Lecture 10: Model-Free Prediction

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Outline

Outline

Model-Free Reinforcement Learning

- Last lecture:
  - Planning by dynamic programming
  - Solve a *known* MDP
- This lecture:
  - Model-free prediction
  - Estimate the value function of an *unknown* MDP
- Next lecture:
  - Model-free control
  - Optimize the value function of an *unknown* MDP

Outline

1. Introduction
2. Monte-Carlo Learning
3. Temporal-Difference Learning
4. n-step TD Methods
5. TD(λ)
6. References

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is *model-free*: no knowledge of MDP transitions / rewards
- MC learns from *complete* episodes: no bootstrapping
- MC uses the simplest possible idea: value = mean return
- To learn values & policies, MC can be used in two ways:
  - *model-free*: no model necessary and still attains optimality
  - *simulated*: needs only a simulation, not a full model
- Caveat: can only apply MC to *episodic* MDPs
  - All episodes must terminate

Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of experience under policy π

$$S_1, A_1, R_2, \ldots, S_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$

- Monte-Carlo policy evaluation uses *empirical mean* return instead of *expected* return

Monte-Carlo Policy Evaluation

- Goal: learn $v_\pi(s)$
- Given: some number of episodes under π which contains s
- Idea: average returns observed after visits to s
- Every-Visit MC: average returns for every time s is visited in an episode
- First-visit MC: average returns only for first time s is visited in an episode
- Both converge asymptotically

First-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- The first time-step t that state s is visited in an episode
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- By law of large numbers, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

First-Visit Monte-Carlo Policy Evaluation

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated
Initialize:
    $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$
    $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):
    Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:
            Append G to $Returns(S_t)$
            $V(S_t) \leftarrow \text{average}(Returns(S_t))$

Every-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- Every time-step t that state s is visited in an episode
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- Again, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

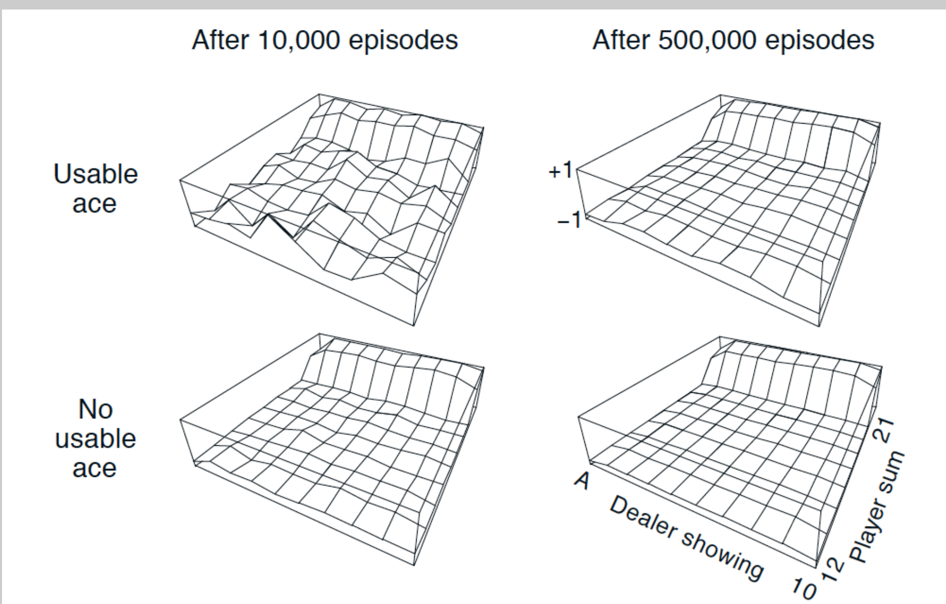
Blackjack Example

- States (200 of them):
  - Current sum (12-21)
  - Dealer's showing card (ace-10)
  - Do I have a "useable" ace? (yes-no)

Blackjack Example

- Action stick: Stop receiving cards (and terminate)
- Action twist: Take another card (no replacement)
- Reward for stick:
  - $+1$ if sum of cards $>$ sum of dealer cards
  - 0 if sum of cards $=$ sum of dealer cards
  - -1 if sum of cards $<$ sum of dealer cards
- Reward for twist:
  - -1 if sum of cards $>$ 21 (and terminate)
  - 0 otherwise
- Transitions: automatically twist if sum of cards $<$ 12

Blackjack Value Function after Monte-Carlo Learning



Policy: stick if sum of cards ≥ 20, otherwise twist

Incremental Mean

The mean $\mu_1, \mu_2, \ldots$ of a sequence $x_1, x_2, \ldots$ can be computed incrementally,

$$\begin{aligned}
\mu_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\
&= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right) \\
&= \frac{1}{k}(x_k + (k-1)\mu_{k-1}) \\
&= \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})
\end{aligned}$$

Incremental Monte-Carlo Updates

- Update $V(s)$ incrementally after episode $S_1, A_1, R_2, \ldots, S_T$
- For each state S_t with return G_t

$$\begin{aligned}
N(S_t) &\leftarrow N(S_t) + 1 \\
V(S_t) &\leftarrow V(S_t) + \frac{1}{N(S_t)}(G_t - V(S_t))
\end{aligned}$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes.

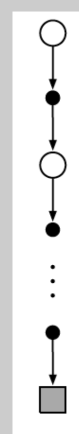
$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

Monte-Carlo Estimation of Action Values

- Monte Carlo is most useful when a model is not available: we want to learn q_*
- $q_\pi(s, a)$: average return starting from state s and action a following policy π
- Converges asymptotically if every state-action pair is visited
- Exploring Starts: every state-action pair has a non-zero probability of being the starting pair

Backup Diagram for Monte-Carlo

- Entire rest of episode included
- Only one choice considered at each state (unlike DP)
- thus, there will be an explore/exploit dilemma
- Does not bootstrap from successor states's values (unlike DP)
- Time required to estimate one state does not depend on the total number of states



Outline

Temporal-Difference Learning

- TD methods learn directly from episodes of experience
- TD is *model-free*: no knowledge of MDP transitions / rewards
- TD learns from *incomplete* episodes, by *bootstrapping*
- TD updates a guess towards a guess

MC and TD

- Goal: learn v_π online from experience under policy π
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ toward *actual* return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

- Simplest temporal-difference learning algorithm: TD(0)
  - Update value $V(S_t)$ toward *estimated* return $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

  - $R_{t+1} + \gamma V(S_{t+1})$ is called the *TD target*
  - $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the *TD error*

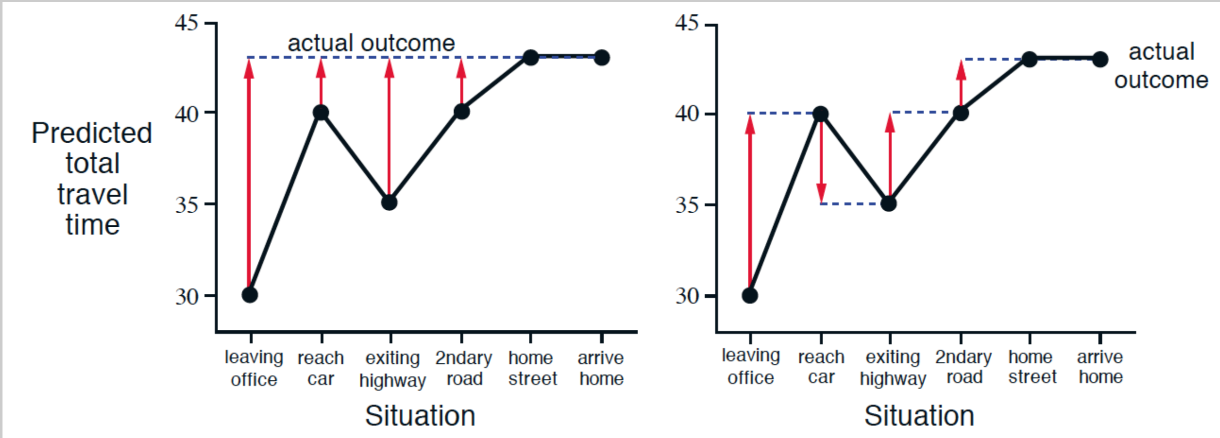
Driving Home Example

State	*Elapsed Time (minutes)*	*Predicted Time to Go*	*Predicted Total Time*
leaving office, friday at 6	0	30	30
reach car, raining	5	35	40
exiting highway	20	15	35
2ndary road, behind truck	30	10	40
entering home street	40	3	43
arrive home	43	0	43

Driving Home Example: MC vs. TD

Changes recommended by MC methods ($\alpha = 1$)

Changes recommended by TD methods ($\alpha = 1$)



Advantages and Disadvantages of MC vs. TD

- TD can learn *before* knowing the final outcome
  - TD can learn online after every step (less memory & peak computation)
  - MC must wait until end of episode before return is known
- TD can learn *without* the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments

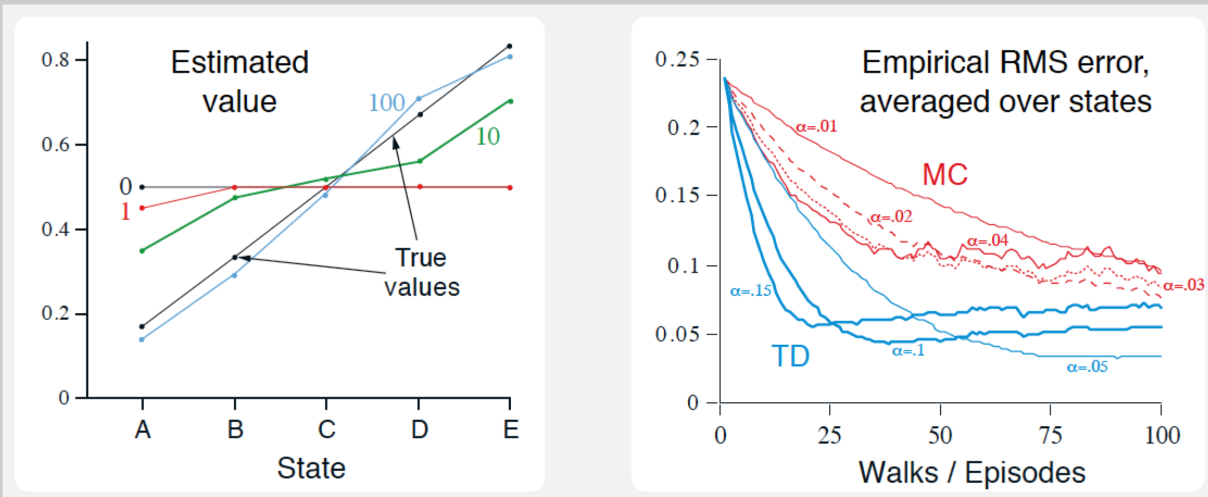
Bias/Variance Trade-Off

- Return $G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T$ is *unbiased* estimate of $v_\pi(S_t)$
- True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ is *unbiased* estimate of $v_\pi(S_t)$
- TD target $R_{t+1} + \gamma V(S_{t+1})$ is *biased* estimate of $v_\pi(S_t)$
- TD target is much lower variance than the return:
  - Return depends on *many* random actions, transitions, rewards
  - TD target depends on *one* random action, transition, reward

Advantages and Disadvantages of MC vs. TD (2)

- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not very sensitive to initial value
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to $v_\pi(s)$
  - (but not always with function approximation)
  - More sensitive to initial value

Random Walk Example: MC vs. TD



Batch MC and TD

- MC and TD converge: $V(s) \to v_\pi(s)$ as experience $\to \infty$
- But what about batch solution for finite experience?

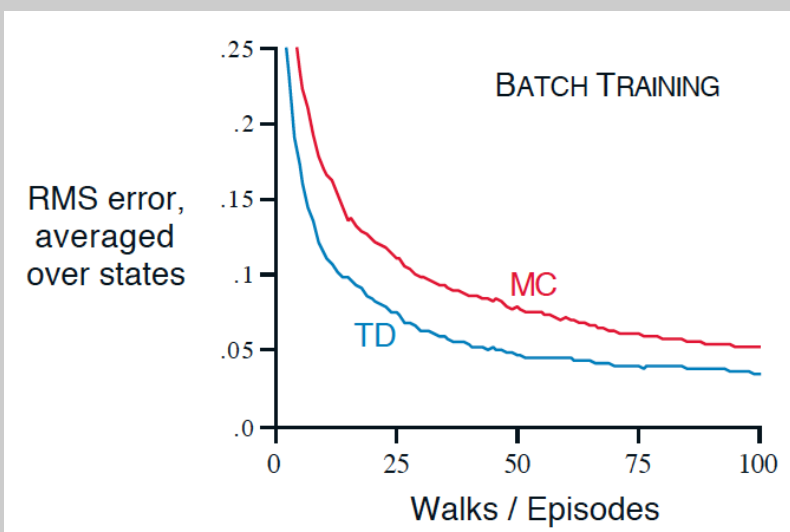
$$s_1^1, a_1^1, r_2^1, \ldots, s_{T_1}^1$$
$$\vdots$$
$$s_1^K, a_1^K, r_2^K, \ldots, s_{T_K}^K$$

  - e.g. Repeatedly sample episode $k \in [1, K]$
  - Apply MC or TD(0) to episode k

Batch MC and TD

- For any finite Markov prediction task, under batch updating, TD converges for sufficiently small step-size parameter α
- Constant-α MC also converges under these conditions, but to a different answer!

Random Walk under Batch Updating



AB Example

Two states A, B; no discounting; 8 episodes of experience

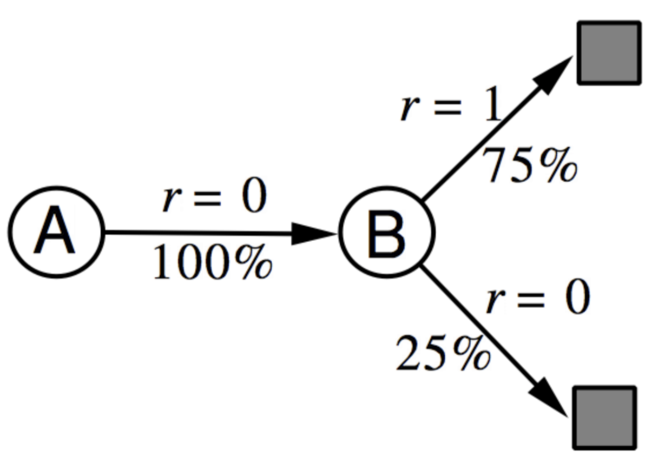
A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0

What is $V(A), V(B)$?

AB Example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0



What is $V(A), V(B)$?

AB Example

- The prediction that best matches the training data is $V(A) = 0$
  - This minimizes the mean-square-error on the training set
  - This is what a batch Monte Carlo method gets
- If we consider the sequentiality of the problem, then we would set $V(A) = 0.75$
  - This is correct for the maximum likelihood estimate of a Markov model generating the data
  - i.e, if we do a best fit Markov model, and assume it is exactly correct, and then compute what it predicts
  - This is called the certainty-equivalence estimate
  - This is what TD gets

Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^{K} \sum_{t=1}^{T_k} (G_t^k - V(s_t^k))^2$$

  - In the AB example, $V(A) = 0$
- TD(0) converges to solution of maximum likelihood Markov model
  - Solution to the MDP $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$ that best fits the data

$$\hat{\mathcal{P}}_{s,s'}^a = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s')$$

$$\hat{\mathcal{R}}_s^a = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k = s, a) r_t^k$$

  - In the AB example, $V(A) = 0.75$

Advantages and Disadvantages of MC vs. TD (3)

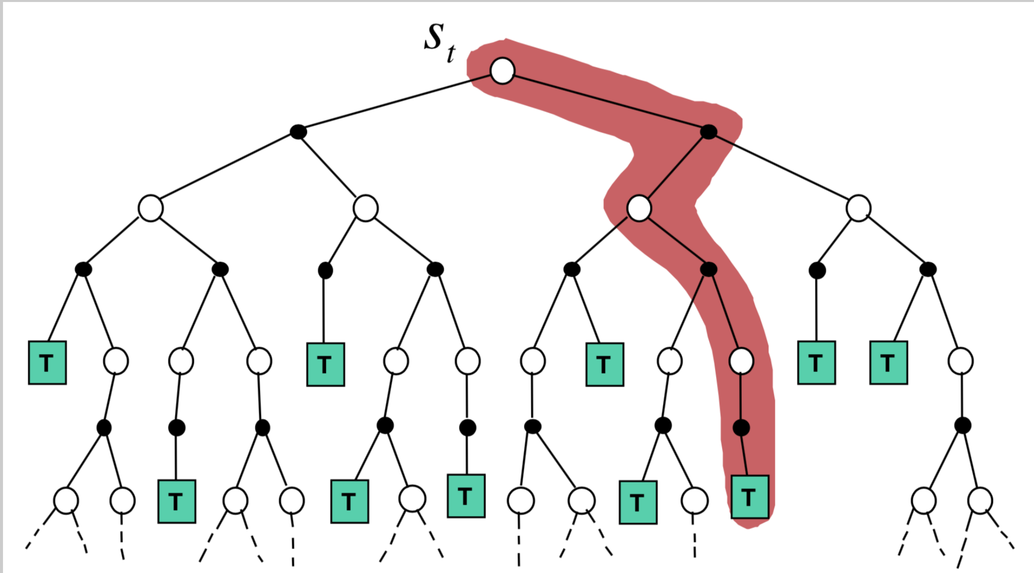
- TD exploits Markov property
  - Usually more efficient in Markov environments
- MC does not exploit Markov property
  - Usually more effective in non-Markov environments
- MC has lower error on past data, but higher error on future data

Bellman Backup

- The term "Bellman backup" comes up quite frequently in the RL literature.
- The Bellman backup for a state (or a state-action pair) is the right-hand side of the Bellman equation:
  the reward-plus-next-value.
- Under different algorithms, we obtain
  - Monte-Carlo Backup
  - Temporal-Difference Backup
  - Dynamic Programming Backup

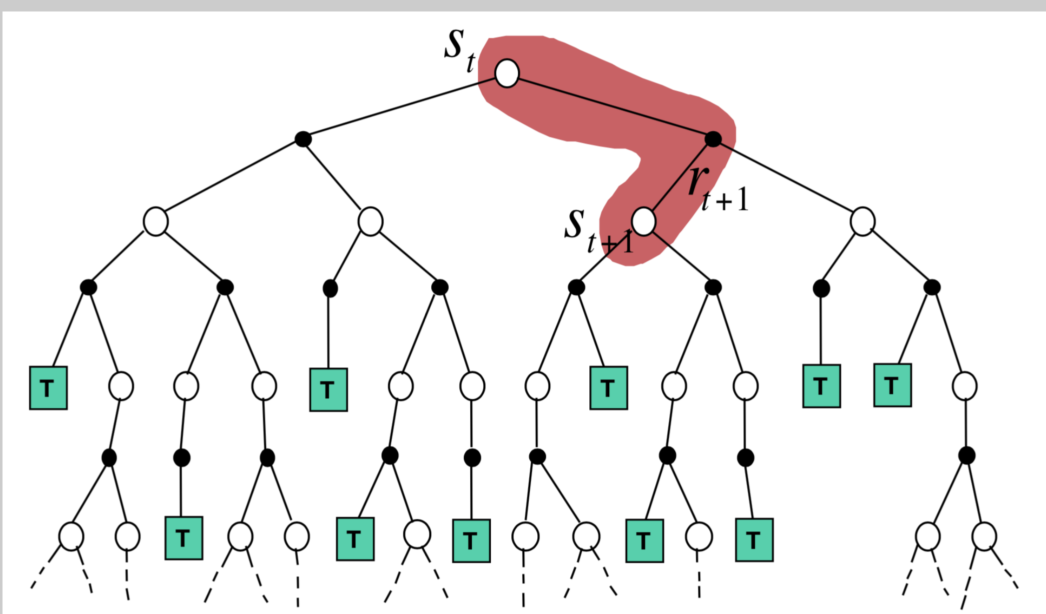
Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$



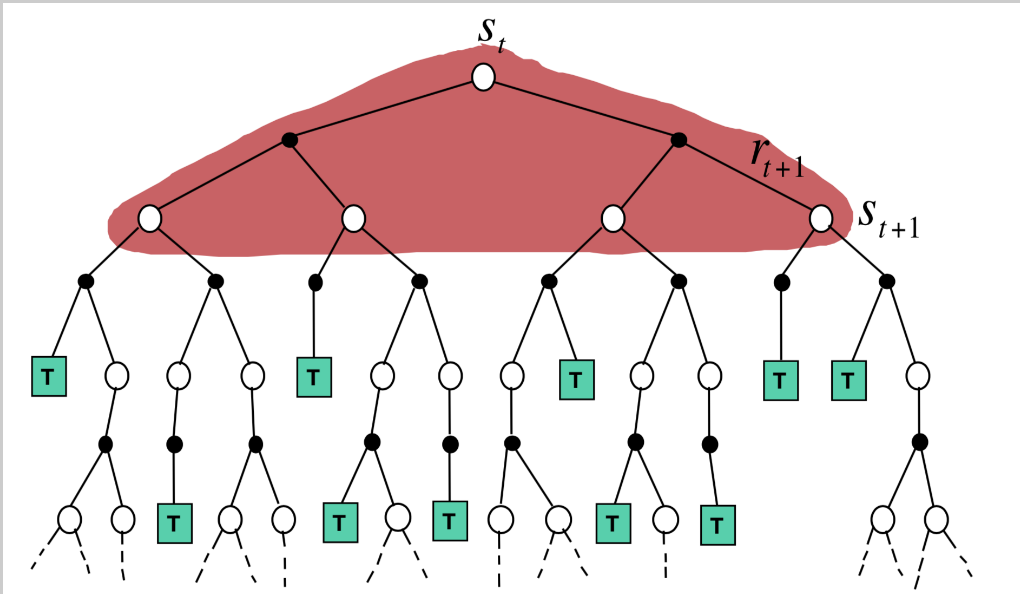
Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$



Dynamic Programming Backup

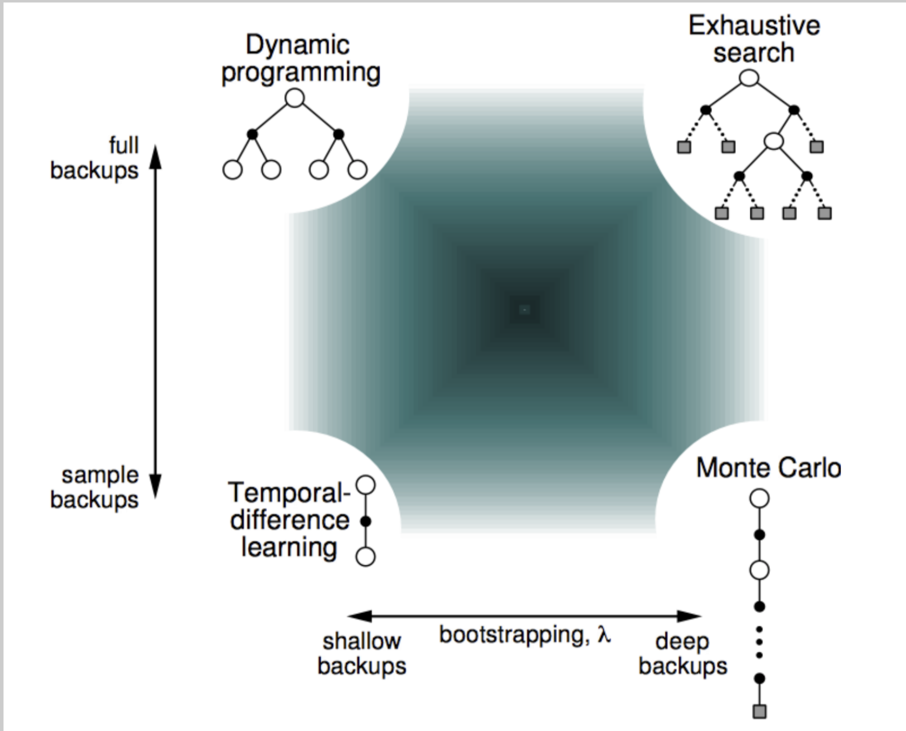
$$V(S_t) \leftarrow \mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1})]$$



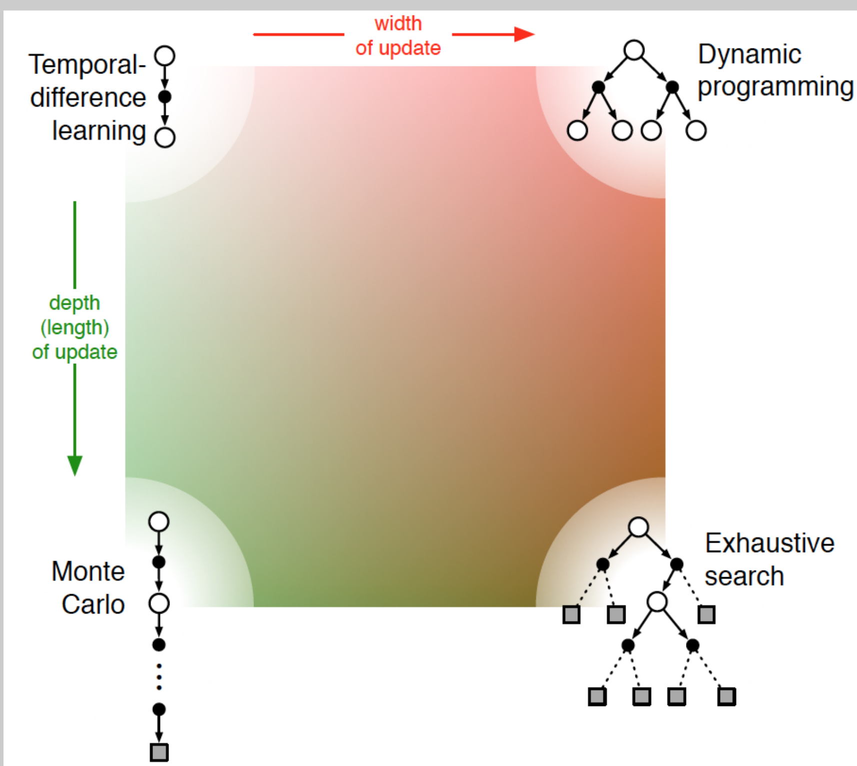
Bootstrapping and Sampling

- Bootstrapping: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- Sampling: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning



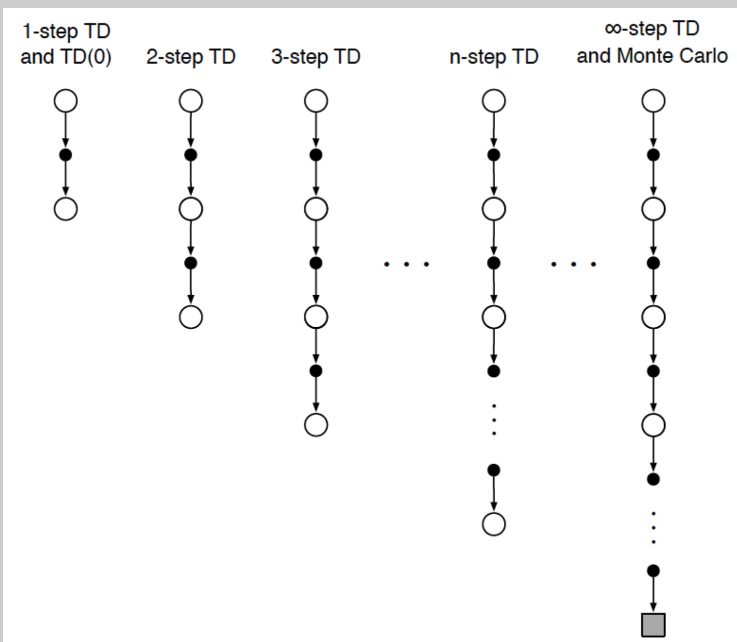
Unified View of Reinforcement Learning



Outline

n-Step Prediction

- Let TD target look *n* steps into the future



n-Step Return

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{aligned}
&n = 1 \quad (TD) \quad && G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\
&n = 2 && G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
&\vdots && \vdots \\
&n = \infty \quad (MC) && G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \ldots \gamma^{T-1} R_T
\end{aligned}
$$

- Define the n-step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- n-step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t^{(n)} - V(S_t))$$

n-step TD

- Recall the n-step return:

$$G_t^{(n)} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V_{t+n-1}(S_{t+n}), \; n \geq 1, 0 \leq t < T - n$$

- Of course, this is not available until time $t + n$
- The natural algorithm is thus to wait until then

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \alpha \left[G_t^{(n)} - V_{t+n-1}(S_t) \right], \quad 0 \leq t < T$$

- This is called n-step TD

n-step TD Algorithm

n-step TD for estimating $V \approx v_\pi$

Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0, 1]$, a positive integer n
All store and access operations (for S_t and R_t) can take their index mod n

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$:
    | If $t < T$, then:
    | Take an action according to $\pi(\cdot|S_t)$
    | Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
    | If S_{t+1} is terminal, then $T \leftarrow t + 1$
    | $\tau \leftarrow t - n + 1$ (τ is the time whose state's estimate is being updated)
    | If $\tau \geq 0$:
    | $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    | If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$ $(G_\tau^{(n)})$
    | $V(S_\tau) \leftarrow V(S_\tau) + \alpha [G - V(S_\tau)]$
    Until $\tau = T - 1$

Error Reduction Property

- Error reduction property of n-step returns

$$\underbrace{\max_s \left| \mathbb{E}_\pi\left[G_t^{(n)} \middle| S_t = s \right] - v_\pi(s) \right|}_{\text{Maximum error using } n\text{-step return}} \leq \underbrace{\gamma^n \max_s \left| V_t(s) - v_\pi(s) \right|}_{\text{Maximum error using V}}$$

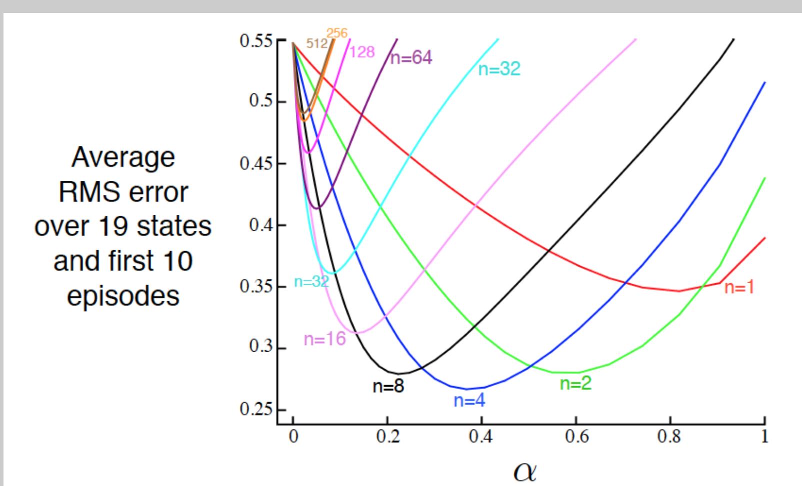
- Using this property, we can show that n-step TD methods converge
- n-step TD methods: a family of sound methods including one-step TD methods & MC methods as extreme members

Random Walk Examples

- How does 2-step TD work here?
- How about 3-step TD?

Large Random Walk Example

- n-step TD for 19-state random walk
- An intermediate α is the best
- An intermediate n is the best

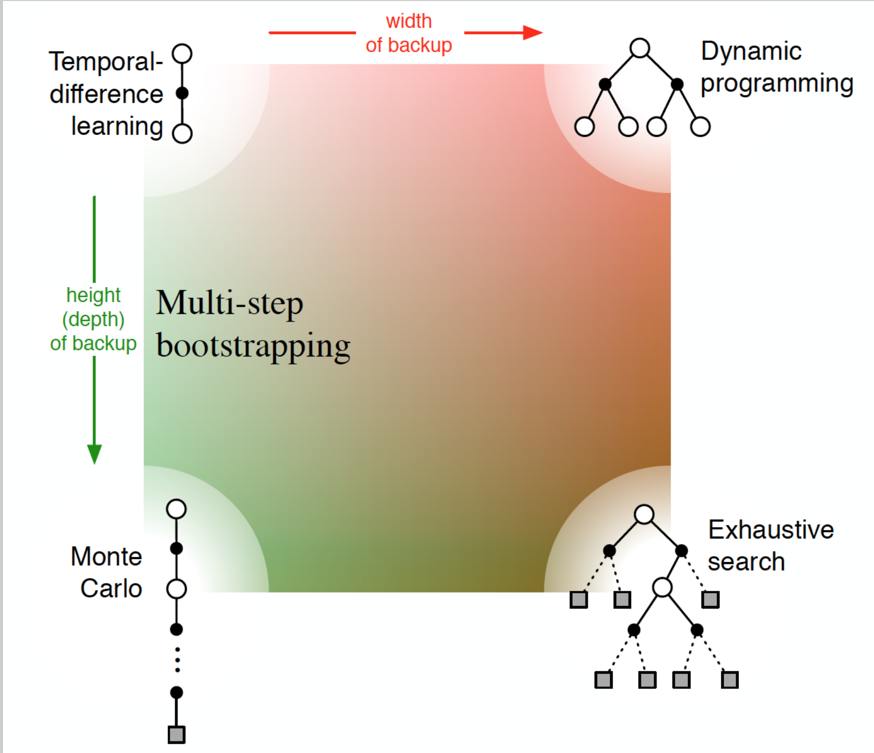


Summary of n-step TD Methods

- Generalize Temporal-Difference and Monte Carlo learning methods, sliding from one to the other as n increases
  - $n = 1$ is TD
  - $n = \infty$ is MC
  - an intermediate n is often much better than either extreme
  - applicable to both continuing and episodic problems
- There is some cost in computation
  - need to remember the last n states
  - learning is delayed by n steps
  - per-step computation is small and uniform, like TD
- Everything generalizes nicely: error-reduction theory

Outline

Unified View

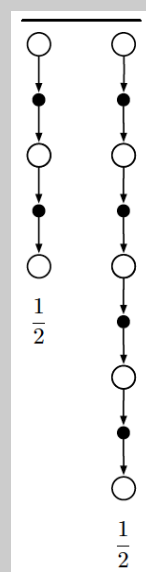


Averaging n-Step Returns

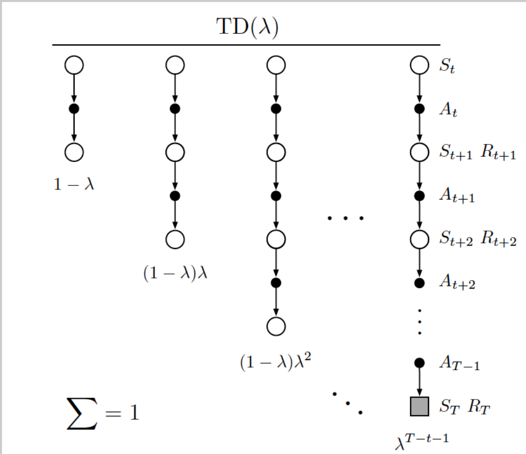
- We can average n-step returns over different n
- e.g. average the 2-step and 4-step returns

$$\frac{1}{2}G^{(2)} + \frac{1}{2}G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps?



λ-return



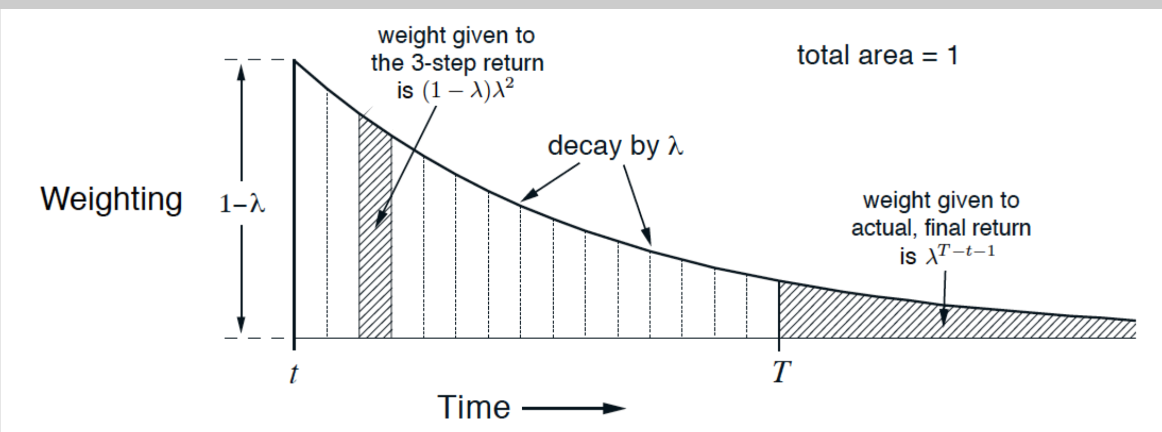
- The *λ-return* G_t^λ combines all n-step returns $G_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- Forward-view TD(λ)

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

Relation to TD(0) & MC

- The λ-return can be rewritten as:

$$G_t^\lambda = \underbrace{(1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_t^{(n)}}_{\text{Until termination}} + \underbrace{\lambda^{T-t-1}G_t}_{\text{After termination}}$$

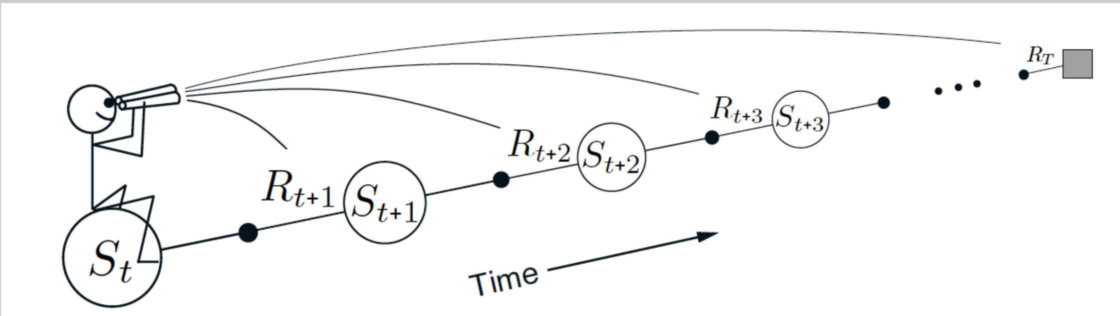
- if $\lambda = 1$, you get the MC target:

$$G_t^\lambda = (1-1)\sum_{n=1}^{T-t-1}1^{n-1}G_t^{(n)} + 1^{T-t-1}G_t = G_t$$

- If $\lambda = 0$, you get the TD(0) target:

$$G_t^\lambda = (1-0)\sum_{n=1}^{T-t-1}0^{n-1}G_t^{(n)} + 0^{T-t-1}G_t = G_t^{(1)}$$

Forward-View TD(λ)



- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes

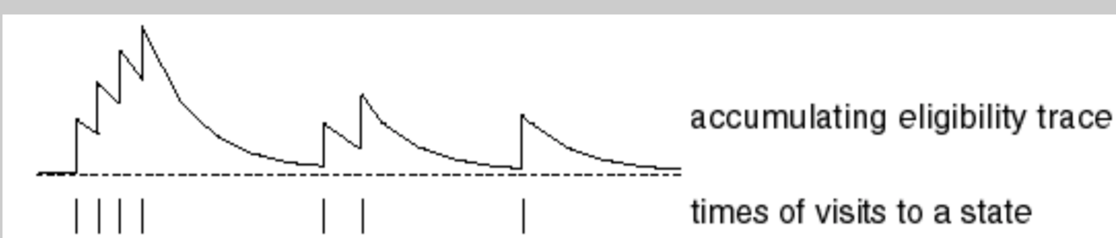
Backward View TD(λ)

- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

Eligibility Traces

- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- *Eligibility traces* combine both heuristics

$$E_0(s) = 0$$
$$E_t(s) = \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s)$$

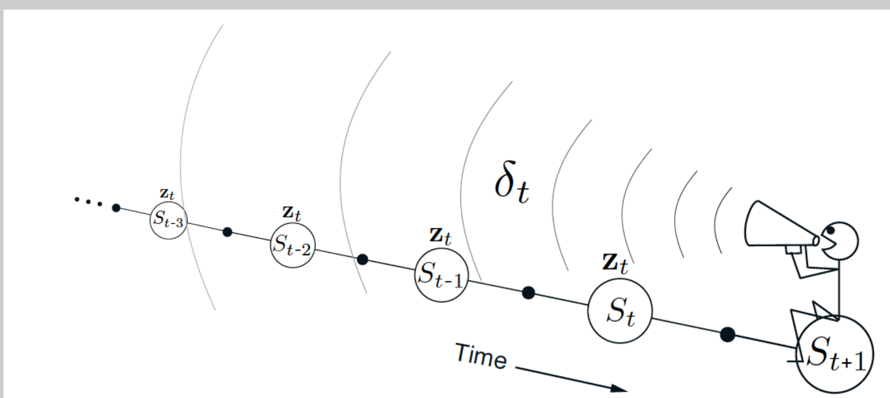


Eligibility Traces

- Eligibility trace keep track of the last time we visited a particular state
- The current reward is then assigned to recently visited states.
- States that have not been visited for a long time are not given much credit for the current reward.
- The eligibility trace for each state decays as $\gamma\lambda$
- The eligibility trace for the state just visited is increased by 1

$$
\begin{aligned}
E_t(s) &= \gamma\lambda E_{t-1}(s) && \text{if } S_t \neq s \\
E_t(s) &= \gamma\lambda E_{t-1}(s) + 1 && \text{if } S_t = s
\end{aligned}
$$

Backward View TD(λ)



- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$
\begin{aligned}
\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\
V(s) &\leftarrow V(s) + \alpha\delta_t E_t(s)
\end{aligned}
$$

TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

$$E_t(s) = \mathbf{1}(S_t = s)$$
$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha\delta_t$$

TD(λ) and MC

- When $\lambda = 1$, credit is deferred until end of episode
- Consider episodic environments with offline updates
- Over the course of an episode, total update for TD(1) is the same as total update for MC

Theorem

The sum of offline updates is identical for forward-view and backward-view TD(λ)

$$\sum_{t=1}^{T} \alpha\delta_t E_t(s) = \sum_{t=1}^{T} \alpha \left(G_t^{\lambda} - V(S_t) \right) \mathbf{1}(S_t = s)$$

Summary of TD(λ) algorithms

- Another way of interpolating between MC and TD methods
- A way of implementing compound λ-return targets
- A basic mechanistic idea: a short-term, fading memory
- A new style of algorithm development & analysis
  - the forward-view & backward-view transformation
  - Forward view: conceptually simple, good for theory & intuition
  - Backward view: computationally congenial implementation of the forward view

Outline

Main References

- Reinforcement Learning: An Introduction (second edition), R. Sutton & A. Barto, 2018.
- RL course slides from Richard Sutton, University of Alberta.
- RL course slides from David Silver, University College London.