Poisson Process

SI252 Reinforcement Learning

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Outline

1. Poisson Distribution

2. Poisson Process

Poisson Distribution

Definition

An r.v. X has the **Poisson distribution** with parameter λ if the PMF of X is

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!},\ k = 0, 1, 2, \cdots$$

We write this as $X \sim \mathrm{Pois}(\lambda)$.

Example: Poisson Expectation & Variance

Example (Poisson Expectation & Variance)

Consider an r.v. $X \sim \mathrm{Pois}(\lambda)$, find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Example: Poisson Expectation & Variance (Solution)

Example: Poisson Expectation & Variance (Solution)

Poisson Approximation

Theorem (Poisson Approximation)

Let $A_1, A_2, \cdots, A_n$ be events with $p_j = P(A_j)$, where n is large, the p_j are small, and the A_j are independent or weakly dependent. Let

$$X = \sum_{j=1}^{n} I(A_j)$$

count how many of the A_j occur. Then X is approximately $\mathrm{Pois}(\lambda)$, with $\lambda = \sum_{j=1}^{n} p_j$.

Example: Birthday Problem Revisited

Example (Birthday Problem Revisited)

There are m people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

Example: Birthday Problem Revisited (Solution)

Sum of Independent Poissons

Theorem (Sum of Independent Poissons)

If $X \sim \mathrm{Pois}(\lambda_1)$, $Y \sim \mathrm{Pois}(\lambda_2)$, *and* X *is independent of* Y, *then* $X + Y \sim \mathrm{Pois}(\lambda_1 + \lambda_2)$.

Poisson Given A Sum of Poissons

Theorem (Poisson Given A Sum of Poissons)

If $X \sim \mathrm{Pois}(\lambda_1)$, $Y \sim \mathrm{Pois}(\lambda_2)$, *and* X *is independent of* Y, *then the conditional distribution of* X *given* $X + Y = n$ *is* $\mathrm{Bin}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

Poisson Approximation to Binomial

Theorem (Poisson Approximation to Binomial)

If $X \sim \mathrm{Bin}(n, p)$ and we let $n \to \infty$ and $p \to 0$ such that $\lambda = np$ remains fixed, then the PMF of X converges to the $\mathrm{Pois}(\lambda)$ PMF. More generally, the same conclusion holds if $n \to \infty$ and $p \to 0$ in such a way that np converges to a constant λ.

Poisson Approximation to Binomial (Proof)

Example: Visitors to A Website

Example (Visitors to A Website)

The owner of a certain website is studying the distribution of the number of visitors to the site. Every day, a million people independently decide whether to visit the site, with probability $p = 2 \times 10^{-6}$ of visiting. Give a good approximation for the probability of getting *at least three* visitors on a particular day.

Outline

Definition (Poisson Process-Definition 1)

A Poisson process with parameter λ is a counting process $(N_t)_{t\geq 0}$ with the following properties:

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with parameter λt.
3. (Stationary increments) For all $s, t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t. That is,

$$P(N_{t+s} - N_s = k) = P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad \text{for} \quad k = 0, 1, \ldots$$

4. (Independent increments) For $0 \leq q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent random variables.

Example

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon and 70 texts by 5 p.m.

Definition (Translated Poisson Process)

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. For $s > 0$, let $\tilde{N}_t = N_{t+s} - N_s$, for $t \geq 0$. Then we have

- $(\tilde{N}_t)_{t\geq 0}$ is called "Translated Poisson Process".
- $(\tilde{N}_t)_{t\geq 0}$ is a Poisson process with parameter λ.

Example

Example

On election day, people arrive at a voting center according to a Poisson process. On average, 100 voters arrive every hour. If 150 people arrive during the first hour, what is the probability that at most 350 people arrive before the third hour?

Definition (Poisson Process-Definition 2)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables with parameter λ. For $t > 0$, let

$$N_t = max\{n : X_1 + \ldots + X_n \leq t\},$$

with $N_0 = 0$. Then, $(N_t)_{t \geq 0}$ defines a Poisson process with parameter λ.

Definition 2 $\Rightarrow$ Definition 1

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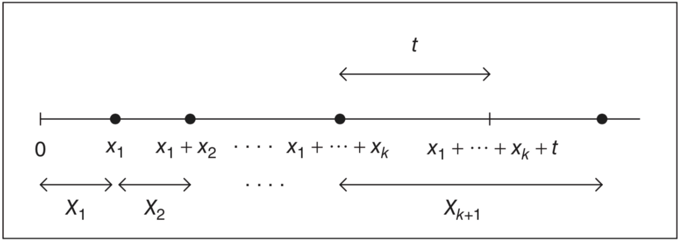
Definition 1 $\Rightarrow$ Definition 2



Definition 1 $\Rightarrow$ Definition 2

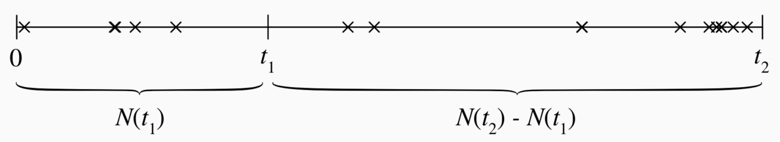
Conditional Counts

Theorem (Conditional Counts)

Let $\{N(t), t > 0\}$ be a Poisson process with rate λ, and let $t_1 < t_2$. Then the conditional distribution of $N(t_1)$ given $N(t_2) = n$ is

$$N(t_1) \,|\, N(t_2) = n \sim \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right).$$

Conditional Counts



Arrival Times & Uniform Distribution

Theorem (Arrival Times & Uniform Distribution)

Let $S_1, S_2, \ldots,$ be the arrival times of a Poisson process with parameter λ. Conditional on $N_t = n$, the joint distribution of $(S_1, \ldots, S_n)$ is the distribution of the order statistics of n i.i.d. uniform random variables on $[0, t]$. That is, the joint density function of $S_1, \ldots, S_n$ is

$$f(s_1, \ldots, s_n) = \frac{n!}{t^n}, \quad \text{for } 0 < s_1 < \ldots < s_n < t.$$

Equivalently, let $U_1, \ldots, U_n$ be an i.i.d.sequence of random variables uniformly distributed on $[0, t]$. Then, conditional on $N_t = n$,

$$(S_1, \ldots, S_n) \ \text{ and } \ (U_{(1)}, \ldots, U_{(n)})$$

have the same distribution.

Arrival Times & Uniform Distribution (Proof)

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Example

Example

Students enter a campus building according to a Poisson process $(N_t)_{t\geq 0}$ with parameter λ. The times spent by each student in the building are i.i.d. random variables with continuous cumulative distribution function $F(t)$. Find the probability mass function of the number of students in the building at time t, assuming there are no students in the building at time 0.

Example (Solution)

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