CS290: Introduction to Algorithmic Game Theory

Week 4.1, VCG (Dengji ZHAO)

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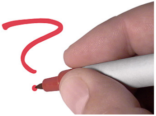
Recap: Game Theory

	Participants		Game		Outcome
Game Theory	✓	⇒		⇒	
Mechanism Design	✓	⇒		⇒	✓

Recap: The General Setting of Mechanism Design

- A set of n participants/players, denoted by N.
- A mechanism needs to choose some alternative from A (allocation space), and to decide a payment for each player.
- Each player $i \in N$ has a private valuation function $v_i : A \to \mathbb{R}$, let V_i denote all possible valuation functions for i.
- Let $v = (v_1, \cdots, v_n)$, $v_{-i} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$.
- Let $V = V_1 \times \cdots \times V_n$, $V_{-i} = V_1 \times \cdots V_{i-1} \times V_{i+1} \times \cdots \times V_n$.

Recap: A Definition of a Mechanism (with Money)

Definition

A (direct revelation) mechanism is a social choice function $f : V_1 \times \cdots \times V_n \to A$ and a vector of payment functions $p_1, \ldots, p_n$, where $p_i : V_1 \times \cdots \times V_n \to \mathbb{R}$ is the amount that player i pays.

- direct revelation: *the mechanism requires each player to report her valuation function to the mechanism.*

Definition

Given a mechanism $(f, p_1, \ldots, p_n)$, and players' valuation report profile $v' = (v'_1, \cdots, v'_i, v'_n)$, player i's utility is defined by $v_i(f(v')) - p_i(v')$, where v_i is i's true valuation function.

Recap: Properties of a Mechanism

Truthfulness A mechanism $(f, p1, \ldots, p_n)$ is called truthful (*incentive compatible*) if for every player i, every $v_1 \in V_1, \ldots, v_n \in V_n$ and every $v_i' \in V_i$, if we denote $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$, then $v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v_i', v_{-i})$.

Efficiency We say a social choice function f is efficient if it maximises social welfare for all valuation reports. That is, for all $v \in V$, $f \in \arg\max_{f' \in F} \sum_{i \in N} v_i(f'(v))$ where F is the set of all feasible social choice functions.

Individual Rationality We say a mechanism $(f, p_1, \ldots, p_n)$ is individually rational if for every player i, every $v \in V$, we have $u_i(f, p_1, \ldots, p_n, v, v_i) \geq 0$.

Vickrey-Clarke-Groves Mechanisms

Definition 9.16 A mechanism $(f, p_1, \ldots, p_n)$ is called a Vickrey–Clarke–Groves (VCG) mechanism if

- $f(v_1, \ldots, v_n) \in \operatorname{argmax}_{a \in A} \sum_i v_i(a)$; that is, f maximizes the social welfare, and
- for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \to \Re$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$: $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$.

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- Definition of $h_{-i} : V_{-i} \to \mathbb{R}$
  - $h_{-i}(.) = 0$
  - $h_{-i}(v_{-i}) = \sum_{j \in N \setminus \{i\}} v_j(f(v_{-i}))$, the maximum social welfare without i's participation.
  - ...

Examples of Applying VCG

A seller sells m (heterogeneous) items:

- A set of m items to be allocated (denoted by M)
- A set of n players (denoted by N)
- Each player i has a valuation function $v_i : 2^M \to \mathbb{R}$

Question

What is size of the allocation space?

Properties of VCG

Is VCG truthful, efficient and individually rational?

How to verify a mechanism is truthful or not?

Theorem

A mechanism is truthful if and only if it satisfies the following conditions for every i and every v_{-i}:

1. *The payment p_i does not depend on v_i, but only on the alternative chosen $f(v_i, v_{-i})$. That is, for every v_{-i}, there exist prices $p_a \in \mathbb{R}$, for every $a \in A$, such that for all v_i with $f(v_i, v_{-i}) = a$ we have that $p(v_i, v_{-i}) = p_a$.*
2. *The mechanism optimizes for each player. That is, for every v_i, we have that $f(v_i, v_{-i}) \in \arg\max_a(v_i(a) - p_a)$, where the quantification is over all alternatives in the range of $f(\cdot, v_{-i})$.*

Advanced Reading

- Introduction to Mechanism Design [AGT Chapter 9]
- Vickrey-Clarke-Groves mechanisms [AGT Chapter 9.3]