(注意，要么写 nonzero，要么像 Problem1（a）将=0 的情况写明，不然要扣分)

Problem 1

(10 points)

For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Solution:

The given signal is:

$$\begin{aligned} x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

So, $a_0 = 2,\ a_2 = a_{-2} = \frac{1}{2},\ a_5 = -2j,\ a_{-5} = 2j,\ a_k = 0\ (\text{for } k \neq 0, 2, -2, 5, -5),\ \omega_0 = 2\pi/6 = \pi/3$

Problem 2

(20 points)
Suppose we are given the following information about a signal x(t):
1. x(t) is real and odd.
2. x(t) is periodic with period T = 2 and has Fourier coefficients a_k.
3. $a_k = 0$ for |k| > 1.
4. $\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$.

Specify two different signals that satisfy these conditions.

Solution:

From the problem description we know that signal x(t), is real and odd. Using the properties of continuous-time Fourier series (refer to table 3.1 in the book) we know that Fourier coefficients are purely imaginary and odd:

$$a_k = -a_{-k} \text{ and } a_0 = 0 \quad (1)$$

For real signal x(t) Fourier coefficients become:

$$|a_{-k}| = |a_k| \quad (2)$$

Furthermore, from problem description we know that $a_k = 0$ for |k| > 1, i.e.:

$$a_k = 0, \text{ for } k > 1$$

$$a_k = 0, \text{ for } k < -1$$

$$a_{-1} \neq 0, a_0 = 0, a_1 \neq 0$$

Then we can use Parseval's relation for continuous-time periodic signal:

$$\frac{1}{T}\int_0^T |x(t)^2|\,dt = \sum_{-\infty}^{\infty} |a_k|^2$$

In our case previous equation will become:

$$\frac{1}{2}\int_0^2 |x(t)^2|\,dt = \sum_{-1}^{1} |a_k|^2 = 1$$

Furthermore, we can write:

$$|a_{-1}|^2 + |a_0|^2 + |a_1|^2 = 1$$

$$|a_{-1}|^2 + |a_1|^2 = 1$$

Using an property (2) of real signal we have:

$$|a_{-1}| = |a_1| \rightarrow |a_{-1}|^2 + |a_1|^2 = 2|a_1|^2 = 1$$

Then, there are two possible solutions for coefficients a_{-1} and a_1:

$$-a_{-1} = a_1 = \frac{1}{j\sqrt{2}} \rightarrow a_{-1} = -\frac{1}{j\sqrt{2}}$$

$$-a_{-1} = a_1 = -\frac{1}{j\sqrt{2}} \rightarrow a_{-1} = \frac{1}{j\sqrt{2}}$$

Recall that signal x(t) can be shown in form:

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

The first solution is:

$$x_1(t) = -\frac{1}{j\sqrt{2}} e^{-j\frac{2\pi}{2}t} + \frac{1}{j\sqrt{2}} e^{j\frac{2\pi}{2}t}$$

$$= \frac{\sqrt{2}}{2j}(e^{j\pi t} - e^{-j\pi t})$$

$$= \sqrt{2}\sin(\pi t)$$

The second solution is:

$$x_2(t) = \frac{1}{j\sqrt{2}} e^{-j\frac{2\pi}{2}t} - \frac{1}{j\sqrt{2}} e^{j\frac{2\pi}{2}t}$$

$$= -\frac{\sqrt{2}}{2j}(e^{j\pi t} - e^{-j\pi t})$$

$$= -\sqrt{2}\sin(\pi t)$$

Problem 3

(20 points)
Consider the following three discrete-time signals with a fundamental period of 6:

$$x[n] = 1 + \cos(\tfrac{2\pi}{6}n), \qquad y[n] = \sin(\tfrac{2\pi}{6}n + \tfrac{\pi}{4}), \qquad z[n] = x[n]y[n].$$

(a) Determine the Fourier series coefficients of x[n].

(b) Determine the Fourier series coefficients of y[n].

(c) Use the results of parts (a) and (b), along with the multiplication property of the discrete-time Fourier series, to determine the Fourier series coefficients of z[n] = x[n]y[n].

(d) Determine the Fourier series coefficients of z[n] through direct evaluation, and compare your result with that of part (c).

Solution:

(a) Given input,

$$x[n] = 1 + \cos(\frac{2\pi}{6}n)$$

$$x[n] = 1 + \frac{1}{2}(e^{j\frac{2\pi}{6}n} + e^{-j\frac{2\pi}{6}n})$$

So, the nonzero FS coefficients of x[n] are $a_0=1$, $a_1 = a_{-1} = \frac{1}{2}$

(b) Given input,

$$y[n] = \sin(\frac{2\pi}{6}n + \frac{\pi}{4})$$

$$y[n] = \frac{1}{2j}(e^{j(\frac{2\pi}{6}n+\frac{\pi}{4})} - e^{-j(\frac{2\pi}{6}n+\frac{\pi}{4})})$$

So, the nonzero FS coefficients of y[n] are $b_1 = \frac{1}{2j}e^{j\pi/4}$, $b_{-1} = -\frac{1}{2j}e^{-j\pi/4}$

(c) Using the multiplication property of FS from section 3.5.5,

$$x[n] \overset{F.S}{\Leftrightarrow} a_k$$

$$y[n] \overset{F.S}{\Leftrightarrow} b_k$$

$$z[n] = x[n]y[n] \overset{F.S}{\Leftrightarrow} h_k = \sum_{l=<N=6>} a_l b_{k-l}$$

Therefore, $h_k = a_k * b_k$, i.e., convolution of the FS coefficients,where $a_k = \delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$ and $b_k = \frac{1}{2j}e^{j\pi/4}\delta[k-1] - \frac{1}{2j}e^{-j\pi/4}\delta[k+1]$

$$h_k = a_k * b_k$$

$$= (\delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]) * (\frac{1}{2j}e^{j\pi/4}\delta[k-1] - \frac{1}{2j}e^{-j\pi/4}\delta[k+1])$$

$$= \frac{1}{2j}e^{j\pi/4}\delta[k-1] - \frac{1}{2j}e^{-j\pi/4}\delta[k+1] + \frac{1}{4j}e^{j\pi/4}\delta[k-2] - \frac{1}{4j}e^{-j\pi/4}\delta[k] + \frac{1}{4j}e^{j\pi/4}\delta[k] - \frac{1}{4j}e^{-j\pi/4}\delta[k+2]$$

$$= \frac{1}{2}\sin(\pi/4)\delta[k] + \frac{1}{2j}e^{j\pi/4}\delta[k-1] - \frac{1}{2j}e^{-j\pi/4}\delta[k+1] + \frac{1}{4j}e^{j\pi/4}\delta[k-2] - \frac{1}{4j}e^{-j\pi/4}\delta[k+2] \tag{4}$$

From equation(4), the non-zero FS coefficients of z(t) i.e. h_k are,

$$h_0 = \frac{1}{2}\sin(\pi/4),\ \ h_1 = \frac{1}{2j}e^{j\pi/4},\ h_{-1} = -\frac{1}{2j}e^{-j\pi/4},\ h_2 = \frac{1}{4j}e^{j\pi/4},\ h_{-2} = -\frac{1}{4j}e^{-j\pi/4}$$

(d) Given,

$$z[n] = x[n]y[n]$$

$$= [1 + \cos(\frac{2\pi}{6}n)][\sin(\frac{2\pi}{6}n + \frac{\pi}{4})]$$

$$= \sin(\frac{2\pi}{6}n + \frac{\pi}{4}) + \sin(\frac{2\pi}{6}n + \frac{\pi}{4})\cos(\frac{2\pi}{6}n)$$

$$= \sin(\frac{2\pi}{6}n + \frac{\pi}{4}) + \frac{1}{2}[\sin(\frac{2\pi}{6}n + \frac{\pi}{4} + \frac{2\pi}{6}n) + \sin(\frac{2\pi}{6}n + \frac{\pi}{4} - \frac{2\pi}{6}n)]$$

$$= \sin(\frac{2\pi}{6}n + \frac{\pi}{4}) + \frac{1}{2}[\sin(\frac{4\pi}{6}n + \frac{\pi}{4}) + \sin(\frac{\pi}{4})]$$

So, the nonzero FS coefficients of z[n] are

$$h_0 = \frac{1}{2}\sin(\pi/4),\ \ h_1 = \frac{1}{2j}e^{j\pi/4},\ h_{-1} = -\frac{1}{2j}e^{-j\pi/4},\ h_2 = \frac{1}{4j}e^{j\pi/4},\ h_{-2} = -\frac{1}{4j}e^{-j\pi/4}$$

Problem 4

(25 points)
Consider a discrete-time LTI system with impulse response

$$h[n] = (\frac{1}{4})^n u[n]$$

Find the Fourier series representation of the output y[n] for each of the following inputs:

(a) $x[n] = \sin(\frac{3\pi}{4}n)$

(b) $x[n] = \cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n)$

Solution:

Given the h[n],we get,

$$H(e^{jw}) = \frac{4}{4 - e^{-jw}}$$

From, the section3.8,

$$y[n] = \sum_{k=<N>} a_k H(e^{j2\pi k/N}) e^{j(2\pi k/N)n}$$

Where $\omega = 2\pi/N$, and the input x[n] is given as,

$$x[n] = \sum_{k=<N>} a_k e^{j(2\pi k/N)n}$$

(a) Given input $x[n] = \sin(\frac{3\pi}{4}n)$, which implies that N=8, m=3,

$$x[n] = \frac{e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}}{2j}$$

$$\Rightarrow a_3 = \frac{1}{2j}, a_{-3} = -\frac{1}{2j}$$

Let the FS coefficients of y[n] be b_k then,

$$b_3 = a_3 H(e^{j3\pi/4}) = \frac{4}{2j(4 - e^{-j3\pi/4})}$$

$$b_{-3} = a_{-3} H(e^{-j3\pi/4}) = \frac{-4}{2j(4 - e^{j3\pi/4})}$$

(b) Given input,

$$x[n] = \cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n)$$

$$x[n] = \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2} + 2\frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}$$

$$\Rightarrow a_1 = a_{-1} = \frac{1}{2}, \ a_2 = a_{-2} = 1$$

Let the FS coefficients of y(t) be $b_k$then,

$$b_1 = a_1 H(e^{j\pi/4}) = \frac{4}{2(4 - e^{-j\pi/4})}$$

$$b_{-1} = a_{-1} H(e^{-j\pi/4}) = \frac{4}{2(4 - e^{j\pi/4})}$$

$$b_2 = a_2 H(e^{j\pi/2}) = \frac{4}{(4 - e^{-j\pi/2})}$$
$$b_{-2} = a_{-2} H(e^{-j\pi/2}) = \frac{4}{(4 - e^{j\pi/2})}$$

Problem 5

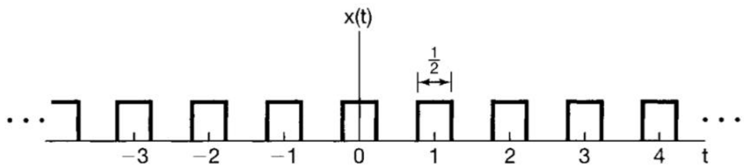
(25 points)
Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}$$

Find the Fourier series representation of the output y(t) for each of the following inputs:

(a) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$

(b) x(t) is the periodic wave depicted showed below:



Solution:

(a) Given that $h(t) = e^{-4|t|}$, so we get,

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-4|t|}e^{-jwt}\,dt$$

$$= \int_{-\infty}^{0} e^{4t}e^{-jwt}\,dt + \int_{0}^{\infty} e^{-4t}e^{-jwt}\,dt$$

$$= \int_{-\infty}^{0} e^{(4t-jwt)}\,dt + \int_{0}^{\infty} e^{-(4t+jwt)}\,dt$$

$$= \frac{1}{4-jw} + \frac{1}{4+jw}$$

$$= \frac{8}{16+w^2}$$

Let the FS coefficients of y(t) be b_k then,

$$b_k = a_k H(j2\pi k) = \frac{8}{16+(2\pi k)^2}$$

(b) From the figure, we can get that the period of x(t) is T=1, which implies that $\omega = 2\pi/T = 2\pi/1 = 2\pi$.Therefore , its FS coefficients are,

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega t}\,dt$$

For k = 0, we get,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \int_{-1/4}^{1/4} 1\,dt$$

$$= \frac{1}{2}$$

For $k \neq 0$, we get,

$$a_k = \int_{-1/4}^{1/4} e^{-jk\omega t} dt$$

$$= \frac{1}{j2\pi k}[-e^{-j\pi k/2} + e^{j\pi k/2}]$$

Therefore,

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \ \text{even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \ \text{odd} \end{cases}$$

Therefore, let the FS coefficients of y(t) be b_k then,

$$b_k = a_k H(jk\omega) = \begin{cases} \frac{1}{4}, & k = 0 \\ 0, & k \ \text{even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}[\frac{8}{16 + (2\pi k)^2}], & k \ \text{odd} \end{cases}$$