1 Constant Sum Coalitional Game

A constant sum coalitional game (N, v) is one in which for every coalition C:

$$v(C) + v(N \backslash C) = c > 0$$

where c is a constant.

1.1 Essential Game(1pt)

A coalitional game (N, v) is essential if:

$$\sum_{i \in N} v(i) \neq v(N)$$

Prove that the core of any essential constant sum coalitional game with $|N| > 2$ is empty.

Suppose x is an allocation in core. Then $\forall i$, $x(i) \geq v(i)$.
If $\exists i$, s.t. $x(i) > v(i)$:
Since $x(i) + \sum_{j \in N \backslash \{i\}} x(j) = c$,
then $\sum_{j \in N \backslash \{i\}} x(j) = c - x(i) < c - v(i) = v(N \backslash \{i\})$, which makes x not in core.
Thus, $\forall i$, $x(i) = v(i)$.
However, now $\sum_{i \in N} x(i) = \sum_{i \in N} v(i) \neq v(N)$, which makes x not in core (contradiction).
Therefore, the core of the given game is empty.

2 Shapley Value

Consider the following characteristic form game with three players:

$$\begin{aligned} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = a \quad v(\{1,3\}) = b \quad v(\{2,3\}) = c \\ v(\{1,2,3\}) = 1 \end{aligned}$$

Assume that $0 \leq a, b, c \leq 1$.

2.1 (1pt)

Find the conditions about a, b, c under which the core is non-empty.

$a + b + c \leq 2$.

2.2 (1pt)

Compute the Shapley value of the game.

$$\phi_1 = \frac{1}{6}(a + b - 2c + 2)$$
$$\phi_2 = \frac{1}{6}(a + c - 2b + 2)$$
$$\phi_3 = \frac{1}{6}(b + c - 2a + 2)$$

2.3 (2pt)

Assuming the core is non-empty, does the Shapley value belong to the core? Under what conditions will the Shapley value belong to the core of this game?

If the core is non-empty, the Shapley value belong to the core if and only if:

$$\begin{cases} 4a + b + c \leq 4 \\ 4b + a + c \leq 4 \\ 4c + a + b \leq 4 \end{cases}$$

3 Core

Consider the game with 5 players, where player L_1, L_2 and L_3 each have one left-hand glove, and player R_1 and R_2 each have one right-hand glove. The value of a coalition is the number of pairs of gloves it has.

3.1 (1pt)

Find the Shapley value of this game. Is the Shapley value in the core?

The Shapley value is given by

$$\phi = (7/30, 7/30, 7/30, 13/20, 13/20)$$

3.2 (2pt)

Find the core of this game. Prove that there is a unique solution in the core.

The core contains a unique solution $(0, 0, 0, 1, 1)$.
Suppose the solution (a, b, c, d, e) is in the core. First, $v(\{i\}) = 0$ for all i and $v(N) = 2$, so that $a, b, c, d, e \geq 0$ and $a + b + c + d + e = 0$. If $a > 0$, then $b + c + d + e < 2$. However, $v(\{L_2, L_3, R_1, R_2\}) = 2$. Hence $a = 0$. Similarly, $a = b = c = 0$. Now we have $d + e = 2$. If $d > e$, then $a + e = e < 1$. However, $v(\{L_1, R_2\}) = 1$. Thus $d \leq e$. Similarly, we have $e \leq d$. Therefore $d = e = 1$. The only solution in the core is $(0, 0, 0, 1, 1)$.

4 Javelin Competition Prediction

Alice and Bob are watching javelin competition together, and they are trying to predict the score of athletes. For simplicity, we treat the athletes' score as a continuous random variable X in interval $[0, 1]$. However, their belief on distribution of athletes' score are different: Alice is optimistic about the scores, while Bob is relatively pessimistic. Suppose Alice considers the score's cumulative distribution function[1] to be $F_A(x) = x^2$, and Bob considers that to be $F_B(x) = \sqrt{x}$. Assume this function is private and everyone doesn't know other's function. Alice and Bob decides to play a game: Alice first give a demarcation point $a \in [0, 1]$, then Bob guess whether $X > a$ or $X < a$. If he is right, then Bob wins, otherwise Alice wins.

4.1 Cut and Choose (0.5pt)

To ensure a winning rate[2] (under her own belief) at least half, what demarcation point a should Alice give?

$$F_A(a) = 0.5 \Rightarrow a = \frac{1}{\sqrt{2}} \approx 0.7071$$

She should give the demarcation point $a = 1/\sqrt{2}$.

[1]For a continuous variable X and its cumulative distribution function $F(\cdot)$, the probability that X falls in interval $[a, b]$ is $\Pr(a < X \leq b) = F(b) - F(a)$.
[2]The rate represents the win probability under her own belief, instead of the real winning rate.

4.2 Cut and Choose with Knowing Others' Valuation (1pt)

If Alice knows the distribution of Bob F_B secretly, what demarcation point a should she give to maximize her winning rate (under her own belief)? What is the upper bound of the winning rate?

$$F_B(a) = 0.5 \Rightarrow a = 0.25$$
$$F_A(1) - F_A(0.25) = 15/16 = 0.9375$$

She should give the demarcation point $a = 0.25 + \epsilon$, where ϵ is a small offset to incentivize Bob to guess $X < a$. Her winning rate canbe infinitely close to $15/16$.

4.3 Moving-Knife Protocol (1.5pt)

Suppose Charlie's distribution function is $F_C(x) = x$, and he wants to join their game. Now the game is finding two demarcation points $0 < a < b < 1$, then each person choose one of the intervals: $[0, a]$, $(a, b]$ and $(b, 1]$, the person whose interval contains X wins. You should design a process and guarantee each player's winning rate (under their own beliefs) at least $1/3$. Perform Moving-knife protocol and calculate the demarcation points a, b and the allocation of three intervals.

$$F_B\left(\frac{1}{9}\right) = \frac{1}{3} \qquad F_C\left(\frac{4}{9}\right) - F_C\left(\frac{1}{9}\right) = \frac{1}{3}$$

Therefore, allocate $[0, 1/9]$ to B, $(1/9, 4/9]$ to C and $(4/9, 1]$ to A.