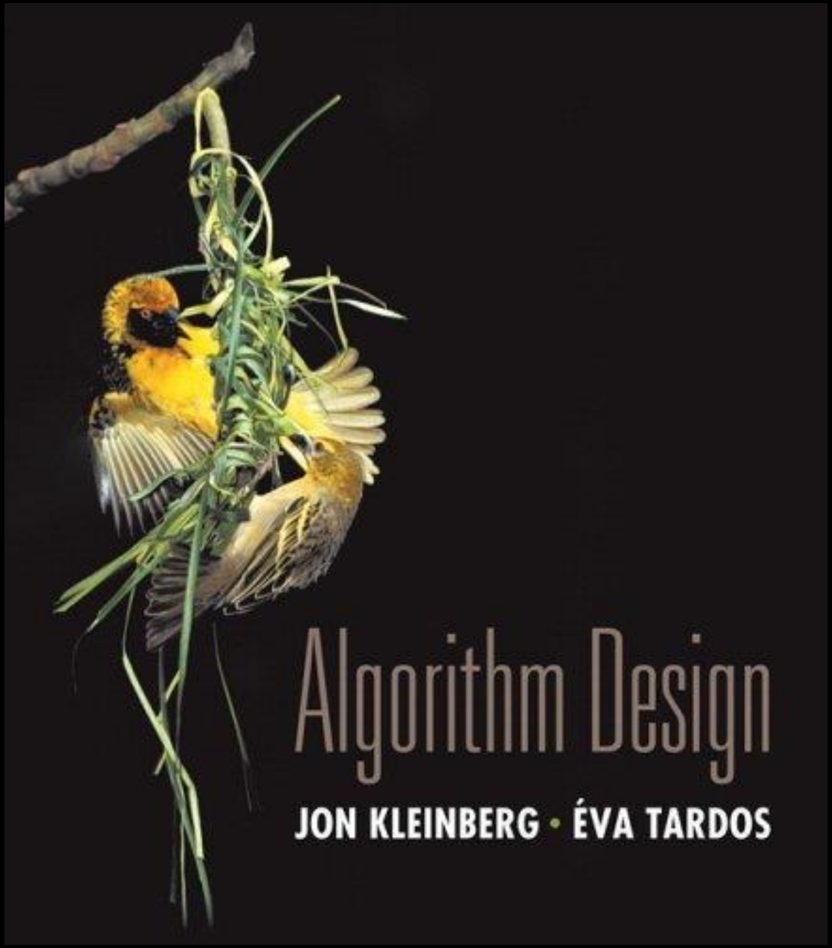


Chapter 5

Divide and Conquer

Slides by Kevin Wayne.

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}$n in linear time.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

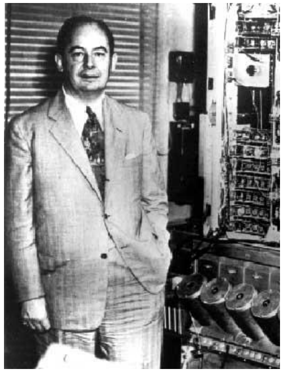
- Divide-and-conquer: Θ(n log n)

5.1 Mergesort

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann (1945)

A	L	G	O	R	I	T	H	M	S	
A	L	G	O	R	I	T	H	M	S	divide
A	G	L	O	R	H	I	M	S	T	sort
A	G	H	I	L	M	O	R	S	T	merge

Analysis of Mergesort Recurrence

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort recurrence.

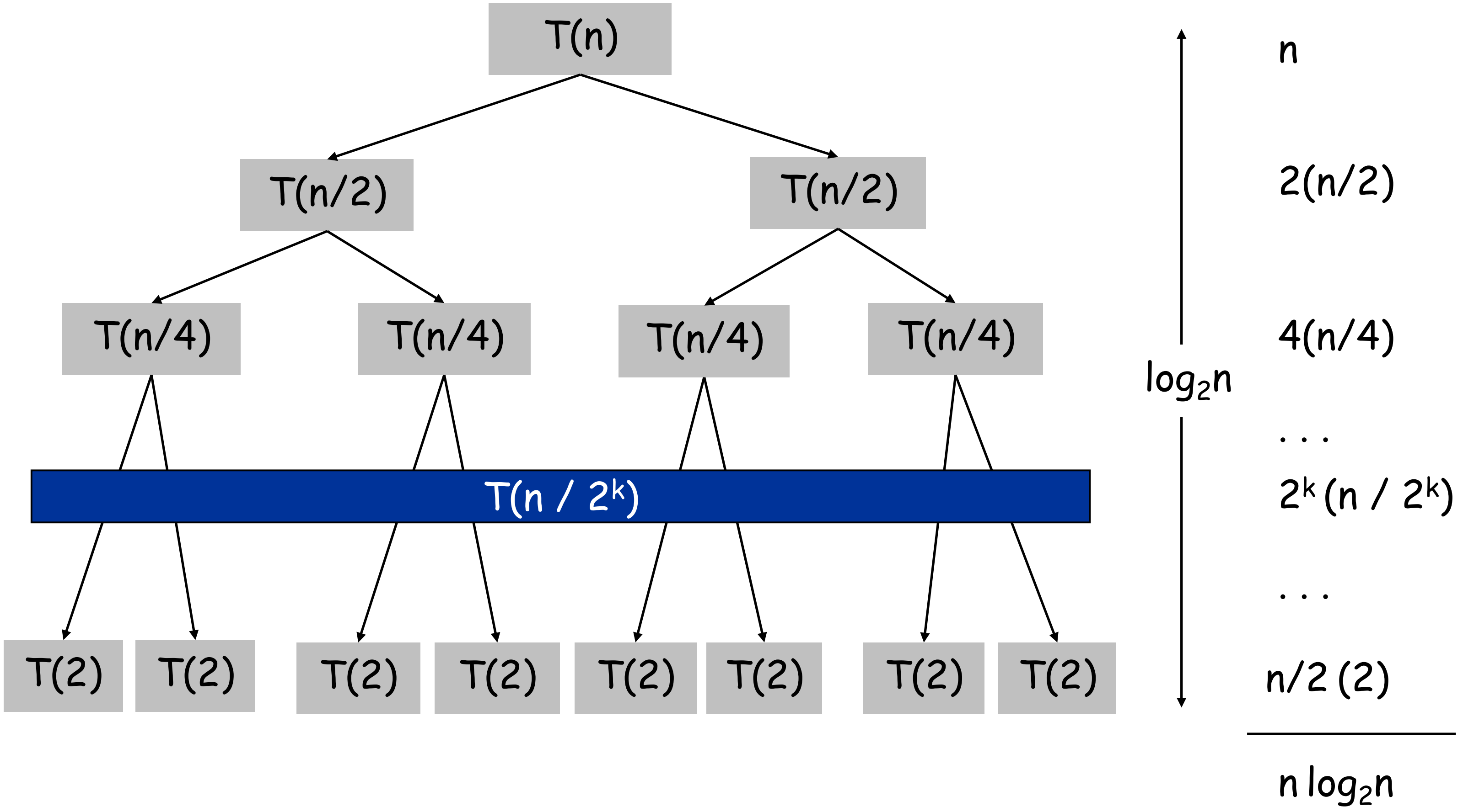
$$\mathrm{T}(n) \le \begin{cases} 0 & \text{if } n=1 \\ \underbrace{T\left(\lceil n/2 \rceil\right)}_{\text{solve left half}} + \underbrace{T\left(\lfloor n/2 \rfloor\right)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace ≤ with =.

Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by Induction

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: T(n) = n $\log_2$ n.
- Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n\log_2 n + 2n \\ &= 2n\left(\log_2(2n) - 1\right) + 2n \\ &= 2n\log_2(2n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then T(n) $\leq n \lceil \log_2 n \rceil$.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n-1.

$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + n \\ &\leq n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\leq n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \\ &\leq n(\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\leq \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \\ \Rightarrow \log_2 n_2 &\leq \lceil \log_2 n \rceil - 1 \end{aligned}$$

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Fast closest pair inspired fast algorithms for these problems: nearest neighbor, Euclidean MST, Voronoi.

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

Assumption. No two points have same x coordinate.

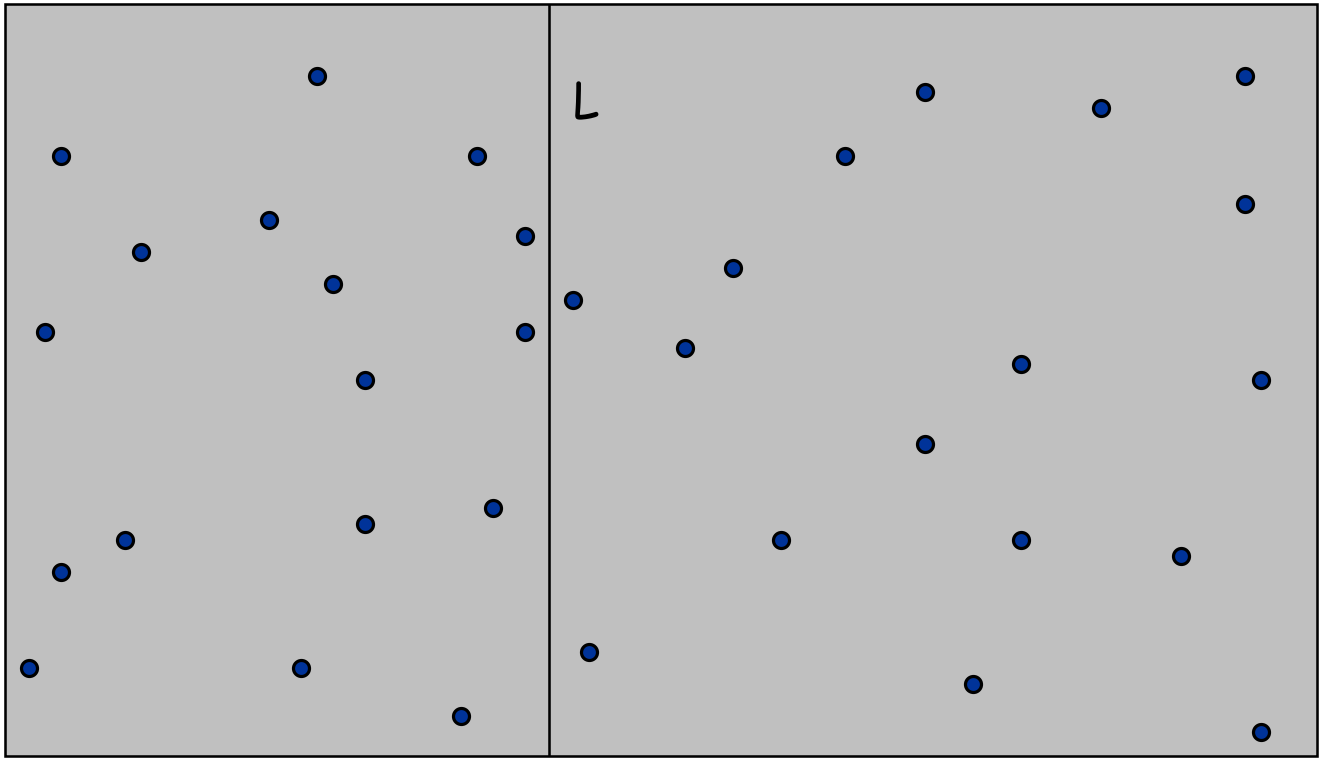
↑

to make presentation cleaner

Closest Pair of Points

Algorithm.

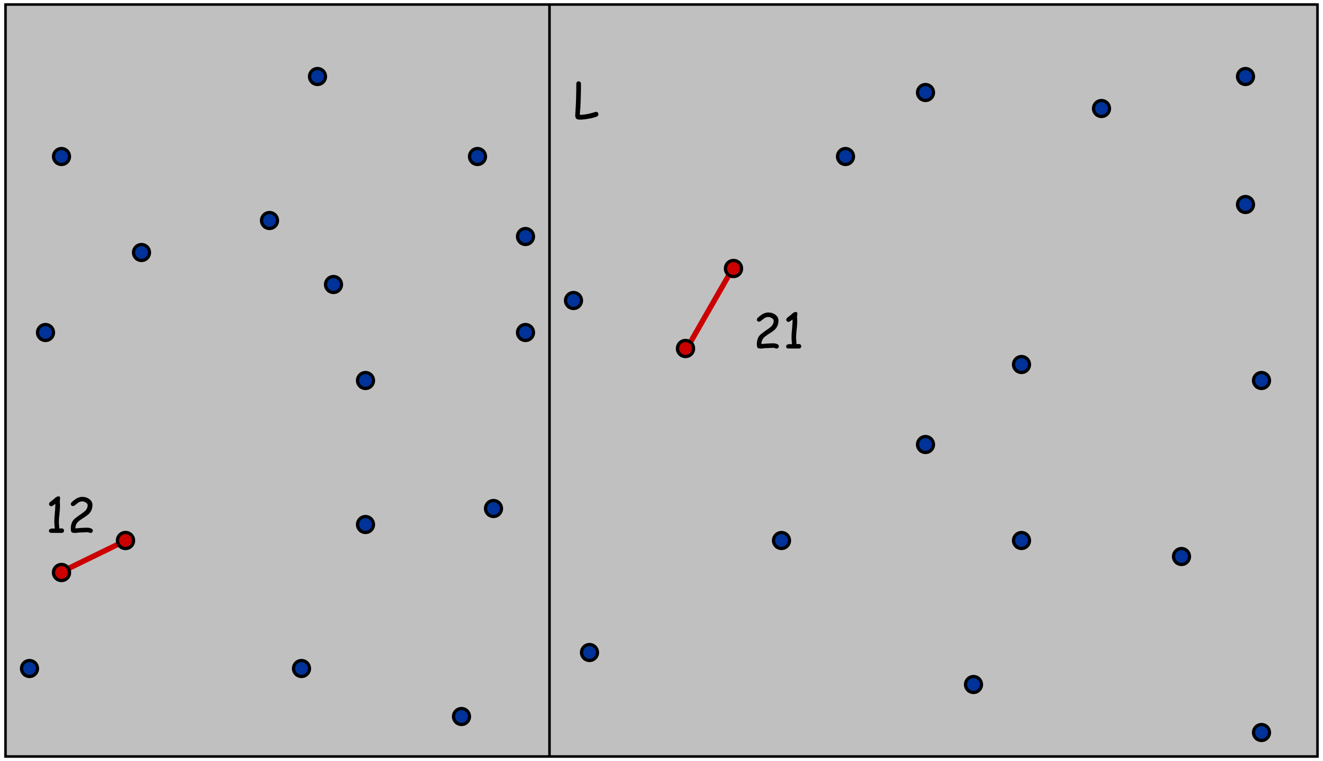
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

Algorithm.

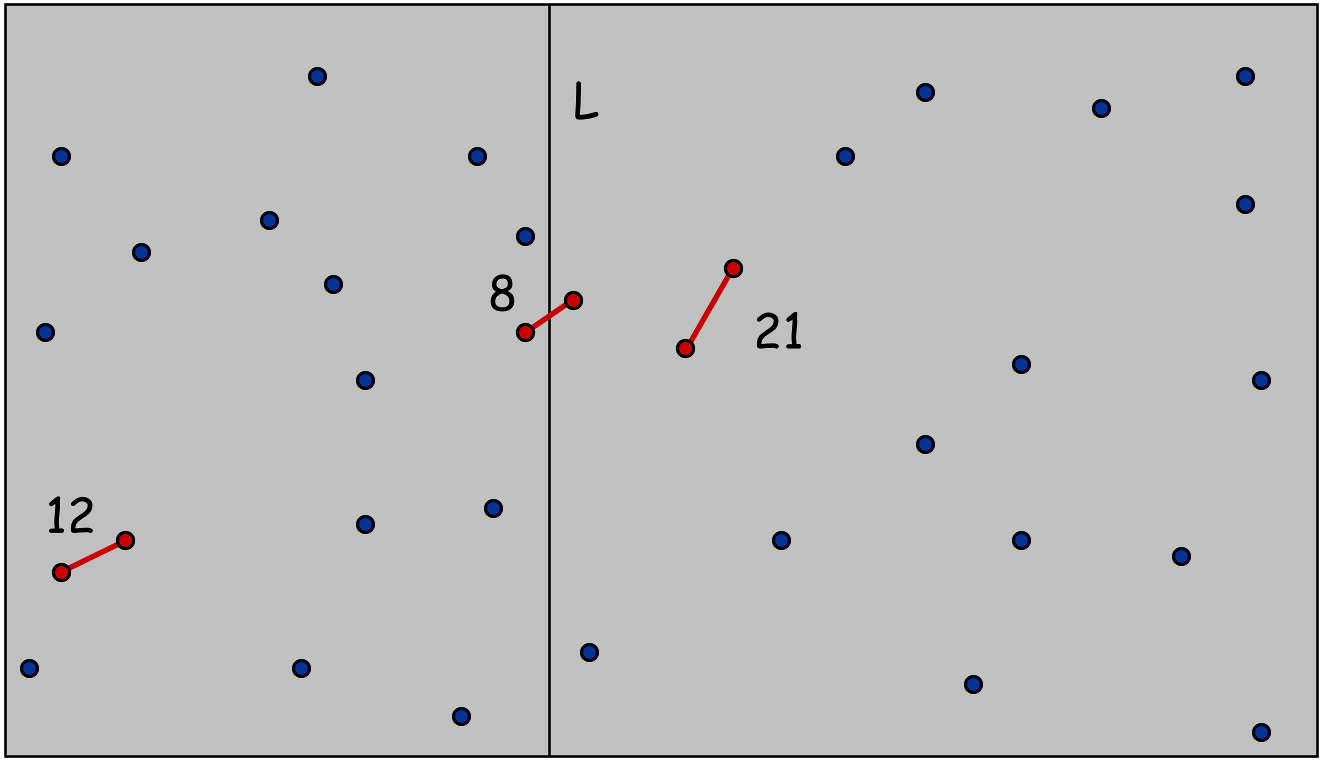
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

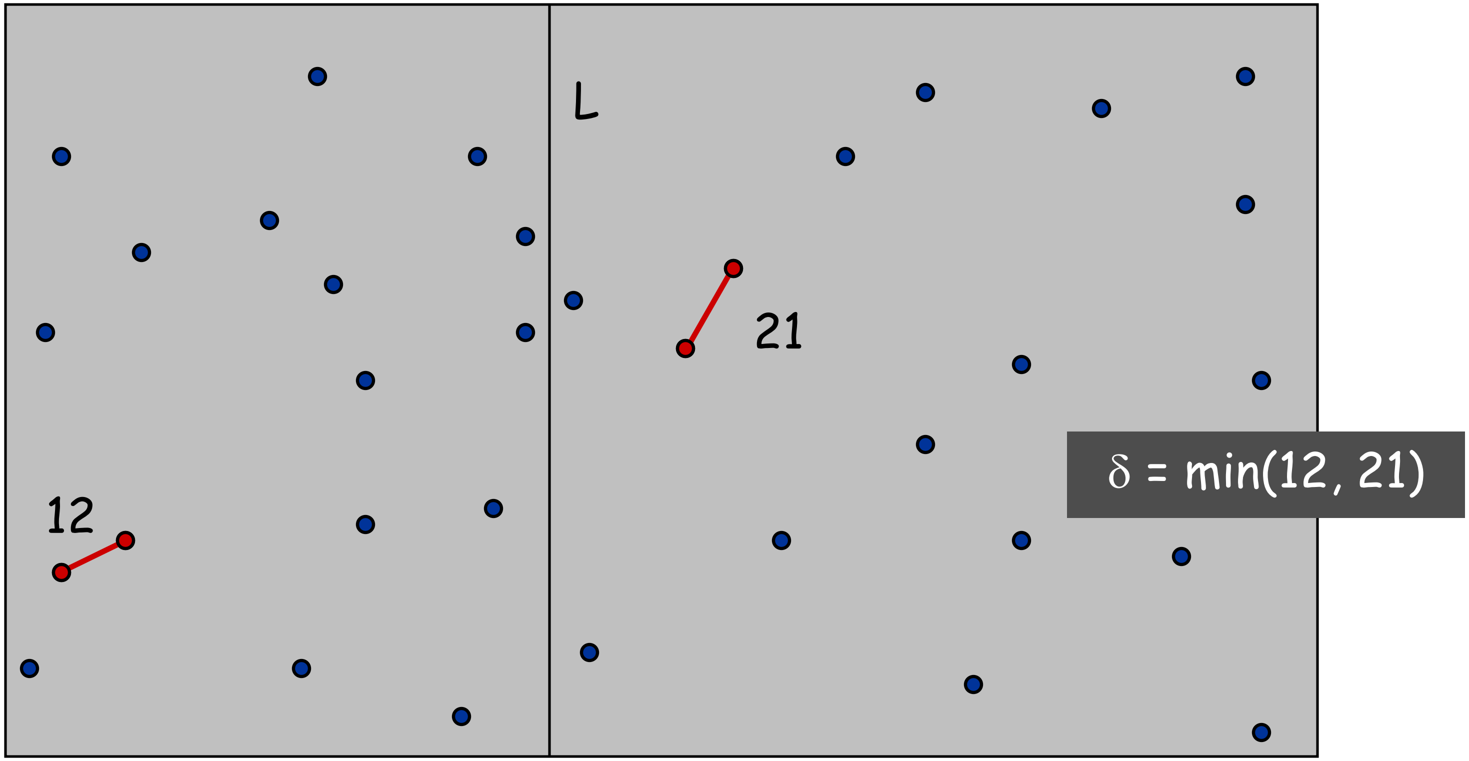
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

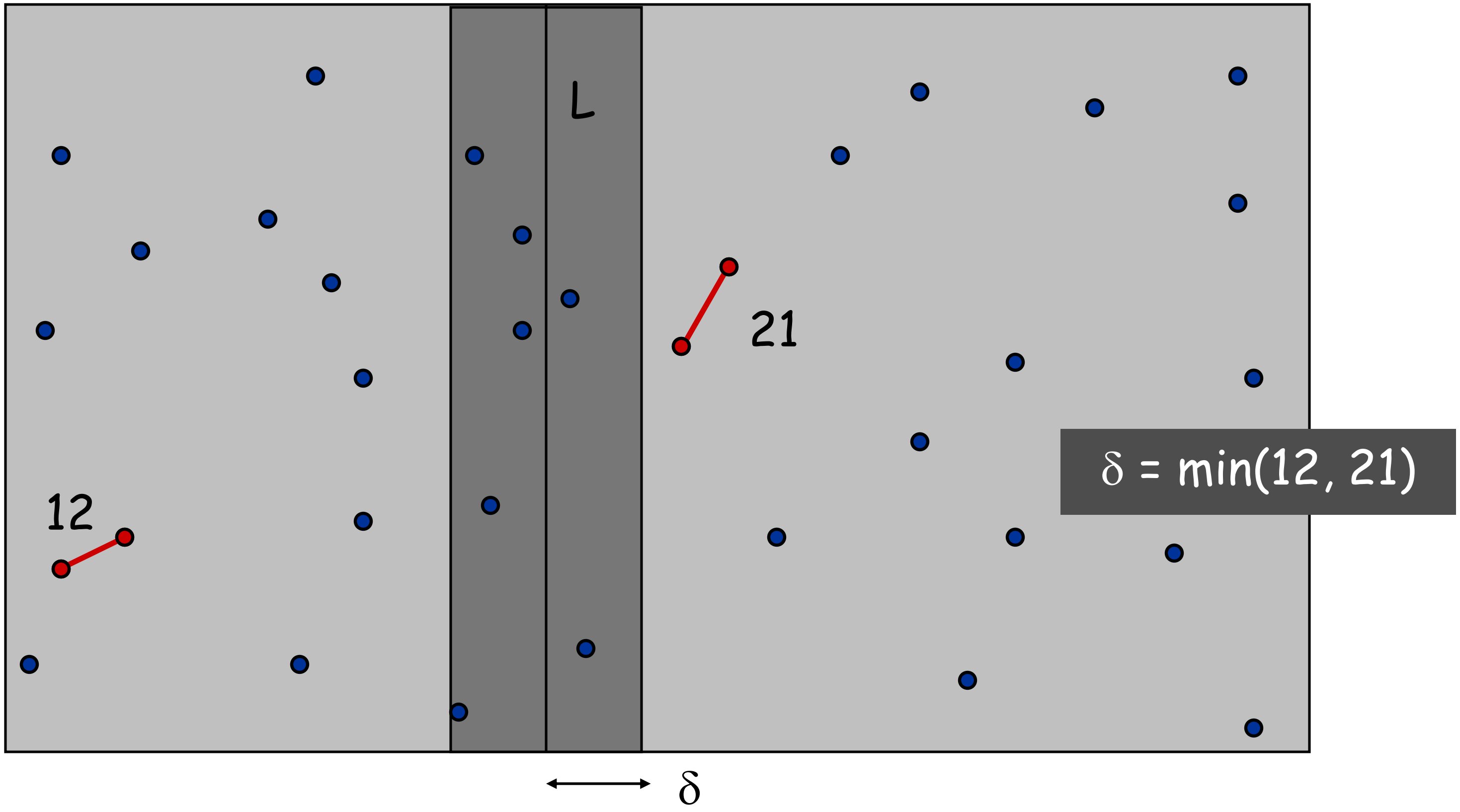


δ = min(12, 21)

Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

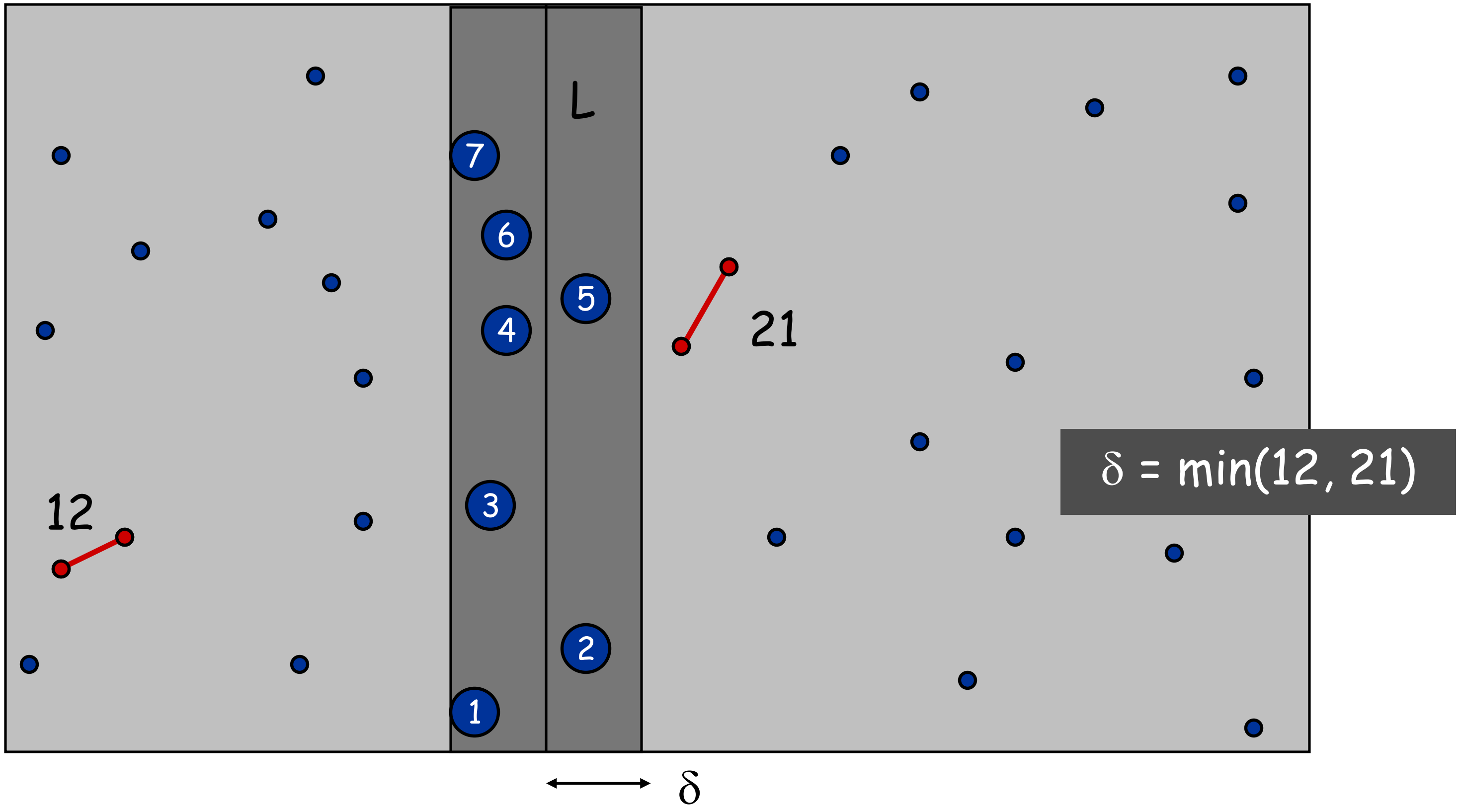
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.

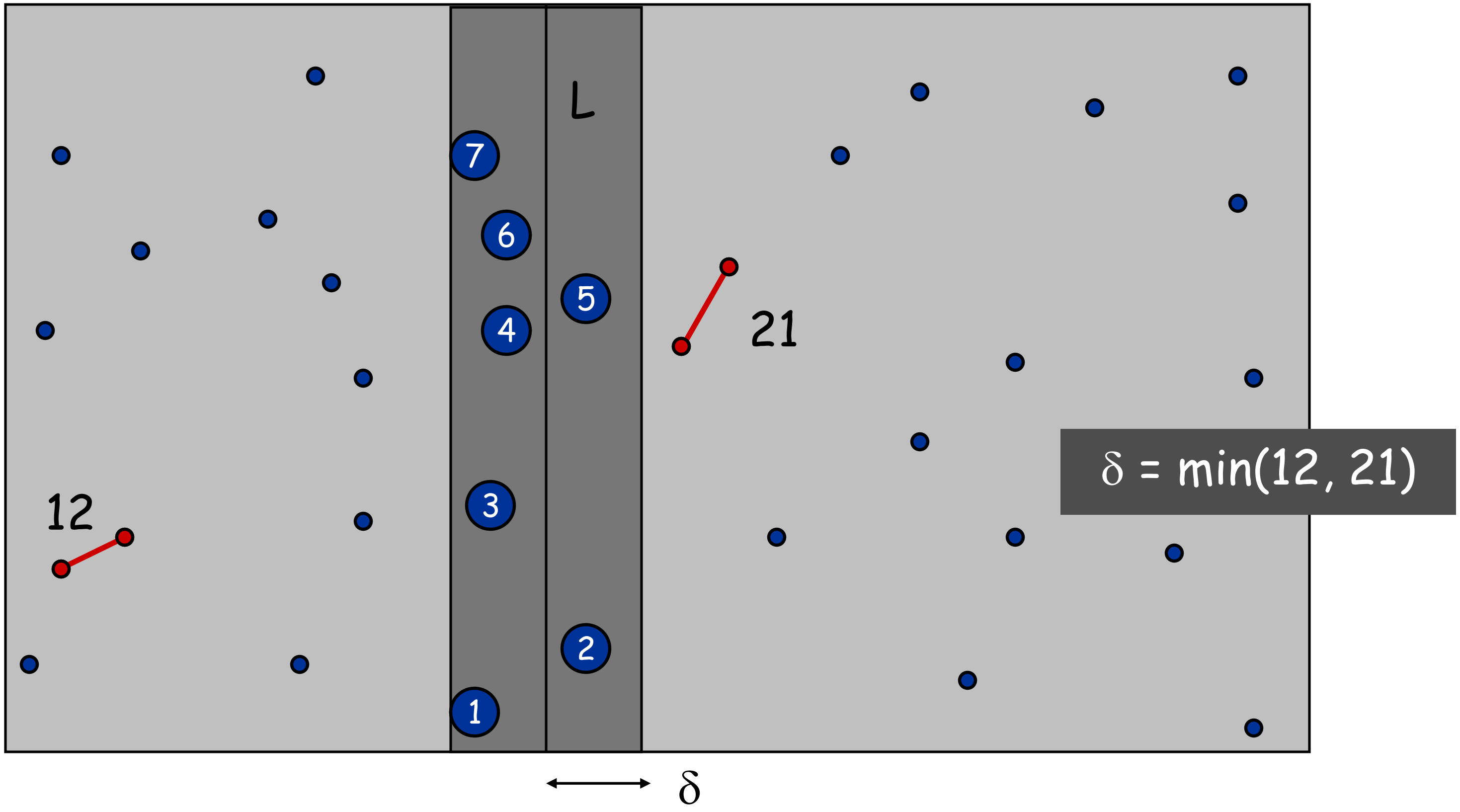


δ = min(12, 21)

Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



δ = min(12, 21)

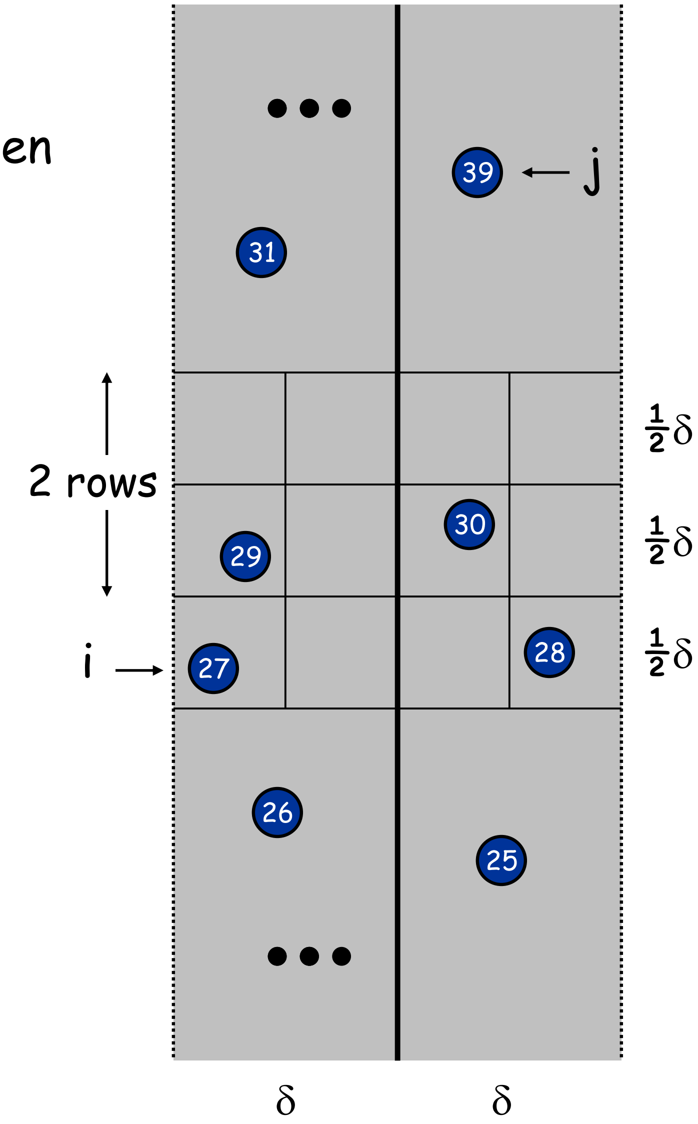
Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪



Fact. Still true if replace 12 with 8. Why?

Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Linear time algorithm!

Without this assumption?

- Run the same algorithm
- If assumption true, we will find the right closest pair with one point in each side
- If false, the algorithm will find a pair with distance $\geq \delta$, and then the combine step will correctly return δ as distance of closest pair

Closest Pair Algorithm

```
Closest-Pair($p_1$, …, $p_n$) {
   Compute separation line L such that half the points        $O(n \log n)$
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)                        $2T(n / 2)$
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L   $O(n)$

   Sort remaining points by y-coordinate.                      $O(n \log n)$

   Scan points in y-order and compare distance between         $O(n)$
   each point and next 11 neighbors. If any of these
   distances is less than $\delta$, update $\delta$.

   return $\delta$.
}
```

Closest Pair of Points: *Analysis*

Running time.

$$\mathrm{T}(n) \le 2T\big(n/2\big) + O(n \log n) \;\Rightarrow\; \mathrm{T}(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

- Sort all the points twice before recursive call, once by x coordinate and once by y coordinate
- Reuse the sorted sequences when needed (linear time)

$$T(n) \le 2T\big(n/2\big) + O(n) \;\Rightarrow\; \mathrm{T}(n) = O(n \log n)$$

5.5 Integer Multiplication

Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- Brute force solution: $\Theta(n^2)$ bit operations.

	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

								1	1	0	1	0	1	0	1	0
								*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1	0
							0	0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1	0		
					1	1	0	1	0	1	0	1	0			
				1	1	0	1	0	1	0	1	0				
			1	1	0	1	0	1	0	1	0					
		1	1	0	1	0	1	0	1	0						
	0	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0

Multiply

Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:

- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$x = \underbrace{1\,0\,0\,0}_{x_1}\ \underbrace{1\,1\,0\,1}_{x_0}$$

$$\begin{aligned} x &= 2^{n/2} \cdot x_1 + x_0 \\ y &= 2^{n/2} \cdot y_1 + y_0 \\ xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) = 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \end{aligned}$$

$$\mathrm{T}(n) = \underbrace{4T\left(n/2\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)$$

↑

assumes n is a power of 2

Karatsuba Multiplication

To multiply two n-digit integers:

- Add two ½n digit integers.
- Multiply three ½n-digit integers.
- Add, subtract, and shift ½n-digit integers to obtain result.

$$\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^n \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \big(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0}\big) + \underset{C}{x_0 y_0}
\end{aligned}$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

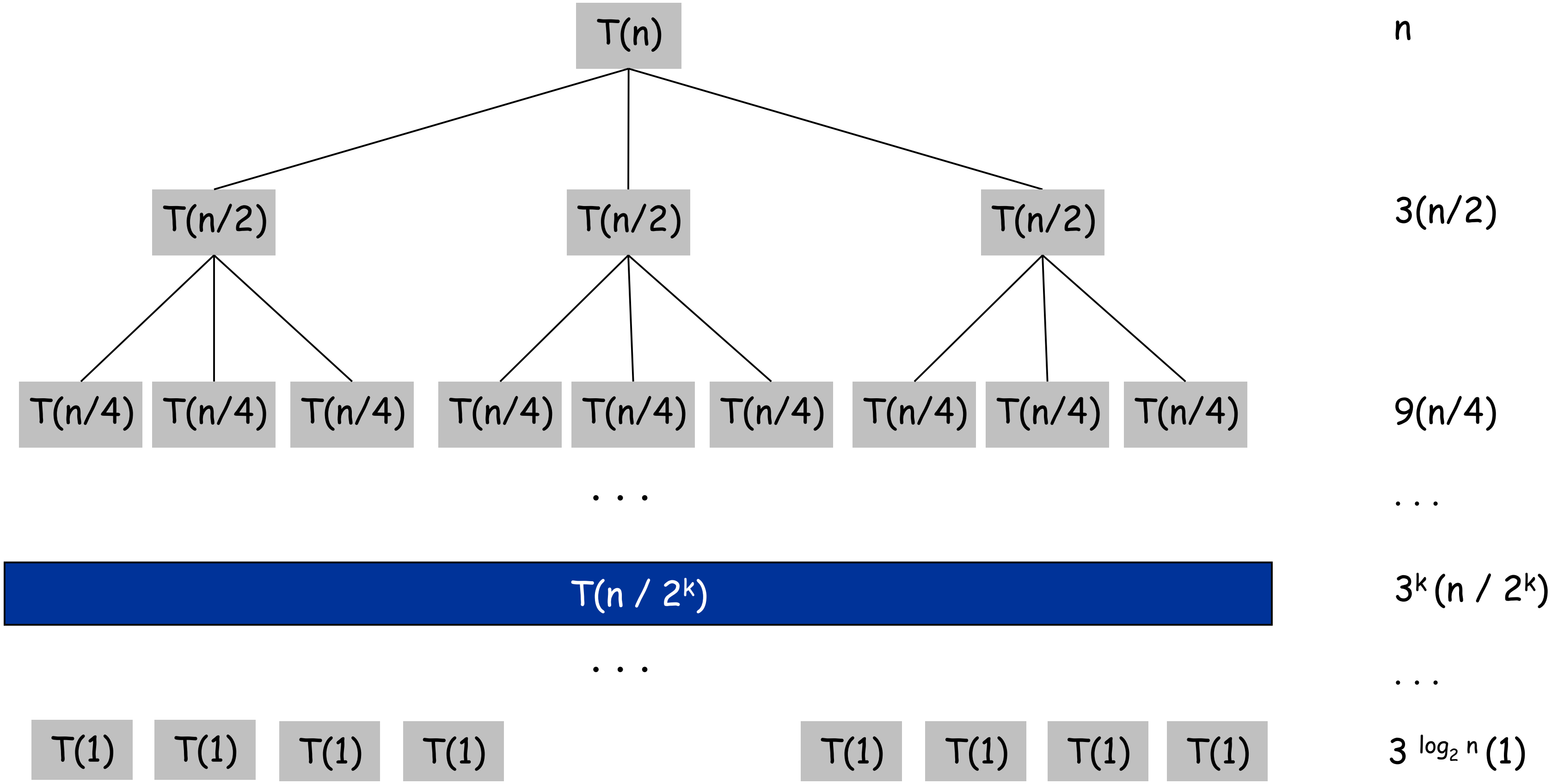
$$\mathrm{T}(n) \le \underbrace{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1 + \lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract,shift}}$$

$$\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\tfrac{3}{2}\right)^k = \frac{\left(\tfrac{3}{2}\right)^{1+\log_2 n} - 1}{\tfrac{3}{2} - 1}\, n = 3n^{\log_2 3} - 2n$$



Matrix Multiplication

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute C = AB.

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Brute force. $\Theta(n^3)$ arithmetic operations.

Fundamental question. Can we improve upon brute force?

Matrix Multiplication: Warmup

Divide-and-conquer.

- Divide: partition A and B into ½n-by-½n blocks.
- Conquer: multiply 8 ½n-by-½n recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$\mathrm{T}(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow \mathrm{T}(n) = \Theta(n^3)$$

Matrix Multiplication: Key Idea

Key idea. multiply 2-by-2 block matrices with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

- 7 multiplications.
- 18 = 10 + 8 additions (or subtractions).

Fast Matrix Multiplication

Fast matrix multiplication. (Strassen, 1969)

- Divide: partition A and B into ½n-by-½n blocks.
- Compute: 14 ½n-by-½n matrices via 10 matrix additions.
- Conquer: multiply 7 ½n-by-½n matrices recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- T(n) = # arithmetic operations.

$$\mathrm{T}(n) = \underbrace{7T\big(n/2\big)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \quad \Rightarrow \quad \mathrm{T}(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Fast Matrix Multiplication in Practice

Common misperception: "Strassen is only a theoretical curiosity."

- Advanced Computation Group at Apple Computer reports 8x speedup on G4 Velocity Engine when n ~ 2,500.
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize" Ax=b, determinant, eigenvalues, and other matrix ops.

Fast Matrix Multiplication in Theory

Q. Multiply two 2-by-2 matrices with only 7 scalar multiplications?

A. Yes! [Strassen, 1969]

$$\Theta(n^{\log_2 7}) = O(n^{2.81})$$

Q. Multiply two 2-by-2 matrices with only 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr, 1971]

$$\Theta(n^{\log_2 6}) = O(n^{2.59})$$

Q. Two 3-by-3 matrices with only 21 scalar multiplications?

A. Also impossible.

$$\Theta(n^{\log_3 21}) = O(n^{2.77})$$

Q. Two 70-by-70 matrices with only 143,640 scalar multiplications?

A. Yes! [Pan, 1980]

$$\Theta(n^{\log_{70} 143640}) = O(n^{2.80})$$

Fast Matrix Multiplication in Theory

Decimal wars.

- December, 1979: $O(n^{2.521813})$.
- January, 1980: $O(n^{2.521801})$.
- …
- 1987: $O(n^{2.375477})$.
- 2010: $O(n^{2.374})$.
- 2011: $O(n^{2.3728642})$.
- 2014: $O(n^{2.3728639})$.

Best known. $O(n^{2.3728596})$ *[Alman & Williams, 2020]*

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. Theoretical improvements to Strassen are progressively less practical.

5.6 Convolution and FFT

Fast Fourier Transform: Applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, control systems, signal processing, speech recognition, data compression, image processing.
- DVD, JPEG, MP3, MRI, CAT scan.
- Numerical solutions to Poisson's equation.

> The FFT is one of the truly great computational developments of this [20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. *-Charles van Loan*

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add: O(n) arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate: O(n) using Horner's method.

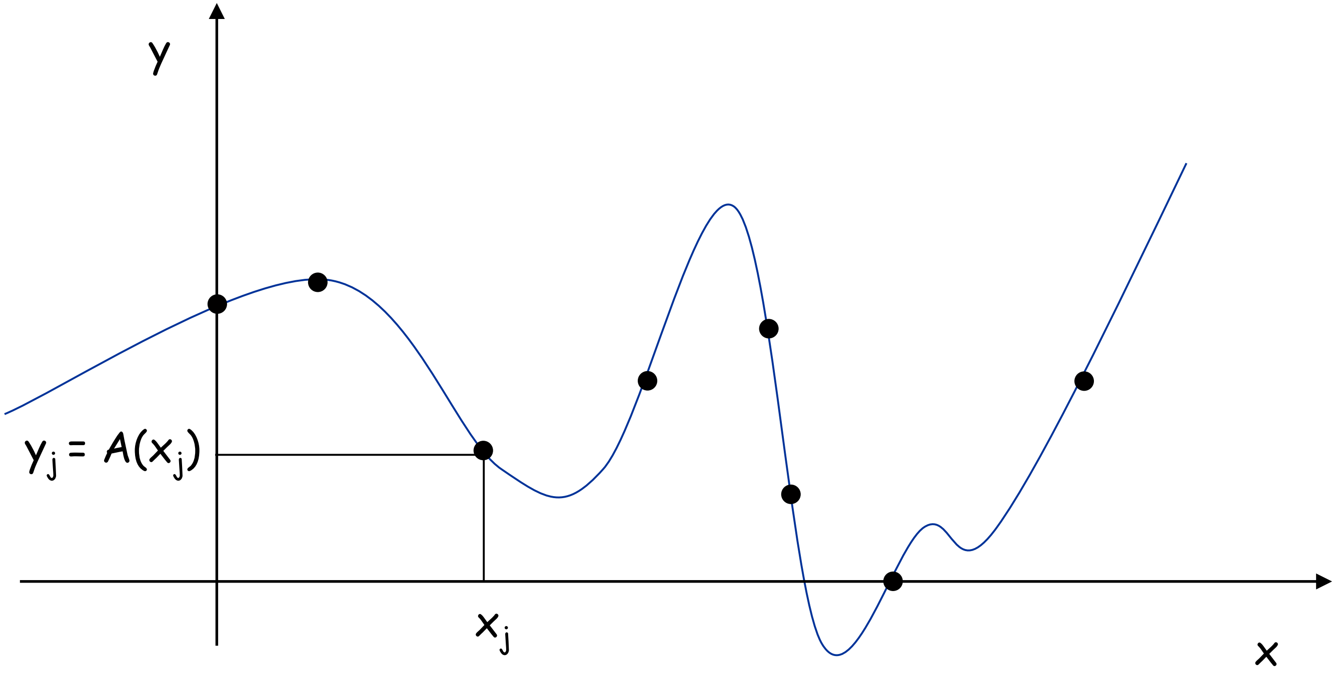
$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

Multiply (convolve): O(n^2) using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i\, x^i, \quad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

A degree n-1 polynomial A(x) is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): (\mathrm{x}_0, y_0), \ldots, (\mathrm{x}_{\text{n-1}}, y_{n-1})$$
$$B(x): (\mathrm{x}_0, z_0), \ldots, (\mathrm{x}_{\text{n-1}}, z_{n-1})$$

Add: O(n) arithmetic operations.

$$A(x)+B(x): (\mathrm{x}_0, y_0+z_0), \ldots, (\mathrm{x}_{\text{n-1}}, y_{n-1}+z_{n-1})$$

Multiply: O(n), but need 2n-1 points.

$$A(x)\times B(x): (\mathrm{x}_0, y_0\times z_0), \ldots, (\mathrm{x}_{\text{2n-1}}, y_{2n-1}\times z_{2n-1})$$

Evaluate: $O(n^2)$ using Lagrange's formula.

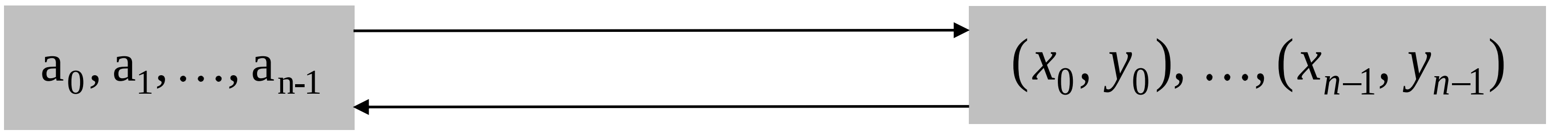
$$A(x)=\sum_{k=0}^{n-1} y_k \frac{\prod_{j\neq k}(x-x_j)}{\prod_{j\neq k}(x_k-x_j)}$$

Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

Representation	Multiply	Evaluate
Coefficient	$O(n^2)$	$O(n)$
Point-value	$O(n)$	$O(n^2)$

Goal. Make all ops fast by efficiently converting between two representations.



$a_0, a_1, \ldots, a_{n-1}$

coefficient representation

$(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$

point-value representation

Converting Between Two Polynomial Representations: Brute Force

Coefficient ➔ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^2)$ for matrix-vector multiplication or n times Horner's method

Converting Between Two Polynomial Representations: Brute Force

Point-value ➔ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^3)$ for Gaussian elimination.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

We can choose which points!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Intuition. Choose two points to be ±1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.

Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \dots, x_{n-1}$.

We can choose which points!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\,A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\,A_{odd}(x^2)$.

Intuition. Choose four complex points to be ±1, ±i.

- $A(1) = A_{even}(1) + 1\,A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\,A_{odd}(1)$.
- $A(i) = A_{even}(-1) + i\,A_{odd}(-1)$.
- $A(-i) = A_{even}(-1) - i\,A_{odd}(-1)$.

Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 point.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

We can choose which points!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Goal. Choose n points s.t.

- Can evaluate polynomial of degree $\leq$ n at n points by evaluating two polynomials of degree $\leq \frac{1}{2}$n at $\frac{1}{2}$n point.
- But also: can evaluate polynomial of degree $\leq \frac{1}{2}$n at $\frac{1}{2}$n points by evaluating two polynomials of degree $\leq \frac{1}{4}$n at $\frac{1}{4}$n point, and so on.

Roots of Unity

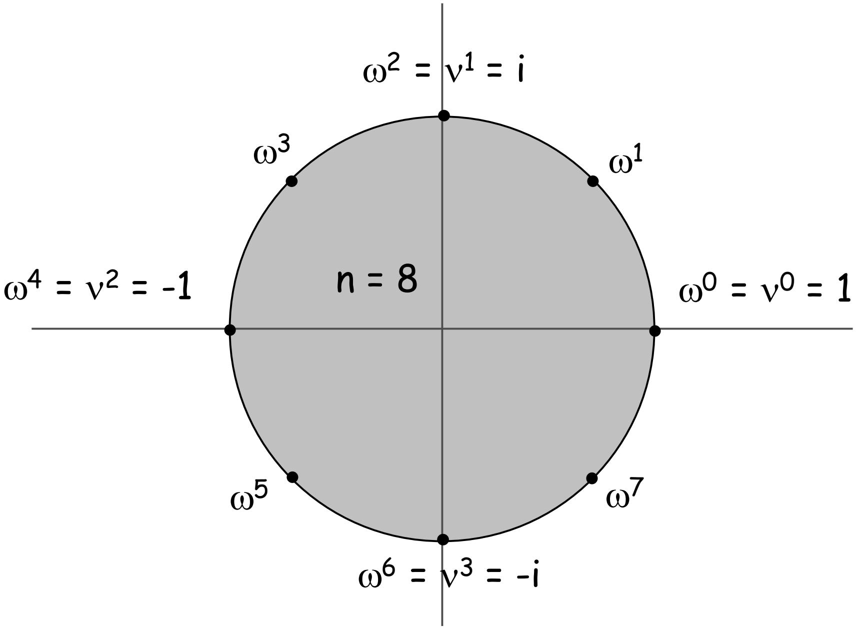
Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1.$

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.

Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Discrete Fourier Transform

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea: choose $x_k = \omega^k$ where ω is principal nth root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Discrete Fourier transform

Fourier matrix F_n

Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at its nth roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2} x^{(n-1)/2}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1} x^{(n-1)/2}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate degree $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}$nth roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \leq k < n/2$
- $A(\omega^{k+n/2}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \leq k < n/2$

$\uparrow$

$\omega^{k+n/2} = -\omega^k$

$\nu^k = (\omega^k)^2 = (\omega^{k+n/2})^2$

FFT Algorithm

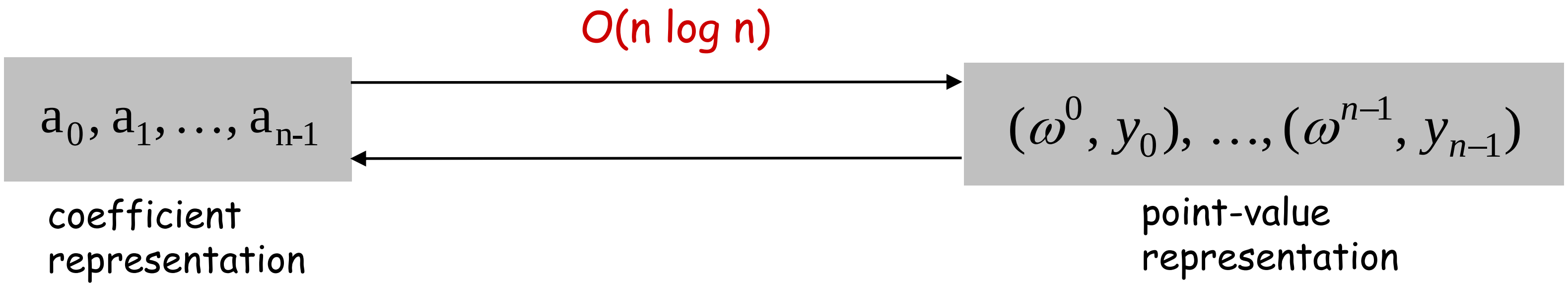
```
FFT(n, $a_0, a_1, \ldots, a_{n-1}$) {
    if (n == 1) return $a_0$

    ($e_0, e_1, \ldots, e_{n/2-1}$) $\leftarrow$ FFT(n/2, $a_0, a_2, a_4, \ldots, a_{n-2}$)
    ($d_0, d_1, \ldots, d_{n/2-1}$) $\leftarrow$ FFT(n/2, $a_1, a_3, a_5, \ldots, a_{n-1}$)

    for k = 0 to n/2 - 1 {
        $\omega^k \leftarrow e^{2\pi ik/n}$
        $y_k \leftarrow e_k + \omega^k d_k$
        $y_{k+n/2} \leftarrow e_k - \omega^k d_k$
    }

    return ($y_0, y_1, \ldots, y_{n-1}$)
}
```

FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the nth roots of unity in O(n log n) steps.

↑ assumes n is a power of 2

Pf. $T(2n) = 2T(n) + O(n) \Rightarrow T(n) = O(n \log n)$.



Point-Value to Coefficient Representation: Inverse DFT

Goal. Given the values $y_0, \ldots, y_{n-1}$ of a degree n-1 polynomial at the n points $\omega^0, \omega^1, \ldots, \omega^{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

↑ Inverse DFT

↑ Fourier matrix inverse $(F_n)^{-1}$

Inverse FFT

Claim. Inverse of Fourier matrix is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n \, G_n\right)_{k\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{k\,j} \, \omega^{-j\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} \underset{\substack{\uparrow \\ \text{summation lemma}}}{=} \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{k\,j} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ sums to n.
- Else: $\omega^k \neq 1$
  - $x^n - 1 = (x - 1)\,(1 + x + x^2 + \ldots + x^{n-1})$
  - Let $x = \omega^k$, we have $0 = \omega^{kn} - 1 = (\omega^k - 1)\,(1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)})$
  - Therefore, $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ sums to 0. ▪

Point-Value to Coefficient Representation: Inverse DFT

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Observation. Almost the same form as DFT.

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal nth root of unity (and divide by n).

Inverse FFT: Algorithm

```
IFFT(n, $a_0, a_1, \ldots, a_{n-1}$) {
    if (n == 1) return $a_0$

    $(e_0, e_1, \ldots, e_{n/2-1}) \leftarrow$ IFFT(n/2, $a_0, a_2, a_4, \ldots, a_{n-2}$)
    $(d_0, d_1, \ldots, d_{n/2-1}) \leftarrow$ IFFT(n/2, $a_1, a_3, a_5, \ldots, a_{n-1}$)

    for k = 0 to n/2 - 1 {
        $\omega^k \leftarrow e^{-2\pi ik/n}$
        $y_k \leftarrow (e_k + \omega^k d_k)$
        $y_{k+n/2} \leftarrow (e_k - \omega^k d_k)$
    }

    return $(y_0, y_1, \ldots, y_{n-1})$
}
```

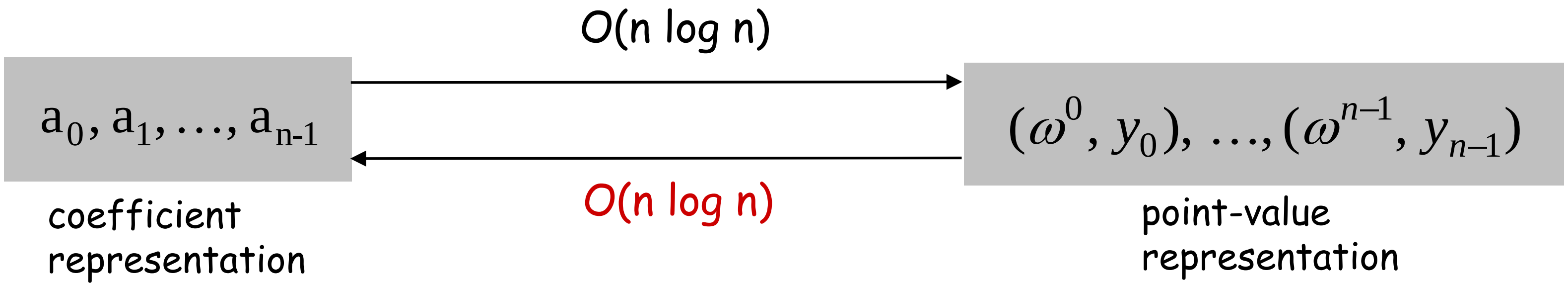
Note. Need to divide the final result by n.

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the nth roots of unity in O(n log n) steps.

↑

assumes n is a power of 2



O(n log n)

$a_0, a_1, \ldots, a_{n-1}$

$(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$

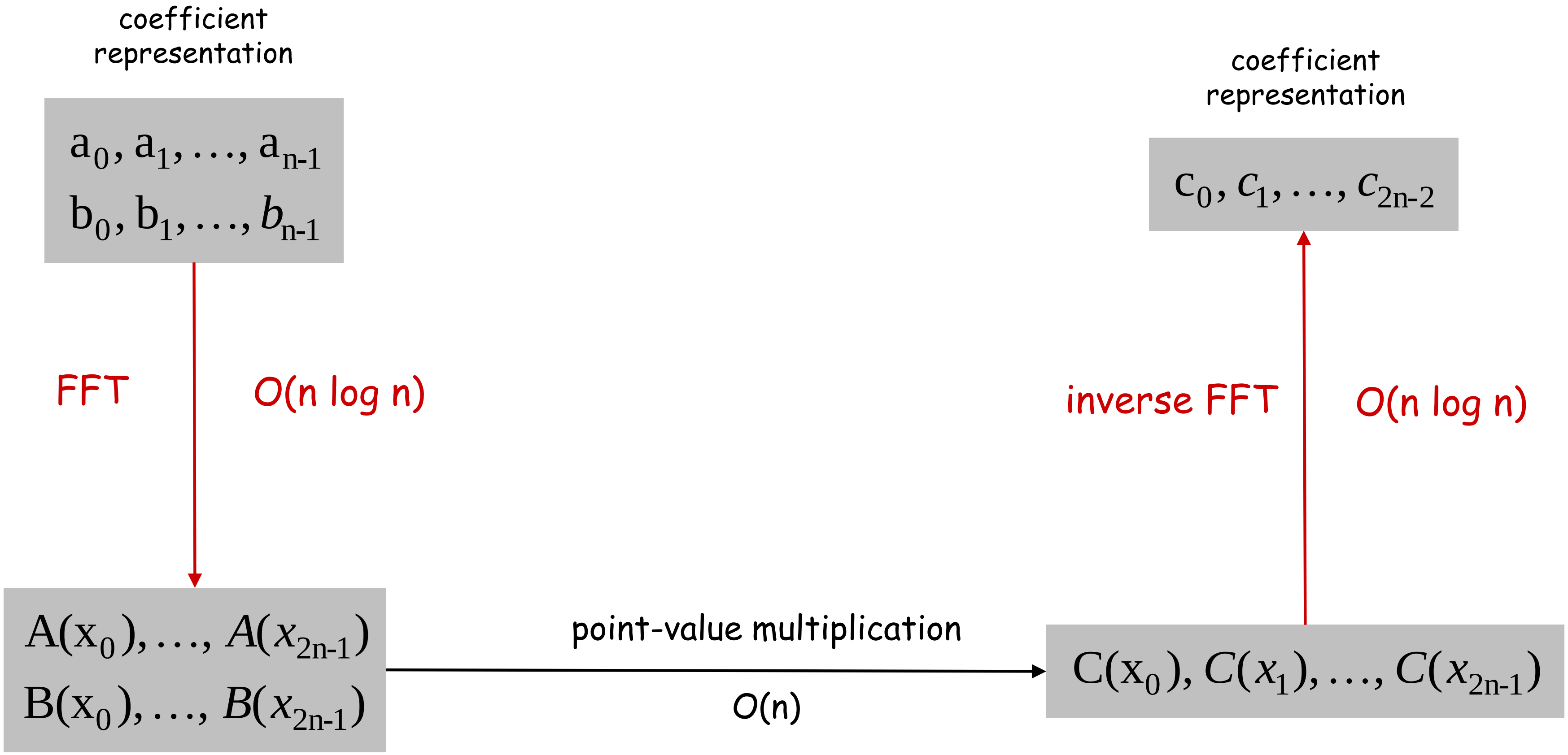
O(n log n)

coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in O(n log n) steps.



coefficient representation

$a_0, a_1, \ldots, a_{n-1}$

$b_0, b_1, \ldots, b_{n-1}$

FFT

O(n log n)

$A(x_0), \ldots, A(x_{2n-1})$

$B(x_0), \ldots, B(x_{2n-1})$

point-value multiplication

O(n)

$C(x_0), C(x_1), \ldots, C(x_{2n-1})$

inverse FFT

O(n log n)

coefficient representation

$c_0, c_1, \ldots, c_{2n-2}$

Integer Multiplication

Integer multiplication. Given two n bit integers a = $a_{n-1} \ldots a_1 a_0$ and b = $b_{n-1} \ldots b_1 b_0$, compute their product c = a × b.

Convolution algorithm.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- Form two polynomials.
- Note: a = A(2), b = B(2).
- Compute C(x) = A(x) × B(x).
- Evaluate C(2) = a × b.
- Running time: O(n log n) complex arithmetic steps.

Practice. [GNU Multiple Precision Arithmetic Library] GMP proclaims to be "the fastest bignum library on the planet." It uses brute force, Karatsuba, and FFT, depending on the size of n.

Divide-and-Conquer: Chapter Summary

Divide-and-Conquer

Basic idea

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Algorithms

- Mergesort
    - Divide a sequence into two of same size
- Closest Pair of Points
    - Vertically divide the space
- Integer Multiplication
    - Divide each n-digit integer into two $\frac{1}{2}$n-digit integers
- Matrix Multiplication
    - Divide each n-by-n matrix into four $\frac{1}{2}$n-by-$\frac{1}{2}$n blocks
- Fast Fourier Transform
    - Divide a polynomial into two with even and odd powers