Signals & Systems: Homework #4

Problem 1

(15 points) Compute the Fourier transform of each of the following signals

(a)

$$\sum_{n=-\infty}^{+\infty} e^{-|t-2n|}$$

(b)

$$x(t) = [te^{-2t} sin(4t)]u(t)$$

(c)

$$x(t) = \begin{cases} 1-t^2, & 0 < t < 1 \\ 0, & otherwise \end{cases}$$

Solution

(a)

x(t) is periodic with period 2. Therefore,

$$\omega_0 = \frac{2\pi}{T} = \pi$$

and

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \text{ and let } x'(t) = e^{-|t|}$$

$$\Rightarrow a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-\infty}^{+\infty} x'(t)e^{jk\pi t}dt$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{0} e^{(1-jk\pi)t}dt + \frac{1}{2}\int_{0}^{+\infty} e^{(-1-jk\pi)t}dt = \left(\frac{1}{1-jk\pi} + \frac{1}{1+jk\pi}\right) = \frac{1}{1+k^2\pi^2}$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{+\infty} \frac{1}{1+k^2\pi^2}\delta(\omega - k\pi)$$

another answer:

$$a_k = \frac{1}{1-e^{-2}}\left[\frac{1-e^{-2(1+j\omega)}}{1+j\omega} - \frac{e^{-2}\left[1-e^{-2(1+j\omega)}\right]}{1-j\omega}\right]$$

(b)

$$x(t) = te^{-2t}u(t)[\frac{1}{2j}(e^{j4t} - e^{-4jt})] = \frac{tu(t)}{2j}(e^{-(2-j4)t} - e^{-(2+j4)})$$

$$Let \quad y(t) = e^{-(2-j4)t} - e^{-(2+j4)}$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} y(t)e^{-j\omega t}dt = \int_{0}^{+\infty} e^{-(2-j4)t} - e^{-(2+j4)}e^{-j\omega t}dt = \frac{1}{2-j4+j\omega} - \frac{1}{2+j4+j\omega}$$

$$\Rightarrow X(j\omega) = \frac{1}{2j}\cdot j\frac{d}{d\omega}Y(j\omega) = \frac{16+8j\omega}{(20-\omega^2+j4\omega)^2}$$

(c)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_0^1 (1-t^2)e^{-j\omega t}dt = \int_0^1 e^{-j\omega t}dt - \int_0^1 t^2 e^{-j\omega t}dt = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{\omega^2} - \frac{2-2e^{-j\omega}}{j\omega^3}$$

Problem 2

(15 points) Consider a signal p(t) = $\sum_{k=-\infty}^{+\infty} \delta(t-kT)$ and a signal s(t) with spectrum S(jω), where 3Tω_1 = 2π

(a) Determine the FT of p(t)

(b) Dentermine and sketch the FT of r(t) = p(t)s(t)

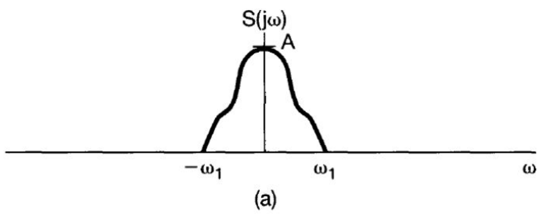


Figure 1 2(a)

Solution

(a)

The signal is periodic with fundamental period T and fundamental frequency: $\omega_0 = \frac{2\pi}{T}$

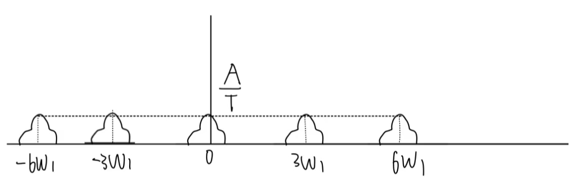
$$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} = \frac{3\omega_1}{2\pi}$$

$$\Rightarrow P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \frac{2k\pi}{T}) = \sum_{k=-\infty}^{+\infty} 3\omega_1\delta(\omega - 3\omega_1 k)$$

(b)

$$R(j\omega) = \frac{1}{2\pi}S(j\omega) * P(j\omega) = \frac{1}{T}S(j\omega)\sum_{k=-\infty}^{+\infty} \delta(\omega - 3\omega_1 k) = \frac{3\omega_1}{2\pi}\sum_{k=-\infty}^{+\infty} S(j(\omega - 3\omega_1 k))$$



$R(j\omega)$

Problem 3

(20 points) Calculate the Fourier Transform of the following signals:

(a) Calculate the Fourier Transform of x(t) = $\frac{2}{1+(t-5)^2}$

(b) Calculate the inverse Fourier Transform of $X(j\omega) = \frac{1}{(a+j(\omega-3))^2}$

Solution

(a)
We know that $e^{-|t|} \overset{FT}{\leftrightarrow} \frac{2}{1+\omega^2}$

so by the fourier transform's dual property,

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} e^{j\omega t} d\omega$$

$$\Rightarrow 2\pi e^{-|t|} = \int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} e^{-j\omega t} d\omega$$

Exchange t and ω, so that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{+\infty} \frac{2}{1+t^2} e^{-j\omega t} dt$$

$$\Rightarrow F|\frac{2}{1+t^2}| = 2\pi e^{-|\omega|}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{2}{1+(t-5)^2} e^{-j\omega t} dt = e^{-5j\omega}\int_{-\infty}^{+\infty} \frac{2}{1+t^2} e^{-j\omega t} dt = 2\pi e^{-|\omega|-5j\omega}$$

(b)

$$X_1(j\omega) = \frac{1}{(a+j(\omega))^2} \leftrightarrow te^{-at}u(t)$$

$$te^{-at}u(t)e^{3jt} \leftrightarrow \frac{1}{(a+j(\omega-3))^2}$$

$$\Rightarrow x(t) = te^{-at}u(t)e^{3jt}$$

Problem 4

(20 points) Frequency response of a Linear Time-Invariant system is shown below:

$$H(j\omega) = \frac{j\omega+5}{2-\omega^2+3j\omega}$$

(a) Write out the differential equation that associates system input x(t) with output y(t).

(b) Determine the impulse response h(t) of the system.

(c) Determine output of the system with input $x(t) = e^{-5t}u(t)$.

Solution

(a)

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 5x(t)$$

(b)

$$H(j\omega) = \frac{j\omega+5}{2-\omega^2+3j\omega} = \frac{j\omega+5}{(j\omega+2)(\omega+1)} = \frac{4}{j\omega+1} - \frac{3}{j\omega+2}$$

$$\Rightarrow h(t) = 4e^{-4}u(t) - 3e^{-2t}u(t)$$

(c)

$$x(t) = e^{-5t}u(t) \Rightarrow X(j\omega) = \frac{1}{j\omega+5}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{j\omega+2} \cdot \frac{1}{j\omega+1} = \frac{1}{j\omega+1} - \frac{1}{j\omega+2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

Problem 5

(30 points) Let x(t) and y(t) be two real signals. Then the cross-correlation function of x(t) and y(t) is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)d\tau \quad .$$

Similarly, we can define $\phi_{yx}(t), \phi_{xx}(t)$, and $\phi_{yy}(t)$. The last two of these are called the auto-correlation functions of the signals x(t) and y(t), respectively. Let $\Phi_{xy}(j\omega), \Phi_{yx}(j\omega), \Phi_{xx}(j\omega)$ and $\Phi_{yy}(j\omega)$ denote the Fourier transforms of $\phi_{xy}(t), \phi_{yx}(t), \phi_{xx}(t)$, and $\phi_{yy}(t)$, respectively.

(a) Determine the relationship between $\Phi_{xy}(j\omega)$ and $\Phi_{yx}(j\omega)$.
$Hint$: You may need to prove $\phi_{yx}(t) = \phi_{xy}(-t)$ firstly.

(b) Find an expression for $\Phi_{yx}(j\omega)$ in terms of $X(j\omega)$ and $Y(j\omega)$.

(c) Show that $\Phi_{yy}(j\omega)$ is real and non-negative for every ω.

(d) Suppose now that x(t) is the input to an LTI system with a real-valued impulse response and with frequency response $H(j\omega)$ and that y(t) is the output. Find expressions for $\Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ in terms of $\Phi_{xx}(j\omega)$ and $H(j\omega)$.

(e) Let x(t) be as is illustrated in Figure 1, and let the LTI system impulse response be $h(t) = e^{-at}u(t), a > 0$. Compute $\Phi_{xx}(j\omega), \Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ using the results of parts (a)-(d).

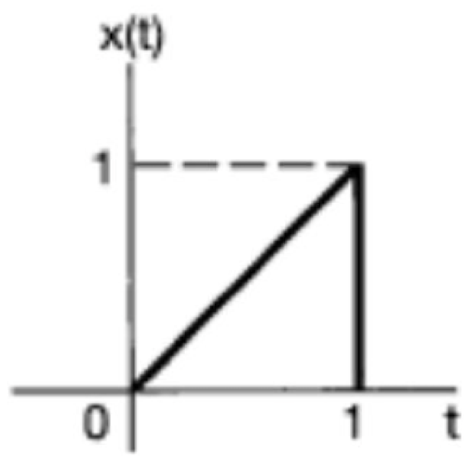


Figure 2 5(e)

Solution

(a)

$$\phi_{yx}(t) = \int_{-\infty}^{+\infty} y(t+\tau)x(\tau)d\tau$$

$$\text{Let } \tau' = \tau + t, \text{ so that } \int_{-\infty}^{+\infty} y(\tau')x(\tau' - t)d\tau'$$

$$\Rightarrow \phi_{yx}(t) = \phi_{xy}(-t)$$

$$\Rightarrow \Phi_{xy}(j\omega) = \Phi_{yx}(-j\omega)$$

$$\text{And they are both real siganls} \Rightarrow \Phi_{xy}(j\omega) = \Phi_{yx}^*(j\omega)$$

(b)

We know that

$$\phi_{yx}(t) = \int_{-\infty}^{+\infty} y(t+\tau)x(\tau)d\tau$$

and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \quad Y(j\omega) = \int_{-\infty}^{+\infty} y(t)e^{-j\omega t}dt$$

And

$$\int_{-\infty}^{+\infty} y(t+\tau)x(\tau)d\tau \quad = \quad \int_{-\infty}^{+\infty} y(k)x(k-t)dk \quad = \quad y(t) * x(-t) \text{ (let } k = t+\tau)$$

$$\Rightarrow \Phi_{yx}(j\omega) = Y(j\omega)X(j\omega)^*$$

(c)

$$\phi_{yy}(t) = \int_{-\infty}^{+\infty} y(t+\tau)y(\tau)d\tau \quad = \quad y(t) * y(-t) \quad = \quad y(t) * y(t)^*$$

$$\Rightarrow \Phi_{yy}(j\omega) = Y(j\omega)Y(j\omega)^* = |Y(j\omega)|^2 \geq 0$$

So it's real and non-negative for every ω.

(d)

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$\Phi_{xy}(j\omega) = X(j\omega)Y(j\omega)^* \quad = \quad X(j\omega)H(j\omega)^*X(j\omega)^* \quad = \quad |X(j\omega)|^2H(j\omega)^* \quad = \quad \Phi_{xx}(j\omega)H(j\omega)^*$$

$$\Phi_{yy}(j\omega) = Y(j\omega)Y(j\omega)^* \quad = \quad H(j\omega)X(j\omega)H(j\omega)^*X(j\omega)^* \quad = \quad |H(j\omega)|^2\Phi_{xx}(j\omega)$$

(e)

From the given information, we have

$$X(j\omega) = \frac{e^{-j\omega}-1}{\omega^2} + j\frac{e^{-j\omega}}{\omega}$$

and

$$H(j\omega) = \frac{1}{a+j\omega}.$$

Therefore,

$$\Phi_{xx}(j\omega) = |X(j\omega)|^2 = \frac{2-2cos\omega}{\omega^4} - \frac{2sin\omega}{\omega^2} + \frac{1}{\omega^2},$$

$$\Phi_{xy}(j\omega) = \Phi_{xx}(j\omega)H(j\omega)^* = \left[\frac{2-2cos\omega}{\omega^4} - \frac{2sin\omega}{\omega^2} + \frac{1}{\omega^2}\right]\left[\frac{1}{a-j\omega}\right],$$

and

$$\Phi_{yy}(j\omega) = \Phi_{xx}(j\omega)|H(j\omega)|^2 = \left[\frac{2-2cos\omega}{\omega^4} - \frac{2sin\omega}{\omega^2} + \frac{1}{\omega^2}\right]\left[\frac{1}{a^2+\omega^2}\right].$$