EE150 Signals and Systems
– Part 5: Discrete-time Fourier Transform (DTFT)

Continuous-time Fourier Transform

Continuous-time Fourier transform of $x(t)$ can be interpreted as

$$e^{j\omega t} \longrightarrow \boxed{\quad x(t) \quad} \longrightarrow X(j\omega)e^{j\omega t}$$

LTI system with impulse response $x(t)$

$X(j\omega)$: eigenvalue of $e^{j\omega t}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Discrete-time Fourier Transform

Discrete-time LTI system with impulse response $x[n]$

$$e^{j\omega n} \longrightarrow \boxed{x[n]} \longrightarrow X(j\omega)e^{j\omega n}$$

$$\begin{aligned} X(j\omega)e^{j\omega n} &= e^{j\omega n} * x[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega(n-m)} \\ &= e^{j\omega n} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \end{aligned}$$

Discrete-time Fourier Transform

- $e^{j\omega n}$ is periodic in ω, with period 2π
- Therefore $X(j\omega) = X(j(\omega + 2\pi))$, i.e. periodic
- Forward and inverse transforms

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(DTFT)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\omega)e^{j\omega n}d\omega \qquad \text{(Inverse DTFT)}$$

Discrete-time FT and FS

- Let $x(t)$ be periodic with period 2π ($\omega_0 = 1$)
- FS and DTFT

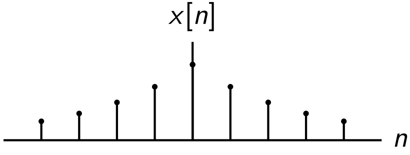
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \qquad X(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

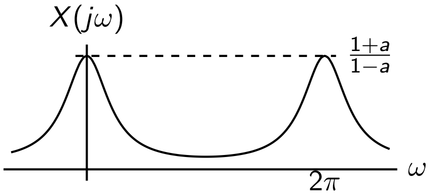
$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega n} d\omega$$

- Really one and the same (time-frequency exchange)

Discrete-time FT and FS

Example (1). (Aperiodic)





$$x[n] = a^{|n|}, \quad 0 < a < 1$$

$$\begin{aligned}
X(j\omega) &= \sum_{n} a^{|n|} e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=1}^{\infty} a^n e^{j\omega n} \\
&= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\
&= \frac{1 - a^2}{1 - a \cdot 2\cos\omega + a^2}
\end{aligned}$$

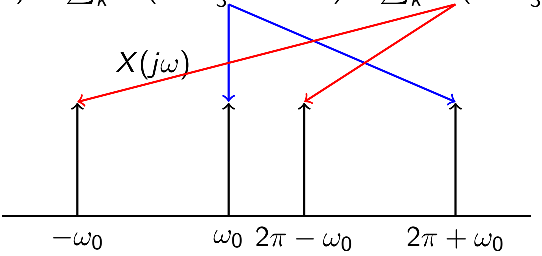
Discrete-time FT and FS

Example (2). (Periodic)

$$x[n] = \cos(\omega_0 n) = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \quad \text{with} \quad \omega_0 = \frac{2\pi}{3}$$

From table: $e^{j\omega_0 n} \longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi k)$

$$X(j\omega) = \sum_k \pi\delta(\omega - \tfrac{2}{3}\pi - 2\pi k) + \sum_k \pi\delta(\omega + \tfrac{2}{3}\pi - 2\pi k)$$



Convolution

- $y[n] = h[n] * x[n]$

$$
\begin{aligned}
Y(j\omega) &= \sum_n y[n] e^{-j\omega n} \\
&= \sum_n \sum_m x[m] h[n-m] e^{-j\omega n} \\
&= \sum_m \left(\sum_n h[n-m] e^{-j\omega(n-m)} \right) x[m] e^{-j\omega m} \\
&= \sum_m H(j\omega) x[m] e^{-j\omega m} \\
&= H(j\omega) X(j\omega)
\end{aligned}
$$

Multiplication

- $y[n] = x[n]h[n]$

$$y[n] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} X(j\omega)e^{j\omega n}d\omega \int_{-\pi}^{\pi} H(j\rho)e^{j\rho n}d\rho$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\omega)H(j(\nu-\omega))d\omega\right)e^{j\nu n}d\nu$$

- So

$$Y(j\nu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\omega)H(j(\nu-\omega))d\omega$$

(periodic convolution)

Properties of DTFT

Differencing:

$x[n] - x[n-1] \xleftrightarrow{FT} (1 - e^{-j\omega})X(j\omega)$

Proof.

$$x[n] - x[n-1] = x[n] * [\delta[n] - \delta[n-1]]$$
$$\delta[n-k] \xleftrightarrow{FT} e^{-j\omega k}$$

$\square$

Properties of DTFT

Time expansion: Consider $y[n] = x[a \cdot n]$ (a non-zero integer)

$$\sum_{k=0}^{a-1} X(j(\omega + k\frac{2\pi}{a})) = \sum_{k=0}^{a-1} \sum_{n} x[n] e^{-j(\omega + k\frac{2\pi}{a})n}$$

$$= \sum_{n} x[n] e^{-j\omega n} \left(\sum_{k=0}^{a-1} e^{-\frac{j2\pi nk}{a}} \right)$$

$$= a \sum_{m} x[am] e^{-j\omega am}$$

$$Y(j\omega) = \frac{1}{a} \sum_{k=0}^{a-1} X(j(\frac{\omega + 2\pi k}{a}))$$

Discrete Fourier Transform (DFT)

Let $x[n]$ be a finite-length sequence of length N.
Suppose $x[n] = 0$ for $n \notin [0 : N-1]$.

The DFT of $x[n]$, denoted as $X[k]$, is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$\text{where} \quad W_N = e^{-j(2\pi/N)}$$

The inverse DFT (IDFT) of $X[k]$ is given by

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

Discrete Fourier Transform (DFT)

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 1} & W_N^{0\cdot 2} & \cdots & W_N^{0\cdot N} \\ W_N^{1\cdot 1} & W_N^{1\cdot 2} & \cdots & W_N^{1\cdot N} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)\cdot 1} & W_N^{(N-1)\cdot 2} & \cdots & W_N^{(N-1)\cdot N} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Important Features of DFT

- one-to-one correspondence between $x[n]$ and $X[k]$.
- an extreme fast algorithm for its calculation, called Fast Fourier Transform (FFT), complexity $O(N \log N)$
- DFT is closely related to discrete Fourier series and Fourier Transform.
- DFT is an appropriate representation for digital computer realization as it is discrete and of finite length in both time and frequency domain.

Cooley-Tukey FFT

Suppose N is even, express $X[k]$ into two parts:
the sum of even-indexed $x[n]$, and that of odd-indexed ones:

$$\begin{aligned} X[k] &= \sum_{m=0}^{\frac{N}{2}-1} x[2m]e^{-\frac{2\pi j}{N}k2m} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]e^{-\frac{2\pi j}{N}k(2m+1)} \\ &= \sum_{m=0}^{\frac{N}{2}-1} x[2m]e^{-\frac{2\pi j}{N/2}km} + e^{-\frac{2\pi j}{N}k} \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]e^{-\frac{2\pi j}{N/2}km} \\ &=: E_k + e^{-\frac{2\pi j}{N}k} O_k \end{aligned}$$

Further for $k + \frac{N}{2}$,

$$X_{k+\frac{N}{2}} = E_k - e^{-\frac{2\pi j}{N}k} O_k$$