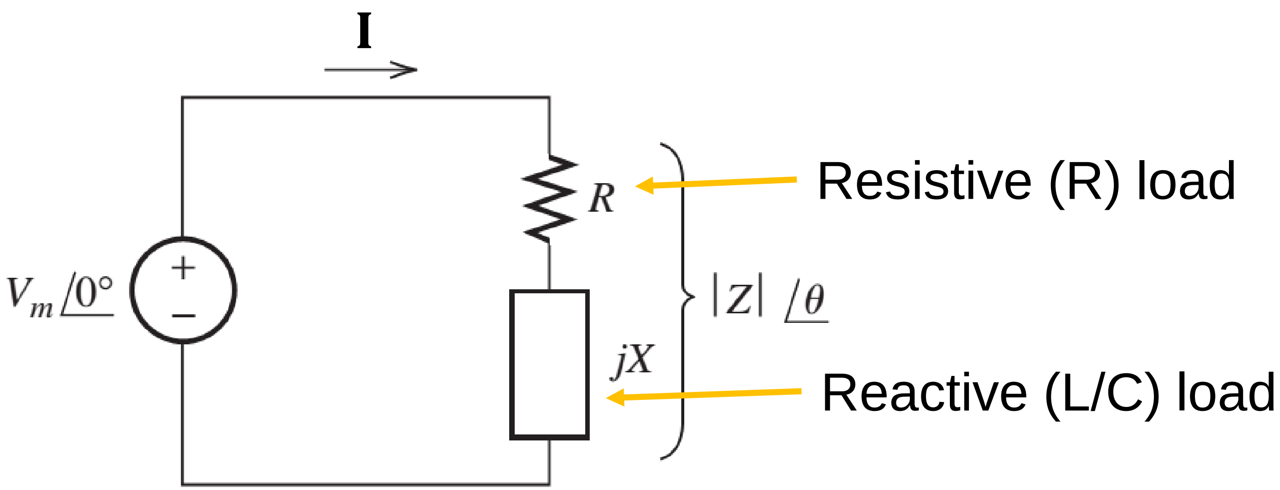
Lecture 9

- AC Power Calculation

Power in AC Circuits

- Consider the situation shown below: A voltage $v(t) = V_m \cos(\omega t)$ is applied to an RLC network.

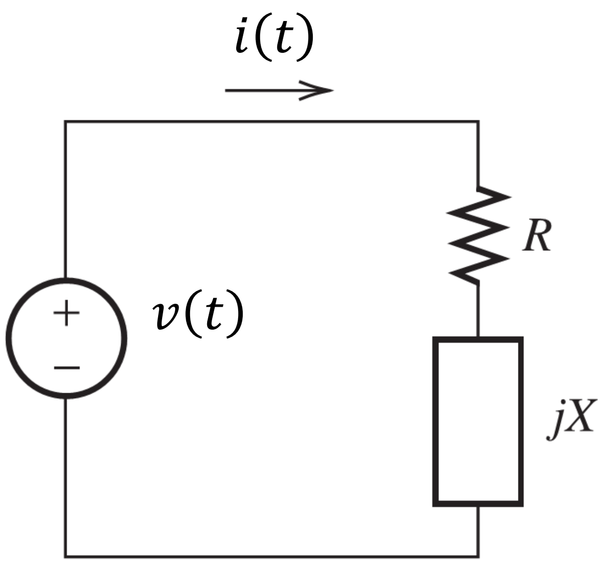


Outline

- Instantaneous power
  - A function of time
- Average power
  - For R
- Reactive power
  - For L/C
- Complex power
  - Apparent power
  - Power factor

AC Power in Time Domain: Instantaneous

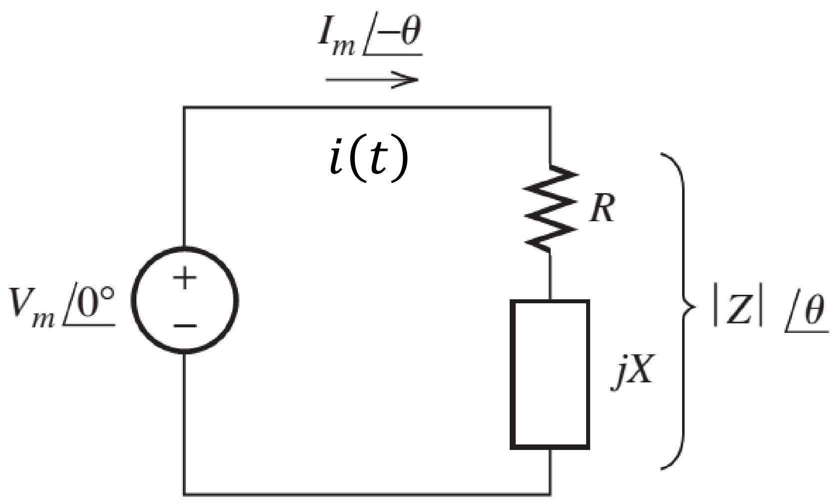
$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$



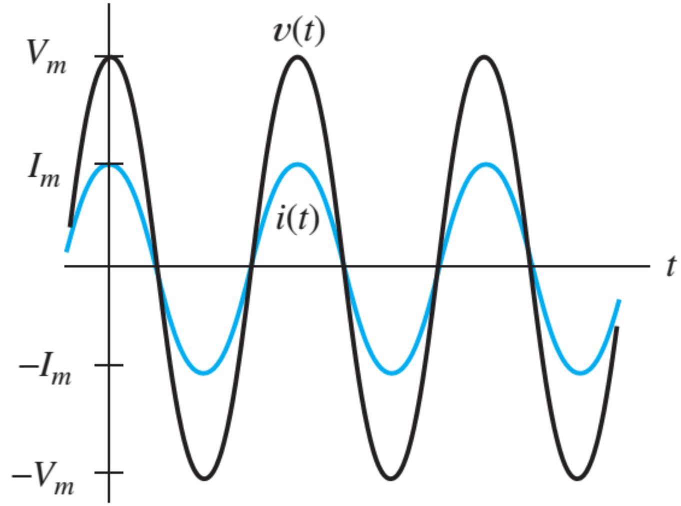
Instantaneous power:
power at any instant of time.

$$p(t) = v(t)i(t) =$$

Instantaneous Power: Resistive Load



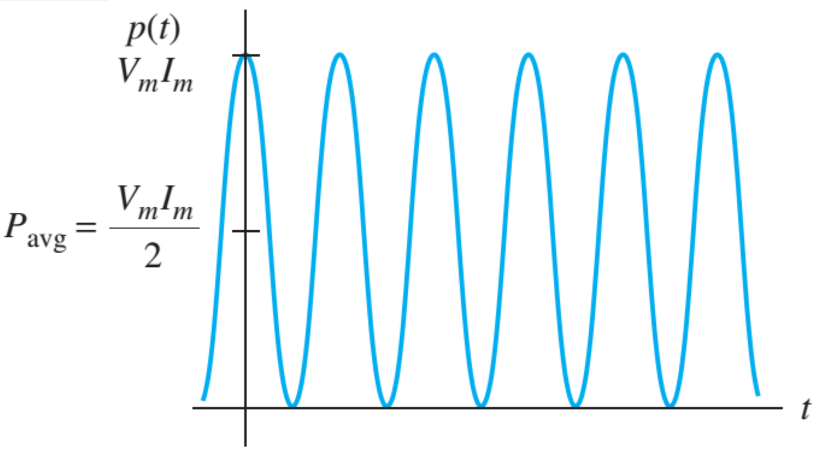
$v(t) = V_m \cos(\omega t) \quad \theta_v = 0^\circ$



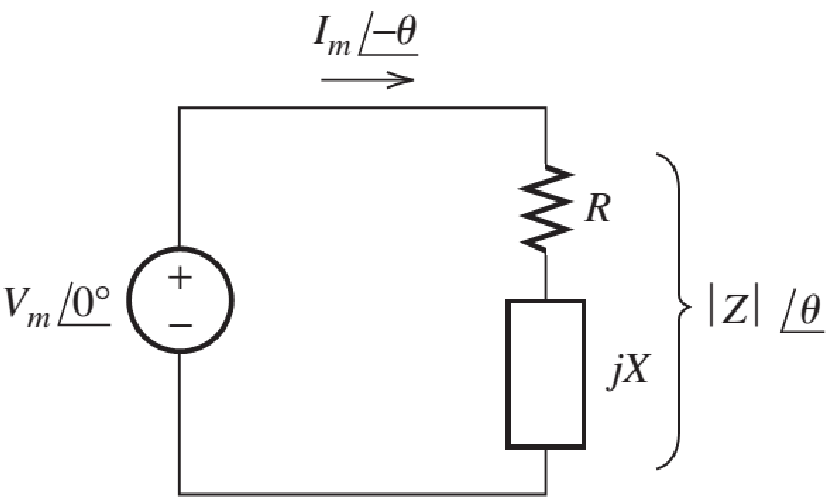
$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

$Z = R, \theta = \theta_v - \theta_i =$

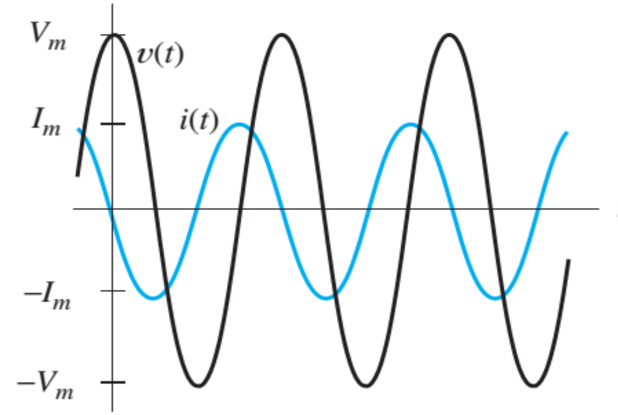
$p(t) = v(t)i(t) =$



Instantaneous Power: Capacitive Load



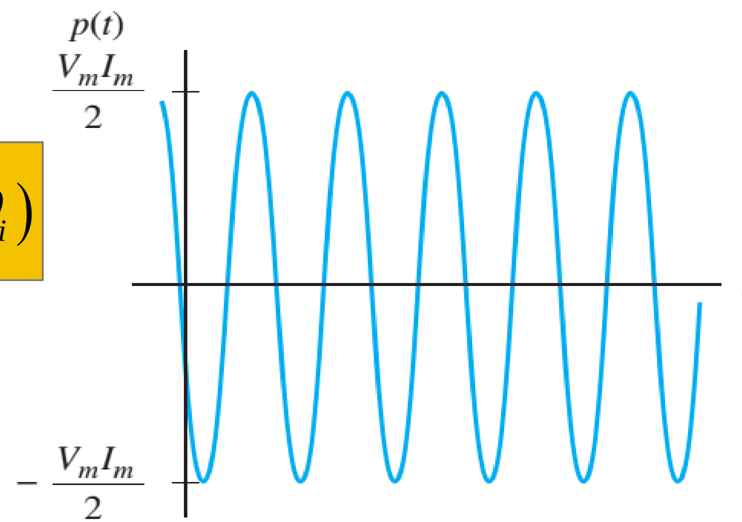
$$v(t) = V_m \cos(\omega t) \quad \theta_v = 0^\circ$$



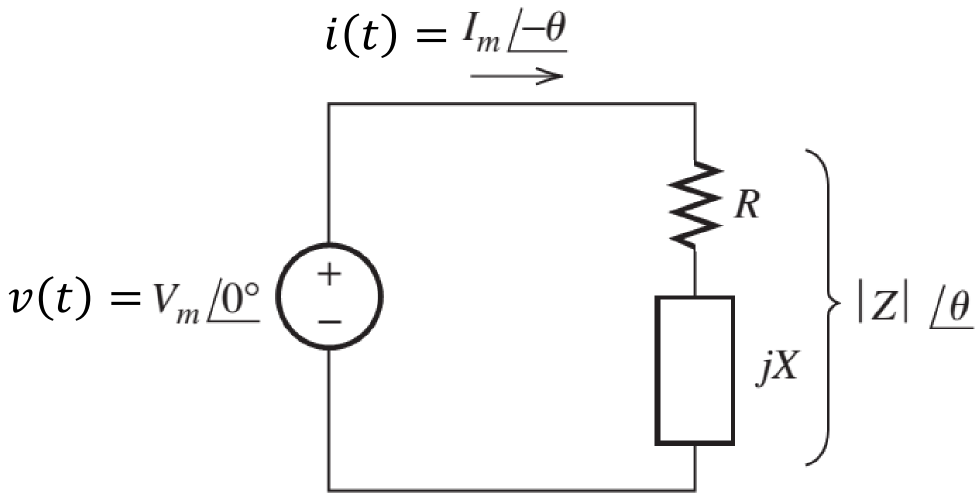
$$Z = \frac{1}{\omega C}\angle -90^\circ, \theta = \theta_v - \theta_i =$$

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

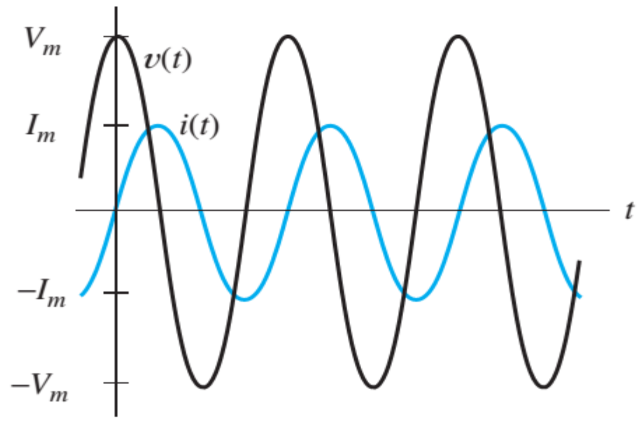
$$p(t) = v(t)i(t) = -\frac{V_m I_m}{2}\sin(2\omega t)$$



Instantaneous Power: Inductive Load



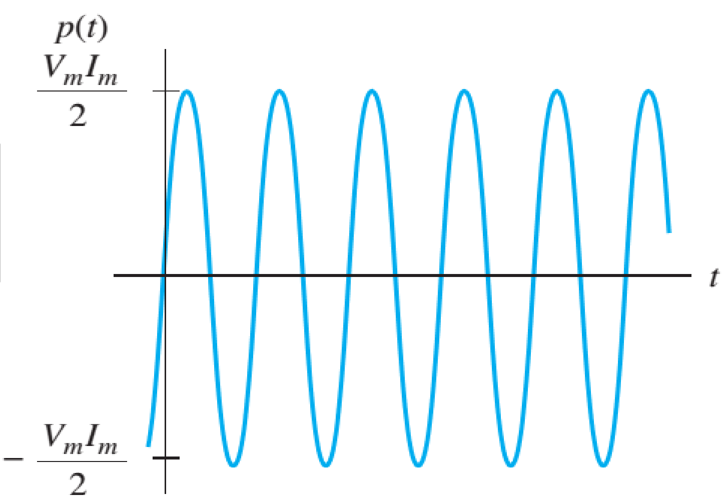
$v(t) = V_m \cos(\omega t) \quad \theta_v = 0^{\circ}$



$Z = \omega L \angle 90^{\circ}, \theta = \theta_v - \theta_i =$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

$$p(t) = v(t) i(t) = \frac{V_m I_m}{2} \sin(2\omega t)$$



Outline

- Instantaneous power
  - A function of time
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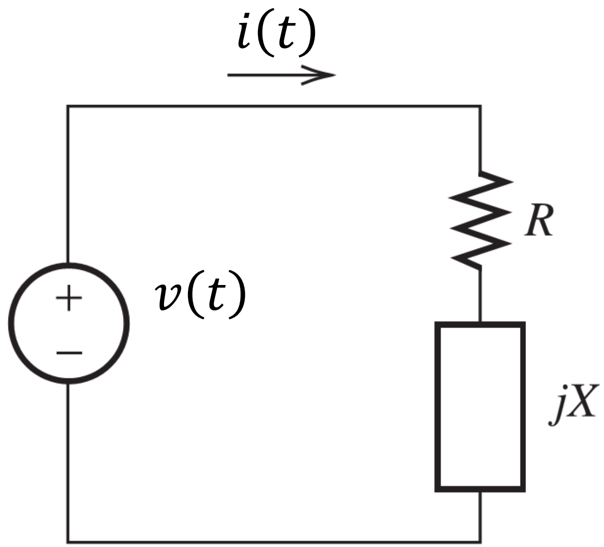
  - Power factor

Average Power *P*: Real Dissipated Power

$$v(t)=V_m\cos(\omega t+\theta_v) \quad i(t)=I_m\cos(\omega t+\theta_i)$$

$$\boxed{p(t)=\frac{1}{2}V_mI_m\cos(\theta_v-\theta_i)+\frac{1}{2}V_mI_m\cos(2\omega t+\theta_v+\theta_i)}$$



<u>Average (or real)</u> power (unit: watts):

$$P=\frac{1}{T}\int_0^T p(t)\,dt$$

p(t): First item + second item

$$\theta_v-\theta_i=\begin{matrix} 0 \quad \text{for a resistor} \\ 90 \text{ degrees for inductor} \\ -90 \text{ degrees for capacitor} \end{matrix}$$

$P =$

$=$

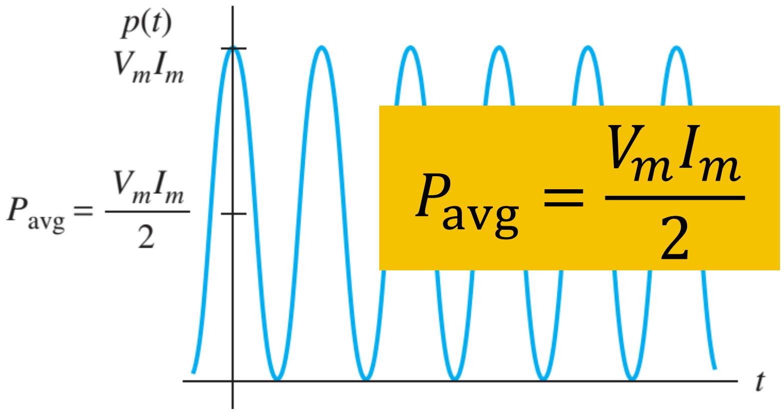
Resistive vs. Inductive

$$p(t)=\frac{1}{2}V_m I_m \cos(\theta_v-\theta_i)+\frac{1}{2}V_m I_m \cos(2\omega t+\theta_v+\theta_i)$$

$$\theta_v = 0^\circ \quad \theta_v - \theta_i = 90 \quad \theta_v + \theta_i = -90$$

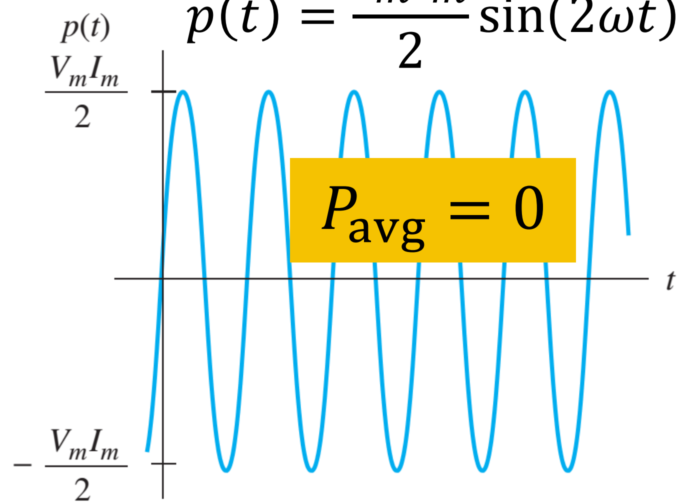
$$\theta_v - \theta_i = 0$$

$$p(t) = \frac{V_m I_m}{2}\sin(2\omega t)$$



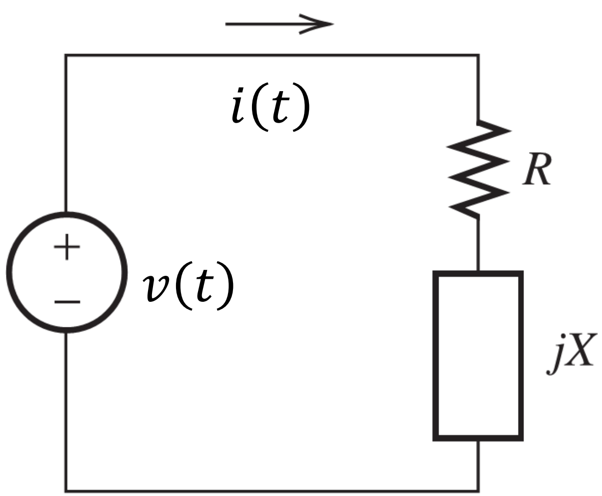
$$P_{avg} = \frac{V_m I_m}{2}$$

Pure resistive load



$$P_{avg} = 0$$

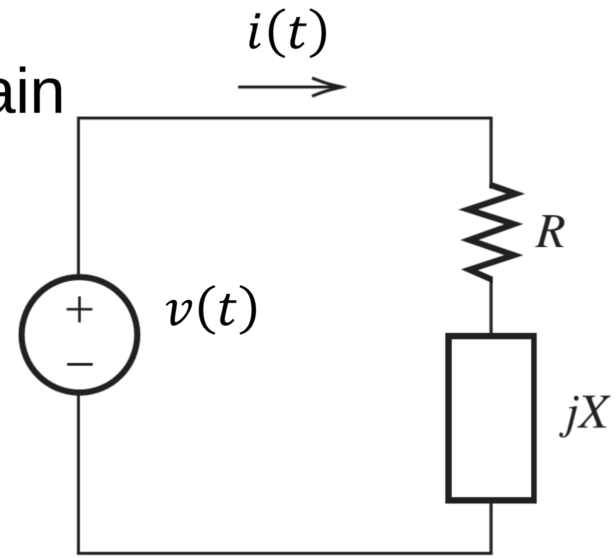
Pure inductive load



- Half the time, the energy is delivered to the inductance;
- The other half time, energy is returned to the source.

Reactive Power *Q*

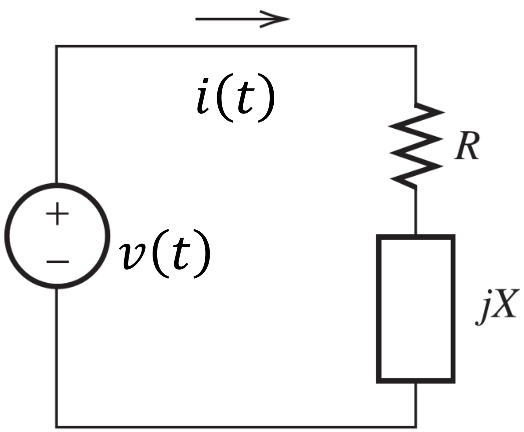
Let us look at Instantaneous power again



$$p(t) = v(t)i(t)$$
$$p(t) = p_R(t)+p_X(t)$$
$$p_R(t) =$$
$$p_X(t) =$$

Reactive Power Q: Peak Exchanged Power

- Definition: The peak instantaneous power associated with the energy storage elements contained in a general load.



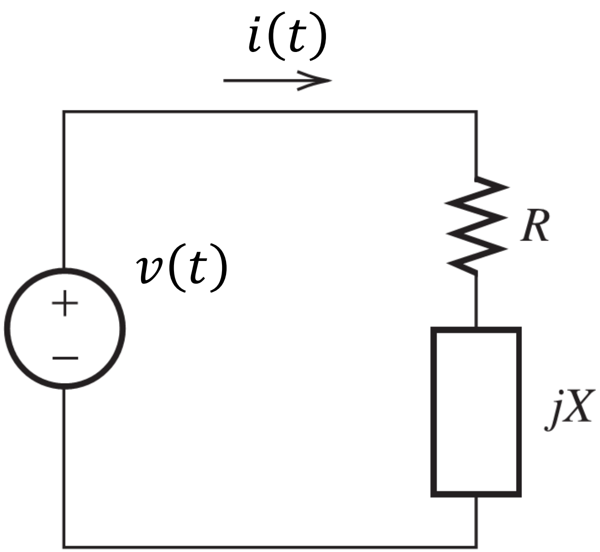
$$\boxed{Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)}$$

$$Q = \begin{cases} 0 & \text{for a resistive } (\theta_v - \theta_i = 0^\circ) \\ \frac{1}{2} V_m I_m & \text{for inductive } (\theta_v - \theta_i = 90^\circ) \\ -\frac{1}{2} V_m I_m & \text{for capacitive } (\theta_v - \theta_i = -90^\circ) \end{cases}$$

- Reactive power is still of concern to power-system engineers
  - Transmission lines/transformers/fuses et al. must be capable of withstanding the current associated with reactive power.

Example

- Find the average power and reactive power absorbed by an impedance $Z = 30 - j70\Omega$, when a voltage $\mathbf{V} = 120\angle 0^{\circ}$ is applied across it.



$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{120\angle 0^{\circ}}{30 - j70} = \frac{120\angle 0^{\circ}}{76.16\angle -66.8^{\circ}}$$
$$= 1.576\angle 66.8^{\circ}\text{ A}$$

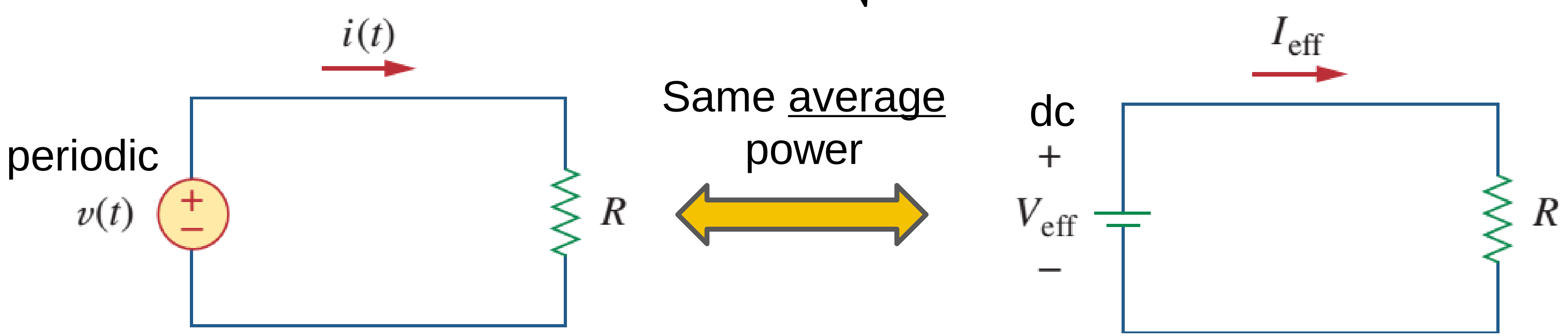
$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = 37.24\text{W}$$

$$Q = \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i) = -86.91\text{VAR}$$

Review: Effective (=RMS) Value

- For any periodic function $x(t)$ in general, its rms value is

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt}$$



$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \qquad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$

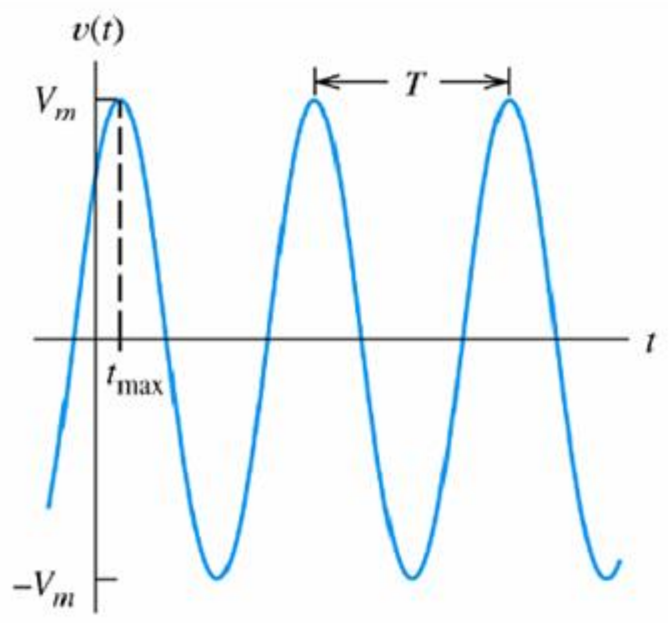
Example: RMS of a Sinusoidal

- The RMS value of $v(t) = V_m \cos(\omega t + \phi)$ is

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt}$$

$$= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cdot \cos^2(\omega t + \phi)dt}$$

$$=$$



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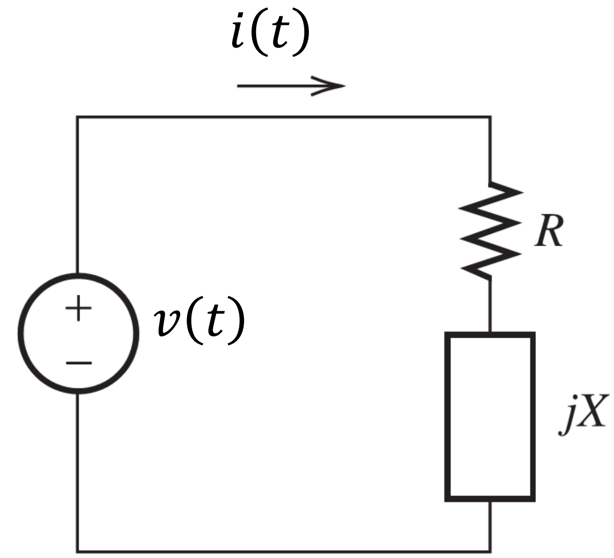
AC Power for General Load: Phasor Domain

$$v(t) = V_m \cos(\omega t + \theta_v) \Longrightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Longrightarrow \mathbf{I} = I_m \angle \theta_i$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \quad Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i)$$



We observe that:

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i)$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + j\frac{1}{2} V_m I_m \sin(\theta_v - \theta_i)$$

Average (or real) power

$$P = \mathrm{Re}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: Watts

Reactive power

$$Q = \mathrm{Im}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

Unit: Volt Amperes Reactive (VARs)

Complex Power *S*

- Define a *single* power metric to include both *P* and *Q*.

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = P + jQ$$

Unit: volt-amp (VA)

Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

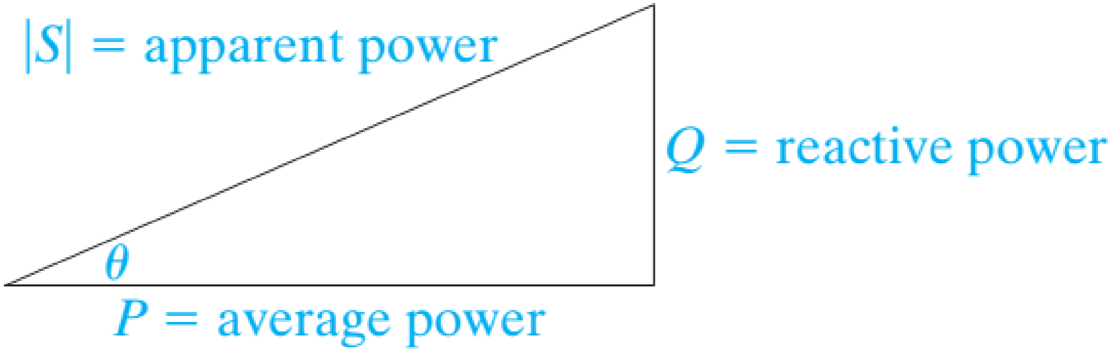
- Apparent power

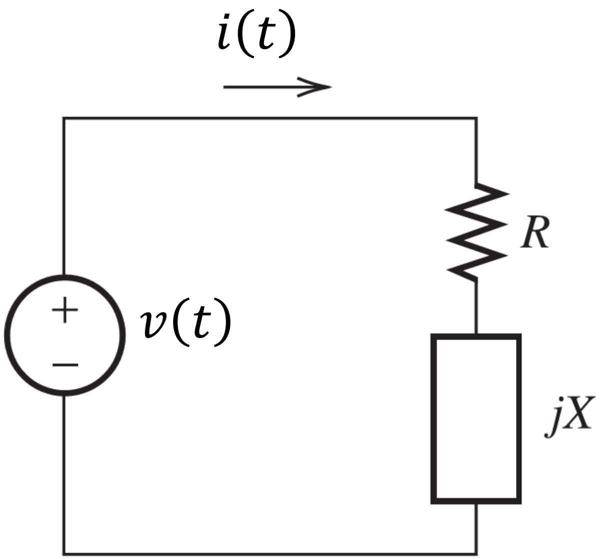
$$s = |\mathbf{S}| = \frac{1}{2}V_m I_m = V_{rms}I_{rms}$$

Unit: volt-amp (VA)

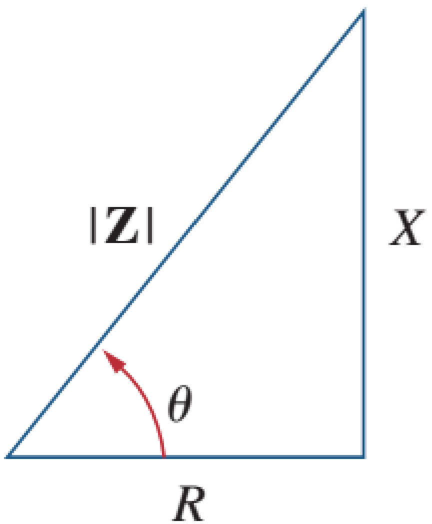
it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits.

Power Triangle





Quantity	Units
Complex power	volt-amps
Average power	watts
Reactive power	var

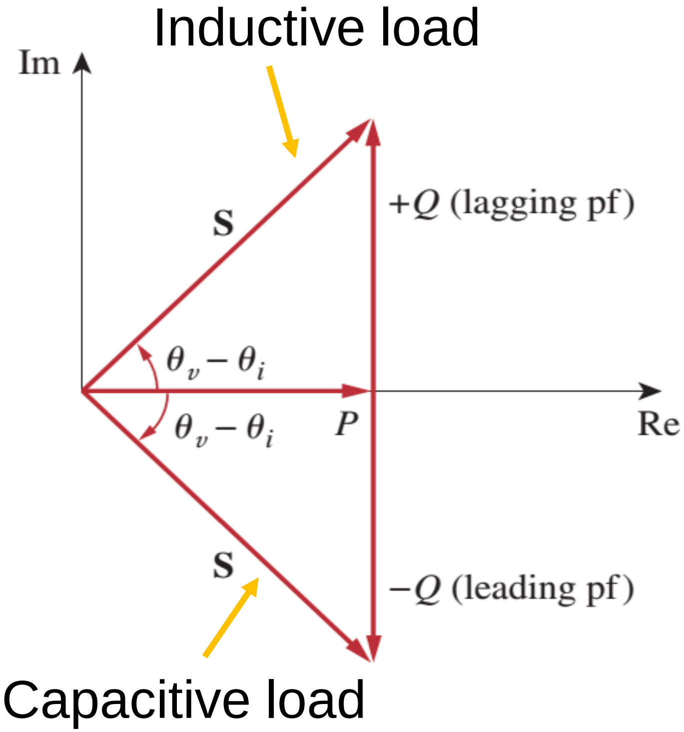


Power Factor

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$= V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$= S \cos(\theta_v - \theta_i)$$

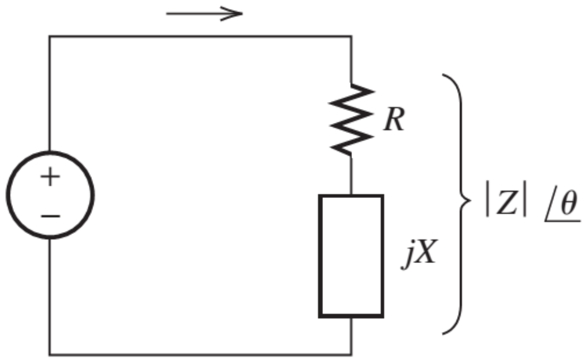


- The power factor

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)}$$

  - $(\theta_v - \theta_i)$ is called <u>power factor angle</u>.
    - >0 means a *lagging* pf (current lags voltage)
    - <0 means a *leading* pf (current leads voltage)
  - pf ranges from 0 to 1.

Power Factor



Power factor leading and lagging relationships for a load $\mathbf{Z} = R + jX$.

Load Type	$\phi_z = \phi_v - \phi_i$	I-V Relationship	*pf*
Purely Resistive ($X = 0$)	$\phi_z = 0$	**I** in-phase with **V**	1
Inductive ($X > 0$)	$0 < \phi_z \le 90°$	**I** lags **V**	lagging
Purely Inductive ($X > 0$ and $R = 0$)	$\phi_z = 90°$	**I** lags **V** by 90°	lagging
Capacitive ($X < 0$)	$-90° \le \phi_z < 0$	**I** leads **V**	leading
Purely Capacitive ($X < 0$ and $R = 0$)	$\phi_z = -90°$	**I** leads **V** by 90°	leading

Example

- A series-connected load draws a current

$$i(t) = 4\cos(100\pi t + 10°)\text{A}$$

when the applied voltage is

$$v(t) = 120\cos(100\pi t - 20°)\text{V}$$

  - Find the apparent power and the power factor of the load.
  - Determine the values that form the series-connected load.

$$V_{\text{rms}} I_{\text{rms}} = 240 \text{ VA}$$

$$\text{pf} = \cos(\theta_v - \theta_i) = 0.866 \qquad \text{(leading)}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 25.98 - j15\ \Omega$$

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Exercise

- The voltage across a load is $v(t) = 60\cos(\omega t - 10°)\mathrm{V}$, and the current through the load is $i(t) = 1.5\cos(\omega t + 50°)$. Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = 45\angle{-60°}\ \text{VA}$$

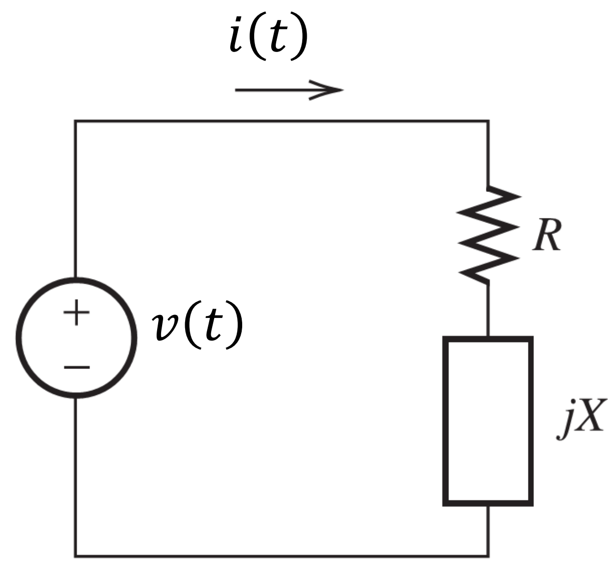
$$\text{pf} = 0.5\ \text{(leading)}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 40\angle{-60°}\ \Omega$$

Quick Summary – Power Calculation

$$v(t) = V_m \cos(\omega t + \theta_v) \Longrightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Longrightarrow \mathbf{I} = I_m \angle \theta_i$$



$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

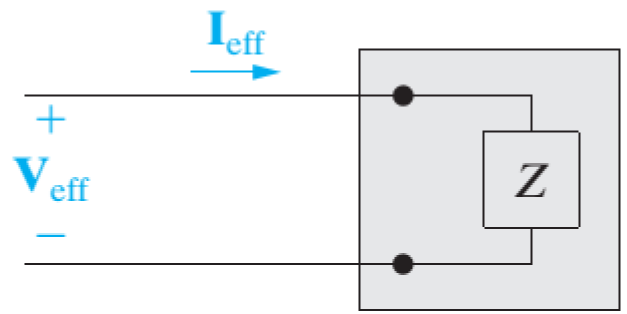
$$S = \frac{1}{2} V_m I_m = V_{rms} I_{rms}$$

$$\boxed{\mathbf{S} = S \angle (\theta_v - \theta_i) = P + jQ}$$

Another Way to Calculate Complex Power

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^{*}$$
$$= \mathbf{V}_{\text{rms}}\left(\frac{\mathbf{V}_{\text{rms}}}{Z}\right)^{*}$$
$$= \frac{|\mathbf{V}_{\text{rms}}|^2}{Z^*}$$

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^{*}$$
$$= \mathbf{I}_{\text{rms}}Z\mathbf{I}_{\text{rms}}^{*}$$
$$= |\mathbf{I}_{\text{rms}}|^2 Z$$
$$= |\mathbf{I}_{\text{rms}}|^2 (R + jX)$$
$$= |\mathbf{I}_{\text{rms}}|^2 R + j|\mathbf{I}_{\text{rms}}|^2 X$$
$$= I_{rms}^2 R + jI_{rms}^2 X$$



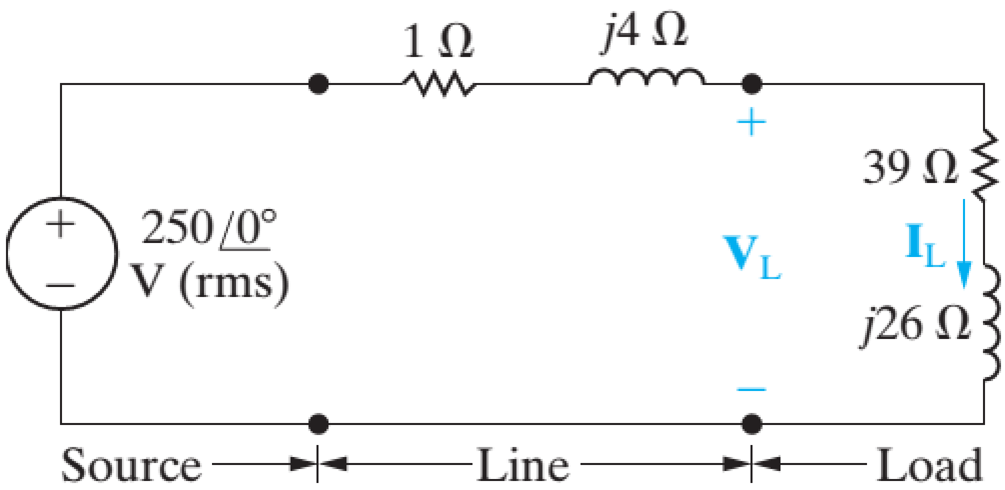
$$\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}} Z$$

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X$$

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^{*}}$$

Example



- Find $\mathbf{V}_\mathrm{L}$ and $\mathbf{I}_\mathrm{L}$.
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

$$\mathbf{I}_\mathrm{L} = \frac{250\angle 0^\circ}{40+j30} = 4 - j3$$
$$= 5\angle -36.87^\circ \text{ (rms)}$$

$$\mathbf{V}_\mathrm{L} = \mathbf{I}_\mathrm{L}(39 + j26)$$
$$= 234 - j13$$
$$= 234.36\angle -3.18^\circ$$

Load:

$$\mathbf{V}_\mathrm{L}\mathbf{I}_\mathrm{L}^* = 975 + \mathrm{j}650 \text{ VA}$$

Line:

$$P = (5)^2(1) = 25 \text{ W}$$

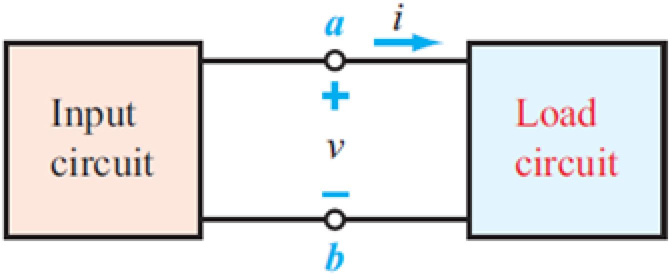
$$Q = (5)^2(4) = 100 \text{ VAR}$$

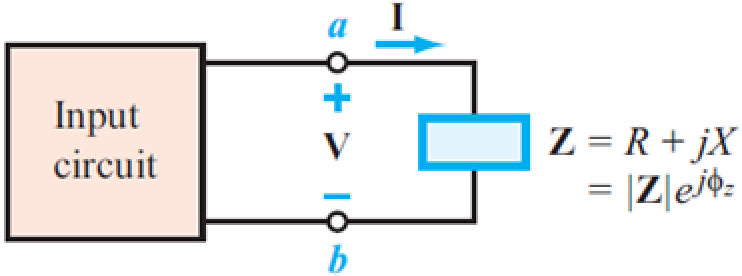
Source:

$$250\angle 0^\circ \mathbf{I}_\mathrm{L}^* = 1000 + \mathrm{j}750 \text{ VA}$$

Complex Power

Time Domain | **Phasor Domain**





$$v(t) = V_m \cos(\omega t + \phi_v) \quad\longleftrightarrow\quad \mathbf{V} = V_m e^{j\phi_v}$$

$$i(t) = I_m \cos(\omega t + \phi_i) \quad\longleftrightarrow\quad \mathbf{I} = I_m e^{j\phi_i}$$

$$V_{rms} = V_m/\sqrt{2} \quad\longleftrightarrow\quad \mathbf{V}_{rms} = V_{rms} e^{j\phi_v}$$

$$I_{rms} = I_m/\sqrt{2} \quad\longleftrightarrow\quad \mathbf{I}_{rms} = I_{rms} e^{j\phi_i}$$

Complex Power

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = P + jQ$$

Real Average Power

$$\begin{aligned} P &= \mathfrak{Re}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \cos(\phi_v - \phi_i) \\ &= I^2_{rms} R \end{aligned}$$

Reactive Power

$$\begin{aligned} Q &= \mathfrak{Im}\,[\mathbf{S}] \\ &= V_{rms} I_{rms} \sin(\phi_v - \phi_i) \\ &= I^2_{rms} X \end{aligned}$$

Apparent Power

$$\begin{aligned} S = |\mathbf{S}| &= \sqrt{P^2 + Q^2} \\ &= V_{rms} I_{rms} \\ &= I^2_{rms} |\mathbf{Z}| \end{aligned}$$

$$\mathbf{S} = S e^{j\phi_s}$$

$$\phi_s = \phi_v - \phi_i = \phi_z$$

Power Factor

$$\begin{aligned} pf &= \frac{P}{S} \\ &= \cos(\phi_v - \phi_i) \\ &= \cos \phi_z \end{aligned}$$