Approximation algorithms 2
Scheduling, Knapsack

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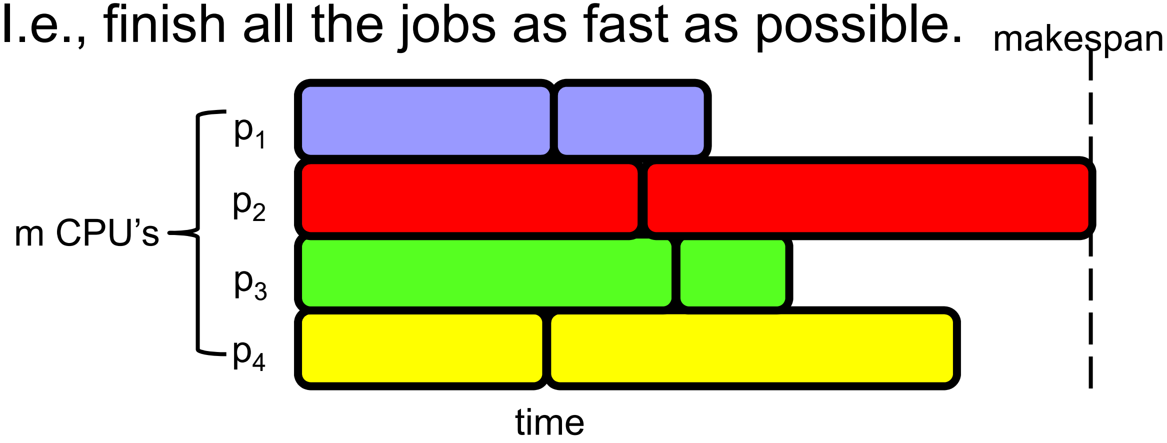
Parallel computing and scheduling

- Computers today are parallel.
  - Multiple processors in a system.
  - Multiple tasks for the processors to run.
- Multiprocessor scheduling is the problem of deciding which tasks to run on which processors at what time.
- Many possible objectives.
  - Throughput, fairness, energy usage.
  - Latency, i.e. finishing all jobs as fast as possible.

Makespan scheduling

- n independent jobs.
  - Jobs have different sizes, i.e. time needed to perform job.
  - Jobs can be done in any order.
  - Any job can be done on any machine.
- m processors.
  - All have the same speed.
  - Each processors can do one job at a time.
- Assign the jobs to the processors.
- Makespan is when the last processor finishes all its jobs.
- Minimize the makespan.
  - I.e., finish all the jobs as fast as possible.



Minimizing makespan is NPC

- The decision version of scheduling is obviously in NP.
- SUBSET-SUM: given a set of numbers S and target t, is there a subset of S summing to t?
  - Ex $S=\{1,3,8,9\}$. $t=9$, yes. $t=14$, no.
  - This is NP-complete. We reduce SUBSET-SUM to scheduling.
- Let (S,t) be an instance of SUBSET-SUM.
  - Let s be sum of all elements in S.
- Make a set of jobs $J = S \cup \{s-2t\}$, and schedule them on 2 processors.

Minimizing makespan is NPC

- Claim If some subset of S sums to t, then min makespan is s-t.
- Proof Say S'$\subseteq$S sums to t. Schedule the jobs in S' and job s-2t on processor 1. So proc 1 finishes at time t+s-2t=s-t. Proc 2 does the jobs in S-S', so it finishes at time s-t as well.
- Claim If the min makespan is s-t, there exists a subset of S that sums to t.
- Proof Suppose WLOG proc 1 does the s-2t job. Since makespan is s-t, the other jobs proc 1 does must have total size s-t-(s-2t)=t.
- So (S,t) is yes instance of SUBSET-SUM iff makespan = s-t.
  - So SUBSET-SUM $\leq_p$ scheduling, and scheduling is NP-complete.

Graham's list scheduling

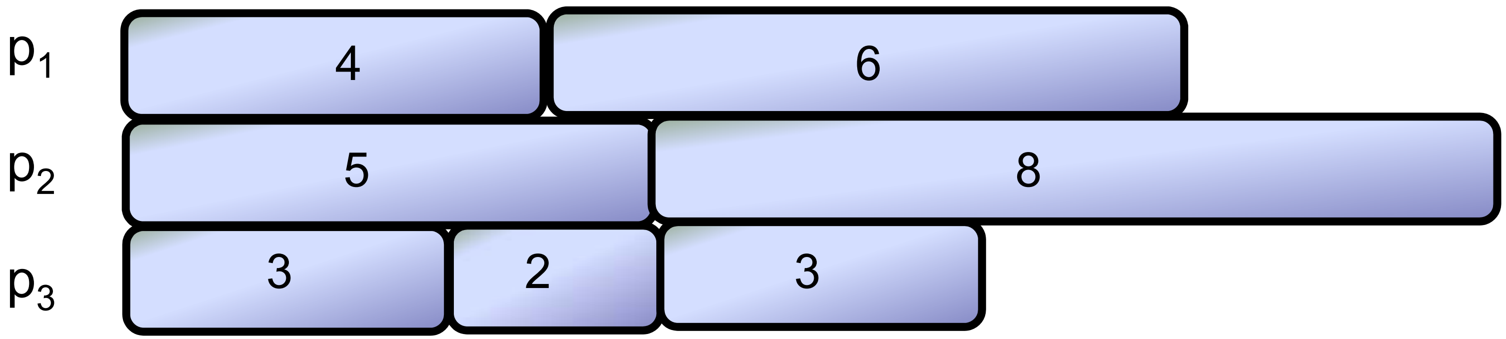
- Since scheduling is NPC, it's unlikely we can find the min makespan in polytime.
- List scheduling is a simple greedy algorithm.
  - Finds a schedule with makespan at most twice the minimum.
  - A 2-approximation.
- If there are n tasks and m processors, list scheduling only takes O(n log n) time.
  - Compare this to n! C(n+m-1, m-1) time to try all possible schedules and pick the best.

Graham’s list scheduling

- List the jobs in any order.
- As long as there are unfinished jobs.
  - If any processor doesn’t have a job now, give it the next job in the list.

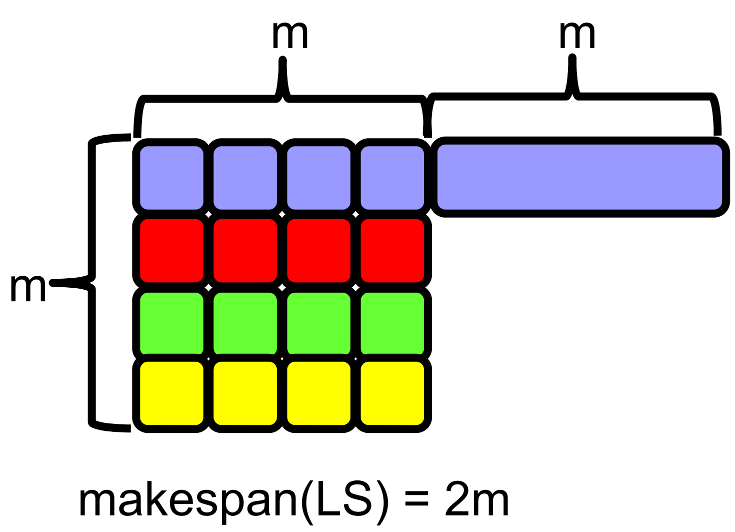
Example

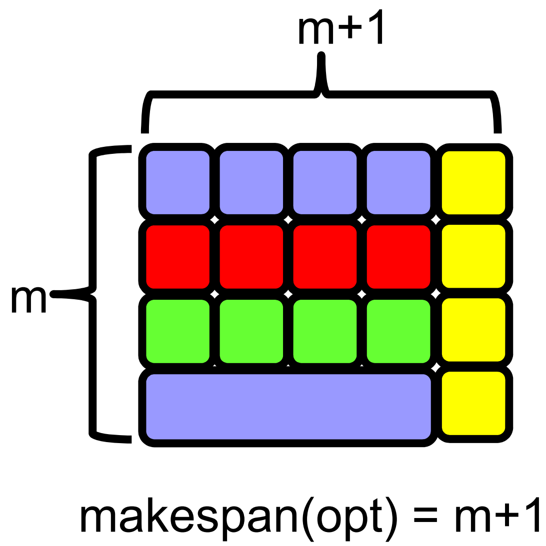
- 3 processors. The jobs have length 2, 3, 3, 4, 5, 6, 8.
- List them in any order. Say 4, 5, 3, 2, 6, 8, 3.
- Initially, no proc has a job. Give first 3 jobs to the 3 procs.
- At time 3, proc 3 is done. Give it next job in list, 2.
- At time 4, proc 2 is done. Give it next job in list, 6.
- At time 5, both 1, 3 are done. Give them next jobs in list, 8,3.
- Everybody finishes by time 13.
  - The makespan of this schedule is 13.



The worst case for LS

- How badly can list scheduling do compared to optimal?
- Say there are m^2 jobs with length 1, and one job with length m.
  - Suppose they're listed in the order 1,1,1,...,1,m.
  - LS has makespan 2m. Optimal makespan is m+1.
  - makespan(LS) / makespan(opt) = 2m/(m+1) $\approx$ 2.
- This is worst possible case for list scheduling.

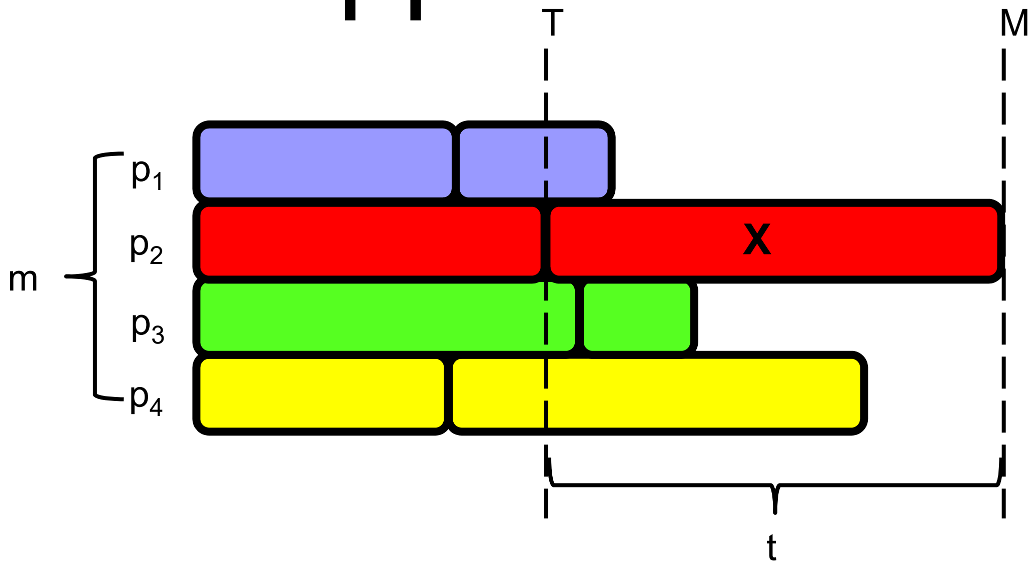




LS is a 2-approximation

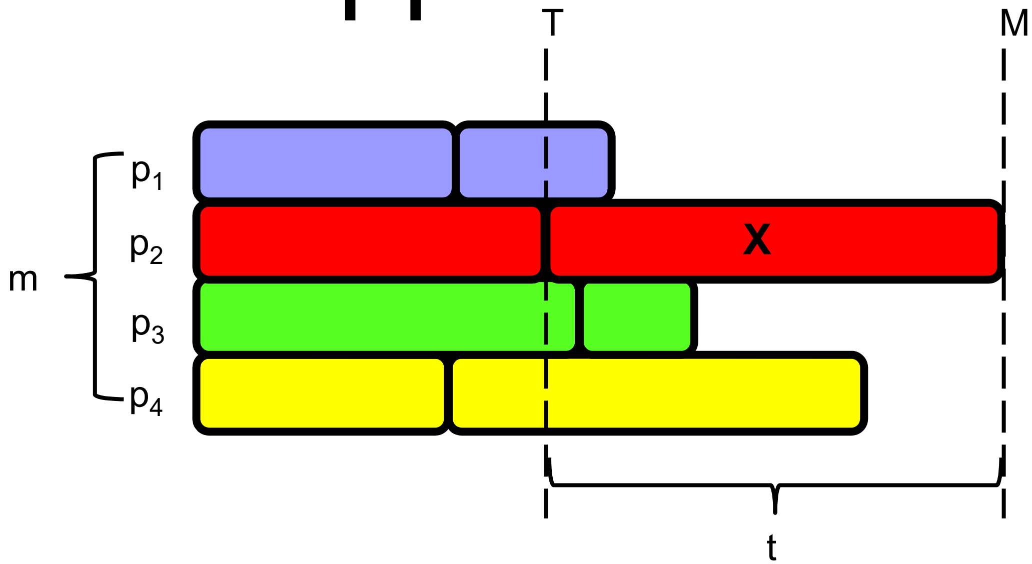
- Next, we prove LS always gives a schedule at most twice the optimal.
- Suppose LS gives makespan of M.
- Let the optimal schedule have makespan M*.
- We prove that $M \leq 2M^*$.

LS is a 2-approximation



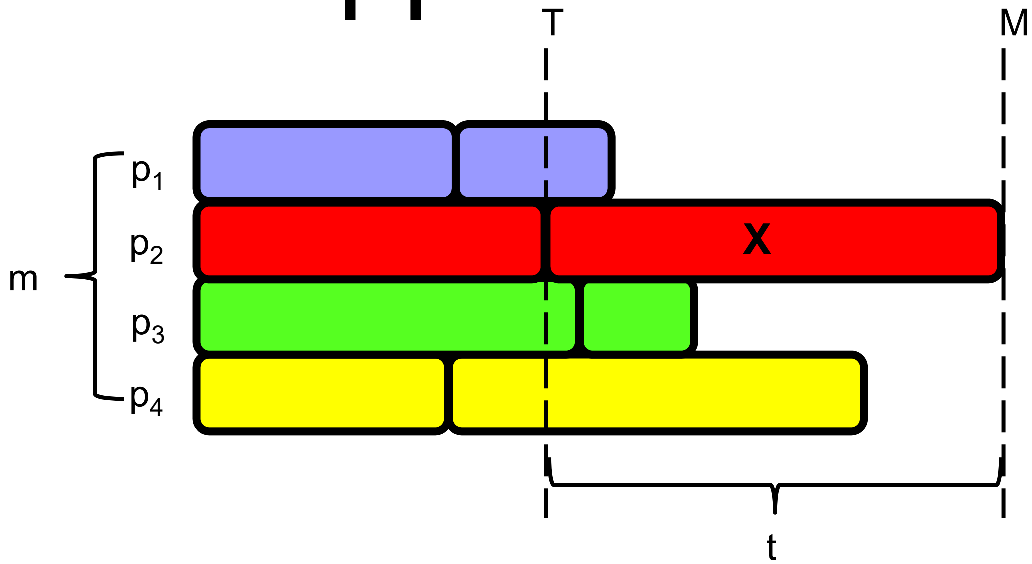
- The picture above is the schedule produced by list scheduling.
- Consider task X that finishes last.
  - Say X starts at time T, and has length t.
- Claim 1 $M^* \geq t$.
  - In any schedule, X has to run on some process.
  - Since X takes t time, every schedule, including the opt, takes $\geq$ t time.

LS is a 2-approximation



- Claim 2 M* ≥ T.
  - Up to time T, no processor is ever idle.
    - Up to T, there's always some unfinished job.
    - As soon as a processor finishes one job, it's assigned another one.
  - So at time T, each processor completed T units of work.
  - So total amount of work in all the jobs is ≥ mT.
  - In the opt schedule, m processors complete at most m units of work per time unit.
  - So length of opt schedule is ≥ (total work)/m ≥ mT/m = T.

LS is a 2-approximation



- From Claims 1 and 2, we have $M^* \geq t$ and $M^* \geq T$.
- So $M^* \geq \max(T,t)$.
- $M = T + t$, because X is last job to finish.
- So $M/M^* \leq (T+t)/\max(T,t) \leq 2$.

LPT scheduling

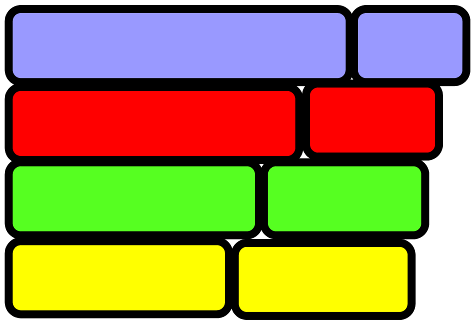
- Worst case for LS occurred when longest job was scheduled last.
  - Large jobs are “dangerous” at end.
- Let’s try to schedule longest jobs first.
- Longest processing time (LPT) schedule is just like list scheduling, except it first sorts tasks by nonincreasing order of size.
- Ex For three processors and tasks with sizes 2, 3, 3, 4, 5, 6, 8, LPT first sorts the jobs as 8,6,5,4,3,3,2. Then it assigns p_1 tasks 8,3, p_2 tasks 6,3, p_3 tasks 5,4,2, for a makespan of 11.
- LPT has an approximation ratio of 4/3.

LPT is a 4/3-approximation

- Thm Suppose the optimal makespan is M*, and LPT produces a schedule with makespan M. Then $M \leq 4/3\ M^*$.
- Let X be the last job to finish. Assume it starts at time T and has size t.
- Assume WLOG that X is the last job to start.
  - If not, then say Y starts after T.
  - Y finishes before T+t. So we can remove Y without increasing the makespan.
- Cor 1 X is the smallest job.
  - X is the last job to start, so due to LPT scheduling it's the smallest.

LPT is a 4/3-approximation

- Claim 1 LPT's makespan = T+t $\leq$ M*+t.
  - As in LS, no processor is idle up to time T, so M* $\geq$ T.
- Case 1 t $\leq$ M*/3.
  - Then LPT's makespan $\leq$ M* + t $\leq$ M* + M*/3 = 4/3 M*.
- Case 2 t > M*/3.
  - Since X is the smallest task, all tasks have size > M*/3.
  - So the optimal schedule has at most 2 tasks per processor. So n $\leq$ 2m.
  - If 1 $\leq$ n $\leq$ m, then LPT and optimal schedule both put one task per processor.
  - If m < n $\leq$ 2m, then optimal schedule is to put tasks in nonincreasing order on processors 1,...,m, then on m,...,1.
    - LPT also schedules tasks this way, so it's optimal.

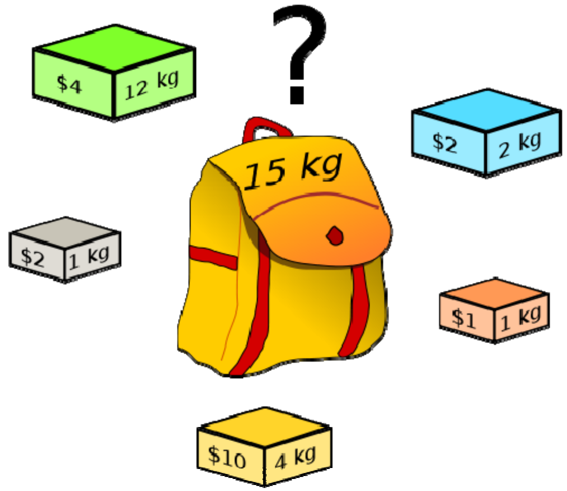


LS vs LPT

- LPT gives better approx ratio, has same running time. Why bother with LS?
- LS is online.
  - Imagine the jobs are coming one by one.
    - LS just puts them on any idle computer.
- LPT is offline
  - It needs to know all the jobs that will ever arrive, in order to sort them.
- In a realistic parallel computation, you get jobs on the fly.
  - Online is more realistic.
  - LS is usually more useful.

The knapsack problem

- We have a set of items, each having a weight and a value.
- We have a knapsack that can carry up to W amount of weight.
- We want to put items in the knapsack to maximize the total value, but not exceed the weight limit.
- Ex Items 3 and 4 are the highest value items with weight ≤ 11.
- Assume all items have weight ≤ W, i.e. any single item fits in knapsack.



W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

A dynamic program for knapsack

- Let OPT(i,v) = minimum weight of a subset of items 1,...,i that has value $\geq$ v.
- If optimal solution uses item i.
  - Then we pay w_i weight for item i, and need to achieve value $\geq$ $v-v_i$ using items 1,...,i-1 using min weight.
  - So $OPT(i,v)=w_i+OPT(i-1,v-v_i)$.
- If optimal solution doesn't use item i.
  - Then we need to achieve value $\geq$ v using items 1,...,i-1.
  - So OPT(i,v)=OPT(i-1,v).
- Choose the case that gives smaller weight.
- $$OPT(i,v) = \begin{cases} 0 & \text{if } v=0 \\ \infty & \text{if } i=0,\ v>0 \\ \min(OPT(i-1,v),\ w_i+OPT(i-1,v-v_i)) & \text{otherwise} \end{cases}$$

Running time of dynamic program

- Say there are n items, and the largest value of any item is v*.
- The max value we can pack into the knapsack is nv*, where v* is the largest v value.
- Solve all subproblems of the form OPT(i,v), where $i \leq n$ and $v \leq nv^*$.
  - This is a total of $O(n^2v^*)$ subproblems.
- The solution to Knapsack is the max value V that can be packed with weight $\leq W$.
- Having solved all the subproblems, we can find V by finding the subproblem with the largest value that has optimum weight $\leq W$.
  - $V = \max_{v \leq nv^*} OPT(n,v) \leq W$.
- So solving Knapsack takes total time $O(n^2v^*)$.

Running time of dynamic program

- The DP gives an optimal solution to Knapsack and takes $O(n^2v^*)$ time. Have we found a polytime algorithm for an NP-complete problem?
- No. The problem size is $O(n \log(v^*))$, because it takes $\log(v^*)$ bits to express each item's value. But $O(n^2v^*)$ is not polynomial in $n \log(v^*)$.
- To make this DP fast, we have to make the largest value small.

PTAS

- Let $\varepsilon>0$ be any number. We'll give a $(1+\varepsilon)$-approximation for knapsack.
- By setting ε sufficiently small, we can get as good an approximation as we want!
  - This type of algorithm is called a polynomial time approximation scheme, or PTAS.
- Contrast this with earlier algs we studied, which had worse approx ratios, e.g. 2 or log n.
- But the running time will be $O(n^3/\varepsilon)$. Hence we can't set $\varepsilon=0$ get the optimal solution.
- We're trading accuracy for time. The more accurate (smaller ε), the more time the algorithm takes.

Main idea: rounding

- Since we only need an approximate solution, we can change the values of the items a little (round the values) and not affect the solution much.
- We scale and round the original values to make them small.
- The previous DP took $O(n^2v^*)$ time. So if the rounded values are small, this DP is fast.

W = 11

Item	Value	Weight
1	134,221	1
2	656,342	2
3	1,810,013	5
4	22,217,800	6
5	28,343,199	7

→

W = 11

Item	Value	Weight
1	2	1
2	7	2
3	19	5
4	23	6
5	29	7

Rounding

- Let $\varepsilon>0$ be the precision we want.
- Set $\theta = \varepsilon v^*/2n$ to be a scaling factor.
  - v^* is the largest value of any item.
- Scale all values down by θ then round up.
  - $v' = \lceil v/\theta \rceil$.
- Make a problem where each value v_i is replaced by v'_i.
  - Call this the scaled rounded problem.
- Let v^ be max value in the scaled rounded problem. Then $\hat{v} = \lceil v^*/\theta \rceil = \lceil v^*/(\varepsilon v^*/2n) \rceil = \lceil 2n/\varepsilon \rceil$.
- Running time of DP on scaled rounded problem is $O(n^2 \hat{v}) = O(n^3/\varepsilon)$.

Solving the original problem

- Make another new problem in which each value v_i is replaced by $u_i = \lceil v_i/\theta \rceil * \theta$.
  - Call this the rounded problem.
  - We have $u_i \geq v_i$, and $u_i \leq v_i + \theta$.
- Note u values are equal to v' values multiplied by θ.
  - Thus, the optimal solution for the rounded problem and the scaled rounded problem are the same.
- We now have 3 problems, the original problem, the scaled rounded problem, and the rounded problem.
- Let S be the optimal solution to the scaled rounded problem, which we can find in time $O(n^3/\varepsilon)$. S is also optimal for the rounded problem.
- We'll show S is a $1+\varepsilon$ approximation for the original problem.

Correctness

- Thm Let S* be the optimal solution to the original problem. Then $(1+\varepsilon)\sum_{i \in S} v_i \geq \sum_{i \in S^*} v_i$. Hence S is a (1+ε)-approximate solution.

- Proof

$\sum_{i \in S^*} v_i \leq \sum_{i \in S^*} u_i$	$u_i \geq v_i$
$\leq \sum_{i \in S} u_i$	S is opt soln for rounded problem
$\leq \sum_{i \in S} (v_i + \theta)$	$u_i \leq v_i + \theta$
$\leq \sum_{i \in S} v_i + n\theta$	$\lvert S \rvert \leq n$

Correctness

- Suppose item j has the largest value, so $v^* = v_j$. Then $n\theta = \frac{\varepsilon}{2} v_j \le \frac{\varepsilon}{2} u_j \le \frac{\varepsilon}{2} \sum_{i \in S} u_i$
  - Last inequality because item j itself is feasible solution, so opt solution S is no smaller.
- So $\sum_{i \in S} v_i \ge \sum_{i \in S} u_i - n\theta \ge \left(\frac{2}{\varepsilon} - 1\right) n\theta$, where first inequality comes inequalities on last page.
- Assuming $\varepsilon \le 1$, then $n\theta \le \varepsilon \sum_{i \in S} v_i$
- Finally, we have

$$\sum_{i \in S^*} v_i \le \sum_{i \in S} v_i + n\theta \le \sum_{i \in S} v_i + \varepsilon \sum_{i \in S} v_i = (1 + \varepsilon) \sum_{i \in S} v_i$$

Summary

- We gave a DP for Knapsack.
- We scale and round to reduce number of different item values.
- Running the DP on the scaled rounded problem and using the solution for the original problem leads to an arbitrarily good approximation for Knapsack, a PTAS.
- There are PTAS's for a number of other problems.
  - Multiprocessor scheduling.
  - Bin packing.
  - Euclidean TSP.
- However, there are also many problems for which PTAS's do not exist, unless P=NP.