SI151A
Convex Optimization and its Applications in Information Science, Fall 2021
Homework 1

Due on Oct 11, 2021, 23:59 UTC+8

1. Specify whether each of the following statements is true or false. A proof or a counterexample is required.

(1) The set of points that are closer to a point $\overline{x} \in \mathbb{R}^n$ than a set $S \subseteq \mathbb{R}^n$ is convex. (10 points)

Solution:

True.
Proof: Denote $A = \{x | \|x - \overline{x}\|_2 < \inf\limits_{s \in S} \|x - s\|_2\}$, which can be rewritten as

$$\begin{aligned}
A &= \bigcap_{s \in S} \{x | \|x - \overline{x}\|_2 < \|x - s\|_2\} \\
&= \bigcap_{s \in S} \{x | (x - \overline{x})^\top (x - \overline{x}) < (x - s)^\top (x - s)\} \\
&= \bigcap_{s \in S} \{x | x^\top x - 2x^\top \overline{x} < +\overline{x}^\top \overline{x} < x^\top x - 2x^\top s + s^\top s\} \\
&= \bigcap_{s \in S} \{x | 2(s - \overline{x})^\top x < s^\top s - \overline{x}^\top \overline{x}\}
\end{aligned}$$

This set is the intersection of halfspaces, thus it is convex.

(2) The set of points that are farther from a point $\overline{x} \in \mathbb{R}^n$ than a set $S \subseteq \mathbb{R}^n$ is convex. (10 points)

Solution:

False.
Counter example: For simplicity, take $n = 1$.
Given $\overline{x} = 0, S = \{-2, 2\}$, then the required set $A = (-\infty, -1) \cup (1, \infty)$, which is not convex.

(3) The set of points that are closer to a set $S \subseteq \mathbb{R}^n$ than another set $T \subseteq \mathbb{R}^n$ is convex. (10 points)

Solution:

False.
Counter example: For simplicity, take $n = 1$.
Given $S = \{-2, 2\}, T = \{0\}$, then the required set $A = (-\infty, -1) \cup (1, \infty)$, which is not convex.

2. *Polyhedra.*

(1) Show that if $P \subseteq \mathbb{R}^n$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A(P) = \{Ax : x \in P\}$ is a polyhedron. (10 points) Hint: you may use the fact that

$$P \subseteq \mathbb{R}^{m+n} \text{ is a polyhedron } \Rightarrow \{x \in \mathbb{R}^n : (x, y) \in P \text{ for some } y \in \mathbb{R}^m\} \text{ is a polyhedron.}$$

Solution:

Denote the polyhedron $P = \{x : A_p x \preceq b_p, C_p x = d_p\}$.
Consider the set $S = \{(x, y) : A_p x \preceq b_p, C_p x = d_p, y = Ax\}$, which is also a polyhedron.
Since $S \subseteq \mathbb{R}^{m+n}$ is a polyhedron, then from the given fact we know that

$$\begin{aligned} A(P) =& \{Ax : x \in P\} \\ =& \{y : y = Ax, A_p x \preceq b_p, C_p x = d_p\} \\ =& \{y \in \mathbb{R}^m : (x, y) \in S \text{ for some } x \in \mathbb{R}^n\} \end{aligned}$$

is a polyhedron.

(2) Show that if $Q \subseteq \mathbb{R}^m$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A^{-1}(Q) = \{x \in \mathbb{R}^n : Ax \in Q\}$ is a polyhedron. (10 points)

Solution:

Denote the polyhedron $Q = \{y : A_q y \preceq b_q, C_q y = d_q\}$.
Consider the set $S = \{(x, y) : A_q y \preceq b_q, C_q y = d_q, y = Ax\}$, which is also a polyhedron.
Since $S \subseteq \mathbb{R}^{m+n}$ is a polyhedron, then from the given fact we know that

$$\begin{aligned} A^{-1}(Q) =& \{x : Ax \in Q\} \\ =& \{x : y = Ax, A_q y \preceq b_q, C_q y = d_q\} \\ =& \{x \in \mathbb{R}^n : (x, y) \in S \text{ for some } y \in \mathbb{R}^m\} \end{aligned}$$

is a polyhedron.

3. Let A and B be two compact (*i.e.* closed and bounded) sets in $\mathbb{R}^n$. Show that there exists a nonzero vector $\mathbf{a} \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that

$$\mathbf{a}^\top \mathbf{x} - b \leq -1 \ \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} - b \geq 1 \ \forall \mathbf{x} \in B,$$

if and only if the intersection of the convex hull of A and the convex hull of B is empty. (10 points)

Solution:

"if":
According to separating hyperplane theorem, if $\mathbf{conv}(A) \cap \mathbf{conv}(B) = \emptyset$, then there exists $\mathbf{a} \in \mathbb{R}^n (\mathbf{a} \neq 0)$ and $b \in \mathbb{R}$ such that

$$\mathbf{a}^\top \mathbf{x} \leq b \ \forall \mathbf{x} \in \mathbf{conv}(A) \text{ and } \mathbf{a}^\top \mathbf{x} \geq b \ \forall \mathbf{x} \in \mathbf{conv}(B).$$

Since $A \subseteq \mathbf{conv}(A)$ and $B \subseteq \mathbf{conv}(B)$, we can further conclude that

$$\mathbf{a}^\top \mathbf{x} \leq b \ \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} \geq b \ \forall \mathbf{x} \in B.$$

By the compactness of A and B, we can claim that there must exist $\mathbf{a} \in \mathbb{R}^n (\mathbf{a} \neq 0)$ and $b \in \mathbb{R}$ such that

$$\mathbf{a}^\top \mathbf{x} < b \ \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} > b \ \forall \mathbf{x} \in B.$$

Then for a sufficiently small ϵ, we have

$$\mathbf{a}^\top \mathbf{x} - b \leq -\epsilon \ \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} - b \geq \epsilon \ \forall \mathbf{x} \in B.$$

By substituting $\mathbf{a}$ with $\epsilon \mathbf{a}$ and $\mathbf{b}$ with $\epsilon \mathbf{b}$, we can obtain

$$\mathbf{a}^\top \mathbf{x} - b \leq -1 \ \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} - b \geq 1 \ \forall \mathbf{x} \in B.$$

"only if":
We use proof by contradiction. Suppose $\mathbf{conv}(A) \cap \mathbf{conv}(B) \neq \emptyset$, then we take $x_0 \in \mathbf{conv}(A) \cap \mathbf{conv}(B)$.
We can express x_0 as

$$x_0 = \sum_{i=1}^{m} \theta_i x_i \text{ with } \sum_{i=1}^{m} \theta_i = 1, \theta_i \geq 0, x_i \in A,$$

and

$$x_0 = \sum_{i'=1}^{n} \theta_{i'} x_{i'} \text{ with } \sum_{i'=1}^{n} \theta_{i'} = 1, \theta_{i'} \geq 0, x_{i'} \in B.$$

Since there exists a nonzero vector $\mathbf{a} \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that

$$\mathbf{a}^\top \mathbf{x} - b \leq -1 \; \forall \mathbf{x} \in A \text{ and } \mathbf{a}^\top \mathbf{x} - b \geq 1 \; \forall \mathbf{x} \in B,$$

we can obtain

$$\mathbf{a}^\top \mathbf{x}_i - b \leq -1 \text{ for } \mathbf{x}_i \in A,$$
$$\mathbf{a}^\top \mathbf{x}_{i'} - b \geq 1 \text{ for } \mathbf{x}_{i'} \in B,$$

which is equivalent to

$$\mathbf{a}^\top (\theta_i \mathbf{x}_i) - \theta_i b \leq -\theta_i \text{ for } \mathbf{x}_i \in A,$$
$$\mathbf{a}^\top (\theta_{i'} \mathbf{x}_{i'}) - \theta_{i'} b \geq \theta_{i'} \text{ for } \mathbf{x}_{i'} \in B.$$

Take sum of the first inequality w.r.t m, we have

$$\mathbf{a}^\top \sum_{i=1}^{k} (\theta_i \mathbf{x}_i) - \sum_{i=1}^{k} \theta_i b \leq -\sum_{i=1}^{k} \theta_i \text{ for } \mathbf{x}_i \in A \Longleftrightarrow \mathbf{a}^\top \mathbf{x}_0 - b \leq -1$$

Similarly, we can obtain $\mathbf{a}^\top \mathbf{x}_0 - b \geq 1$, which leads to a contradiction.

4. An $n \times n$ real symmetric matrix Q is said to be *copositive* if $\mathbf{x}^\top Q \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \geq 0$. (The inequality on $\mathbf{x}$ is elementwise.)

(1) Prove that the set of $n \times n$ copositive matrices is convex. Show that the set of $n \times n$ noncopositive matrices is nonconvex unless $n = 1$. (10 points)

Solution:

We denote by K the set of copositive matrices in $\mathbb{S}^n$. K is a convex cone because it is the intersection of (infinitely many) halfspaces defined by homogeneous inequalities

$$\mathbf{x}^\top Q \mathbf{x} = \sum_{i,j} x_i x_j Q_{ij} \geq 0.$$

If $n = 1$, then the set of 1×1 noncopositive matrices is simply composed of negative real numbers, which is clearly convex.
If $n = 2$, consider the following counter example:

$$Q_1 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

It can be easily verified that Q_1 and Q_2 are noncopositive matrices. However, $\frac{1}{2}Q_1 + \frac{1}{2}Q_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ is a copositive matrix. Therefore, the set of 2×2 noncopositive matrices is nonconvex.
If $n > 2$, we can generalize the counter example of the "$n = 2$" case:

$$Q_1 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}.$$

which leads to the same result as $n = 2$.

(2) Give an example of a matrix that is copositive but neither positive semidefinite nor elementwise nonnegative. (You have to prove all claims about the example that you provide.) (10 points)

Solution:

An example can be

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

To verify that Q is copositive, we can do the calculation $\mathbf{x}^\top Q\mathbf{x} = (x_1 - x_2)^2 + 2x_2x_3$, which is nonnegative for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \geq 0$. It's not positive semidefinite because it has a negative eigenvalue. Clearly it's not elementwise nonnegative.

5. Describe the dual cone for each of the following cones.

(1) $K = \{0\}$. (5 points)

(2) $K = \mathbb{R}^2$. (5 points)

(3) $K = \{(x_1, x_2) \mid |x_1| \leq x_2\}$. (5 points)

(4) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$. (5 points)

Solution:

(1) $K^* = \mathbb{R}$.

(2) $K^* = \{\mathbf{0}\}$.

(3) $K^* = \{(x_1, x_2) \mid |x_1| \leq x_2\}$. (This cone is self-dual.)

(4) $K^* = \{(x_1, x_2) \mid x_1 - x_2 = 0\}$.