Name: ___________ Student ID:___________

Signals and Systems Homework 6

Due Time: 23:59 April 27, 2018

Submitted to blackboard online (photos or electronic documents) and to the box in front of SIST 1C 403E (the instructor's office).

Throughout this problem set, the n-th Fourier series coefficient of some function f of period T means $\frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-j2\pi nt/T}\mathrm{d}t$

1. (20') Review the trick to solve the Problem 4 in Mid-Term Exam and solve the following questions:
   (a) (10') Find the Fourier series coefficients a_n of function $f(t)$ of period 2 with $f(t) = \frac{1}{2}(t|t| - t)$ for $t \in [-1, 1]$. You can use, without proof, the fact that the Fourier series coefficients of $g(t)$ of period 2 with $g(t) = |t|$ for $t \in [-1, 1]$ are:
   $$b_n = \begin{cases} \dfrac{-2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \neq 0 \text{ is even} \\ \dfrac{1}{2} & \text{if } n = 0 \end{cases}$$
   if necessary.
   (b) (10') Evaluate $\sum_{m=1}^{\infty} m^{-6}$. (**Provide your reasoning, or you will receive 0 credits**)

2. (40') Let $g_{\alpha,\beta}(t) = \alpha e^{-\beta t^2}$, where α and β are positive real numbers. The following 2 facts can be used without proof when solving the following questions:
   - For all positive real numbers α and β, there exist $\alpha' > 0$ and $\beta' > 0$ such that $g_{\alpha',\beta'}(\omega)$ is equal to the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}(t)$;
   - $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.

   (a) (5') Establish a differential equation containing $g_{\alpha,\beta}(t)$ and its derivative $g'_{\alpha,\beta}(t)$.
   (b) (5') Establish a differential equation containing the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}(t)$ and its derivative $G'_{\alpha,\beta}(j\omega)$.
   (c) (10') Compare the results from (a) and (b) and determine the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}$ by finding $\alpha' > 0$ and $\beta' > 0$ such that $g_{\alpha',\beta'} = G_{\alpha,\beta}$.
   (d) (20') Verify your answers obtained in the previous questions by computing $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ in the following two ways:
      i. (10') Compute $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ by definition;
      ii. (10') Compute $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ by the convolution property.

3. (40') Let $f_a(x) = e^{-a|x|}$ where $a > 0$.
   (a) (10') Determine the Fourier transform $F_a(j\omega)$ of $f_a(x)$.
   (b) (10') Consider $\widetilde{f}_a(x) = \sum_{n=-\infty}^{\infty} f_a(x + n)$. Derive the expression of $\widetilde{f}_a(x)$ and write down the fundamental period of $\widetilde{f}_a(x)$ if it exists.
   (c) (10') Decide the Fourier series coefficients c_n of $\widetilde{f}_a(x)$ if $\widetilde{f}_a(x)$ is periodic.
   (d) (10') How are c_n and $F_a(j\omega)$ related? (*Hint: Observe* $F_a(j2\pi n)$)
   (e) (0') If you have taken a course on *Mathematical Analysis*, think about why the your observation is valid. (*Hint: Weierstrass M-test*)