Discussion 5

- Second-Order Circuits

11/03/2016

Outline

- Review & Extension
  - Solving First-Order equation
  - Solving Second-Order equation
  - Mid-term analysis
- Q&A

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    - Cases analysis
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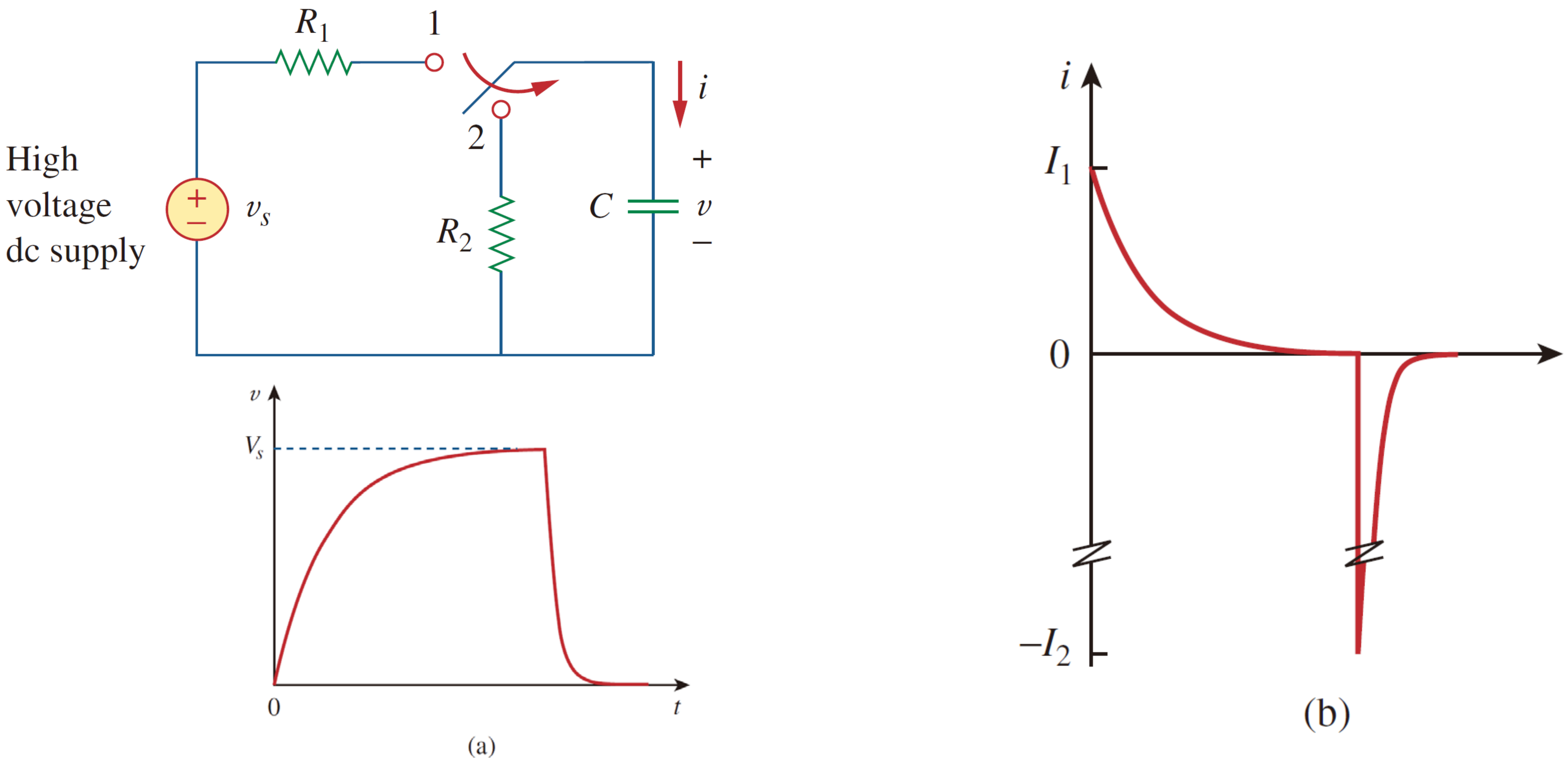
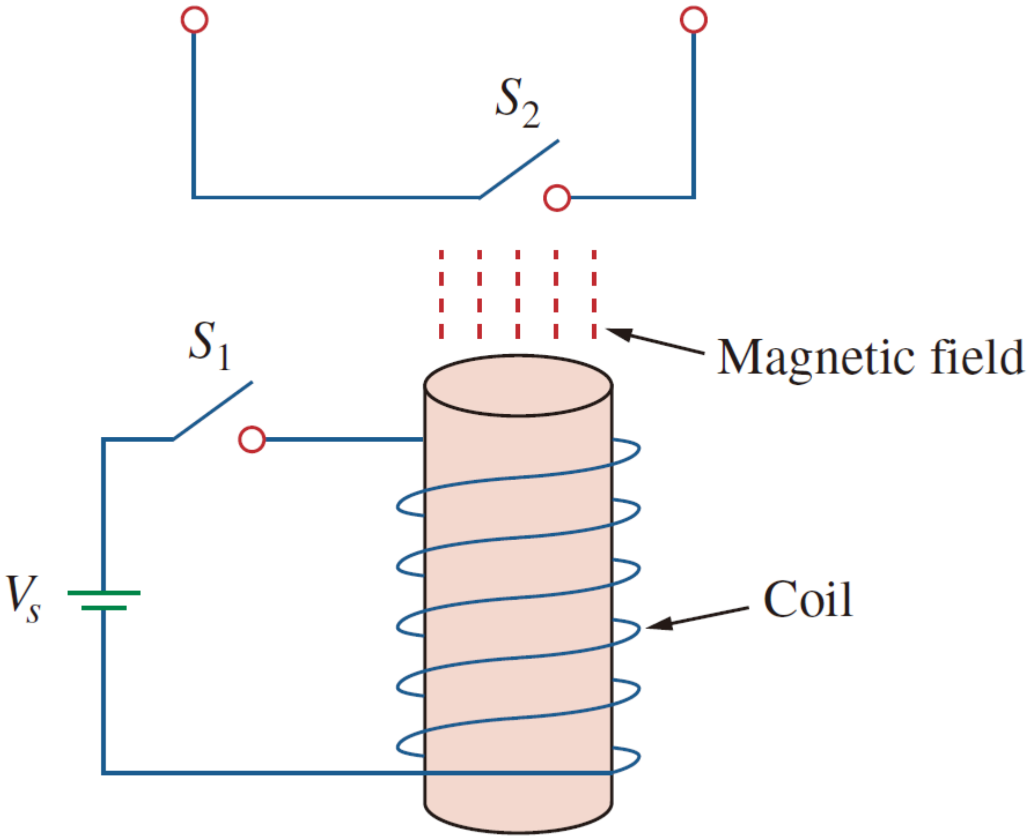
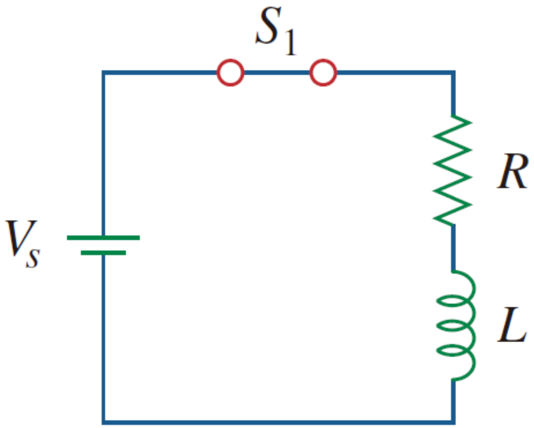
Photoflash Unit



Relay Circuits



(a)



(b)

First Order Linear Equations

- Consider $$\dot{x} + p(t)x = q(t) \qquad (1)$$
  - Note:
  - 1. the equation is inhomogeneous when $q(t) \neq 0$ and
  - and when homogeneous when $q(t) = 0$

$$\dot{x} + p(t)x = 0 \qquad (2)$$

  - We will call this the associated homogeneous equation to the inhomogeneous equation (1)

First Order Linear Equations

- Solutions to the Homogeneous Equation
  - $\dot{x} + p(t)x = 0 \qquad (2)$
  - Separate variables $\frac{dx}{x} = -p(t)dt$
  - Integrate $ln|x| = -\int p(t)dt + C_1$
  - Exponentiate $|x| = e^{C_1}e^{-\int p(t)dt}$
  - $|x| = Ce^{-\int p(t)dt} \qquad C > 0$
  - Drop the absolute value and recover the lost solution $x(t) = 0$
  - This gives the general solution to (2)
    - $x(t) = Ce^{-\int p(t)dt}\ where\ C = any\ value$

First Order Linear Equations

- Solutions to the Homogeneous Equation
- Example-3 solve $\dot{x} + 2tx = 0$
  - Separate variables $\frac{dx}{x} = -2tdt$
  - Integrate $ln|x| = -\int 2tdt = -t^2 + C_1$
  - Exponentiate $|x| = e^{C_1}e^{-t^2} = Ce^{-t^2}$
  - Drop the absolute value and recover the lost solution $x(t) = Ce^{-t^2}$
  - This gives the general solution to
    - $x(t) = Ce^{-t^2} where\ C = any\ value$

First Order Linear Equations

- Solutions to the Inhomogeneous Equation
- integrating factors formula: the general solution to the inhomogeneous first order linear ODE($\dot{x} + p(t)x = q(t)$) is
    - $$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), where\ u(t) = e^{\int p(t)dt}$$

First Order Linear Equations

- Example-4: solve ODE $\dot{x} + 2x = e^{3t}$ using the method of integrating factors
- Solution: Multiply both sides by u
  - $u(t)\dot{x}(t) + 2u(t)x(t) = u(t)e^{3t}$ (3)
  - Next, find an integrating factor u so that the left-hand side is equal to $\frac{d}{dt}(ux)$
    - $u\dot{x} + \dot{u}x = \dot{u}x + 2ux \Rightarrow \dot{u} = 2u \quad we\ choose\ u(t) = e^{2t}$

First Order Linear Equations

- Example-4: continued
  - Now substitute $u(t) = e^{2t}$ into (3) replace the left-hand side by
  - $\frac{d}{dt}(ux)\ and\ solve\ for\ x, \frac{d}{dt}(e^{2t}x) = e^{2t}e^{3t} \quad \Rightarrow e^{2t}x = \frac{1}{5}e^{5t} + C$
  - $\Rightarrow x(t) = \frac{1}{5}e^{3t} + Ce^{-2t}$

First Order Linear Equations

- Example-4: continued
  - When we use integrating factors directly.
    - Integrating factor: $u(t) = e^{\int 2dt} = e^{2t}$
    - Solution :$x(t) = \frac{1}{u(t)}\int u(t)e^{3t}dt = e^{-2t}\int e^{5t}dt = e^{-2t}\left(\frac{1}{5}e^{5t} + C\right) = \frac{1}{5}e^{3t} + Ce^{-2t}$

First Order Linear Equations

- Other methods -1
  - For $y' + p(x)y = q(x)\ and\ y' + p(x)y = 0$
    - $y = e^{-\int p(x)dx}\left[C + \int q(x)e^{\int p(x)dx}dx\right]$ $\quad (inhomogeneous\ equation)$
    - $y = Ce^{-\int p(x)dx}$ $\quad (homogeneous\ equation)$

First Order Linear Equations

- Other methods -2
  - For $y' + p(x)y = q(x)\ and\ y' + p(x)y = 0$
    - $First\ get\ the\ general\ solution\ of\ \ homogeneous\ equation\ y = Ce^{-\int p(x)dx}$
    - $Next\ let\ C = C(x)\ then\ y = C(x)e^{-\int p(x)dx}, substitute\ into\ the\ original\ equation$
    - $\Rightarrow C'(x)e^{-\int p(x)dx} = q(x)\ \ then\ integrate\ to\ get\ C(x)$

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Second Order Linear Equations

- Homogeneous ODEs

$$mx'' + bx' + kx = 0 \quad where\ m, b, k\ constant, m \neq 0$$

Note:

- The general solution is of the form $x(t) = c_1x_1(t) + c_2x_2(t)$ where x_1 and x_2 are two linearly independent solutions(none of them can be written as a constant multiple of the other)
- The characteristic polynomial of this equations is $\mathrm{p}(s) = ms^2 + bs + k$
- The exponential solutions of this equation are $c_1e^{r_1t}\ and\ c_2e^{r_2t}$,where $r_1, r_2\ are\ the\ roots\ of$ the characteristic polynomial and c_1 and c_2 are arbitrary constants. If $r_1 = r_2 = \mathrm{r}$, there is only one family of exponential solutions, namely ce^{rt}

Second Order Linear Equations

- Homogeneous ODEs

$$mx'' + bx' + kx = 0 \quad where\ m, b, k\ constant, m \neq 0$$

Solve :

- 1. Write down the characteristic equation $ms^2 + bs + k = 0$
- 2. Compute its discriminant $\Delta = b^2 - 4mk$
- 3.There are three possible situations:
  - $\Delta > 0 \quad \Rightarrow$ two distinct real solutions $\Rightarrow x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - $\Delta = 0 \quad \Rightarrow$ only one real root r $\Rightarrow x = c_1 e^{rt} + c_2 t e^{rt}$
  - $\Delta < 0 \quad \Rightarrow$ two complex conjugate roots $\Rightarrow x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - $r_{1,2} = \alpha \pm i\beta\ where\ \alpha = -\frac{b}{2m}\ and\ \beta = \frac{\sqrt{|\Delta|}}{2m}$ the general solution
    - $x = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$

Second Order Linear Equations

- Inhomogeneous ODEs

$$mx'' + bx' + kx = f(t) \; where \; f(t) \neq 0$$

Solve :

- We solve these by finding a particular solution x_p of the given ODE and the solution x_h to the corresponding homogeneous equation $mx'' + bx' + kx = 0$
- The general solution is given by $\mathrm{x} = x_h + x_p$
- But how to find x_p?

Second Order Linear Equations

- $mx'' + bx' + kx = f(t)\ where\ f(t) \neq 0$

$f(t)$	x_p
$f(t) = P_n(t)$, P_n is polynomial	$0\ isn't\ the\ root \Rightarrow x_p = H_n(t)$ $0\ is\ the\ simple\ root \Rightarrow x_p = tH_n(t)$ $0\ is\ one\ twofold\ root \Rightarrow x_p = t^2 H_n(t)$
$f(t) = P_n(t)e^{\alpha t}$	$\alpha\ isn't\ the\ root \Rightarrow x_p = H_n(t)\, e^{\alpha t}$ $\alpha\ is\ the\ simple\ root \Rightarrow x_p = tH_n(t)\, e^{\alpha t}$ $\alpha\ is\ one\ twofold\ rootroot \Rightarrow x_p = t^2 H_n(t)\, e^{\alpha t}$
$f(t) = e^{\alpha t}\begin{bmatrix} P_n(t)\sin\beta t \\ + \\ Q_m(t)\cos\beta t \end{bmatrix}$	$\alpha \pm i\beta\ isn't\ the\ root \Rightarrow x_p = e^{\alpha t}[R_l(t)cos\beta t + S_l(t)sin\beta t]$ $\alpha \pm i\beta\ is\ the\ root \Rightarrow x_p = te^{\alpha t}[R_l(t)cos\beta t + S_l(t)sin\beta t]$

Second Order Linear Equations

Example -5 Solve $y'' - 3y' = 2 - 6x$

Solution:

- 1.get the general solution corresponding to homogeneous ODE
  - $\lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0 \quad \Rightarrow y_h\,(x) = C_1 + C_2 e^{3x}$
- 2.get the $y_p\,(x)$ according to the text in previous page, let $y_p\,(x) = x(Ax + B)$
  - Then $[y_p\,(x)]'' - 3[y_p\,(x)]' = 2A - 3(2Ax + B) = -6Ax + 2A - 3B = 2 - 6x$
  - $\begin{cases} 2A - 3B = 2 \\ 6A = 6 \end{cases} yields\ A = 1, B = 0 \quad \Rightarrow y_p\,(x) = x^2$
  - General solution : $y(x) = x^2 + C_1 + C_2 e^{3x}$ $where\ C_1\ and\ C_2\ are\ constants$

Solving second order equation

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}}v = \frac{v_s}{LC}$$

First solving the Eigen-function of and Eigenvalues of a second order formula

And there are THREE cases you should know

Case 1: Overdamped ($\alpha > \omega_0$)

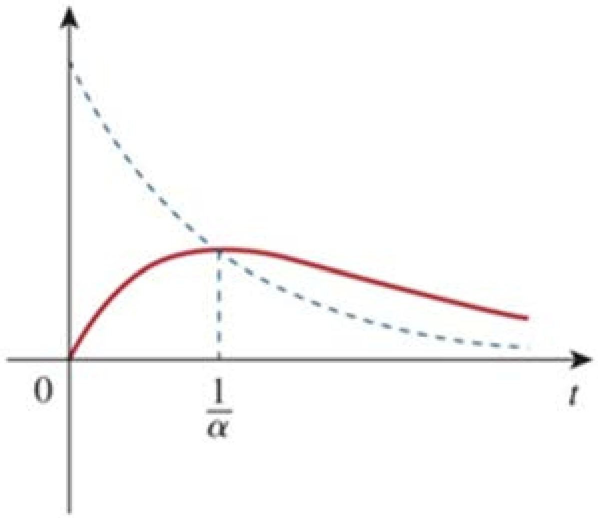
$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Case 2: Critically Damped ($\alpha=\omega_0$)

$$v(t) = (A_1 t + A_2)e^{-\alpha t} \qquad \alpha = \frac{R}{2L}$$



Case 3: Underdamped ($\alpha<\omega_0$)

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

- ω_0 is often called the undamped natural frequency.
- ω_d is called the damped natural frequency.

The natural response

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Recall Euler's formula $e^{ix} = cosx + isinx$

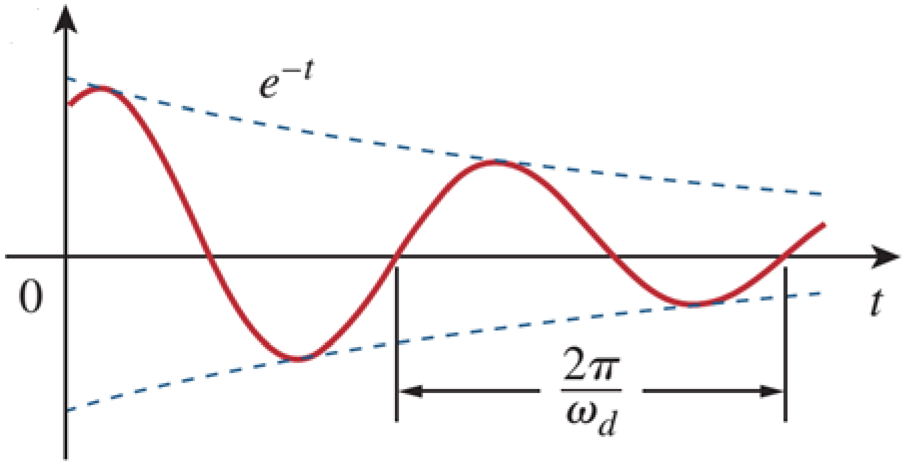
becomes

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Case 3: Underdamped ($\alpha<\omega_0$)

$$\mathrm{v}(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

- Exponential $e^{-\alpha t} *$ Sine/Cosine term
  - Exponentially damped, time constant = $1/\alpha$
  - Oscillatory, period $T = \frac{2\pi}{\omega_d}$



Think: what if the resistance is zero in the circuit?

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