EE150 Signals and Systems – Part 4: Continuous-time Fourier Transform (C-T F.T.)

$\downarrow$ Week 5, Thu, 20180329

Aperiodic signals

- Aperiodic signal $x(t)$ (periodic with $T_0 \to \infty$)
- Eigenfunctions (LTI system): $e^{j\omega t}$ all ω
- Dot-product (Inner-product)

$$< x_t(t), x_2(t) >= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_1(t) x_2^*(t) dt$$

- (Show) $e^{j\omega t}$ are orthonormal

Fourier Transform of $x(t)$

- Consider LTI system with impulse response $x(t)$
  -Know: $e^{j\omega t}$ is an eigen function
- Fourier transform: $X(j\omega)$ is eigenvalue corresponding to $e^{j\omega t}$
- Therefore

$$
\begin{aligned}
X(j\omega)e^{j\omega t} &= e^{j\omega t} * x(t) \\
&= \int_{-\infty}^{\infty} x(\tau) e^{j\omega(t-\tau)} d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau
\end{aligned}
$$

Fourier Transform pair

- Fourier transform defines a bijection (one-to-one, invertible) mapping (via):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{(Fourier Transform)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{(Inverse F.T.)}$$

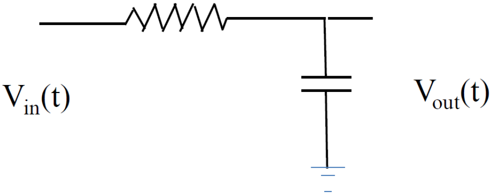
- This is valid as long as $x(t)$ is well-behaved, e.g. Schwartz function (wiki: Schwartz class)

Fourier Transform pair

$X(j\omega)$ "spectrum" of $x(t)$

- Eigenvalueof $e^{j\omega t}$
  - tells the amplification for frequency ω

Example: Consider the LTI system



F.T. of system (freq. response)

$$V_R(t) = Ri(t)$$
$$C\frac{dV_C}{dt} = i(t)$$
$$V_{out}(t) = V_C(t)$$
$$V_{in}(t) = V_R(t) + V_C(t)$$

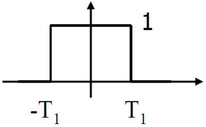
We know if $V_{in}(t) = e^{j\omega t}$ then $V_{out}(t) = H(j\omega)e^{j\omega t}$. Therefore

$$i(t) = j\omega C H(j\omega)e^{j\omega t}$$
$$e^{j\omega t} = (1 + j\omega RC)H(j\omega)e^{j\omega t}$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

Square pulse and "sinc" function

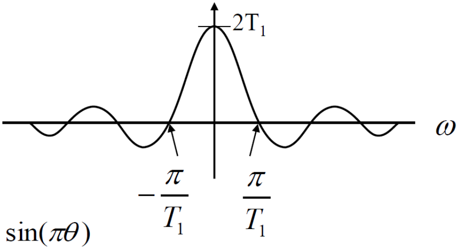
Example (1). Square Pulse



$\xleftrightarrow{\;\;F\;\;}$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt$$

$$= \frac{2\sin\omega T_1}{\omega}$$

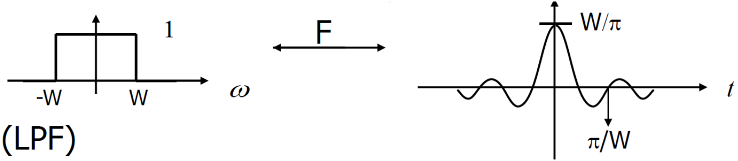


Define $\quad \mathrm{sinc}(\theta) \equiv \dfrac{\sin(\pi\theta)}{\pi\theta}$

Then $\quad X(j\omega) = 2T_1\,\mathrm{sinc}(\dfrac{\omega T_1}{\pi})$ $\quad$ for square pulse.

Square pulse and "sinc" function

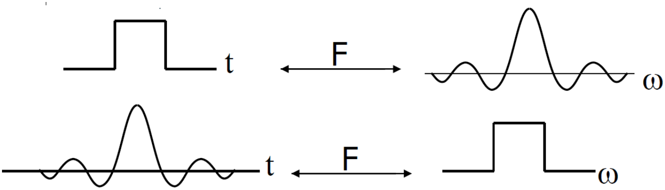
Example (2). Frequency-domain



$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi} sinc\left(\frac{Wt}{\pi}\right)$$

Duality property of Fourier Transform

Note:



"Duality property" of Fourier Transform

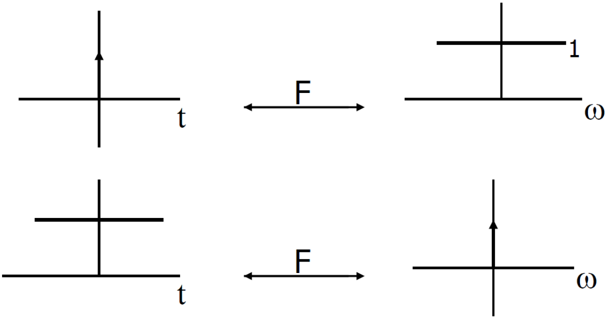
$$\mathcal{F}(\mathcal{F}(x(t))) = \int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}d\omega$$
$$= 2\pi x(-t)$$

Duality property of Fourier Transform

Example (3).

$$x(t) = \delta(t) = \xleftrightarrow{F} X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$(Note : \int_{-\infty}^{\infty} \delta(t - t_0) \cdot f(t) dt = f(t_0))$$



$$\boxed{\delta(t) \xleftrightarrow{F} 1}$$

Fourier Transform for Periodic Signal

Fourier transform can be applied to periodic signal

Consider $x(t)$ and its F.T., $X(j\omega)$.

Assume $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. Find $x(t)$.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_0) e^{j\omega t} d\omega \\ &= e^{j\omega_0 t} \end{aligned}$$

Fourier Transform for Periodic Signal

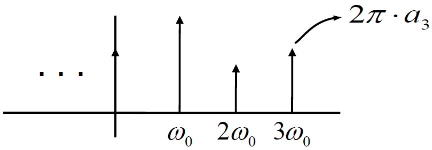
Now for more general case,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

exactly Fourier Series representation of a <u>periodic</u> signal.

$\Rightarrow$ We can find the F.T. for a periodic signal by

$$x(t) = \xrightarrow{F.S.} a_k \rightarrow X(j\omega) = \sum_{-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)$$

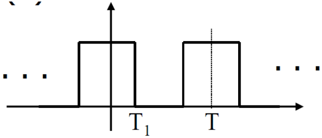


Fourier Transform for Periodic Signal

Note: If $x(t)$ is periodic with period T

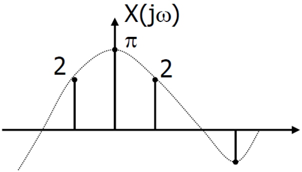
$\rightarrow X(j\omega)$ is discrete, with frequency spacing$= \omega_0 = \frac{2\pi}{T}$

e.g. (1)



$$\xleftrightarrow{\text{F.S.}} a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2 \cdot \frac{\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$



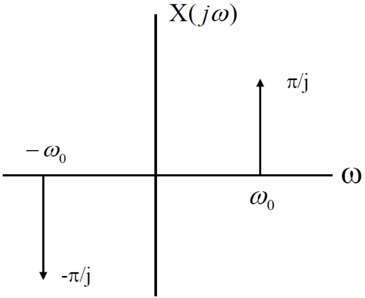
(for T_1=T/4 case)

Fourier Transform of $\sin(\omega_0 t)$

E.g. (2)

$$x(t) = \sin(\omega_0 t) \xrightarrow{\quad F.S. \quad} a_1 = \frac{1}{2j}, a_{-1} = \frac{1}{-2j}$$

$$\& \ a_k = 0 \ \textit{for all other } k$$

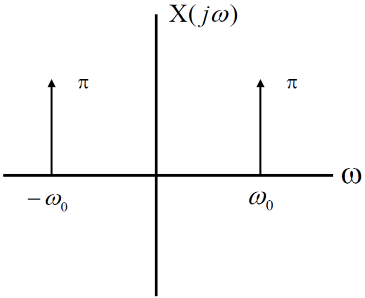


Fourier Transform of $\cos(\omega_0 t)$

E.g. (3)

$$x(t) = \cos(\omega_0 t) \xrightarrow{F.S.} a_1 = a_{-1} = \frac{1}{2}$$

$$\& \ a_k = 0 \ \textit{for all other } k$$

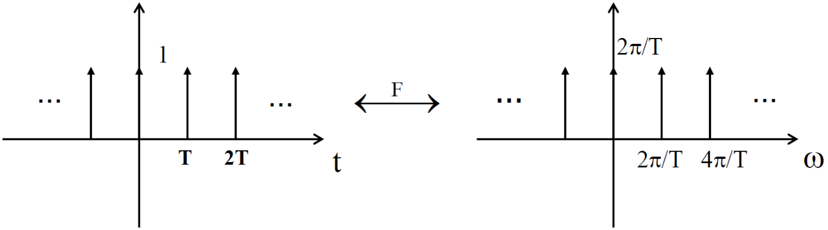


Fourier Transform of unit impulse function

E.g. (4)

$$x(t) = \sum_{-\infty}^{\infty} \delta(t - kT) \xleftrightarrow{F} a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) \cdot e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$\therefore X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$



Properties of Fourier Transform

Notation: $X(j\omega) = \mathcal{F}(x(t))$ *or* $x(t) \overset{F}{\longleftrightarrow} X(j\omega)$

1. Linearity $a \cdot x(t) + b \cdot y(t) \overset{F}{\longleftrightarrow} a \cdot X(j\omega) + b \cdot Y(j\omega)$
2. Time-shift $x(t - t_0) \overset{F}{\longleftrightarrow} e^{-j\omega t_0} \cdot X(j\omega)$
3. Conjugation $x^*(t) \overset{F}{\longleftrightarrow} X^*(-\omega)$

   Conjugate symmetry: if $x(t)$ is real $\rightarrow X(-j\omega) = X^*(j\omega)$

↑ Week 5, Thu, 20180329

↓ Week 6, Tue, 20180403

Properties of Fourier Transform

4. Differentiation & integration

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$\therefore \frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} j\omega \cdot X(J\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Properties of Fourier Transform

5. Time and Frequency Scaling

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{j\omega}{a})$$

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

5. Duality

$$g(t) \xleftrightarrow{F} f(j\omega) \Rightarrow f(t) \xleftrightarrow{F} 2\pi \cdot g(-j\omega)$$

5. Parsevals Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Properties of Fourier Transform

Proof of (7) Parsevals Relation:

Proof.

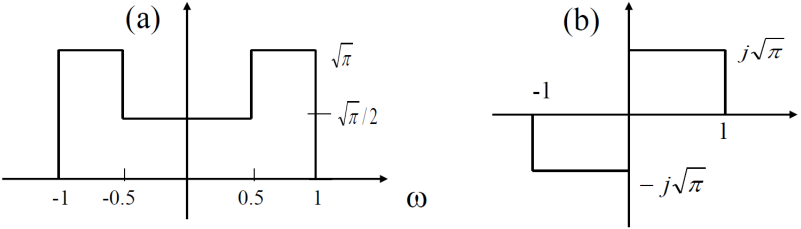
$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \\
&= \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\
Change\ order: &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \cdot \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}
$$

□

Properties of Fourier Transform

Ex. Find $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$ and $D = \frac{dx(t)}{dt}\Big|_{t=0}$

for the following two $X(j\omega)$.



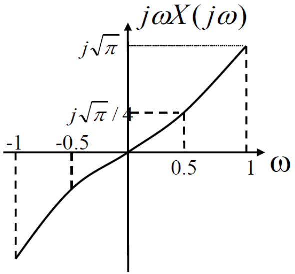
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \begin{cases} \frac{5}{8} & for\ (a) \\ 1 & for\ (b) \end{cases}$$

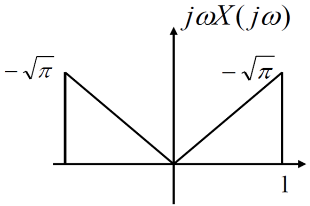
Properties of Fourier Transform

For D, remember $g(t) = \frac{d}{dt}x(t) \overset{F}{\longleftrightarrow} j\omega \cdot X(j\omega) = G(j\omega)$
Also note that

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega = D$$

$$\therefore D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)d\omega = \begin{cases} 0 & \textit{for } (a) \\ -\frac{\sqrt{\pi}}{2\pi} & \textit{for } (b) \end{cases}$$





Convolution property

$y(t) = h(t) * x(t) \overset{F}{\longleftrightarrow} Y(j\omega) = H(j\omega) \cdot X(j\omega)$
where $h(t)$ is system impulse response

$$
\begin{aligned}
Y(j\omega) &= \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) e^{-j\omega t} d\tau dt \\
&= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} \left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega(t-\tau)} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(j\omega) d\tau \\
&= X(j\omega) H(j\omega)
\end{aligned}
$$

Utilization of Convolution property

Ex. Assume $x(t) = \frac{\sin(\omega_i t)}{\pi t}$ is the input and is filtered by an ideal LPF with cut-off frequency ω_c. Find the output, y(t).
Idela LPF: $h(t) = \frac{\sin(\omega_c t)}{\pi t}$

$$y(t) = h(t) * x(t) = \frac{\sin(\omega_c t)}{\pi t} * \frac{\sin(\omega_i)t}{\pi t} \Rightarrow \textit{difficult to find}.$$

On the other hand, $Y(j\omega) = H(j\omega) \cdot X(j\omega)$

Utilization of Convolution property

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_i \\ 0 & elsewhere \end{cases} \quad H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & elsewhere \end{cases}$$

$$\rightarrow Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & elsewhere \end{cases}$$

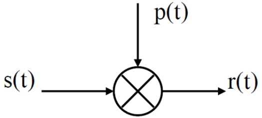
where ω_i is the smaller of ω_0 & ω_c .

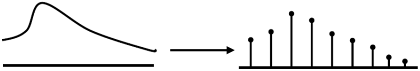
$$\Rightarrow y(t) = \begin{cases} \dfrac{\sin(\omega_c t)}{\pi t} & if \ \omega_c \leq \omega_i \\ \dfrac{\sin(\omega_i t)}{\pi t} & if \ \omega_i \leq \omega_c \end{cases}$$

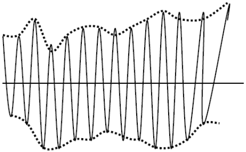
Multiplication

(multiplication in time $\leftrightarrow$ convolution in frequency)

$$r(t) = s(t) \cdot p(t) \xleftrightarrow{F} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$



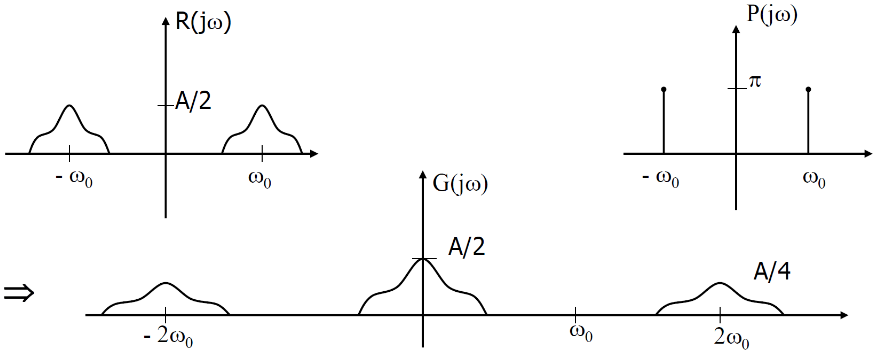
e.g. 1). sampling process: 

2). amplitude modulation (AM): 

Multiplication

Ex: Assume $g(t) = r(t) \cdot p(t)$ where:

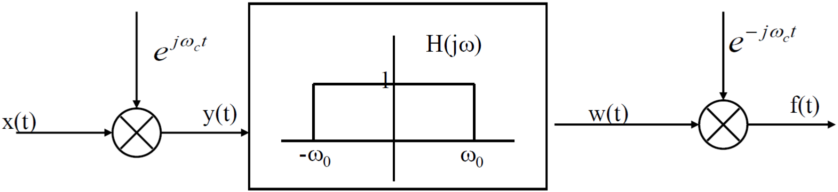
the F.T. of $r(t)$ is: the F.T. of $p(t) = \cos(\omega_0 t)$ is:



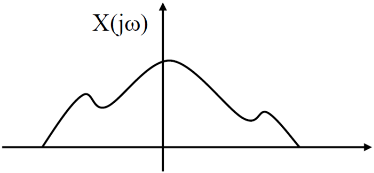
(This problem illustrates the "demodulation process" that is discussed in Principle Comm.)

Multiplication

Ex: (Frequency Selective Filtering with variable Central Frequency)

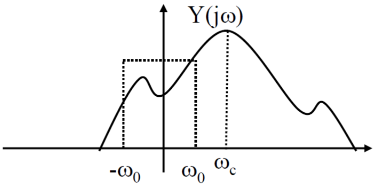


Assume

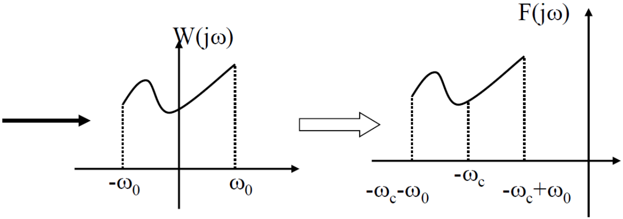


Multiplication

Then



$$Y(j\omega) = \delta(\omega - \omega_c) * X(j\omega)$$
$$= X(j(\omega - \omega_c))$$



$$F(j\omega) = W(j(\omega + \omega_c))$$

Remarks

1. Compared $X(j\omega)$ & $F(j\omega)$, we found that the overall response is equivalent to a BPF center at $-\omega_c$ with bandwidth $2\omega_0$
2. By adjusting ω_c we can achieve a tunable BPF using a LPF and a frequency-tunable complex exponential signal $e^{j\omega_c t}$
3. Q:What if $e^{j\omega_c t}$ is replaced with a sinusoidal signal, such as $\sin(\omega_c t)$ or $\cos(\omega_c t)$?

Summary

- Developed Fourier transformation representation of continuous-time signals.
- Aperiodic signal as the limit of periodic signal with period $\rightarrow \infty$
- Derive F.T. from F.S. for periodic signal.
- Properties of C-T Fourier Transform.
- Basic F.T. pair.