Name: ___________ **Student ID:**___________

Signals and Systems Homework 6

Due Time: 23:59 April 27, 2018

Submitted to blackboard online (photos or electronic documents) and to the box in front of SIST 1C 403E (the instructor's office).

Throughout this problem set, the n-th Fourier series coefficient of some function f of period T means $\frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-j2\pi nt/T}\mathrm{d}t$

1. (20') Review the trick to solve the Problem 4 in Mid-Term Exam and solve the following questions:

   (a) (10') Find the Fourier series coefficients a_n of function $f(t)$ of period 2 with $f(t) = \frac{1}{2}(t|t| - t)$ for $t \in [-1, 1]$. You can use, without proof, the fact that the Fourier series coefficients of $g(t)$ of period 2 with $g(t) = |t|$ for $t \in [-1, 1]$ are:

   $$b_n = \begin{cases} \dfrac{-2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \neq 0 \text{ is even} \\ \dfrac{1}{2} & \text{if } n = 0 \end{cases}$$

   if necessary.

   (b) (10') Evaluate $\sum_{m=1}^{\infty} m^{-6}$. (**Provide your reasoning, or you will receive 0 credits**)

   Solution

   (a) Let $h(t) := g(t) - \frac{1}{2}$, and it follows from the Fourier series coefficients of $g(t)$ that the Fourier series coefficients c_n of $h(t)$ equal to $-2/(n^2\pi^2)$ for odd n and 0 otherwise. Note that $f(t)$ is continuous and $f'(t) = h(t)$, and thus,

   $$a_n = \frac{c_n}{jn\pi} = \begin{cases} \dfrac{-2}{jn^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \neq 0 \text{ is even} \end{cases}$$

   Because $a_0 = \frac{1}{2}\int_{-1}^{1} f(t)\mathrm{d}t = 0$,

   $$a_n = \frac{c_n}{jn\pi} = \begin{cases} \dfrac{-2}{jn^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

   (b) By Parseval's theorem,

   $$\frac{1}{120} = \frac{1}{2}\int_{-1}^{1} |f(t)|^2\mathrm{d}t = \sum_{n=-\infty}^{\infty} |a_n|^2 = \sum_{n \text{ is odd}} \frac{4}{n^6\pi^6} = \sum_{m=1}^{\infty} \frac{8}{(2m-1)^6\pi^6}$$

   and thus,

   $$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^6} = \frac{\pi^6}{960}.$$

   Therefore, $\sum_{m=1}^{\infty} \frac{1}{m^6} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^6} + \sum_{m=1}^{\infty} \frac{1}{(2m)^6} = \pi^6/960 + \frac{1}{64}\sum_{m=1}^{\infty} \frac{1}{m^6}$. Hence,

   $$\sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{64}{63} \cdot \frac{\pi^6}{960} = \frac{\pi^6}{945}.$$

2. (40') Let $g_{\alpha,\beta}(t) = \alpha e^{-\beta t^2}$, where α and β are positive real numbers. The following 2 facts can be used without proof when solving the following questions:

   - For all positive real numbers α and β, there exist $\alpha' > 0$ and $\beta' > 0$ such that $g_{\alpha',\beta'}(\omega)$ is equal to the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}(t)$;
   - $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.

(a) (5') Establish a differential equation containing $g_{\alpha,\beta}(t)$ and its derivative $g'_{\alpha,\beta}(t)$.

(b) (5') Establish a differential equation containing the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}(t)$ and its derivative $G'_{\alpha,\beta}(j\omega)$.

(c) (10') Compare the results from (a) and (b) and determine the Fourier transform $G_{\alpha,\beta}(j\omega)$ of $g_{\alpha,\beta}$ by finding $\alpha' > 0$ and $\beta' > 0$ such that $g_{\alpha',\beta'} = G_{\alpha,\beta}$.

(d) (20') Verify your answers obtained in the previous questions by computing $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ in the following two ways:

i. (10') Compute $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ by definition;

ii. (10') Compute $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ by the convolution property.

Solution

(a) $g'_{\alpha,\beta}(t) = -2\alpha\beta t e^{-\beta t^2} = -2\beta t g_{\alpha,\beta}(t)$.

(b) On one hand, the Fourier transform of $g'_{\alpha,\beta}(t)$ is $j\omega G_{\alpha,\beta}(j\omega)$. On the other hand, the Fourier transform of $g'_{\alpha,\beta}(t)$ also equals to $-2j\beta G'_{\alpha,\beta}(j\omega)$ as $g'_{\alpha,\beta}(t) = -2\beta t g_{\alpha,\beta}(t)$.
Thus, $-2j\beta G'_{\alpha,\beta}(j\omega) = j\omega G_{\alpha,\beta}(j\omega) \Rightarrow G'_{\alpha,\beta}(j\omega) = -\frac{\omega}{2\beta} G_{\alpha,\beta}(j\omega)$.

(c) If $g_{\alpha',\beta'}(\omega) = G_{\alpha,\beta}(j\omega)$, then

$$-2\beta'\omega g_{\alpha',\beta'}(\omega) = g'_{\alpha',\beta'}(\omega) = G'_{\alpha,\beta}(j\omega) = -\frac{\omega}{2\beta} G_{\alpha,\beta}(j\omega) = -\frac{\omega}{2\beta} g_{\alpha',\beta'}(\omega).$$

Thus, $\boxed{\beta' = \frac{1}{4\beta}.}$

Note that

$$\int_{-\infty}^{\infty} |g_{\alpha,\beta}(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{\alpha,\beta}(j\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_{\alpha',\beta'}(\omega)|^2 \mathrm{d}\omega$$

and

$$\int_{-\infty}^{\infty} |g_{\alpha,\beta}(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} \alpha^2 e^{-2\beta t^2} \mathrm{d}t = \frac{\alpha^2}{\sqrt{2\beta}}\int_{-\infty}^{\infty} e^{-s^2} \mathrm{d}s = \alpha^2\sqrt{\frac{\pi}{2\beta}},$$

implying

$$(\alpha')^2\sqrt{\frac{\pi}{2\beta'}} = 2\pi\alpha^2\sqrt{\frac{\pi}{2\beta}} \Rightarrow (\alpha')^2 = 2\pi\alpha^2\sqrt{\frac{\beta'}{\beta}} = \frac{\pi\alpha^2}{\beta}.$$

Consequently,

$$\boxed{\alpha' = \sqrt{\frac{\pi}{\beta}}\alpha.}$$

(d) i. By definition:

$$\begin{aligned}
(g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2})(t) &= \alpha_1\alpha_2 \int_{-\infty}^{\infty} e^{-\beta_1 s^2} e^{-\beta_2 (t-s)^2} \mathrm{d}s \\
&= \alpha_1\alpha_2 e^{-\beta_2 t^2} \int_{-\infty}^{\infty} e^{-(\beta_1+\beta_2)s^2 + 2\beta_2 ts} \mathrm{d}s \\
&= \alpha_1\alpha_2 e^{-\beta_2 t^2 + \frac{\beta_2^2 t^2}{\beta_1+\beta_2}} \int_{-\infty}^{\infty} e^{-(\beta_1+\beta_2)\left(s - \frac{\beta_2}{\beta_1+\beta_2}t\right)^2} \mathrm{d}s \\
&= \alpha_1\alpha_2 e^{-\frac{\beta_1\beta_2 t^2}{\beta_1+\beta_2}} \int_{-\infty}^{\infty} e^{-(\beta_1+\beta_2)s^2} \mathrm{d}s \\
&= \alpha_1\alpha_2 \sqrt{\frac{\pi}{\beta_1+\beta_2}} e^{-\frac{\beta_1\beta_2 t^2}{\beta_1+\beta_2}} \\
&= g_{\alpha_1\alpha_2\sqrt{\pi/(\beta_1+\beta_2)}, \beta_1\beta_2/(\beta_1+\beta_2)}(t).
\end{aligned}$$

ii. Note that the Fourier transform of $g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$ is

$$\begin{aligned}
G_{\alpha_1,\beta_1} G_{\alpha_2,\beta_2} &= g_{\alpha_1\sqrt{\pi/\beta_1},(4\beta_1)^{-1}} g_{\alpha_2\sqrt{\pi/\beta_2},(4\beta_2)^{-1}} \\
&= g_{\alpha_1\alpha_2\pi/\sqrt{\beta_1\beta_2},(\beta_1+\beta_2)(4\beta_1\beta_2)^{-1}}
\end{aligned}$$

Let $g_{\alpha_0,\beta_0} = g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2}$, we have

$$\begin{cases} \alpha_0 = \sqrt{\dfrac{\beta_0}{\pi}} \dfrac{\alpha_1\alpha_2\pi}{\sqrt{\beta_1\beta_2}} \\ \beta_0 = \dfrac{1}{4}\left(\dfrac{\beta_1+\beta_2}{4\beta_1\beta_2}\right)^{-1} = \dfrac{\beta_1\beta_2}{\beta_1+\beta_2} \end{cases}$$

and thus,

$$\begin{cases} \alpha_0 = \alpha_1\alpha_2\sqrt{\dfrac{\pi}{\beta_1+\beta_2}} \\ \beta_0 = \dfrac{\beta_1\beta_2}{\beta_1+\beta_2} \end{cases}$$

In brief,

$$g_{\alpha_1,\beta_1} * g_{\alpha_2,\beta_2} = g_{\alpha_1\alpha_2\sqrt{\pi/(\beta_1+\beta_2)},\beta_1\beta_2/(\beta_1+\beta_2)}$$

3. (40') Let $f_a(x) = e^{-a|x|}$ where $a > 0$.
   (a) (10') Determine the Fourier transform $F_a(j\omega)$ of $f_a(x)$.
   (b) (10') Consider $\widetilde{f}_a(x) = \sum_{n=-\infty}^{\infty} f_a(x+n)$. Derive the expression of $\widetilde{f}_a(x)$ and write down the fundamental period of $\widetilde{f}_a(x)$ if it exists.
   (c) (10') Decide the Fourier series coefficients c_n of $\widetilde{f}_a(x)$ if $\widetilde{f}_a(x)$ is periodic.
   (d) (10') How are c_n and $F_a(j\omega)$ related? (*Hint: Observe* $F_a(j2\pi n)$)
   (e) (0') If you have taken a course on *Mathematical Analysis*, think about why the your observation is valid. (*Hint: Weierstrass M-test*)

Solution

(a) Fourier transform of $f_a(x)$:

$$F_a(j\omega) = \int_{-\infty}^{0} e^{(a-j\omega)x}\mathrm{d}x + \int_{0}^{\infty} e^{-(a+j\omega)x}\mathrm{d}x = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}.$$

(b) $\boxed{\text{Period: 1.}}$

$$\begin{aligned} \widetilde{f}_a(x) &= \sum_{n=-\infty}^{\infty} f_a(x+n) \\ &= \sum_{n=-\infty}^{\infty} e^{-a|x+n|} \\ &= \sum_{x+n\geq 0} e^{-a(x+n)} + \sum_{x+n<0} e^{a(x+n)} \\ &= \sum_{\lfloor x\rfloor+n\geq 0} e^{-a(x-\lfloor x\rfloor)-a(n+\lfloor x\rfloor)} + \sum_{\lfloor x\rfloor+n\leq -1} e^{a(x-\lfloor x\rfloor)+a(\lfloor x\rfloor+n)} \\ &= \sum_{m=0}^{\infty} e^{-a(x-\lfloor x\rfloor+m)} + \sum_{m=1}^{\infty} e^{a(x-\lfloor x\rfloor-m)} \\ &= \frac{1}{1-e^{-a}} e^{-a(x-\lfloor x\rfloor)} + \frac{e^{-a}}{1-e^{-a}} e^{a(x-\lfloor x\rfloor)} \\ &= \frac{e^a}{e^a-1} e^{-a(x-\lfloor x\rfloor)} + \frac{1}{e^a-1} e^{a(x-\lfloor x\rfloor)}. \end{aligned}$$

(c) The n-th Fourier series coefficient of $\widetilde{f_a}$:

$$\begin{aligned}
c_n &= \int_0^1 \widetilde{f_a}(x)e^{-j2\pi nx}\mathrm{d}x \\
&= \frac{e^a}{e^a-1}\int_0^1 e^{-(a+j2\pi n)x}\mathrm{d}x + \frac{1}{e^a-1}\int_0^1 e^{(a-j2\pi n)x}\mathrm{d}x \\
&= \frac{e^a}{e^a-1}\cdot\frac{1-e^{-a}}{a+j2\pi n} + \frac{1}{e^a-1}\cdot\frac{e^a-1}{a-j2\pi n} \\
&= \frac{1}{a+j2\pi n} + \frac{1}{a-j2\pi n} \\
&= \frac{2a}{a^2+4\pi^2 n^2}.
\end{aligned}$$

(d) Observation: $c_n = F_a(j2\pi n)$.

(e) Note that, by triangle inequality, $|x+m|+|x| = |x+m|+|-x| \leq |m| \Rightarrow |x+m| \leq |m|-|x|$, and thus for any $x \in [0,1]$,

$$\begin{aligned}
|f_a(x+m)e^{-j2\pi nx}| &= |f_a(x+m)| \\
&= e^{-a|x+m|} \\
&\leq e^{-a(|m|-|x|)} \\
&= e^{a|x|}e^{-a|m|} \\
&\leq e^a e^{-a|m|}.
\end{aligned}$$

Since $\sum_m e^a e^{-a|m|}$ is a geometric series with common ratio $e^{-a} < 1$, implying that the series converges, by Weierstrass M-test, $\sum_{m=0}^{N} f(x+m)e^{-j2\pi nx}$ and $\sum_{m=-M}^{-1} f(x+m)e^{-j2\pi nx}$ both uniformly converge on $[0,1]$ and thus the summation and integration below can be interchanged:

$$\begin{aligned}
\int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-j2\pi nx}\mathrm{d}x &= \sum_{m=-\infty}^{\infty}\int_0^1 f(x+m)e^{-j2\pi nx}\mathrm{d}x \\
&= \sum_{m=-\infty}^{\infty}\int_m^{m+1} f(y)e^{-j2\pi n(y-m)}\mathrm{d}y \\
&= \sum_{m=-\infty}^{\infty}\int_m^{m+1} f(y)e^{-j2\pi ny}\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(y)e^{-j2\pi ny}\mathrm{d}y = F(j2\pi n).
\end{aligned}$$