Time and frequency characterization of signals and systems (ch.6)

❑ **The magnitude-phase representation of Fourier Transform**

❑ The magnitude-phase representation of the frequency response of LTI systems

❑ Time-domain properties of ideal frequency-selective filters

❑ Time-domain and frequency-domain aspects of non-ideal filters

❑ First-order system

Magnitude and phase spectrum

□ Continuous FT $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$

□ Discrete FT $\quad x[n] \longleftrightarrow X(e^{j\omega}) \quad X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

□ Amplitude spectrum: $|X(j\omega)|$ and $|X(e^{j\omega})|$

□ Phase spectrum (angle): $\angle X(j\omega)$ and $\angle X(e^{j\omega})$
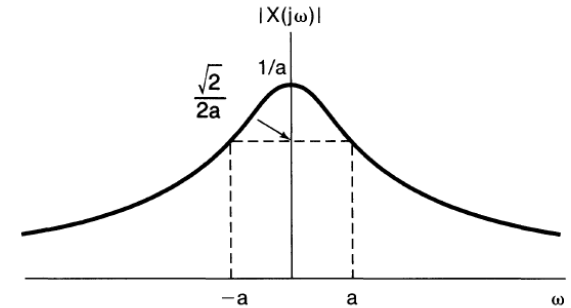
Magnitude spectrum Continuous time as an example

$$\text{IFT: } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

|X(jω)|

$\frac{\sqrt{2}}{2a}$ 1/a

−a   a   ω

- ❑ IFT: decomposition of the signal $x(t)$ into a "sum" of complex exponentials at different frequencies

- ❑ $|X(e^{j\omega})|$: describes the basic frequency content of a signal, and the relative magnitude of the each frequency (complex exponential)

- ❑ $|X(j\omega)|^2$: energy-density spectrum of $x(t)$

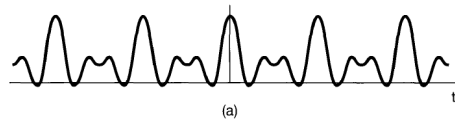- ❑ $|X(j\omega)|^2 d\omega/2\pi$: energy in the signal between ω and $\omega + d\omega$
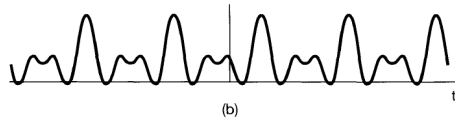
Phase spectrum

- ❑ $\angle X(j\omega)$
  - relative phase of the each complex exponential
  - significant effect on the nature of the signal
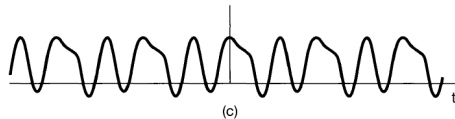 - changes in $\angle X(j\omega)$ lead to phase distortion

❑ Example 1: $x(t) = 1 + \frac{1}{2}\cos(2\pi t + \varphi_1) + \frac{1}{2}\cos(4\pi t + \varphi_2) + \frac{1}{2}\cos(6\pi t + \varphi_3)$

(a)
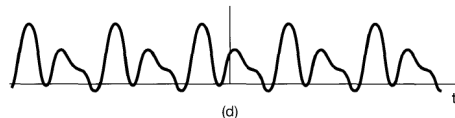
$\varphi_1 = \varphi_2 = \varphi_3 = 0$

(b)

$\varphi_1 = 4rad, \varphi_2 = 8rad, \varphi_3 = 12rad$

(c)

$\varphi_1 = 6rad, \varphi_2 = -2.7rad, \varphi_3 = 0.93rad$

(d)

$\varphi_1 = 1.2rad, \varphi_2 = 4.1rad, \varphi_3 = -7.02rad$

3

Time and frequency characterization of signals and systems (ch.6)

❑ The magnitude-phase representation of Fourier Transform

❑ **The magnitude-phase representation of the frequency response of LTI systems**

❑ Time-domain properties of ideal frequency-selective filters

❑ Time-domain and frequency-domain aspects of non-ideal filters

❑ First-order system

Gain and phase shift

❑ For LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

❑ The frequency response $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$

❑ $|H(j\omega)|$: Gain of the LTI system; $\angle H(j\omega)$: phase shift of the LTI system
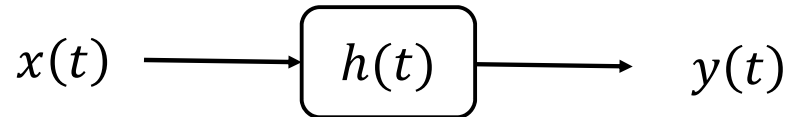
$$Y(j\omega) = H(j\omega)X(j\omega) = |H(j\omega)||X(j\omega)|e^{j(\angle H(j\omega)+\angle X(j\omega))}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \quad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Linear phase system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$
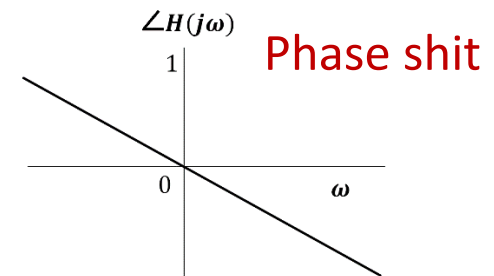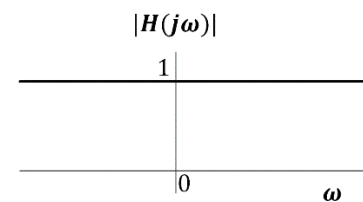
For $H(j\omega) = e^{-j\omega t_0}$

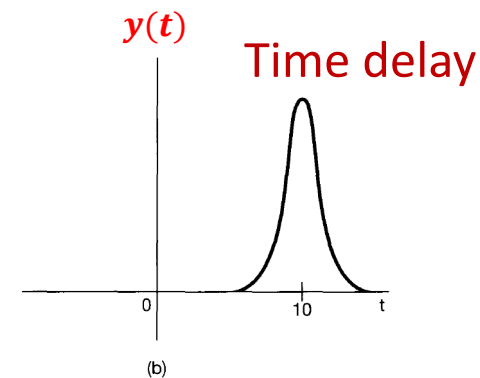$$|H(j\omega)| = 1 \qquad \angle H(j\omega) = -\omega t_0$$

All-pass system

$|H(j\omega)|$

$\angle H(j\omega)$

Phase shit

Output of system：

$x(t)$

$y(t)$

Time delay

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= X(j\omega)e^{-j\omega t_0}$$

$$y(t) = x(t - t_0)$$

(a)

(b)
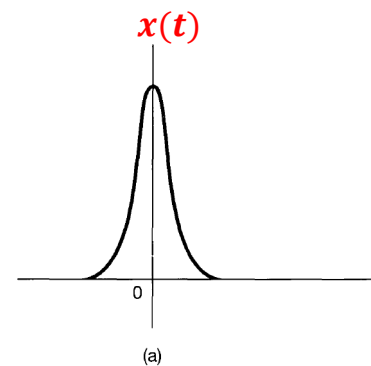
Non-linear phase system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

For $H(j\omega) = H_1(j\omega)H_2(j\omega)$

$$H_1(j\omega) = e^{-j\omega t_0}$$

$$H_2(j\omega) = e^{\angle H_2(j\omega)}$$

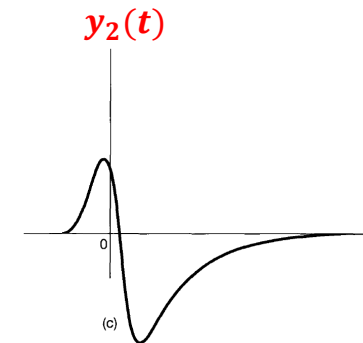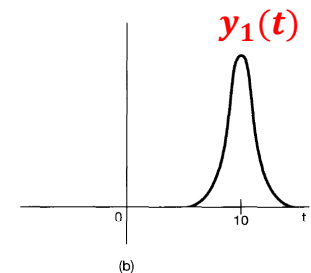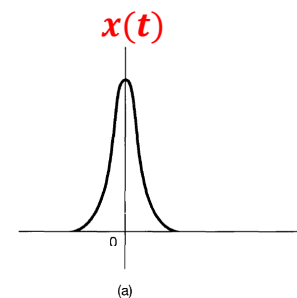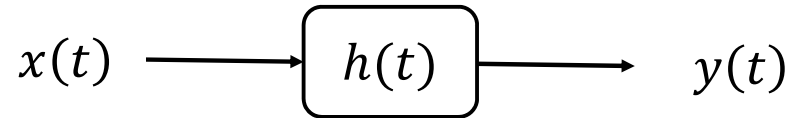$\angle H_2(j\omega)$ is a nonlinear function of ω

$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0 + \angle H_2(j\omega)$$

$x(t)$

(a)

$y_1(t)$

(b)

$y_2(t)$

(c)

$y(t)$

(d)

7

Group delay

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

❑ Consider a system with $\angle H(j\omega)$ a nonlinear function of ω

❑ For a narrow band input $x(t)$, $\angle H(j\omega) \simeq -\phi - \alpha\omega$

$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\alpha\omega}$$

❑ The time delay α is referred to as the group delay at $\omega = \omega_0$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\}$$

$X(j\omega)$

ω_0

ω

8
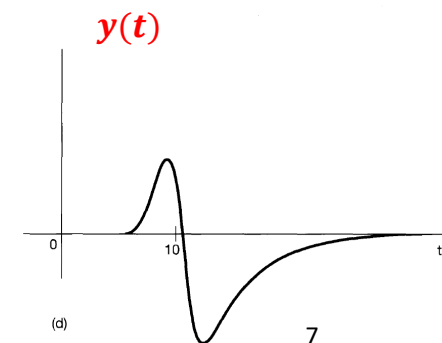
The magnitude-phase representation of LTI

Group delay: example

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

❑ Consider

$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega)$$
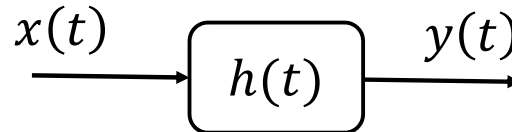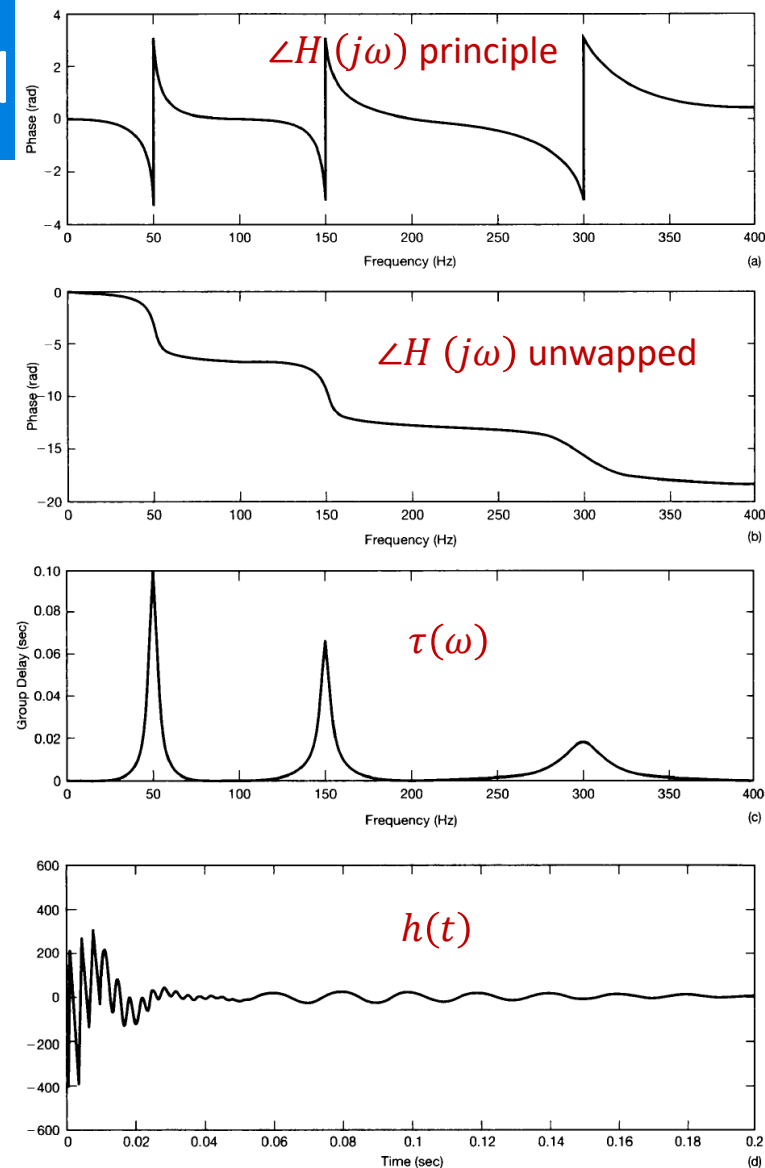
$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec and } \zeta_1 = 0.066, \\ \omega_2 = 943 \text{ rad/sec and } \zeta_2 = 0.033, \\ \omega_3 = 1888 \text{ rad/sec and } \zeta_3 = 0.058. \end{cases}$$

$$|H_i(j\omega)| = 1 \Rightarrow |H(j\omega)| = 1$$

$$\angle H_i(j\omega) = -2\arctan\left[\frac{2\zeta_i(\omega/\omega_i)}{1 - (\omega/\omega_i)^2}\right]$$
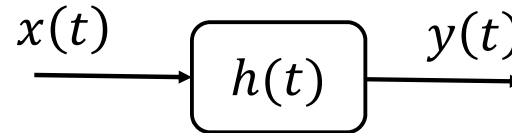
$$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i(j\omega)$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\}$$

∠$H(j\omega)$ principle

∠$H(j\omega)$ unwrapped

$\tau(\omega)$

$h(t)$

Log-Magnitude and Bode Plots

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

Time domain: $\qquad y(t) = x(t) * h(t)$ $\qquad$ Convolution

Frequency domain: $\qquad Y(j\omega) = H(j\omega)X(j\omega)$ $\qquad$ Multiplication

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Logarithmic amplitude: $\quad \log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$ $\qquad$ Summation
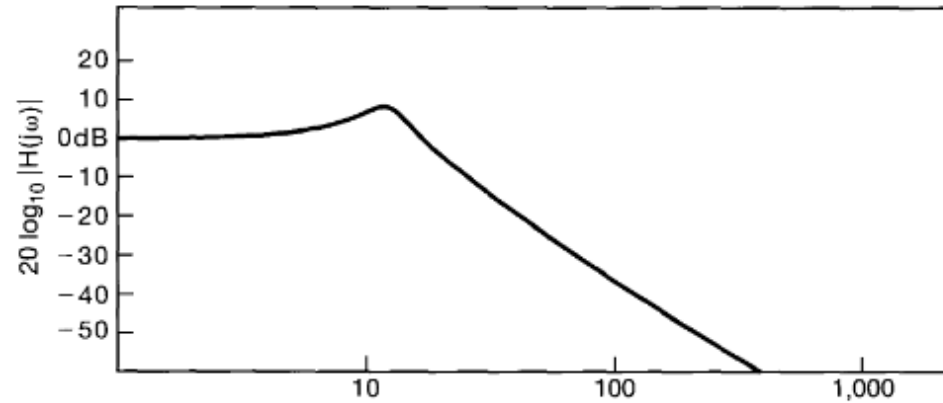
Logarithmic amplitude scale: $20 \log_{10}$, referred to as _decibels_ (dB).

Bode plots: Plots of $20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ versus $\log_{10}(\omega)$

Log-Magnitude and Bode Plots

Plot of $20\log_{10}|H(j\omega)|$ vs $\log_{10}(\omega)$
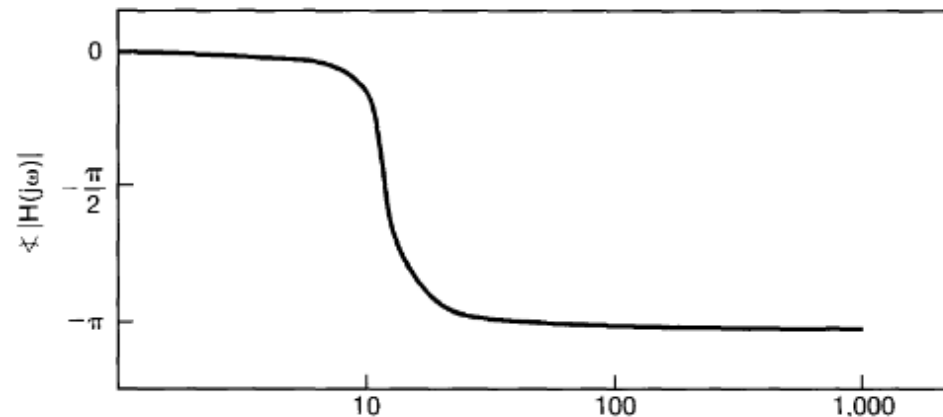
Plot of $\angle H(j\omega)$ vs. $\log_{10}(\omega)$

Figure 6.8 A typical Bode plot. (Note that ω is plotted using a logarithmic scale.)

11

Time and frequency characterization of signals and systems (ch.6)

❑ The magnitude-phase representation of Fourier Transform

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❑ Time-domain properties of ideal frequency-selective filters

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❑ First-order system

Frequency-selective filters

Low-pass filter

High-pass filter

Band-pass filter

We focus on low-pass filter, similar concepts and results hold for high-pass and band pass filter.

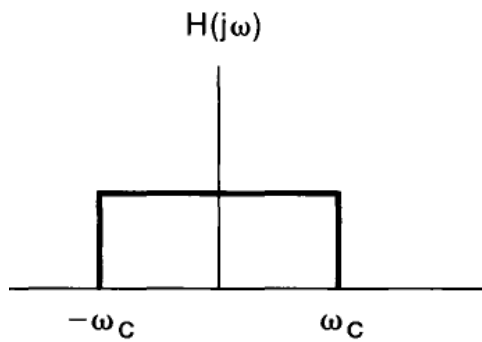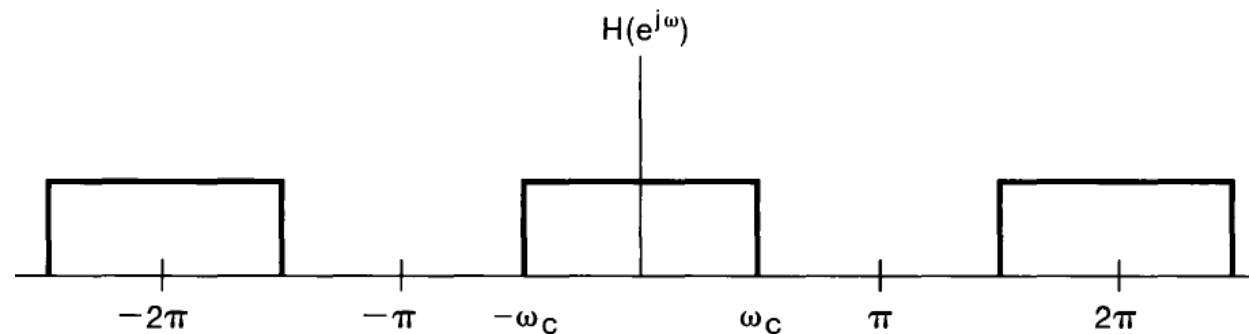## *Ideal low-pass filters: zero phase*

CT

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

DT

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Ideal low-pass filters: zero phase

❑ Impulse response:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{1}{jt} e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{jt} \cdot 2j\sin(\omega_c t) = \frac{\sin \omega_c t}{\pi t}$$

$$h(n) = \frac{\sin \omega_c n}{\pi n}$$
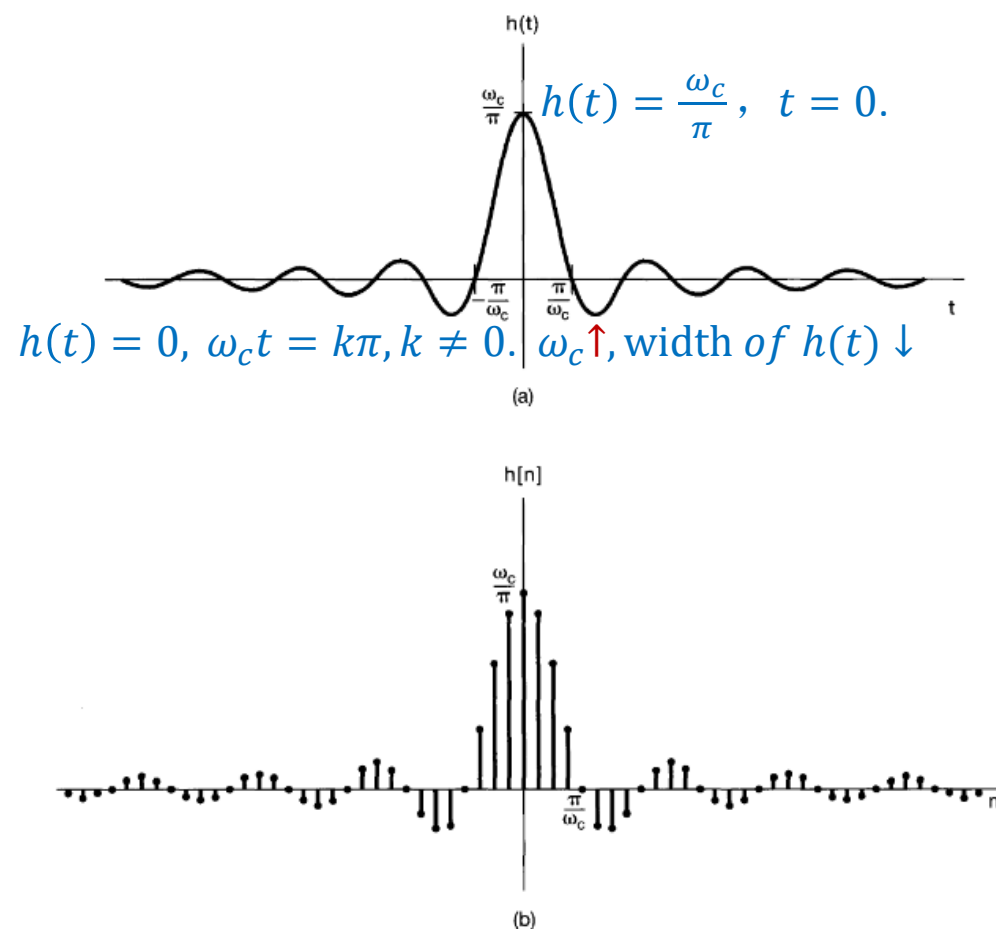
$h(t) = \frac{\omega_c}{\pi}, \quad t = 0.$

$h(t) = 0, \ \omega_c t = k\pi, k \neq 0.$ $\omega_c \uparrow, \text{width of } h(t) \downarrow$

(a)

(b)

Ideal low-pass filters: zero phase

❑ Step response:

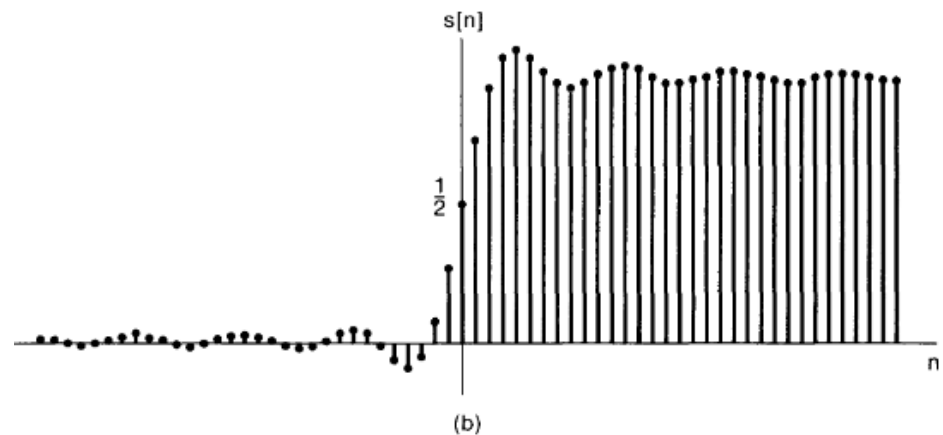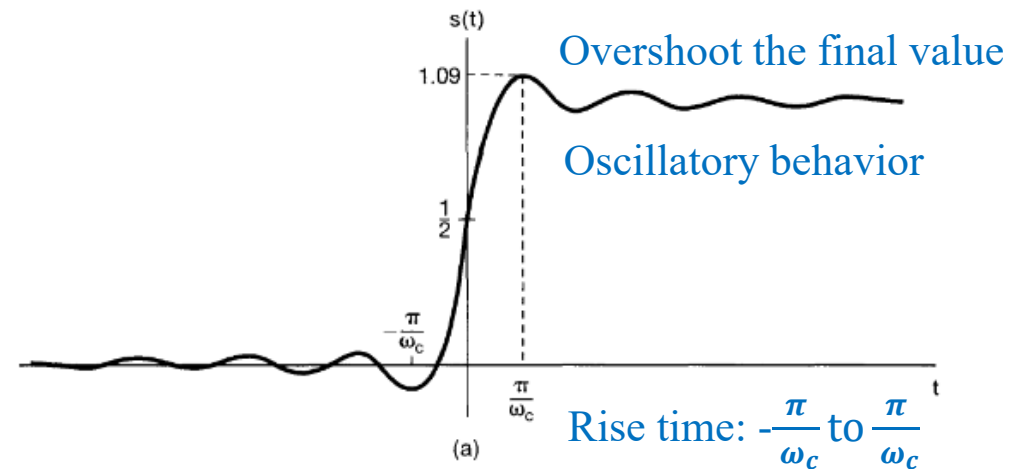$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(n) = \sum_{m=-\infty}^{n} h(m)$$

Overshoot the final value

Oscillatory behavior

Rise time: $-\dfrac{\pi}{\omega_c}$ to $\dfrac{\pi}{\omega_c}$

16

Ideal low-pass filters: linear phase

❑ Impulse response:
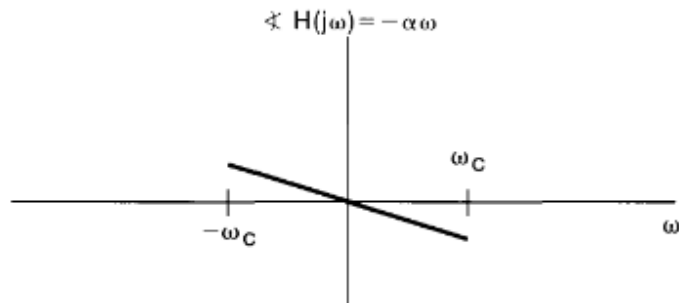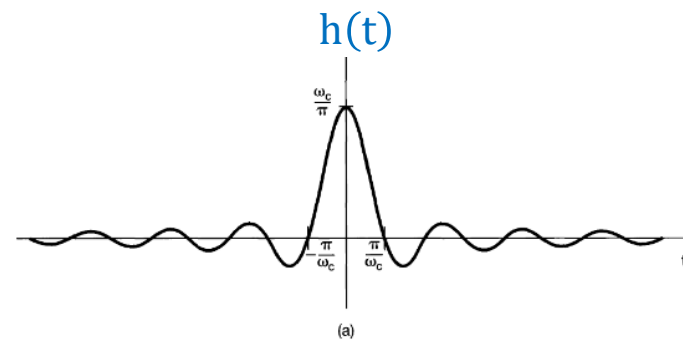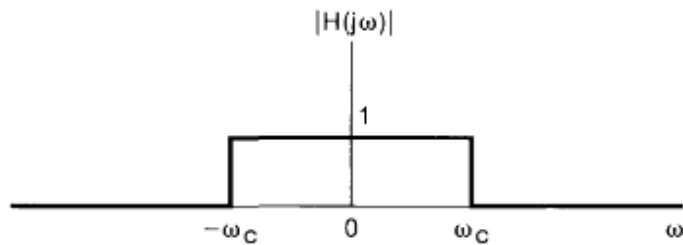
Time and frequency characterization of signals and systems (ch.6)

❑ The magnitude-phase representation of Fourier Transform

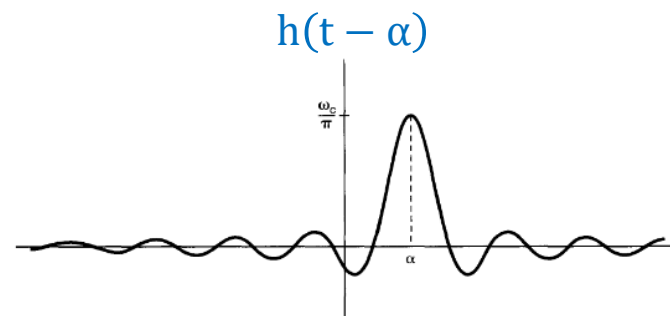❑ The magnitude-phase representation of the frequency response of LTI systems

❑ Time-domain properties of ideal frequency-selective filters

❑ Time-domain and frequency-domain aspects of non-ideal filters

❑ First-order system

Frequency domain (low-pass)

- Idea Low-pass filter is not implementable

- Gradual transition band is sometimes preferable
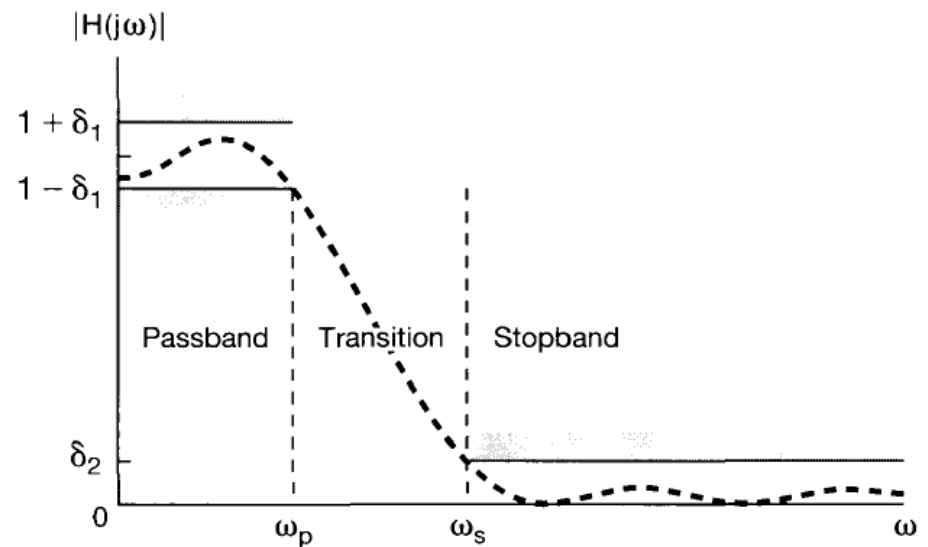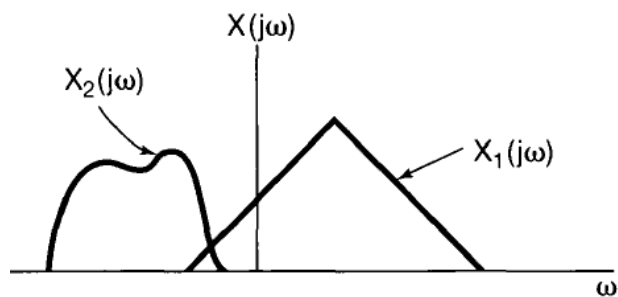
- Pass band: $0 - \omega_p$, stop band: $\omega > \omega_s$, transition: $\omega_s - \omega_p$
- Pass-band ripple: δ_1, stop-band ripple: δ_2
- Linear (nearly) linear phase over the passband is desirable.

19

Time domain (low-pass)
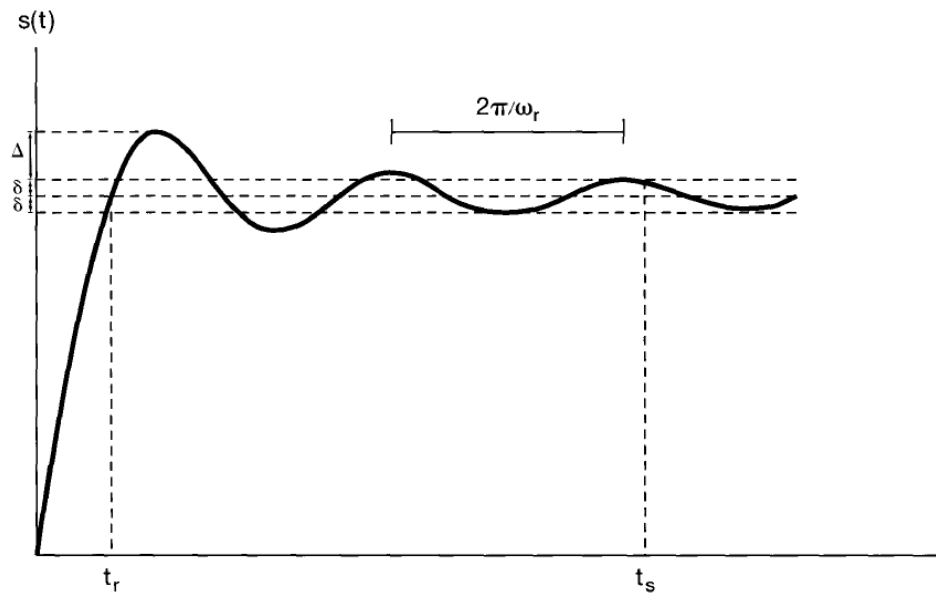
Step response of a CT low-pass filter

- Rise time: t_r
- Overshoot: Δ
- Ringing frequency: ω_r
- Settling time: t_s

An example

Magnitude

Step response
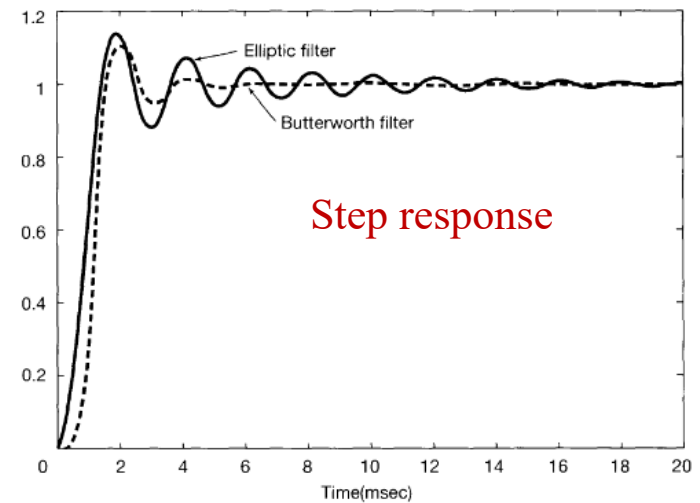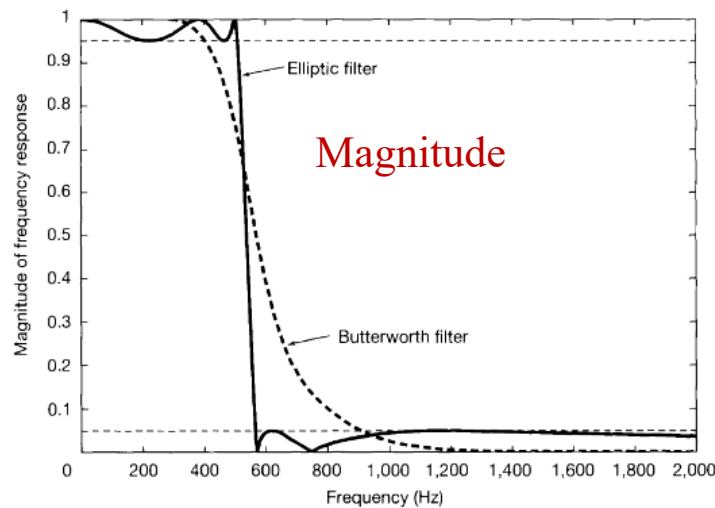
- Fifth-order Butterworth filter and a fifth-order elliptic filter
- Same cutoff frequency
- Same passband and stopband ripple

Trade-off between time-domain (t_s) and frequency-domain (ω_s - ω_p).

21

# Time and frequency characterization of signals and systems (ch.6)

❑ The magnitude-phase representation of Fourier Transform

❑ The magnitude-phase representation of the frequency response of LTI systems

❑ Time-domain properties of ideal frequency-selective filters

❑ Time-domain and frequency-domain aspects of non-ideal filters

❑ First-order system

First-order system (Continuous time)

- Differential equation:
$$C\frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}$$

$$\tau\frac{dy(t)}{dt} + y(t) = x(t), \tau = RC$$

- Frequency response:
$$\tau j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega\tau + 1}$$

23

First-order systems

First-order system (Continuous time)

☐ Impulse response $\quad H(j\omega) = \dfrac{1}{j\omega\tau + 1} = \dfrac{1/\tau}{j\omega + 1/\tau}$

$$e^{-at}u(t), a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{j\omega + a}$$

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

☐ Step response

$$s(t) = \int_{-\infty}^{t} h(t')\,dt' = \frac{1}{\tau}\int_{0}^{t} e^{-t'/\tau}\,dt' = \begin{cases} 0, t < 0 \\ 1-e^{-t/\tau}), t \geq 0 \end{cases}$$

$$s(t) = (1-e^{-t/\tau})\,u(t)$$

- τ: time constant
- $t = \tau$, $h(t) = 1/(\tau e)$
  $\quad\quad s(t) = 1 - 1/e$
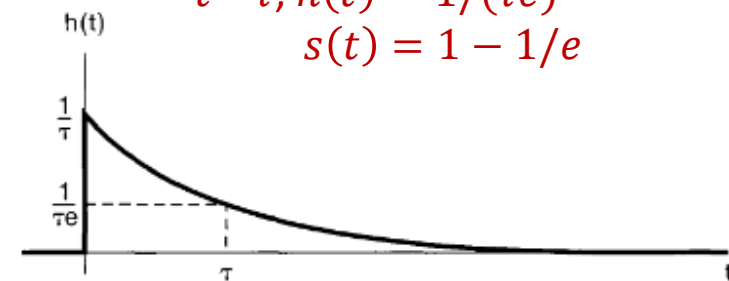
(a)

- $\tau \downarrow$, $h(t)$ decays more sharply
  $\quad\quad s(t)$ rises more sharply

(b)

Bold Plots (Continuous time) $H(j\omega) = \dfrac{1}{j\omega\tau + 1}$

❑ $20\log_{10}|H(j\omega)| = -10\log_{10}[(\omega\tau)^2 + 1]$

$$\simeq \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}$$

$\omega = 1/\tau, \ 20\log_{10}|H(j\omega)| = -10\log_{10}(2) \simeq -3dB$

$\omega = 1/\tau$：break frequency

❑ $\angle H(j\omega) = -\tan^{-1}(\omega\tau)$

$$\simeq \begin{cases} 0, & \omega \le 0.1/\tau \\ -\dfrac{\pi}{4}[\log_{10}(\omega\tau) + 1], & 0.1/\tau \le \omega \le 10/\tau \\ -\pi/2, & \omega \ge 10/\tau \end{cases}$$

$\omega = 1/\tau, \ \angle H(j\omega) = -\pi/4$

20-dB-per-decade
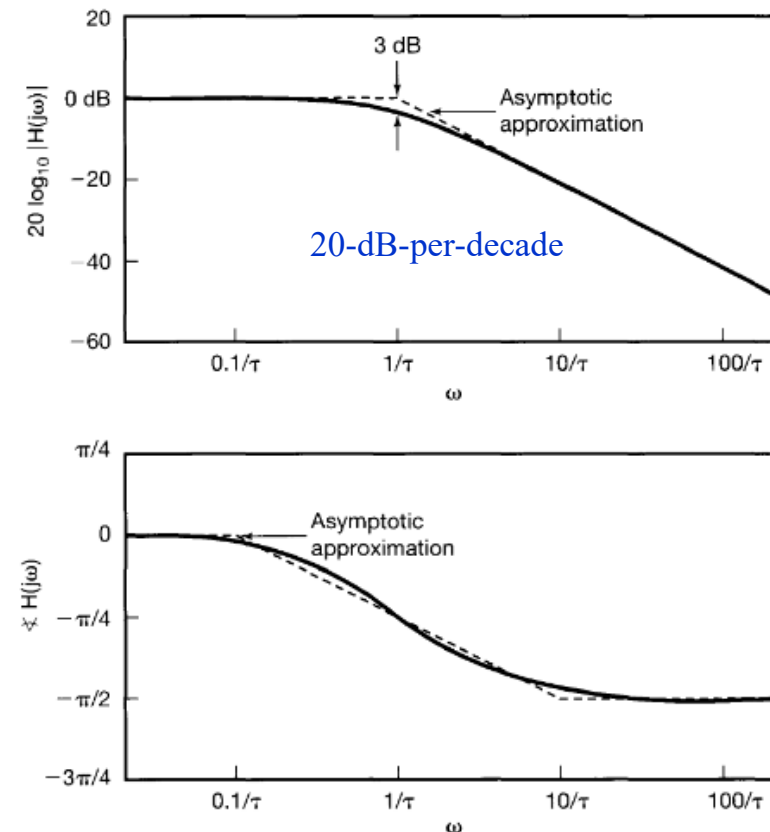
$\tau \downarrow$, $h(t)$ and $s(t)$ more sharply, break frequency $\uparrow$.

Differential equation

$$m\frac{d^2y(t)}{dt} = x(t) - ky(t) - b\frac{dy(t)}{dt}$$

y(t) (displacement)

Spring k

Mass m

x(t) (applied force)

Dashpot b

$$\frac{d^2y(t)}{dt} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t)$$

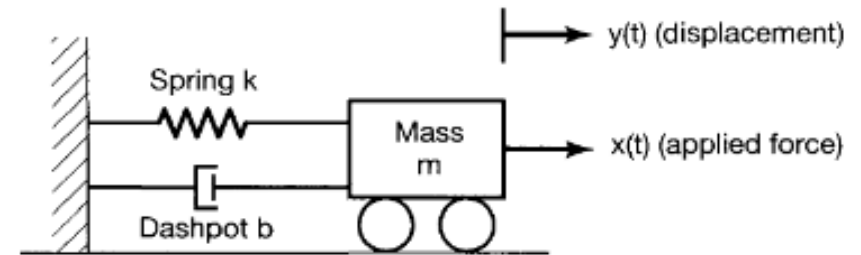$$\omega_n{}^2 = \frac{k}{m} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}} \qquad 2\zeta\omega_n = \frac{b}{m}$$

$$\frac{d^2y(t)}{dt} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n{}^2y(t) = \omega_n{}^2x(t)$$

26

Second-order systems

❑ Frequency response: $\dfrac{d^2y(t)}{dt} + 2\zeta\omega_n\dfrac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n(j\omega)Y(j\omega) + \omega_n^2 Y(j\omega) = \omega_n^2 X(j\omega)$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

❑ Impulse response: $H(j\omega) = \dfrac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)} = \dfrac{M_1}{(j\omega - c_1)} - \dfrac{M_2}{(j\omega - c_2)}$

$\zeta \neq 1$

c_1, c_2: roots of $(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2 = 0$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M_1 = M_2 = M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \quad \longrightarrow \quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

27

❑ Impulse response:

$\zeta = 1$ $\qquad c_1 = c_1 = -\omega_n$ $\qquad H(j\omega) = \dfrac{\omega_n^2}{(j\omega + \omega_n)^2}$

Critically damped

$$te^{-at}u(t) \xrightarrow{\quad \mathcal{F} \quad} H(j\omega) = \dfrac{1}{(j\omega + a)^2} \qquad \therefore h(t) = \omega_n^2 te^{-\omega_n t}u(t)$$

❑ Recall $\zeta \neq 1$ $\quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$

$0 < \zeta < 1$ $\qquad h(t) = \dfrac{\omega_n}{2\sqrt{\zeta^2 - 1}}[e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}]\, u(t)$
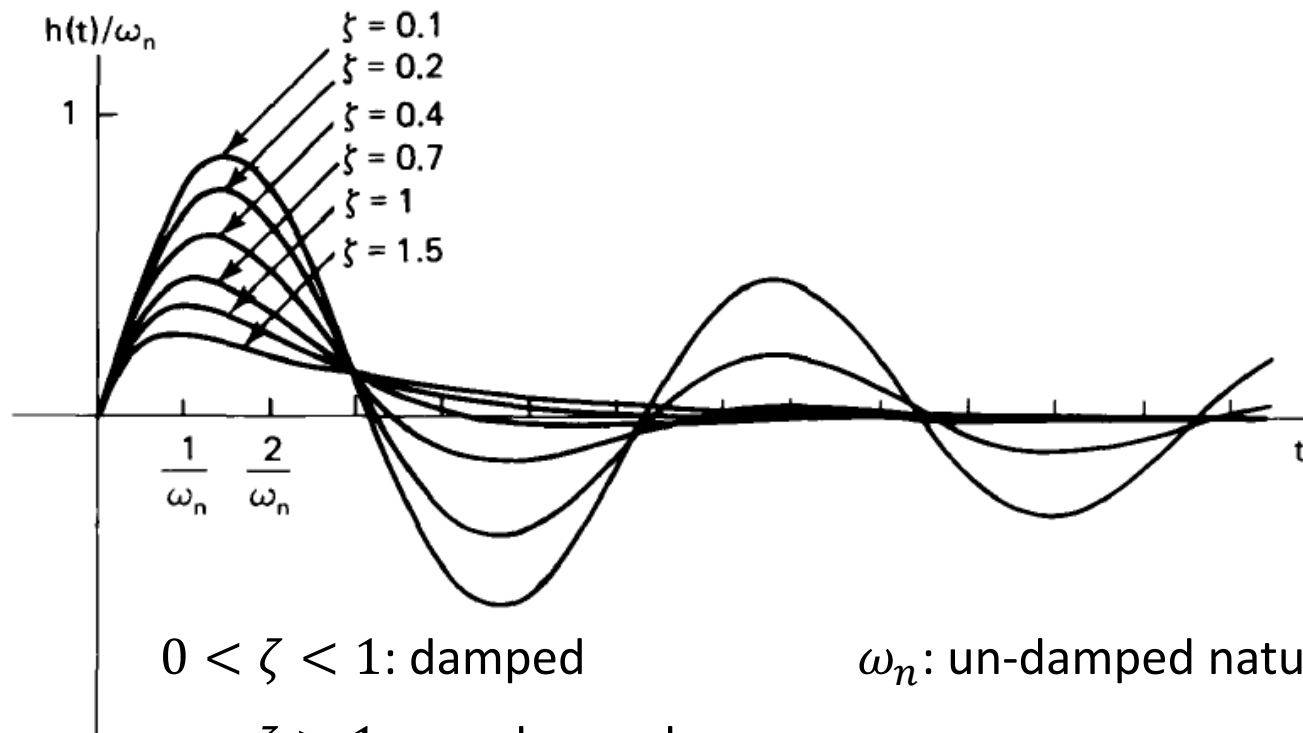
$$= \dfrac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left[e^{j\omega_n\sqrt{1-\zeta^2}t} - e^{-j\omega_n\sqrt{1-\zeta^2}t}\right]u(t)$$

$\zeta > 1$: over damped

$$= \dfrac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}[2j\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t) = \dfrac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t)$$

28

❑ Impulse response:

$0 < \zeta < 1$: damped ω_n: un-damped natural frequency

$\zeta > 1$: overdamped

$\zeta = 1$: critically damped ζ: damping ratio

❑ Step response

$\zeta \neq 1$

$$s(t) = \int_{-\infty}^{t} h(t')\,dt' = M\int_{0}^{t} e^{c_1 t'} - e^{c_2 t'}\,dt'$$

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$= \left\{ \begin{array}{c} 0, t < 0 \\ M(\dfrac{e^{c_1 t'}}{c_1} - \dfrac{e^{c_2 t'}}{c_2})\Big|_{0}^{t} = 1 + M\left[\dfrac{e^{c_1 t}}{c_1} - \dfrac{e^{c_2 t}}{c_2}\right], t \geq 0 \end{array} \right\} = \left\{ 1 + M\left[\dfrac{e^{c_1 t}}{c_1} - \dfrac{e^{c_2 t}}{c_2}\right] \right\} u(t)$$

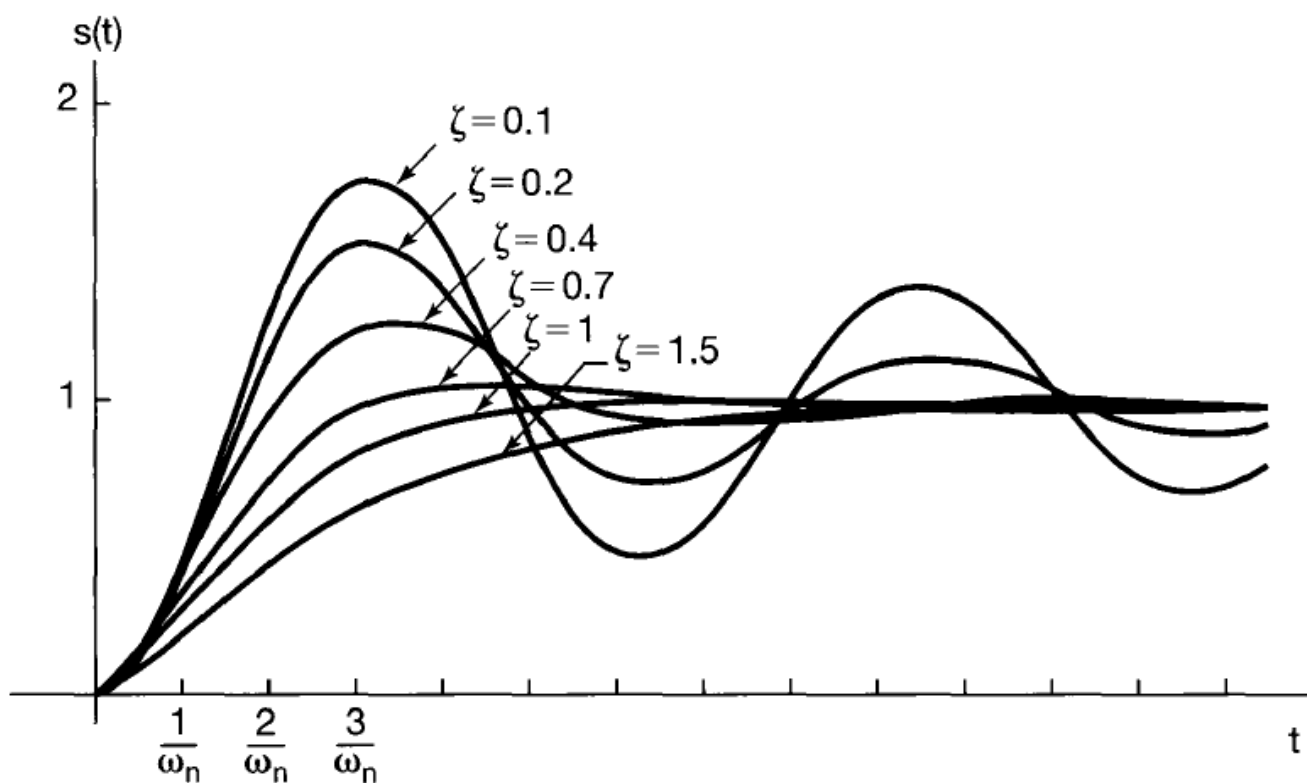$\zeta = 1$ $h(t) = \omega_n{}^2 t e^{-\omega_n t} u(t)$

$$s(t) = \int_{0}^{t} \omega_n{}^2 t' e^{-\omega_n t'}\,dt' = -\omega_n \int_{0}^{t} t'\,d e^{-\omega_n t'}$$

$$= \left\{ \begin{array}{c} 0, t < 0 \\ -\omega_n t' e^{-\omega_n t'}\Big|_{0}^{t} - \int_{0}^{t} e^{-\omega_n t'}\,d(-\omega_n t') = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}, t \geq 0 \end{array} \right\}$$

$$s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$$

❑ Step response

Second-order systems

❑ Bold plots

$$H(j\omega) = \frac{\omega_n{}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

$$20\log_{10}|H(j\omega)| = -20\log_{10}|(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1|$$

$$= -10\log_{10}\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

$$\simeq \begin{cases} 0, & \omega \ll \omega_n \\ -40\log_{10}\omega + 40\log_{10}\omega_n, & \omega \gg \omega_n \end{cases}$$
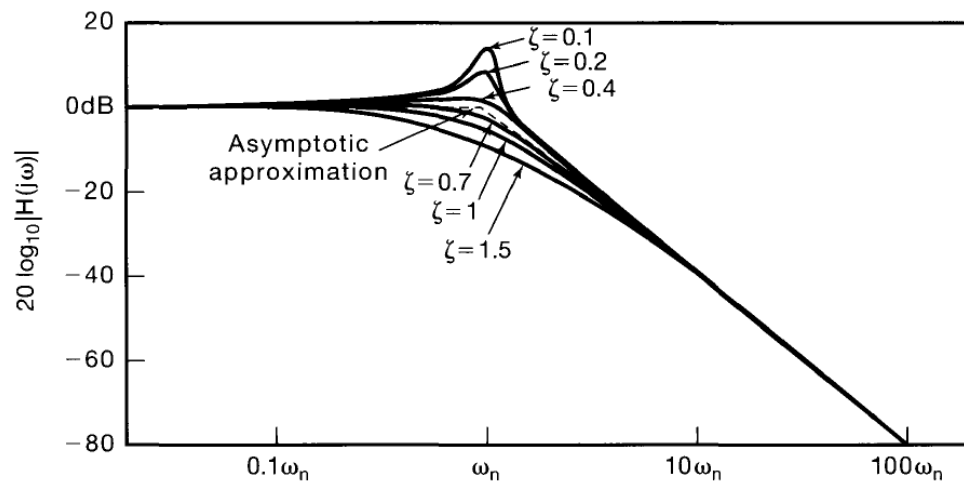
$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right] \simeq \begin{cases} 0, & \omega \leq 0.1\omega_n \\ -\frac{\pi}{2}\left[\log_{10}\left(\frac{\omega}{\omega_n}\right) + 1\right], & 0.1\omega_n \leq \omega \leq 10\omega_n \\ -\pi, & \omega \geq 10\omega_n \end{cases}$$

❑ Bold plots

$$20\log_{10}|H(j\omega)|$$

$$\angle H(j\omega)$$