CS101 Algorithms and Data Structures

Minimum Spanning Tree

Textbook Ch 23

Outline

- Definition and applications
- Prim’s algorithm
- Kruskal’s algorithm

Spanning trees

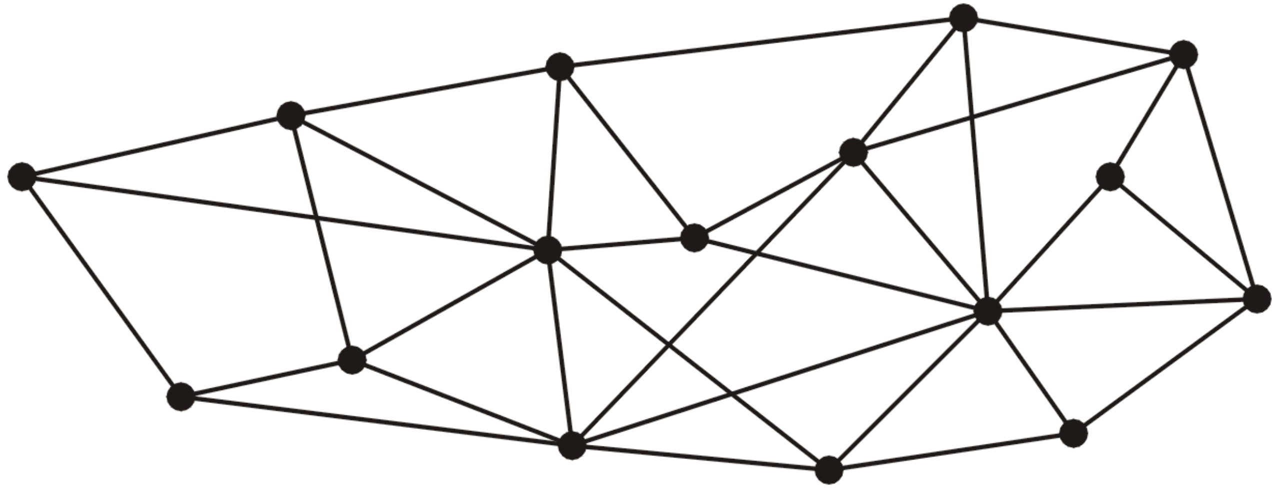
Given a connected graph with n vertices, a spanning tree is defined as a subgraph that is a tree and includes all the n vertices

- It has $n - 1$ edges

A spanning tree is not necessarily unique

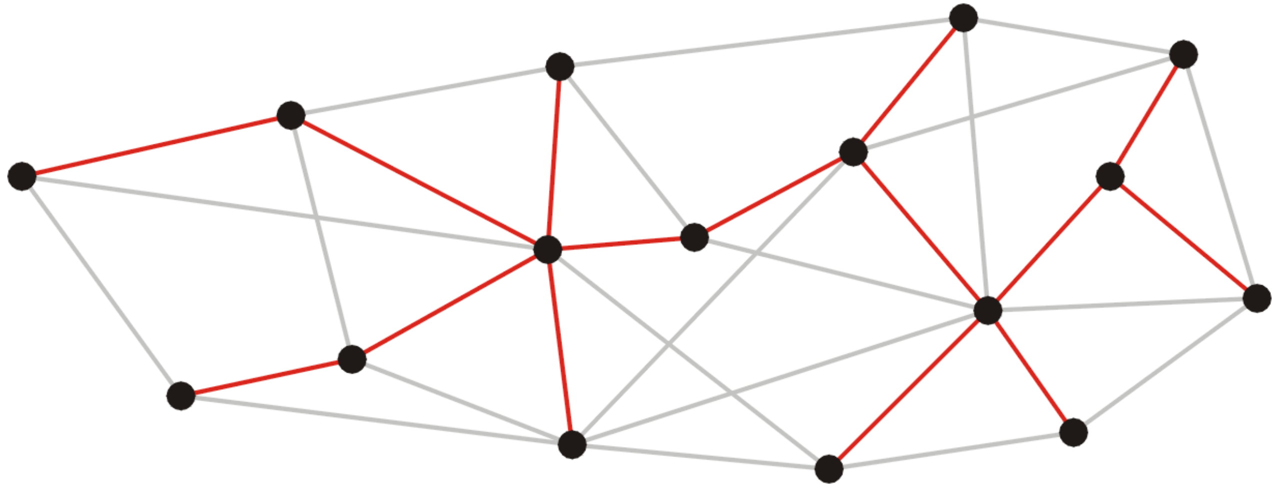
Spanning trees

This graph has 16 vertices and 35 edges



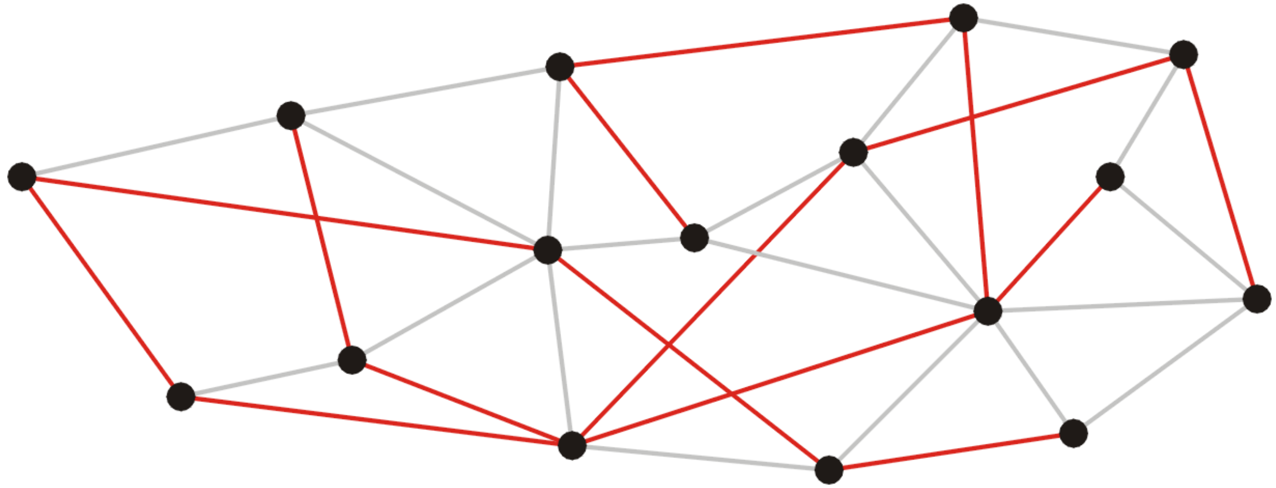
Spanning trees

These 15 edges form a spanning tree

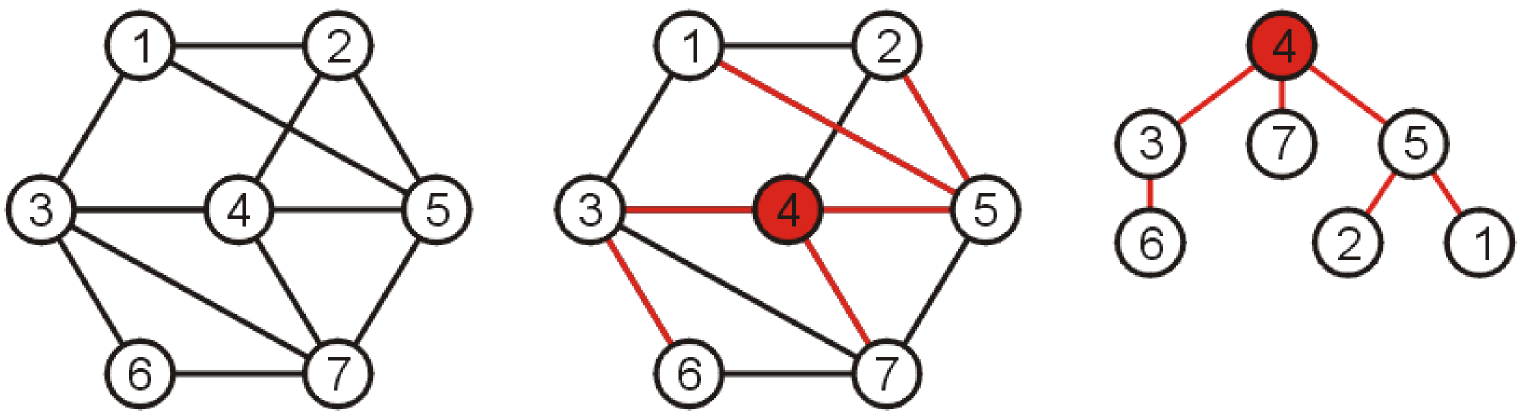


Spanning trees

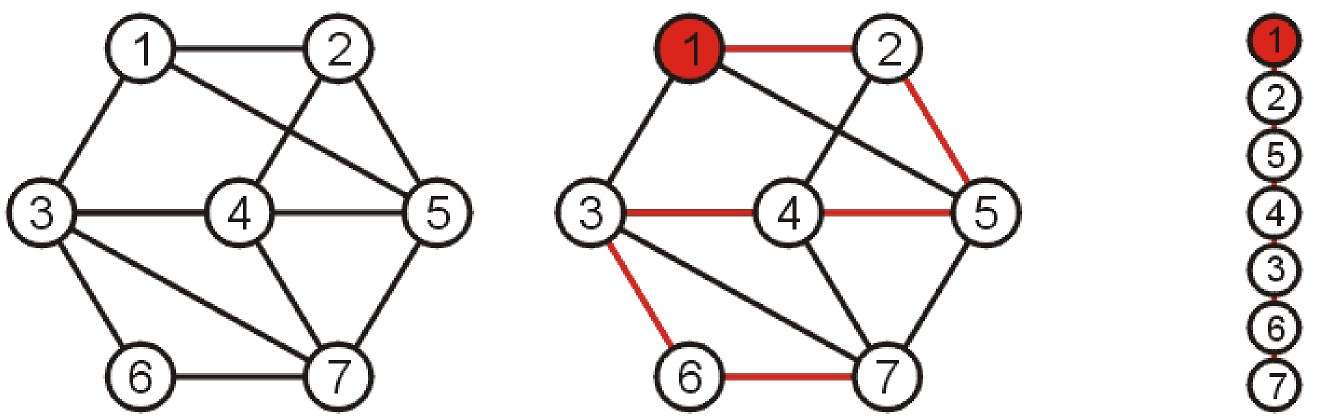
As do these 15 edges:



Spanning trees



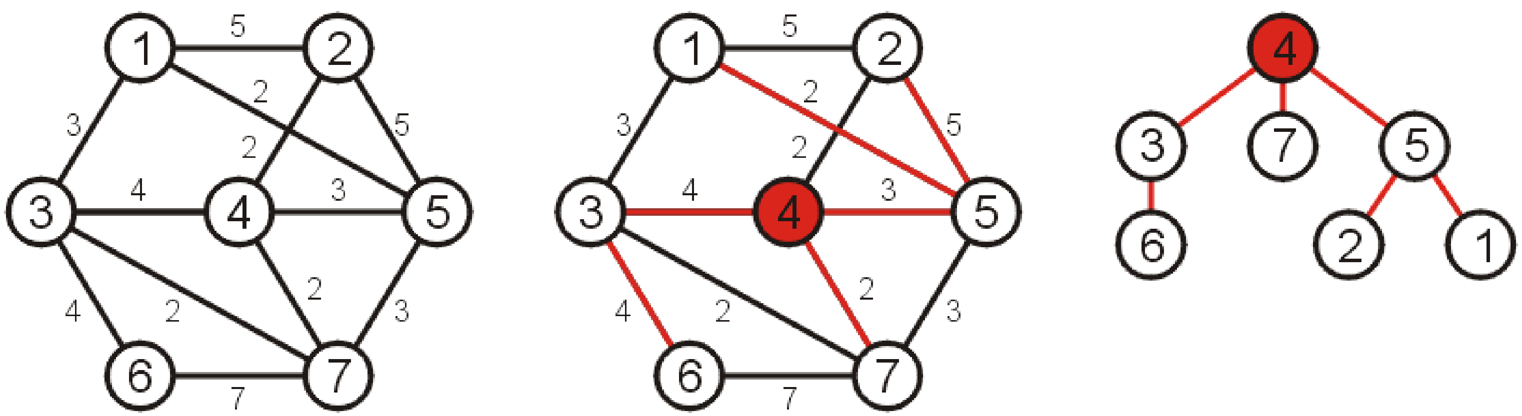
Spanning trees



Spanning trees on weighted graphs

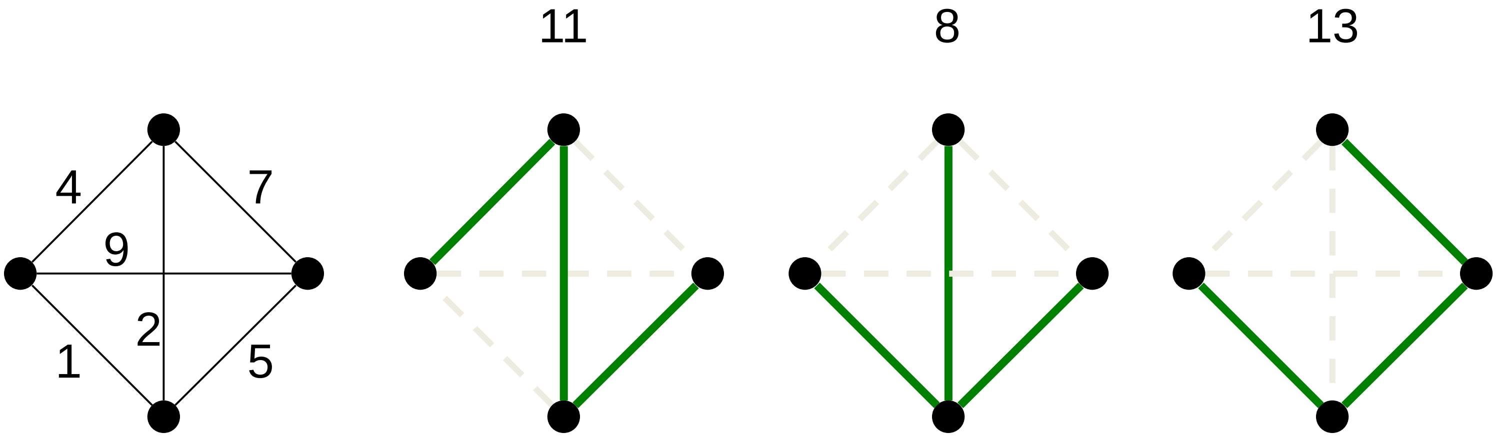
The weight of a spanning tree is the sum of the weights on all the edges which comprise the spanning tree

– The weight of this spanning tree is 20



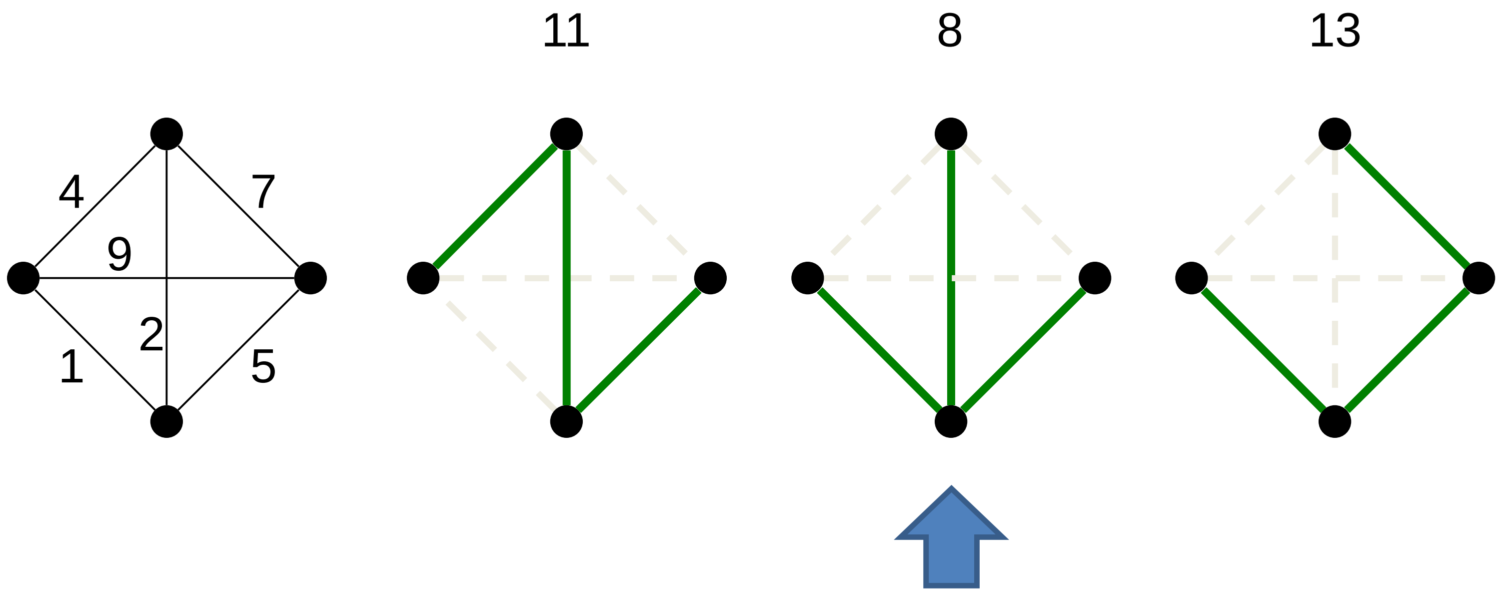
Spanning trees on weighted graphs

The weight of a spanning tree is the sum of the weights on all the edges which comprise the spanning tree



Minimum Spanning Trees

- Which spanning tree minimizes the weight?
  - Such a tree is termed a *minimum spanning tree*



Application

Consider supplying power to

- All circuit elements on a board
- A number of loads within a building

A minimum spanning tree will give the lowest-cost solution

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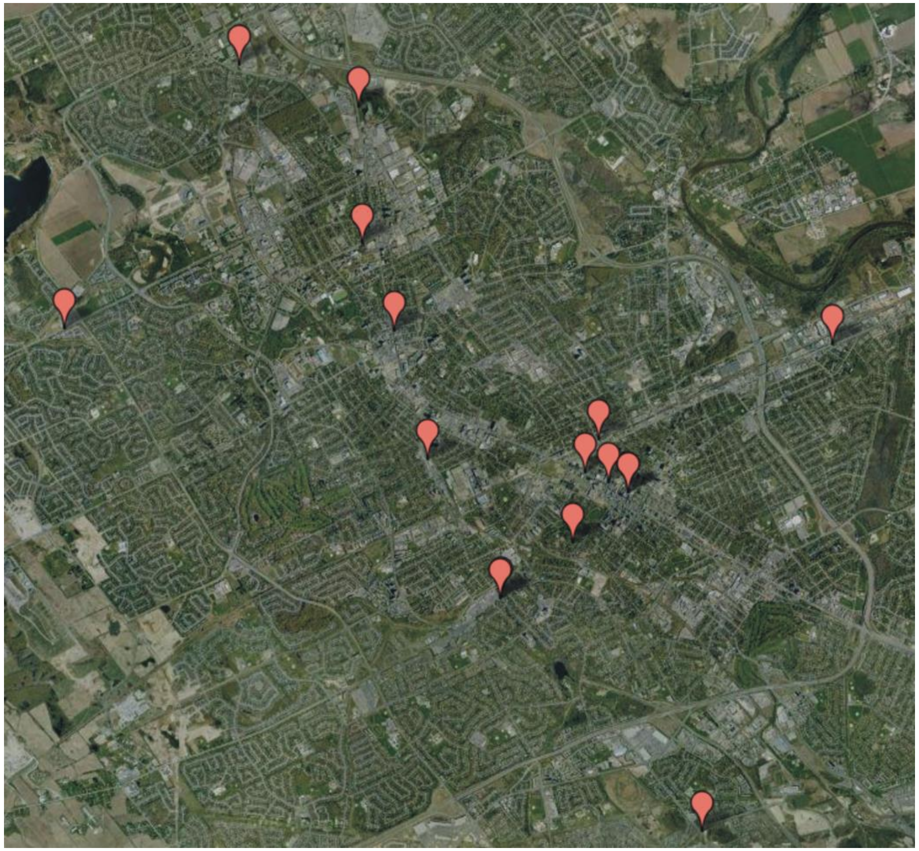
Application

The first application of a minimum spanning tree algorithm was by the Czech mathematician Otakar Borůvka who designed electricity grid in Moravia in 1926

Application

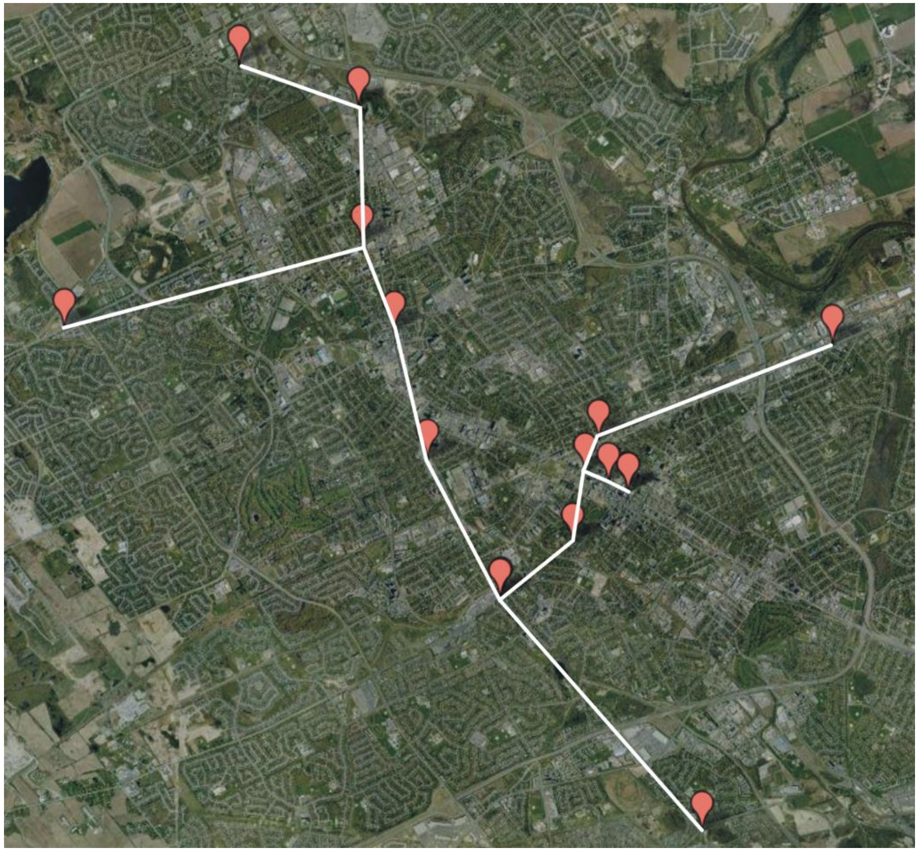
Consider attempting to find the best means of connecting a number of houses

- Minimize the length of transmission lines



Application

A minimum spanning tree will provide the optimal solution



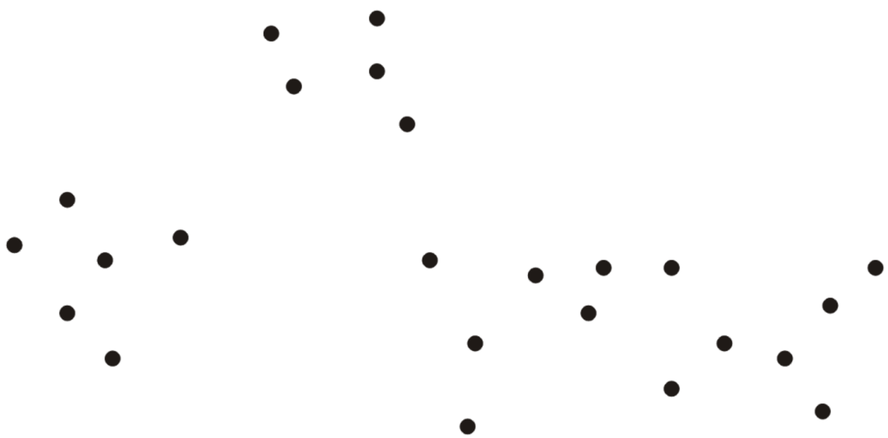
Application

Consider an *ad hoc* wireless network

- Any two terminals can connect with any others

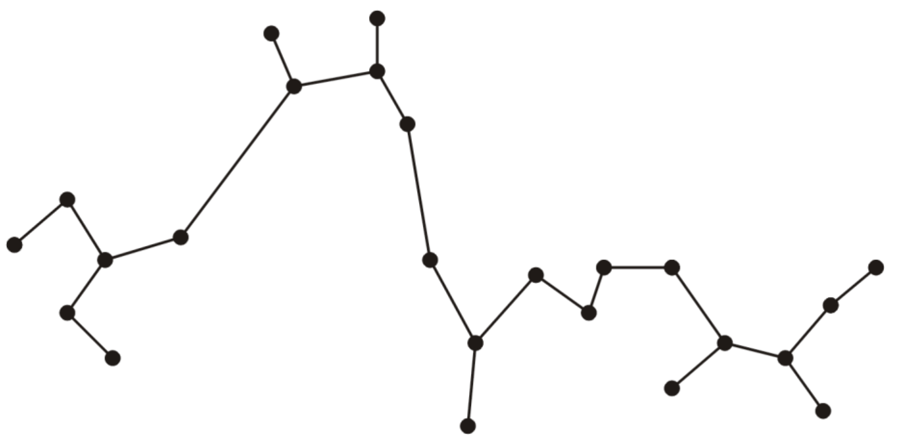
Problem:

- Errors in transmission increase with transmission length
- Can we find clusters of terminals which can communicate safely?



Application

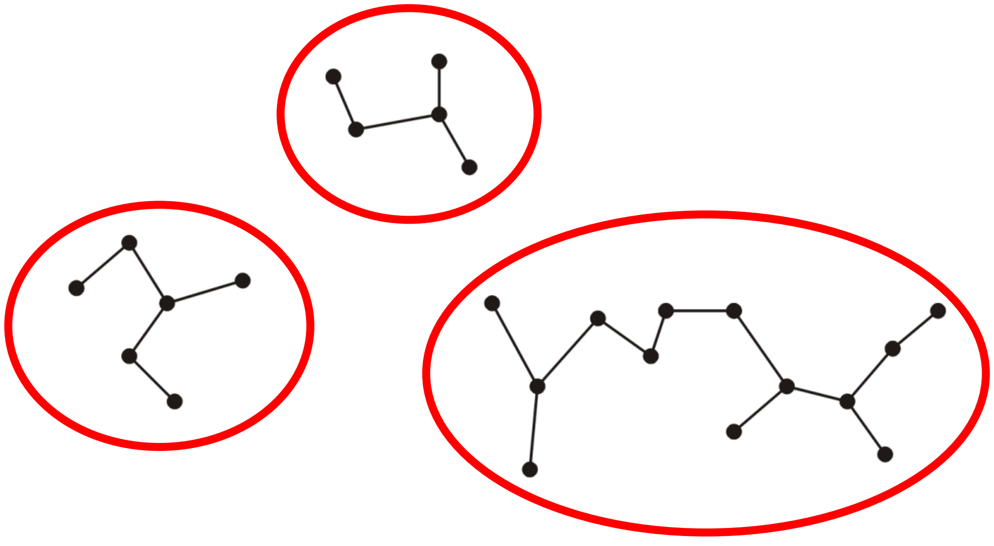
Find a minimum spanning tree



Application

Remove connections which are too long

This *clusters* terminals into smaller and more manageable sub-networks



Minimum Spanning Trees

Simplifying assumption:

- All edge weights are distinct

This guarantees that given a graph, there is a unique minimum spanning tree.

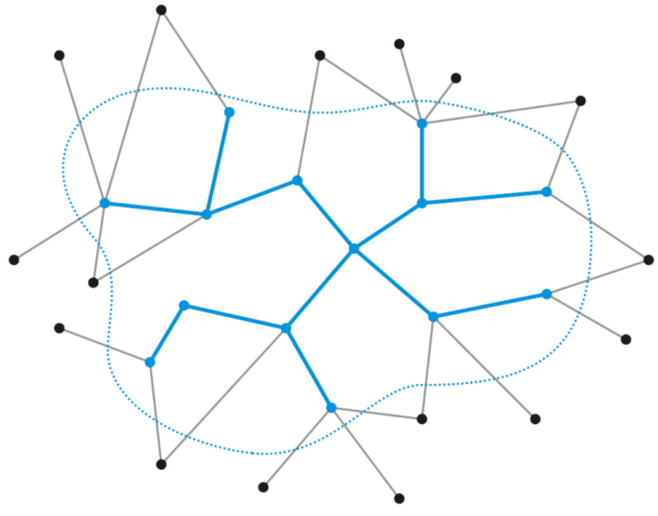
Outline

- Definition and applications
- Prim's algorithm
- Kruskal's algorithm

Strategy

Strategy:

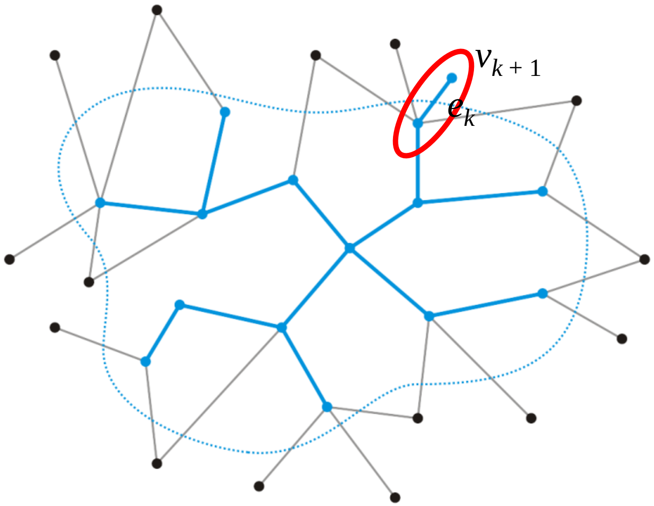
- Suppose we have a known minimum spanning tree on $k < n$ vertices
- How could we extend this minimum spanning tree?



Strategy

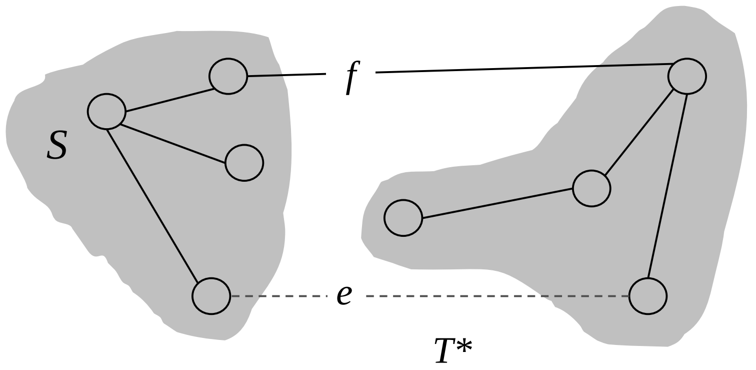
Add the edge e_k with least weight that connects this minimum spanning tree to a new vertex v_{k+1}

- This does create a minimum spanning tree on the $k + 1$ nodes—there is no other edge that extends the tree with less weight
- Does the new edge belong to the minimum spanning tree on all n vertices?
  - Yes! The cut property.



Cut property

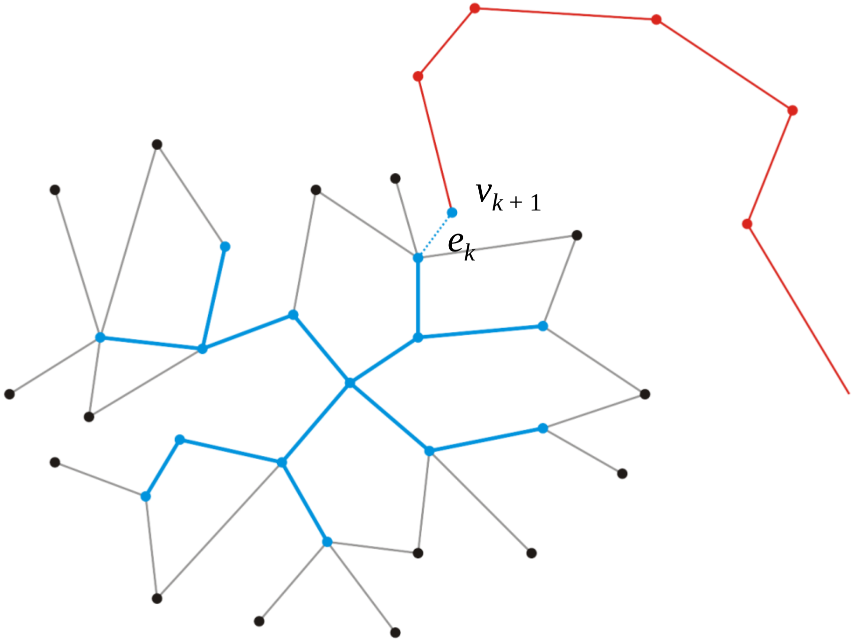
- Let S be any subset of nodes, and let e be the least weight edge with exactly one endpoint in S. Then the MST T^* contains e.
- Proof
  - Suppose e does not belong to T^*.
  - Adding e to T^* creates a cycle C in T^*.
  - e is in a cycle C with exactly one endpoint in S $\Rightarrow$ there exists another edge f in C with exactly one endpoint in S.
  - $T' = T^* \cup \{e\} - \{f\}$ is also a spanning tree.
  - Since $w_e < w_f$, the weight of T' is smaller than that of T^*.
  - This is a contradiction



Proof

Suppose it does not

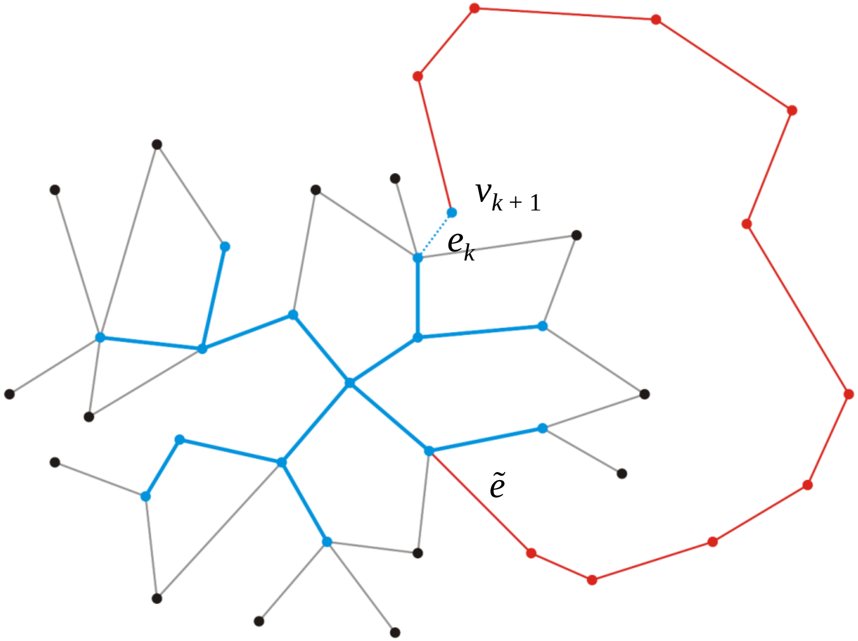
Thus, vertex v_{k+1} is connected to the minimum spanning tree via another sequence of edges



Proof

Because a minimum spanning tree is connected, there must be a path from vertex v_{k+1} back to our existing minimum spanning tree

- Let the last edge in this path be $\tilde{e}$

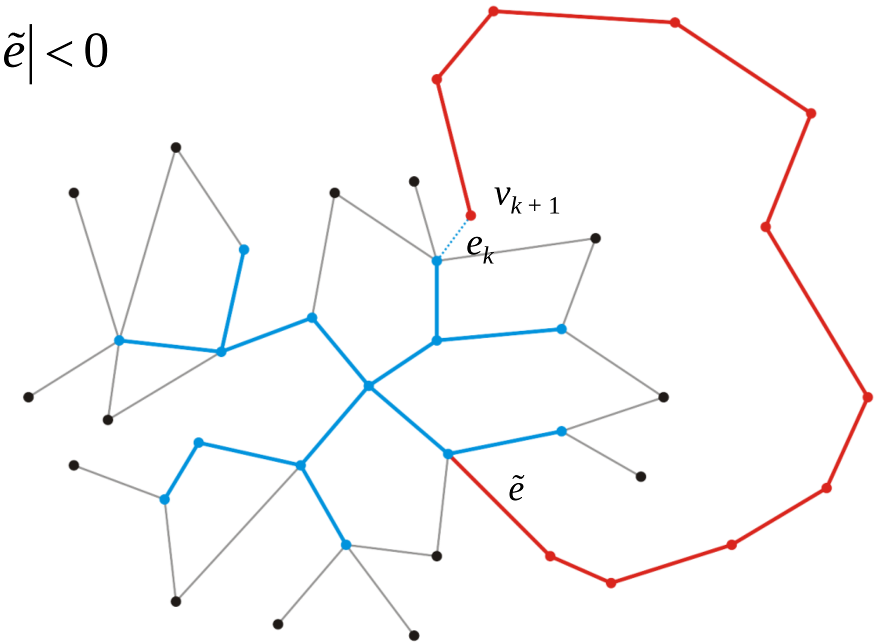


Proof

Let w be the weight of this minimum spanning tree

- Recall, however, that when we chose to add v_{k+1}, it was because e_k was the edge connecting an adjacent vertex with **least** weight
- Therefore $|\tilde{e}| > |e_k|$ where $|e|$ represents the weight of the edge e

$$|e_k| - |\tilde{e}| < 0$$

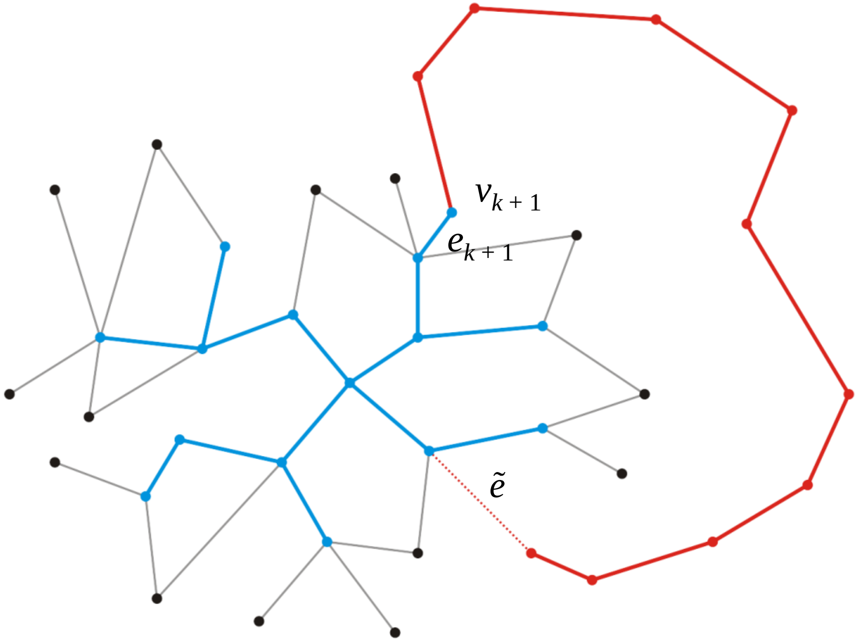


Proof

Suppose we swap edges and choose to include e_k and exclude $\tilde{e}$

– The result is still a spanning tree, but the weight is now

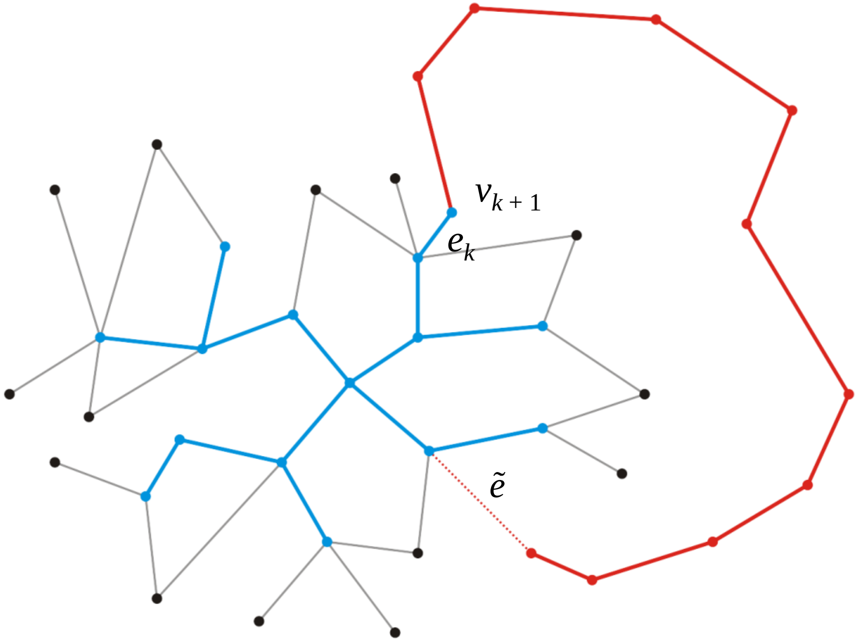
$$w + |e_{k+1}| - |\tilde{e}| \leq w$$



Proof

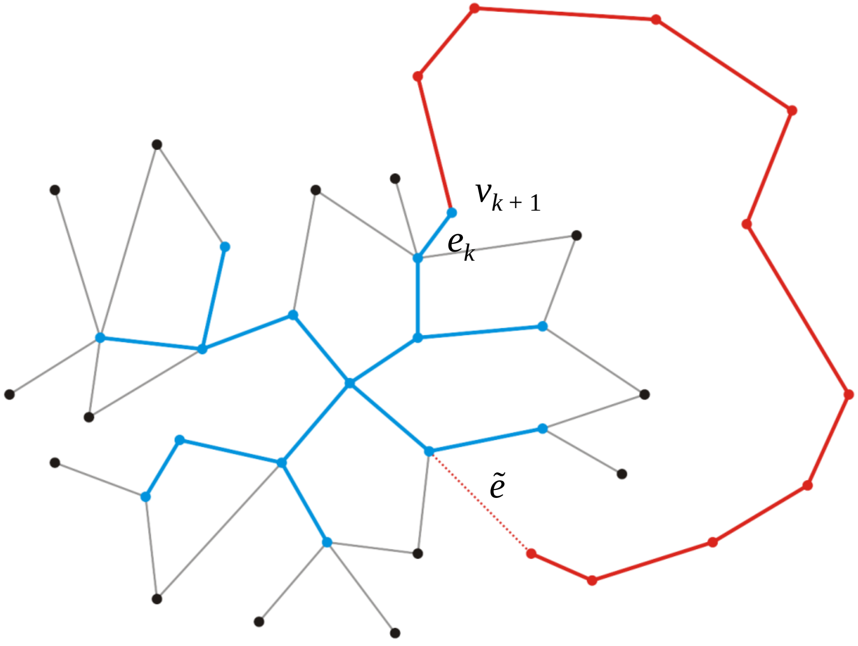
Thus, by swapping e_k for $\tilde{e}$, we have a spanning tree that has less weight than the so-called minimum spanning tree containing $\tilde{e}$

- This contradicts our assumption that the spanning tree containing $\tilde{e}$ was minimal
- Therefore, we have proved that our minimum spanning tree must contain e_k



Strategy

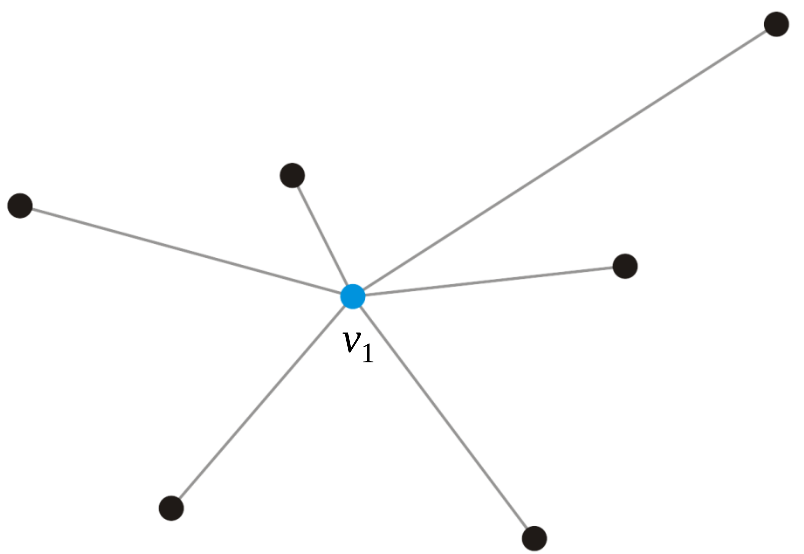
Recall that we did not prescribe the value of k, and thus, k could be any value, including $k = 1$



Strategy

Recall that we did not prescribe the value of k, and thus, k could be any value, including $k = 1$

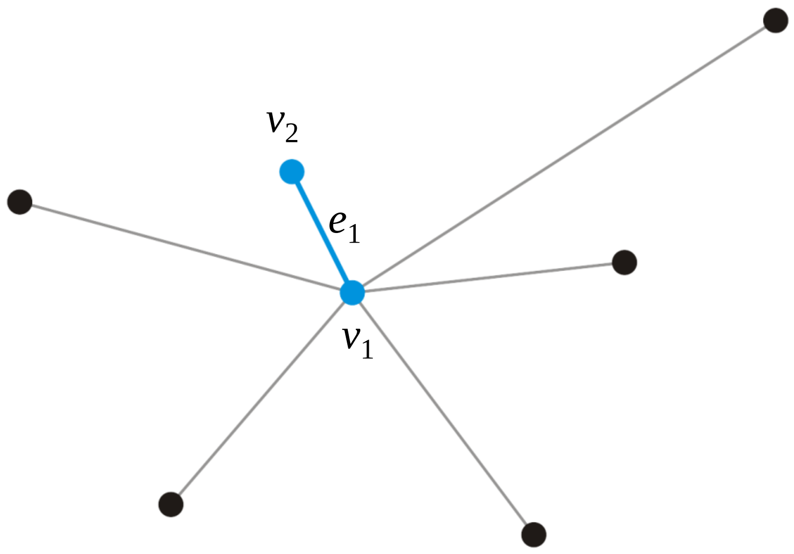
- Given a single vertex e_1, it forms a minimum spanning tree on one vertex



Strategy

Add that adjacent vertex v_2 that has a connecting edge e_1 of minimum weight

- This forms a minimum spanning tree on our two vertices and e_1 must be in any minimum spanning tree containing the vertices v_1 and v_2

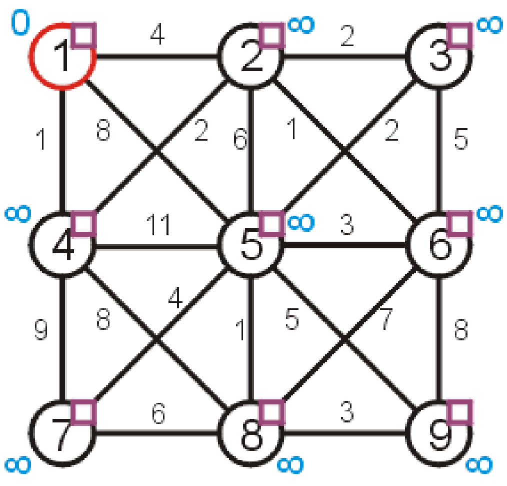


Prim’s Algorithm

Prim’s algorithm for finding the minimum spanning tree states:

- Start with an arbitrary vertex to form a minimum spanning tree on one vertex
- At each step, add the edge with least weight that connects the current minimum spanning tree to a new vertex
- Continue until we have $n - 1$ edges and n vertices

Prim's Algorithm



Visited or not

		Distance	Parent
1			
2			
3			
4			
5			
6			
7			
8			
9			

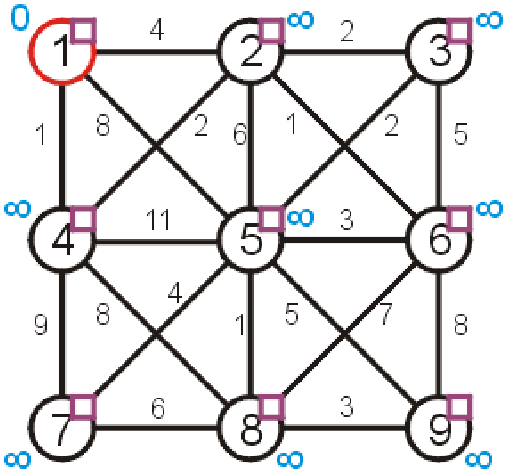
Prim’s Algorithm

Initialization:

- Select a root node and set its distance as 0
- Set the distance to all other vertices as ∞
- Set all vertices to being unvisited
- Set the parent pointer of all vertices to 0

Prim’s Algorithm

First we initialize the table



		Distance	Parent
1	F	0	0
2	F	∞	0
3	F	∞	0
4	F	∞	0
5	F	∞	0
6	F	∞	0
7	F	∞	0
8	F	∞	0
9	F	∞	0

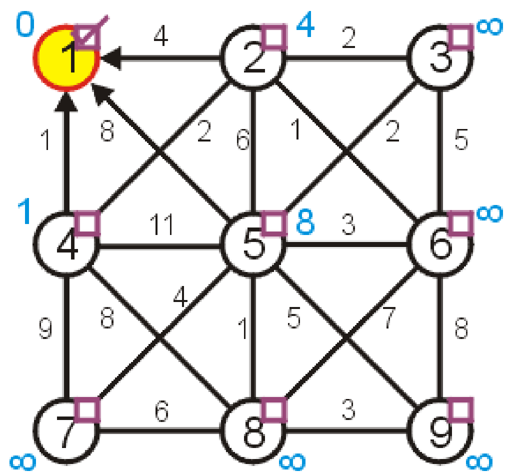
Prim's Algorithm

Iterate while there exists an unvisited vertex with distance < ∞

- Select the unvisited vertex v with minimum distance
- Mark v as having been visited
- For each unvisited adjacent vertex of v, if the weight of the connecting edge is less than the current distance to that vertex:
  - Update the distance to the weight of the edge
  - Set v as the parent of the vertex

Prim's Algorithm

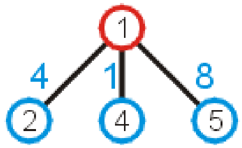
Visiting vertex 1, we update vertices 2, 4, and 5



		Distance	Parent
1	T	0	0
2	F	4	1
3	F	∞	0
4	F	1	1
5	F	8	1
6	F	∞	0
7	F	∞	0
8	F	∞	0
9	F	∞	0

Prim’s Algorithm

What these numbers really mean is that at this point, we could extend the trivial tree containing just the root node by one of the three possible children:

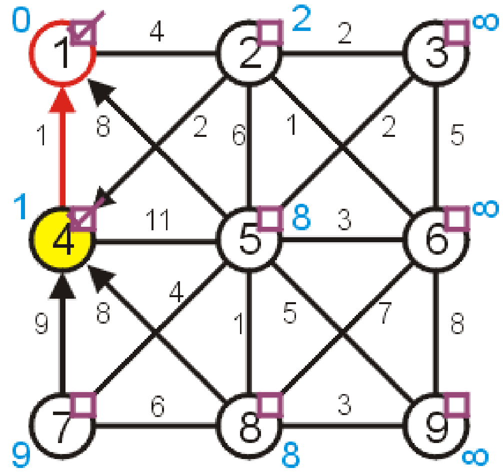


As we wish to find a *minimum* spanning tree, it makes sense we add that vertex with a connecting edge with least weight

Prim's Algorithm

The next unvisited vertex with minimum distance is vertex 4

– Update vertices 2, 7, 8
– Don't update vertex 5



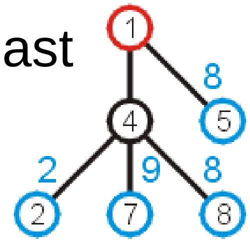
		Distance	Parent
1	T	0	0
2	F	2	4
3	F	∞	0
4	T	1	1
5	F	8	1
6	F	∞	0
7	F	9	4
8	F	8	4
9	F	∞	0

Prim's Algorithm

Now that we have updated all vertices adjacent to vertex 4, we can extend the tree by adding one of the edges

(1, 5), (4, 2), (4, 7), or (4, 8)

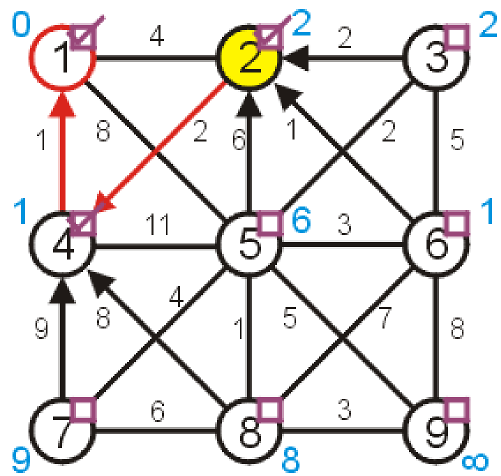
We add that edge with the least weight: (4, 2)



Prim’s Algorithm

Next visit vertex 2

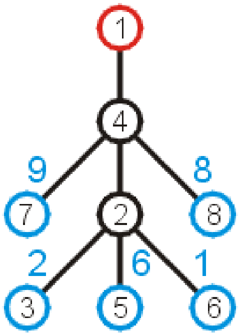
– Update 3, 5, and 6



		Distance	Parent
1	T	0	0
2	T	2	4
3	F	2	2
4	T	1	1
5	F	6	2
6	F	1	2
7	F	9	4
8	F	8	4
9	F	∞	0

Prim's Algorithm

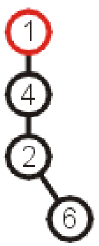
Again looking at the shortest edges to each of the vertices adjacent to the current tree, we note that we can add (2, 6) with the least increase in weight

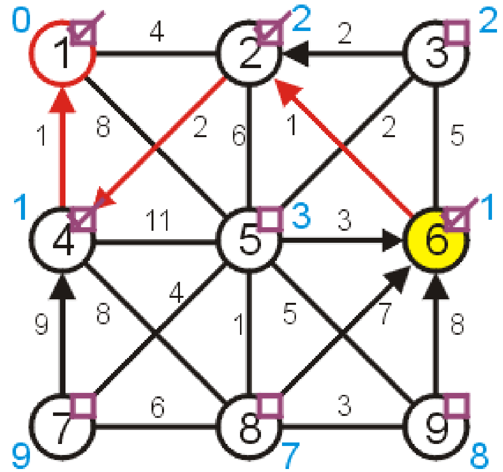


Prim's Algorithm

Next, we visit vertex 6:

– update vertices 5, 8, and 9



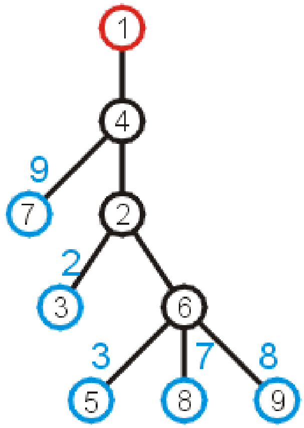


		Distance	Parent
1	T	0	0
2	T	2	4
3	F	2	2
4	T	1	1
5	F	3	6
6	T	1	2
7	F	9	4
8	F	7	6
9	F	8	6

Prim's Algorithm

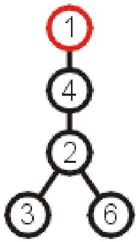
The edge with least weight is (2, 3)

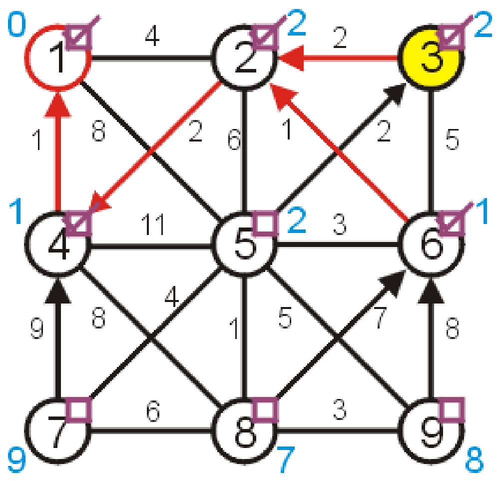
– This adds the weight of 2 to the weight minimum spanning tree



Prim's Algorithm

Next, we visit vertex 3 and update 5

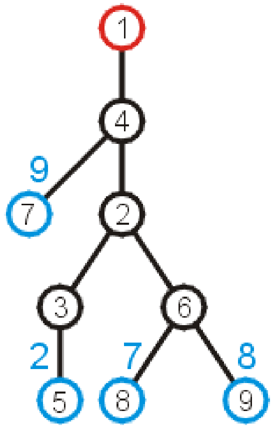




		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	F	2	3
6	T	1	2
7	F	9	4
8	F	7	6
9	F	8	6

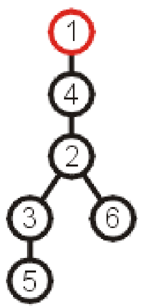
Prim's Algorithm

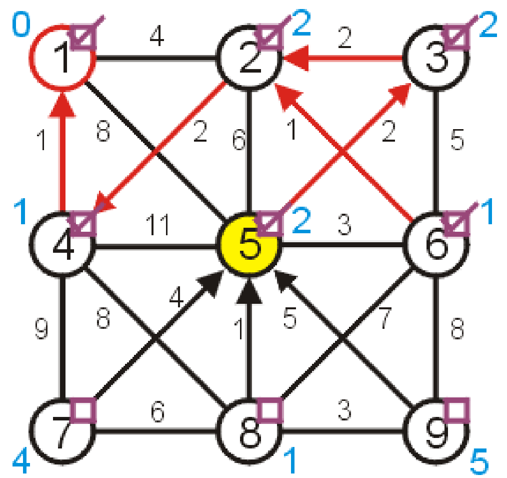
At this point, we can extend the tree by adding the edge (3, 5)



Prim’s Algorithm

Visiting vertex 5, we update 7, 8, 9



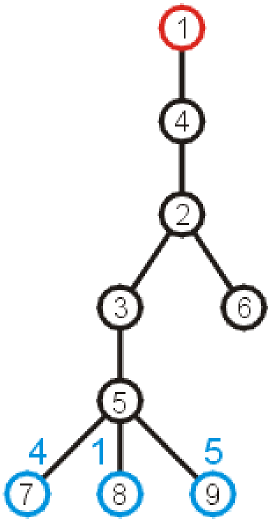


		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	T	2	3
6	T	1	2
7	F	4	5
8	F	1	5
9	F	5	5

Prim’s Algorithm

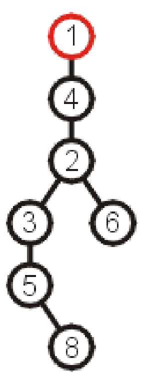
At this point, there are three possible edges which we could include which will extend the tree

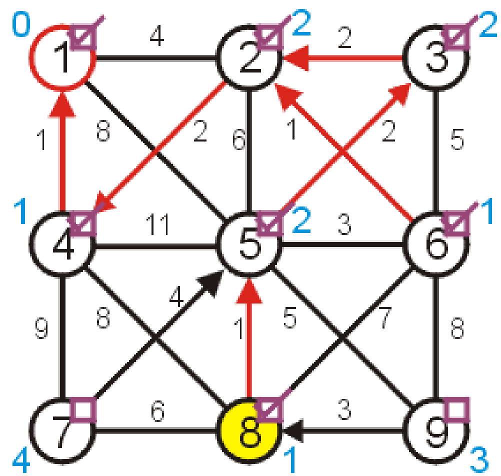
The edge to 8 has the least weight



Prim's Algorithm

Visiting vertex 8, we only update vertex 9

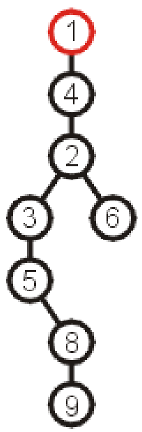


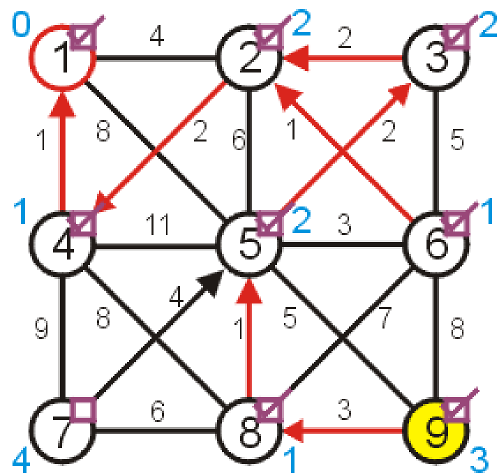


		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	T	2	3
6	T	1	2
7	F	4	5
8	T	1	5
9	F	3	8

Prim's Algorithm

There are no other vertices to update while visiting vertex 9

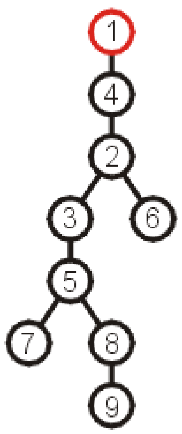


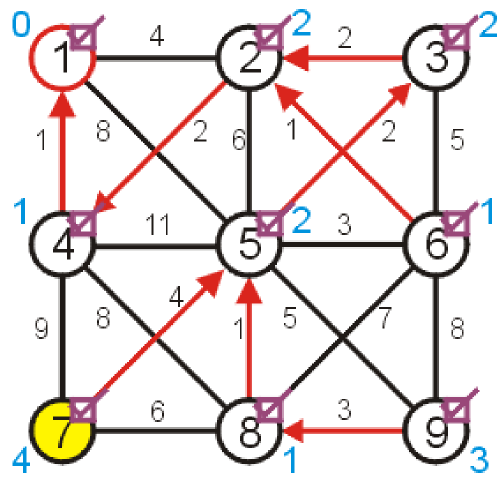


		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	T	2	3
6	T	1	2
7	F	4	5
8	T	1	5
9	T	3	8

Prim’s Algorithm

And neither are there any vertices to update when visiting vertex 7





		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	T	2	3
6	T	1	2
7	T	4	5
8	T	1	5
9	T	3	8

Prim’s Algorithm

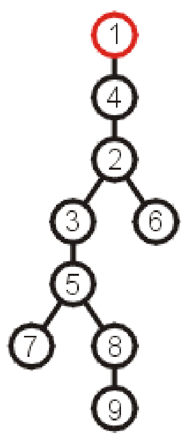
At this point, there are no more unvisited vertices, and therefore we are done

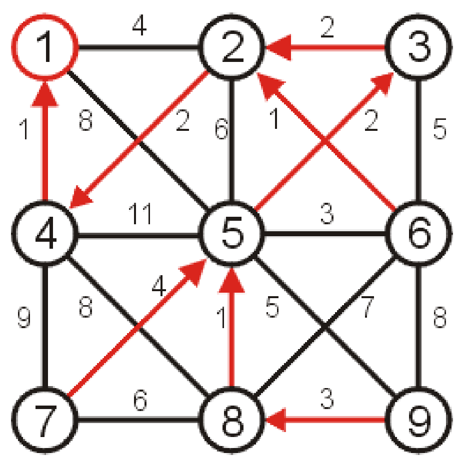
If at any point, all remaining vertices had a distance of ∞, this would indicate that the graph is not connected

- in this case, the minimum spanning tree would only span one connected sub-graph

Prim’s Algorithm

Using the parent pointers, we can now construct the minimum spanning tree





		Distance	Parent
1	T	0	0
2	T	2	4
3	T	2	2
4	T	1	1
5	T	2	3
6	T	1	2
7	T	4	5
8	T	1	5
9	T	3	8

Prim’s Algorithm

To summarize:

– we begin with a vertex which represents the root
– starting with this trivial tree and iteration, we find the shortest edge which we can add to this already existing tree to expand it

Implementation and analysis

The initialization requires $\Theta(|V|)$ memory and run time

We iterate $|V| - 1$ times, each time finding the *closest* vertex

- Iterating through the table requires is $\Theta(|V|)$ time
- Each time we find a vertex, we must check all of its neighbors

With an adjacency list, the run time is $\Theta(|V|^2 + |E|) = \Theta(|V|^2)$ as $|E| = O(|V|^2)$

Can we do better?

- At each iteration, we need to find the shortest edge
- How about a priority queue?
  - Assume we are using a binary heap

Implementation and analysis

The initialization still requires $\Theta(|V|)$ memory and run time

– The priority queue will also requires $O(|V|)$ memory

We iterate $|V| - 1$ times, each time finding the *closest* vertex

– The size of the priority queue is $O(|V|)$

– Pop the closest vertex from the priority queue is $O(\ln(|V|))$

– For each of its neighbors, we may update the distance, which is $O(\ln(|V|))$

With an adjacency list, the total run time is $O(|V| \ln(|V|) + |E| \ln(|V|)) = O(|E| \ln(|V|))$

Implementation and analysis

We could use a different heap structure:

- A Fibonacci heap is a node-based heap
- Pop is still $O(\ln(|V|))$, but inserting and moving a key is $\Theta(1)$
- Thus, the overall run-time is $O(|E| + |V| \ln(|V|))$

Implementation and analysis

Thus, we have two run times when using

- A binary heap: $\mathrm{O}(|E| \ln(|V|))$
- A Fibonacci heap: $\mathrm{O}(|E| + |V| \ln(|V|))$

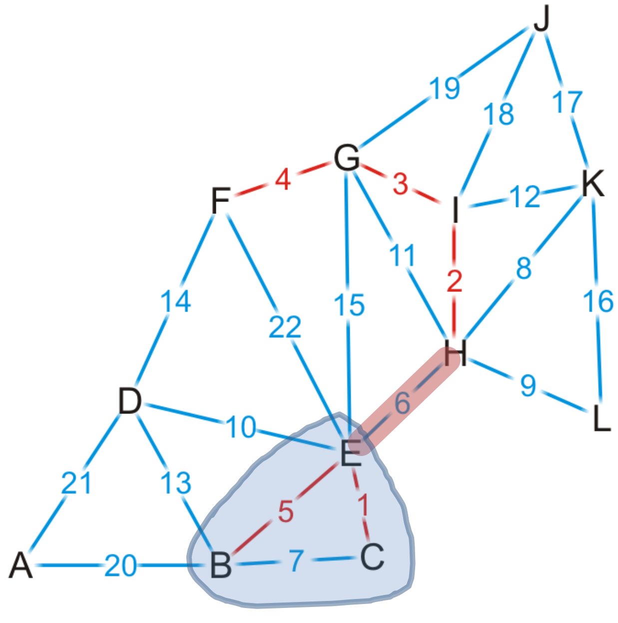
Questions: Which is faster if $|E| = \Theta(|V|)$? How about if $|E| = \Theta(|V|^2)$?

Outline

- Definition and applications
- Prim’s algorithm
- Kruskal’s algorithm

Kruskal's Algorithm

- Sort the edges by weight
- Go through the edges from least weight to greatest weight
  - add the edges to the spanning tree so long as the addition does not create a cycle
  - Does this edge belong to the minimum spanning tree?
    - Yes! The cut property (consider the subtree connected to one end of the edge as the set *S*).



Kruskal's Algorithm

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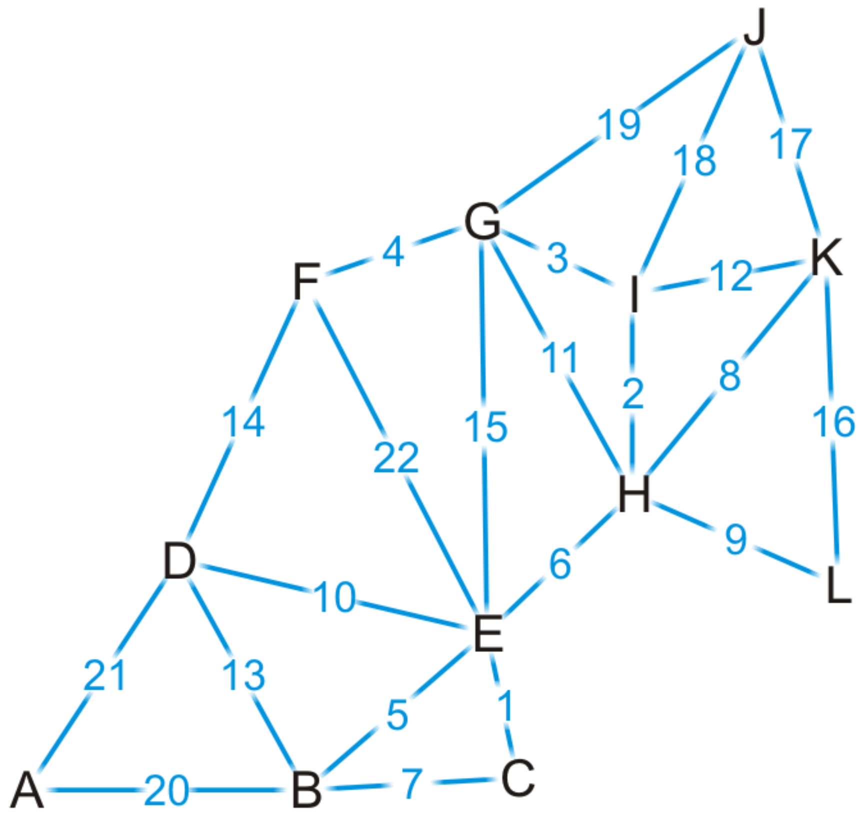
- Go through the edges from least weight to greatest weight
  - add the edges to the spanning tree so long as the addition does not create a cycle
  - Does this edge belong to the minimum spanning tree?
    - Yes! The cut property (consider the subtree connected to one end of the edge as the set *S*).

- Repeatedly add more edges until:
  - $|V| - 1$ edges have been added, then we have a minimum spanning tree
  - Otherwise, if we have gone through all the edges, then we have a forest of minimum spanning trees on all connected sub-graphs

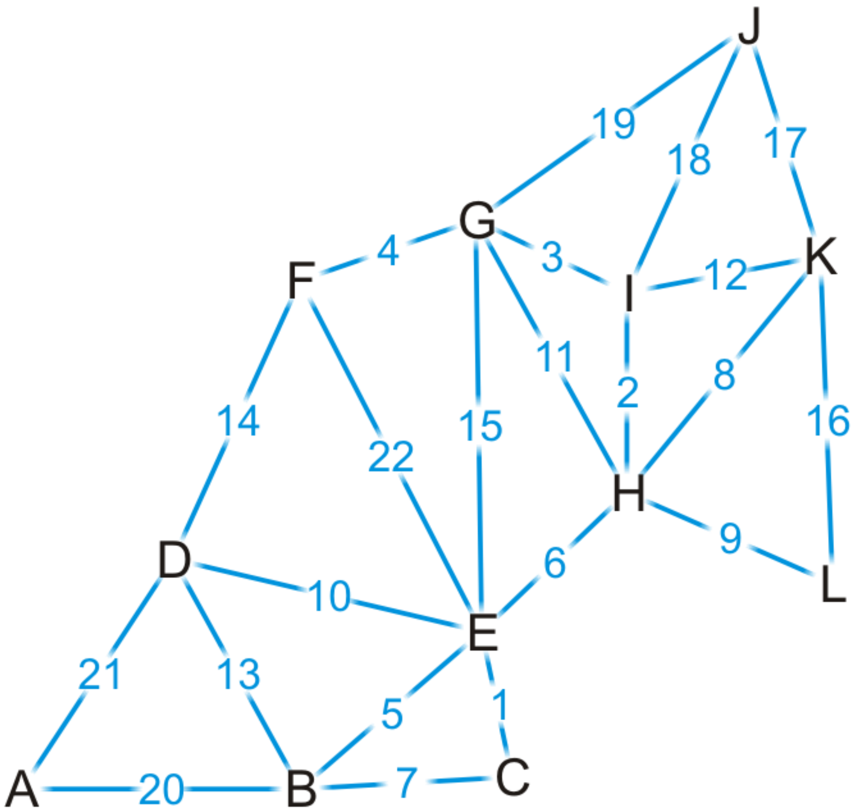
Example

Here is an example graph



Example

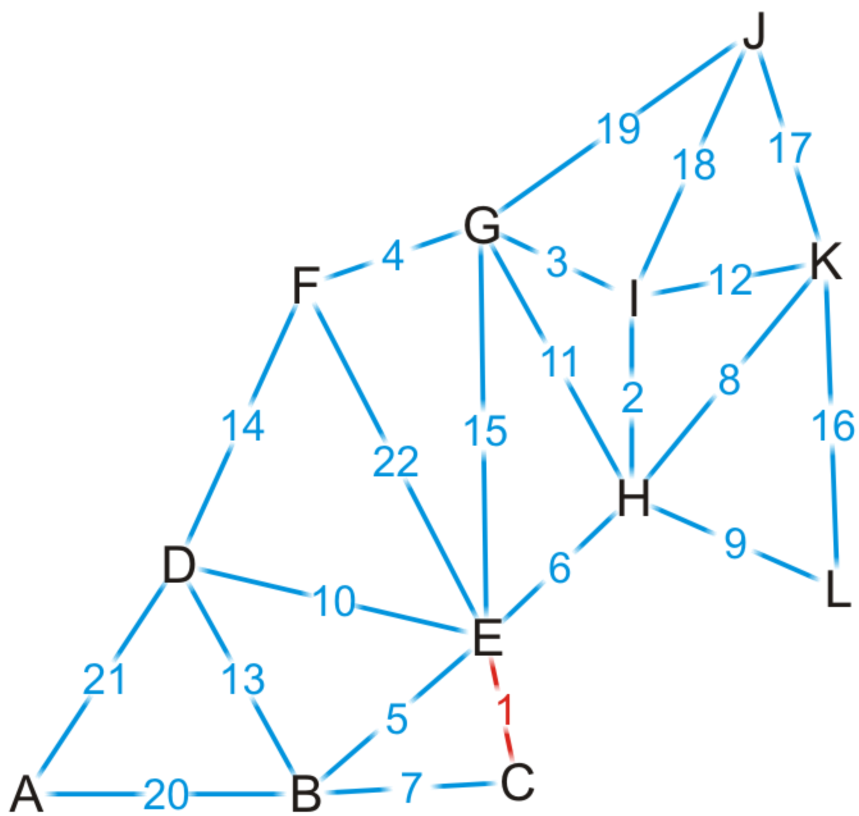
First, we sort the edges based on weight



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

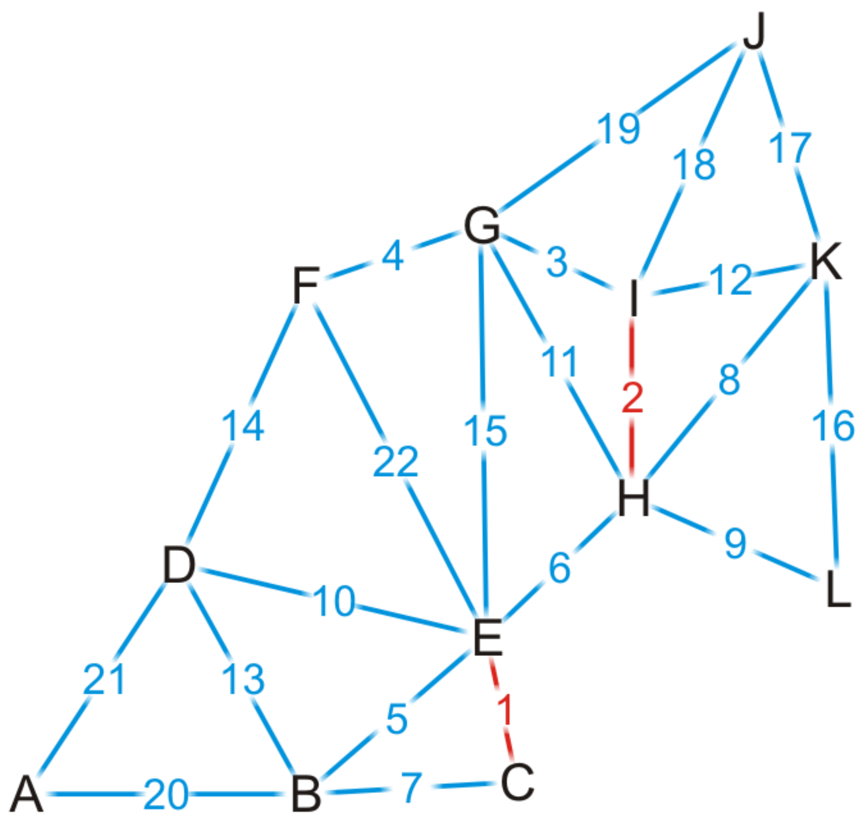
We start by adding edge {C, E}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

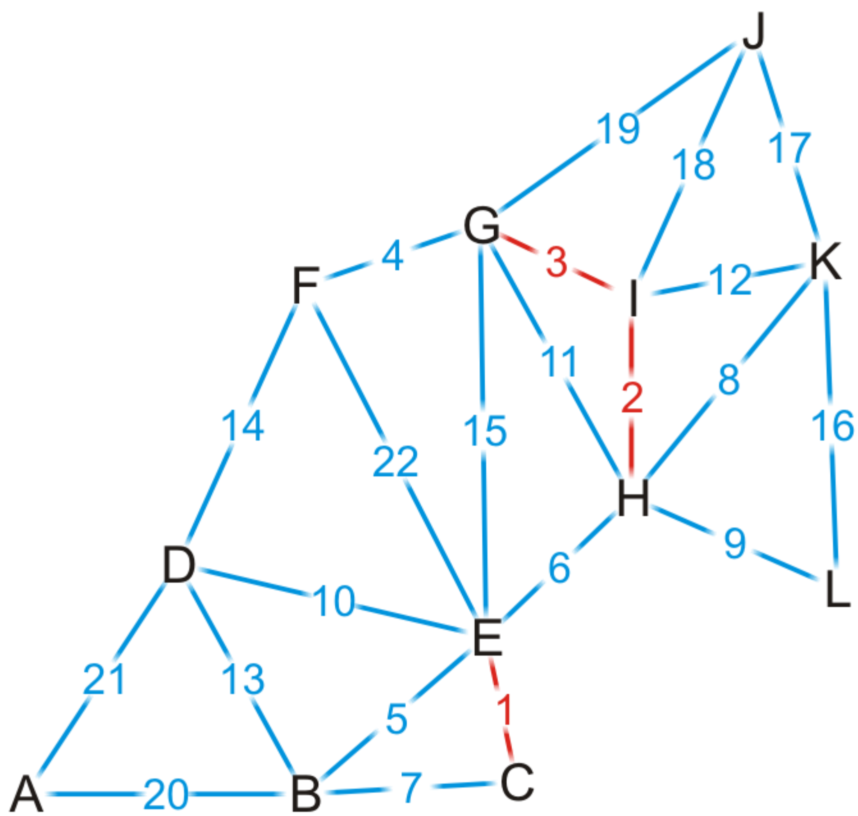
We add edge {H, I}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

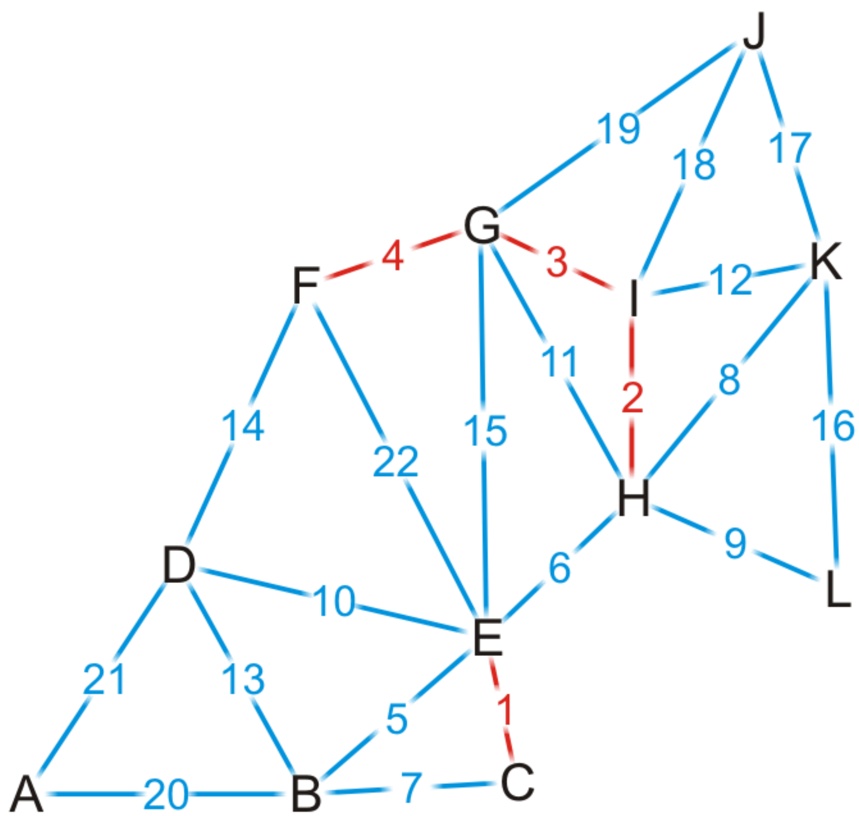
We add edge {G, I}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

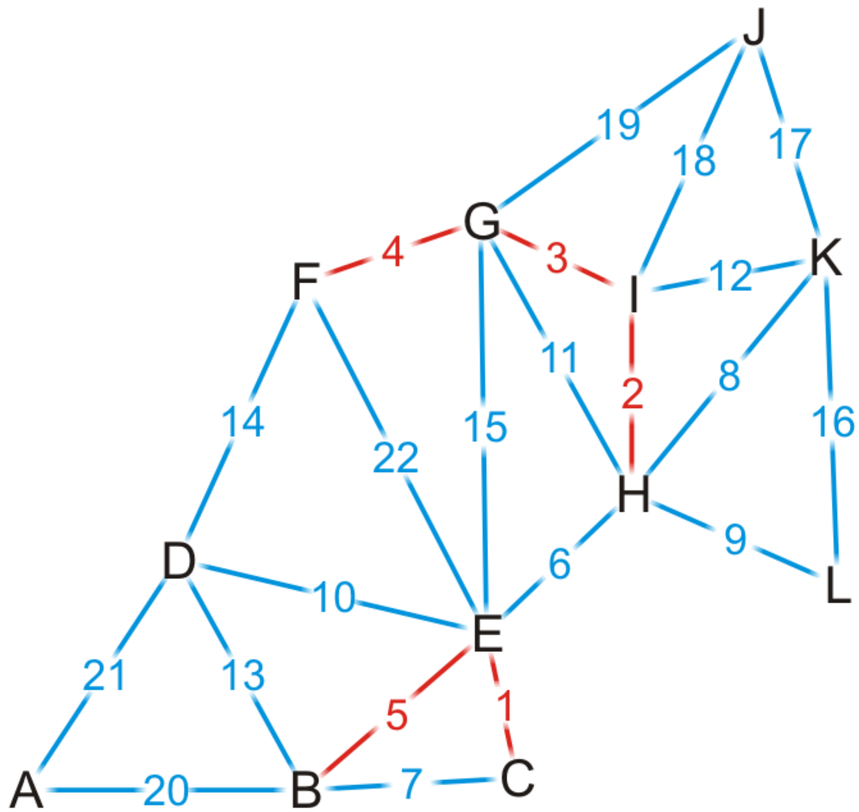
We add edge {F, G}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We add edge {B, E}

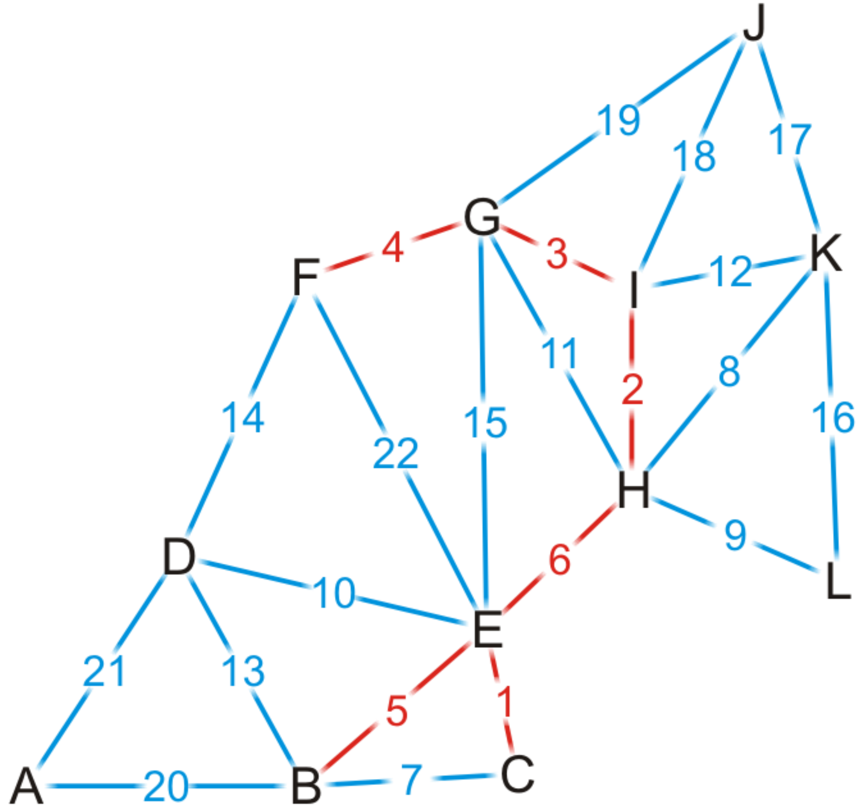


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We add edge {E, H}

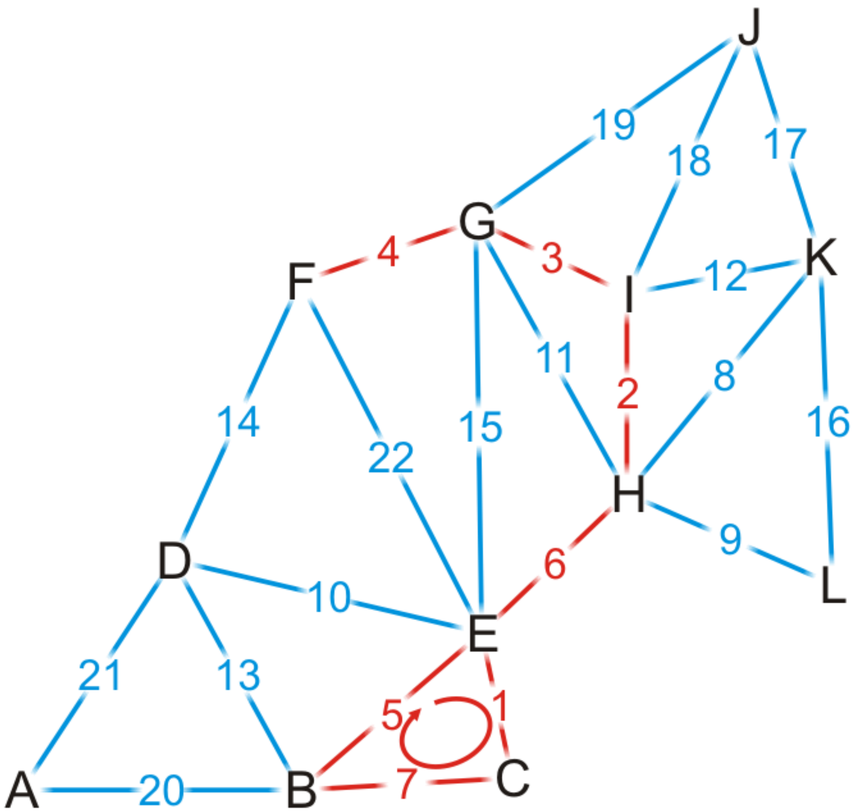
- This coalesces the two spanning sub-trees into one



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

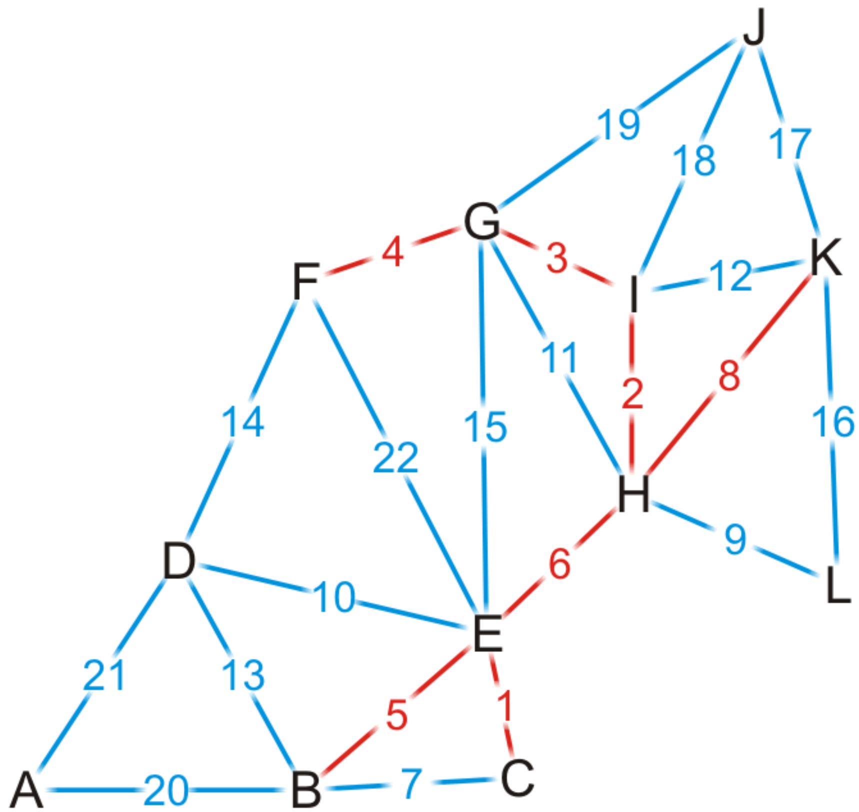
We try adding {B, C}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

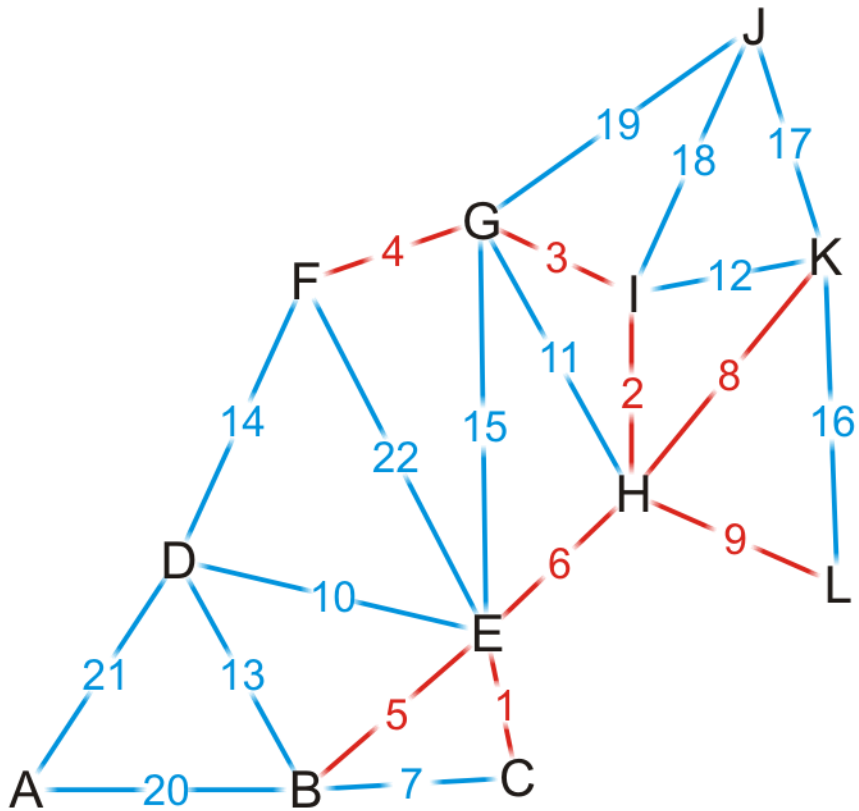
We add edge {H, K}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

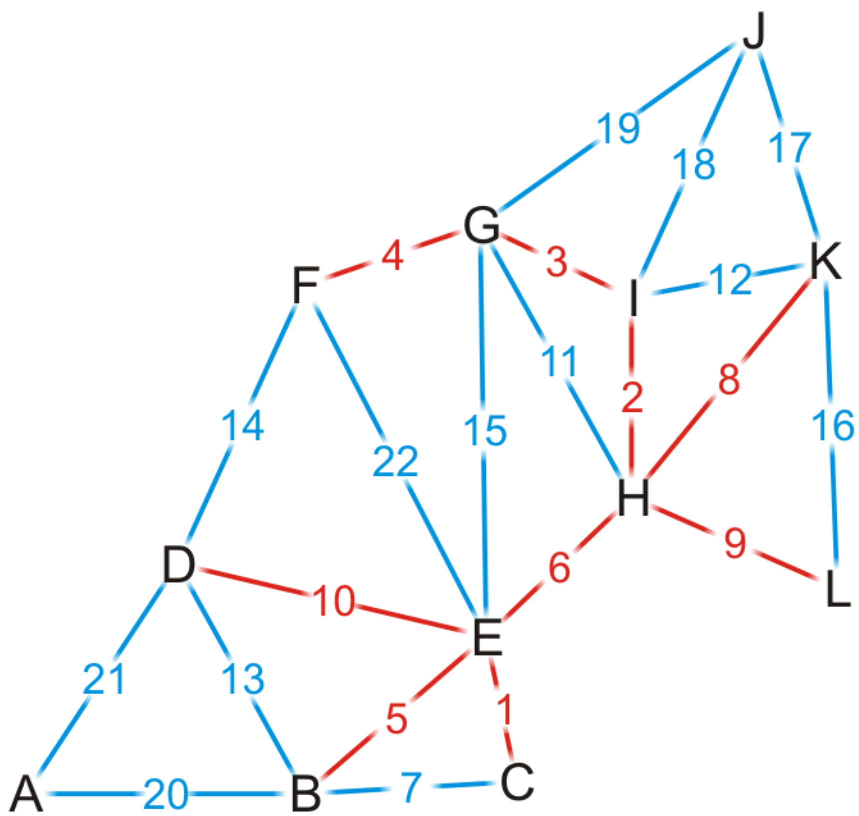
We add edge {H, L}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

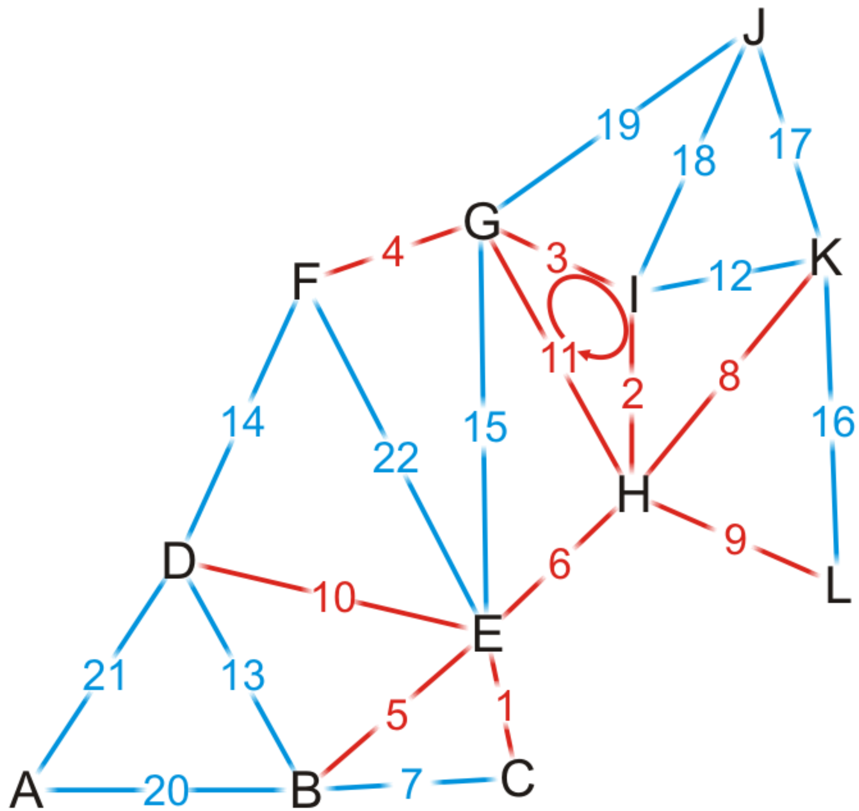
We add edge {D, E}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

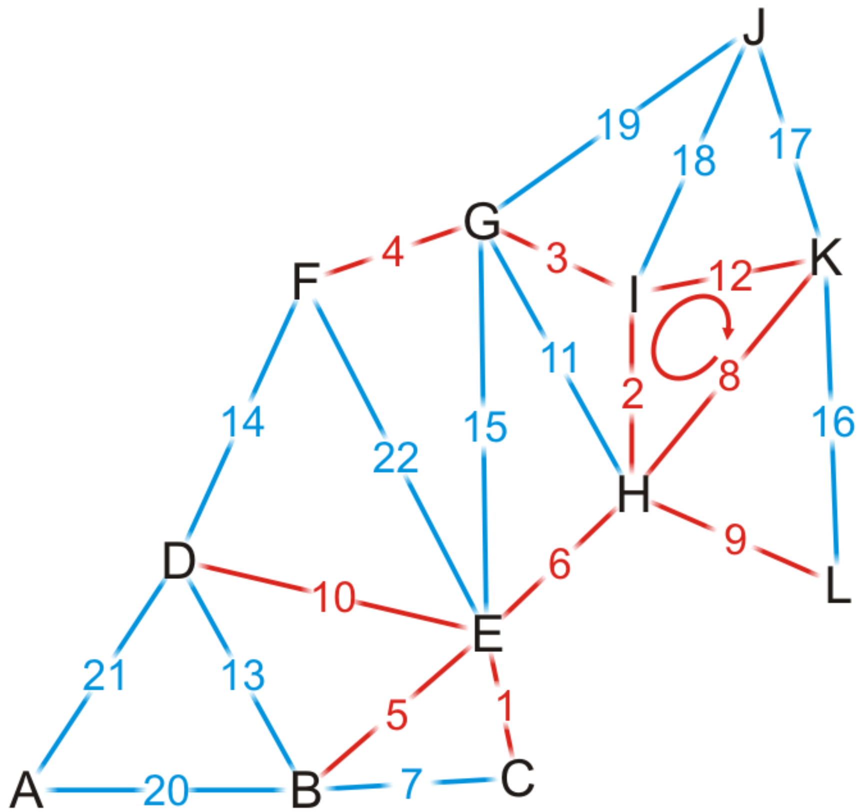
We try adding {G, H}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

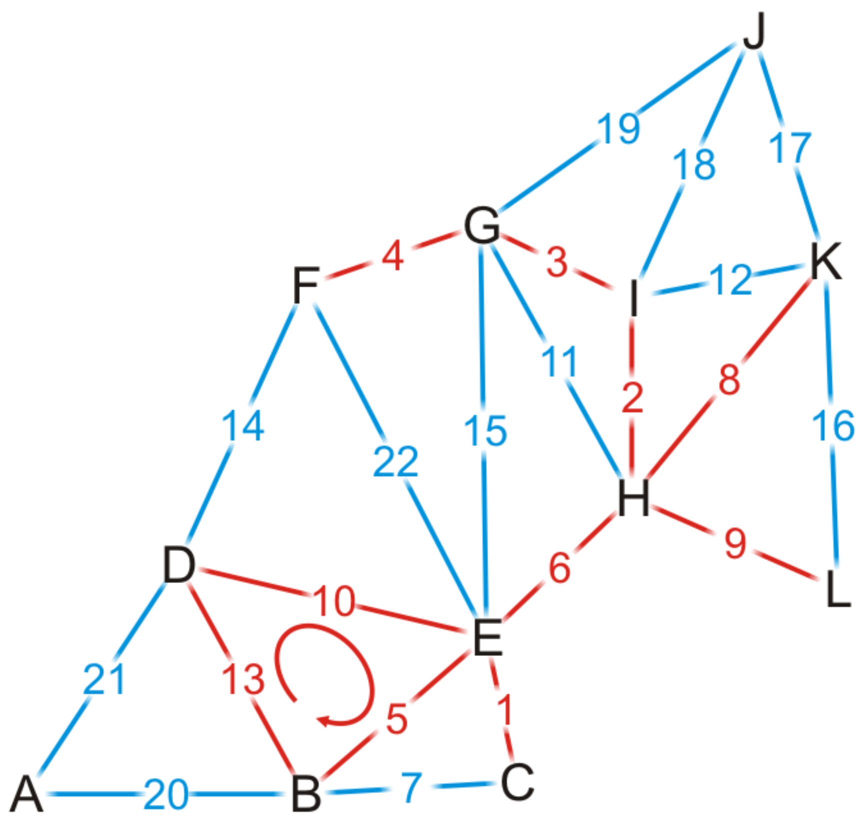
We try adding {I, K}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

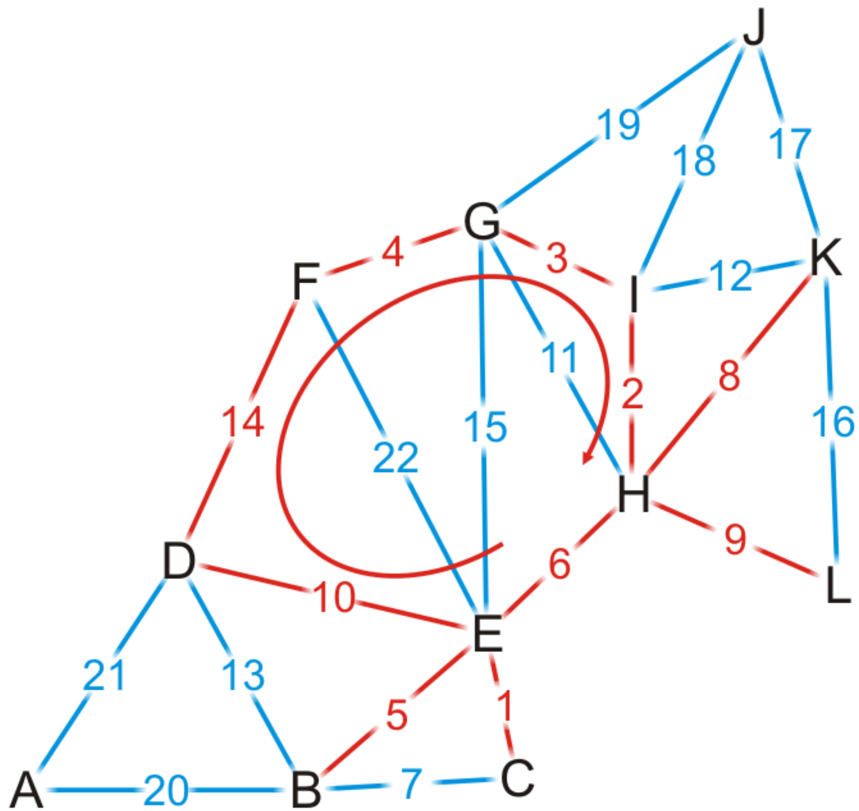
We try adding {B, D}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

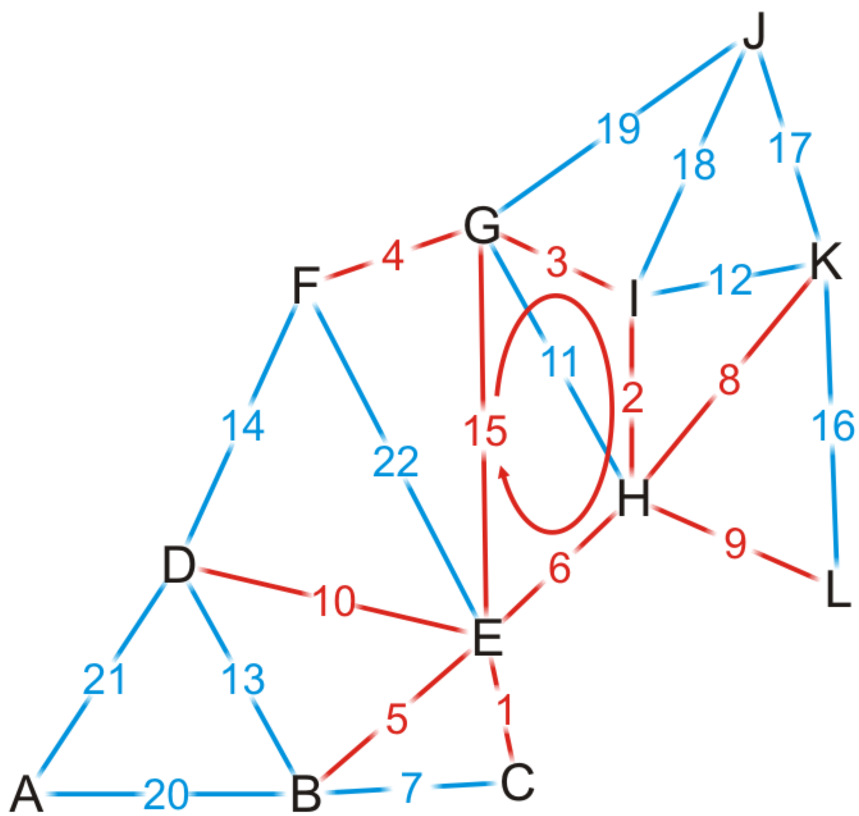
We try adding {D, F}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

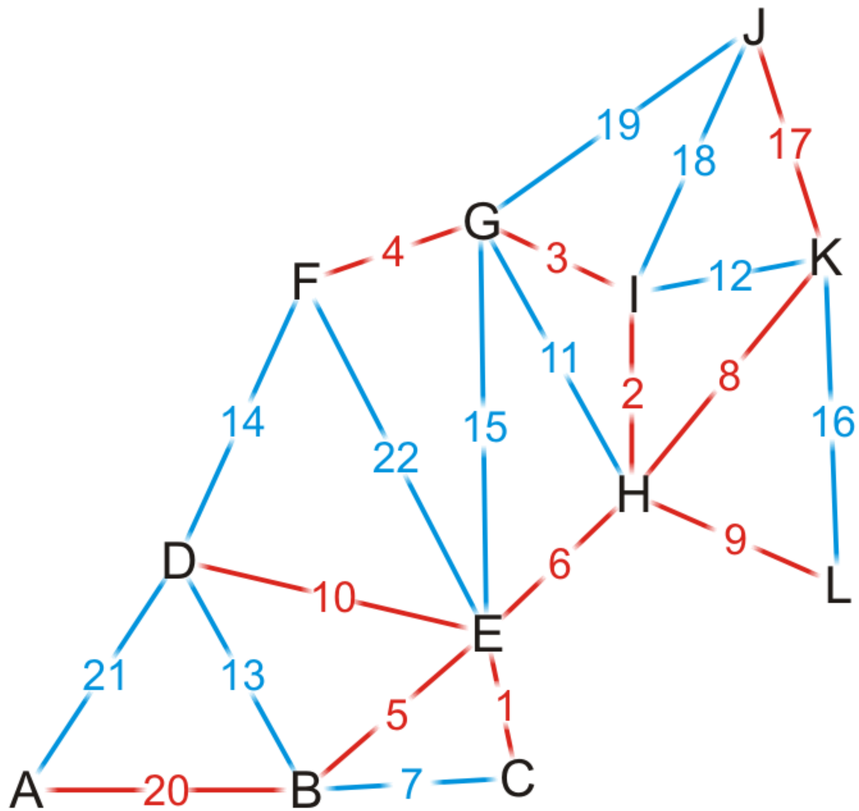
We try adding {E, G}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

By observation, we can still add edges {J, K} and {A, B}

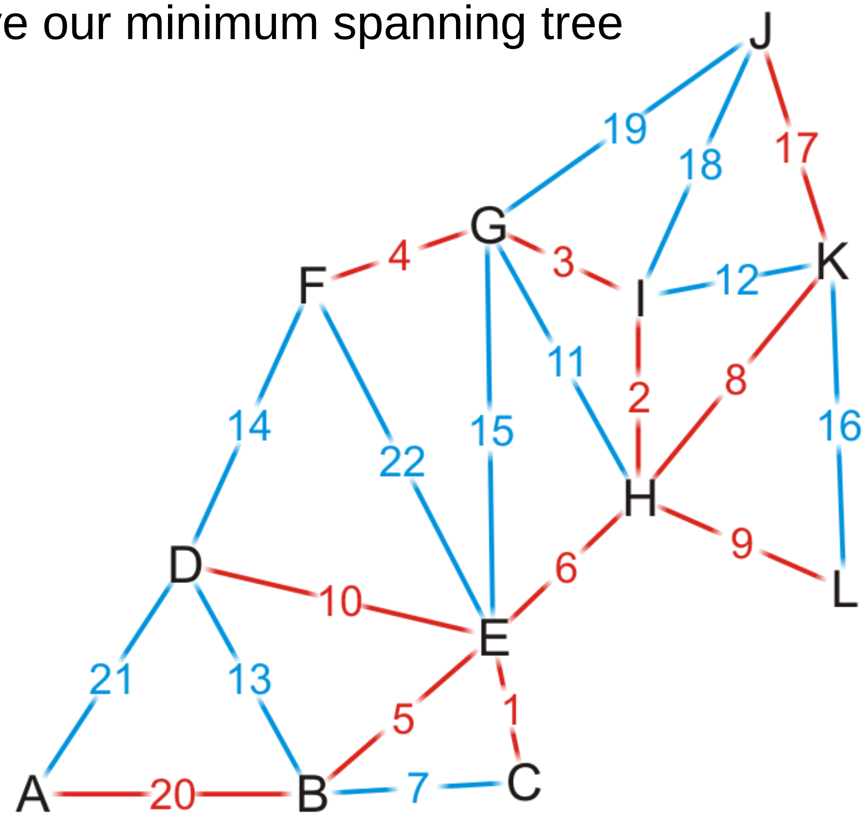


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Having added {A, B}, we now have 11 edges

- We terminate the loop
- We have our minimum spanning tree



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Analysis

Implementation

- We would store the edges and their weights in an array
- We would sort the edges using some sorting algorithm: $O(|E| \ln(|E|))$
- For each edge, add it if no cycle is created.
    - How do we determine if a cycle would be created?
    - Check if the two vertices of the edge are already connected by the added edges.

The critical operation is determining if two vertices are connected

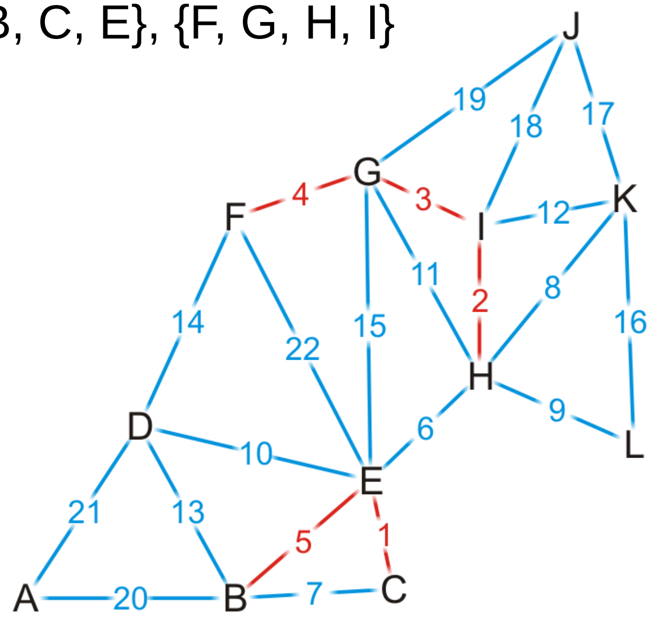
- If we perform a traversal on the added edges, it is $O(|V|)$. Consequently, the total run-time would be $O(|E| \ln(|E|) + |E|\cdot|V|) = O(|E|\cdot|V|)$
- Better solution?

Analysis

Instead, we could use disjoint sets

– Consider edges in the same connected sub-graph as forming a set

{B, C, E}, {F, G, H, I}



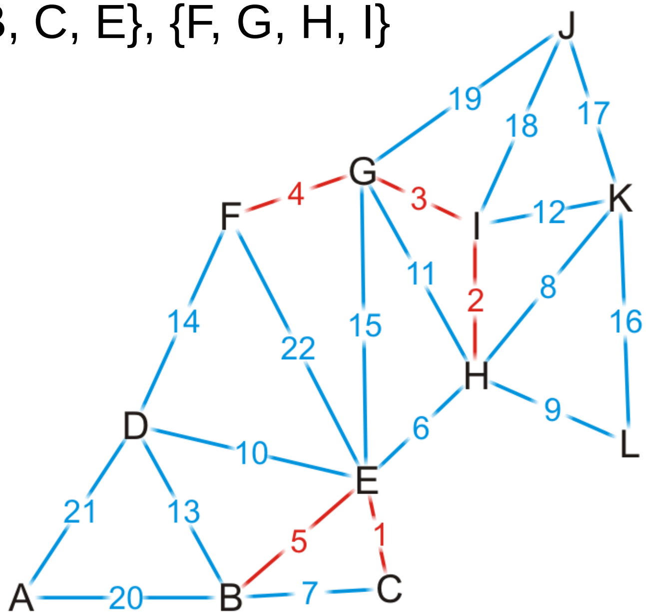
Analysis

Instead, we could use disjoint sets

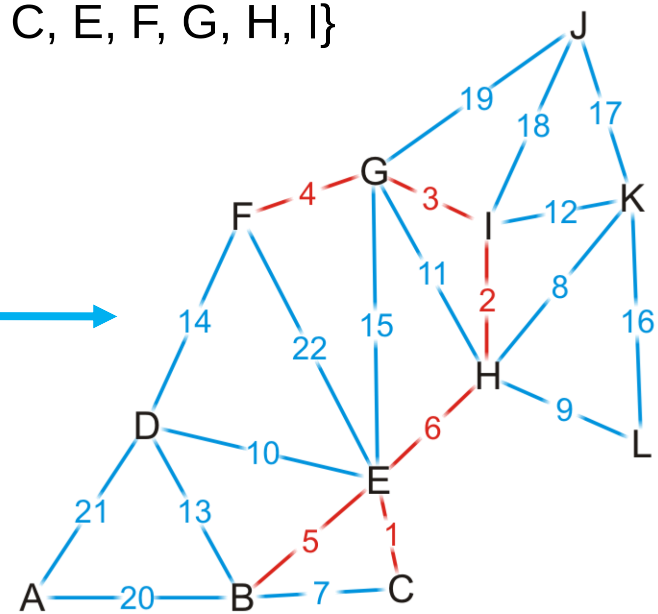
- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets, take the union of the two sets

Add edge (E, H)?

{B, C, E}, {F, G, H, I}



{B, C, E, F, G, H, I}



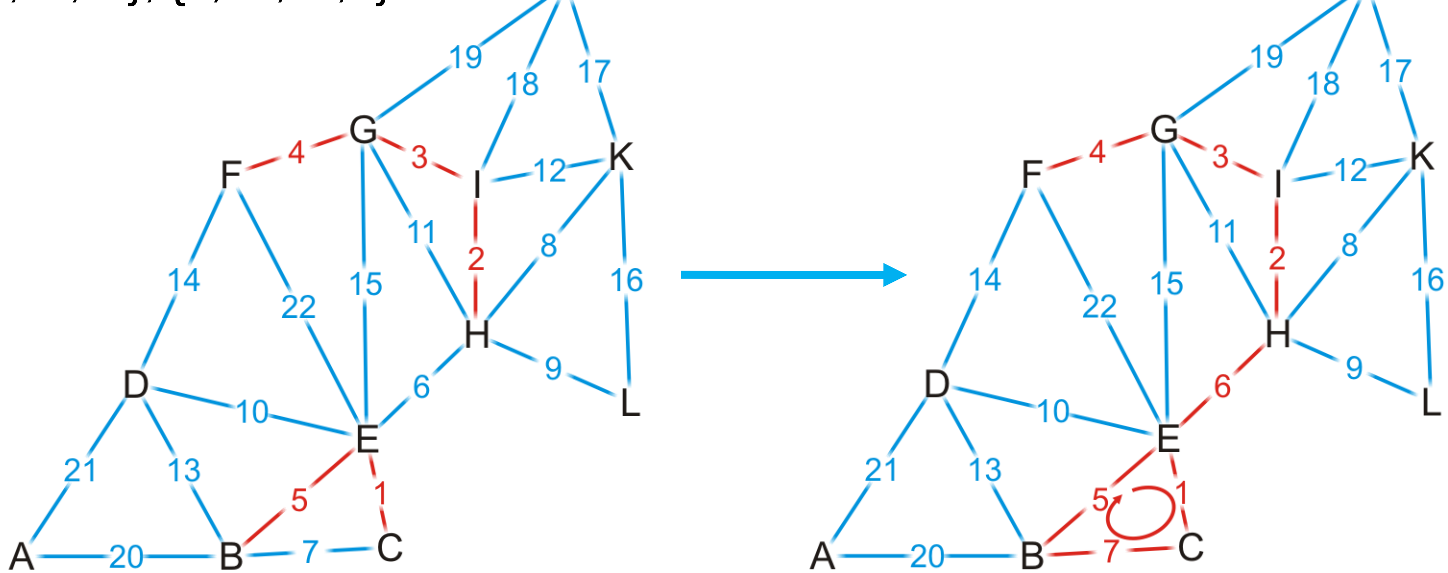
Analysis

Instead, we could use disjoint sets

- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets, take the union of the two sets
- Do not add an edge if both vertices are in the same set

Add edge (B, C)?

{B, C, E}, {F, G, H, I}



Analysis

The disjoint set data structure has run-time $\mathbf{O}(\alpha(n))$, which is effectively a constant

Thus, checking and building the minimum spanning tree is now $\mathrm{O}(|E|)$

The dominant time is now the time required to sort the edges, which is $\mathrm{O}(|E| \ln(|E|)) = \mathrm{O}(|E| \ln(|V|))$

– If there is an efficient $\Theta(|E|)$ sorting algorithm, the run-time is then $\Theta(|E|)$

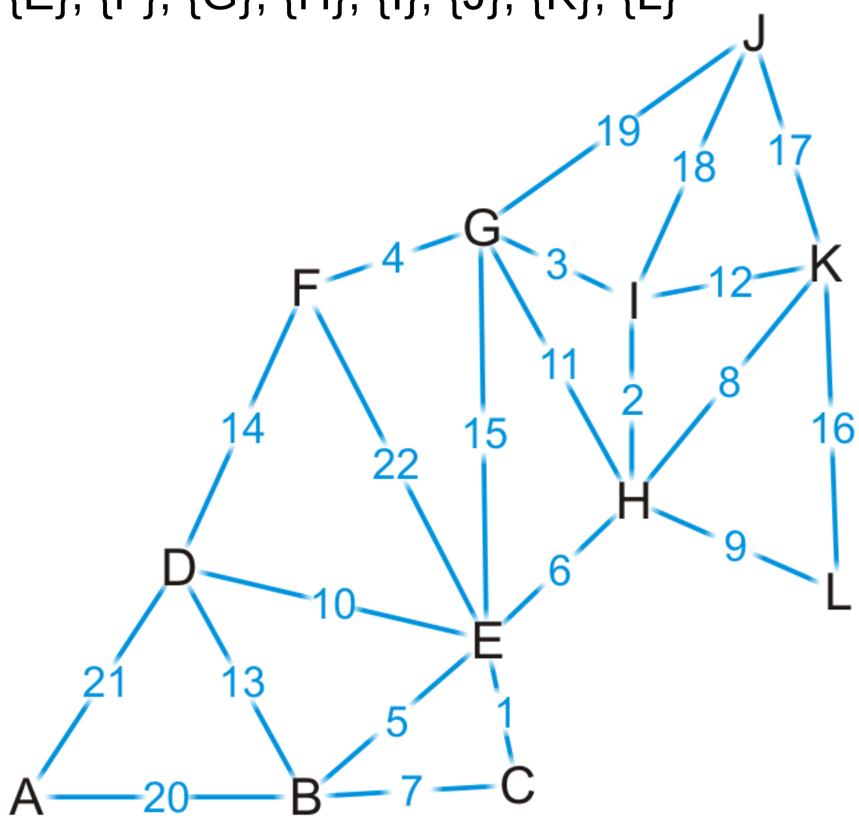
Example

Going through the example again with disjoint sets

Example

We start with twelve singletons

{A}, {B}, {C}, {D}, {E}, {F}, {G}, {H}, {I}, {J}, {K}, {L}

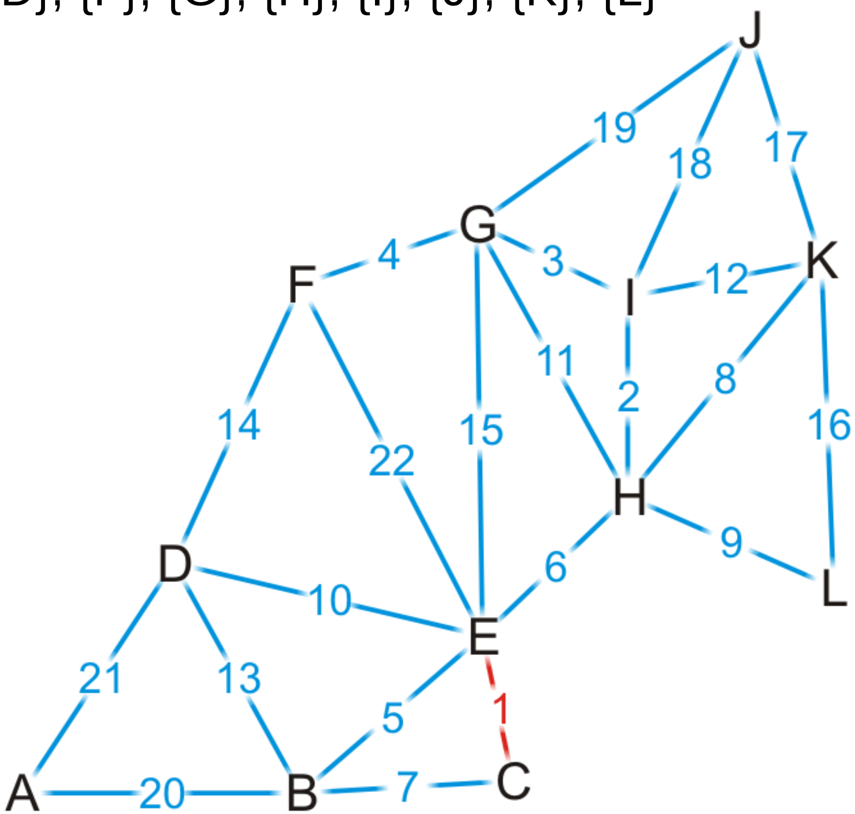


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We start by adding edge {C, E}

{A}, {B}, {C, E}, {D}, {F}, {G}, {H}, {I}, {J}, {K}, {L}

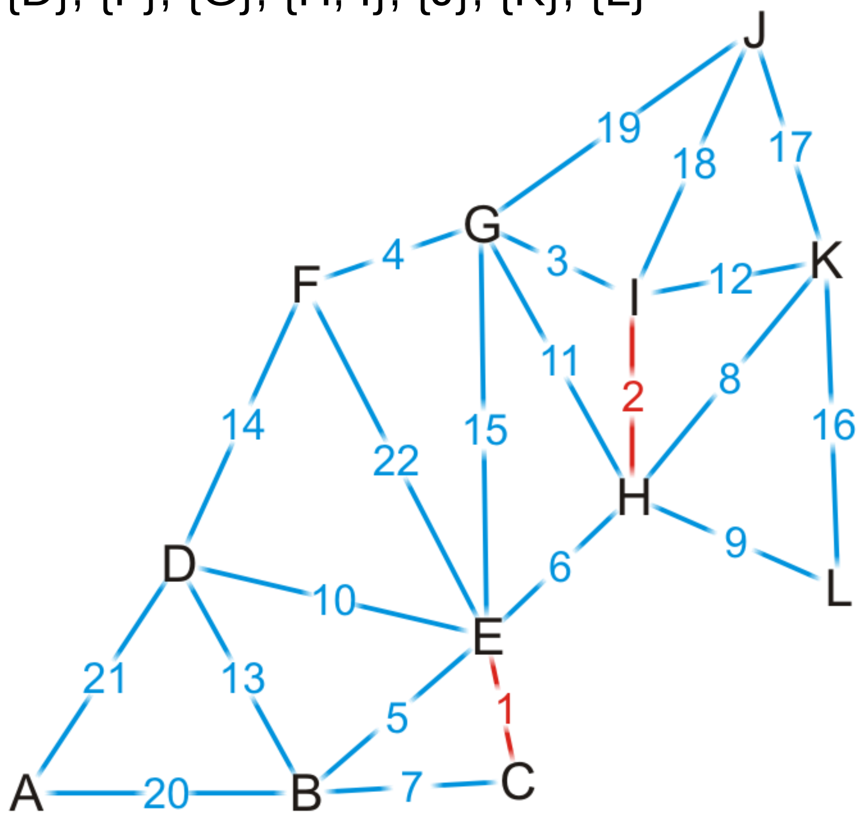


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We add edge {H, I}

{A}, {B}, {C, E}, {D}, {F}, {G}, {H, I}, {J}, {K}, {L}

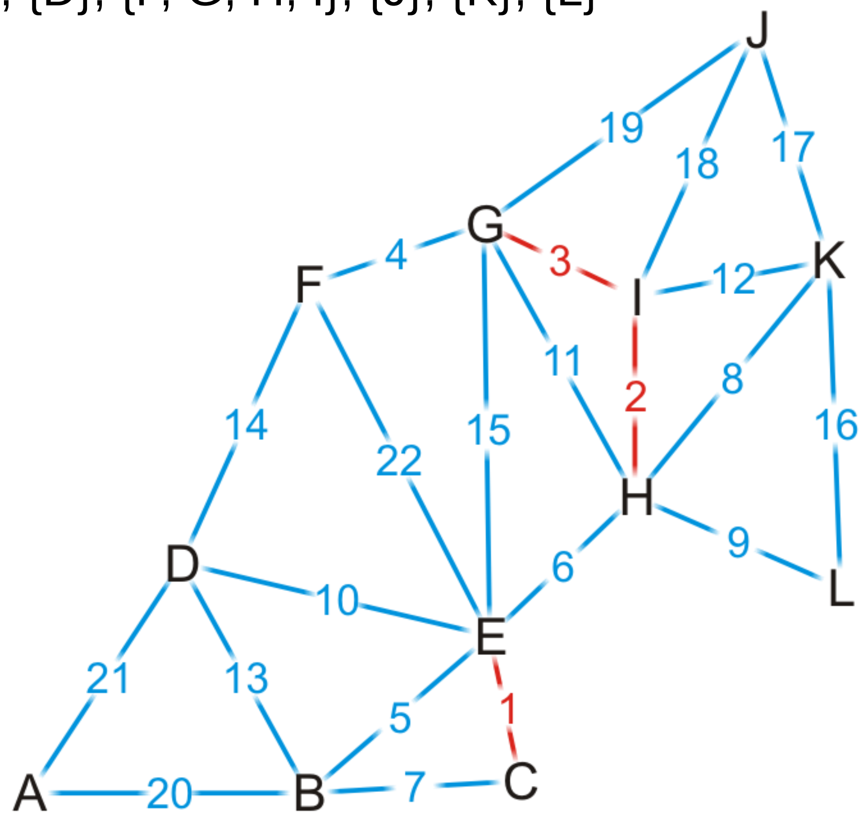


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Similarly, we add {G, I}, {F, G}, {B, E}

{A}, {B, C, E}, {D}, {F, G, H, I}, {J}, {K}, {L}

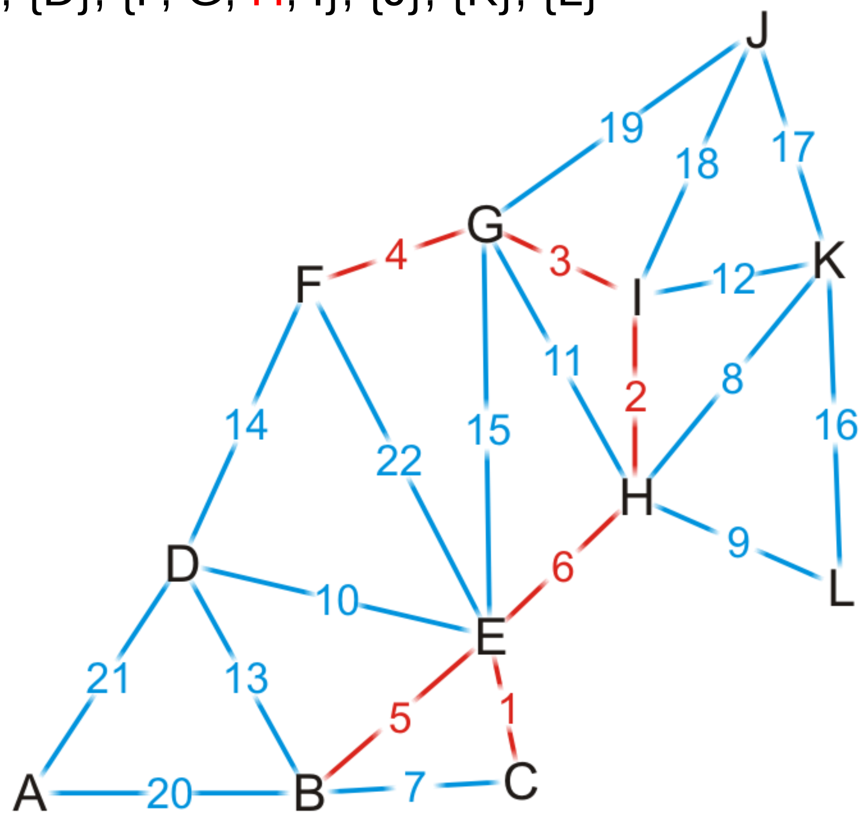


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

The vertices of {E, H} are in different sets

{A}, {B, C, E}, {D}, {F, G, H, I}, {J}, {K}, {L}

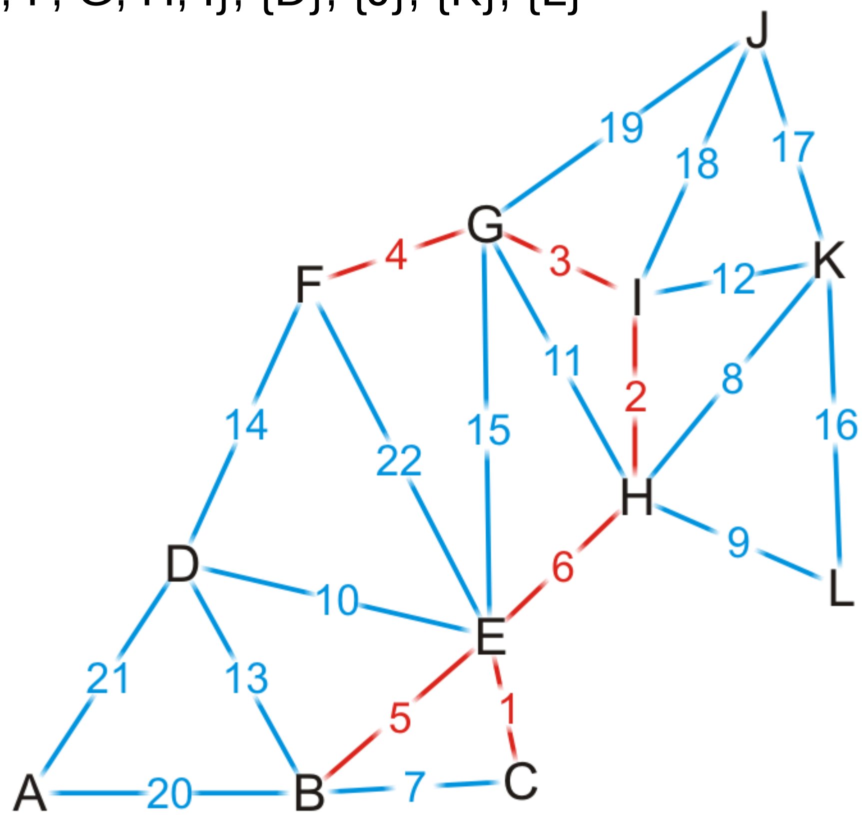


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Adding edge {E, H} creates a larger union

{A}, {B, C, E, F, G, H, I}, {D}, {J}, {K}, {L}

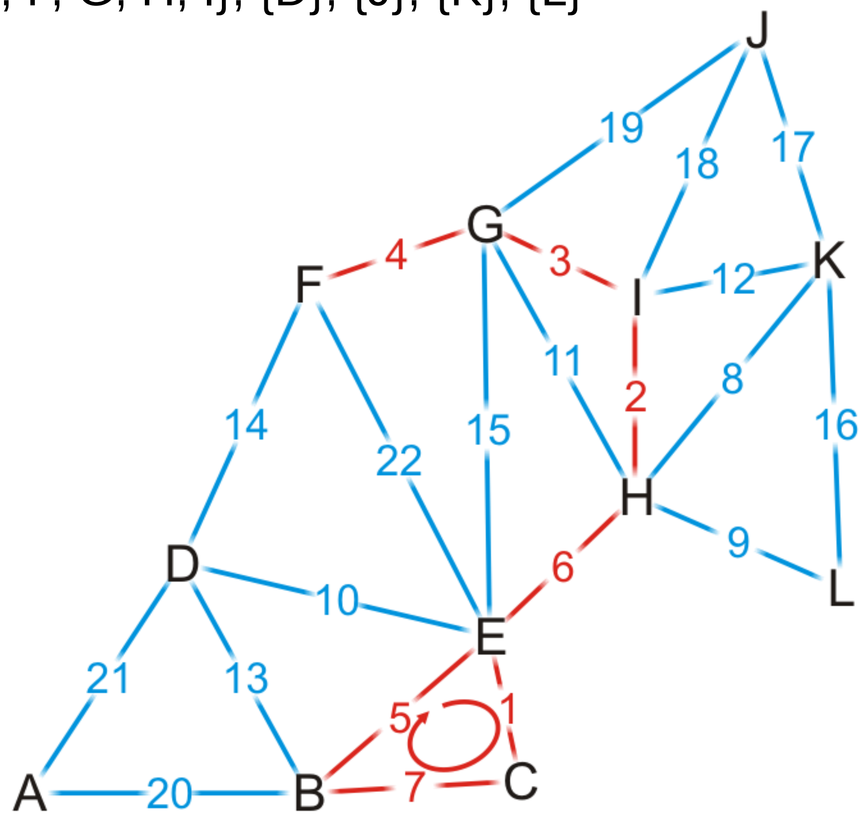


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We try adding {B, C}, but it creates a cycle

{A}, {B, C, E, F, G, H, I}, {D}, {J}, {K}, {L}

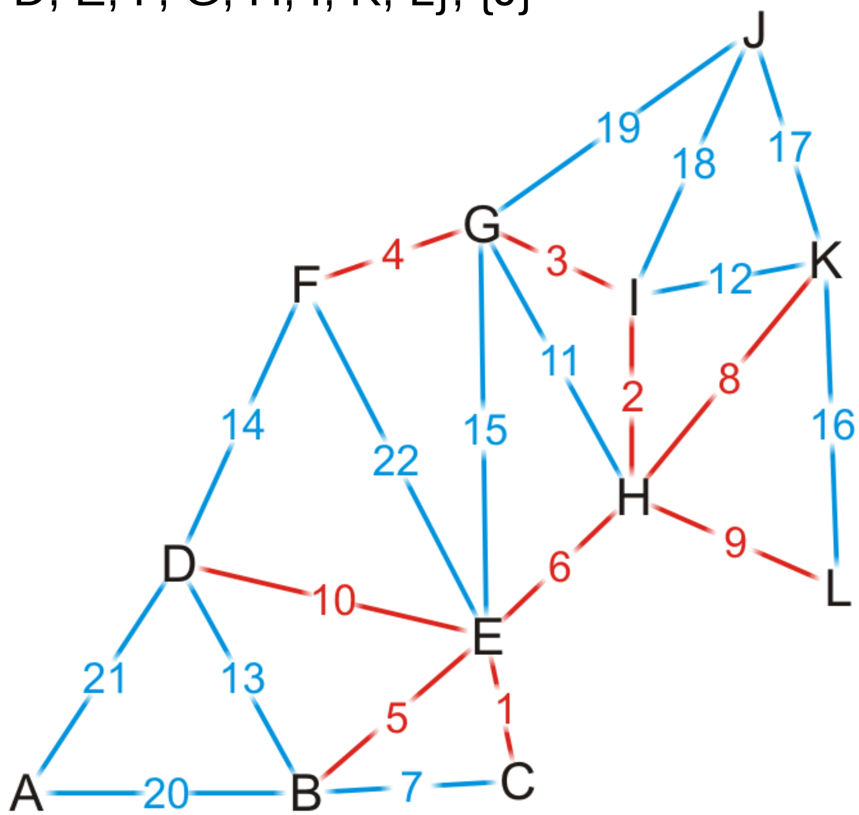


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We add edge {H, K}, {H, L} and {D, E}

{A}, {B, C, D, E, F, G, H, I, K, L}, {J}

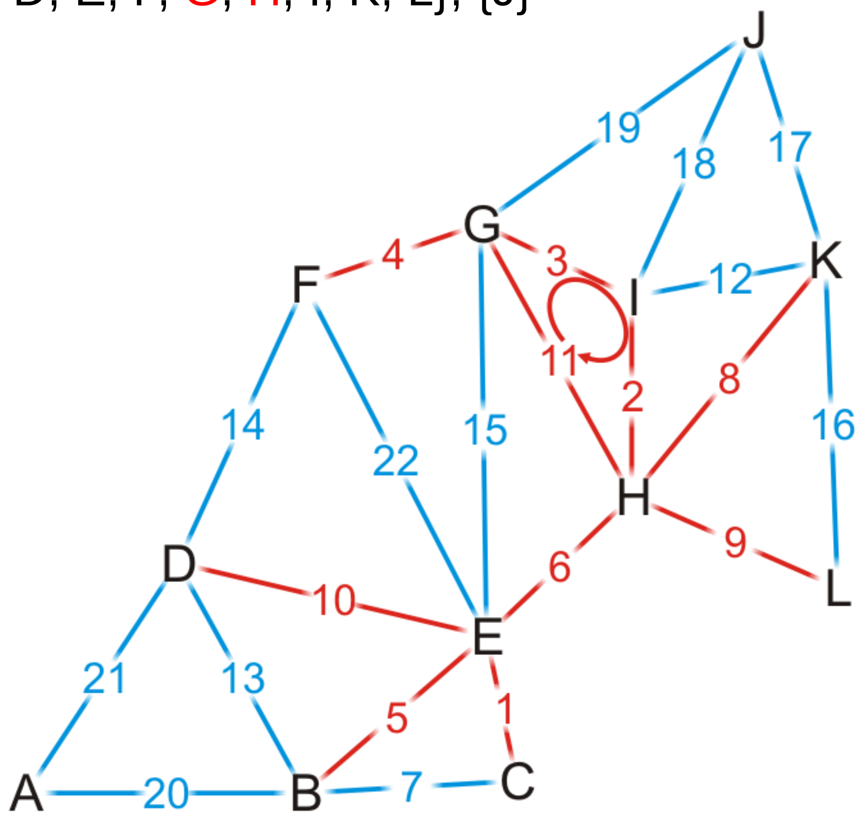


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Both G and H are in the same set

{A}, {B, C, D, E, F, G, H, I, K, L}, {J}

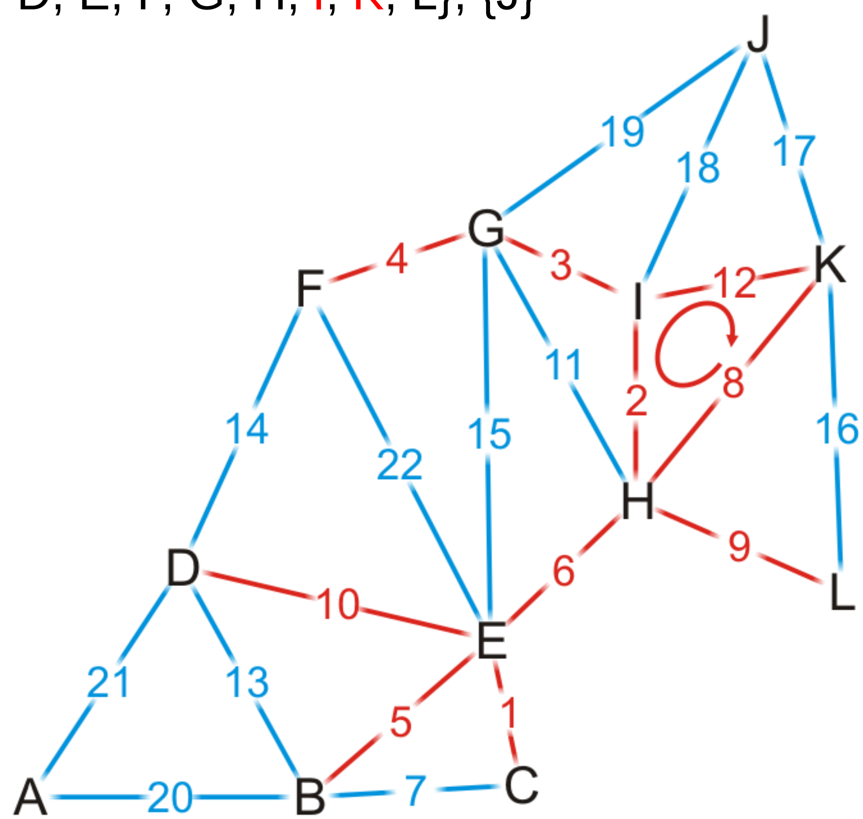


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Both {I, K} are in the same set

{A}, {B, C, D, E, F, G, H, I, K, L}, {J}

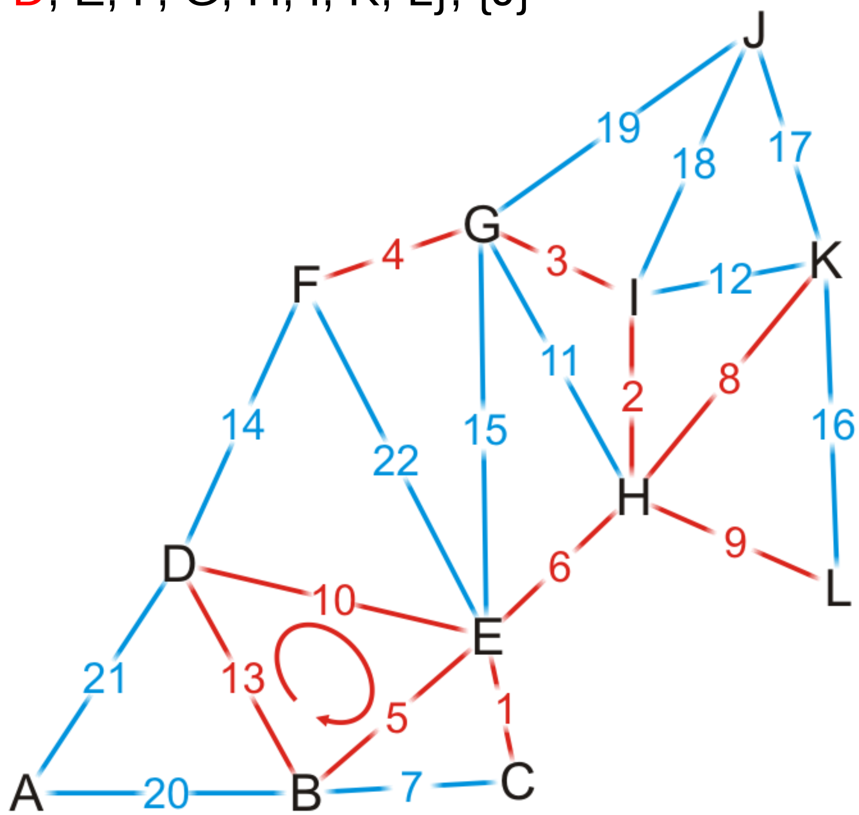


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Both {B, D} are in the same set

{A}, {B, C, D, E, F, G, H, I, K, L}, {J}

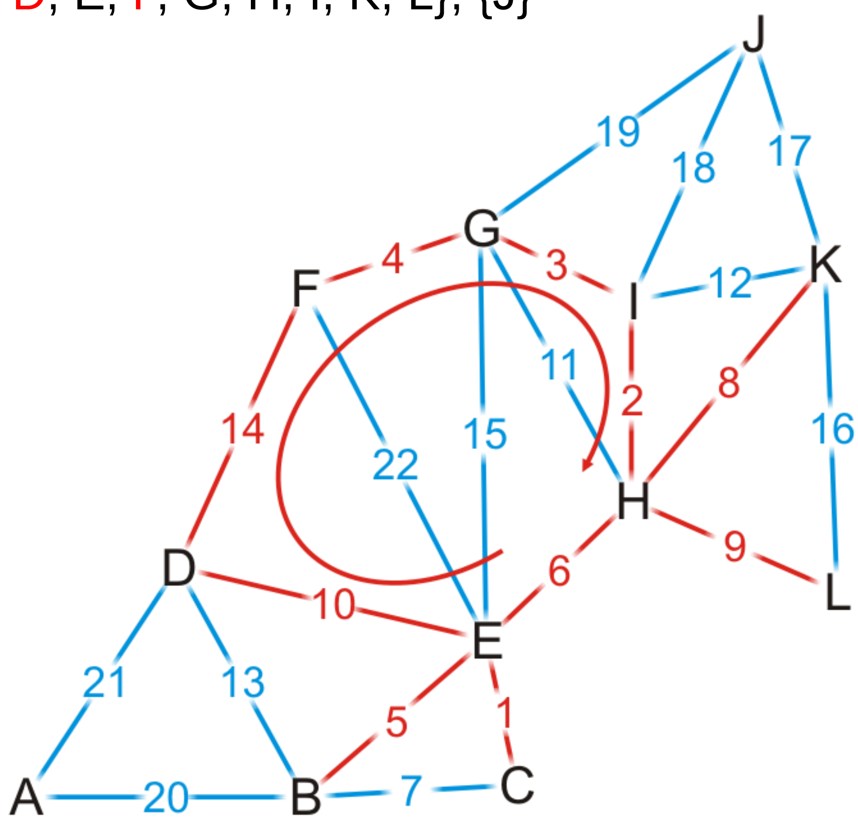


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

Both {D, F} are in the same set

{A}, {B, C, D, E, F, G, H, I, K, L}, {J}

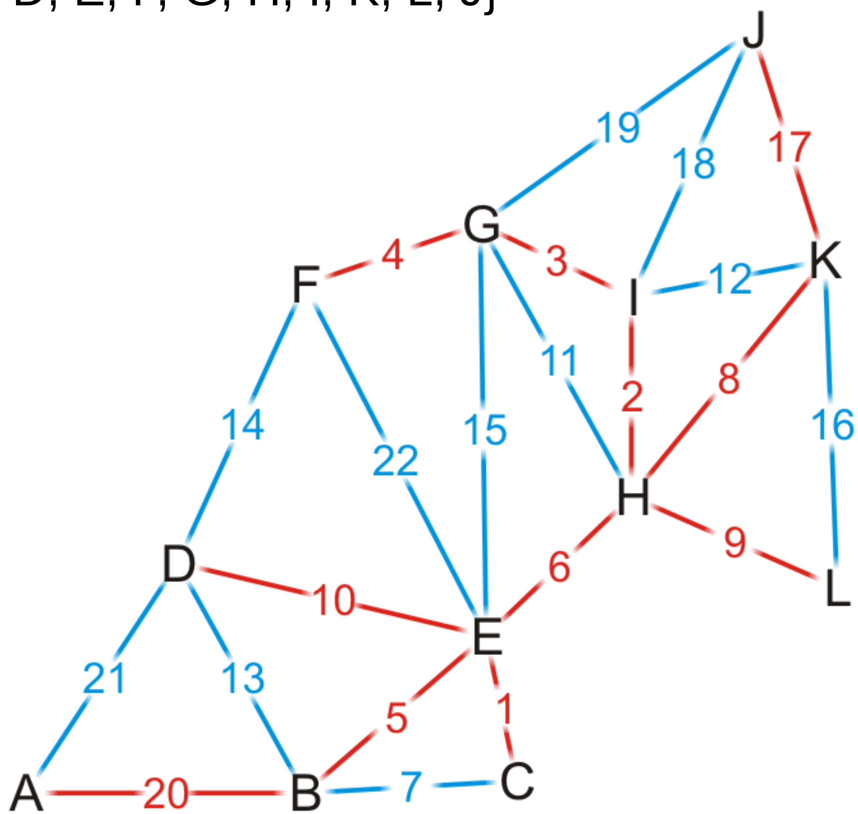


{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Example

We end when there is only one set, having added (A, B)

{A, B, C, D, E, F, G, H, I, K, L, J}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Summary

This topic has covered Kruskal's algorithm

- Sort the edges by weight
- Create a disjoint set of the vertices
- Begin adding the edges one-by-one checking to ensure no cycles are introduced
- The result is a minimum spanning tree
- The run time is $\mathrm{O}(|E| \ln(|V|)$

Summary

- Definition and applications
- Prim’s algorithm
  - Start with a trivial minimum spanning tree and grow it by adding edges with least weight
- Kruskal’s algorithm
  - Go through the edges from least weight to greatest weight, adding an edge if it does not create a cycle