

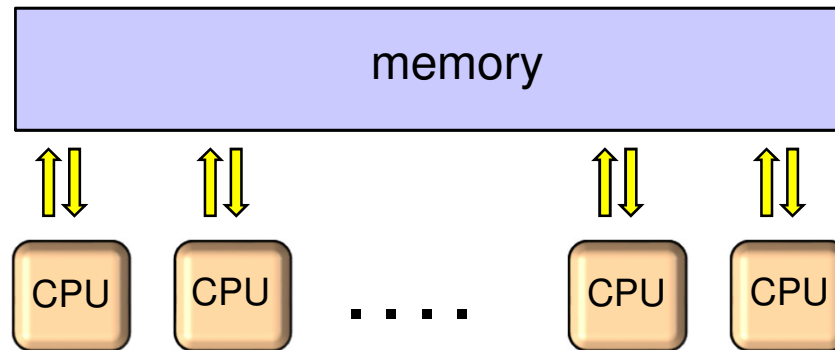


PRAM 1

Model and basic algorithms

CS121 Parallel Computing
Spring 2018

PRAM



- Parallel Random Access Machine, generalizes von Neumann model for sequential computing.
 - Given input of size n , we have $f(n)$ processors accessing a shared memory.
 - $f(n)$ can be very large, even larger than n .
 - All processors execute in synchronized steps.
 - In each step, each processor reads a memory location, computes, then writes a memory location.



PRAM

- Theoretically interesting model, but not practical.
 - Assumes unrealistically large number of processors.
 - Also assumes all processors can communicate every time step; ignores memory latency and bandwidth.
- PRAM's main use is as a simple, clean model to develop parallel algorithms.
 - First maximize parallelism inherent in problem using PRAM.
 - Then simulate the algorithm with real hardware, i.e. map it onto hardware with limited processors / communication.
 - Ex Some GPU algorithms are adaptations of PRAM algorithms.



Memory conflicts

- What if processors read / write to the same memory location in same time step?
- EREW Exclusive read exclusive write.
 - Most restrictive model. Algorithm returns error if processors read/write same location simultaneously.
- CREW Concurrent read exclusive write.
 - Several processors can read same location simultaneously, but error if they write.
- ERCW Exclusive read concurrent write.
 - Uncommon.
- CRCW Concurrent read concurrent write.
 - If multiple writes to same location, can either
 - Let an arbitrary write succeed.
 - Choose a write according to some priority to succeed.



Work and depth

- Depth is the number of (parallel) steps till a PRAM algorithm terminates.
 - Polylogarithmic depth means the algorithm terminates in $O(\log(n)^k)$ steps, where n is input size and k is constant.
 - Goal for PRAM algorithms is often polylog depth using $O(n^k)$ number of processors.
- Work is (number of processors used) x depth.
 - Work of sequential algorithm is just its depth, i.e. time complexity.
 - Work of parallel algorithm $\geq O(\text{work of best sequential algorithm})$.
 - If the work is equal, the parallel algorithm is work-efficient.
- In practice, minimizing work of PRAM algorithm is more important than minimizing depth.

Parallel carry lookahead addition

$$\begin{array}{r} a \quad \quad 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \\ b \quad + \quad 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \\ \hline \text{carry} \quad 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \\ \text{sum} \quad 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \end{array}$$

- Suppose we want to add two n -digit binary numbers, but we can only add a single digit at a time and compute its carry.
 - This is what's provided by full adders in a CPU.
- If we add digit by digit using the grade school method, it takes $O(n)$ time.
 - For $n=32$ or $n=64$, this is much too slow.
- Each digit in the sum depends on the digit from the summands, but also a carry bit from the previous digit.
 - The summand digits can be added in parallel, but it seems the carry bits must be computed sequentially.

a_i	b_i	c_i	s_i	c_{i+1}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



Parallel carry lookahead addition

- We'll show how to compute all the carry bits in parallel in $O(\log n)$ time using n processors.
- After this, all the sum bits can be computed in $O(1)$ parallel time, since $s_i = a_i \oplus b_i \oplus c_i$.
- Denote bitwise AND and OR by \cdot and $+$.
- Define $g_i = a_i \cdot b_i$ as i 'th “carry generate” bit.
 - If $a_i = b_i = 1$, $c_{i+1} = 1$ no matter what c_i is.
- Define $p_i = a_i \oplus b_i$ as i 'th “carry propagate” bit.
 - If $p_i = 1$, then $c_{i+1} = c_i$.

Parallel carry lookahead addition

- We have $c_{i+1} = g_i + c_i p_i$.
- Carry the $i+1$ 'st bit if
 - i 'th bit of a and b generate a carry, OR
 - We carried the i 'th bit, and this was propagated by a and b 's i 'th bit.
- We can also verify $c_{i+1} = g_i + c_i p_i$ directly.

a_i	b_i	c_i	g_i	p_i	c_{i+1}
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	0	1

Parallel carry lookahead addition

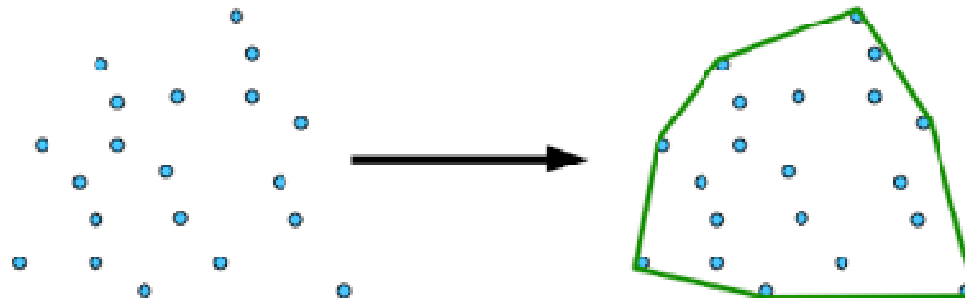
- **Observation** Can write $\begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} c_i \\ T \end{bmatrix} = \begin{bmatrix} c_i p_i + g_i \\ T \end{bmatrix} = \begin{bmatrix} c_{i+1} \\ T \end{bmatrix}$
 - Recall \cdot and $+$ represent AND and OR.
 - Boolean matrix multiplication done same way as for reals.
- Applying this repeatedly, we get

$$\begin{aligned} \begin{bmatrix} c_{i+1} \\ T \end{bmatrix} &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} c_i \\ T \end{bmatrix} \\ &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} p_{i-1} & g_{i-1} \\ F & T \end{bmatrix} \begin{bmatrix} c_{i-1} \\ T \end{bmatrix} = \dots \\ &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \dots \begin{bmatrix} p_1 & g_1 \\ F & T \end{bmatrix} \begin{bmatrix} c_0 \\ T \end{bmatrix} \end{aligned}$$

- Since all the p_i and g_i values are known, the final product can be computed using prefix sum in $O(\log n)$ time with n processors.
- This algorithm or variants are implemented in most real CPUs.

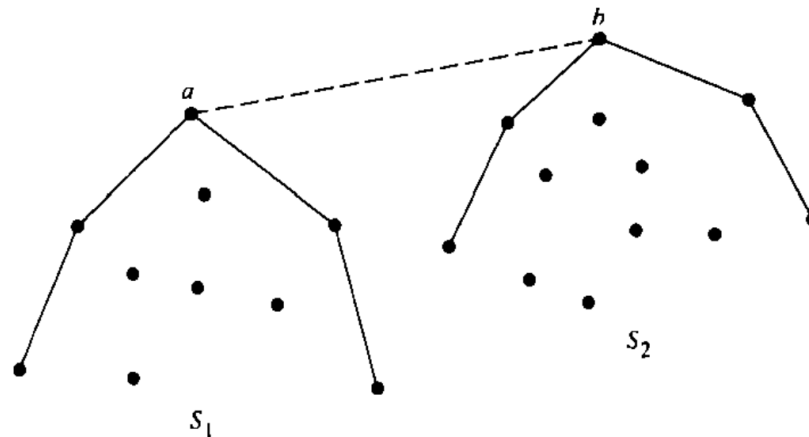
Convex hull

- The convex hull of a set of 2D points is the smallest convex set including all the points.
 - A convex set if, for any two points in the set, the line connecting the point lines in the set.
- Applications in computer graphics, path planning, image processing, pattern recognition, etc.
- Convex hull of n 2D points can be computed in $\Theta(n \log n)$ time.
 - Points on the convex hull output in clockwise order.
 - Sorting can be reduced to convex hull.
- The leftmost and rightmost points, p and q , in the set S must be on the convex hull.
 - Upper hull $UH(S)$ are all points on hull from p to q in clockwise direction.
 - Lower hull $LH(S)$ are all points on hull from q to p in clockwise direction.



Divide and conquer

- Sort the n input points by their x coordinate.
 - Sorting can be done in $O(\log n)$ parallel time with $O(n \log n)$ work.
 - Let sorted points be p_1, p_2, \dots, p_n .
- Let $S_1 = p_1, \dots, p_{n/2}$ and $S_2 = p_{n/2+1}, \dots, p_n$.
- Define upper common tangent of $UH(S_1)$ and $UH(S_2)$ as the common tangent line lying above both $UH(S_1)$ and $UH(S_2)$.
- Given $UH(S_1)$ and $UH(S_2)$, their upper common tangent can be found in $O(\log n)$ time and $O(n)$ work using binary search.
 - Use property that the angle from any point in one set to the points in the other set first increases then decreases.
- The overall upper hull consists of points of S_1 from p_1 to a , and points of S_2 from b to p_n .





Parallel convex hull

- The divide and conquer approach gives an $O(\log^2 n)$ time and $O(n \log n)$ work parallel convex hull algorithm.
 - There are $O(\log n)$ stages of dividing the input. Each stage takes $O(\log n)$ time to find the upper common tangent.
 - Finding the upper common tangent takes $O(n)$ amount of work.
 - The total amount of work satisfies $W(n) = 2W\left(\frac{n}{2}\right) + O(n)$, so $W(n) = O(n \log n)$.

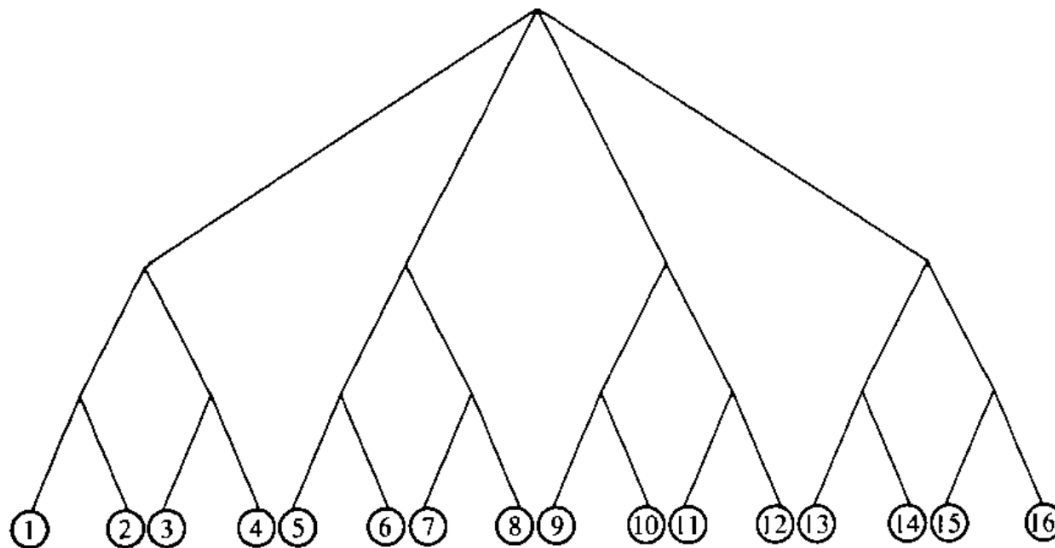


Constant time max finding

- Using a balanced binary tree we can find the max of n numbers in $O(\log n)$ time and $O(n)$ work.
- We show how to find max in $O(\log \log n)$ time using $O(n)$ work on a min priority CRCW PRAM.
 - I.e. when multiple values are written to same location, the min value wins.
- First, we can find the max of p numbers x_1, \dots, x_p in $O(1)$ time and $O(p^2)$ work on the CRCW PRAM.
 - For $1 \leq i, j \leq p$, in parallel set $B(i, j) = 1$ if $x_i \geq x_j$, and $B(i, j) = 0$ otherwise.
 - Uses p^2 processors.
 - For $1 \leq i \leq p$, in parallel set $M_i = B(i, 1) \wedge B(i, 2) \wedge \dots \wedge B(i, p)$.
 - $M_i = 1$ iff x_i is the max value.
 - This requires that when 0's and 1's are written to the same M_i , the minimum value (i.e. 0) gets written.

Doubly logarithmic tree

- Create a tree with the x_i 's at the leaves.
- For each internal node u , let n_u be the number of leaves in the subtree rooted at u . Make the degree of u be $\lceil \sqrt{n_u} \rceil$.
 - For simplicity, assume $n = 2^{2^k}$. Then the tree has $k = \log \log n$ levels.
 - The root of the tree has degree $2^{2^{k-1}} = \sqrt{n}$.
 - Each child of the root has degree $2^{2^{k-2}}$.
 - In general, at level $0 \leq i \leq k-1$, each node has degree $2^{2^{k-i-1}}$, and there are $2^{2^k - 2^{k-i}}$ nodes total at the level.



Source: Introduction to
Parallel Algorithms, Jaja



Superfast max finding

- Suppose each node computes the max of all its children.
 - Then each node has the max of all the leaf nodes in its subtree, and the root has the overall max value.
 - To compute the max of p children takes $O(p^2)$ work.
- Total time for algorithm is $O(\log \log n)$.
- Total work per level is $O\left(\left(2^{2^{k-i-1}}\right)^2 \cdot 2^{2^k - 2^{k-i}}\right) = O\left(2^{2^k}\right) = O(n)$.
 - Total overall work is $O(n \log \log n)$. So the algorithm isn't work efficient.



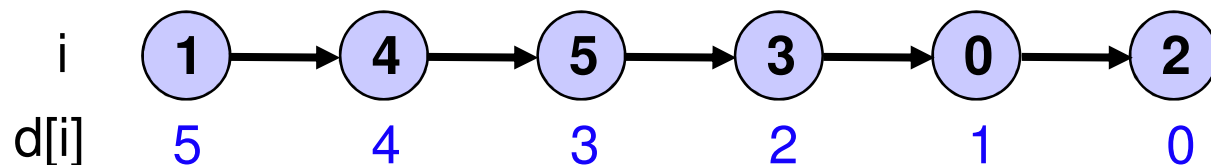
Superfast max finding

- To make the previous algorithm work efficient, we use a technique called accelerated cascading.
 - Start with an optimal algorithm until problem size is sufficiently small.
 - Then switch to fast but nonoptimal algorithm.
- First, use a binary max tree on the inputs for $\lceil \log \log \log n \rceil$ levels.
 - This produces $n' = O\left(\frac{n}{\log \log n}\right)$ values, each the max of $\log \log n$ input values.
 - It also does $O(n)$ amount of work.
 - Then use the doubly logarithmic tree on the n' values.
 - This runs for $O(\log \log n') = O(\log \log n)$ time.
 - It does $O(n' \log \log n') = O(n)$ work.

List ranking

- Given a linked list, compute the distance of each node to the end.
 - Linked list is represented by an array `next`, where `next[i]` initially points to node following node `i`.
 - `next[i]=NULL` for the last node.
- Let `d[i]` be `i`'s estimate of its distance to the end.
 - Initially `d[i]=0` for the last node, and `d[i]=1` for all other nodes.

i	0	1	2	3	4	5
$next[i]$	2	4	\perp	0	5	3

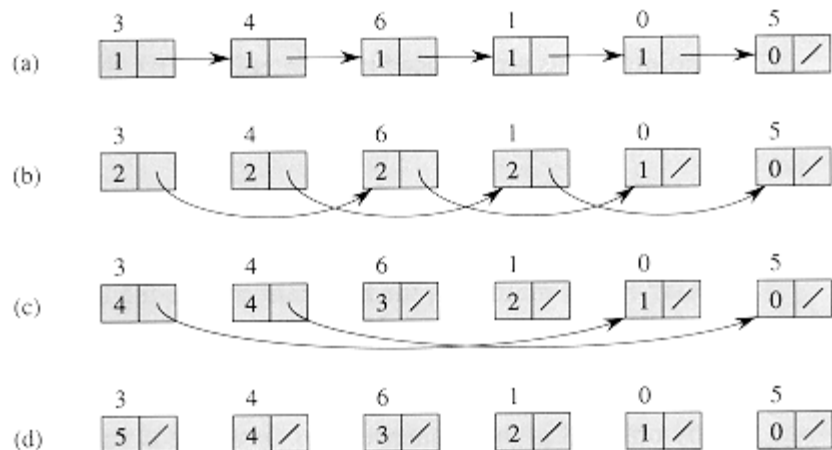


List ranking

- Repeatedly apply pointer jumping.
 - If currently $i \rightarrow j$ and $j \rightarrow k$, set $i \rightarrow k$.
- Let k be a node that's distance m away from the end, for some m .
 - After i steps, $d[k] = \min(m, 2^i)$, and $\text{next}[k]$ points $\min(m, 2^i)$ distance away.
- Since $d[*] \leq n$, algorithm terminates in $O(\log n)$ steps.
- Work is $O(n \log n)$.
 - Not efficient, since sequential list ranking takes $O(n)$ work.
- List ranking has many applications, including Euler tour technique, connected components, expression tree evaluation, ear decomposition, etc.

```

while next[i] ≠ NULL for some i
  do parallel for all i
    if next[i] ≠ NULL
      d[i] = d[i] + d[next[i]]
      next[i] = next[next[i]]
  
```

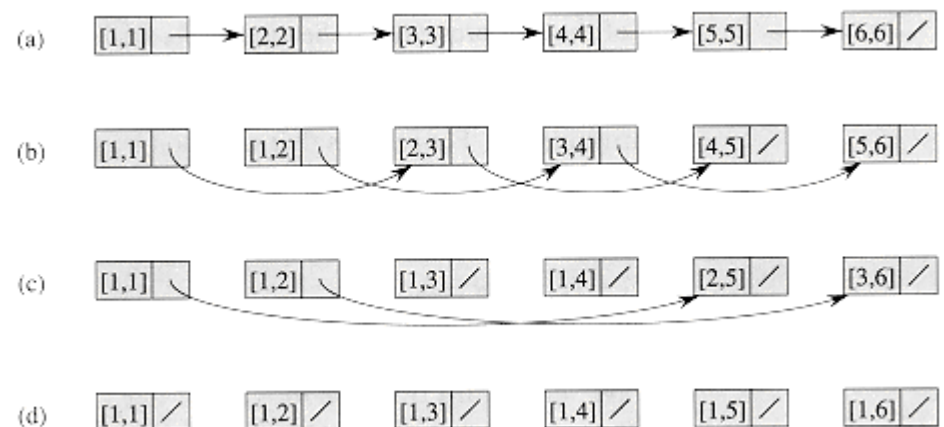


Prefix sum on linked list

- We've seen how to do prefix sum on an array.
- Using pointer jumping, can also do prefix sum on a linked list.
 - Initially each node i has a value $x[i]$.
 - The output, i.e. prefix sum of node i is stored in $d[i]$.
 - Only difference with list ranking is update $d[\text{next}[i]]$ instead of $d[i]$.
- After i steps, first 2^i nodes have correct prefix sum, and other nodes have the sum of the preceding 2^i values.
- Takes $O(\log n)$ time, does $O(n \log n)$ work.

```

do parallel for all i
  d[i]=x[i]
while next[i]≠NULL for some i
  do parallel for all i
    if next[i]≠NULL
      d[next[i]]=d[i]+d[next[i]]
      next[i]=next[next[i]]
    
```



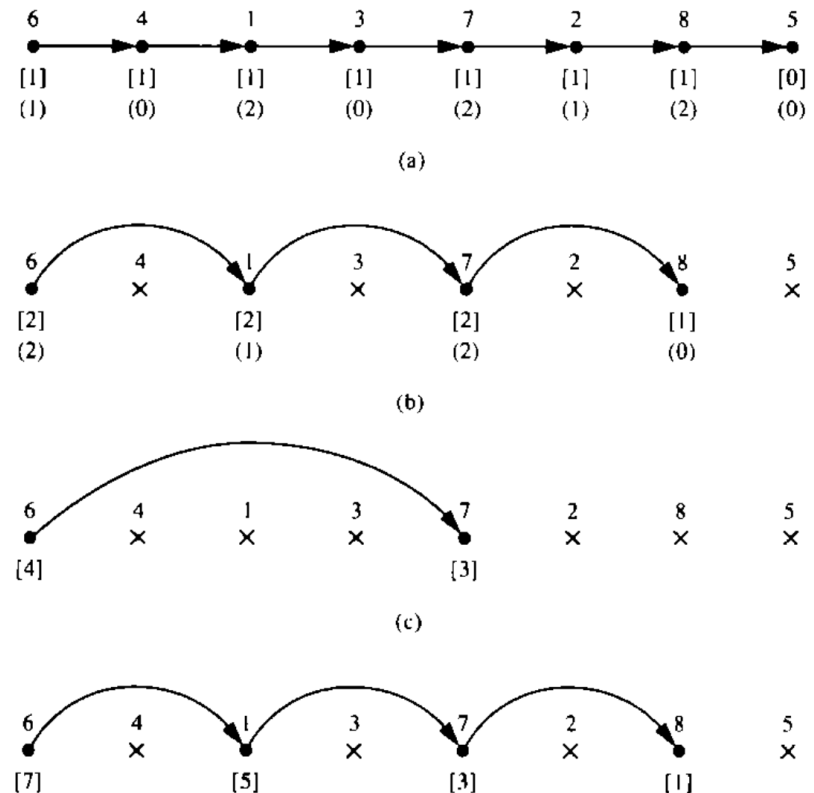


Work efficient list ranking

- List ranking using pointer jumping does $O(n \log n)$ work.
- To make list ranking efficient, we can
 - Shrink the list until only $O(n / \log n)$ nodes remain.
 - Apply pointer jumping to remaining nodes.
 - Restore the removed nodes and determine their ranks.
- Assume first and third steps take $O(n)$ work.
- Then second step takes $O\left(\frac{n}{\log n} \log\left(\frac{n}{\log n}\right)\right) = O(n)$ work, so total work is $O(n)$.

Work efficient list ranking

- To shrink the list, we repeatedly remove an independent set of nodes.
 - A set of nodes I is independent if $\forall i \in I: (prev(i) \notin I) \wedge (next(i) \notin I)$.
 - Suppose we have a set of n nodes. We show next lecture how find $\Omega(n)$ independent nodes in $O(\log n)$ time and $O(n)$ work.
- Given an independent set I , for each $i \in I$ set $dist[prev[i]] = dist[prev[i]] + dist[i]$.
- To compute distance of a removed node i , set $dist[i] = dist[i] + dist[next[i]]$.



- values in parentheses are used to find independent set.
- dist values are shown in brackets.



Work efficient list ranking

- Since each round we remove $\Omega(n)$ number of remaining nodes, it takes $O(\log \log n)$ rounds to shrink the list to size $O(n / \log n)$.
 - After this the pointer jumping takes $O(\log n)$ time.
- Each round takes $O(\log n)$ time to find the independent set.
- So total time is $O(\log n \log \log n)$.
 - Time can be reduced to $O(\log n)$ using more efficient algorithm.
- For total work, we have that in round k , number of remaining nodes is $O(c^k n)$ for some $c < 1$.
- So total work to find independent sets in all rounds is $\sum_{c=0}^{\log \log n} O(c^k n) = O(n)$. Since work from pointer jumping is also $O(n)$, total work is $O(n)$.