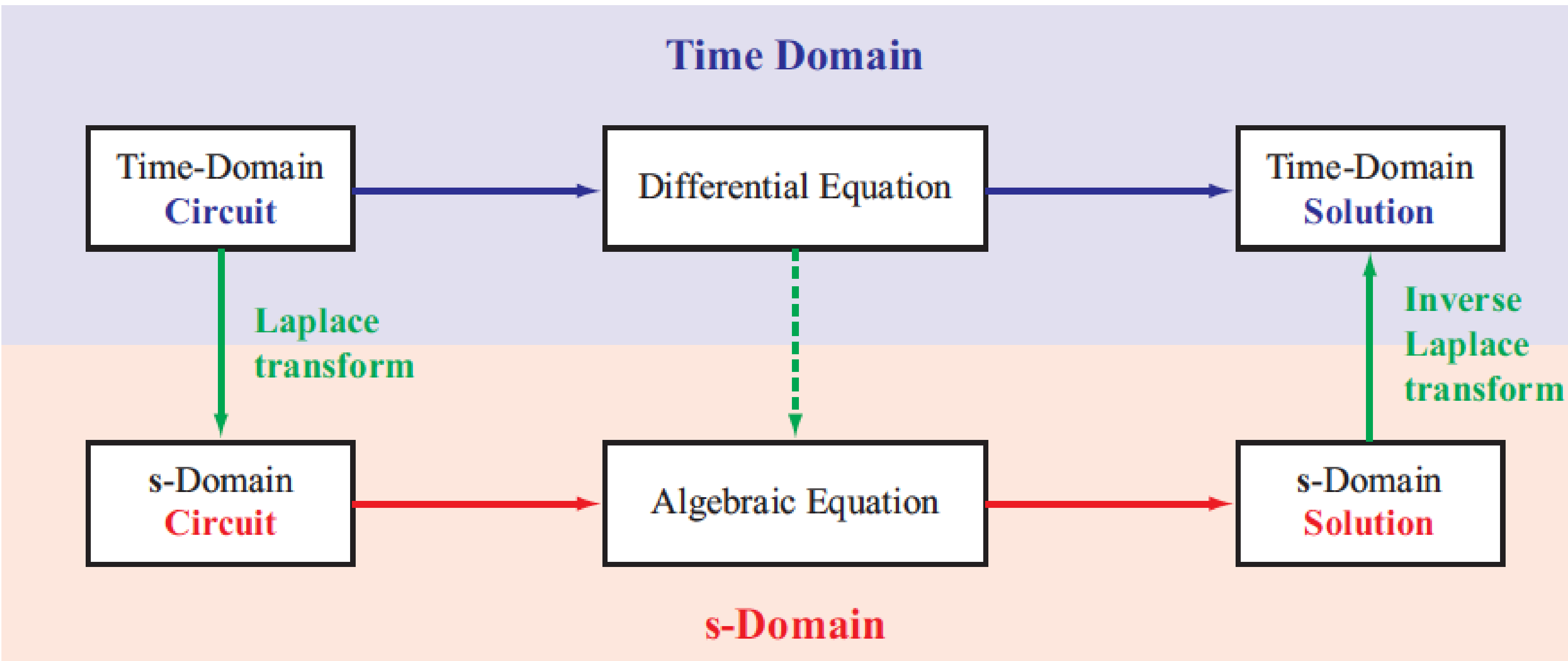
Lecture 13
- Laplace Transform

Laplace Transform Technique



[Source: Berkeley]

Analysis Techniques

Circuit Excitation	Method of Solution
dc (w/ switches)	DC/Transient analysis
ac	Phasor-domain analysis (Steady state only)
Periodic waveform	Fourier series + Phasor-domain (Steady state only)
Waveform (single-sided)	Laplace transform (transient + steady state)

Single-sided: defined over $[0, \infty]$

The French Newton Pierre-Simon Laplace (Late 1700)

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
  - One of the first scientists to suggest the existence of black holes

https://en.wikipedia.org/wiki/Pierre-Simon_Laplace

What are Laplace Transforms?

$$F(s) = \int_{0_-}^{\infty} f(t) e^{-st} dt \qquad F(s) = \mathcal{L}[f(t)]$$

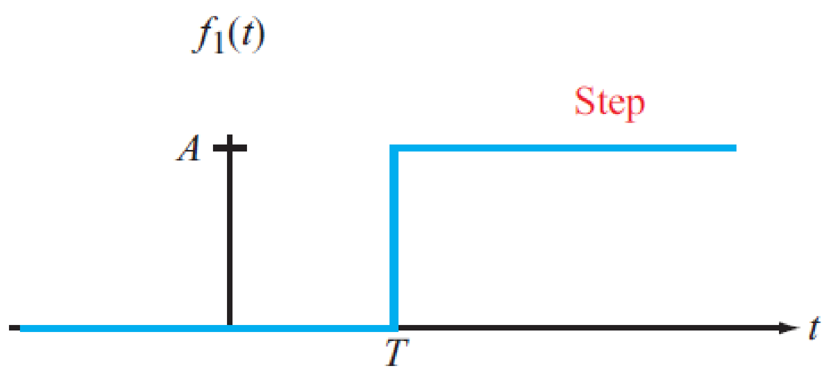
$\Updownarrow$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds \qquad f(t) = \mathcal{L}^{-1}[F(s)]$$

- *t* is real, s is complex! $s = \sigma + j\omega$
- Ignore f(t) for all $t < 0$
- Note in f(*t*) $\rightarrow$ F(*s*), *t* is integrated and *s* is variable.
- Conversely, F(*s*) $\rightarrow$ f(*t*), *t* is variable and s is integrated.

Example: Step Function

$$u(t) \quad \longleftrightarrow \quad \frac{1}{\mathbf{s}}.$$



$$f_1(t) = A\,u(t-T).$$

$$\mathbf{F}(\mathbf{s}) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\,e^{-\mathbf{s}t}\,dt,$$

$$\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{\infty} f_1(t)\,e^{-\mathbf{s}t}\,dt = \int_{0^-}^{\infty} A\,u(t-T)\,e^{-\mathbf{s}t}\,dt$$

$$= A\int_{T}^{\infty} e^{-\mathbf{s}t}\,dt = -\frac{A}{\mathbf{s}}\,e^{-\mathbf{s}t}\Big|_{T}^{\infty} = \frac{A}{\mathbf{s}}\,e^{-\mathbf{s}T}.$$

Example: Exponential Function

$$f(t) = e^{-at}$$

$$\mathcal{L}[f(t)] = \frac{1}{s+a}$$

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

TABLE 15.2

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Homogeneity and Additivity

$$\mathscr{L}[a_1 f_1(t)] = a_1 \mathscr{L}[f_1(t)] = a_1 F_1(s)$$

$$\mathscr{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathscr{L}[f_1(t)] + a_2 \mathscr{L}[f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

here a_1 and a_2 are constants

Important implication:

$$\sum_{k=1}^{k} i_k(t) = 0 \iff \sum_{k=1}^{k} I_k(s) = 0$$

$$\sum_{k=1}^{k} u_k(t) = 0 \iff \sum_{k=1}^{k} U_k(s) = 0$$

Differentiation

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0\_)$$

$$\mathcal{L}[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0\_)-s^{n-2}f^{(1)}(0\_)-\cdots-f^{(n-1)}(0\_)$$

$$=s^nF(s)-\sum_{k=0}^{n-1}s^kf^{(n-1-k)}(0\_)$$

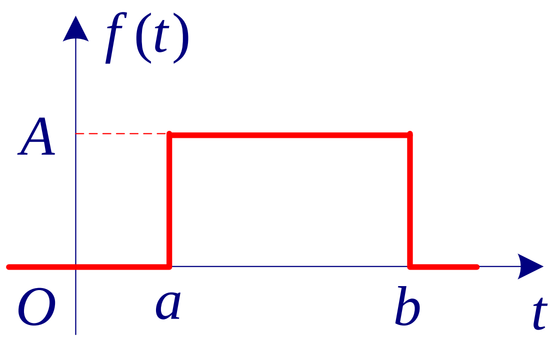
Integration

$$\mathcal{L}\left[\int_{0_-}^{t} f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Translation in the Time Domain

$$\mathcal{L}[f(t-\tau)] = e^{-s\tau}F(s)$$

- Example



$$f(t) = A[\mathrm{u}(\mathrm{t}-\mathrm{a}) - \mathrm{u}(\mathrm{t}-\mathrm{b})]$$

$$F(s) = A\mathcal{L}[\mathrm{u}(\mathrm{t}-\mathrm{a}) - \mathrm{u}(\mathrm{t}-\mathrm{b})] = \frac{A}{s}(e^{-as} - e^{-bs})$$

Translation in Frequency domain

$$\mathcal{L}[e^{\alpha t} f(t)] = F(s-\alpha)$$

- Example

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \qquad \mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

TABLE 12.2 An Abbreviated List of Operational Transforms

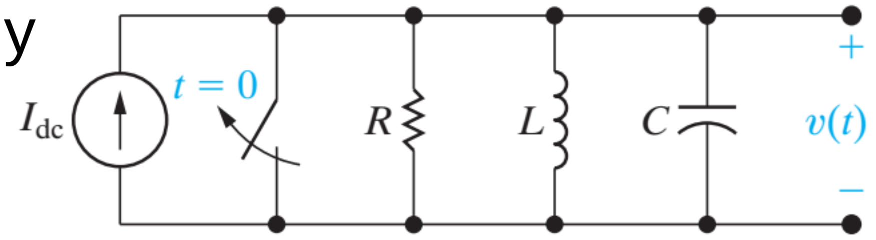
Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

V-I relations of R,L,C

- R $$U_R(s) = RI_R(s)$$
- L $$I_L(s) = \frac{i_L(0_-)}{s} + \frac{1}{sL} U_L(s)$$
- C $$I_C(s) = sCU_C(s) - Cu_C(0_-)$$

Applying the Laplace Transform

- We assume no initial energy stored at t=0



$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right)$$

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s}$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}. \longrightarrow \boxed{v(t) = \mathscr{L}^{-1}\{V(s)\}.}$$

Inverse Transforms

In principle, we can recover f from F via

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(S)e^{st}\,ds$$

Surprisingly, this formula isn't really useful!

What is more common/useful as follows:

$$\boxed{F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_n}{s-p_n} = \sum_{j=1}^{n}\frac{K_j}{s-p_j}}$$

Generally

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

a_i and b_i are real constants, and the exponents m,n are positive integers

- If m<n, proper rational function
- If m>n, improper rational function

$$F(s)=\frac{P(s)}{Q(s)}=A(s)+\frac{B(s)}{Q(s)}$$

$$F(s)=\frac{s^3+5s^2+10s+16}{s+3}=s^2+2s+4+\frac{4}{s+3}$$

Partial Fraction Expansion

- Let F(s) be proper rational function, then

$$F(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\cdots+\frac{K_n}{s-p_n}=\sum_{j=1}^{n}\frac{K_j}{s-p_j}$$

p_j(j=1， 2， …， n) are the roots of equation $Q(s)$=0

- Distinct and multiple (real) roots

Example:

$\frac{s+6}{s(s+3)(s+1)^2}$ has

2 distinct roots: s=0 and s=-3,

1 multiple root of multiplicity 2 occurs at s=-1

Partial Fraction Expansion with Real Distinct Roots

$$F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_n}{s-p_n} = \sum_{j=1}^{n} \frac{K_j}{s-p_j}$$

$K_j(j=1，2，…，n)$ are unknown constants

If the roots are real, $p_i \neq p_j \;\; for\, \forall i \neq j$

$$K_j = \lim_{s \to p_j} (s-p_j)F(s) = (s-p_j)F(s)\Big|_{s=p_j}$$

Exercise

$$F(s)=\frac{s^2+3s+5}{s^3+6s^2+11s+6}$$

$$F(s)=\frac{s^2+3s+5}{(s+1)(s+2)(s+3)}=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+3}$$

Partial Fraction Expansion with Multiple Roots

- If Q(s) has r^{th} multiple roots at p_1, *while others are distinct single root:*

$$F(s)=\frac{K_{11}}{s-p_1}+\frac{K_{12}}{(s-p_1)^2}+\cdots+\frac{K_{1r}}{(s-p_1)^r}+\frac{K_{r+1}}{s-p_{r+1}}\cdots+\frac{K_n}{s-p_n}$$

$$K_{1r}=(s-p_1)^r F(s)\Big|_{s=p_1}$$

$$K_{1(r-1)}=\frac{d}{ds}[(s-p_1)^r F(s)]_{s=p_1}$$

$$K_{1(r-2)}=\frac{1}{2!}\frac{d^2}{ds^2}[(s-p_1)^r F(s)]_{s=p_1}$$

$$\vdots$$

$$K_{11}=\frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}[(s-p_1)^r F(s)]_{s=p_1}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{K_{11}}{s-p_1}+\frac{K_{12}}{(s-p_1)^2}+\cdots+\frac{K_{1r}}{(s-p_1)^r}+\frac{K_{r+1}}{s-p_{r+1}}\cdots+\frac{K_n}{s-p_n}\right]$$

$$=\left[(K_{11}+K_{12}t+\cdots+\frac{1}{(r-1)!}K_{1r}t^{r-1})e^{p_1 t}+(K_{r+1}e^{p_{r+1}t}+\cdots+K_n e^{p_n t})\right]u(t)$$

Exercise

$$F(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}$$

$$F(s)=\frac{K_{11}}{s}+\frac{K_{21}}{s+1}+\frac{K_{31}}{s+2}+\frac{K_{32}}{(s+2)^2}$$

$$F(s)=\frac{K_{11}}{s}+\frac{K_{21}}{s+1}+\frac{K_{31}}{s+2}+\frac{K_{32}}{(s+2)^2}$$

$$K_{11}=sF(s)\big|_{s=0}=\left.\frac{10s^2+4}{(s+1)(s+2)^2}\right|_{s=0}=1$$

$$K_{21}=(s+1)F(s)\big|_{s=-1}=\left.\frac{10s^2+4}{s(s+2)^2}\right|_{s=-1}=-14$$

$$K_{31}=\frac{d}{ds}\left[(s+2)^2F(s)\right]_{s=-2}=\frac{d}{ds}\left[\frac{10s^2+4}{s^2+s}\right]_{s=-2}$$

$$=\left.\frac{20s(s^2+s)-(10s^2+4)(2s+1)}{(s^2+s)^2}\right|_{s=-2}=13$$

$$K_{32}=(s+2)^2F(s)\big|_{s=-2}=\left.\frac{10s^2+4}{s(s+1)}\right|_{s=-2}=22$$

$$f(t)=[1-14e^{-t}+(13+22t)e^{-2t}]u(t)$$

Partial Fraction Expansion with Complex Roots

If F(s) has a pole of P_1 expressed by a complex number, then it must have a complex root P_2 as a conjugate of P_1

$$p_1 = \alpha + j\omega \quad p_2 = p_1^* = \alpha - j\omega$$

$$F(s) = \frac{K_1}{s-(\alpha + j\omega)} + \frac{K_2}{s-(\alpha - j\omega)}$$

$$K_1 = [s-(\alpha + j\omega)]F(s)|_{s=\alpha+j\omega}$$

$$K_2 = [s-(\alpha - j\omega)]F(s)|_{s=\alpha-j\omega}$$

$$K_2 = K_1^* = |K_1| e^{-j\varphi_K}$$

$$f(t) = K_1 e^{(\alpha+j\omega)t} + K_2 e^{(\alpha-j\omega)t} = |K_1| e^{\alpha t}[e^{j(\omega t+\varphi_K)} + e^{-j(\omega t+\varphi_K)}]$$

$$= 2|K_1| e^{\alpha t}\cos(\omega t + \varphi_K)\, u(t)$$

Partial Fraction Expansion with Complex Roots

- Example：

$$F(s)=\frac{s^2+3s+7}{(s^2+4s+13)(s+1)}$$

$$p_1=-2+j3,\quad p_2=-2-j3,\quad p_3=-1$$

$$F(s)=\frac{K_1}{s-(-2+j3)}+\frac{K_1^*}{s-(-2-j3)}+\frac{K_3}{s+1}$$

$$K_1=\left.\frac{s^2+3s+7}{[s-(-2-j3)](s+1)}\right|_{s=-2+j3}=\frac{4+j3}{18+j6}=0.264e^{-j18.4^\circ}$$

$$K_3=\left.\frac{s^2+3s+7}{s^2+4s+13}\right|_{s=-1}=0.5$$

$$f(t)=[0.528e^{-2t}\cos(3t-18.4^\circ)+0.5e^{-t}]\,u(t)$$

TABLE 12.3 Four Useful Transform Pairs

Pair Number	Nature of Roots	$F(s)$	$f(t)$
1	Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$
4	Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$

Note: In pairs 1 and 2, K is a real quantity, whereas in pairs 3 and 4, K is the complex quantity $\lvert K\rvert \angle\theta$.

- it is important to note that *K* is defined as the coefficient associated with the denominator term $s + \alpha - j\beta$

Application to Integrodifferential Equations

- The Laplace transform is useful in solving linear integrodifferential equations.
  - Initial conditions are automatically taken into account.

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

subject to $v(0) = 1, v'(0) = -2$.

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4} \qquad v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$