Poisson Process

SI252 Reinforcement Learning

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Outline

Poisson Distribution

Definition

An r.v. X has the **Poisson distribution** with parameter λ if the PMF of X is

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \ \ k = 0, 1, 2, \cdots$$

We write this as $X \sim \mathrm{Pois}(\lambda)$.

Example: Poisson Expectation & Variance

Example (Poisson Expectation & Variance)

Consider an r.v. $X \sim \mathrm{Pois}(\lambda)$, find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Example: Poisson Expectation & Variance (Solution)

Mean:

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

Variance: $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\boxed{\sum_{k=1}^{\infty} k^2\frac{\lambda^k}{k!}}$$

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k\frac{\lambda^{k-1}}{k!} = e^{\lambda} \Rightarrow \sum_{k=1}^{\infty} k\frac{\lambda^k}{k!} = \lambda e^{\lambda}$$

Example: Poisson Expectation & Variance (Solution)

$$\sum_{k=1}^{\infty} k^2 \frac{\lambda^{k-1}}{k!} = e^{\lambda} + \lambda e^{\lambda} = (1+\lambda) e^{\lambda}$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} = e^{\lambda} \lambda (1+\lambda)$$

$$E(X^2) = e^{-\lambda} \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} = \lambda (1+\lambda)$$

$$Var(X) = \lambda(1+\lambda) - \lambda^2 = \lambda$$

Poisson Approximation

Theorem (Poisson Approximation)

Let $A_1, A_2, \cdots, A_n$ be events with $p_j = P(A_j)$, where n is large, the p_j are small, and the A_j are independent or weakly dependent. Let

$$X = \sum_{j=1}^{n} I(A_j)$$

I: indicator

count how many of the A_j occur. Then X is approximately $\mathrm{Pois}(\lambda)$, with $\lambda = \sum_{j=1}^{n} p_j$.

Example: Birthday Problem Revisited

Example (Birthday Problem Revisited)

There are m people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

Example: Birthday Problem Revisited (Solution)

$A_{ij} = \{$ person i and j have the same birthday $\}$

$X \triangleq$ number of birthday matches

$X = \sum_{i \neq j} I(A_{ij}) \approx \mathrm{Pois}\left(\frac{1}{365}\binom{m}{2}\right)$

$P_{ij} = P_r(A_{ij}) = \sum_{d=1}^{365} P_r(i\text{'s birthday is } d) P_r(j\text{'s birthday is } d)$

$= \sum_{d=0}^{365} \frac{1}{365} \times \frac{1}{365} = \frac{1}{365}$ $\quad \left(\lambda = \sum_{i \neq j} P_{ij}\right)$

$P(X \geq 1) = 1 - P(X=0) \approx 1 - e^{-\lambda}$

$m = 23 \quad 1 - e^{-\lambda} \approx 0.500002$

True value : 0.507

Sum of Independent Poissons

$$P(X+Y=k) = \sum_{j=0}^{k} P(X+Y=k \mid X=j)\, P(X=j) \qquad (\text{LOTP})$$

$$= \sum_{j=0}^{k} P(Y=k-j \mid X=j)\, P(X=j)$$

$$= \sum_{j=1}^{k} P(Y=k-j)\, P(X=j)$$

Theorem (Sum of Independent Poissons)

If $X \sim \mathrm{Pois}(\lambda_1)$, $Y \sim \mathrm{Pois}(\lambda_2)$, *and* X *is independent of* Y, *then* $X + Y \sim \mathrm{Pois}(\lambda_1 + \lambda_2)$.

$$= \sum_{j=1}^{k} \frac{e^{-\lambda_2}\lambda_2^{k-j}}{(k-j)!} \cdot \frac{e^{-\lambda_1}\lambda_1^{j}}{j!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^{j} \lambda_2^{k-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1+\lambda_2)^k \quad \Rightarrow \quad X+Y \sim \mathrm{Pois}(\lambda_1+\lambda_2)$$

$$\forall k = 0, 1, 2, \ldots$$

Poisson Given A Sum of Poissons

$$P(X=k \mid X+Y=n) = \frac{P(X+Y=n \mid X=k)\,P(X=k)}{P(X+Y=n)}$$

$$= \frac{P(Y=n-k)\,P(X=k)}{P(X+Y=n)}$$

$X+Y \sim \mathrm{Pois}(\lambda_1+\lambda_2)$

Theorem (Poisson Given A Sum of Poissons)

If $X \sim \mathrm{Pois}(\lambda_1)$, $Y \sim \mathrm{Pois}(\lambda_2)$, and X is independent of Y, then the conditional distribution of X given $X+Y=n$ is $\mathrm{Bin}(n, \lambda_1/(\lambda_1+\lambda_2))$.

$$= \frac{\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\cdot\frac{e^{-\lambda_1}\lambda_1^{k}}{k!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

PMF

$\mathrm{Bin}\left(n, \frac{\lambda_1}{\lambda_2+\lambda_2}\right)$

Poisson Approximation to Binomial

Theorem (Poisson Approximation to Binomial)

If $X \sim \mathrm{Bin}(n, p)$ and we let $n \to \infty$ and $p \to 0$ such that $\boxed{\lambda = np}$ remains fixed, then the PMF of X converges to the $\mathrm{Pois}(\lambda)$ PMF. More generally, the same conclusion holds if $n \to \infty$ and $p \to 0$ in such a way that np converges to a constant λ.

Poisson Approximation to Binomial (Proof)

$\lambda = np$ is fixed while $n \to \infty$, $p \to 0$.

$X \sim Bin(n, p)$

$\lambda = np$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\to 1} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\to 1}$$

Letting $n \to \infty$ with k fixed:

$$\frac{n(n-1)\cdots(n-k+1)}{n^k} \to 1$$

$$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

$$\left(1-\frac{\lambda}{n}\right)^{-k} \to 1$$

$$P(X=k) \to e^{-\lambda}\frac{\lambda^k}{k!} \quad \forall k = 0, 1, 2, \cdots$$

$Pois(\lambda)$

Example: Visitors to A Website

$X \triangleq$ number of visitors $\quad X \sim Bin(n, p) \quad n = 10^6 \quad p = 2 \times 10^{-6}$

$Pois(np) \Rightarrow Pois(2)$

Example (Visitors to A Website)

The owner of a certain website is studying the distribution of the number of visitors to the site. Every day, a million people independently decide whether to visit the site, with probability $p = 2 \times 10^{-6}$ of visiting. Give a good approximation for the probability of getting *at least three* visitors on a particular day.

$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2) \approx 1 - 5e^{-2}$

Outline

Definition (Poisson Process-Definition 1)

A Poisson process with parameter λ is a counting process $(N_t)_{t\geq 0}$ with the following properties:

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with parameter λt.
3. (Stationary increments) For all $s, t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t. That is,

$$P(N_{t+s} - N_s = k) = P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad \text{for} \quad k = 0, 1, \ldots$$

4. (Independent increments) For $0 \leq q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent random variables.

Example

$N_t \triangleq$ number of texts in a interval of t hours

$N_t \sim \text{Pois}(\lambda t)$ $\quad (N_t)_t \quad \lambda = 10$

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon and 70 texts by 5 p.m. $\lambda = 10$

12:00

$$P(N_2 = 18, N_7 = 70) = P(N_2 = 18, N_7 - N_2 = 52)$$

$$= P(N_2 = 18)\, P(N_7 - N_2 = 52)$$

$$= P(N_2 = 18)\, P(N_5 = 52)$$

$$= 0.0095$$

Definition (Translated Poisson Process)

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. For $s > 0$, let $\tilde{N}_t = N_{t+s} - N_s$, for $t \geq 0$. Then we have

- $(\tilde{N}_t)_{t\geq 0}$ is called "Translated Poisson Process".
- $(\tilde{N}_t)_{t\geq 0}$ is a Poisson process with parameter λ.

Example

$N_t \triangleq$ number of arrivals in the first t hours

$(N_t)_{t\geq 0}$ $\quad \lambda = 100$

Example

$\lambda = 100$

On election day, people arrive at a voting center according to a Poisson process. On average, 100 voters arrive every hour. If 150 people arrive during the first hour, what is the probability that at most 350 people arrive before the third hour?

$$P(N_3 \leq 350 \mid N_1 = 150) = P(N_3 - N_1 \leq 200 \mid N_1 = 150)$$

$$= P(N_3 - N_1 \leq 200) = P(N_2 \leq 200)$$

$$= \sum_{k=0}^{200} P(N_2 = k) = \sum_{k=0}^{200} e^{-100(2)} \frac{(100(2))^k}{k!}$$

Definition (Poisson Process-Definition 2)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables with parameter λ. For $t > 0$, let

$X_i \sim Expo(\lambda)$

$$N_t = max\{n : X_1 + \ldots + X_n \leq t\},$$

with $N_0 = 0$. Then, $(N_t)_{t \geq 0}$ defines a Poisson process with parameter λ.

Definition 2 $\Rightarrow$ Definition 1

Assume Definition 2.

$X_1, X_2, \cdots$ a sequence of i.i.d. r.v.s $X_i \sim Expo(\lambda)$

$S_n \triangleq X_1 + \cdots + X_n \quad S_0 \triangleq 0$

$N_t = \max\{n : S_n \le t\}$

We need to show that N_t has a Poisson distribution with para. λt.

$N_t = k \Leftrightarrow S_k \le t < S_{k+1} \qquad S_{k+1} = S_k + X_{k+1}$

S_k is independent of X_{k+1}. $\quad S_k \sim Gamma(k, \lambda) \quad X_{k+1} \sim Expo(\lambda)$

$$f_{S_k, X_{k+1}}(s, x) = f_{S_k}(s) f_{X_{k+1}}(x) = \frac{\lambda^k s^{k-1} e^{-\lambda s}}{(k-1)!} \lambda e^{-\lambda x} \qquad s, x > 0$$

Definition 2 $\Rightarrow$ Definition 1

$\forall\, k \geq 0$

$$P(N_t = k) = P(S_k \leq t < S_k + X_{k+1})$$

$$= P(S_k \leq t,\ X_{k+1} > t - S_k)$$

(LOTP)
$$= \int_0^\infty \int_0^\infty P(S_k \leq t,\ X_{k+1} > t - S_k \mid S_k = s,\ X_{k+1} = x)\, f_{S_k, X_{k+1}}(s, x)\, dx\, ds$$

$$= \int_0^t \int_{t-s}^\infty f_{S_k, X_{k+1}}(s, x)\, dx\, ds$$

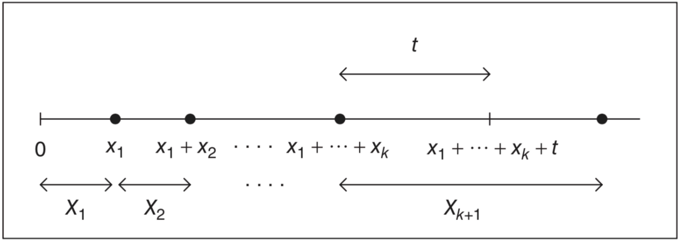
$$= \frac{e^{-\lambda t} (\lambda t)^k}{k!} \Rightarrow N_t \sim \mathrm{Pois}(\lambda t)$$

Definition 1 $\Rightarrow$ Definition 2

Assume Definition 1. $N_t \sim Pois(\lambda t)$

We need to show $X_1, X_2, \ldots$ is a sequence of iid r.v.s

$X_i \sim Expo(\lambda)$



For X_1: $P(X_1 > t) = P(N_t = 0) = e^{-\lambda t}$ $\quad X_1 \sim Expo(\lambda)$

For X_2: $P(X_2 > t \mid X_1 = s) = P(\text{no arrival during interval } (s, s+t] \mid X_1 = s)$

$= P(N_{s+t} - N_s = 0 \mid X_1 = s)$

$= P(N_{s+t} - N_s = 0) = P(N_t = 0) = e^{-\lambda t}$

$X_2 \sim Expo(\lambda)$

Definition 1 $\Rightarrow$ Definition 2

$P(X_{k+1} > t \mid X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k)$

$= P(\text{no arrival during interval } (x_1 + \cdots + x_k, x_1 + \cdots + x_k + t] \mid \cdots)$

$= P(N_{x_1 + \cdots + x_k + t} - N_{x_1 + \cdots + x_k} = 0 \mid \cdots)$

$= P(N_{x_1 + \cdots + x_k + t} - N_{x_1 + \cdots + x_k} = 0)$

$= P(N_t = 0) = e^{-\lambda t}$

X_{k+1} is independent of $X_1, \cdots, X_k$ $\quad$ $X_{k+1} \sim \text{Expo}(\lambda)$

$\forall k \geq 2$

$X_1, X_2, \cdots$ i.i.d. $\text{Expo}(\lambda)$ (Definition 2)

Conditional Counts

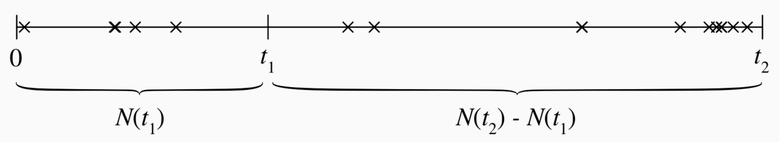
Theorem (Conditional Counts) $N(t) = N_t$

Let $\{N(t), t > 0\}$ be a Poisson process with rate λ, and let $t_1 < t_2$. Then the conditional distribution of $N(t_1)$ given $N(t_2) = n$ is

$$N(t_1) \,|\, N(t_2) = n \sim \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right).$$

Conditional Counts

$$N(t_1) \mid \{N(t_2)=n\} \sim \underline{\mathrm{Bin}\left(n, \frac{t_1}{t_2}\right)}$$



$$X \triangleq N(t_1) \sim \mathrm{Pois}(\lambda t_1) \qquad Y \triangleq N(t_2) - N(t_1) \sim \mathrm{Pois}(\lambda(t_2 - t_1))$$

$$N(t_1) \mid \{N(t_2)=n\} = X \mid \{X+Y=n\} \sim \mathrm{Bin}\left(n, \frac{\lambda t_1}{\lambda t_1 + \lambda(t_2 - t_1)}\right)$$

$$= \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right)$$

Arrival Times & Uniform Distribution

Theorem (Arrival Times & Uniform Distribution)

Let $S_1, S_2, \ldots,$ be the arrival times of a Poisson process with parameter λ. Conditional on $N_t = n$, the joint distribution of $(S_1, \ldots, S_n)$ is the distribution of the order statistics of n i.i.d. uniform random variables on $[0, t]$. That is, the joint density function of $S_1, \ldots, S_n$ is

$$f(s_1, \ldots, s_n) = \frac{n!}{t^n}, \quad \text{for } 0 < s_1 < \ldots < s_n < t.$$

Equivalently, let $U_1, \ldots, U_n$ be an i.i.d.sequence of random variables uniformly distributed on $[0, t]$. Then, conditional on $N_t = n$,

$$(S_1, \ldots, S_n) \quad \text{and} \quad (U_{(1)}, \ldots, U_{(n)})$$

have the same distribution.

Arrival Times & Uniform Distribution (Proof)

$$f_{S_1,\cdots,S_n}(s_1,\cdots,s_n \mid N_t=n)$$

$$= \frac{\displaystyle\lim_{\varepsilon_1\to 0}\cdots\lim_{\varepsilon_n\to 0}\frac{P(s_1\le S_1\le s_1+\varepsilon_1,\cdots,s_n\le S_n\le s_n+\varepsilon_n \mid N_t=n)}{\varepsilon_1\cdots\varepsilon_n}}{0<s_1<\cdots<s_n<t}$$

$$\frac{P(s_1\le S_1\le s_1+\varepsilon_1,\cdots,s_n\le S_n\le s_n+\varepsilon_n,\ N_t=n)}{P(N_t=n)}$$

$$= \frac{P(\text{one arrival in each interval } \underbrace{[s_k,s_k+\varepsilon_k]} \text{ and no arrival in rest intervals})}{P(N_t=n)}$$

$$= \frac{P(N_{s_1+\varepsilon_1}-N_{s_1}=1)\cdots P(N_{s_n+\varepsilon_n}-N_{s_n}=1)P(N_{t-\varepsilon_1-\cdots-\varepsilon_n}=0)}{P(N_t=n)}$$

$$= \frac{P(N_{\varepsilon_1}=1)\cdots P(N_{\varepsilon_n}=1)\ P(N_{t-\varepsilon_1-\cdots-\varepsilon_n}=0)}{} \qquad N_t\sim Pois(\lambda t)$$

$$= \frac{n!}{t^n}\varepsilon_1\cdots\varepsilon_n$$

Arrival Times & Uniform Distribution (Proof)

$$\underbrace{f_{S_1,\cdots,S_n}(s_1,\cdots,s_n \mid N_t = n)}$$

$$= \lim_{\varepsilon_1 \to 0} \cdots \lim_{\varepsilon_n \to 0} \frac{\frac{n!}{t^n}\varepsilon_1 \cdots \varepsilon_n}{\varepsilon_1 \cdots \varepsilon_n} = \underbrace{\frac{n!}{t^n}}$$

$$f_{U_1 \cdots U_n}(u_1, \cdots, u_n) = \frac{1}{t^n} \qquad U_i \sim \mathrm{Unif}(0,t)$$

$$f_{U_{(1)} \cdots U_{(n)}}(u_1, \cdots, u_n) = \underbrace{\frac{1}{t^n} \cdot n!}$$

Arrival Times & Uniform Distribution (Proof)

Example

Example

Students enter a campus building according to a Poisson process $(N_t)_{t\geq 0}$ with parameter λ. The times spent by each student in the building are i.i.d. random variables with continuous cumulative distribution function $F(t)$. Find the probability mass function of the number of students in the building at time t, assuming there are no students in the building at time 0.

$N_0 = 0$

Example (Solution)

$B_t \triangleq$ number of students in the building at time t.

By LOTP:

$(N_t)_t$ Poisson process

$N_t \sim \text{Pois}(\lambda t)$

$$P(B_t = k) = \sum_{n=k}^{\infty} P(B_t = k \mid N_t = n)\, P(N_t = n)$$

$$= \sum_{n=k}^{\infty} P(B_t = k \mid N_t = n) \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Assume that n students enter the building by time t, with arrival times $S_1, \cdots, S_n$.

$Z_k \triangleq$ length of time spent in the building by the k-th student

Students leave the building at times $S_1 + Z_1, \cdots, S_n + Z_n$

$$P(B_t = k \mid N_t = n) = P(k \text{ of the } S_1 + Z_1, \cdots, S_n + Z_n \text{ exceed } t \mid N_t = n)$$

$$= P(k \text{ of the } U_{(1)} + Z_1, \cdots, U_{(n)} + Z_n \text{ exceed } t)$$

$U_k \sim \text{Unif}[0, t]$

Example (Solution)

$$\underline{(U_{(1)}, \cdots, U_{(n)})} \qquad \underline{(U_1, \cdots, U_n)}$$

$$P(B_t = k \mid N_t = n) = P(\underline{k \text{ of the } U_1 + Z_1, \cdots, U_n + Z_n \text{ exceed } t})$$

$$B_t \mid \{N_t = n\} \sim Bin(n, p)$$

$$p = P(U_1 + Z_1 > t) = P(Z_1 > t - U_1)$$

$$= \int_0^t P(Z_1 > t - U_1 \mid U_1 = s) \, \underline{f_{U_1}(s)} \, ds$$

$$= \frac{1}{t} \int_0^t (1 - P(Z_1 \le t - s)) \, ds$$

$$= \frac{1}{t} \int_0^t (1 - F(t-s)) \, ds = \underline{\frac{1}{t} \int_0^t (1 - F(s)) \, ds}$$

$$\underline{P(B_t = k \mid N_t = n)} = \binom{n}{k} p^k (1-p)^{n-k}$$

Example (Solution)

$$P(B_t = k) = \frac{e^{-\lambda p t}(\lambda p t)^k}{k!}$$

$$B_t \sim \mathrm{Pois}(\lambda p t)$$