Signals and Systems

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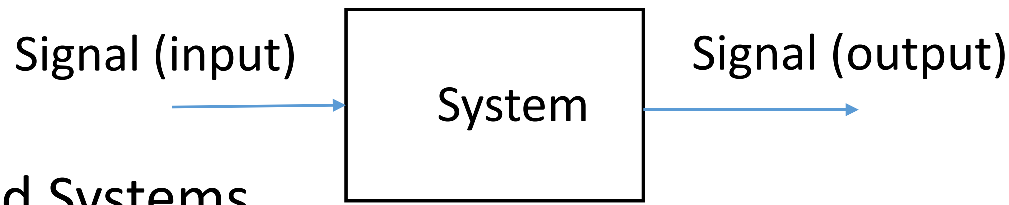
Course Introduction

- ❑ **Global content**
- ❑ **Motivation**
- ❑ **Exams and grades**
- ❑ **Text book and materials**
- ❑ **Organization**
- ❑ **Pre-knowledge**

Course Introduction

Global content



- ❑ Overview of Signals and Systems
- ❑ Linear-Time-Invariant Systems
- ❑ Fourier Series Representation of Periodic Signals
- ❑ The Continues-Time Fourier Transform
- ❑ The Discrete-Time Fourier Transform
- ❑ Time and Frequency Characterization of Signals and Systems
- ❑ Sampling
- ❑ The Laplace Transform
- ❑ The Z-Transform

Course Introduction

Motivation

- ❑ Importance
- ❑ Confidence
- ❑ Math is a important but not everything
- ❑ Focus on big pictures
- ❑ GPA and real knowledge

Rik Vullings

9-year old Laurent Simons got an A after one week study

Course Introduction

Exams and Grades

- ❑ Homework: 20% (Delay <=2 days, *0.8; >2 days, *0)
- ❑ Mid-term (written, close-book): 30%
- ❑ Final Exam (written, close-book): 50%
- ❑ All in English, otherwise *0.8
- ❑ Sign in every lecture, -2 /absence
- ❑ Plagiarism: zero

Course Introduction

Text book and materials

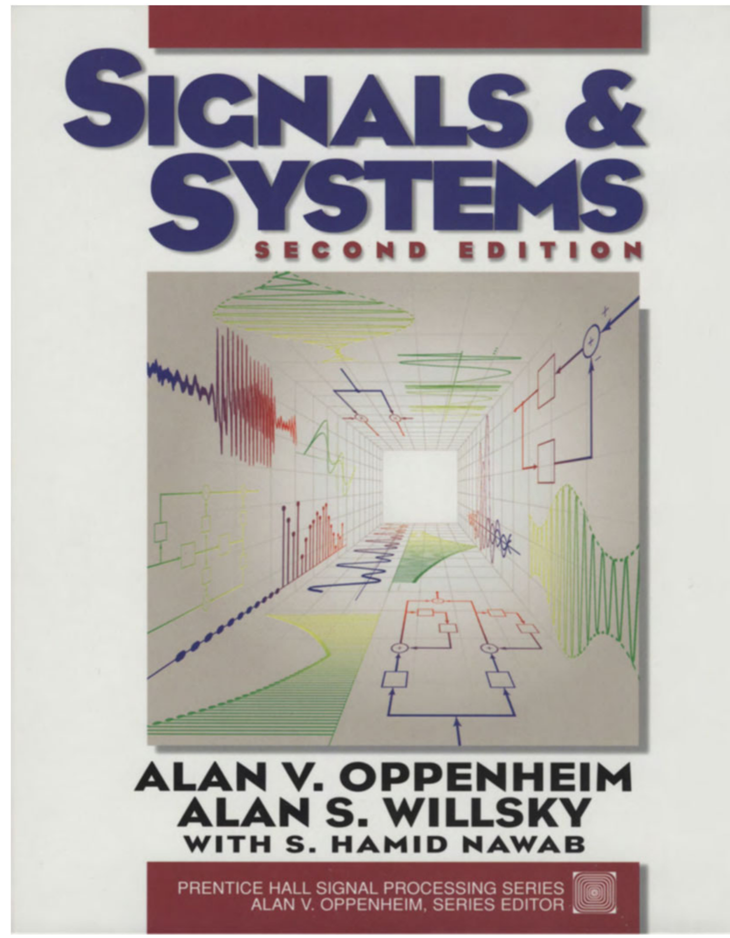
❑ Book

➢ Signals and Systems (2nd Edition), by A. V. Oppenheim, A. S. Willsky, and S. Hamid. ISBN: 978-0138147570.

➢ Signals and Systems using Matlab (2nd Edition), by Luis Chaparro. ISBN: 978-0123948120.

❑ These slides

❑ All materials bill be available in the BB system



Course Introduction

Organization

- ❑ Lecture: week 1-16; teaching center 301; Tue. and Thu. 08:15-9:55
- ❑ Exercise: once per week, time and location TBD
- ❑ Office hour: SIST 2-302I or 3-430, 16:30-18:00
- ❑ Experiment: by Dr. Linyan Lu, start from the 2nd week
- ❑ BB system: Slides and text book
- ❑ Gradescope: homework release and submission, entry code: KYGW3Z

Course Introduction

Pre-knowledge: Complex numbers

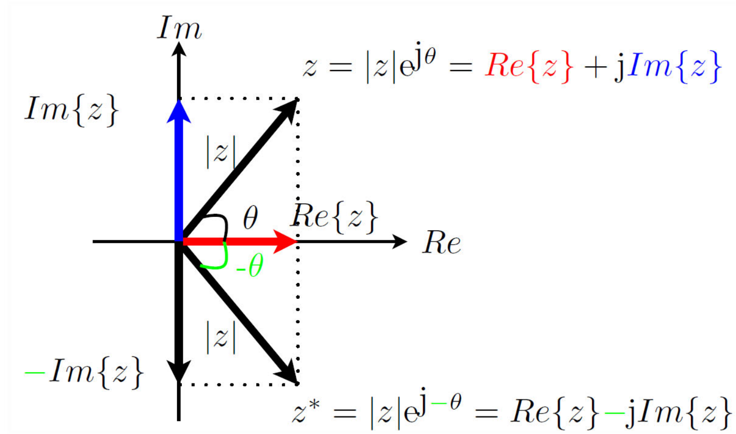
Polar notation:

$$z = |z|e^{j\theta}$$

Cartesian notation:

$$z = Re\{z\} + j \cdot Im\{z\}$$

Complex conjugation: $j \Rightarrow -j$



Euler:

$$
\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\
\cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\
\sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j}
\end{aligned}
$$

Course Introduction

Pre-knowledge: Important geometric series

With z_0 some (possibly complex) number:

$$\boxed{\sum_{n=0}^{\infty}(z_0)^n = \frac{1}{1-z_0}} \quad \text{iff} \quad |z_0| < 1$$

'Proof' via long tail division:

$$\frac{1}{1-z_0} = 1 + z_0 + (z_0)^2 + (z_0)^3 + \cdots = \sum_{n=0}^{\infty}(z_0)^n$$

$$\boxed{\sum_{n=0}^{M-1}(z_0)^n = \frac{1-z_0^M}{1-z_0}}$$

Let $n = M + p$

Proof: $\displaystyle\sum_{n=0}^{M-1}(z_0)^n = \sum_{n=0}^{\infty}(z_0)^n - \sum_{n=M}^{\infty}(z_0)^n = \frac{1}{1-z_0} - (z_0)^M \sum_{p=0}^{\infty}(z_0)^p$

Course Introduction

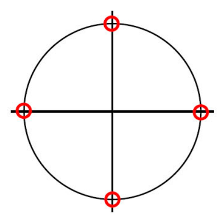
Pre-knowledge: Zeros of a complex equation

With a some (complex) number, find zeros of: $\boxed{z^N - a = 0}$

$$z^N = a = a\mathrm{e}^{\mathrm{j}k\cdot 2\pi} \quad \Rightarrow \quad \boxed{z_k = a^{\frac{1}{N}} \cdot \mathrm{e}^{\mathrm{j}k\cdot\frac{2\pi}{N}}} \text{ for } k = 0, 1, \cdots, N-1$$

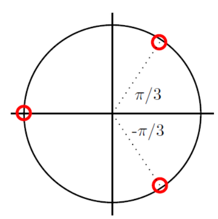
Example: $a = 1$, $N = 4$

$$\Rightarrow \quad z_k = \mathrm{e}^{\mathrm{j}k\cdot\frac{\pi}{2}}$$



Example: $a = -1$, $N = 3$

$$\begin{aligned} \Rightarrow \quad z_k &= (-1)^{\frac{1}{3}} \cdot \mathrm{e}^{\mathrm{j}k\cdot\frac{2\pi}{3}} \\ &= (\mathrm{e}^{\mathrm{j}\pi})^{\frac{1}{3}} \cdot \mathrm{e}^{\mathrm{j}k\cdot\frac{2\pi}{3}} \\ &= \mathrm{e}^{\mathrm{j}\frac{\pi}{3} + k\cdot\frac{2\pi}{3}} \end{aligned}$$

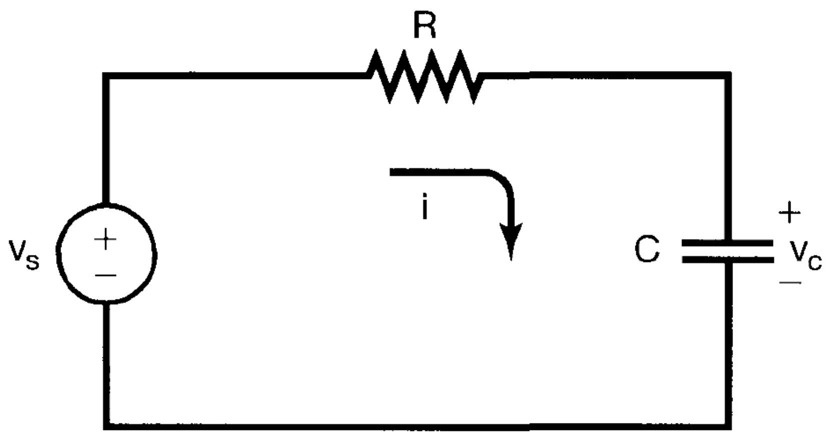


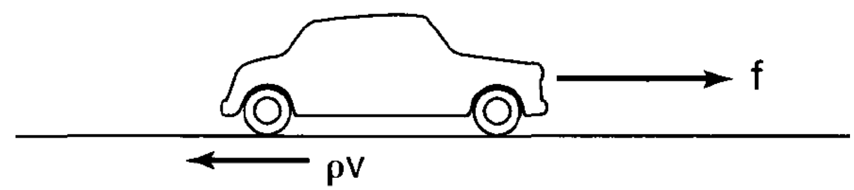
Signals and Systems: An overview (ch.1)

- ❑ Continuous-Time and Discrete-Time Signals
- ❑ Transformations of the Independent Variable
- ❑ Exponential and Sinusoidal Signals
- ❑ The Unit Impulse and Unit Step Functions
- ❑ Continuous-Time and Discrete-Time Systems
- ❑ Basic System Properties

Continuous-Time and Discrete-Time Signals

- ***Signals*** describe a wide variety of physical phenomena





The voltage v_s and v_c are examples of signals.

The force f and velocity v are signals.

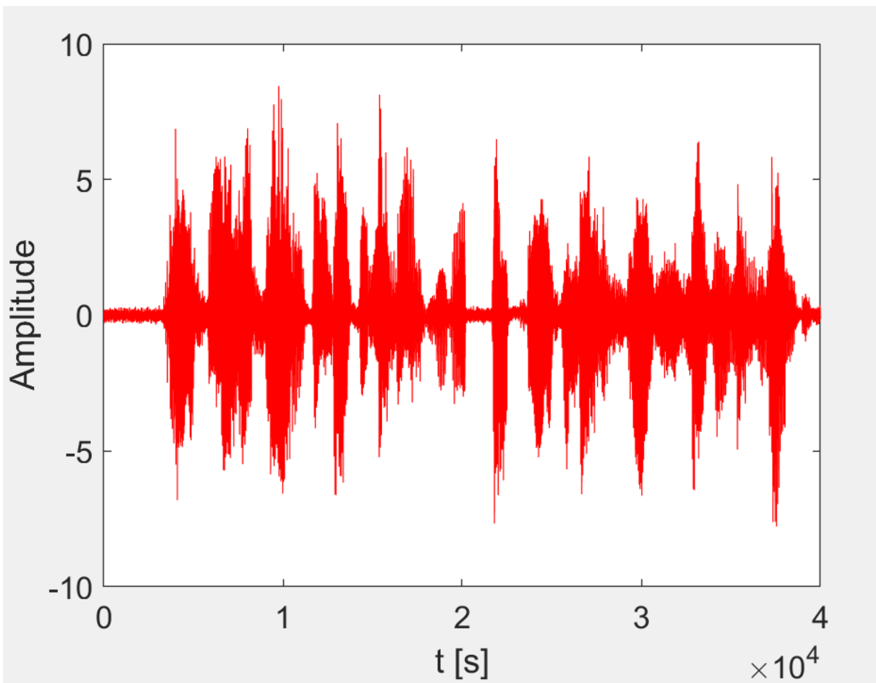
Continuous-Time and Discrete-Time Signals

- ***Mathematically, signals*** are represented as functions of one or more independent variables.

- Example of typical signals
  - Sound
  - Image
  - Video

Continuous-Time and Discrete-Time Signals

- ***Sound:*** represents acoustic pressure as a function of time



$f(t)$

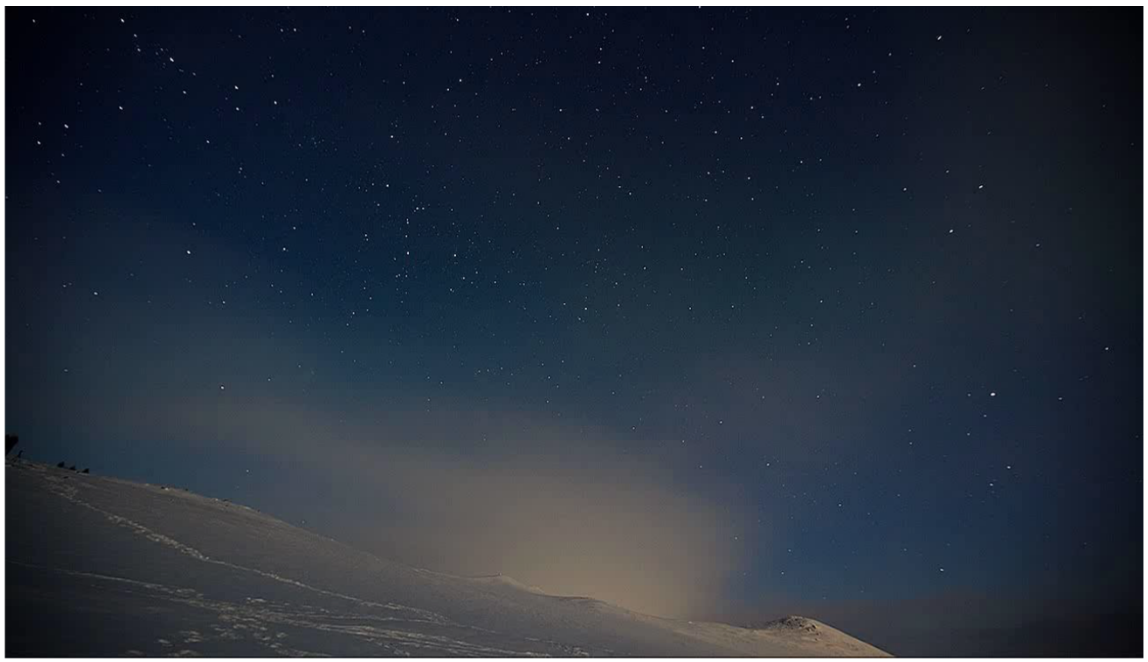
Continuous-Time and Discrete-Time Signals

- ***Picture:*** represents brightness as a function of two spatial variables

$f(x, y)$

Continuous-Time and Discrete-Time Signals

- ***Video:*** consists of a sequence of images, called frames, and is a function of 3 variables: 2 spatial coordinates and time



$f(x, y, t)$

Continuous-Time and Discrete-Time Signals

- Independent variables can be one or more

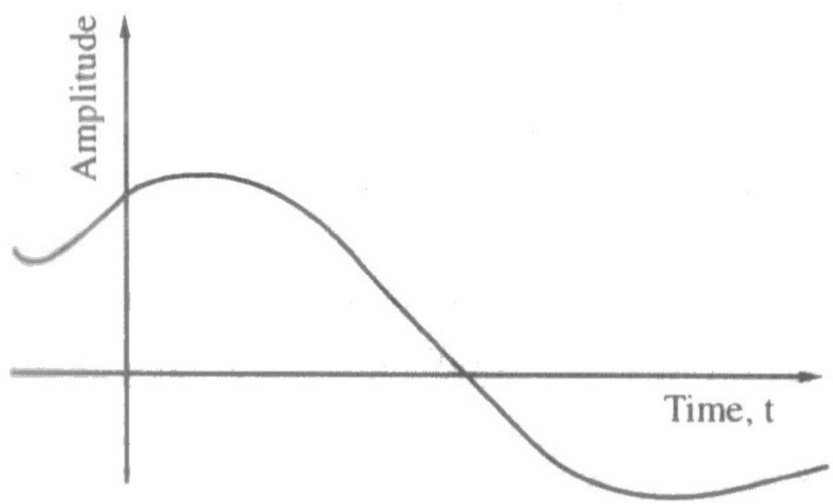
- Focus on signals involving a single independent variable

- Generally refer to the independent variable as time, although it may not in fact represent time in specific applications

- Continues-time and discrete-time signal

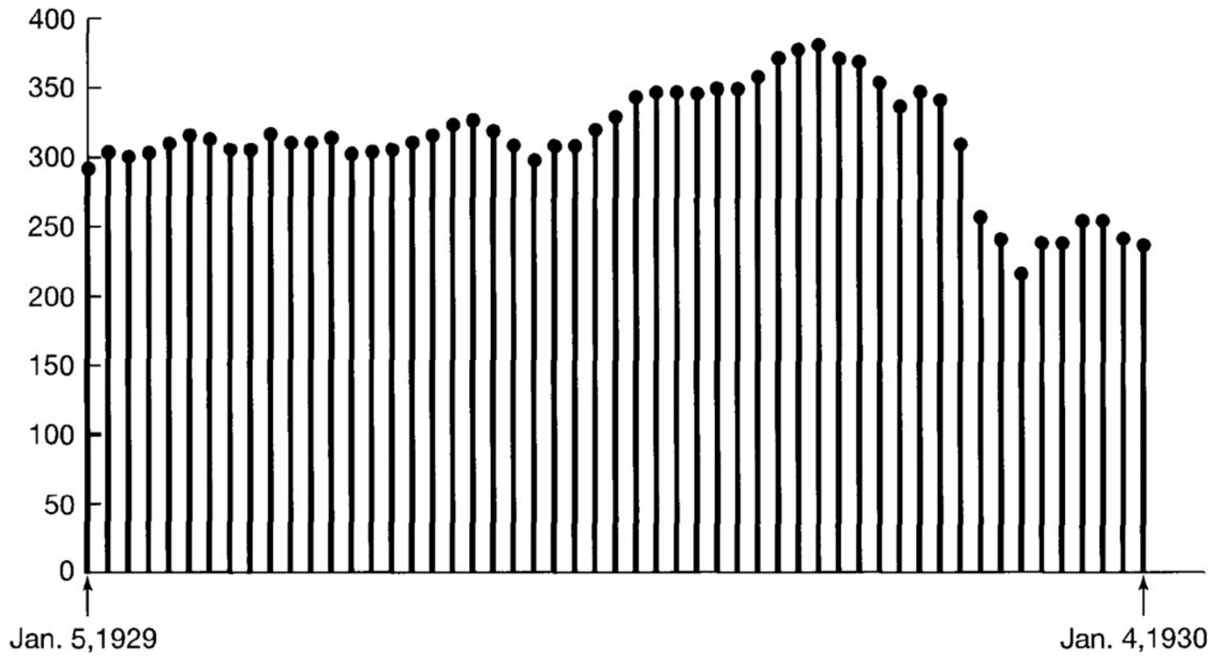
Continuous-Time and Discrete-Time Signals

- ***Continues-time signals:*** the independent variable is continuous, and signals are defined for a continuum of values



Continuous-Time and Discrete-Time Signals

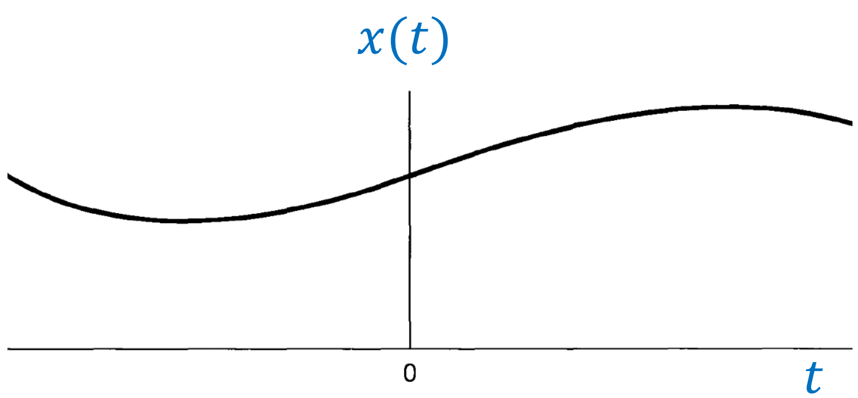
- ***Discrete-time signals:*** defined only at discrete times, and the independent variable takes on only a discrete set of values

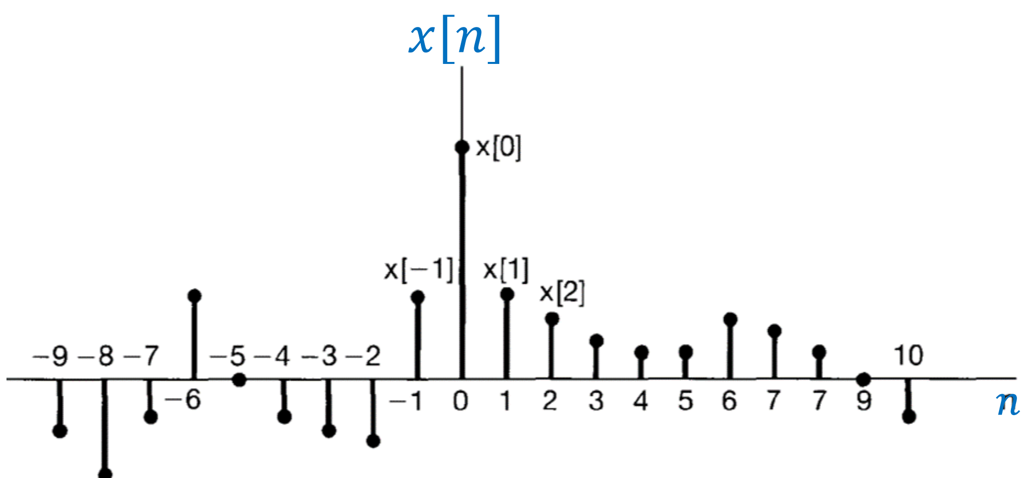


An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

Continuous-Time and Discrete-Time Signals

- ❑ ***Continuous-time signals:*** t denote the independent variable, enclosed in $(\cdot)$
- ❑ ***Discrete-time signals:*** n denote the independent variable, enclosed in $[\cdot]$
- ❑ $x[n]$
  - ➢ discrete in nature; or sampling of continuous-time signal
  - ➢ Focus mainly on the second case, defined only for integer values of n



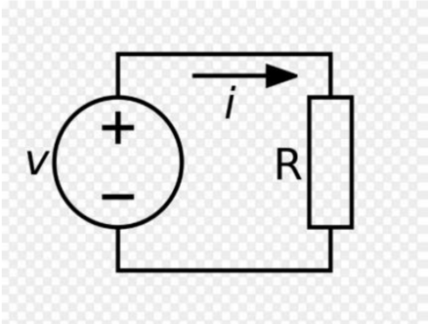


Continuous-Time and Discrete-Time Signals

Signal energy and power

- ❑ $v(t)$ and $i(t)$ are voltage and current across a resistor R, the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$



- ❑ The total energy over the time interval $t_1 \leq t \leq t_2$ is

$$E_R = \int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

- ❑ The average power over the time interval $t_1 \leq t \leq t_2$ is

$$P_R = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Similarly, for any signal $x(t)$ or $x[n]$, the total energy is defined as

$$E = \int_{t_1}^{t_2} |x(t)|^2 \, dt \qquad t_1 \leq t \leq t_2 \qquad \text{Continuous-time signal}$$

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad n_1 \leq n \leq n_2 \qquad \text{Discrete-time signal}$$

- The average power is defined as

$$P = \frac{E}{t_2 - t_1} \quad \text{Continuous} \qquad\qquad P = \frac{E}{n_2 - n_1 + 1} \quad \text{Discrete}$$

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Over infinite time interval $-\infty \leq t \leq \infty$ or $-\infty \leq n \leq \infty$

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Continuous

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Discrete

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

Continuous

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Discrete

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Finite-energy signal: $E_\infty < \infty$

$$P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T} = 0$$

$$P_\infty = \lim_{N \to \infty} \frac{E_\infty}{2N+1} = 0$$

- Finite-power signal: $P_\infty < \infty$, $E_\infty = \infty$

- Infinite energy & power signal $P_\infty \to \infty$, $E_\infty \to \infty$

Signal energy and power

- Examples:

(1) $x(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$ $\quad E_\infty < \infty, P_\infty = 0$

(2) $x[n] = 4$ $\quad P_\infty < \infty, E_\infty = \infty$

(3) $x(t) = t$ $\quad P_\infty \to \infty, E_\infty \to \infty$

- ❑ Continuous-Time and Discrete-Time Signals
- ❑ Transformations of the Independent Variable
- ❑ Exponential and Sinusoidal Signals
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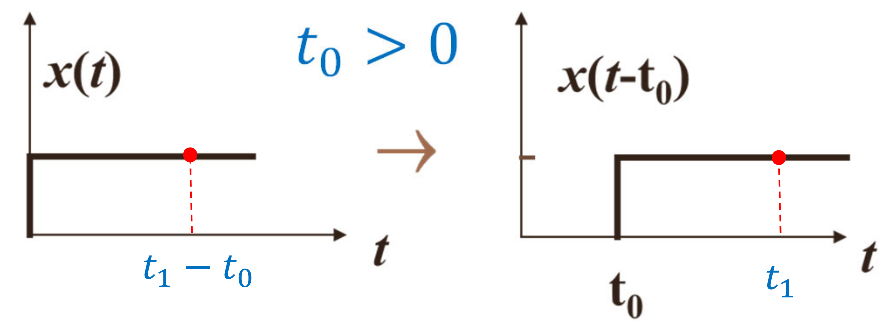
- ❑ Basic System Properties

Transformation of the independent variable

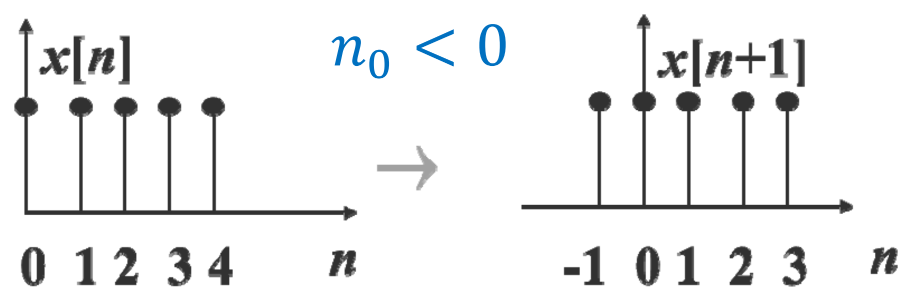
Time shift

$$x(t) \longrightarrow x(t-t_0)=y(t)$$



$$y(t)\Big|_{t=t_1} = x(t-t_0)\Big|_{t=t_1} = x(t_1-t_0) = x(t)\Big|_{t=t_1-t_0}$$

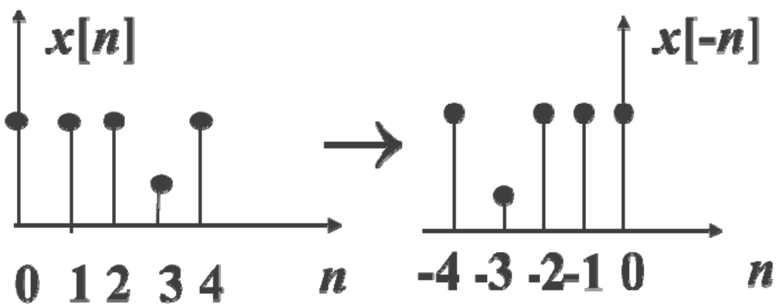
$$x[n] \longrightarrow x[n-n_0]$$



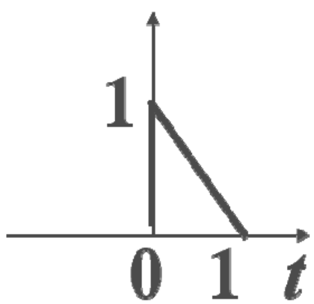
Transformation of the independent variable

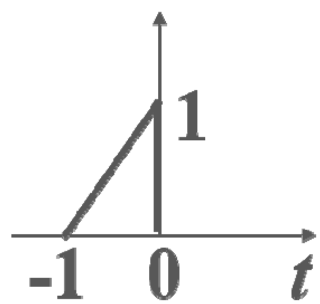
Time reversal

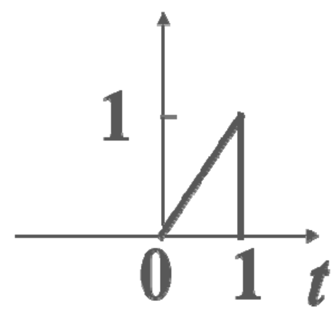
$x[n] \longrightarrow x[-n]$



$x(t) \longrightarrow x(-t) \longrightarrow x(1-t)$





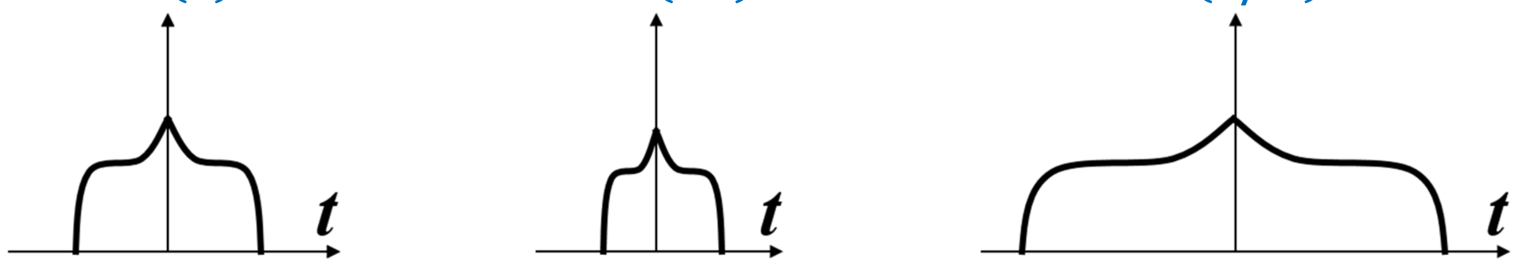


Transformation of the independent variable

Time scaling

$x(t)$	$\longrightarrow$	$x(2t)$	Compressed
$x(t)$	$\longrightarrow$	$x(t/2)$	Stretched

$x(t) \longrightarrow x(2t) \longrightarrow x(t/2)$



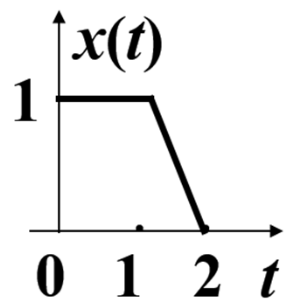
Transformation of the independent variable

General: Let $x(t) \rightarrow x(\alpha t + \beta)$

- if $|\alpha| > 1$, compressed
- if $|\alpha| < 1$, stretched
- if $\alpha < 0$, reversed
- if $\beta \neq 0$, shifted

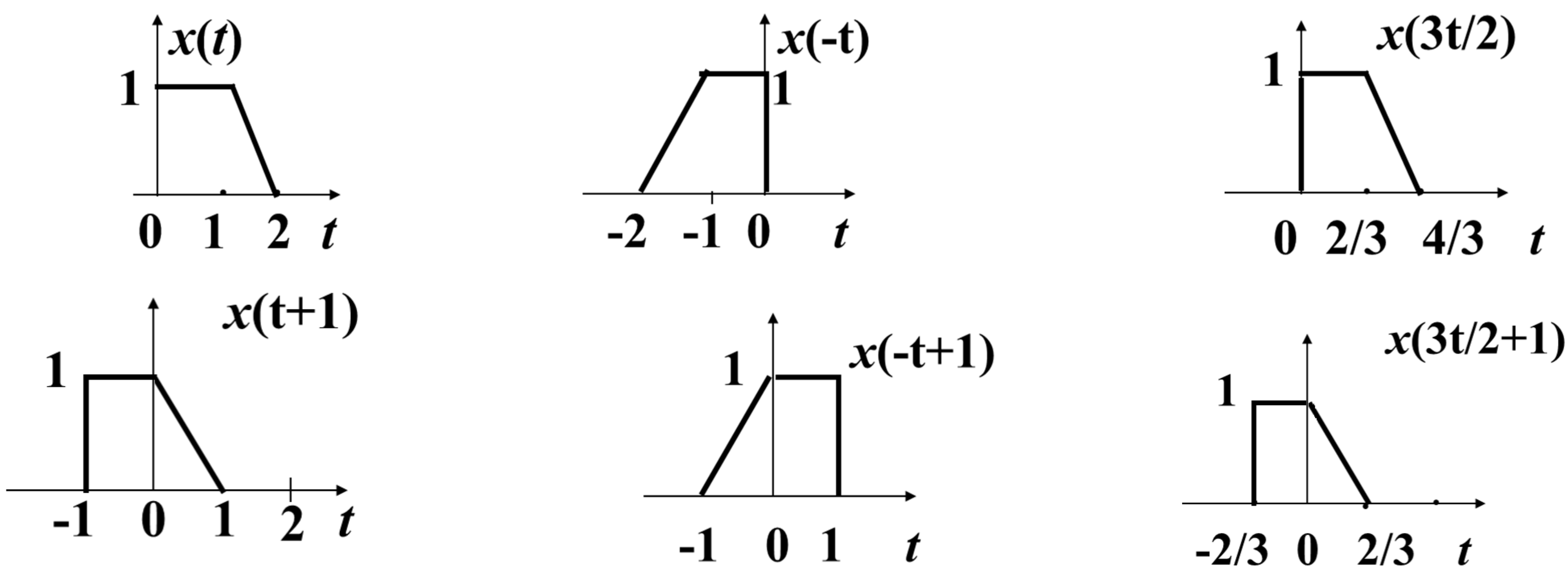
Example1: Given the signal $x(t)$, to illustrate

- $x(t+1)$
- $x(-t+1)$
- $x(3t/2)$
- $x(\frac{3t}{2}+1)$



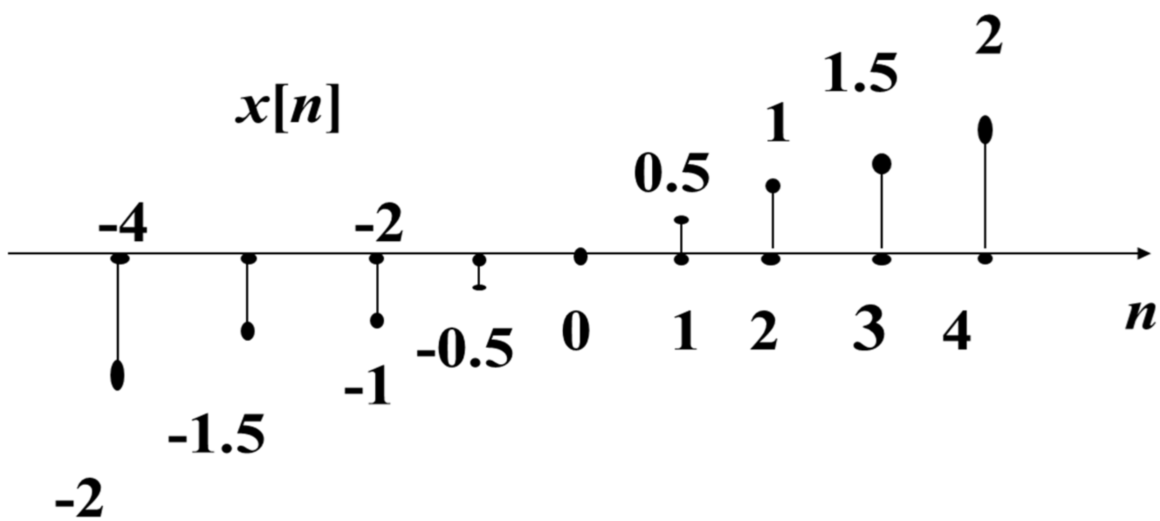
Transformation of the independent variable

- $x(t+1)$ $\quad x(-t+1)$ $\quad x(3t/2)$ $\quad x(\frac{3t}{2}+1)$

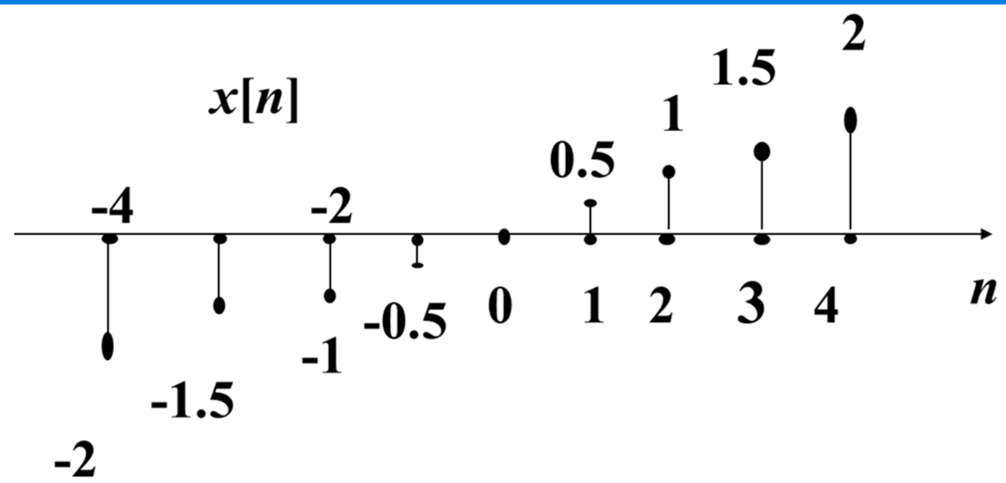


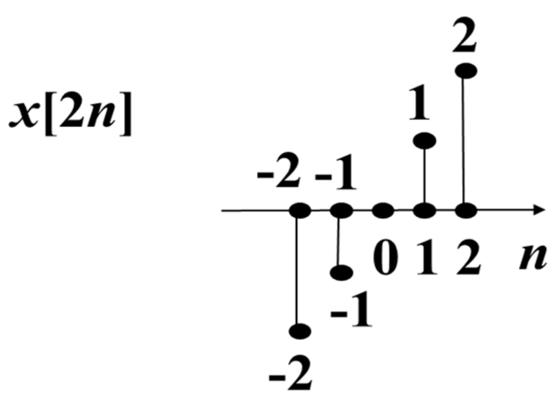
Transformation of the independent variable

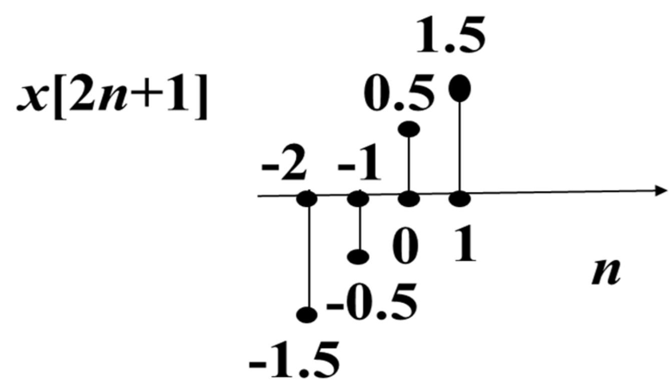
- ***Example2:*** A discrete signal $x[n]$ is shown below, sketch and label following signals:
  - $x[2n]$
  - $x[2n+1]$



Transformation of the independent variable

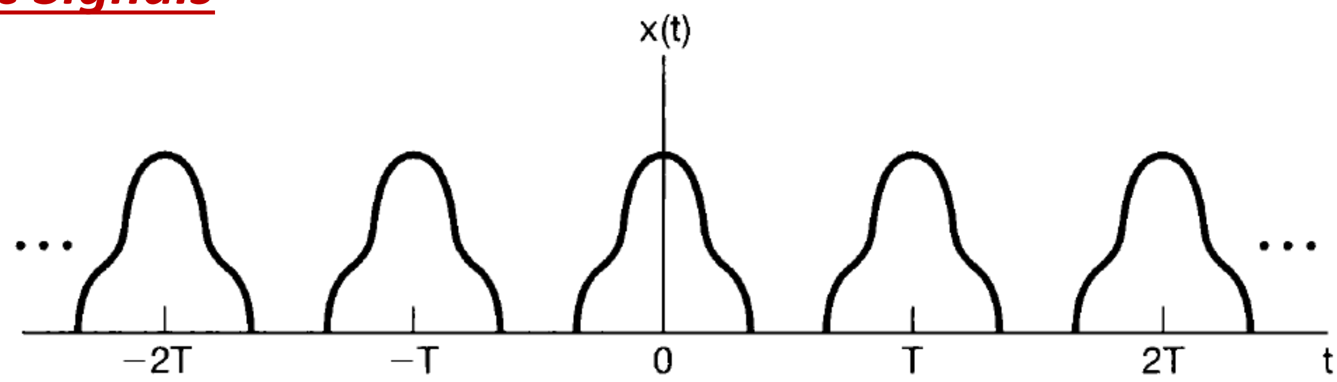






Transformation of the independent variable

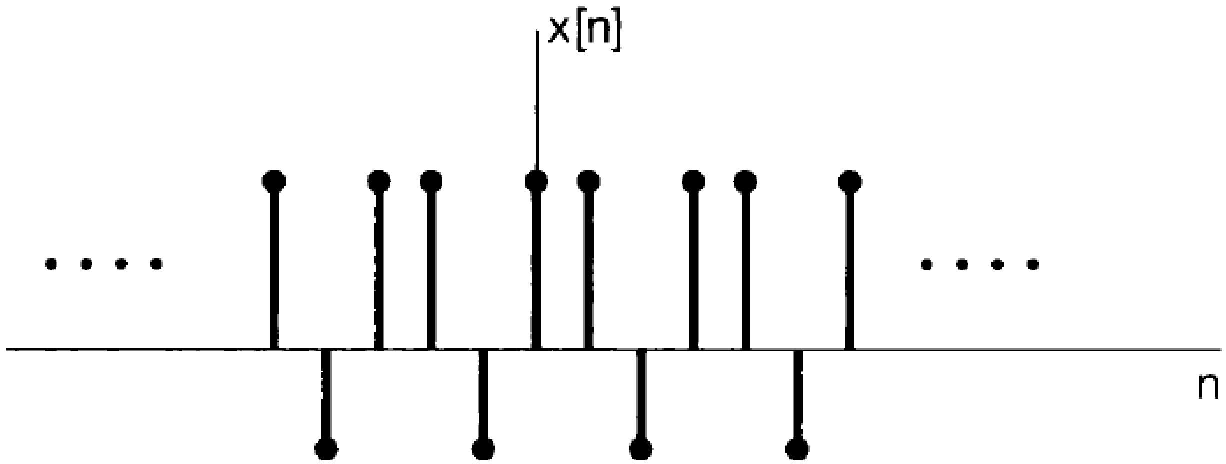
Periodic Signals



- Continuous-time: $x(t) = x(t + T)$ for all t
- Fundamental period
  - The smallest positive value of T for which $x(t) = x(t + T)$ holds

Transformation of the independent variable

Periodic Signals



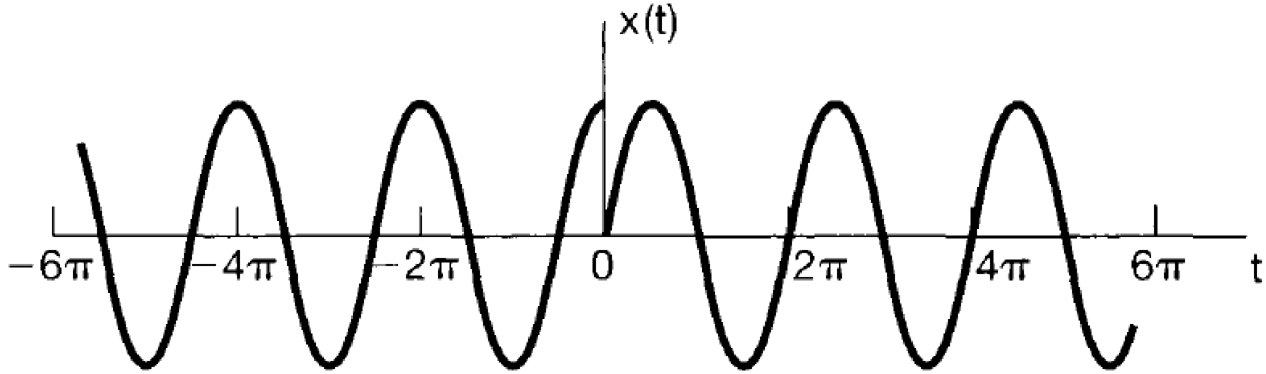
- ❑ Discrete-time: $x[n] = x[n+N]$ for all n
- ❑ Fundamental period
  - The smallest positive value of *N* for which $x[n] = x[n+N]$ holds

Transformation of the independent variable

Periodic Signals

- Example:

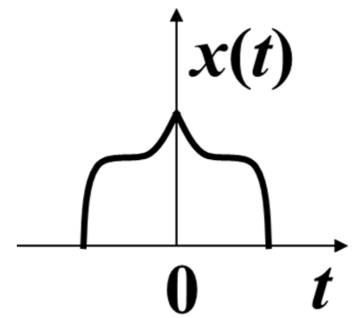
$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$



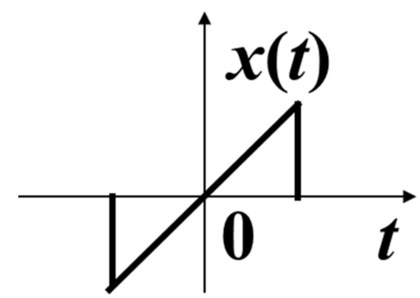
Transformation of the independent variable

Even and Odd Signals

- ❑ Even signal
  - $x(t) = x(-t) \quad x[n] = x[-n]$



- ❑ Odd signal
  - $x(t) = -x(-t) \quad x[n] = -x[-n]$



Transformation of the independent variable

Even and Odd Signals

- Any signal can be broken into a sum of two signals
  - One even and one odd

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

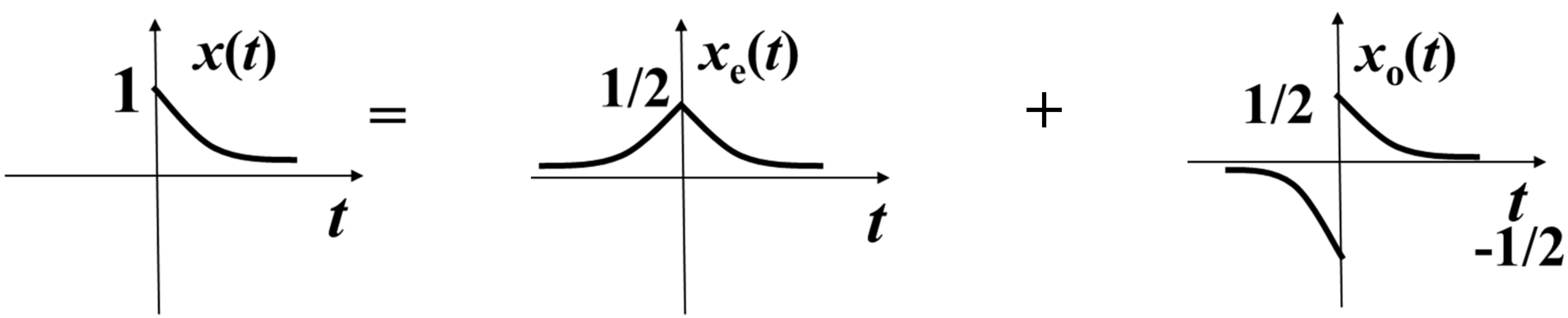
$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Transformation of the independent variable

Even and Odd Signals

$$x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$



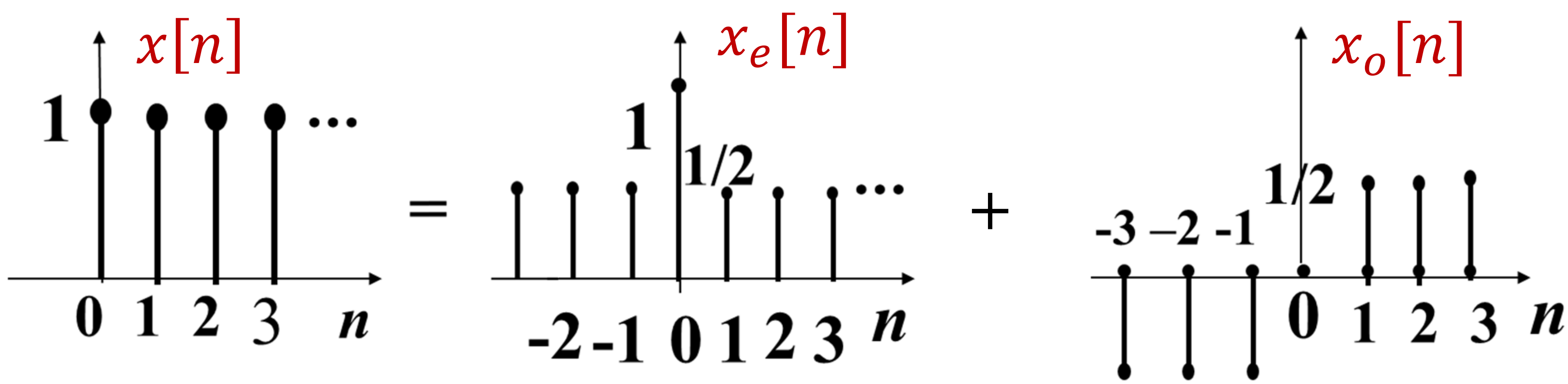
Transformation of the independent variable

Even and Odd Signals

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] = (x[n] + x[-n])/2$$

$$x_o[n] = (x[n] - x[-n])/2$$



Signals and Systems: An overview (ch.1)

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Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

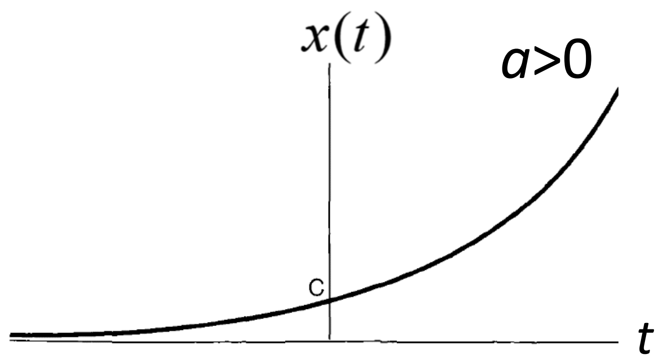
❑ General case

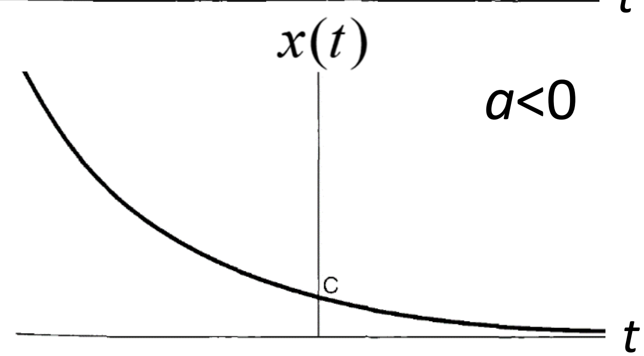
$$x(t) = ce^{at}$$

C and *a* are complex number

❑ Real exponential signal

- *C* and *a* are real
- $a>0$, as $t\uparrow$, $|x(t)|\uparrow$
- $a<0$, as $t\uparrow$, $|x(t)|\downarrow$
- $a=0$, $|x(t)|$ is constant





Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- **Periodic exponential signals**
  - *c* is real, specifically 1
  - *a* is purely imaginary

$$x(t) = e^{j\omega_0 t}$$

  - Fundamental period T_0?

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

$$\Rightarrow \omega_0 T = 2k\pi, k = \pm 1, \pm 2, \ldots \Rightarrow T = \frac{2k\pi}{\omega_0} \Rightarrow T_o = \frac{2\pi}{|\omega_0|}$$

  - T_0 is undefined for $\omega_0 = 0$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

❑ Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \emptyset)$$

- Closely related to complex exponential signals

$$e^{j(\omega_0 t+\emptyset)} = \cos(\omega_0 t + \emptyset) + j\sin(\omega_0 t + \emptyset)$$

$$A\cos(\omega_0 t + \emptyset) = A \cdot Re\{e^{j(\omega_0 t+\emptyset)}\}$$

$$A\sin(\omega_0 t + \emptyset) = A \cdot Im\{e^{j(\omega_0 t+\emptyset)}\}$$

- Fundamental frequency ω_0

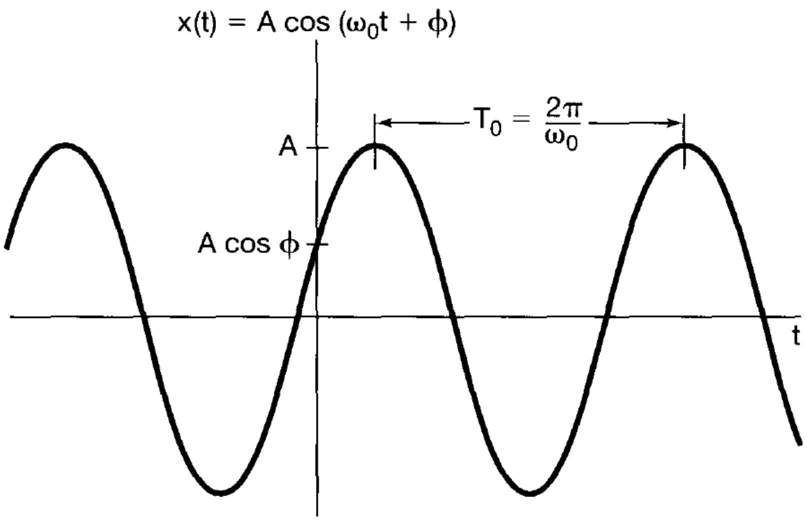
Exponential and Sinusoidal Signals

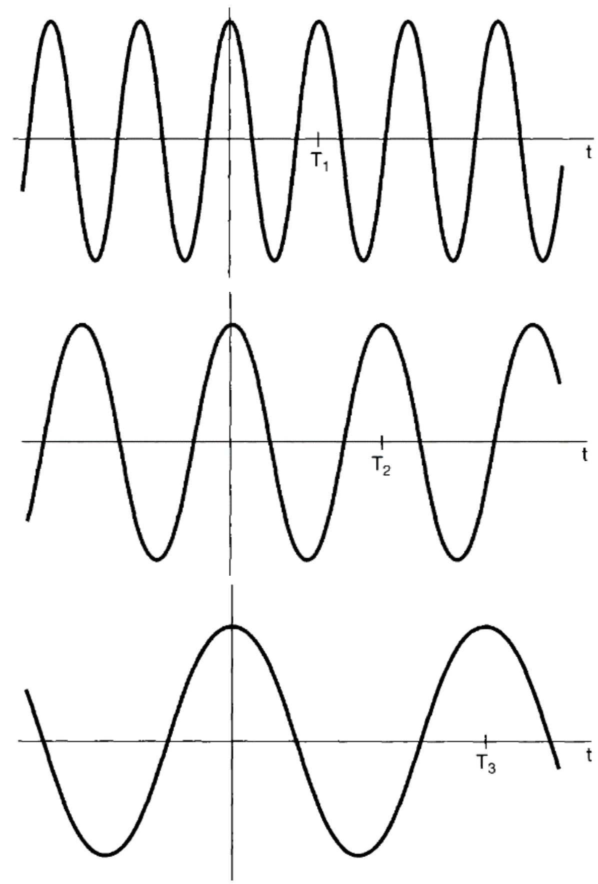
Continuous-Time Complex Exponential and Sinusoidal Signals

❑ Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \emptyset)$$

➢ Fundamental frequency ω_0





$\omega_3 < \omega_2 < \omega_1$

$T_3 > T_2 > T_1$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- $e^{j\omega_0 t}$ and $A\cos(\omega_0 t + \emptyset)$ examples of signals with infinite total energy but finite average power

$$E_{period} = \int_0^{T_0} \left| e^{j\omega_0 t} \right|^2 dt = \int_0^{T_0} 1 dt = T_0$$

$$p_{period} = \frac{1}{T_0} E_{period} = 1$$

  - Total energy: infinite
  - Average power: finite

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

❑ Harmonically related complex exponentials

- Sets of periodic exponentials (with different frequencies), all of which are periodic with a common period T_0

$$e^{j\omega t} = e^{j\omega(t+T_0)} = e^{j\omega t}e^{j\omega T_0}$$

$$\omega T_0 = 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots .$$

$$\omega = 2k\pi/T_0 = k\omega_0, \text{ with } \omega_0 = 2\pi/T_0$$

- $\emptyset_k(t) = e^{jk\omega_0 t}$, $k = 0, \pm 1, \pm 2, \ldots$ is a harmonically related set.
- For any $k \neq 0$, fundamental frequency $|k|\omega_0$; fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- Examples – Periodic or not?

(1) $x_1(t) = je^{j10t}$ $\qquad \omega_0 = 10,\ T_0 = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

(2) $x_2(t) = e^{(-1+j)t}$ $\qquad$ Aperiodic

(3) $x_3(t) = 2\cos(3t + \dfrac{\pi}{4})$ $\qquad \omega_0 = 3,\ T_0 = \dfrac{2\pi}{3}$

(4) $x(t) = 2\cos(3t + \dfrac{\pi}{4}) + 3\cos(2t - \dfrac{\pi}{6})$

$$T_{01} = \frac{2\pi}{3},\quad T_{02} = \pi \qquad T_0 = SCM(T_{01}, T_{02}) = 2\pi$$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- General case

$$x(t) = Ce^{at}$$

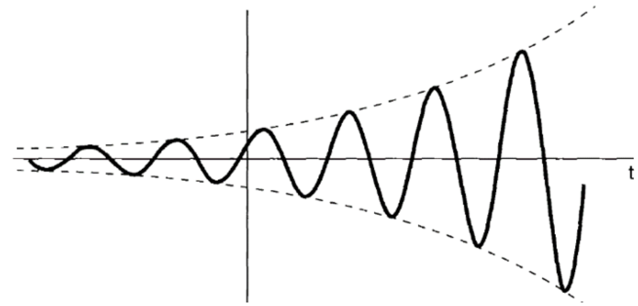
C and a are complex numbers

$$C = |C|e^{j\theta}, a = r + j\omega_0$$

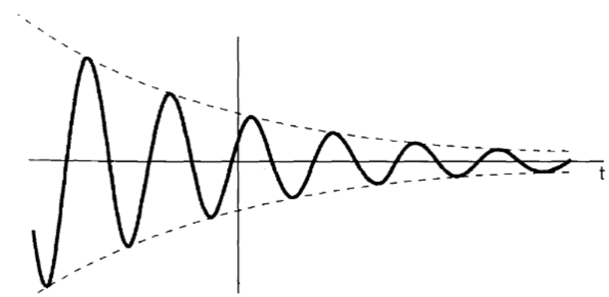
$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta), r > 0$



$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta), r < 0$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

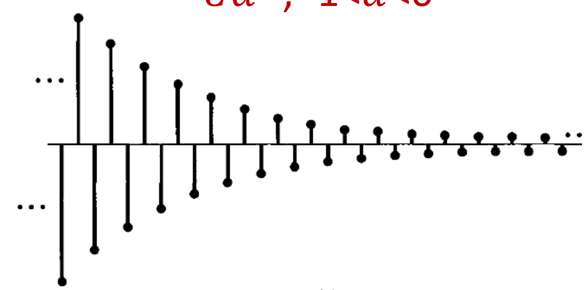
- General case

$$x[n] = C\alpha^n$$

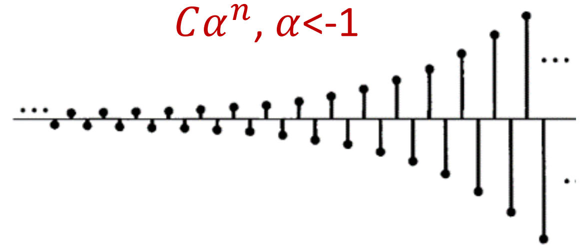
c and α are complex numbers

$$x[n] = Ce^{\beta n} \qquad \alpha = e^{\beta}$$

$C\alpha^n$, -1<α<0



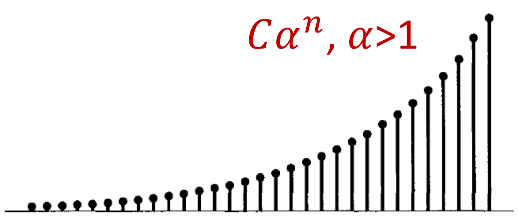
$C\alpha^n$, α<-1



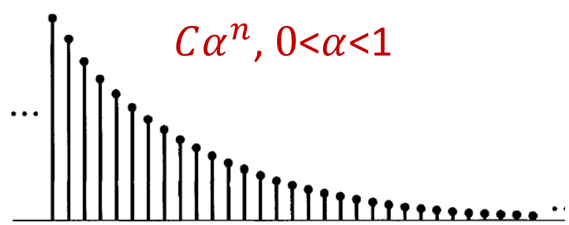
- Real Exponential Signals

C and α are real numbers

$C\alpha^n$, α>1



$C\alpha^n$, 0<α<1



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Sinusoidal signals
  - c is real, specifically 1; β is purely imaginary

$$x[n] = e^{j\omega_0 n}$$

Closely related $$A\cos(\omega_0 n + \emptyset)$$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

$$A\cos(\omega_0 n + \emptyset) = A/2 \cdot e^{j\emptyset} e^{j\omega_0 n} + A/2 \cdot e^{-j\emptyset} e^{-j\omega_0 n}$$

  - Infinite total energy but finite average power

$$|e^{j\omega_0 n}|^2 = 1$$

Exponential and Sinusoidal Signals

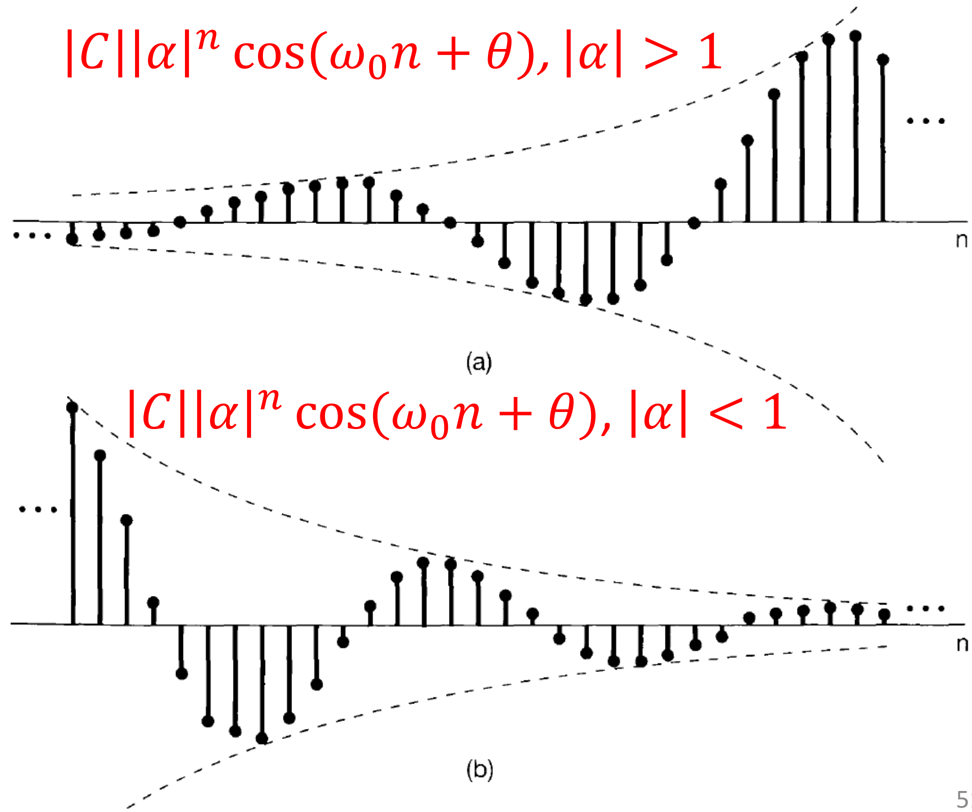
Discrete-Time Complex Exponential and Sinusoidal Signals

- General Signals

$$x[n] = C\alpha^n$$

$$C = |C|e^{j\theta}, \alpha = |\alpha|e^{j\omega_0}$$

$$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta) \\ +j\,|C||\alpha|^n \sin \omega_0 n + \theta$$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties $x[n] = e^{j\omega_0 n}$ **Focusing on ω_0**
  - ω_0, same value at ω_0 and $\omega_0 + 2k\pi$

$$e^{j(\omega_0+2k\pi)n} = e^{j2k\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

  - Only consider interval $0 \leq \omega_0 \leq 2\pi$ or $-\pi \leq \omega_0 \leq \pi$

- From 0 to π: $\omega_0 \uparrow$, oscillation rate of $e^{j\omega_0 n} \uparrow$
- From π to 2π: $\omega_0 \uparrow$, oscillation rate of $e^{j\omega_0 n} \downarrow$
- Maximum oscillation rate at $\omega_0 = \pi$

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

Exponential and Sinusoidal Signals

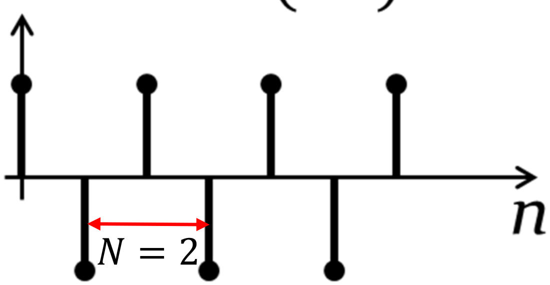
Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

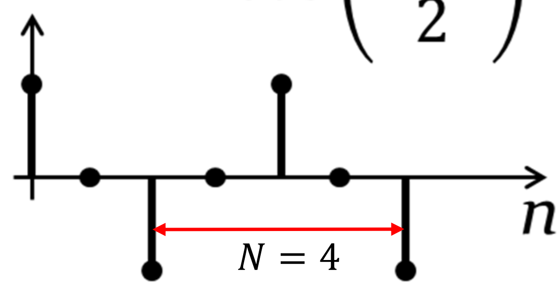
  - Q: Which one is a higher frequency signal?

$$\omega_0 = \pi \qquad \omega_0 = 3\pi/2$$

$$\cos(\pi n)$$



$$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$$

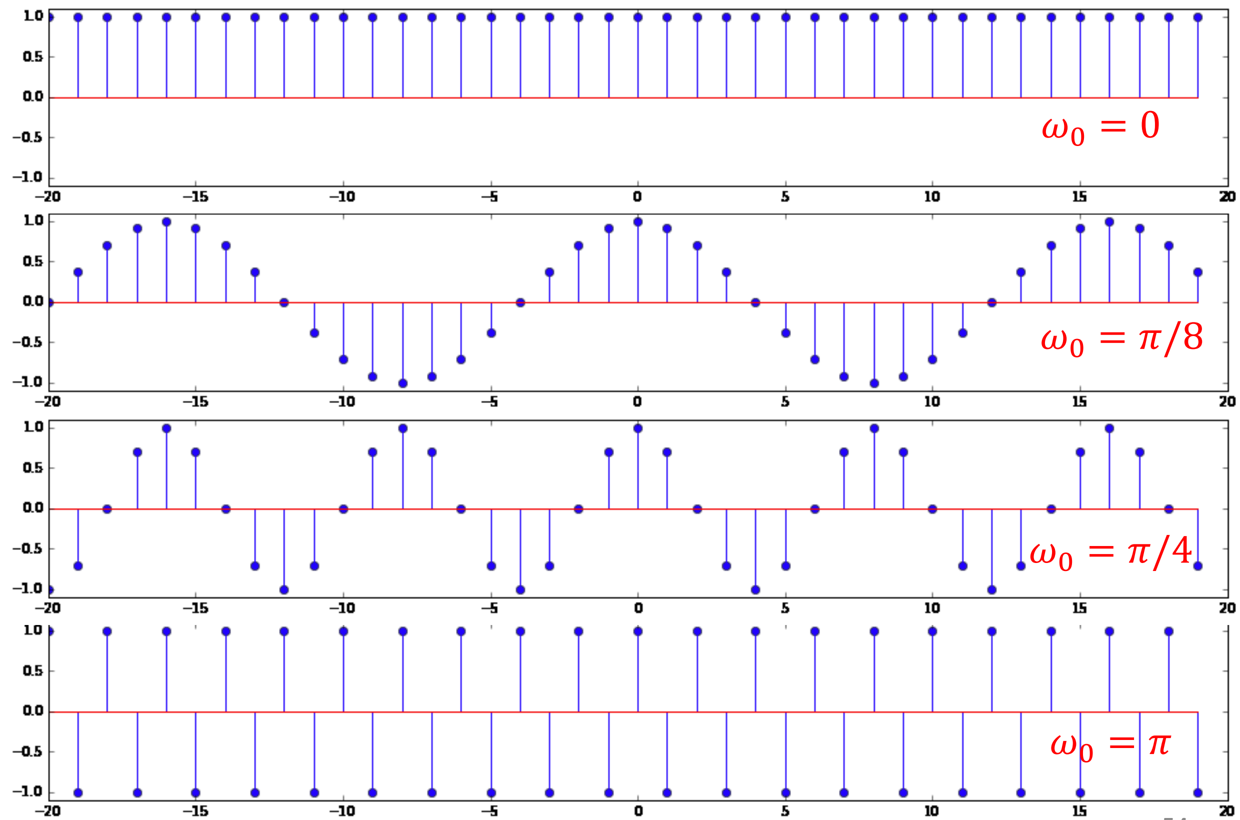


Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

$$\cos(\omega_0 n)$$

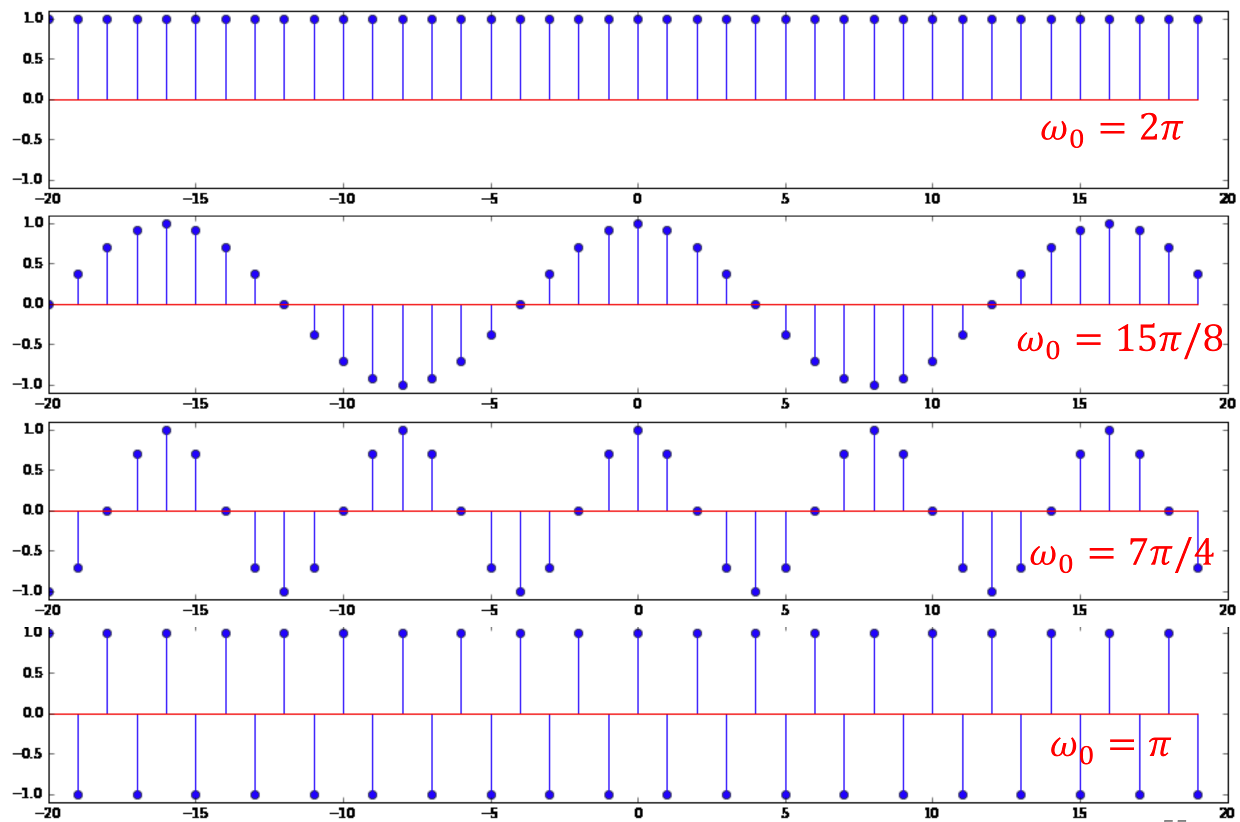


Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

$$\cos(\omega_0 n)$$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties $x[n] = e^{j\omega_0 n}$ **Focusing on n**

  - In order for $e^{j\omega_0 n}$ to be periodic with *N*>0, must

$$e^{j\omega_0 (n+N)} = e^{j\omega_0 N} e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$\omega_0 N = 2\pi m,\ m \text{ integer number}$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

  - $\omega_0/2\pi$: rational number
  - Fundamental frequency: $2\pi/N = \omega_0/m$
  - Fundamental period: $N = m(2\pi/\omega_0)$

Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

$x[n] = \cos(2\pi n/12)$ periodic N=12

$x[n] = \cos(8\pi n/31)$ periodic N=31

$x[n] = \cos(n/6)$ aperiodic

$x[n] = e^{j\left(\frac{2\pi n}{3}\right)} + e^{j\left(\frac{3\pi n}{4}\right)}$ periodic, N=24

Exponential and Sinusoidal Signals

Periodicity properties: discrete-time vs. continuous-time

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct ω_0	**Identical signals for values of ω_0 separated by multiples of 2π**
Periodic for any ω_0	**Only if $\omega_0=2\pi m/N$ for some integers N>0 and m**
Fundamental frequency ω_0	**ω_0/m**
Fundamental period $2\pi/\omega_0$	**$N=m(2\pi/\omega_0)$**

Signals and Systems: An overview (ch.1)

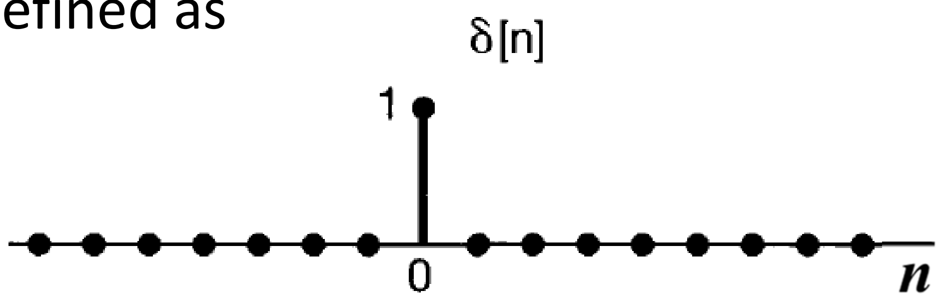
- ❑ Continuous-Time and Discrete-Time Signals
- ❑ Transformations of the Independent Variable
- ❑ Exponential and Sinusoidal Signals
- ❑ The Unit Impulse and Unit Step Functions
- ❑ Continuous-Time and Discrete-Time Systems
- ❑ Basic System Properties

The Unit Impulse and Unit Step Functions

Discrete-time unit impulse and unit step sequences

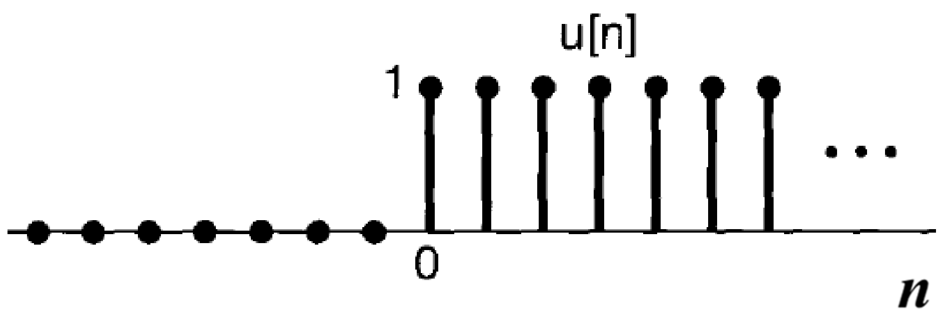
- ❑ Unit impulse (unit sample) is defined as

$$\delta[n]=\begin{cases}0, n\neq 0\\ 1, n=0\end{cases}$$



- ❑ Unit step is defined as

$$u[n]=\begin{cases}0, n<0\\ 1, n\geq 0\end{cases}$$



The Unit Impulse and Unit Step Functions

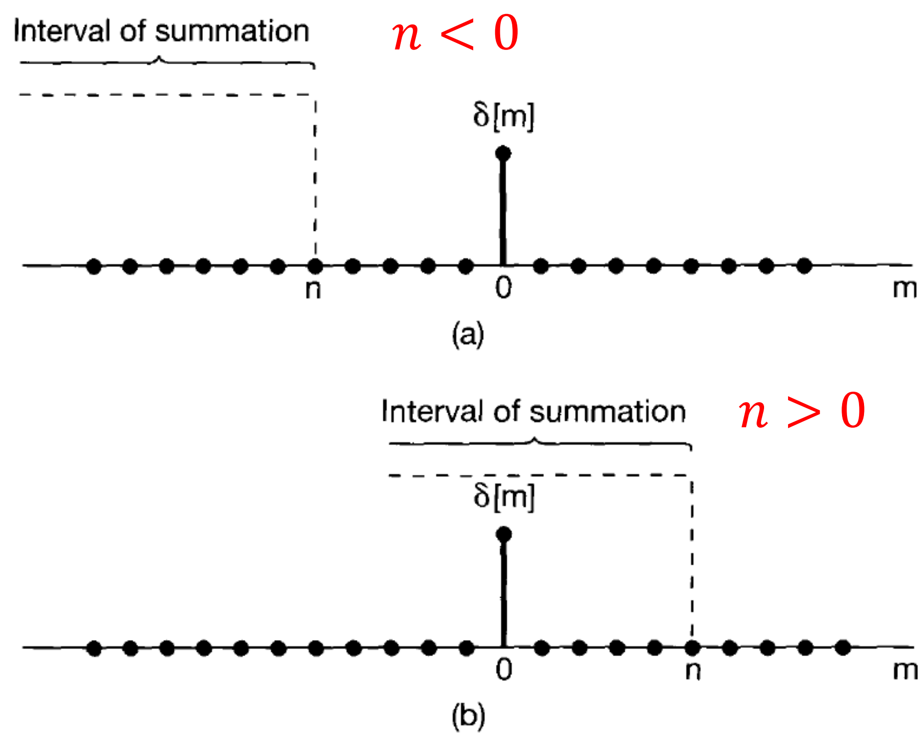
Discrete-time unit impulse and unit step sequences

- The impulse is the first difference of the step

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, the step is the running sum of unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$



The Unit Impulse and Unit Step Functions

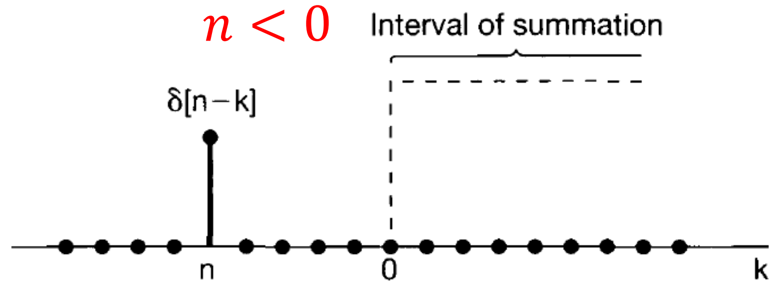
Discrete-time unit impulse and unit step sequences

- Let $m = n - k$,

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

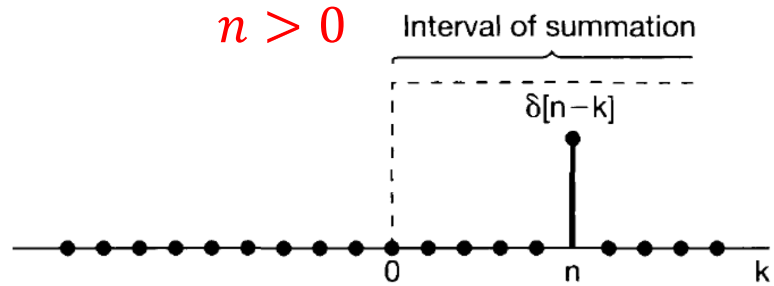
or $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$n < 0$



(a)

$n > 0$



(b)

The Unit Impulse and Unit Step Functions

Discrete-time unit impulse and unit step sequences

- Sampling property

$$x[n]\delta[n] = x[0]\delta[n]$$

- More generally

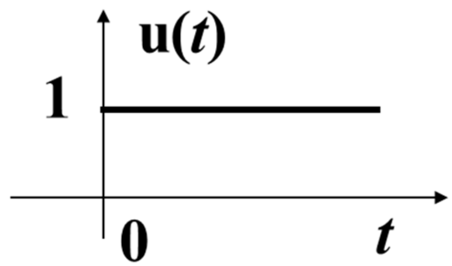
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- Unit step

$$u(t) = \begin{cases} 0, & t < o \\ 1, & t > 0 \end{cases}$$



- The continuous unit step $u(t)$ is the running integral of unit impulse $\delta(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau$$

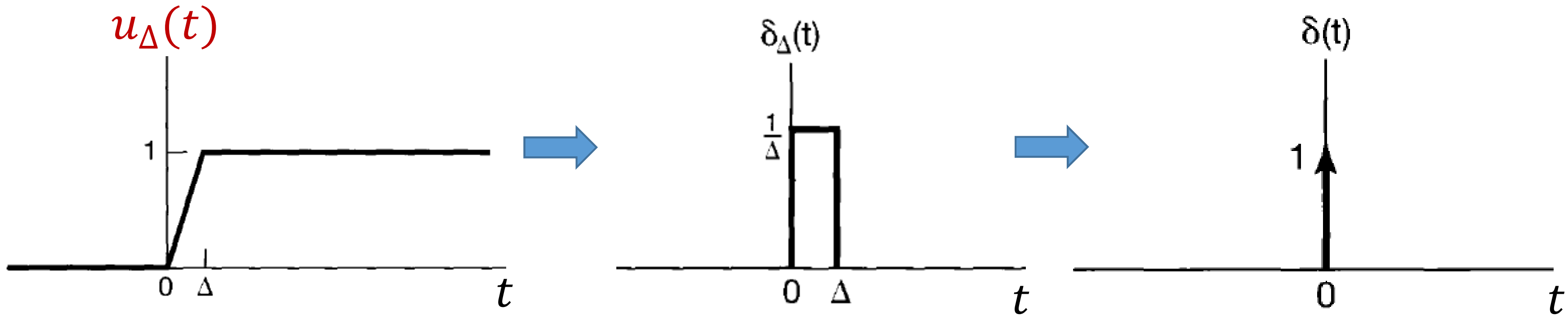
- $\delta(t)$ the first derivative of $u(t)$

$$\delta(t) = \frac{du(t)}{dt}$$

The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- $u(t)$ is discontinuous at $t = 0$, How we get $\delta(t)$?
  - *Consider* $u_\Delta(t)$



$$u(t) = \lim_{\Delta \to 0} u_\Delta(t) \qquad \delta_\Delta(t) = \frac{d\, u_\Delta(t)}{dt} \qquad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

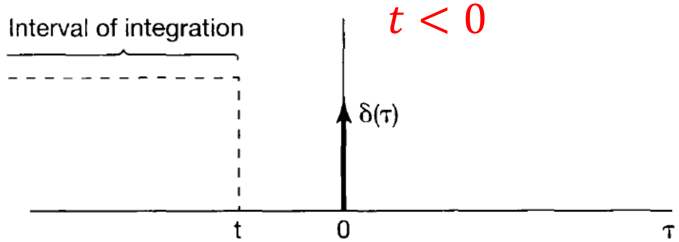
- arrow at $t = 0$: area of the pulse is concentrated at $t = 0$
- arrow height and "1“: *area* of the impulse

The Unit Impulse and Unit Step Functions

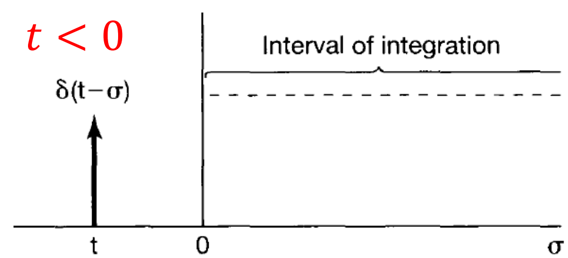
Continuous-time unit impulse and unit step sequences

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad \text{Let } \sigma = t - \tau \qquad u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$

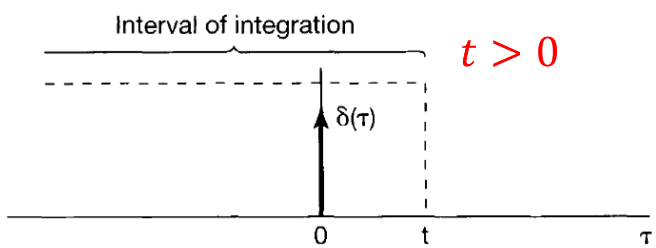
$t < 0$



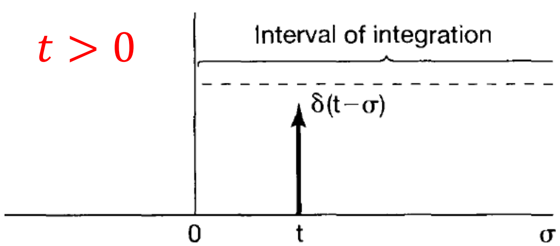
$t < 0$



$t > 0$



$t > 0$



The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- Sampling property

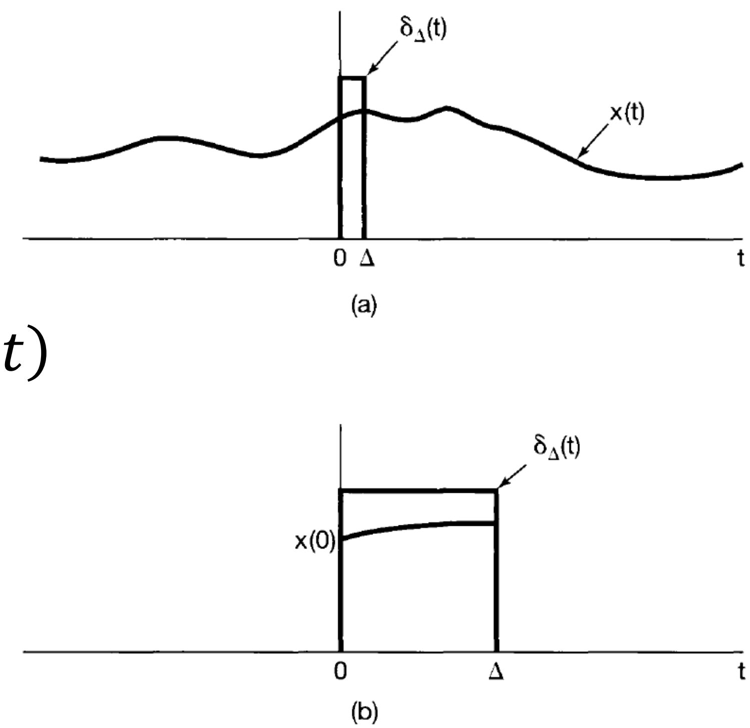
$$x_1(t) = x(t)\delta_\Delta(t)$$

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

$$x(t)\delta(t) = \lim_{\Delta\to 0} x(t)\delta_\Delta(t) = x(0)\delta(t)$$

- More generally

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



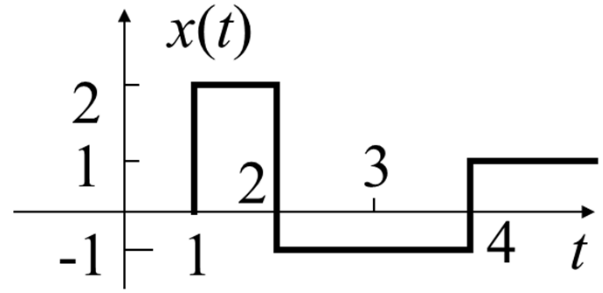
The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- Example:

  (1) Calculate and sketch the $x'(t)$;

  (2) Recover $x(t)$ from $x'(t)$.

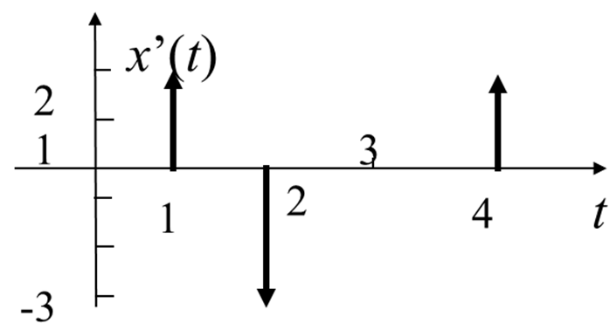


- Solutions:

  (1) $x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$

  $$\therefore \quad x'(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$

  (2) $x(t) = \int_0^{\infty} x'(t)dt$



Signals and Systems: An overview (ch.1)

- ❑ Continuous-Time and Discrete-Time Signals
- ❑ Transformations of the Independent Variable
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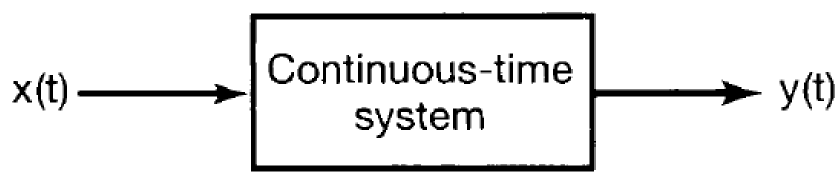
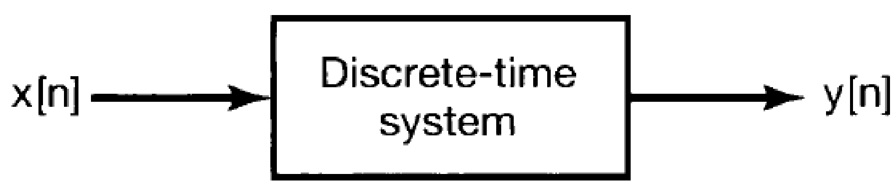
- ❑ Continuous-Time and Discrete-Time Systems
- ❑ Basic System Properties

Continuous-Time and Discrete-Time Systems

- Continuous-Time Systems: Input and output are continuous

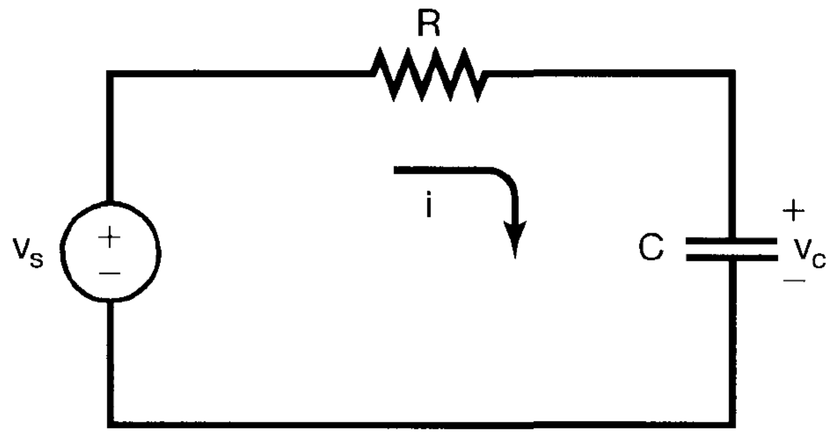


- Discrete-Time Systems: Input and output are discrete



Examples of systems

- RC circuit



$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

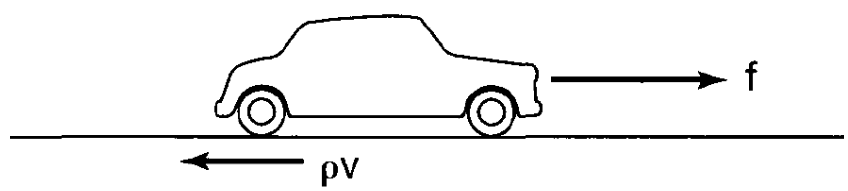
$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t).$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Moving car

$$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - \rho v(t)\right]$$



$$\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

In general: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Balance in a bank account:

$$y[n] = 1.01y[n-1] + x[n]$$

$y[n]$: balance at the end of the nth month; $x[n]$: net deposit; Interest rate: 1%

$$y[n] - 1.01y[n-1] = x[n]$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Digital simulation a differential equation $$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

  - Approximate $dv(t)/dt$ at $t = n\Delta$ by $\frac{v(n\Delta)-v((n-1)\Delta)}{\Delta}$

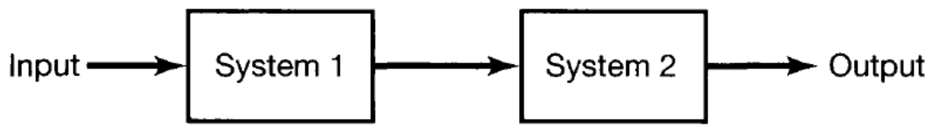
$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta} + \frac{\rho}{m}v(n\Delta) = \frac{1}{m}f(n\Delta)$$

  - Let $v[n] = v(n\Delta)$ $$v[n] - \frac{m}{m+\rho\Delta}v[n-1] = \frac{\Delta}{m+\rho\Delta}f[n]$$

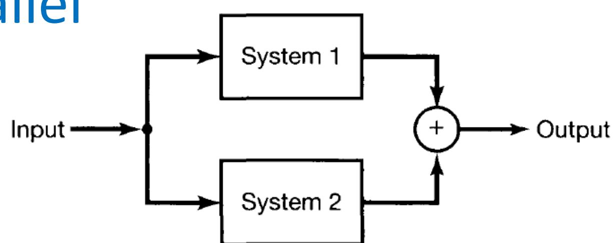
  - In general $$y[n] + ay[n-1] = bx[n]$$

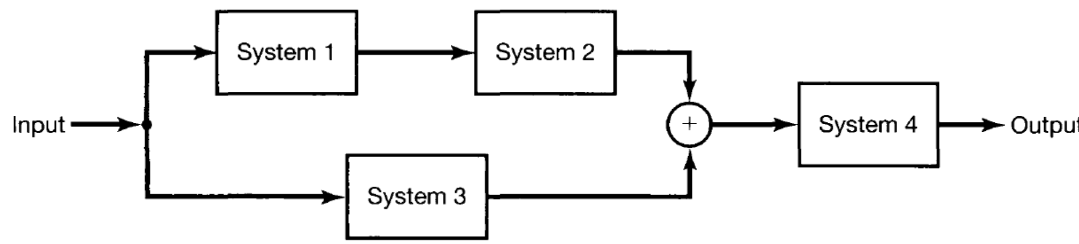
Interconnections of systems

- Series (or cascade)

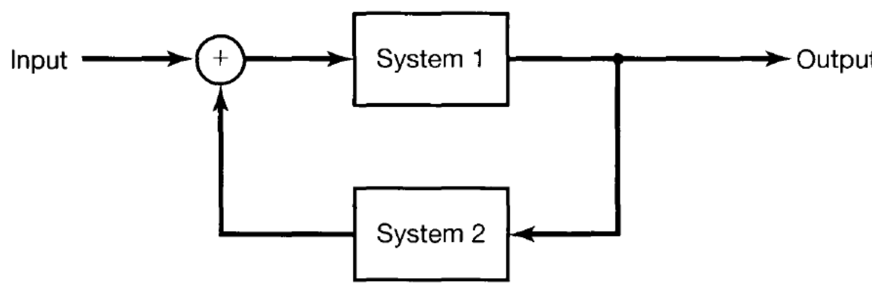


- Parallel





- Feedback



Signals and Systems: An overview (ch.1)

- ❑ Continuous-Time and Discrete-Time Signals
- ❑ Transformations of the Independent Variable
- ❑ Exponential and Sinusoidal Signals
- ❑ The Unit Impulse and Unit Step Functions
- ❑ Continuous-Time and Discrete-Time Systems
- ❑ **Basic System Properties**

Basic System Properties

System with and without memory

- System without memory:
  - Output is dependent only on the current input
  - Examples:

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t)$$

$$y(t) = x(t)$$

$$y[n] = x[n]$$

Basic System Properties

System with and without memory

- System with memory:
  - Output is dependent on the current and previous inputs
  - Examples:

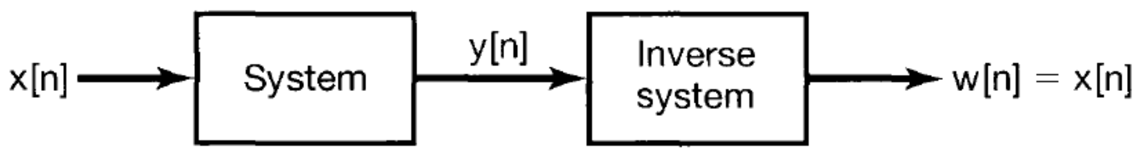
$$y[n] = \sum_{k=-\infty}^{n} x[k], \qquad y[n] = x[n-1] \qquad y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

  - Memory: retaining or storing information about input values at times
  - Physical systems, memory is associated with the storage of energy

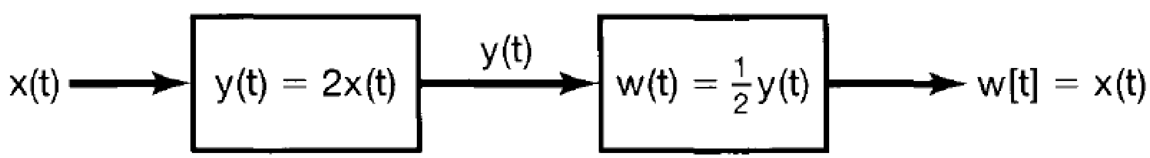
Basic System Properties

Invertibility and inverse system

- Invertible
  - Distinct inputs lead to distinct outputs.



$$y(t) = 2x(t) \qquad w(t) = \frac{1}{2}y(t)$$

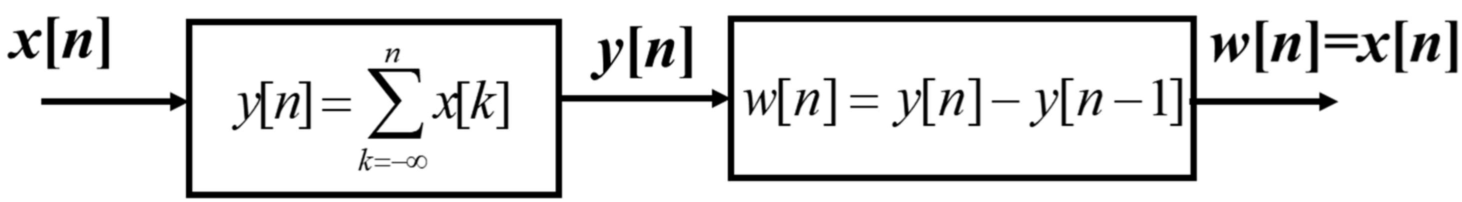


Basic System Properties

Invertibility and inverse system

- Invertible
  - Examples: Accumulator $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
  - The difference between two successive outputs is precisely the inputs

$$y[n] - y[n-1] = x[n]$$



Basic System Properties

Invertibility and inverse system

- Noninvertible

$$y[n] = 0$$

All $x[n]$ leads to the same $y[n]$

$$y(t) = x^2(t)$$

Cannot determine the sign of the inputs

Basic System Properties

Causality

- Causal: the output at any time depends only on the inputs at the present time and in the past

$$y(t) = Rx(t)$$ Causal

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ Causal

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$ Causal

$$y[n] = x[n] - x[n+1]$$ Non-causal

$$y(t) = x(t+1)$$ Non-causal

Basic System Properties

Causality

- Examples

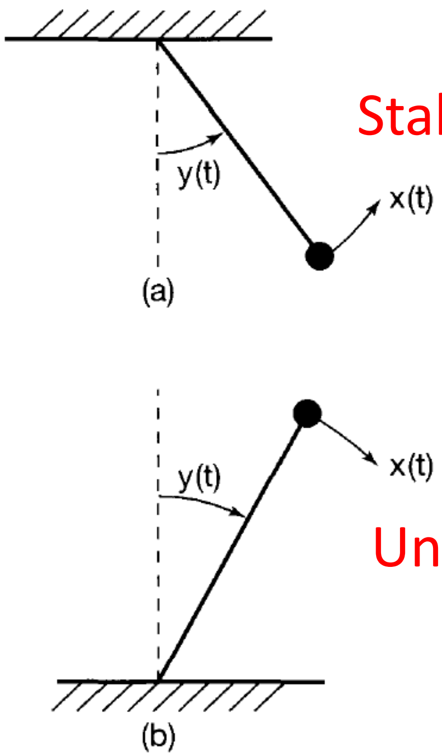
$$y[n] = x[-n]$$ Non-causal

$$y(t) = x(t)\cos(t+1)$$ Causal

Basic System Properties

Stability

- Informally: small inputs lead to responses that do not diverge.



Stable

Unstable

A bank account balance

$$y[n] = x[n] + (1+\alpha) \times y[n-1]$$

Unstable

Basic System Properties

Stability

- Formally: bounded input leads to bounded output
  - Bounded: $|y(t)| < B$

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

Stable

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n]$$

Unstable

Stability

- Examples

$$S_1: y(t) = tx(t)$$ Unstable

$$S_2: y(t) = e^{x(t)}$$ Stable

$$|x(t)| < B \quad \rightarrow \quad -B < x(t) < B \quad \rightarrow \quad e^{-B} < y(t) < e^{B}$$

Basic System Properties

Time Invariance

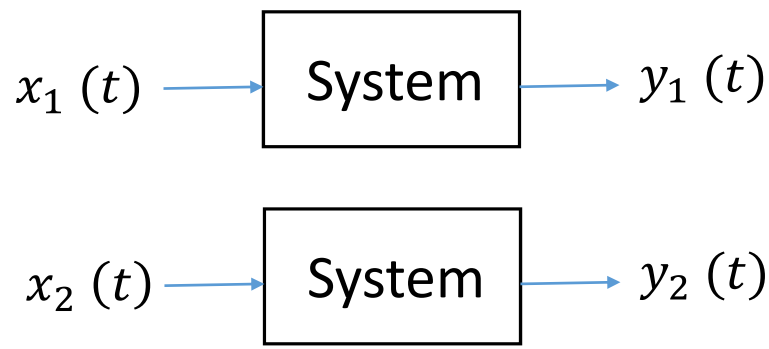
- Time invariant: a time shift in the input signal results in an identical time shift in the output signal

If $\quad x[n] \rightarrow y[n]$

Then $\quad \boldsymbol{x[n-n_0]} \rightarrow \boldsymbol{y[n-n_0]}$

If $\quad x(t) \rightarrow y(t)$

Then $\quad x(t-t_0) \rightarrow y(t-t_0)$



If $x_2(t) = x_1(t-t_0)$

$y_2(t) = f\{x_2(t)\}$

$y_2'(t) = y_1(t-t_0)$

$y_2(t) = y_2'(t)$?

Basic System Properties

Time Invariance

- Examples: $y(t) = \sin[x(t)]$

$$x_1(t) \longrightarrow \boxed{\sin[x(t)]} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\sin[x(t)]} \longrightarrow y_2(t)$$

If $x_2(t) = x_1(t - t_0)$

$$y_2(t) = f\{x_2(t)\}$$

$$f\{\cdot\} = \sin\{\cdot\}$$

$$y_2(t) = \sin[x_1(t - t_0)]$$

$$y_2'(t) = y_1(t - t_0)$$

$$y_1(t) = \sin[x_1(t)]$$

$$y_2'(t) = \sin[x_1(t - t_0)]$$

$$\therefore y_2(t) = y_2'(t)$$

Basic System Properties

Time Invariance

- Examples: $y[n] = nx[n]$

$$x_1[n] \rightarrow \boxed{nx[n]} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{nx[n]} \rightarrow y_2[n]$$

If $x_2[n] = x_1[n - n_0]$

$$y_2[n] = f\{x_2[n]\}$$
$$= n \cdot x_1[n - n_0]$$

$$y_2'[n] = y_1[n - n_0]$$
$$y_1[n] = n \cdot x_1[n]$$
$$y_2'[n] = (n - n_0) \cdot x_1[n - n_0]$$
$$\therefore y_2[n] \neq y_2'[n]$$

Basic System Properties

Time Invariance

- Examples: $y(t) = x(2t)$

$$x_1(t) \longrightarrow \boxed{x(2t)} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{x(2t)} \longrightarrow y_2(t)$$

$$If\ x_2(t) = x_1(t - t_0)$$

$$y_2(t) = f\{x_2(t)\}$$

$$= x_1(2t - t_0)$$

$$y_2'(t) = y_1(t - t_0)$$

$$y_1(t) = x_1(2t)$$

$$y_2'(t) = x_1[2(t - t_0)]$$

$$\therefore y_2(t) \neq y_2'(t)$$

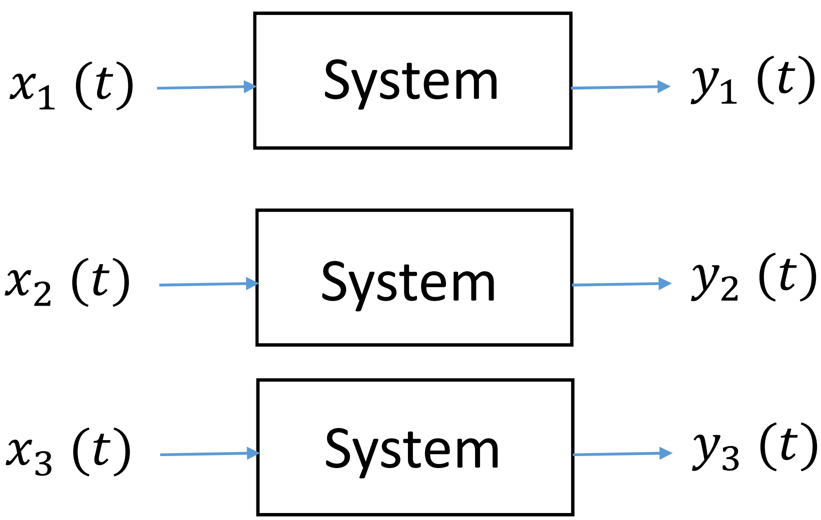
Basic System Properties

Linearity

- Linear $x_1(t) \rightarrow y_1(t), \qquad x_2(t) \rightarrow y_2(t)$

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

Superposition property
(additivity and homogeneity)



If $x_3(t) = a x_1(t) + b x_2(t)$

$$y_3\ (t) = f\{x_3(t)\}$$

$$y_3'(t) = a y_1(t) + b y_2(t)$$

$$y_3\ (t) = y_3'(t)\ \ ?$$

Basic System Properties

Linearity

- Examples $y(t) = tx(t)$

$x_1(t) \rightarrow \boxed{tx(t)} \rightarrow y_1(t) = tx_1(t)$

$x_2(t) \rightarrow \boxed{tx(t)} \rightarrow y_2(t) = tx_2(t)$

$x_3(t) \rightarrow \boxed{tx(t)} \rightarrow y_3(t) = tx_3(t)$

If $x_3(t) = ax_1(t) + bx_2(t)$

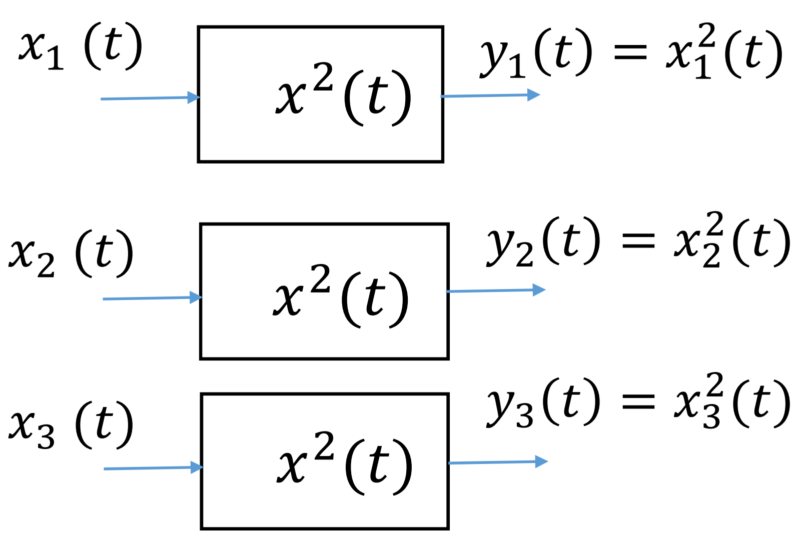
$$y_3(t) = f\{x_3(t)\}$$
$$= t[ax_1(t) + bx_2(t)]$$

$$y_3'(t) = ay_1(t) + by_2(t)$$
$$y_3'(t) = atx_1(t) + btx_1(t)$$

$$y_3(t) = y_3'(t)$$

Linearity

- Examples $y(t) = x^2(t)$



If $x_3(t) = ax_1(t) + bx_2(t)$

$$y_3(t) = f\{x_3(t)\}$$

$$= [ax_1(t) + bx_2(t)]^2$$

$$y_3'(t) = ay_1(t) + by_2(t)$$

$$= ax_1^2(t) + \mathrm{b}x_2^2(t)$$

$$y_3(t) \neq y_3'(t)$$

Basic System Properties

Linearity

- Examples $y[n] = Re\{x[n]\}$

$$x_1[n] \rightarrow \boxed{Re\{x[n]\}} \rightarrow y_1[n] = Re\{x_1[n]\}$$

$$x_2[n] \rightarrow \boxed{Re\{x[n]\}} \rightarrow y_2[n] = Re\{x_2[n]\}$$

$$x_3[n] \rightarrow \boxed{Re\{x[n]\}} \rightarrow y_3[n] = Re\{x_3[n]\}$$

If $x_3[n] = ax_1[n] + bx_2[n]$

$$y_3[n] = f\{x_3[n]\}$$
$$= Re\{ax_1[n] + bx_2[n]\}$$

$$y_3'[n] = ay_1[n] + by_2[n]$$
$$= aRe\{x_1[n]\} + bRe\{x_2[n]\}$$

If a and b are complex numbers

$$y_3[n] \neq y_3'[n]$$

Basic System Properties

Linearity

- Examples $y[n] = 2x[n] + 3$

$$x_1[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_2[n] = 2x_2[n] + 3$$

$$x_3[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_3[n] = 2x_3[n] + 3$$

If $x_3[n] = ax_1[n] + bx_2[n]$

$$y_3[n] = f\{x_3[n]\}$$

$$= 2(ax_1[n] + bx_2[n]) + 3$$

$$y_3'[n] = ay_1[n] + by_2[n]$$

$$= a(2x_1[n] + 3) + b(2x_1[n] + 3)$$

$$y_3[n] \neq y_3'[n]$$