

# The Discrete-Time Fourier Transform (ch.5)

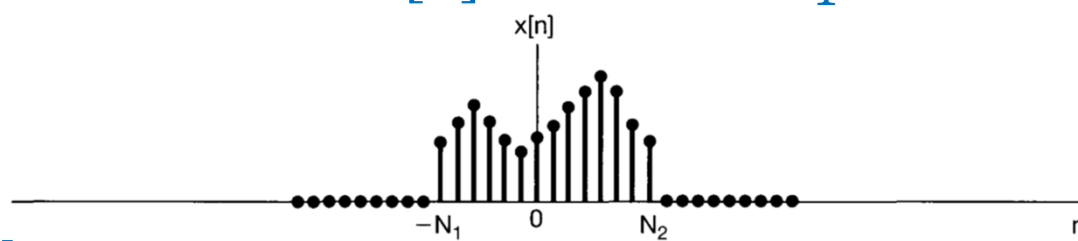
- ☒ Representation of aperiodic signals - Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☐ The multiplication property
- ☐ Duality
- ☐ Systems characterized by difference equations

# Discrete Fourier Transform

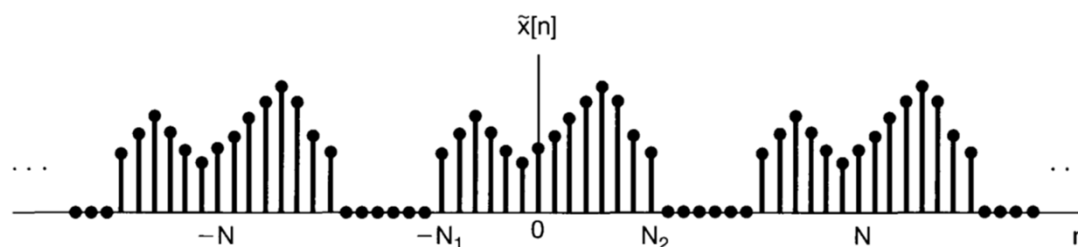


## Development of FT

□ Consider a general sequence of finite duration:  $x[n] = 0$  if  $n < N_1$  or  $n > N_2$



□ Periodic extension of  $x[n]$  with  $N$

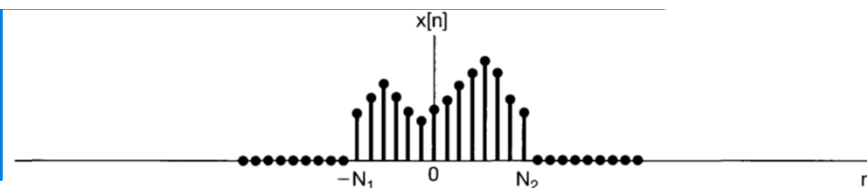


□ FS representation of  $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

# Discrete Fourier Transform



## Development of FT

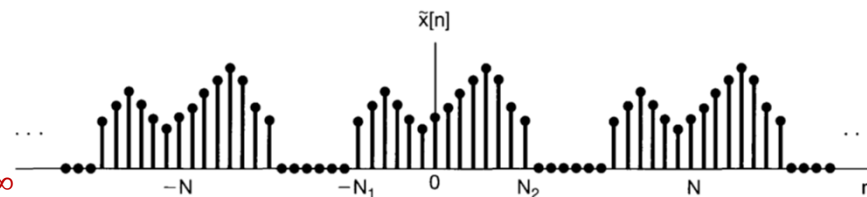
### □ FS coefficients of $\tilde{x}[n]$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n}$$

Define  $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

$$= \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} X(e^{jk\omega_0})$$

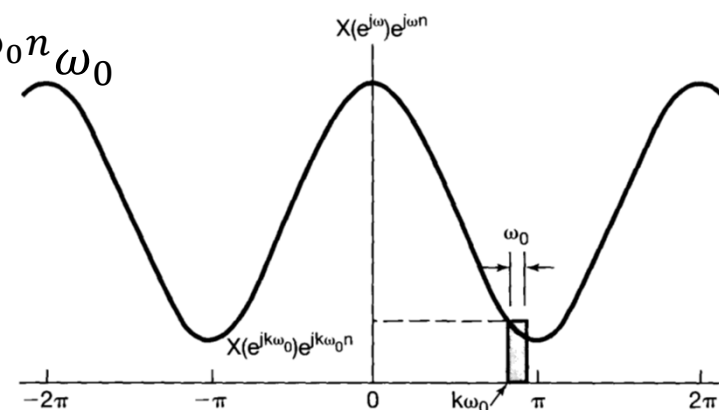


### □ FS of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

### □ $N \rightarrow \infty, \tilde{x}[n] \rightarrow x[n]$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$



# Discrete Fourier Transform



## FT pairs

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Fourier transform (FT)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Inverse Fourier transform

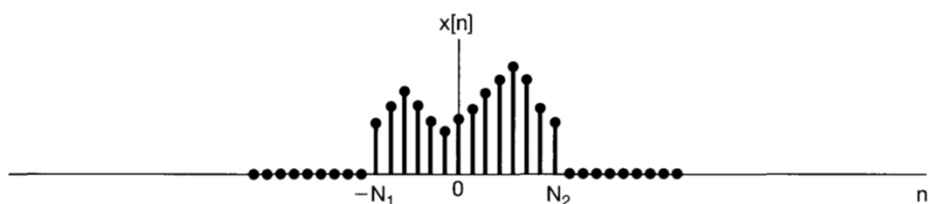
- ❑  $x[n]$  is a linear combination (specifically, an integral) of complex exponentials at different frequencies
- ❑  $X(e^{j\omega})(d\omega/2\pi)$  is the weight for different frequencies
- ❑  $X(e^{j\omega})$  is called the spectrum

# Discrete Fourier Transform



## FT vs. FS

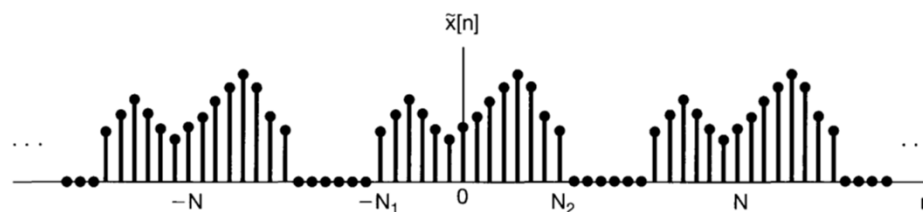
### Fourier transform (FT)



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

### Fourier series (FS)



$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} X(e^{j\omega}) \text{ with } \omega = k(2\pi/N)$$

# Discrete Fourier Transform



## Discrete FT vs. continuous FT

### Discrete FT

$$x[n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

### Continuous FT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

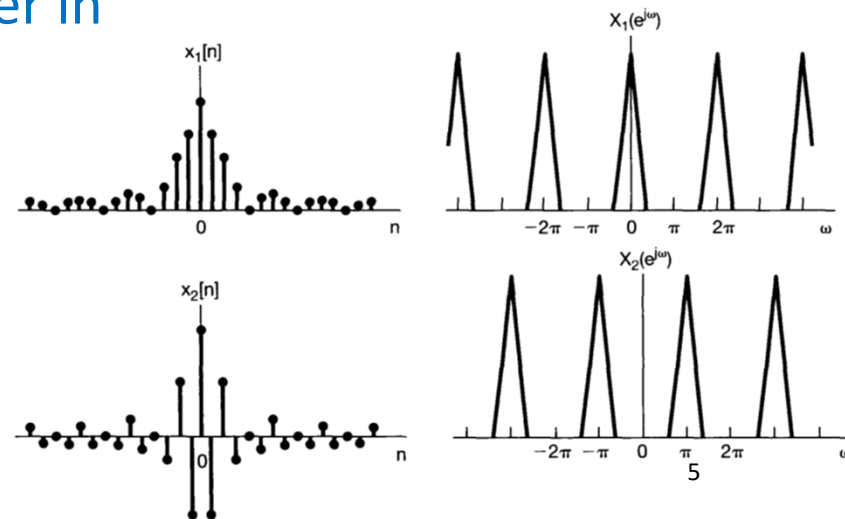
□ Discrete-time complex exponentials that differ in frequency by a multiple of  $2\pi$  are identical

□  $X(e^{j\omega})$  is periodic

□ Finite interval of integration in the synthesis equation for  $x[n]$

□  $\omega = 0, 2\pi, 4\pi, \dots \Rightarrow$  low-frequency

□  $\omega = \pi, 3\pi, 5\pi, \dots \Rightarrow$  high-frequency



# Discrete Fourier Transform



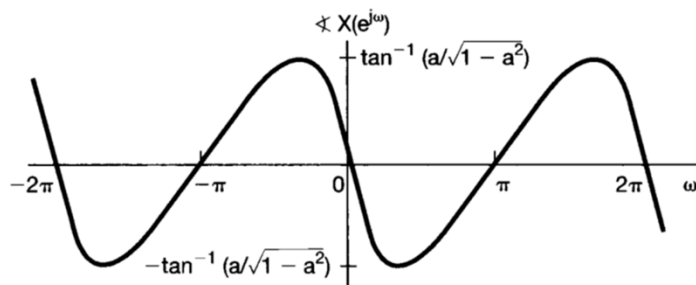
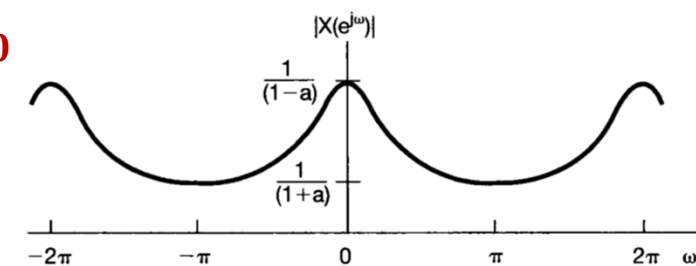
## Examples

$$x[n] = a^n u[n], |a| < 1 \quad X(e^{j\omega}) = ?$$

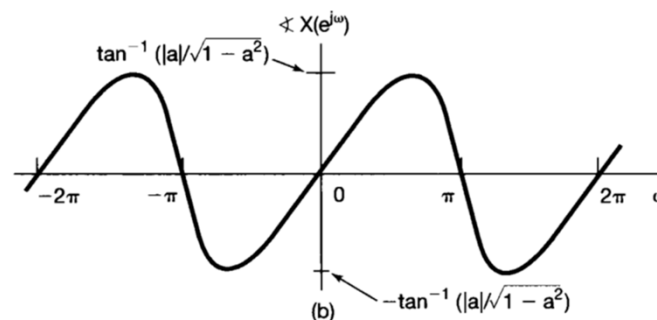
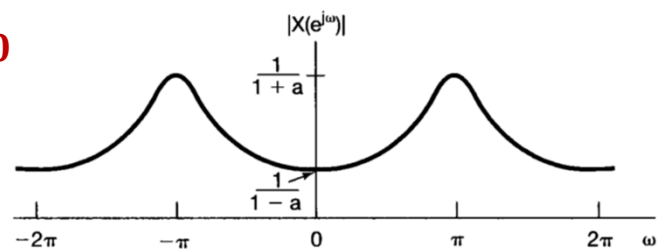
## Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$a > 0$



$a < 0$



# Discrete Fourier Transform

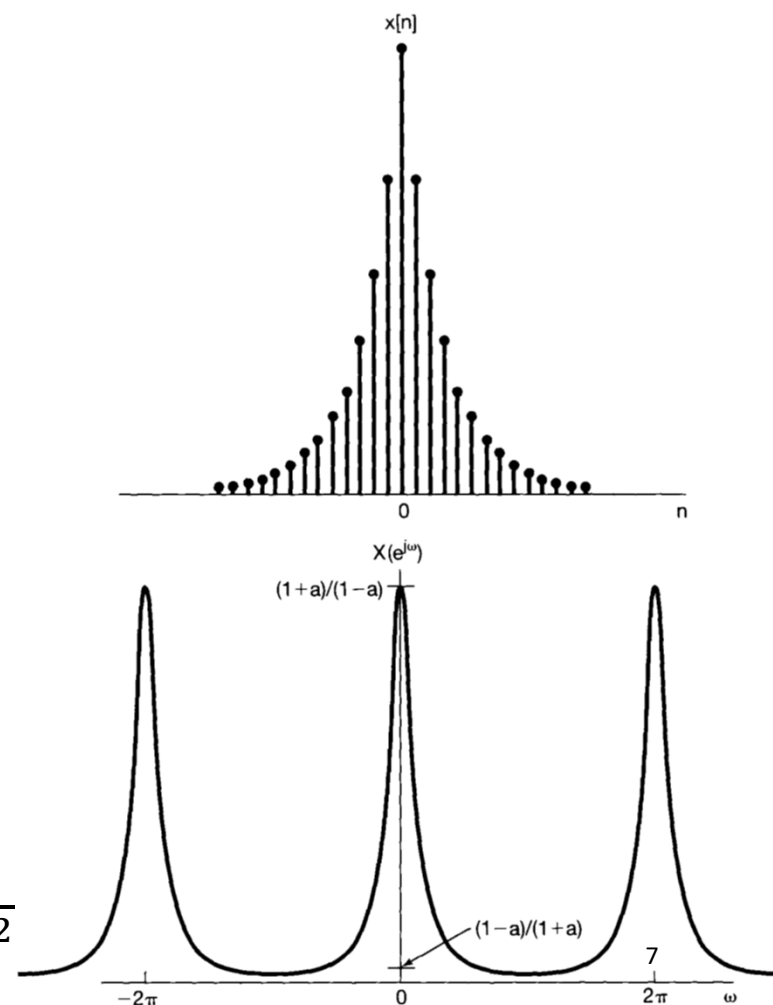


## Examples

$$x[n] = a^{|n|}, |a| < 1 \quad X(e^{j\omega}) = ?$$

## Solution

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m \\ &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2} \end{aligned}$$





# Discrete Fourier Transform



## Examples

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases} \quad X(e^{j\omega}) = ?$$

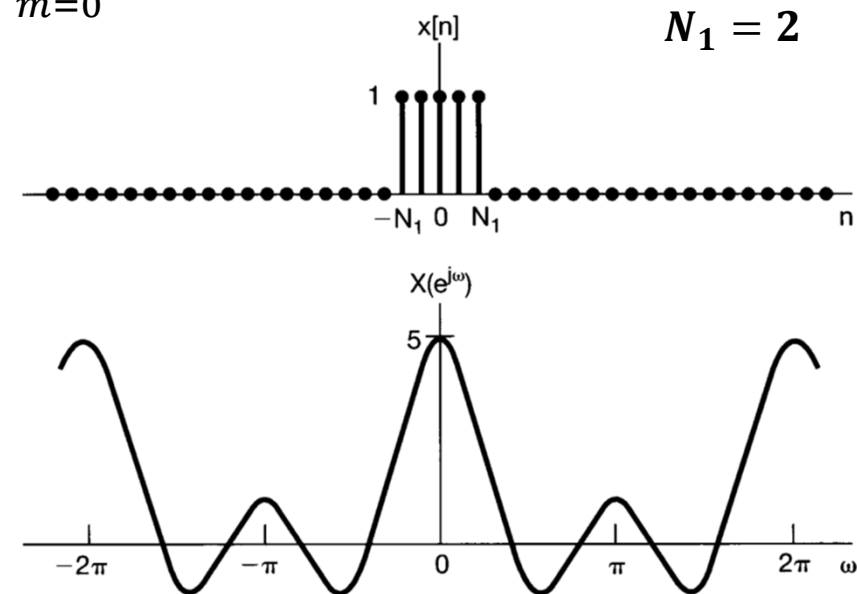
Solution

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)} = e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

$$= e^{j\omega N_1} \left( \frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}} \right)$$

$$= \frac{e^{-j\omega/2} (e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)})}{e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})}$$

$$= \frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$$



# Discrete Fourier Transform



## Convergence of FT

□ For the analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- Finite energy condition

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- Absolutely summable

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

□ For the synthesis equation  $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

- No convergence issues (finite interval of integration)

# Discrete Fourier Transform



## Examples

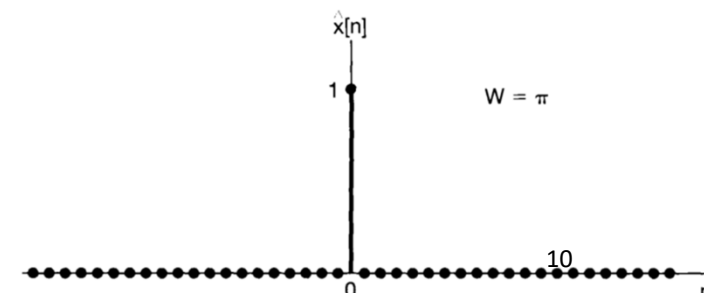
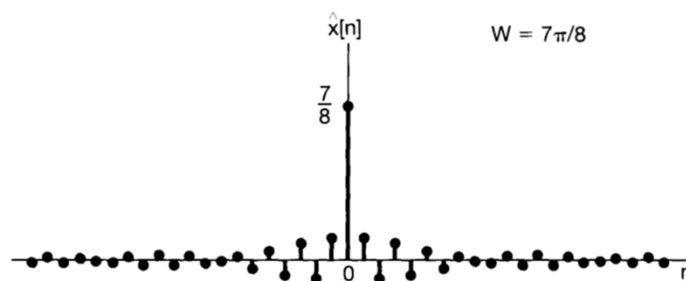
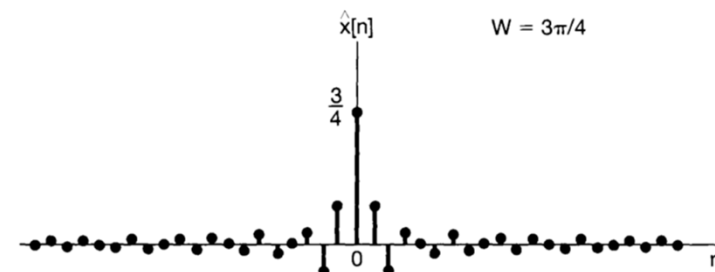
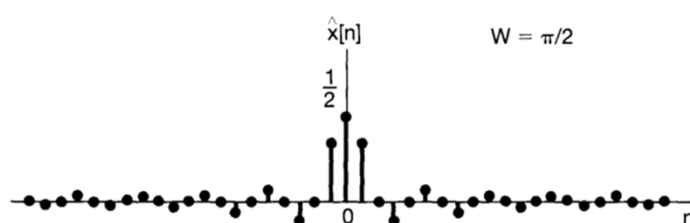
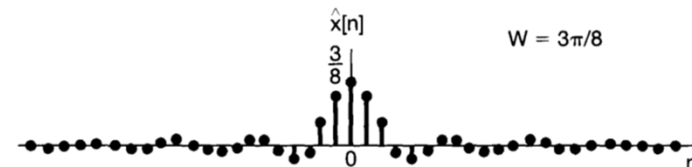
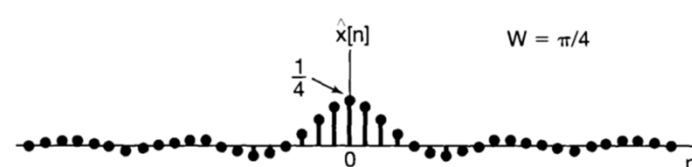
$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = 1$$

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^W e^{j\omega n} d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

$$\lim_{W \rightarrow \pi} \hat{x}[n] = \delta[n]$$



# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
- ☒ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☐ The multiplication property
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# Fourier transform for periodic signals

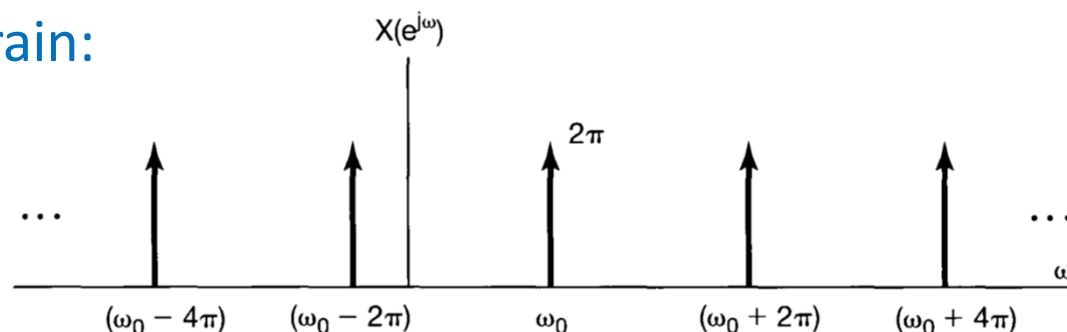


- Consider the sinusoidal signal

$$x[n] = e^{j\omega_0 n}$$

- The FT should be a periodic pulse train:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



- Check validity: evaluate the inverse transform

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega &= \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) e^{j\omega n} d\omega \\ &= e^{j(\omega_0 + 2\pi r)n} \quad \text{Fixed in one period } l = r \\ &= e^{j\omega_0 n} \end{aligned}$$

# Fourier transform for periodic signals



□ Consider a periodic sequence

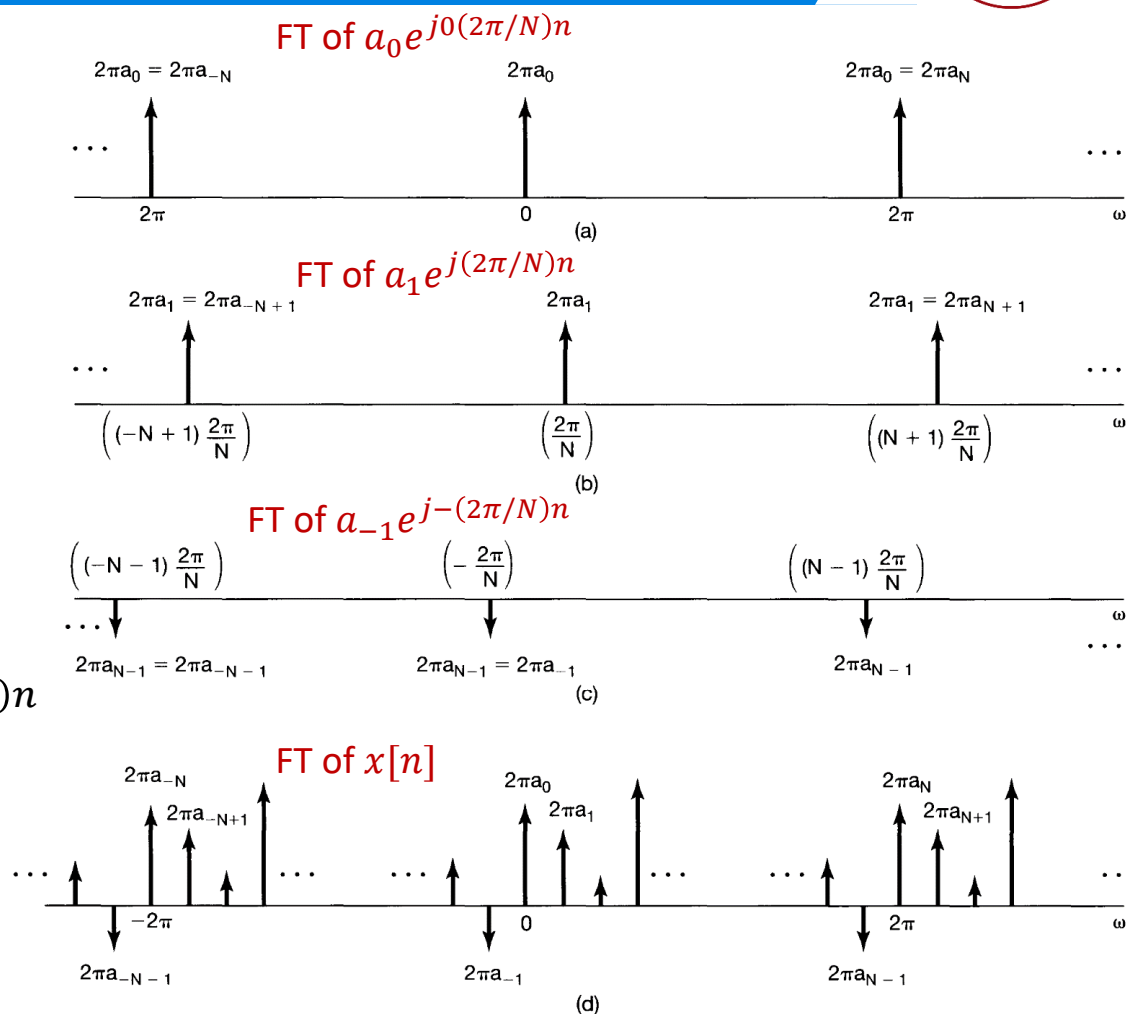
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

□ FT

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - 2\pi k/N)$$

□ Verify

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \dots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$





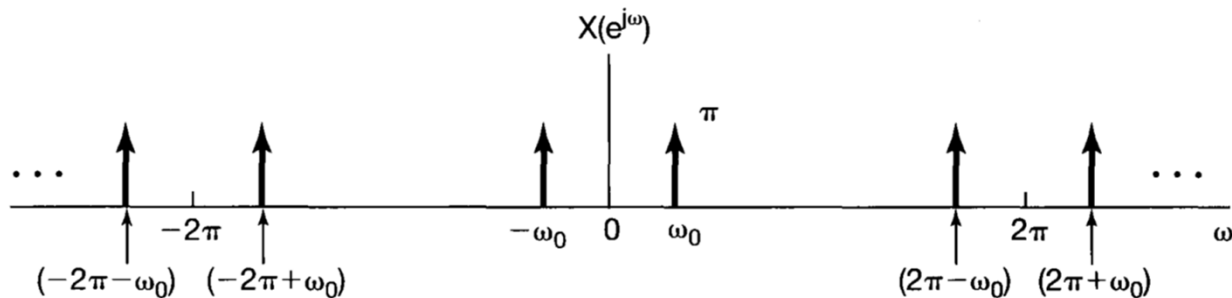
# Fourier transform for periodic signals

## Examples

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \omega_0 = \frac{2\pi}{5} \quad X(e^{j\omega}) = ?$$

## Solution

$$\begin{aligned} X(e^{j\omega}) &= \sum_{l=-\infty}^{+\infty} \pi \delta \left( \omega - \frac{2\pi}{5} - 2\pi l \right) + \sum_{l=-\infty}^{+\infty} \pi \delta \left( \omega + \frac{2\pi}{5} - 2\pi l \right) \\ &= \pi \delta \left( \omega - \frac{2\pi}{5} \right) + \pi \delta \left( \omega + \frac{2\pi}{5} \right), -\pi < \omega < \pi \end{aligned}$$



# Fourier transform for periodic signals



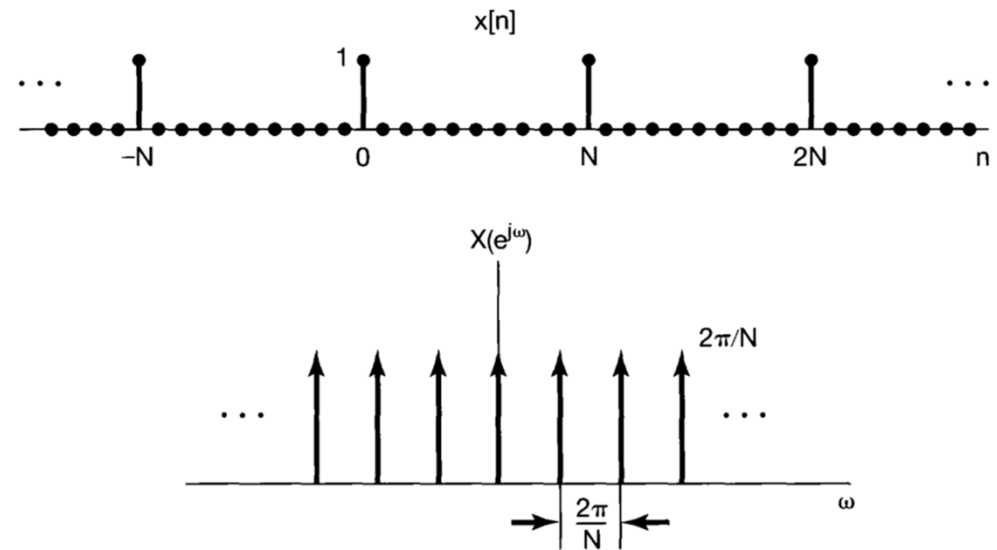
## Examples

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \quad X(e^{j\omega}) = ?$$

## Solution

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$





# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
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- ☐ The convolution property
- ☐ The multiplication property
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- ☐ Systems characterized by difference equations



# Properties of discrete-time Fourier Transform

## Short notation for FT pairs

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$



# Properties of discrete-time Fourier Transform

**Periodicity** In contrast to continuous FT

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

**Linearity**

$$x_1[n] \xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})$$

$$x_2[n] \xleftrightarrow{\mathcal{F}} X_2(e^{j\omega})$$

$$\Rightarrow ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$



# Properties of discrete-time Fourier Transform

## Time shifting and frequency shifting

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \Rightarrow \begin{cases} x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega}) \\ e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)}) \end{cases}$$

## Examples

Low-pass filter  $\Rightarrow$  High-pass filter

$$H_{lp}(e^{j\omega})$$

$$\updownarrow \mathcal{F}$$

$$h_{lp}[n]$$

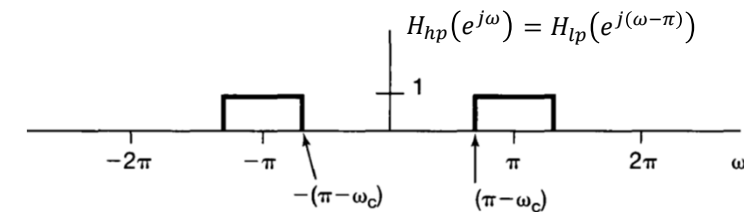
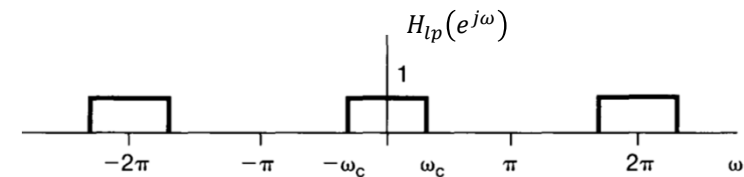
shift by  $\pi$



$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega - \pi)})$$

$$\updownarrow \mathcal{F}$$

$$\begin{aligned} h_{hp} &= e^{j\pi n} h_{lp}[n] \\ &= (-1)^n h_{lp}[n] \end{aligned}$$





# Properties of discrete-time Fourier Transform

## Conjugation and Conjugate Symmetry

### □ Conjugation property

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \quad \Rightarrow \quad x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

### □ Conjugation Symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{ real}] \quad \Rightarrow \quad \begin{cases} \text{Ev}\{x[n]\} \xleftrightarrow{\mathcal{F}} \text{Re}\{X(e^{j\omega})\} \\ \text{Od}\{x[n]\} \xleftrightarrow{\mathcal{F}} j\text{Im}\{X(e^{j\omega})\} \end{cases}$$



# Properties of discrete-time Fourier Transform

## Time reversal

$$\square \quad x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \implies x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

$$\square \quad x[n] \text{ even, } X(e^{j\omega}) \text{ even; } x[n] \text{ odd, } X(e^{j\omega}) \text{ odd}$$

$$\text{Recall: } x[n] \text{ real: } X(e^{j\omega}) = X^*(e^{-j\omega})$$



$$\square \quad x[n] \text{ real and even} \implies X(e^{j\omega}) \text{ real and even}$$

$$x[n] \text{ real and odd} \implies X(e^{j\omega}) \text{ real and odd}$$



# Properties of discrete-time Fourier Transform

## Differencing and accumulation

□ If  $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

□ Then

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} (1 - e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^n x[m] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \underbrace{\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)}_{\text{DC components}}$$

DC components



# Properties of discrete-time Fourier Transform

## Differencing and accumulation

□ Examples Determine FT of unit step  $x[n] = u[n]$

Solution

$$\begin{aligned} g[n] = \delta[n] &\xleftrightarrow{\mathcal{F}} G(e^{j\omega}) = 1 & x[n] &= \sum_{m=-\infty}^n g[m] \\ X(e^{j\omega}) &= \frac{1}{1 - e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \end{aligned}$$





# Properties of discrete-time Fourier Transform

## Time expansion

- Recall the continuous time property

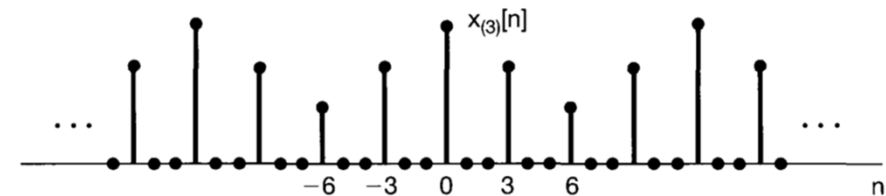
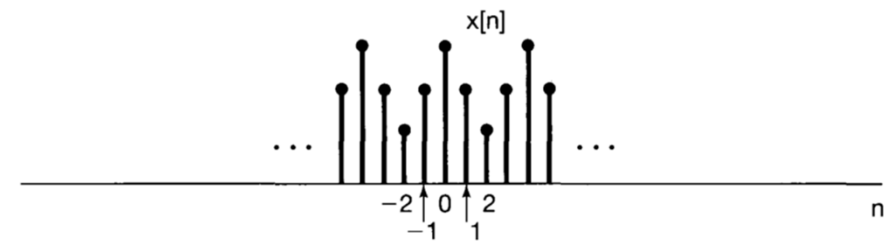
$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Try to define  $x[an]$

- $a$  should be an integer and  $a > 1$
- not merely speed up, but also resample  $x[n]$

- Define instead

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$





# Properties of discrete-time Fourier Transform

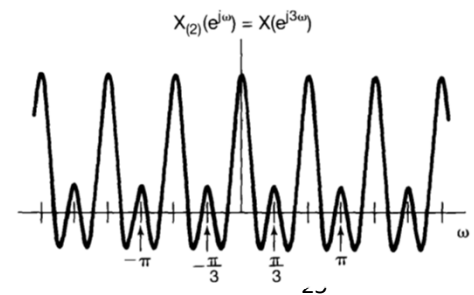
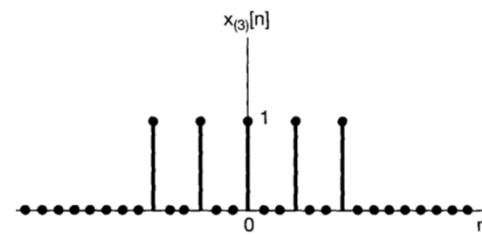
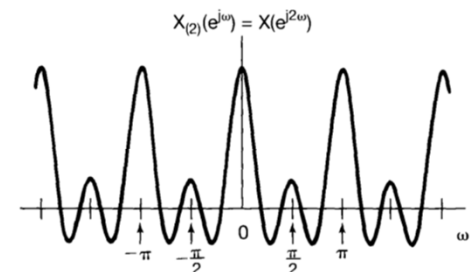
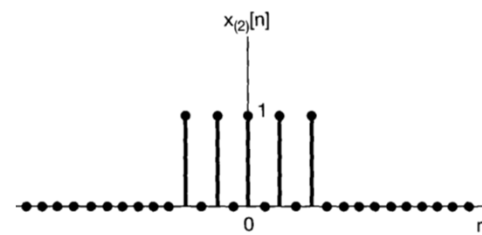
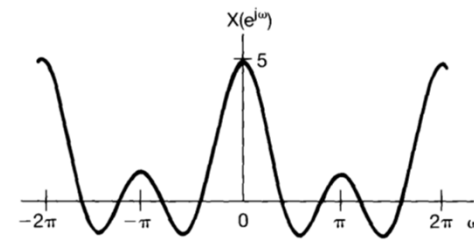
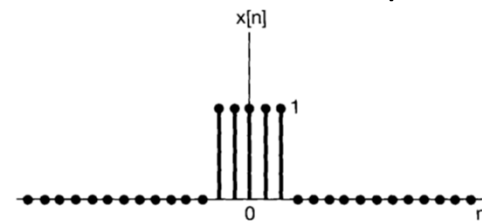
## Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X_{(k)}(e^{j\omega}) = X(e^{jk\omega})$$

$$\begin{aligned} X_{(k)}(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x_{(k)}[n] e^{-j\omega n} \\ &= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk] e^{-j\omega rk} \quad n = rk \\ &= \sum_{r=-\infty}^{+\infty} x[r] e^{-j(k\omega)r} \quad x_{(k)}[rk] = x[r] \\ &= X(e^{jk\omega}) \end{aligned}$$





# Properties of discrete-time Fourier Transform

## Examples

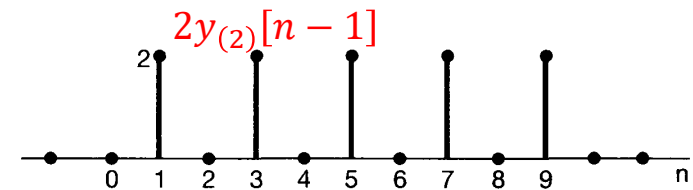
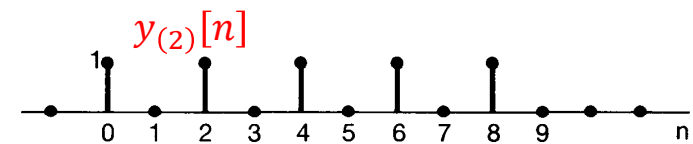
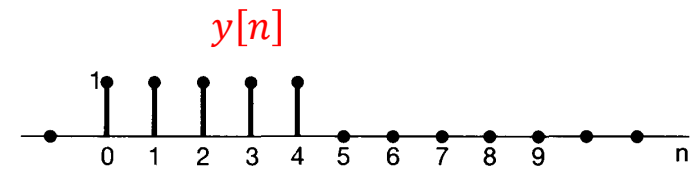
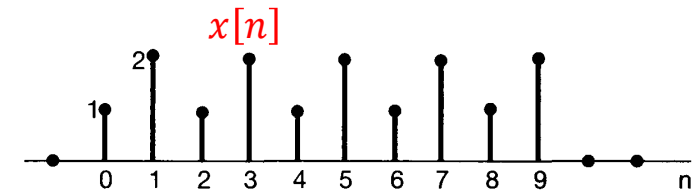
$$X(e^{j\omega}) = ?$$

## Solution

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

where  $y[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{else} \end{cases}$

$$y_2[n] = \begin{cases} y[n/2], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$





# Properties of discrete-time Fourier Transform

## Examples

$$X(e^{j\omega}) = ?$$

## Solution

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

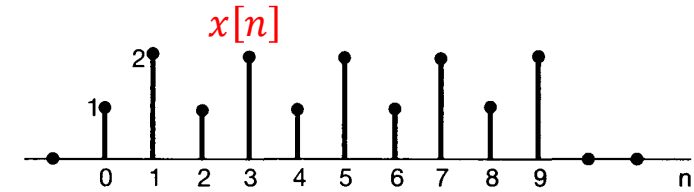
## □ Using the time expansion property

$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} Y_2(e^{j\omega}) = Y(e^{j2\omega}) = e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

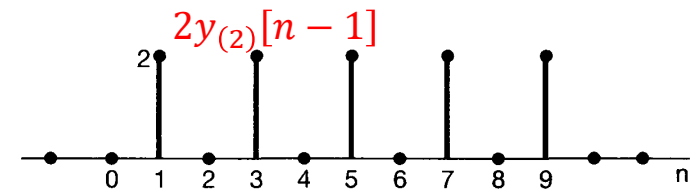
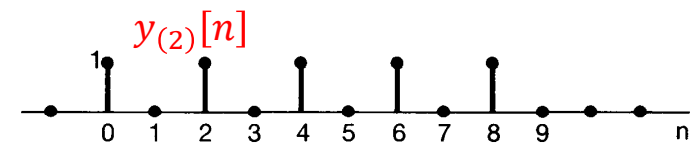
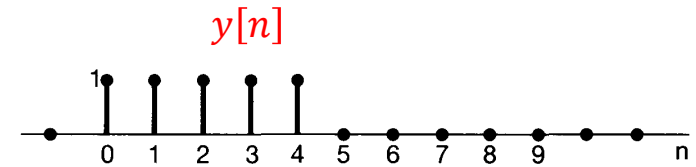
## □ Using the linearity and time-shifting properties

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2e^{-j5\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega} (1 + 2e^{-j\omega}) \left( \frac{\sin(5\omega)}{\sin(\omega)} \right)$$



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$





# Properties of discrete-time Fourier Transform

## Differentiation in frequency

$$nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$$

□ Consider

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} \boxed{-jnx[n]} e^{-j\omega n} \Rightarrow -jnx[n] \xleftrightarrow{\mathcal{F}} \frac{dX(e^{j\omega})}{d\omega}$$

$$\Rightarrow nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$$

## Parseval's relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$



# Properties of discrete-time Fourier Transform

## Examples

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

□  $x[n]$  is

- Periodic?

No

- Real?

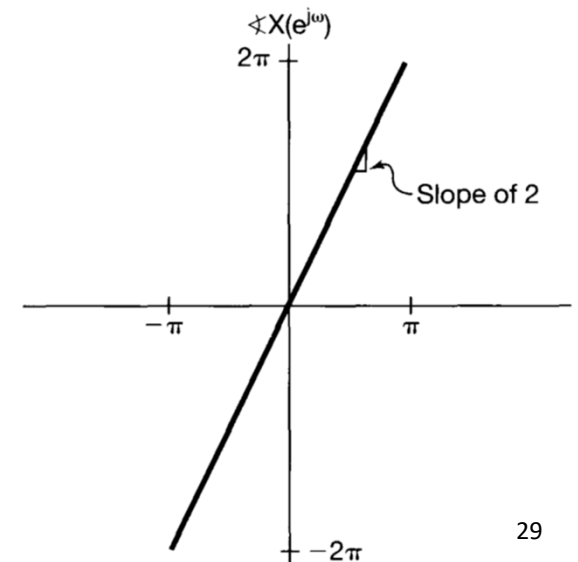
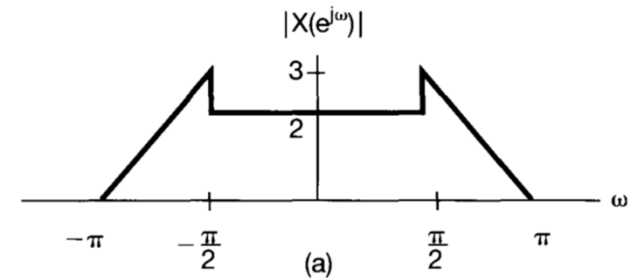
Yes

- Even?

No

- Of finite energy?

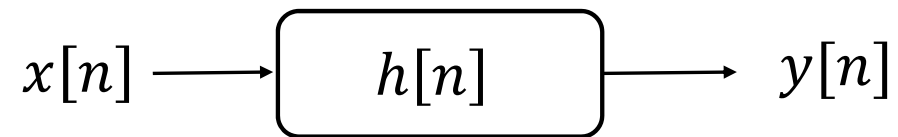
Yes



# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☒ **The convolution property**
- ☐ The multiplication property
- ☐ Duality
- ☐ Systems characterized by difference equations

# The convolution property



$$y[n] = x[n] * h[n] \quad \xleftrightarrow{\mathcal{F}} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

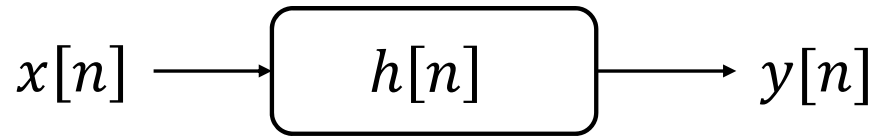
□  $H(j\omega)$ : Frequency response; FT of the impulse response of the system



# The convolution property



## Examples



$$h[n] = \delta[n - n_0] \text{ and } X(e^{j\omega}) = \mathcal{F}\{x[n]\} \quad Y(e^{j\omega}) = ?$$

## Solution

$$\begin{aligned} h[n] &= \delta[n - n_0] \\ \Downarrow & \qquad \qquad \Downarrow \\ H(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} \\ &= e^{-j\omega n_0} \\ \Downarrow & \qquad \qquad \Downarrow \\ Y(e^{j\omega}) &= e^{-j\omega n_0} X(e^{j\omega}) \end{aligned}$$
$$y[n] = x[n - n_0]$$

# The convolution property

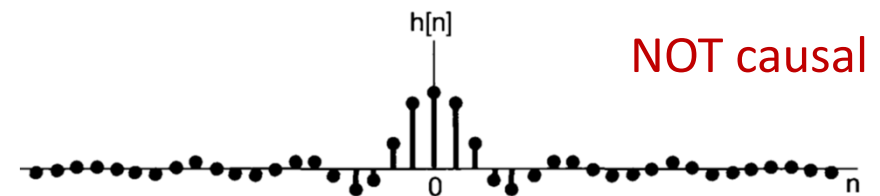
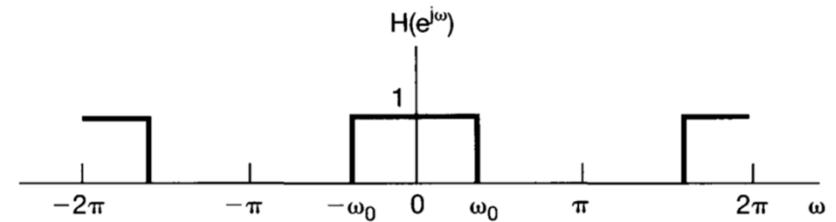


## Examples

Determine the impulse response of an ideal low-pass filter

### Solution

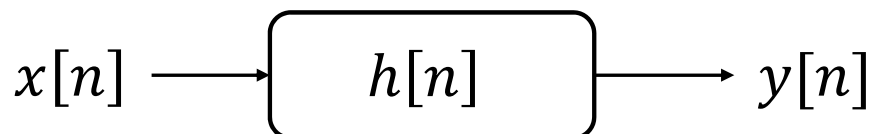
$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega \\ &= \frac{\sin \omega_0 n}{\pi n} \end{aligned}$$



# The convolution property



## Examples



$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

## Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \quad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When  $\alpha \neq \beta$

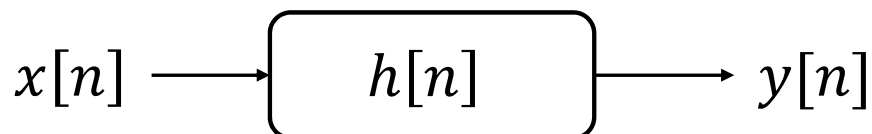
$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} - \frac{B}{1 - \beta e^{-j\omega}} \quad A = \frac{\alpha}{\alpha - \beta} \quad B = \frac{\beta}{\alpha - \beta}$$

$$y[n] = \frac{\alpha}{\alpha - \beta} \alpha^n u[n] - \frac{\beta}{\alpha - \beta} \beta^n u[n] = \frac{1}{\alpha - \beta} (\alpha^{n+1} u[n] - \beta^{n+1} u[n])$$

# The convolution property



## Examples



$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

## Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \quad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When  $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2} = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega} \left( \frac{1}{1 - \alpha e^{-j\omega}} \right)$$

$$y[n] = (n + 1) \alpha^n u[n + 1] = (n + 1) \alpha^n u[n]$$

# The convolution property



## Examples

Consider the ideal band-stop filter,  $Y(e^{j\omega}) = ?$

$$w_1[n] = e^{j\pi n} x[n]$$

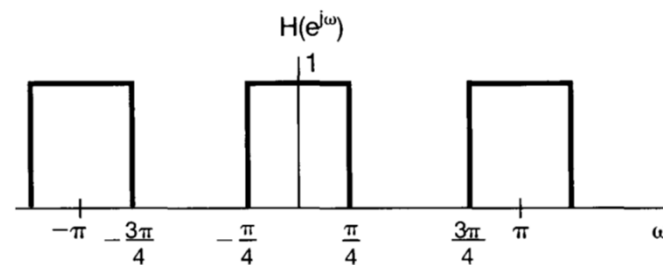
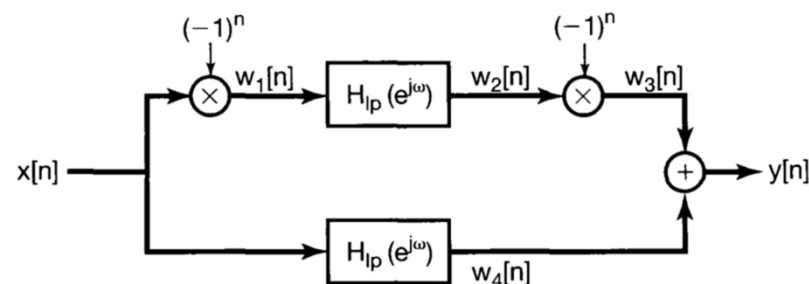
$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$$

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)}) X(e^{j(\omega-2\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega}) \end{aligned}$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})] X(e^{j\omega})$$



# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☒ **The multiplication property**
- ☐ Duality
- ☐ Systems characterized by difference equations

# The multiplication property



$$y[n] = x_1[n]x_2[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \underbrace{\frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}_{\text{Periodic convolution}}$$

Proof

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})e^{j\theta n} d\theta \right\} e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left[ \sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n} \right] d\theta$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})e^{j\theta n} d\theta$$

# The multiplication property



## Examples

$$x_1[n] = \frac{\sin(\pi n/2)}{\pi n} \quad x_2[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x[n] = x_1[n]x_2[n] \quad X(e^{j\omega}) = ?$$

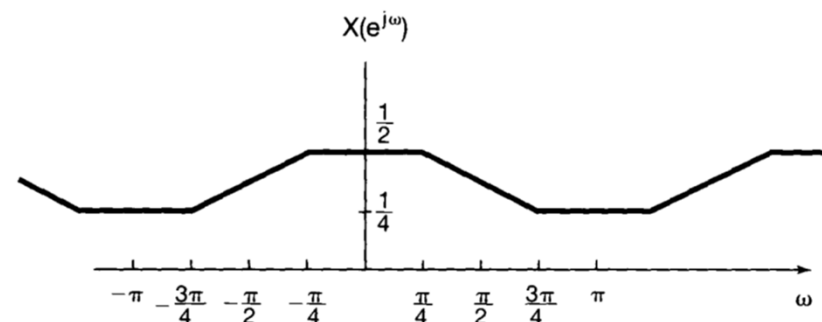
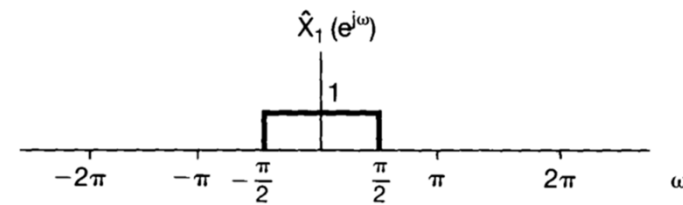
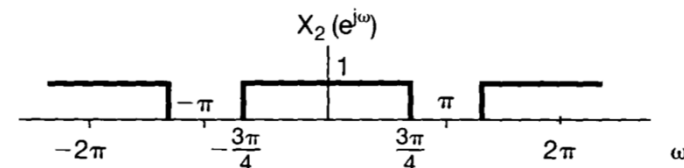
## Solution

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) X_2(e^{j(\omega-\theta)}) d\theta$$

## □ Convert to ordinary convolution

$$\text{Define } \hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), & -\pi < \omega < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$





# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☐ The multiplication property
- ☒ **Duality**
- ☐ Systems characterized by difference equations

# Duality



## Duality in the discrete FS

Consider a periodic signal  $x[n]$  with period of  $N$

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k = g[k] \iff g[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$



$$g[n] = \sum_{k=\langle N \rangle} \left[ \frac{1}{N} x[-k] \right] e^{jk(2\pi/N)n} \iff g[n] = \frac{1}{N} \sum_{k=\langle N \rangle} x[k] e^{-jk(2\pi/N)n}$$

$$\boxed{x[n] \xleftrightarrow{\mathcal{FS}} g[k] \qquad g[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N} x[-k]}$$

Every property of the discrete FS has a dual.

# Duality



## Duality in the discrete FS

□ Every property of the discrete FS has a dual

$$\left\{ \begin{array}{l} x[n - n_0] \xleftrightarrow{\mathcal{FS}} a_k e^{-jk(2\pi/N)n_0} \\ e^{jm(2\pi/N)n} x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m} \end{array} \right.$$

$$\left\{ \begin{array}{l} \sum_{r=\langle N \rangle} x[r]y[n-r] \xleftrightarrow{\mathcal{FS}} Na_k b_k \\ x[n]y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=\langle N \rangle} a_l b_{k-l} \end{array} \right.$$

# Duality

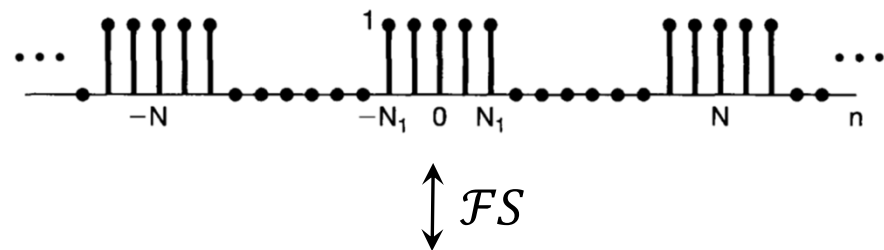


□ **Examples**

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \quad a_k = ?$$

## Solution

- Recall the Fourier coefficients of a periodic square wave



$$\begin{cases} \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \dots \\ (2N_1 + 1)/N, & k = 0, \pm N, \pm 2N, \dots \end{cases}$$

# Duality



## Examples

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \quad a_k = ?$$

## Solution

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases} \quad \xleftrightarrow{\mathcal{FS}} \quad b_k = \begin{cases} \frac{1}{9} \frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ 5/9, & k = \text{multiple of } 9 \end{cases}$$

$g[n]$  is periodic ( $N = 9$ )

⇓ Duality

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \quad \xleftrightarrow{\mathcal{FS}} \quad g[k] = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4 \end{cases}$$

# Duality



## Duality between discrete FT and continuous FS

$$\begin{array}{ccc} x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega & \xleftrightarrow{\mathcal{FT}} & X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \\ & \swarrow \quad \searrow & \\ g(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} & \xleftrightarrow{\mathcal{FS}} & a_k = \frac{1}{T} \int_T g(t) e^{-jk\omega_0 t} dt \end{array}$$



# Properties of discrete-time Fourier Transform

## Examples

Solution

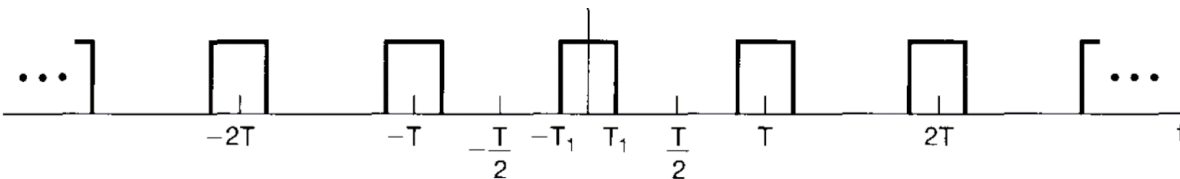
$$x[n] = \frac{\sin(\pi n/2)}{\pi n} \xleftrightarrow{\mathcal{FT}} X(e^{j\omega}) = ? g(-\omega)$$

$$= g(-\omega) = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & \pi/2 < |\omega| \leq \pi \end{cases}$$

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & T_1 < |t| \leq T/2 \end{cases} \xleftrightarrow{\mathcal{FS}} a_k = \frac{\sin(\pi k/2)}{k\pi}$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases} \xleftrightarrow{\mathcal{FS}} \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$\therefore T_1 = T/4$$



# Duality



## Summary FS and FT expressions

	Continuous time		Discrete time	
	Time domain	Frequency domain	Time domain	Frequency domain
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ <p>continuous time periodic in time</p>	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$ <p>discrete frequency aperiodic in frequency</p>	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$ <p>discrete time periodic in time</p>	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$ <p>discrete frequency periodic in frequency</p>
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ <p>continuous time aperiodic in time</p>	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ <p>continuous frequency aperiodic in frequency</p>	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ <p>discrete time aperiodic in time</p>	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ <p>continuous frequency periodic in frequency</p>



# The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
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- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☐ The multiplication property
- ☐ Duality
- ☒ Systems characterized by difference equations



# System characterized by difference equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\sum_{k=0}^N a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^M b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M b_k e^{-jk\omega}}{\sum_{k=0}^N a_k e^{-jk\omega}}$$



# System characterized by difference equations

## Examples

$$y[n] - ay[n-1] = x[n], |a| < 1 \quad h[n] = ?$$

## Solution

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$h[n] = a^n u[n]$$



# System characterized by difference equations

## Examples

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \quad h[n] = ?$$

## Solution

$$\begin{aligned} H(e^{j\omega}) &= \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)} \\ &= \frac{4}{\left(1 - \frac{1}{2}e^{-j\omega}\right)} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)} \end{aligned}$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$



# System characterized by difference equations

## Examples

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \quad x[n] = \left(\frac{1}{4}\right)^n u[n] \quad y[n] = ?$$

## Solution

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) = \left[ \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)} \right] \left[ \frac{1}{1 - \frac{1}{4}e^{-j\omega}} \right] \\ &= \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)^2} = -\frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{\left(1 - \frac{1}{2}e^{-j\omega}\right)} \end{aligned}$$

$$y[n] = \left\{ -4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n \right\} u[n]$$