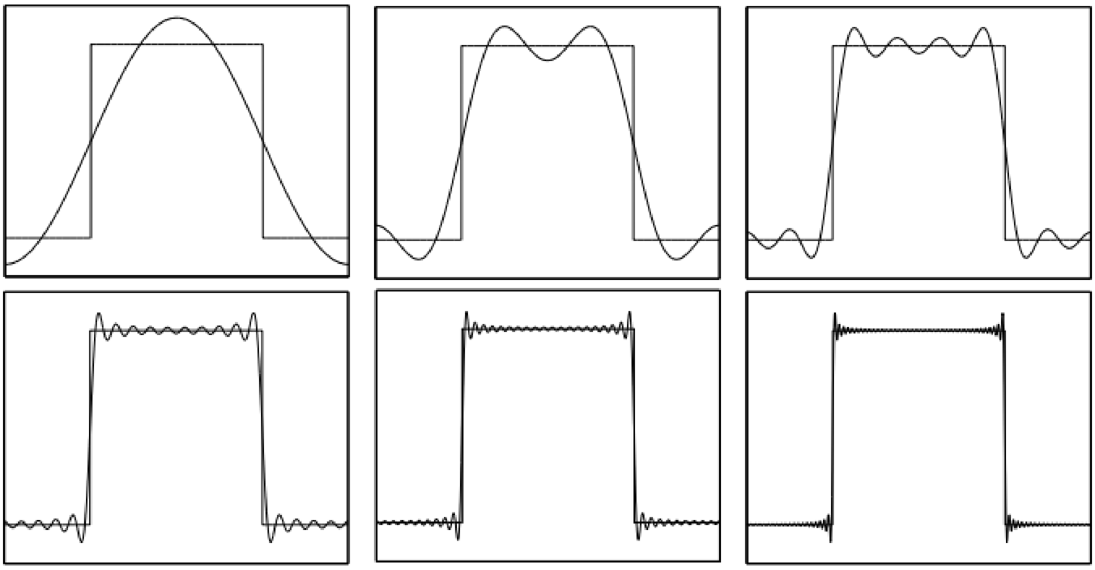

Figure 2 Gibbs Phenomenon

Frequency Analysis of Periodic Signal

Another format of trigonometric Fourier series is like:

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} cos\omega_1 t - \frac{b_n}{\sqrt{a_n^2 + b_n^2}} sin\omega_1 t \right]$$

$$= a_0 + \sum_{n=1}^{\infty} c_n (cos\varphi_n cosn\omega_1 t - sin\varphi_n sinn\omega_1 t)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_1 t + \varphi_n)$$

$$c_0 = a_0$$

$$c_n = \sqrt{a_n^2 + b_n^2}, n = 1,2, \dots \dots$$

$$\varphi_n = -\tan^{-1} \frac{b_n}{a_n}$$

The unilateral spectrum of the signal can be calculated by a trigonometric Fourier series.

Periodic signals can also be expanded into a Fourier series of complex exponential form, which is like:

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \frac{1}{2} \left[e^{jn\omega_1 t} + e^{-jn\omega_1 t} \right] + b_n \cdot \frac{-j}{2} \left[e^{jn\omega_1 t} + e^{-jn\omega_1 t} \right] \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} e^{jn\omega_1 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_1 t} \right)$$

since $a_n = a_{-n}, b_n = -b_{-n}$

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} e^{jn\omega_1 t} + \frac{a_{-n} - jb_{-n}}{2} e^{-jn\omega_1 t} \right)$$

set

$$F_n = \frac{a_n - jb_n}{2}, F_{-n} = \frac{a_{-n} - jb_{-n}}{2}$$

then

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty} \left(F_n e^{jn\omega_1 t} + F_{-n} e^{-jn\omega_1 t} \right) = a_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(F_n e^{jn\omega_1 t} \right)$$

$$F_n = \frac{1}{2} \left[\frac{2}{T_1} \int_{t_0}^{t_0 + T_1} f(t)(cosn\omega_1 t - jsinn\omega_1 t) dt \right] = \frac{1}{T_1} \int_{t_0}^{t_0 + T1} f(t) e^{-jn\omega_1 t} \, dt$$

$$F_0 = \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} f(t) dt = a_0$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_1 t}$$

Different from the trigonometric form of the Fourier series, the exponential form displays the spectral in a bilateral spectral mode.

The relationship of complex exponential Fourier series and trigonometric Fourier series is:

$$F_0 = a_0 = c_0$$

$$F_n = |F_n| e^{j\varphi_n} = \frac{1}{2}(a_n - jb_n) = \frac{1}{2} c_n e^{j\varphi_n}$$

$$|F_n| = \frac{1}{2}\sqrt{{a_n}^2 + {b_n}^2} = \frac{1}{2} c_n$$

$$\varphi_n = -\tan^{-1}\frac{b_n}{a_n}$$

So F_n is the complex spectrum of $f(t)$. From F_n, we can find out the amplitude-frequency and phase-frequency characteristics by using function **abs**() and **angle**().

To find out the amplitude-frequency and phase-frequency characteristics of $f(t)$, follow the steps shown in Figure 3.

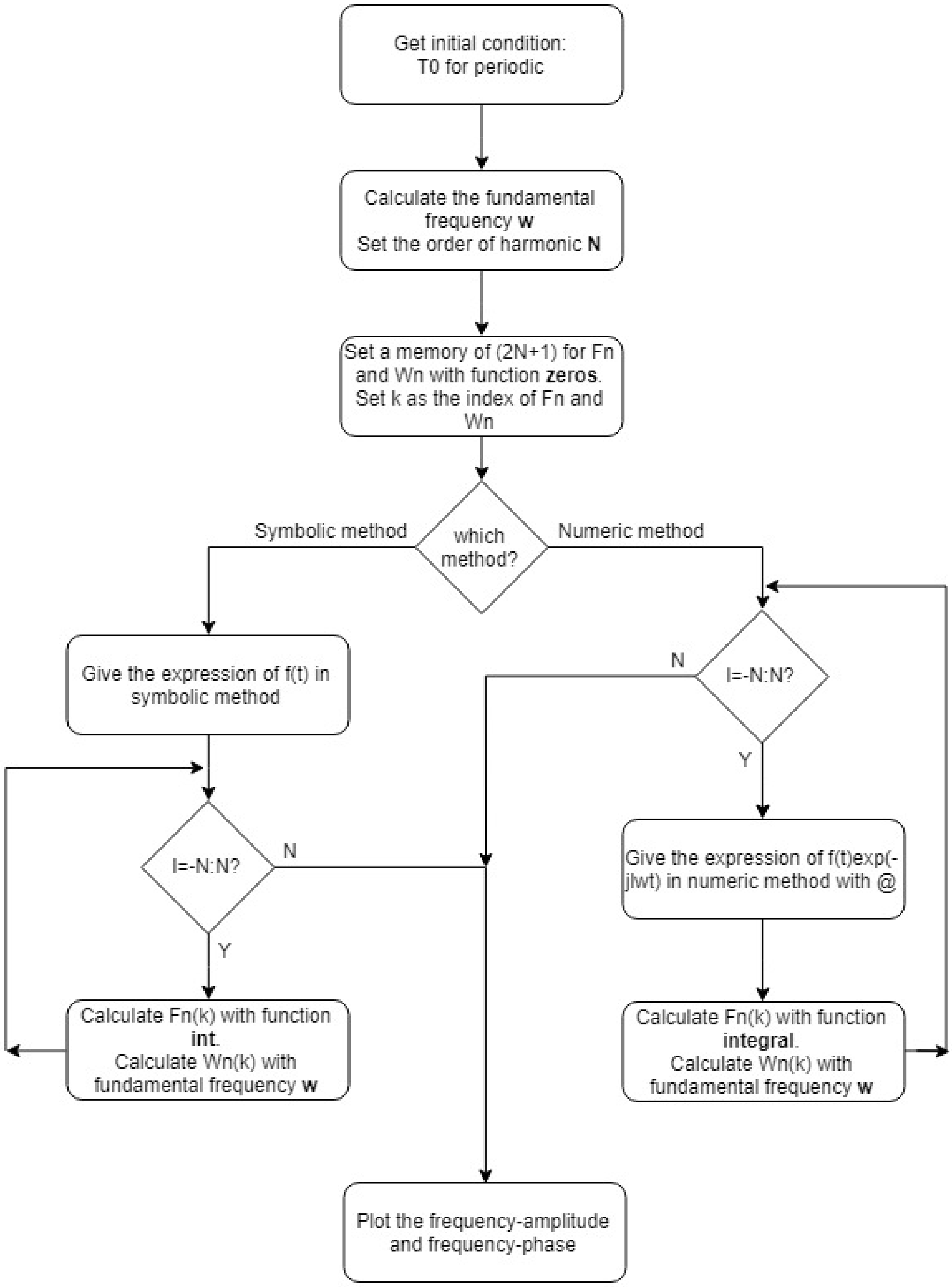

Figure 3 Steps to Analyze the Frequency Characteristic

Tips: when we use the numeric method to find out frequency-phase characteristics with function **angle( )**, since only a finite length of data can be calculated, the introduction of error cannot be avoided. Usually, the data less than 1e-10 can be ignored. So we can solve this problem by multiplying a logical value that determines whether the value is less than 1e-10, like:

```
Fang = angle(Fn).*(abs(Fn)>=1e-10);
```