Problem 1

(20 points) A causal LTI filter has the frequency response $H(j\omega)$ shown in Figure 1. For each of the input signals given below, determine the filtered output signal $y(t)$.

(a) $x(t) = e^{jt}$

(b) $x(t) = (sin\omega_0 t)u(t)$

(c) $X(j\omega) = \frac{1}{(j\omega)(6+j\omega)}$

(d) $X(j\omega) = \frac{1}{2+j\omega}$

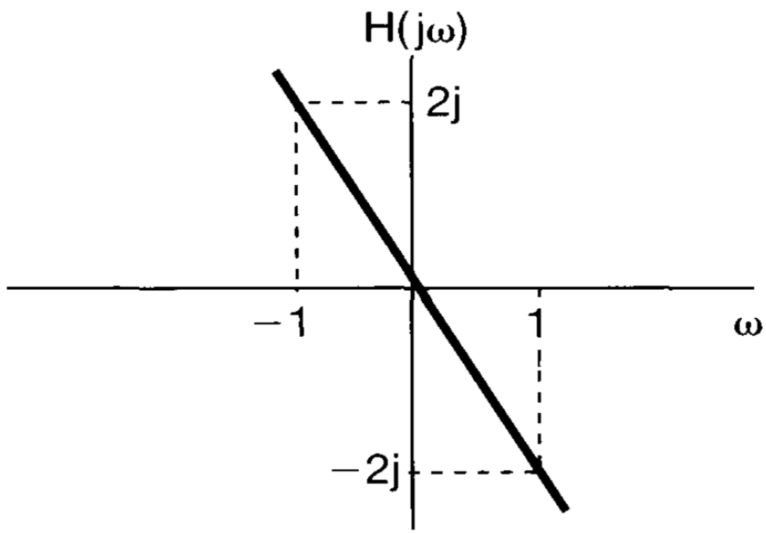


Figure 1: Problem 1

Solution:
Note that in all parts of this problem $Y(j\omega) = H(j\omega)X(j\omega) = -2j\omega X(j\omega)$. Therefore, $y(t) = -2dx(t)/dt$.

(a) Here, $x(t) = e^{jt}$. Therefore, $y(t) = -2dx(t)/dt = -2je^{jt}$. This part could also have been solved by noting that complex exponentials are Eigen functions of LTI systems. Then, when $x(t) = e^{jt}$, $y(t)$ should be $y(t) = H(j1)e^{jt} = -2je^{jt}$.

(b) Here, $x(t) = (sin\omega_0 t)u(t)$. Then, $dx(t)/dt = \omega_0 cos(\omega_0 t)u(t) + sin(\omega_0 t)\delta(t) = \omega_0 cos(\omega_0 t)u(t)$. Therefore, $y(t) = -2dx(t)/dt = -2\omega_0 cos(\omega_0 t)u(t)$.

(c) Here, $Y(j\omega) = H(j\omega)X(j\omega) = \frac{-2}{(6+j\omega)}$. Taking the inverse Fourier transform we obtain $y(t) = -2e^{-6t}u(t)$.

(d) Here, $X(j\omega) = \frac{1}{2+j\omega}$. From this we obtain $x(t) = e^{-2t}u(t)$. Therefore, $y(t) = -2dx(t)/dt = 4e^{-2t}u(t) - 2\delta(t)$.

Problem 2

(20 points) The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{1}$$

(a) Determine the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{2}$$

of the system, and sketch its Bode plot.

(b) Specify, as a function of frequency, the group delay associated with this system.

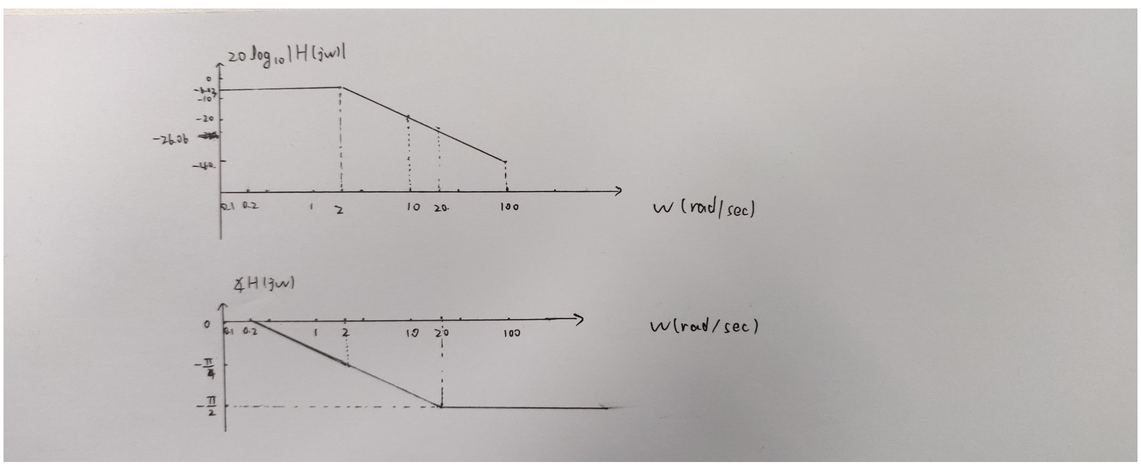
(c) If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output.

Solution:

(a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2+j\omega} \tag{3}$$

The Bode plot is as shown in the Figure below.



(b) From the expression for $H(j\omega)$ we obtain

$$\sphericalangle H(j\omega) = -\tan^{-1}(\omega/2) \tag{4}$$

Therefore,

$$\tau(\omega) = -\frac{d\sphericalangle H(j\omega)}{d\omega} = \frac{2}{4+\omega^2} \tag{5}$$

(c) Since $x(t) = e^{-t}u(t)$,

$$X(j\omega) = \frac{1}{1+j\omega} \tag{6}$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)} \tag{7}$$

Problem 3

(20 points) The frequency which, under the sampling theorem, must be exceeded by the sampling frequency is called the *Nyquist rate*. Determine the Nyquist rate corresponding to each of the following signals:

(a) $x(t) = 1 + cos(2000\pi t) + sin(4000\pi t)$

(b) $x(t) = \frac{sin(4000\pi t)}{\pi t}$

(c) $x(t) = \left(\frac{sin(4000\pi t)}{\pi t}\right)^2$

Solution:

(a) We can easily show that $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$.

(b) We know that $X(j\omega)$ is a rectangular pulse for which $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$

(c) We know that $X(j\omega)$ is the convolution of two rectangular pulses each of which is zero for $|\omega| > 4000\pi$. Therefore, $X(j\omega) = 0$ for $|\omega| > 8000\pi$ and the Nyquist rate for this signal is $\omega_N = 2(8000\pi) = 16000\pi$.

Problem 4

(10 points) Consider the discrete-time sequence $x[n] = cos[n\pi/4]$, find two different continuous-time signals that would produce this sequence when sampled at a frequency of $f_s = 10$ kHz.

Solution

A continuous-time sinusoid

$$x_a(t) = cos(w_0 t) = cos(2\pi f_0 t) \tag{8}$$

that is sampled with a sampling frequency of f_s results in the discrete-time sequence

$$x[n] = x_a(nT_s) = cos(2\pi \frac{f_0}{f_s} n) \tag{9}$$

However, note that for any integer k,

$$cos(2\pi \frac{f_0}{f_s} n) = cos(2\pi \frac{f_0 + kf_s}{f_s} n) \tag{10}$$

Therefore, any sinusoid with a frequency

$$f = f_0 + kf_s \tag{11}$$

will produce the same sequence of samples $x[n]$ when sampled with a sampling frequency f_s. With $x[n] = cos(n\pi/4)$, we want

$$2\pi \frac{f_0}{f_s} = \frac{\pi}{4} \tag{12}$$

or

$$f_0 = \frac{1}{8} f_s = 1250 Hz \tag{13}$$

Therefore, two signals that produce the given sequence are

$$x_1(t) = cos(2500\pi t) \tag{14}$$

and

$$x_2(t) = cos(22500\pi t) \tag{15}$$

Problem 5

(30 points) Suppose that we would like to slow a segment of speech to one-half its normal speed. The speech signal $s_a(t)$ is assumed to have no energy outside of 5 kHz, and is sampled at a rate of 10 kHz, yielding the sequence

$$s[n] = s_a(nT_s) \tag{16}$$

The following system is proposed to create the slowed-down speech signal. Assume that $S_a(\omega)$ is as shown in the following figure:

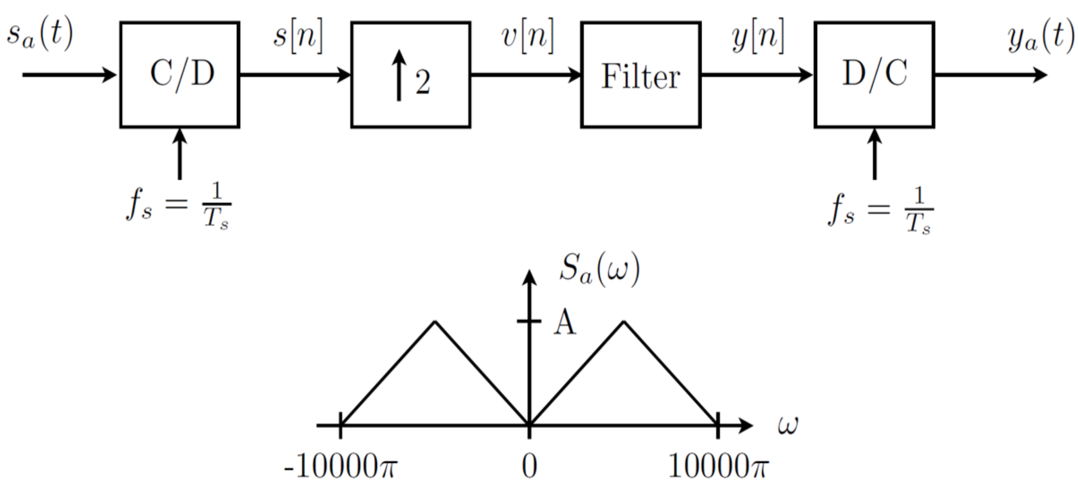


Figure 2: Problem 5

(a) Find the spectrum of $v[n]$.

(b) Suppose that the discrete-time filter is described by the difference equation:

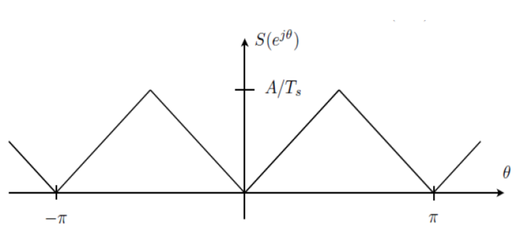
$$y[n] = v[n] + \frac{1}{2}(v[n-1] + v[n+1]) \tag{17}$$

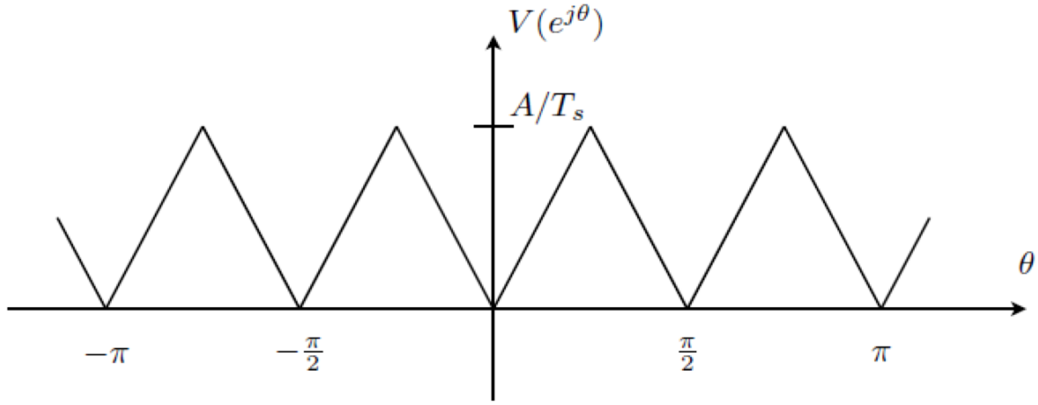
Find the frequency response of the filter and describe its effect on $v[n]$.

(c) What is $Y_a(\omega)$ in terms of $X_a(\omega)$? Does $y_a(t)$ correspond to slowed-down speech?

Solution

(a) Since $s_a(t)$ is sampled at the Nyquist rate, the FTD of the sampled speech signal, $s(n)$, is as follows: Up-sampling by a factor of 2 scales the frequency axis of $S(e^{j\theta})$ by a factor of two as shown below.





(b) The impulse response of the discrete-time filter is

$$h(n) = \frac{1}{2}\delta(n+1) + \delta(n) + \frac{1}{2}\delta(n-1) \tag{18}$$

which has a frequency response

$$H(e^{j\theta}) = 1 + cos\theta \tag{19}$$

To see the effect of this filter on $v(n)$, note that due to the up-sampling, $v(n) = 0$ for n odd. Therefore, with

$$y(n) = v(n) + \frac{1}{2}v(n-1) + \frac{1}{2}v(n+1) \tag{20}$$

it follows that

$$y(t) = \begin{cases} v(n), & n \quad odd \\ \frac{1}{2}v(n-1) + \frac{1}{2}v(n+1), & n \quad even \end{cases} \tag{21}$$

Thus, the even-index values of $v(n)$ are unchanged, and the odd-index values are the average of the two neighboring values. As a result, $h(n)$ performs a linear interpolation between the values of $v(n)$.

(c) The output of the DC converter, $y_a(t)$, has a Fourier transform

$$Y_a(\omega) = \begin{cases} T_s Y(e^{j\omega T_s}), & |\omega| < \pi/T_s \\ 0, & otherwise \end{cases} \tag{22}$$

Since

$$Y(e^{j\theta}) = H(e^{j\theta})V(e^{j\theta}) = (1 + cos\theta)V(e^{j\theta}) \tag{23}$$

and

$$V(e^{j\theta}) = S(e^{2j\theta}) \tag{24}$$

then

$$Y_a(\omega) = \begin{cases} T_s(1+cos\omega T_s)S(e^{j2\omega T_s}), & |\omega| < 10000\pi \\ 0, & otherwise \end{cases} \tag{25}$$

which is the product of $(1+cos\omega T_s)$ and $T_sS(e^{j2\omega T_s})$ as illustrated below.

Thus, $y_a(t)$ does not correspond to slowed-down speech due to the images of $s_a(t)$ that occur in the frequency range $5000\pi < |\omega| < 10000\pi$ and the nonideal linear interpolator.