Problem 1

(20 points) For each pair of sequences in Figure 1, use discrete convolution to find the response to the input $x[n]$ of the linear time-invariant system with impulse response $h[n]$.

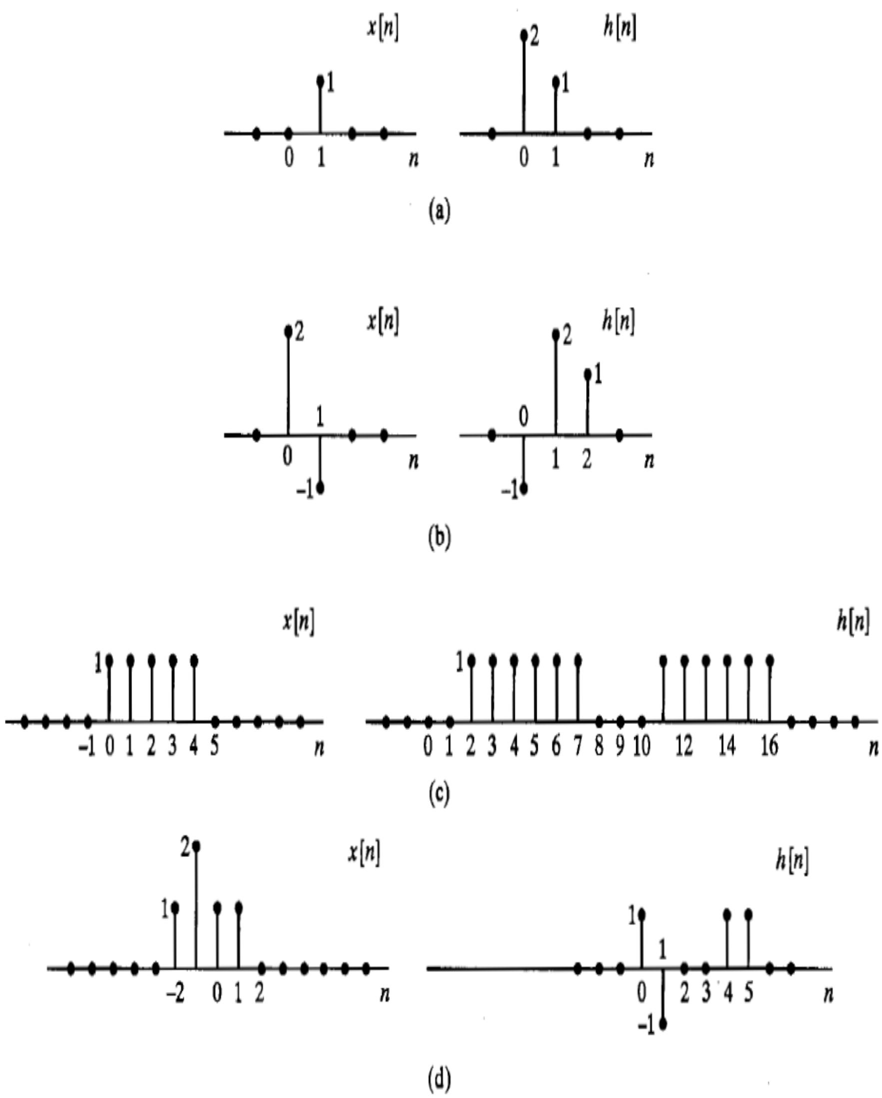


Figure 1: Problem 1(a)

Solution

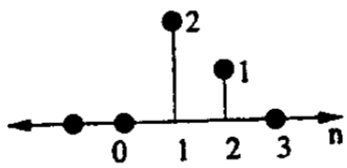
We use the graphical approach to compute the convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(a)

$$y[n] = \delta[n-1] * h[n] = h[n-1]$$

a)算错一个y[n]扣2分



(b)

b)算错一个y[n]扣1分

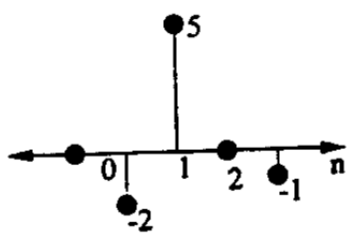
$$y[n] = x[n] * h[n]$$

$$y[0] = \sum_{k=0}^{2} x[k]h[0-k] = x[0]h[0] + x[1]h[-1] + x[2]h[-2] = -2$$

$$y[1] = \sum_{k=0}^{2} x[k]h[1-k] = x[0]h[1] + x[1]h[0] + x[2]h[-1] = 5$$

$$y[2] = \sum_{k=0}^{2} x[k]h[2-k] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 0$$

$$y[3] = \sum_{k=0}^{2} x[k]h[3-k] = x[0]h[3] + x[1]h[2] + x[2]h[1] = -1$$

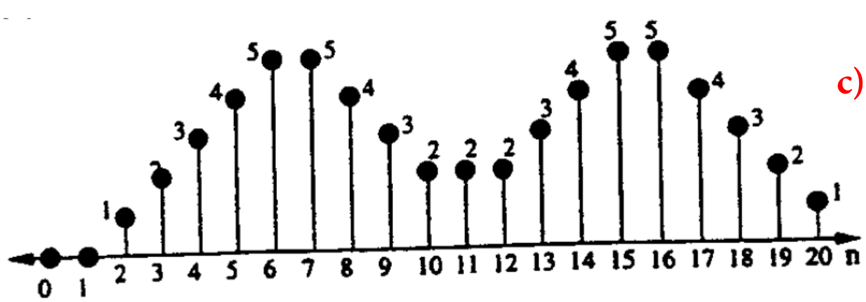


(c)

$$y[n] = x[n] * h[n]$$

$y[1] = 0 \quad y[2] = 1 \quad y[3] = 2 \quad y[4] = 3 \quad y[5] = 4 \quad y[6] = 5 \quad y[7] = 5 \quad y[8] = 4 \quad y[9] = 3 \quad y[10] = 2$

$y[11] = 2 \quad y[12] = 2 \quad y[13] = 3 \quad y[14] = 4 \quad y[15] = 5 \quad y[16] = 5 \quad y[17] = 4 \quad y[18] = 3 \quad y[19] = 2 \quad y[20] = 1$

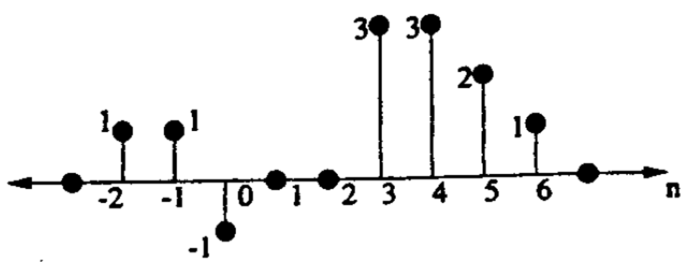


c)算错一个y[n]扣1分

(d)

$$y[n] = x[n] * h[n]$$

$$y[-2] = 1 \quad y[-1] = 1 \quad y[0] = -1 \quad y[1] = 0 \quad y[2] = 0$$
$$y[3] = 3 \quad y[4] = 3 \quad y[5] = 2 \quad y[6] = 1$$



d)算错一个y[n]扣1分

Problem 2

(20 points) For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $x(t)$ and $h(t)$ are as in Figure 2(a).

(b) $x(t)$ and $h(t)$ are as in Figure 2(b).

(c) $x(t)$ and $h(t)$ are as in Figure 2(c).

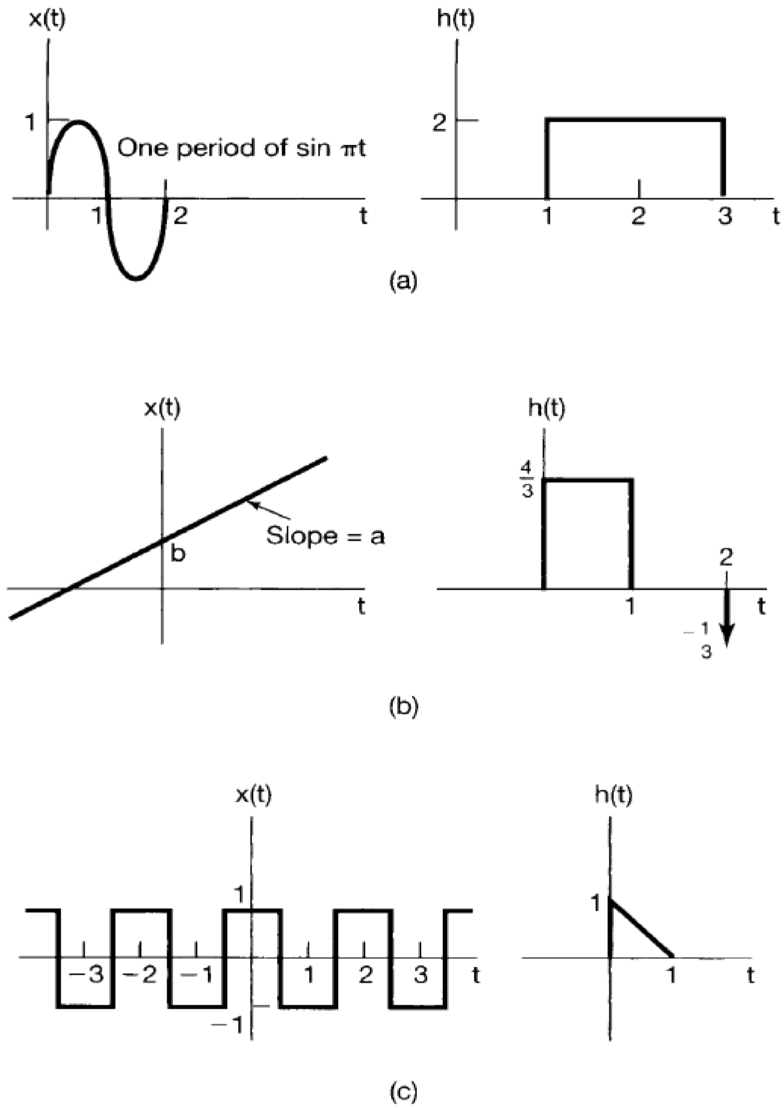


Figure 2: Problem 2

Solution

(a)

$$h(t) = 2u(t-1) - 2u(t-3)$$

$$h(-(\tau - t)) = 2u(-\tau - 1 + t) - 2u(-\tau - 3 + t)$$
$$x(t) = sin(\pi t)[u(t) - u(t-2)]$$

So the disire convolution is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} 2\sin(\pi\tau)[u(\tau) - u(\tau-2)][u(t-\tau-1) - u(t-\tau-3)]d\tau$$

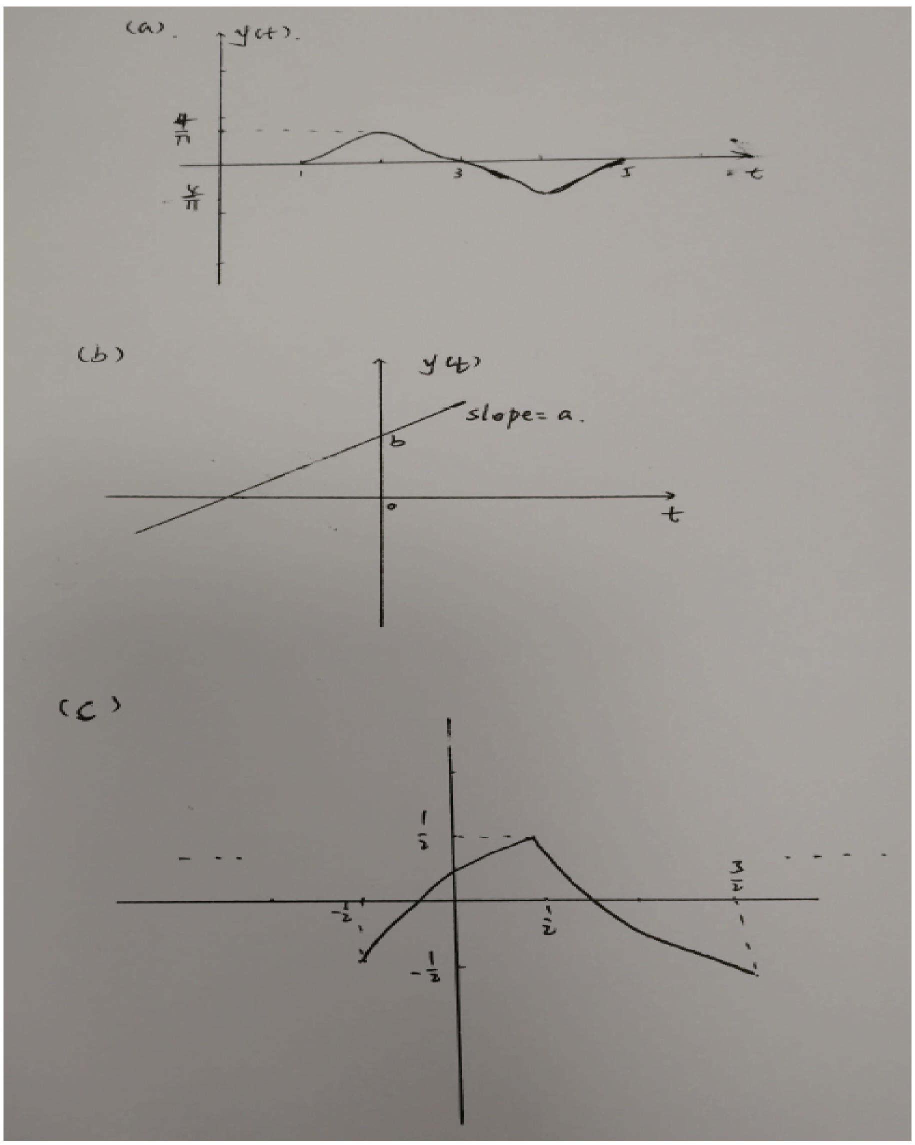
When $1 < t < 3$, we have $y(t) = \int_0^{t-1} 2sin(\pi\tau)d\tau$

When $3 \le t < 5$, we have $y(t) = \int_{t-3}^{2} 2sin(\pi\tau)d\tau$

a)没有画图或者画错扣2分，分类讨论的答案一个2分。

This gives us

$$y(t) = \frac{2}{\pi}[1 - \cos\{\pi(t-1)\}][u(t-1) - u(t-3)] + \frac{2}{\pi}[\cos\{\pi(t-3)\} - 1][u(t-3) - u(t-5)]$$



(b) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2)$$

b)没有画图或者画错扣2分，没有给出最后结果扣2分，过程分3分

where

$$h_1(t) = \frac{4}{3}[u(t) - u(t-1)]$$

Now

$$y(t) = x(t) * h(t) = x(t) * h_1(t) - \frac{1}{3}x(t-2)$$

We have

$$x(t) * h_1(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}at + \frac{4}{3}b - \frac{2}{3}a$$

Therefore,

$$y(t) = \frac{4}{3}at + \frac{4}{3}b - \frac{2}{3}a - \frac{1}{3}[a(t-2)+b] = at + b = x(t)$$

(c) $x(t)$ periodic implies $y(t)$ periodic, so determine 1 period only.

When $-\frac{1}{2} \le t \le \frac{1}{2}$, we have

$$y(t) = \int_{t-1}^{t} x(\tau)h(h-\tau)d\tau$$

$$= \int_{t-1}^{-\frac{1}{2}} -(1-t+\tau)d\tau + \int_{-\frac{1}{2}}^{t} (1-t+\tau)d\tau = \frac{1}{4} + t - t^2$$

When $\frac{1}{2} < t \le \frac{3}{2}$,we have

$$y(t) = \int_{t-1}^{t} x(\tau)h(h-\tau)d\tau$$

$$= \int_{t-1}^{\frac{1}{2}} (1-t+\tau)d\tau + \int_{\frac{1}{2}}^{t} -(1-t+\tau)d\tau = t^2 - 3t + \frac{7}{4}$$

So

$$y(t) = \begin{cases} \int_{t-1}^{-\frac{1}{2}}(t-\tau-1)d\tau + \int_{-\frac{1}{2}}^{t}(1-t+\tau)d\tau = \frac{1}{4} + t - t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-1}^{\frac{1}{2}}(1-t+\tau)d\tau + \int_{\frac{1}{2}}^{t}(t-\tau-1)d\tau = t^2 - 3t + \frac{7}{4}, & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

The period of $y(t)$ is 2.

c)没有画图或者画错扣2分，分类讨论的答案一个2分，过程分1分。

Problem 3

(20 points) Let the signal

$$y[n] = x[n] * h[n]$$

where

$$x[n] = 3^n u[-n-1] + (\frac{1}{3})^n u[n]$$

and

$$h[n] = (\frac{1}{4})^n u[n+3]$$

(a) Determine y[n] without utilizing the distributive property of convolution.

(b) Determine y[n] utilizing the distributive property of convolution.

a)分别写出y[n]的两部分卷积式各给3分，给出结果的两部分各给2分，未进行分类讨论扣2分。
b)分别写出y1[n]和y2[n]的卷积式各给3分，给出y1[n]的结果给2分，给出y2[n]的结果各给1分。

Solution

(a) We may write $x[n]$ as

$$x[n] = (\frac{1}{3})^{|n|}$$

Now the desire convolution is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{-1} (\frac{1}{3})^{-k} (\frac{1}{4})^{n-k} u[n-k+3] + \sum_{k=0}^{\infty} (\frac{1}{3})^{k} (\frac{1}{4})^{n-k} u[n-k+3] \\ &= (\frac{1}{12}) \sum_{k=0}^{\infty} (\frac{1}{3})^{k} (\frac{1}{4})^{n+k} u[n+k+4] + \sum_{k=0}^{\infty} (\frac{1}{3})^{k} (\frac{1}{4})^{n-k} u[n-k+3] \end{aligned}$$

By consider each summation in the above equation separately, we may show that

$$y[n] = \begin{cases} \frac{12^4}{11} 3^n, & n \le -4 \\ \frac{1}{11} 4^{-n} - 3 \cdot 4^{-n} + 256 \cdot 3^{-(n+3)}, & n \ge -3 \end{cases}$$

(b) Now we consider the convolution

$$y_1[n] = \left[(\frac{1}{3})^n u[n]\right] * \left[(\frac{1}{4})^n u[n+3]\right] = \left[-3 \cdot 4^{-n} + 256 \cdot 3^{-(n+3)}\right] u[u+3]$$

Also consider the convolution

$$y_2[n] = [3^n u[-n-1]] * \left[(\frac{1}{4})^n u[n+3]\right] = \begin{cases} \frac{12^4}{11} 3^n, & n \le -4 \\ \frac{1}{11} 4^{-n}, & n \ge -3 \end{cases}$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

Problem 4

(20 points) An analog system has the input-output relation

$$y(t) = \int_0^t e^{-(t-\tau)} x(\tau) d\tau \qquad t > 0$$

and zero otherwise. The input is x(t) and y(t) is the output.

(a) Is this a linear time-invariant system? If so, can you determine without calculating the impulse response of the system? Explain.

(b) Is this system causal? Explain.

(c) Find the unit-step response s(t) and from it find the impulse response h(t). Is this a stable system? Explain.

(d) Find the response due to a pulse x(t) = u(t) - u(t-1).

Solution

(a) **Method1:**

The system is LTI since the input $x(t)$ and the output $y(t)$ are related by a convolution integral with $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$ or $h(t) = e^{-t}u(t)$.If the system is not LTI system, then the relationship between $y(t)$ and $x(t)$ can't be a convolution integral.

Method2:

Linear:

We have two inputs $x_1(t)$ and $x_2(t)$, then

第四题不计入总分，HW2总分为80分，最后成绩归一化到100分

$$y_1(t) = \int_0^t e^{t-\tau} x_1(\tau) d\tau$$

$$y_2(t) = \int_0^t e^{t-\tau} x_2(\tau) d\tau$$

Let $x_3(t) = ax_1(t) + bx_2(t)$

$$y_3(t) = \int_0^t e^{t-\tau}(a \cdot x_1(\tau) + b \cdot x_2(\tau)) d\tau = ay_1(t) + by_2(t)$$

so the system is linear

Time-invariant:

$$y(t-t_0) = \int_0^{t-t_0} e^{-(t-t_0-\tau)} x(\tau) d\tau = e^{-(t-t_0)} \int_0^{t-t_0} e^{\tau} x(\tau) d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} u(-(\tau - t)) x(\tau) u(\tau) d\tau$$

$$y(t) = \int_0^t e^{-(t-\tau)} u(-(\tau - t)) x(\tau) u(\tau) d\tau$$

When input becomes $x(t-t_0)$, then

$$y'(t) = \int_0^t e^{-(t-\tau)} u(-(\tau - t)) x(\tau - t_0) u(\tau - t_0) d\tau$$

Problem 4 continued on next page. . .

because $0 < \tau < t$, let $\tau^{'} = \tau - t_0$, then $-t_0 < \tau^{'} < t - t_0$

$$y'(t) = e^{-(t-t_0)} \int_{-t_0}^{t-t_0} e^{\tau^{'}} u(t - \tau^{'} - t_0)x(\tau^{'})u(\tau^{'})d\tau$$

So

$$y'(t) = e^{-(t-t_0)} \int_{0}^{t-t_0} e^{\tau^{'}} u(t - \tau^{'} - t_0)x(\tau^{'})u(\tau^{'})d\tau = y(t - t_0)$$

So the system is time-invariant.

(b) Yes, this system is causal because the output $y(t)$ depend on present and past values of the input.

(c) Letting $x(t) = u(t)$, the unit-step response is

$$s(t) = \int_0^t e^{-t+\tau} u(\tau)d\tau = e^{-t} \int_0^t e^{\tau} d\tau = 1 - e^{-t}, \quad t > 0$$

and $s(t) = 0$ when $t \leq 0$. The impulse response as indicated before is $h(t) = ds(t)/dt = e^{-t}u(t)$. The BIBO stability of the system is then determined by checking whether the impulse response is absolutely integrable or not,

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t} dt = 1$$

so yes, it is BIBO stable.

(d) Using superposition, the response to the pulse $x(t) = u(t) - u(t-1)$ would be

$$y(t) = s(t) - s(t-1) = \left(1 - e^{-t}\right) u(t) - \left(1 - e^{-(t-1)}\right) u(t-1)$$

Problem 5

(20 points) If the input $x[n]$ and output $y[n]$ for a causal system meet the following difference equation:

$$y[n] = ay[n-1] + x[n]$$

Then the impulse response of the system must be $h[n] = a^n u[n]$.

(a) Determine the value of a when the system is stable.

(b) Consider a casual LTI system, the input-output relation for the system is defined by the following equation:

$$y[n] = ay[n-1] + x[n] - a^N x[n-N]$$

with N is integer(Positive). Find out the impulse response and sketch the figure. Hint:Linear and Time-invariant Properties

(c) Determine whether the system in (b) is IIR system or FIR system? Explain.

(d) Determine the value of a when the system in (b) is stable? Explain.

Solution

(a) LTI system are stable if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$(the summation should converge)

Then

$$S = \sum_{n=-\infty}^{\infty} |a|^n u[n]$$
$$= \sum_{n=0}^{\infty} |a|^n$$

S will converge only when $|a| < 1$ and $S = \frac{1}{1-|a|} < \infty$

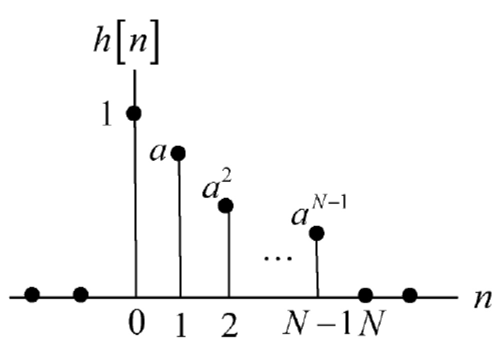
Therefore the system is stable for $|a| < 1$

(b) $y[n] = ay[n-1] + x[n] - a^N x[n-N]$. Therefore $h[n] = ah[n-1] + \delta[n] - a^N \delta[n-N]$.

Since the system is causal,$h[-1] = 0$.Then

$$h[0] = 0 + 1 - 0 = 1$$
$$h[1] = a, h[2] = a^2, h[N] = a^N - a^N = 0$$
$$h[N+1] = a \times 0 + 0 - 0 = 0$$

a)以求绝对可和方式给3分，答案算错扣2分
b)利用递归计算方式得2分，画图1分，结果2分
c）IIR扣2分
d）绝对可和得2分，答案算对得3分



$$h[n] = \begin{cases} a^n & \text{n=0,1,2,...,N-1} \\ 0 & \text{others} \end{cases}$$

Problem 5 continued on next page. . .

(c) We see that though it is a recursive system (with feedback) , its impulse response is finite in length. The length of $h[n]$ is N terms. Hence this system is FIR.

(d) FIR systems are always stable as the sum $\sum_{n=-\infty}^{\infty} |h[n]|$ has at most a finite number of nonzero terms.