Quiz 4

Sampling Suppose we have two independent random variables ξ_1 and ξ_2 that are sampled from the standard uniform distribution $\xi_i \sim U(0,1)$. Given a hemisphere with radius $r = 1$, we wish to have a sampling distribution $p(\omega) \propto \frac{\theta}{\sin\theta}$ with respect to solid angle ω on the hemisphere, where $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi]$ are zenith and azimuth angle, respectively. Please write down the Cartesian coordinate of the sampled point with regard to ξ_1 and ξ_2.

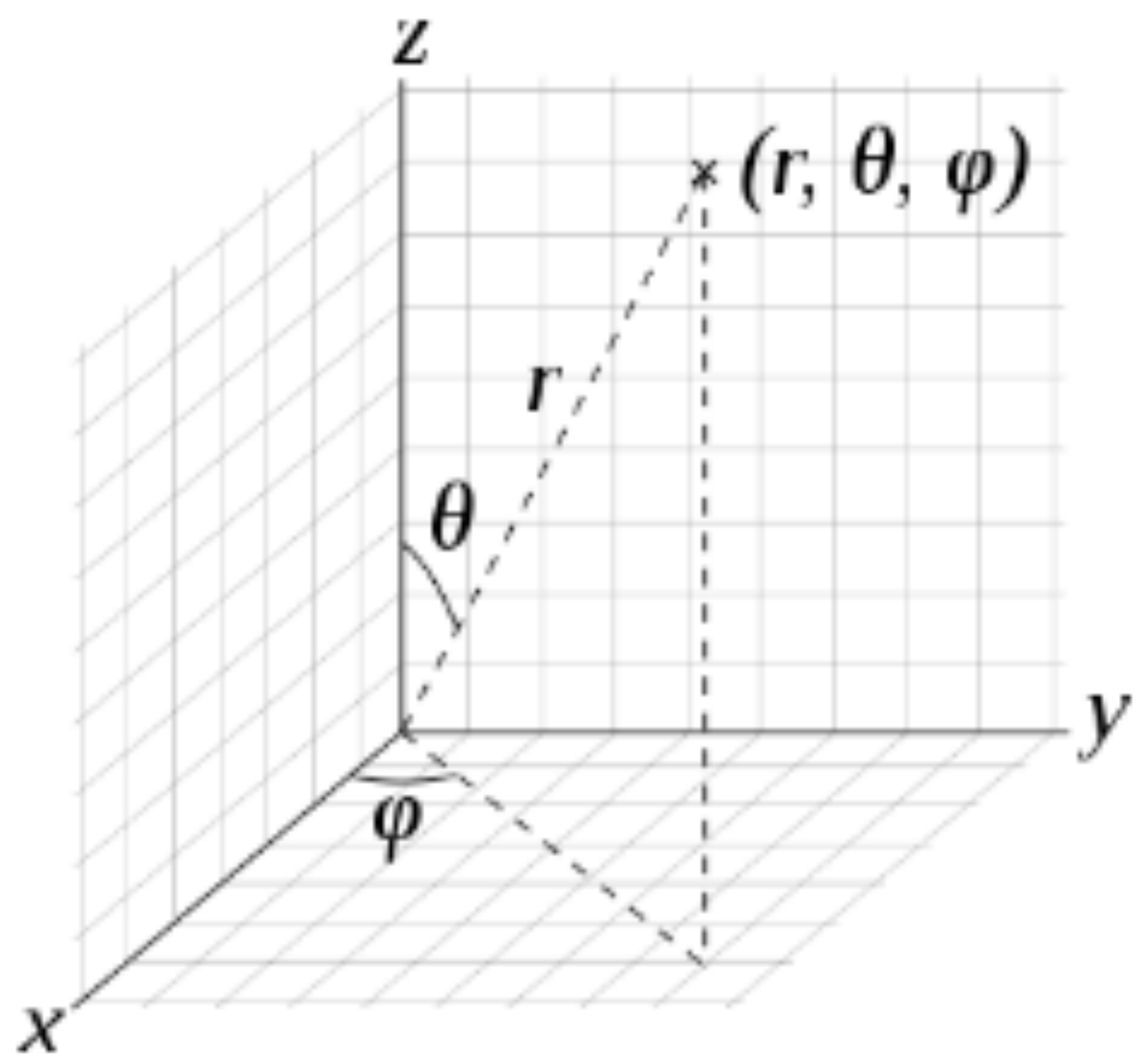
Hints:

(1) $d\omega = \sin\theta d\theta d\phi$

(1) $p(\theta, \phi) = \sin\theta\, p(\omega)$

(3) Spherical coordinates (r, θ, ϕ) to Cartesian coordinates (x, y, z) transformation:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



Quiz 4

Universality of the Uniform

1°. $F(x) \in [0,1]$

2°. $U \sim \text{Unif}(0,1)$ $\left(P(U \le a) = a, \ a \in [0,1] \right)$

① $U \sim \text{Unif}(0,1)$, $X = F^{-1}(U)$, compute CDF of X.

$F_X(x) = P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x))$

$= F(x)$

Theorem

Let F be a CDF which is a continuous function and strictly increasing on the support of the distribution. This ensures that the inverse function F^{-1} exists, as a function from $(0,1)$ to $\mathbb{R}$. We then have the following results.

1. *Let $U \sim \text{Unif}(0,1)$ and $X = F^{-1}(U)$. Then X is an r.v. with CDF F.*
2. *Let X be an r.v. with CDF F. Then $F(X) \sim \text{Unif}(0,1)$.*

$P(X \le a) = F(a)$

② $Y = F(X) \in [0,1]$ $\quad y \in \mathbb{R}$ $\quad P(Y \le y) = 0 \quad y \le 0$

$P(Y \le y) = 1 \quad y \ge 1.$

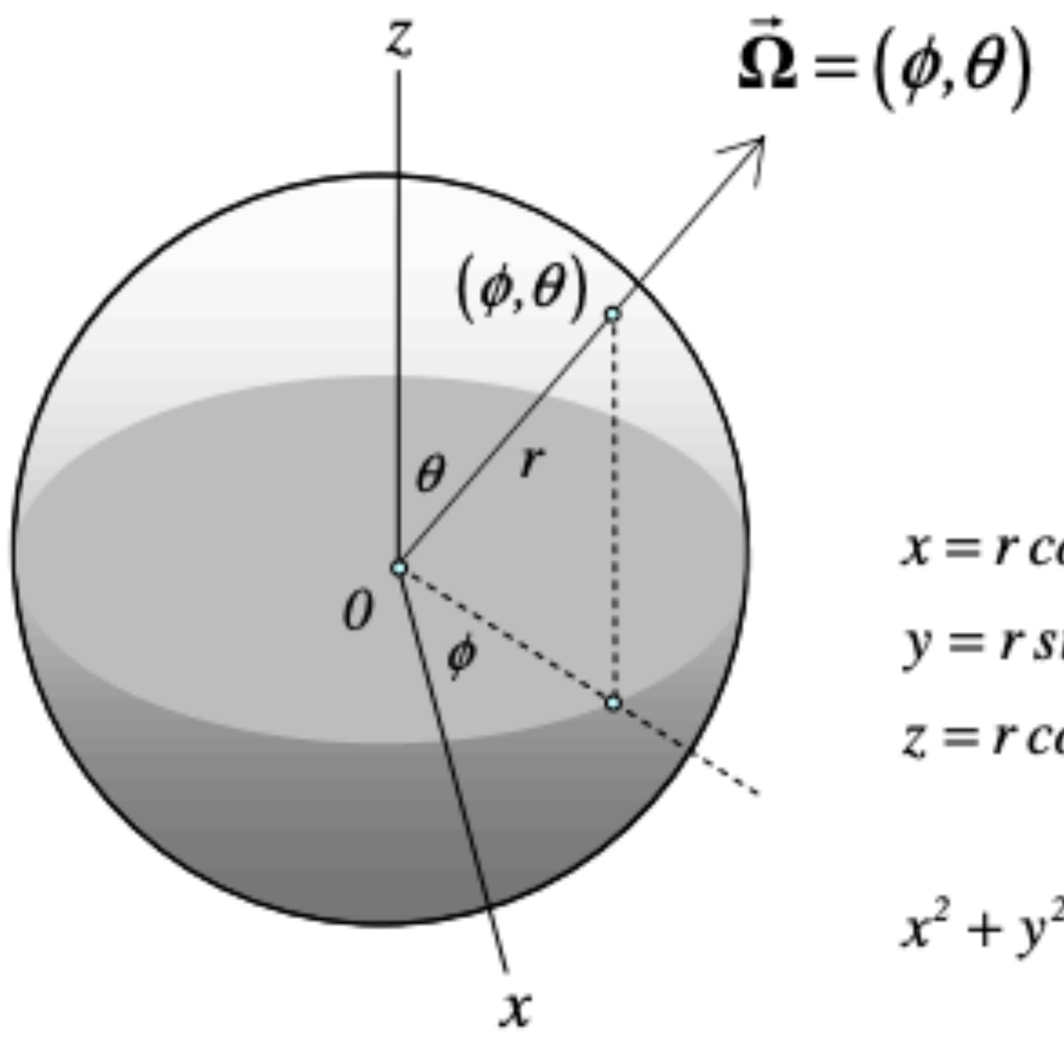
$y \in (0,1)$ $\quad P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F[F^{-1}(y)] = y$

$Y \sim \text{unif}(0,1)$

Quiz 4

Differential Solid Angle in spherical coordinates

Direction is defined by
a pair of angles: $\vec{\Omega} = (\phi, \theta)$
ϕ *is an azimutal angle:* $0 \le \phi \le 2\pi$
θ *is a polar angle:* $0 \le \theta \le \pi$

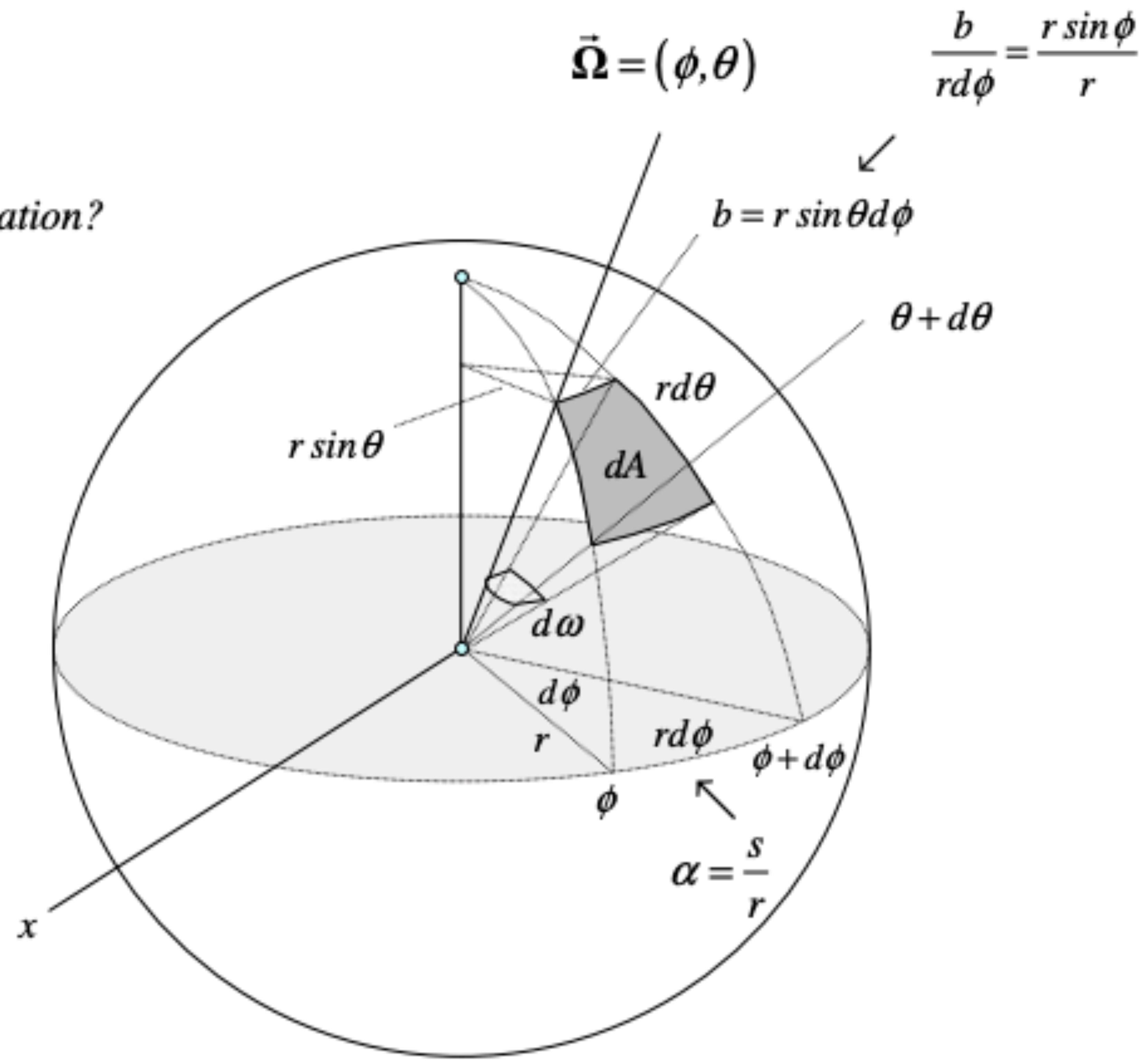


$x = r\cos\phi\sin\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\theta$

$x^2 + y^2 + z^2 = r^2$

Consider a differential
variation of the direction $\vec{\Omega} = (\phi, \theta)$
$\phi + d\phi$ *and* $\theta + d\theta$

What solid anlgle
corresponds to this variation?



$$dA \approx (r\sin\theta d\phi)\cdot(rd\theta) = r^2\sin\theta d\phi d\theta$$

differential solid angle

$$\boxed{d\omega = \frac{dA}{r^2} = \sin\theta d\theta d\phi} \qquad (12.8)$$

Quiz 4

Let $p(\omega) = c\frac{\theta}{sin\theta}$. Denote P as the CDF of $p(\omega)$.

$$\int_{\Omega} p(\omega)d\omega = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} p(\theta,\phi)sin\theta d\theta d\phi = 1 \implies c = \frac{4}{\pi^3} \quad p(\theta,\phi) = \frac{4}{\pi^3}\theta$$

$$p(\theta) = \int_0^{2\pi} p(\theta,\phi)d\phi = \frac{8\theta}{\pi^2} \qquad p(\phi|\theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{1}{2\pi}$$

$$P(\theta) = \int_0^{\pi/2} p(\theta')d\theta' = \frac{4}{\pi^2}\theta^2 \qquad P(\phi|\theta) = \int_0^{2\pi} p(\phi'|\theta)d\phi' = \frac{1}{2\pi}\phi$$

By Universality of Uniform: $P(\theta) = \xi_1 \quad P(\phi|\theta) = \xi_2 \implies \theta = \frac{\pi\sqrt{\xi_1}}{2}, \phi = 2\pi\xi_2$

$$\begin{cases} x = r\cos 2\pi\xi_2 \sin\frac{\pi\sqrt{\xi_1}}{2} \\ y = r\sin 2\pi\xi_2 \sin\frac{\pi\sqrt{\xi_1}}{2} \\ z = r\cos 2\frac{\pi\sqrt{\xi_1}}{2} \end{cases}$$

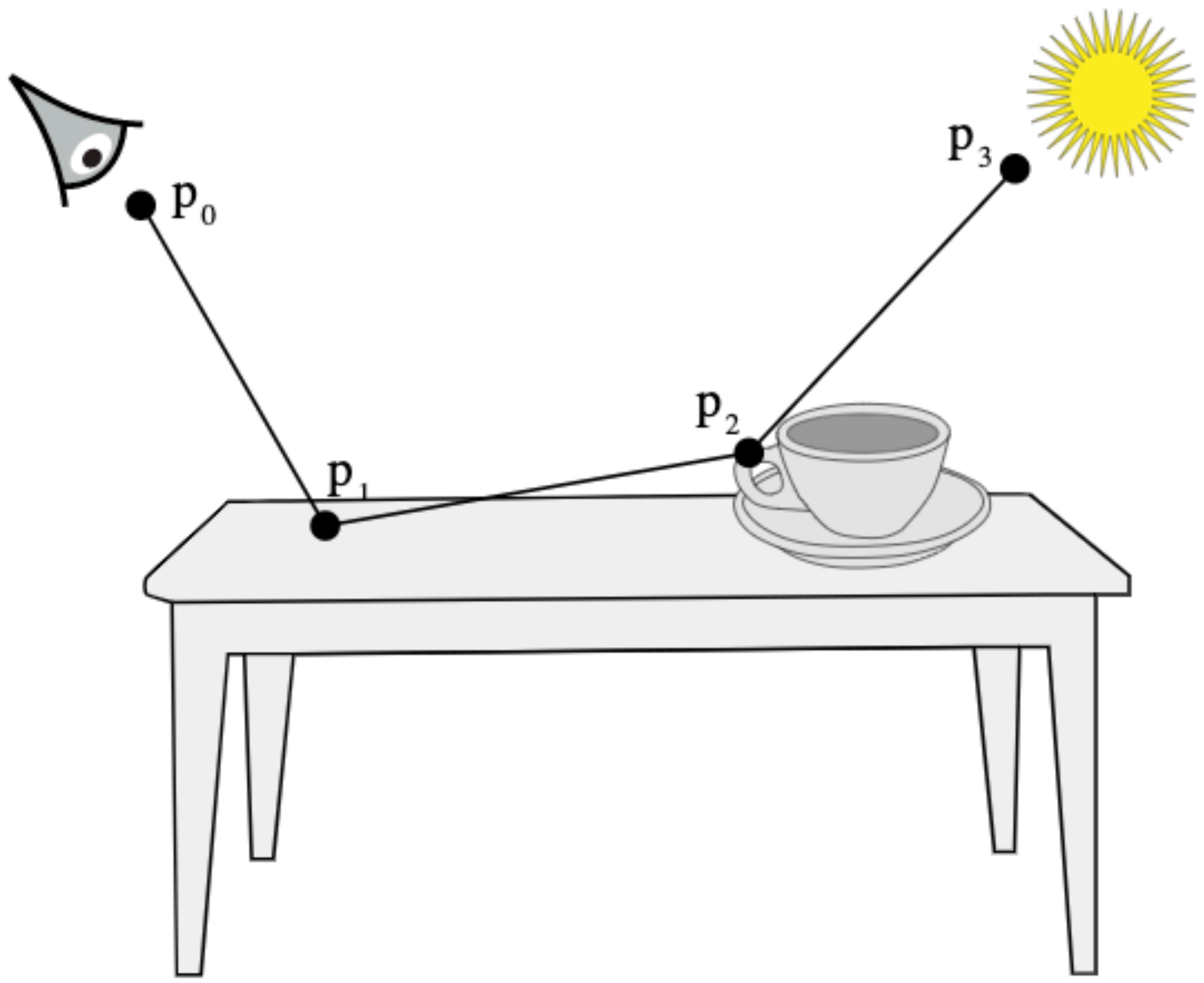
Global illumination

Path tracing & Direct lighting

Yuehao Wang; Apr 29

LTE

- Light transport equation
  - $L_o(p, w_o) = L_e(p, w_o) + \int_\Omega f(p, w_o, w_i)L_i(p, w_i)|cos\theta_i|dw_i$
  - Basic idea: out = reflection + emission
  - Notice that $L_o(p, w_o)$ and $L_i(p, w_i)$ share the same light field but from different solid angles.
  - => Recursive relation



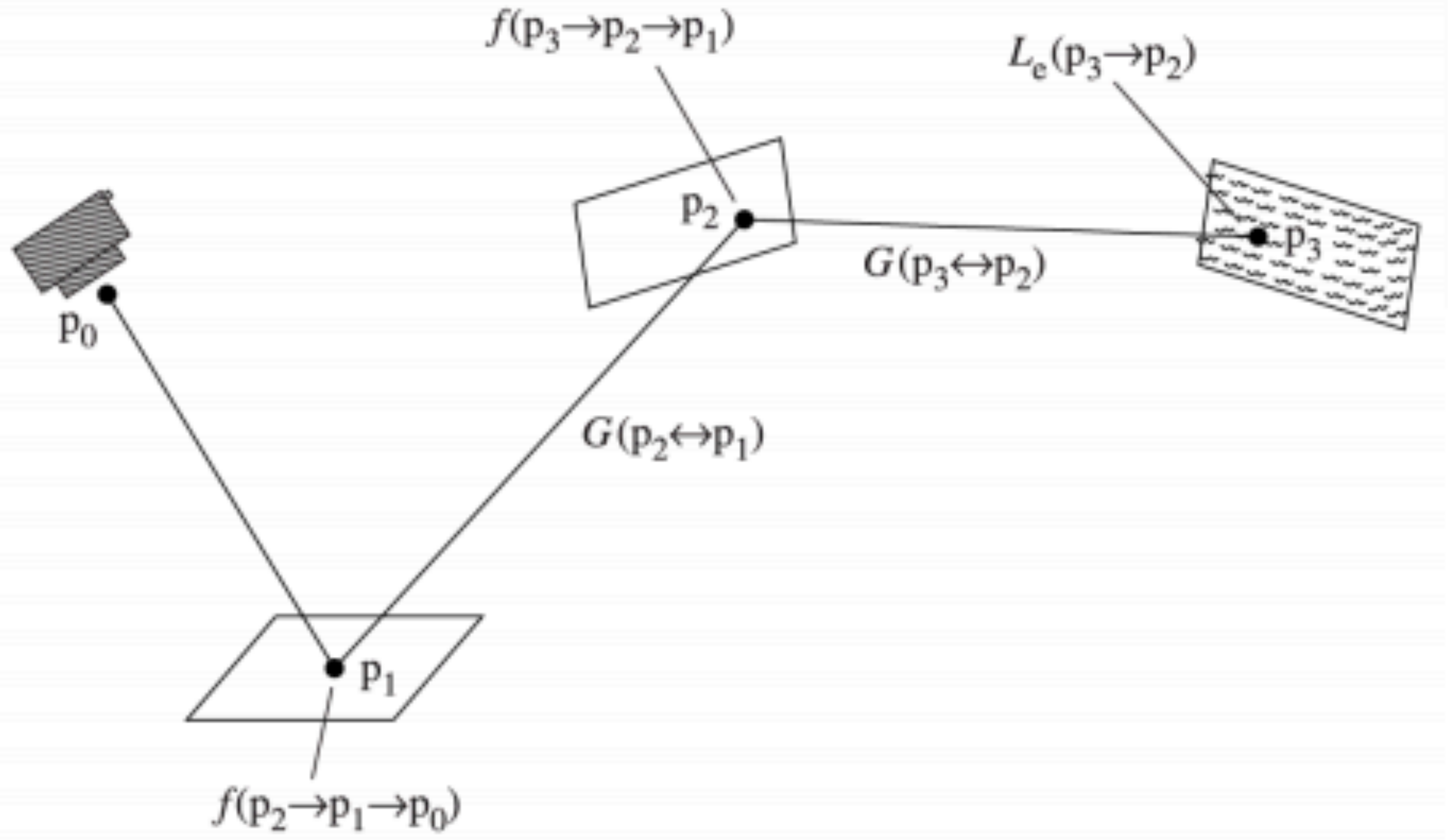
LTE

- In path-tracing, we consider the radiance brought by paths for each pixel.

- Consider a path p0…p3 in the right figure.
  - Compute the radiance from p1 to p0.
  - 2 times reflection

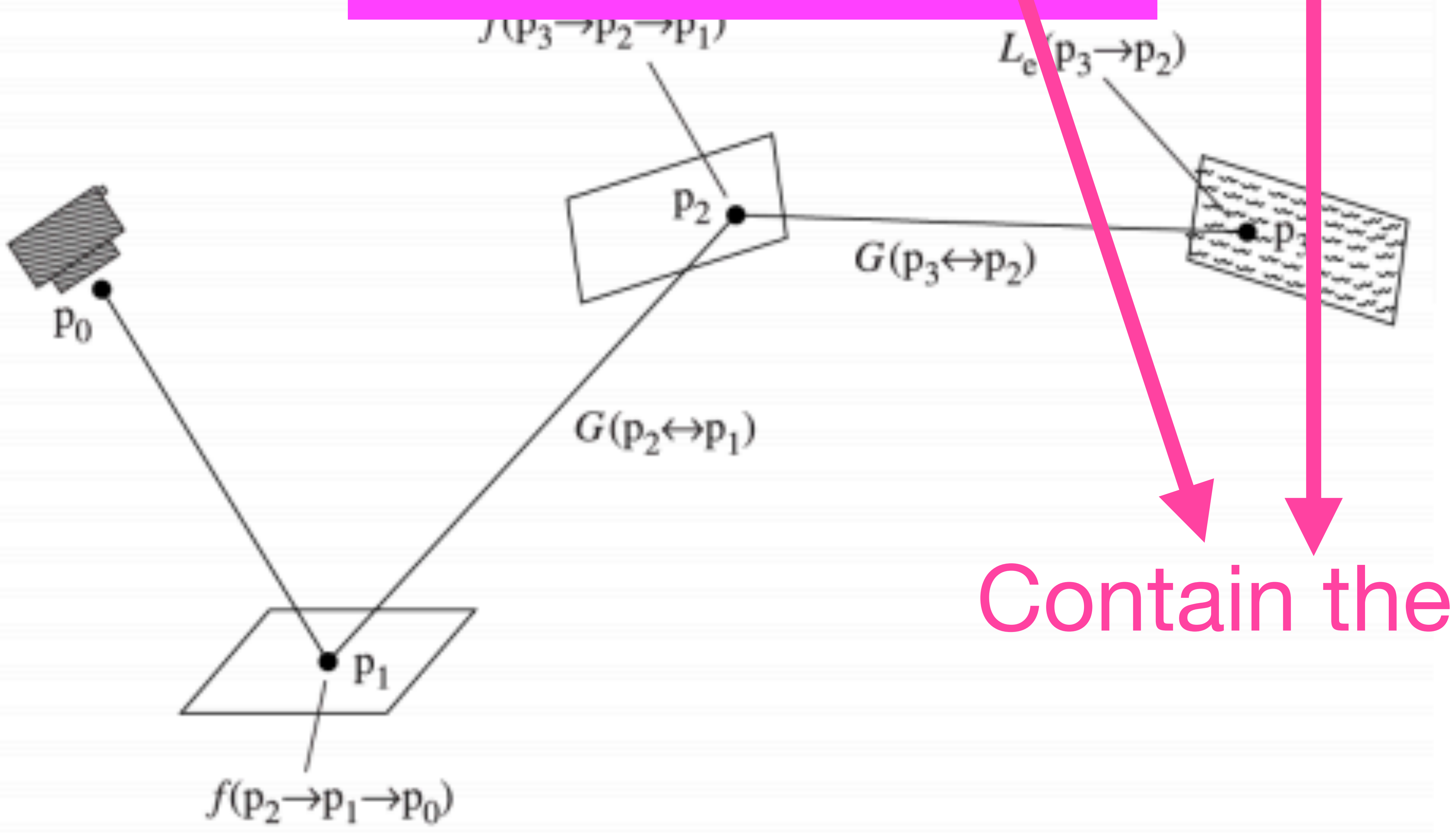
$$
\begin{aligned}
L(\mathrm{p}_1 \to \mathrm{p}_0) = {} & L_e(\mathrm{p}_1 \to \mathrm{p}_0) \\
& + \int_A L_e(\mathrm{p}_2 \to \mathrm{p}_1) f(\mathrm{p}_2 \to \mathrm{p}_1 \to \mathrm{p}_0) G(\mathrm{p}_2 \leftrightarrow \mathrm{p}_1) \mathrm{d}A(\mathrm{p}_2) \\
& + \int_A \int_A L_e(\mathrm{p}_3 \to \mathrm{p}_2) f(\mathrm{p}_3 \to \mathrm{p}_2 \to \mathrm{p}_1) G(\mathrm{p}_3 \leftrightarrow \mathrm{p}_2) \\
& \qquad \times f(\mathrm{p}_2 \to \mathrm{p}_1 \to \mathrm{p}_0) G(\mathrm{p}_2 \leftrightarrow \mathrm{p}_1)\, \mathrm{d}A(\mathrm{p}_3)\, \mathrm{d}A(\mathrm{p}_2) + \cdots
\end{aligned}
$$

Each term represents
A path of increasing length



LTE

$$
\begin{aligned}
L(\mathrm{p}_1 \to \mathrm{p}_0) = {} & L_e(\mathrm{p}_1 \to \mathrm{p}_0) \\
& + \int_A L_e(\mathrm{p}_2 \to \mathrm{p}_1) \boxed{f(\mathrm{p}_2 \to \mathrm{p}_1 \to \mathrm{p}_0) G(\mathrm{p}_2 \leftrightarrow \mathrm{p}_1)} \, dA(\mathrm{p}_2) \\
& + \int_A \int_A L_e(\mathrm{p}_3 \to \mathrm{p}_2) f(\mathrm{p}_3 \to \mathrm{p}_2 \to \mathrm{p}_1) G(\mathrm{p}_3 \leftrightarrow \mathrm{p}_2) \\
& \qquad \times \boxed{f(\mathrm{p}_2 \to \mathrm{p}_1 \to \mathrm{p}_0) G(\mathrm{p}_2 \leftrightarrow \mathrm{p}_1)} \, dA(\mathrm{p}_3) \, dA(\mathrm{p}_2) + \cdots
\end{aligned}
$$



Each term represents
A path of increasing length

Contain the same term

LTE

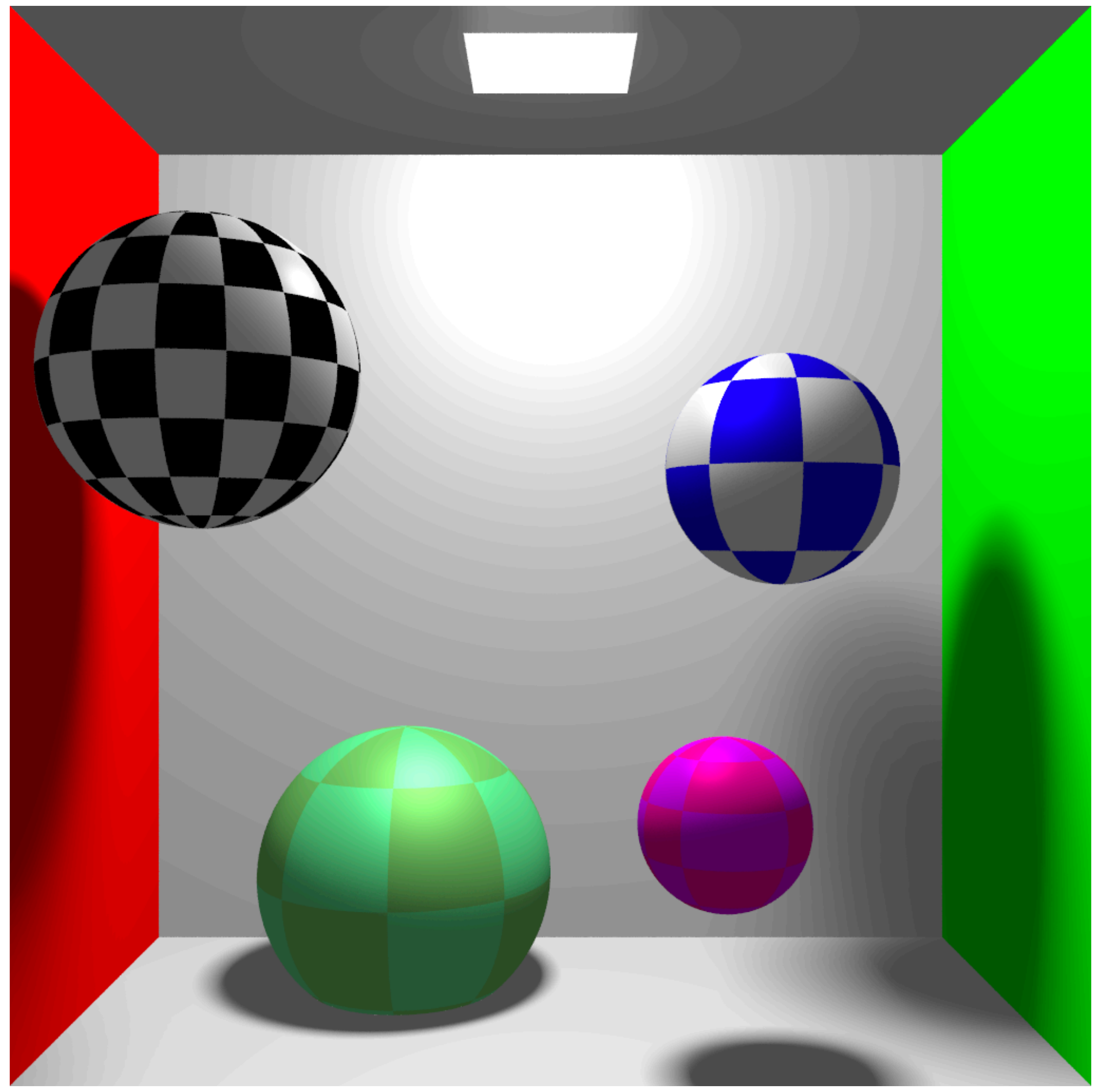
- How to solve the integration?
  - In discrete case, integration is “sum”.
  - If we cloud find all paths, just simply sum the radiance brought by all the paths. But impossible:
    - Consider diffusion: there are infinite possibilities of paths.
  - Monte-Carlo integration:
    - Sample several paths and weighted them by their Pdf.

Path-tracing

- Sample multiple paths for each pixel and compute the contribution of each path. The contribution is weighted by the sampling Pdf.
- Set a max depth. Depth = times of reflection
- Implementation
  - Beta = 1, L = 0
  - For each reflection at p
    - (1) L += Beta * Direct lighting at p
    - (2) Sample the next ray according to BRDF and find the corresponding Pdf
    - (3) Beta *= BRDF *cosΘ / Pdf
    - (4) Spawn the new ray

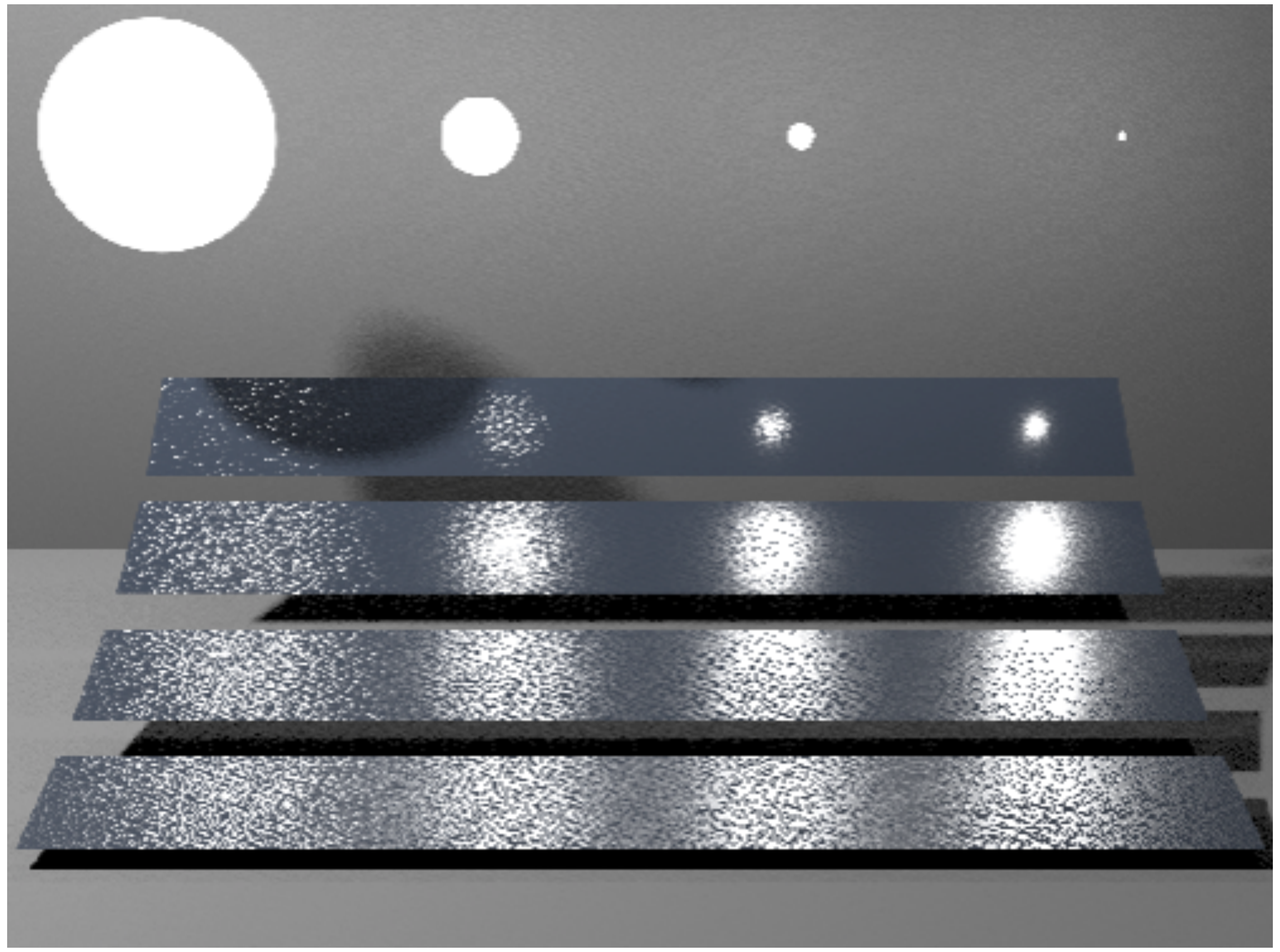
Direct Lighting

- Simplified method:
  - Phong shading at p
    - Generate new rays connecting the point p and light samples.
    - Check whether the new ray is occluded by other objects.
    - If unoccluded, utilize Phong shading to compute the radiance.



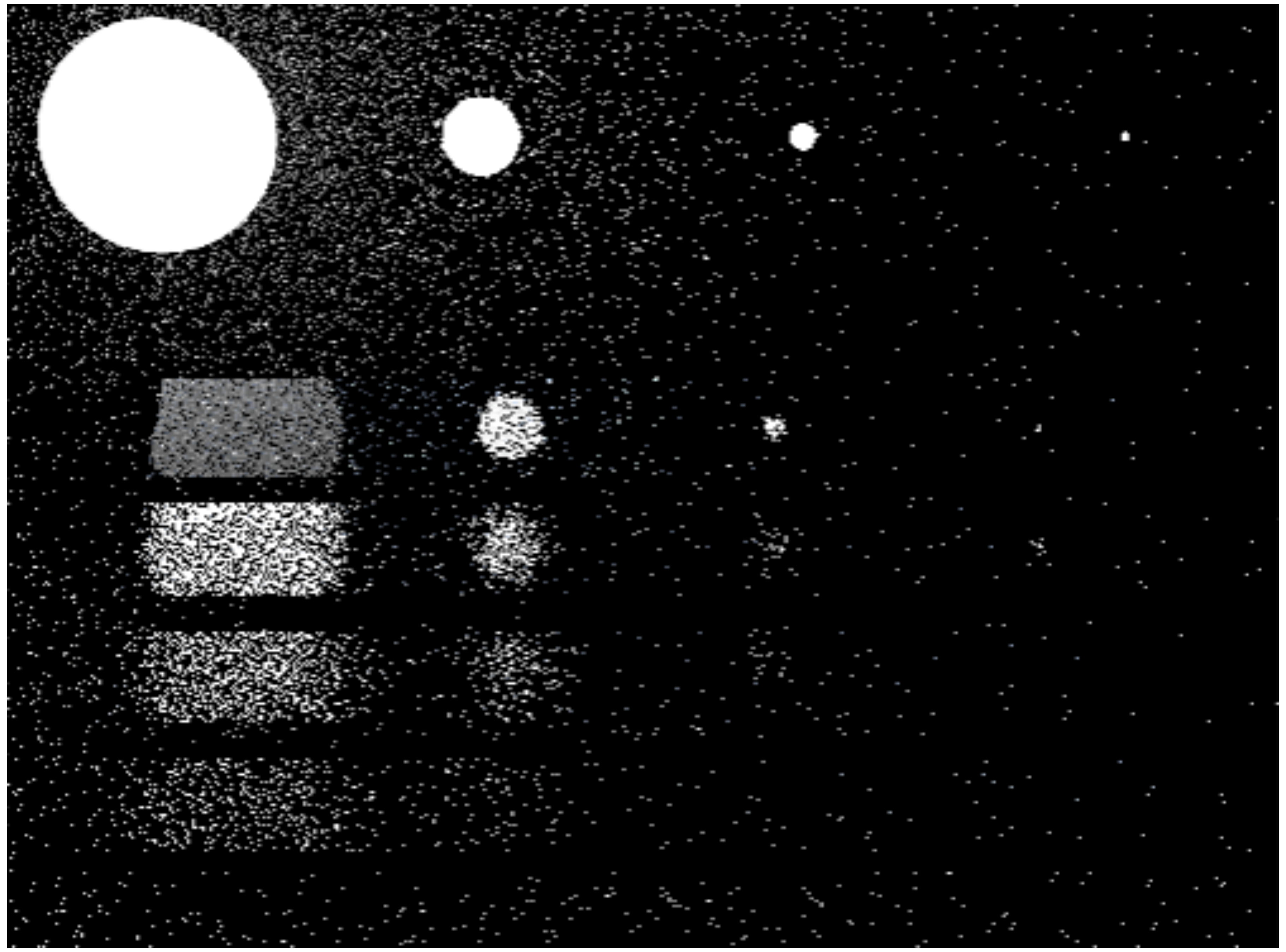
Direct Lighting

- Sampling-based method:
  - Sample light
    - Sample a point from the area light and find the corresponding Pdf.
    - If the light sample is not occluded,
      - L = (Compute the radiance produced by the light sample according to the rendering equation) / Pdf.



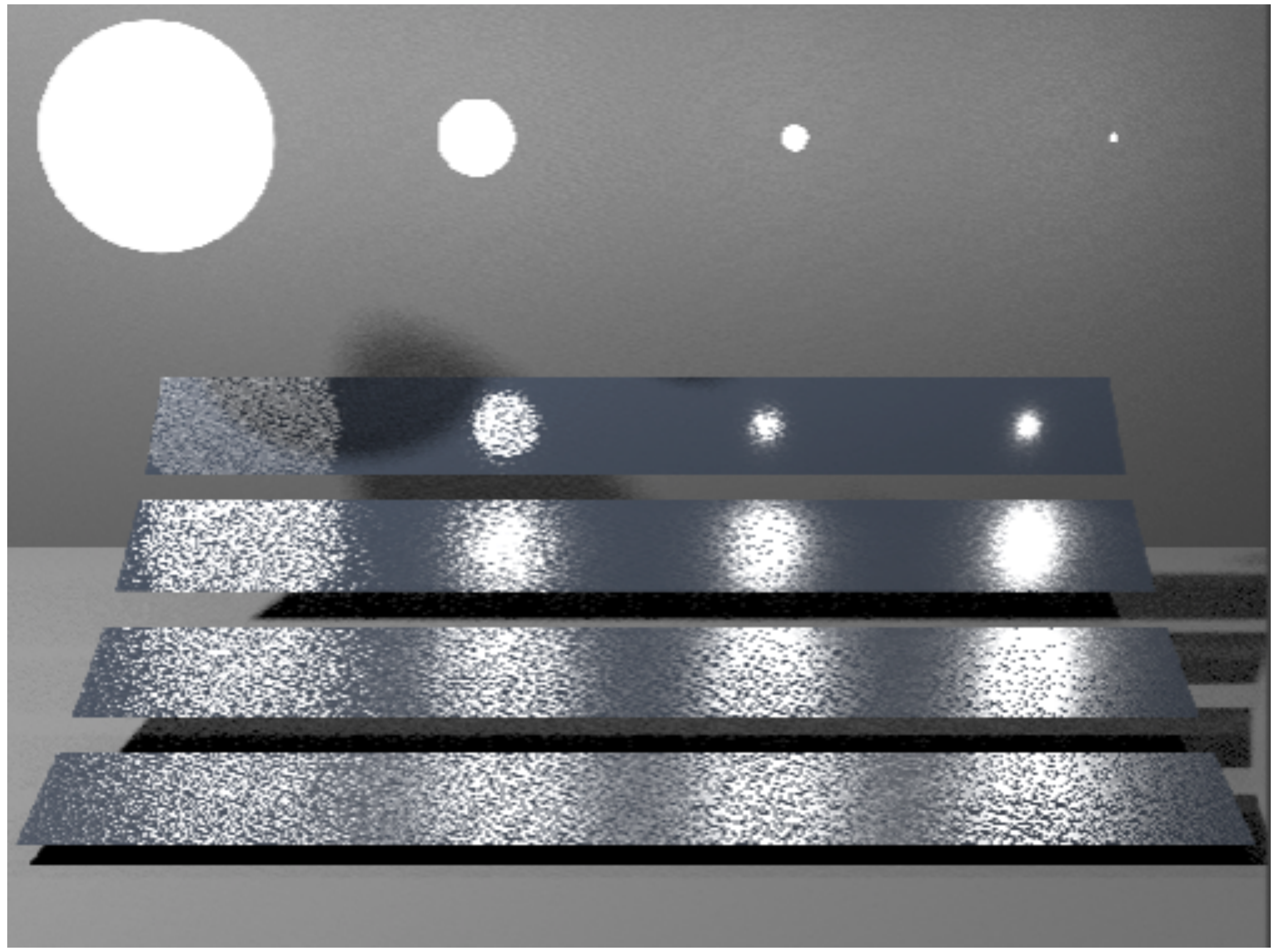
Direct Lighting

- Sampling-based method:
  - Sample BRDF
    - Sample a w_i according to the BRDF and find the corresponding Pdf.
    - If there is a light along the w_i without occlusion,
      - L = (Compute the radiance produced by the light according to the rendering equation) / Pdf



Direct Lighting

- Sampling-based method:
  - Sample light and BRDF (Multiple importance sampling)
    - Find the radiance of sampling light.
    - Find the radiance of sampling birds.
    - Weighted sum ($p_f(x), p_g(x)$ are Pdfs for the two sampling methods):
    - $$\frac{1}{n_f}\sum_{i=1}^{n_f}\frac{f(X_i)g(X_i)w_f(X_i)}{p_f(X_i)} + \frac{1}{n_g}\sum_{j=1}^{n_g}\frac{f(Y_j)g(Y_j)w_g(Y_j)}{p_g(Y_j)}$$



Sampling

- Sampling uniform area light
  - Different shapes require different samplers.
  - For example
    - Disk shape: disk-uniform (see Lecture 13)
- Sampling BRDF
  - Always use cosine-weighted hemisphere sampling (see Lecture 13)
    - $p(\omega) \propto cos\theta$
    - Intuition: less Θ, higher cosine term

HW4 - Global Illumination (tentative)

- [must] Uniform grid
- [must] Monte-Carlo Path Tracing
- [optional] Glossy specular
- [optional] Bidirectional path-tracing
- [optional] KD-tree
- [optional] <u>Metropolis</u> sampling
- [optional] Refraction with handling caustic case
- [optional] Different sampling for indirect/direct lighting

HW4 - Global Illumination