SI251 - Convex Optimization, Spring 2021
Homework 1

Due on Mar 25, 2021, before class

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points ($\leq 20\%$) of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Do your homework by yourself. Any form of plagiarism will lead to 0 point of this homework. If more than one plagiarisms during the semester are identified, we will prosecute all violations to the fullest extent of the university regulations, including but not limited to failing this course, academic probation, or expulsion from the university.
- If you have any doubts regarding the grading, you need to contact the instructor or the TAs within two days since the grade is announced.

I. Convex Set

1. Describe the dual cone for each of the following cones.

(1) $K = \{0\}$. (5 points)

(2) $K = \mathbb{R}^2$. (5 points)

(3) $K = \{(x_1, x_2) \mid |x_1| \leq x_2\}$. (5 points)

(4) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$. (5 points)

2. *Hyperbolic sets.* Show that the *hyperbolic set* $\left\{\mathbf{x} \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\right\}$ is convex. As a generalization, show that $\left\{\mathbf{x} \in \mathbb{R}_+^n \mid \prod_{i=1}^n x_i \geq 1\right\}$ is convex. *Hint.* If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$. (15 points)

3. *Solution set of a quadratic inequality.* Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^{\mathrm{T}} \mathbf{A} \mathbf{x} + \mathbf{b}^{\mathrm{T}} \mathbf{x} + c \leq 0\right\},$$

with $\mathbf{A} \in \mathbb{S}^n, \mathbf{b} \in \mathbb{R}^n,$ and $c \in \mathbb{R}$.

(1) Show that C is convex if $\mathbf{A} \succeq 0$. (5 points)

(2) Show that the intersection of C and the hyperplane defined by $\mathbf{g}^{\mathrm{T}}\mathbf{x} + h = 0$ (where $\mathbf{g} \neq \mathbf{0}$) is convex if $\mathbf{A} + \lambda \mathbf{g}\mathbf{g}^{\mathrm{T}} \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$. (10 points)

II. Convex Function

1. Determine the convexity (i.e., convex, concave, or neither) of the following functions.

(1) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}_{++}^2$ (5 points)

(2) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}_{++}^2$. (5 points)

(3) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$. (5 points)

(4) $f(x_1, x_2) = \sqrt{x_1 x_2}$ on $\mathbb{R}_{++}^2$. (5 points)

2. *Convex hull of functions.* Suppose g and h are convex functions, bounded below, with $\mathbf{dom}\ g = \mathbf{dom}\ h = \mathbb{R}^n$. The convex hull function of g and h is defined as

$$f(\mathbf{x}) = \inf\{\theta g(\mathbf{y}) + (1-\theta) h(\mathbf{z}) \mid \theta \mathbf{y} + (1-\theta)\mathbf{z} = \mathbf{x},\ 0 \leq \theta \leq 1\},$$

where the infimum is over θ, $\mathbf{y}$, $\mathbf{z}$. Show that the convex hull of h and g is convex. Describe $\mathbf{epi}\ f$ in terms of $\mathbf{epi}\ g$ and $\mathbf{epi}\ h$. (10 points)

3. Show that the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are convex.

(1) The difference between the maximum and minimum value of a polynomial on a given interval, as a function of its coefficients:

$$f(\mathbf{x}) = \sup_{t \in [a,b]} p(t) - \inf_{t \in [a,b]} p(t), \text{ where } p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}.$$

a, b are real constants with $a < b$. (5 points)

(2) The 'exponential barrier' of a set of inequalities:

$$f(\mathbf{x}) = \sum_{i=1}^m e^{-1/f_i(\mathbf{x})},\ \mathbf{dom}\ f = \{\mathbf{x} \mid f_i(\mathbf{x}) < 0, i = 1, \ldots, m\}.$$

The functions $f_i(\mathbf{x})$ are convex. (5 points)

(3) The function

$$f(\mathbf{x}) = \inf_{\alpha > 0} \frac{g(\mathbf{y} + \alpha \mathbf{x}) - g(\mathbf{y})}{\alpha}$$

if g is convex and $\mathbf{y} \in \mathbf{dom}\ g$. (It can be shown that this is the directional derivative of g at $\mathbf{y}$ in the direction $\mathbf{x}$.) (10 points)