Tutorial 8

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Sampling: Inversion Method

Universality of the Uniform

1°. $F(x) \in [0,1]$

2°. $U \sim \text{Unif}(0,1)$ $\left(P(U \le a) = a,\ a \in [0,1] \right)$

① $U \sim \text{Unif}(0,1)$, $X = F^{-1}(U)$, compute CDF of X.

$F_X(x) = P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x))$

$= F(x)$

Theorem

Let F be a CDF which is a continuous function and strictly increasing on the support of the distribution. This ensures that the inverse function F^{-1} exists, as a function from $(0,1)$ to $\mathbb{R}$. We then have the following results.

1. *Let $U \sim \text{Unif}(0,1)$ and $X = F^{-1}(U)$. Then X is an r.v. with CDF F.*
2. *Let X be an r.v. with CDF F. Then $F(X) \sim \text{Unif}(0,1)$.*

$P(X \le a) = F(a)$

② $Y = F(X) \in [0,1]$ $\quad y \in \mathbb{R}$ $\quad P(Y \le y) = 0 \quad y \le 0$

$P(Y \le y) = 1 \quad y \ge 1.$

$y \in (0,1)$ $\quad P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F[F^{-1}(y)] = y$

$Y \sim \text{Unif}(0,1)$

Sampling: Inversion Method

– Sampling a unit disk uniformly
  - Wrong approach: $r = \xi_1, \theta = 2\pi\xi_2$
  - PDF p(x,y) by normalization is: $p(x, y) = 1/\pi$
  - Transform into polar coordinate: $p(r, \theta) = r/\pi$
  - Compute the marginal and conditional densities

$$p(r) = \int_0^{2\pi} p(r, \theta)\, \mathrm{d}\theta = 2r$$

$$p(\theta|r) = \frac{p(r, \theta)}{p(r)} = \frac{1}{2\pi}$$

  - Integrating and inverting to find P(r), P^{-1}(r), $\mathrm{P}(\theta|r)\ \mathrm{P}^{-1}(\theta|r)$

$$r = \sqrt{\xi_1}$$

$$\theta = 2\pi\xi_2$$

$$p(r, \theta) = r\, p(x, y)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Suppose we draw samples from some density p(r,θ)
- Computing the Jacobian

$$J_T = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

- The determinant: $r\left(\cos^2\theta + \sin^2\theta\right) = r$
- So

$$p(x, y) = p(r, \theta)/r \quad \Longrightarrow \quad p(r, \theta) = r\, p(x, y)$$

Sampling: Inversion Method

- **Cosine-weighted hemisphere sampling**
  - It is useful to have a cosine distribution over the hemisphere (the incident cosine term)
  - We require: $p(\omega) \propto \cos\theta$
  - Derive as before:

$$\int_{\mathcal{H}^2} c\, p(\omega)\, d\omega = 1 \qquad d\omega = \sin\theta\, d\theta\, d\phi \qquad p(\theta,\phi) = \sin\theta\; p(\omega)$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \cos\theta \sin\theta\, d\theta\, d\phi = 1$$

$$c\, 2\pi \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = 1$$

$$c = \frac{1}{\pi}$$

$$\Rightarrow \quad \boxed{p(\theta,\phi) = \frac{1}{\pi} \cos\theta \sin\theta}$$

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

- Computing the Jacobian determinant: $|J_T| = r^2 \sin\theta$
- The corresponding density function

$$p(r,\theta,\phi) = r^2 \sin\theta\; p(x,y,z)$$

- Solid angle defined with spherical coordinates $d\omega = \sin\theta\, d\theta\, d\phi$
- If we have a density function defined over a solid angle

$$p(\theta,\phi)\, d\theta\, d\phi = p(\omega)\, d\omega \quad \Rightarrow \quad p(\theta,\phi) = \sin\theta\; p(\omega)$$

Sampling: Inversion Method

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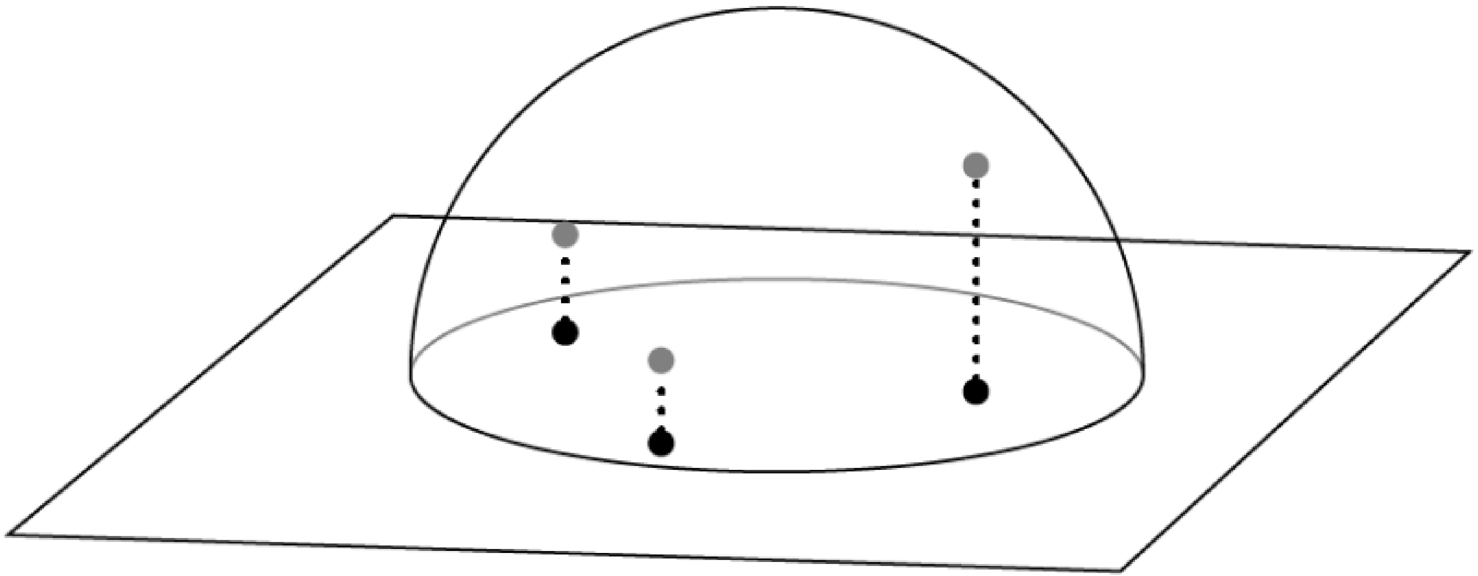
Cosine-weighted hemisphere sampling

- Malley's method
  - Sampling a unit disk and project onto the sphere



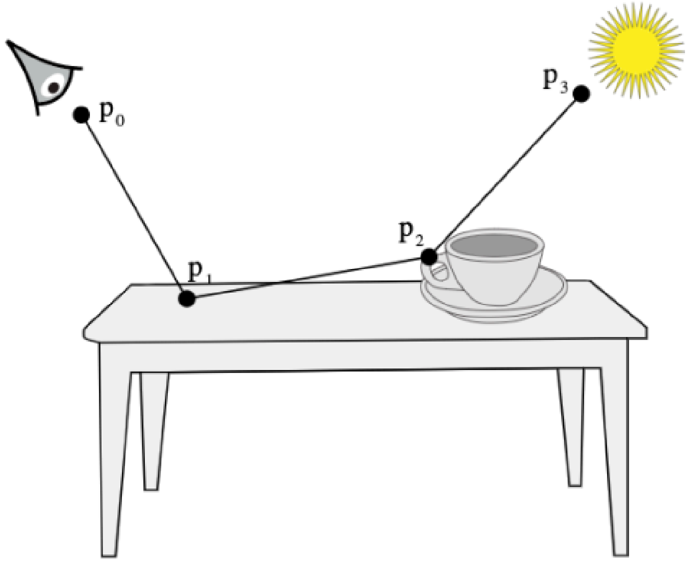
Monte-Carlo Integration

$$I = \int f(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

- p can be arbitrarily chosen
- The shape of p similar to f $\rightarrow$ less variance

LTE

- Light transport equation
  - $L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{\Omega} f(p, \omega_o, \omega_i) L_i(p, \omega_i) |\cos\theta_i| \mathrm{d}\omega_i$
  - Basic idea: out = reflection + emission
  - Notice that $L_o(p, \omega_o)$ and $L_i(p, \omega_i)$ share the same light field but from different solid angles.
  - => Recursive relation



Thanks