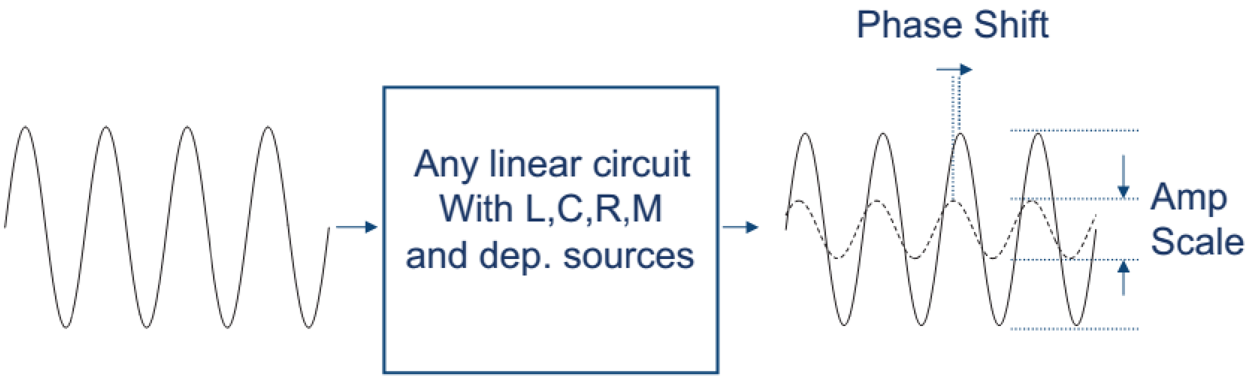
Lecture 11

- Frequency Response

Outline

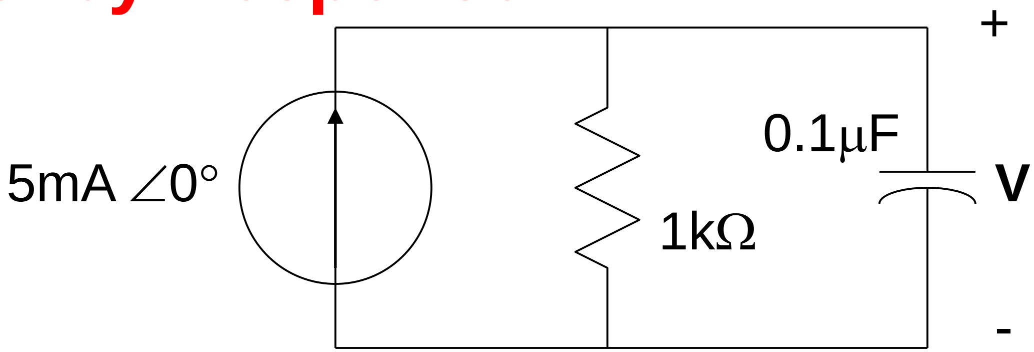
- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

Frequency Response



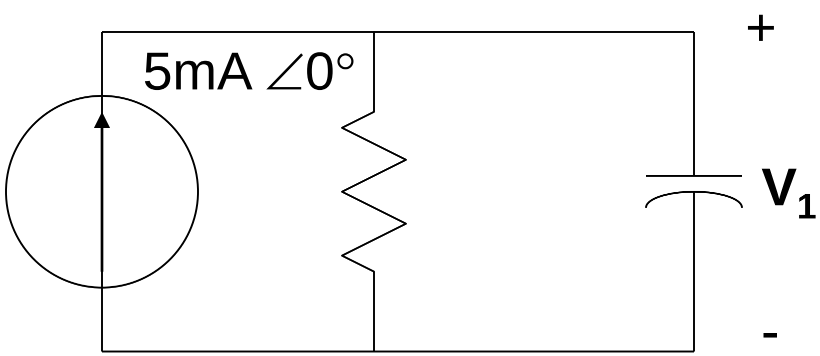
- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it's output is a sinusoid at the *same* frequency.
  - Only the <u>magnitude</u> and <u>phase</u> of the output differ from the input.
- The "Frequency Response" is a characterization of the input-output response for sinusoidal inputs at <u>all</u> frequencies.
  - Significant for applications, esp. in communications and control systems.

Frequency Response



Find $v(t)$

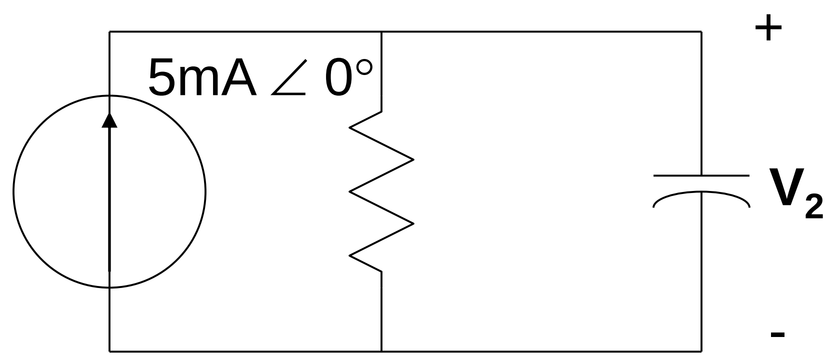
Case 1: $\omega=2\pi \times 3000$



$$\mathbf{V}_1 = 2.34\angle -62.1°\mathrm{V}$$

$$\mathbf{Z}_{eq} = 468.2\angle -62.1°\Omega$$

Case 2: $\omega=2\pi \times 455000$



$$\mathbf{V}_2 = 17.5\angle -89.8°\mathrm{mV}$$

$$\mathbf{Z}_{eq} = 3.5\angle -89.8°\Omega$$

[Source: Berkeley]

Frequency Ranges of Common Signals

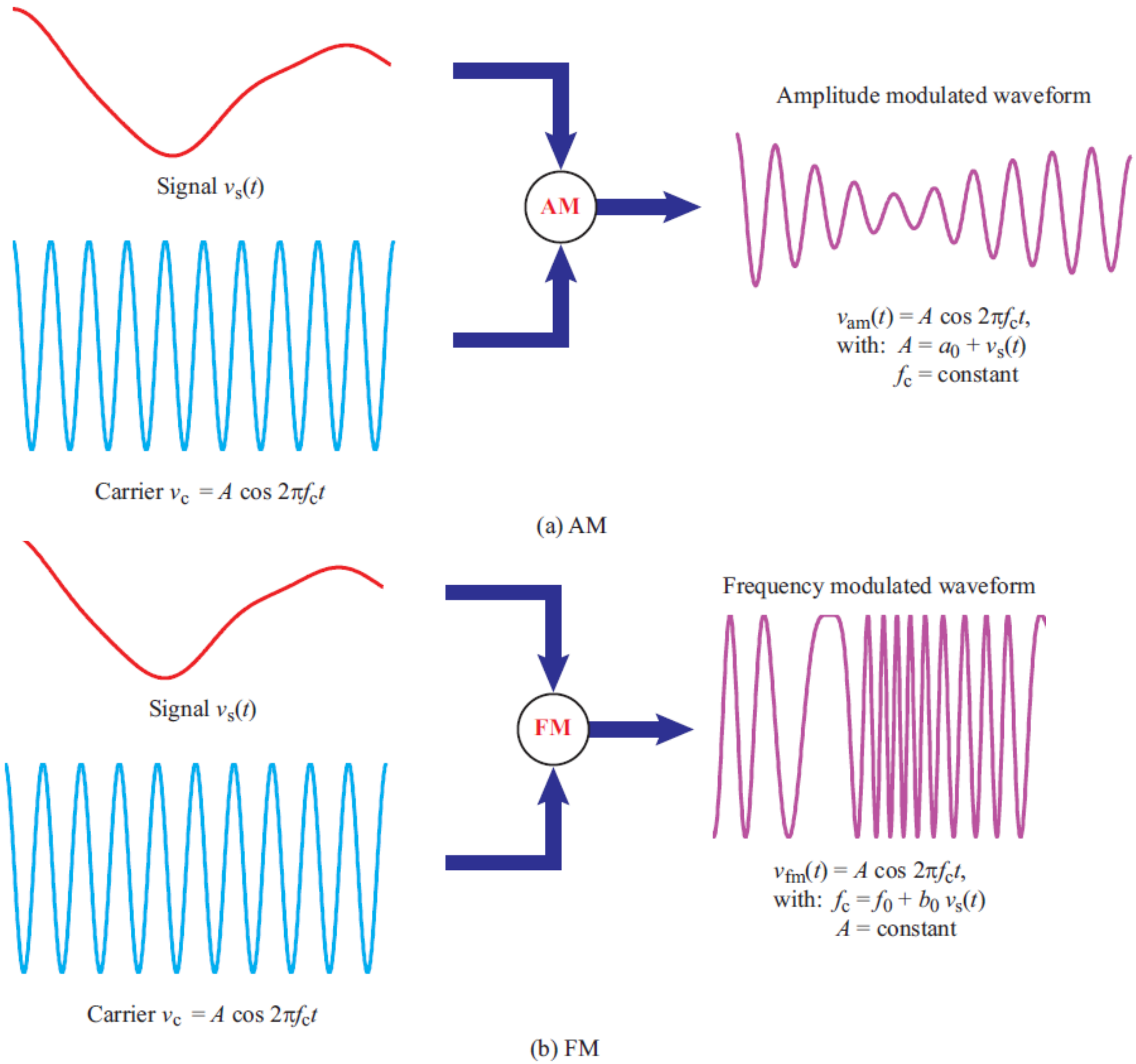
- When we listen to music, our ears respond differently to the various frequency components: some pleasing, whereas others are not.

Frequency Ranges of Selected Signals

Electrocardiogram	0.05 to 100 Hz
Audible sounds	20 Hz to 15 kHz
AM radio broadcasting	540 to 1600 kHz
HD component video signals	Dc to 25 MHz
FM radio broadcasting	88 to 108 MHz
Cellular phone	824 to 894 MHz and 1850 to 1990 MHz
Satellite television downlinks (C-band)	3.7 to 4.2 GHz
Digital satellite television	12.2 to 12.7 GHz

[Allan, 6th ed.]

Signal Modulation



[Source: Berkeley]

Outline

- Frequency response
- Transfer function
- Bode plots (or diagram)
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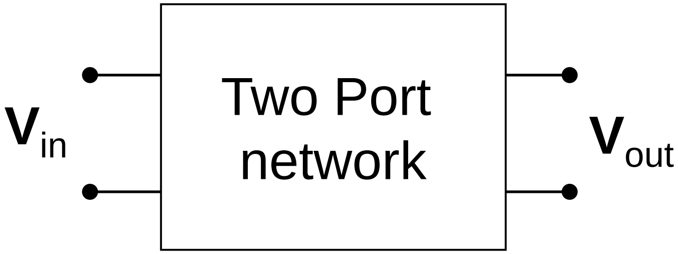
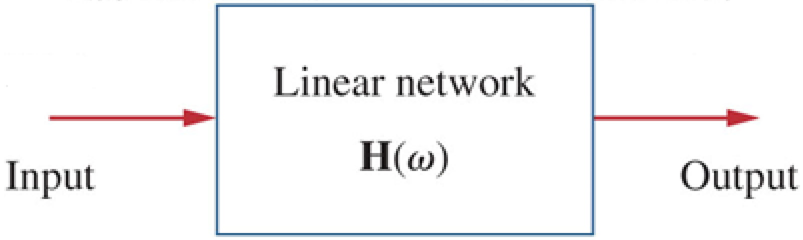
Transfer Function – Voltage Gain

- One useful way to analyze the frequency response of a circuit is the concept of the transfer function.
  - Complex quantity
  - Both magnitude and phase are function of frequency



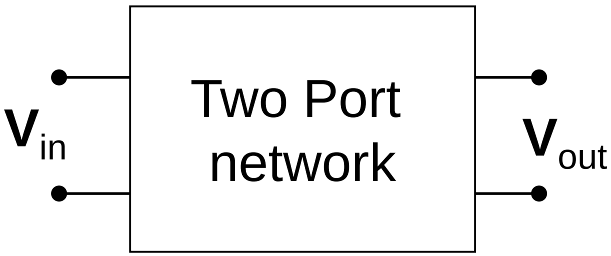


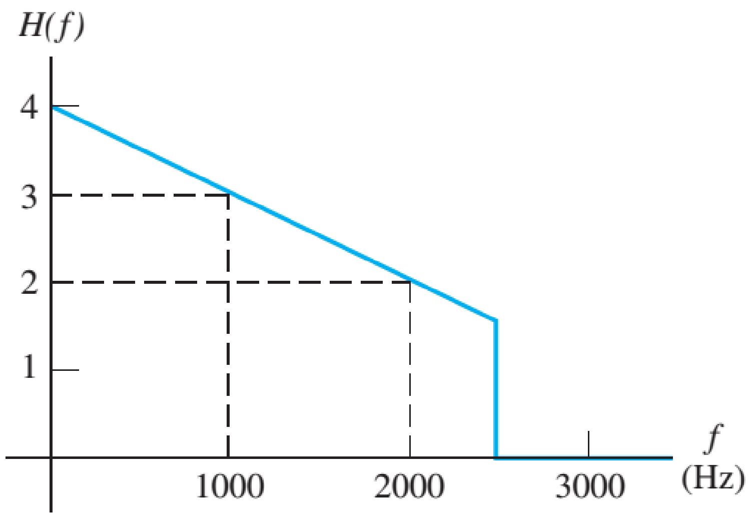
$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle \left(\theta_{out} - \theta_{in}\right)$$

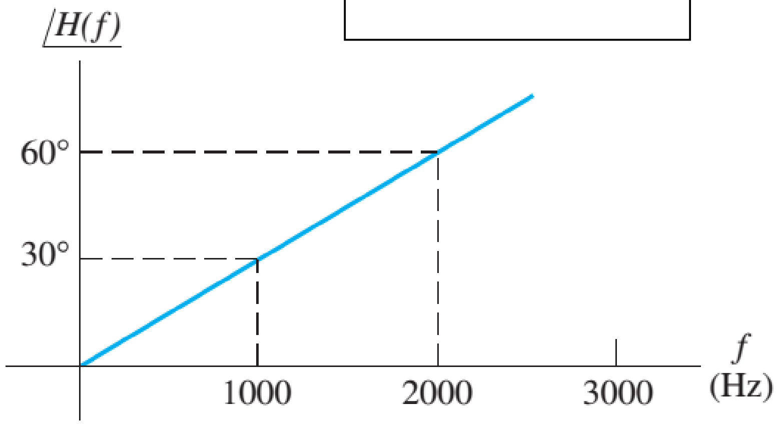
$$\mathbf{H(f)} = H(f) \angle \theta$$

Example

- The transfer function $H(f)$ is shown below. If the input signal is $v_{in}(t) = 2\cos(2000\pi t + 40°)$, find an expression as a function of time for the output.





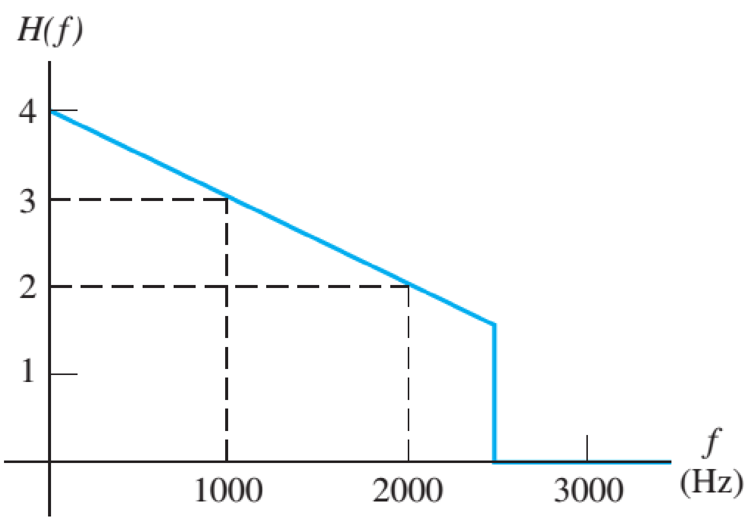


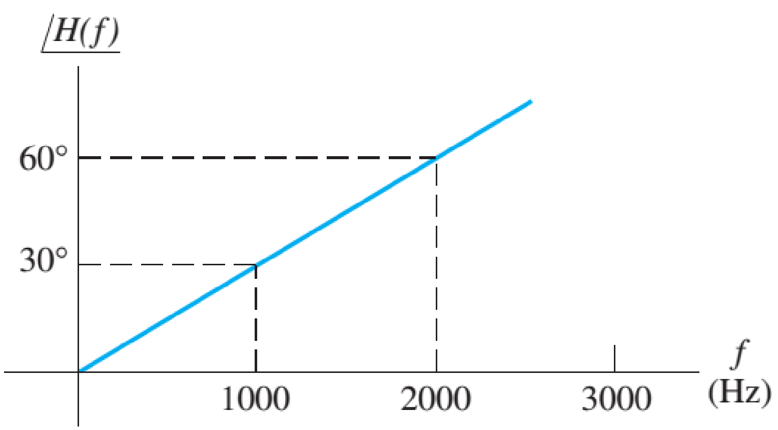
Example

- If the input signal is

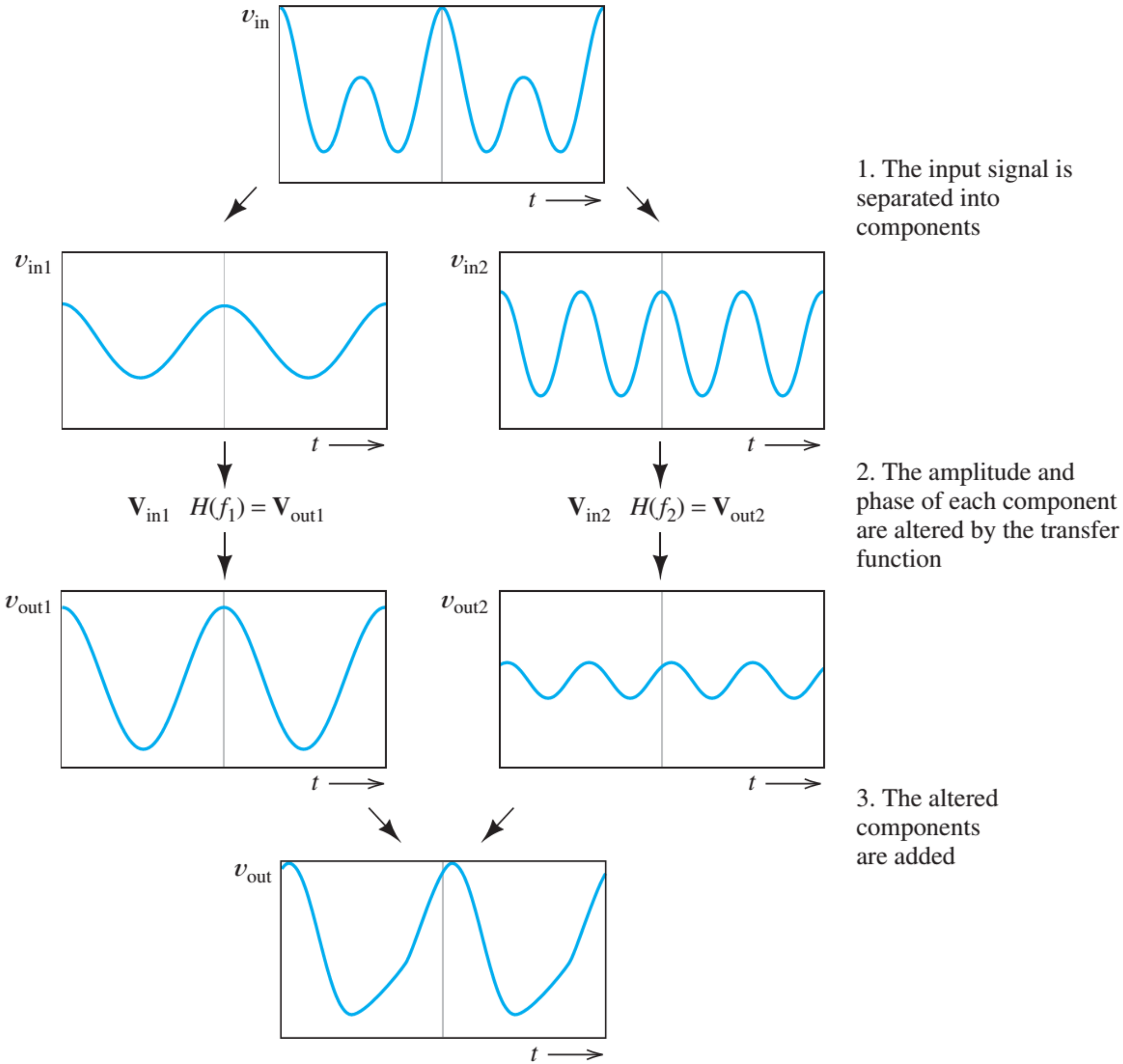
$$v_{in}(t) = 3 + 2\cos(2000\pi t) + \cos(4000\pi t - 70°),$$

find an expression as a function of time for the output.



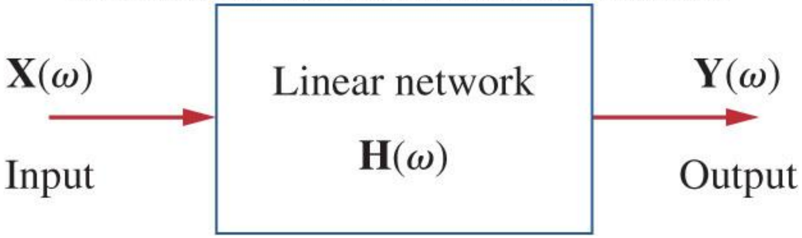


H(f)


v_{in}
$t \longrightarrow$
v_{in1}
v_{in2}
$t \longrightarrow$
$t \longrightarrow$
$\mathbf{V}_{in1} \quad H(f_1) = \mathbf{V}_{out1}$
$\mathbf{V}_{in2} \quad H(f_2) = \mathbf{V}_{out2}$
v_{out1}
v_{out2}
$t \longrightarrow$
$t \longrightarrow$
v_{out}
$t \longrightarrow$
1. The input signal is separated into components
2. The amplitude and phase of each component are altered by the transfer function
3. The altered components are added

Transfer Function – More General Definition

- The transfer function *H(ω)* is the frequency-dependent ratio of a forced function *Y(ω)* to the forcing function *X(ω)*.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

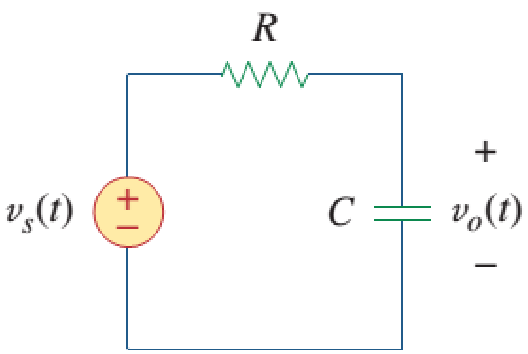
$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

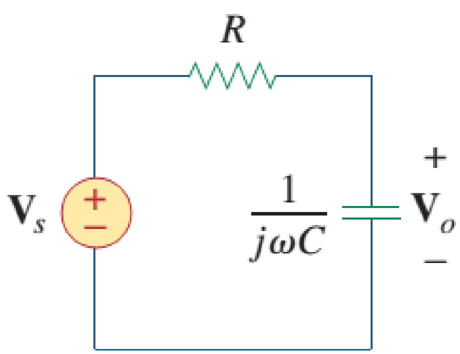
$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

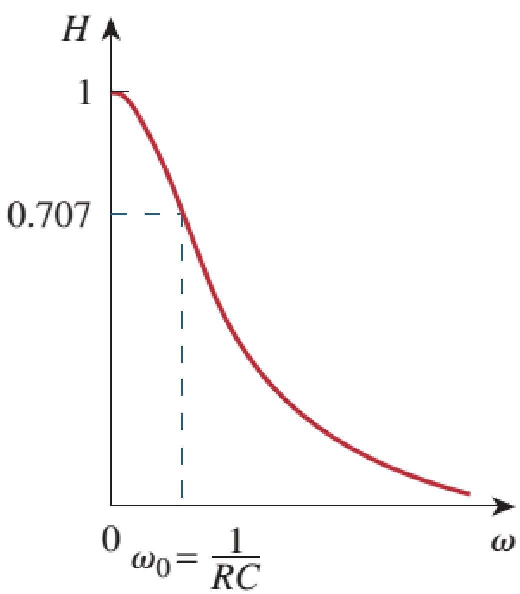
Example

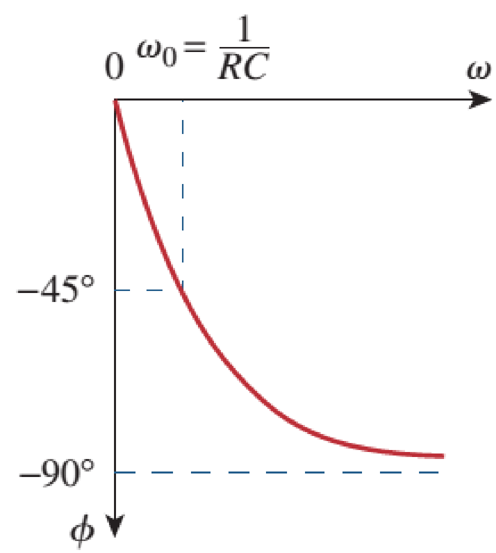




$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

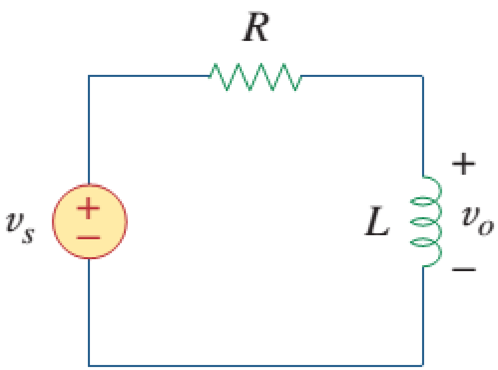
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

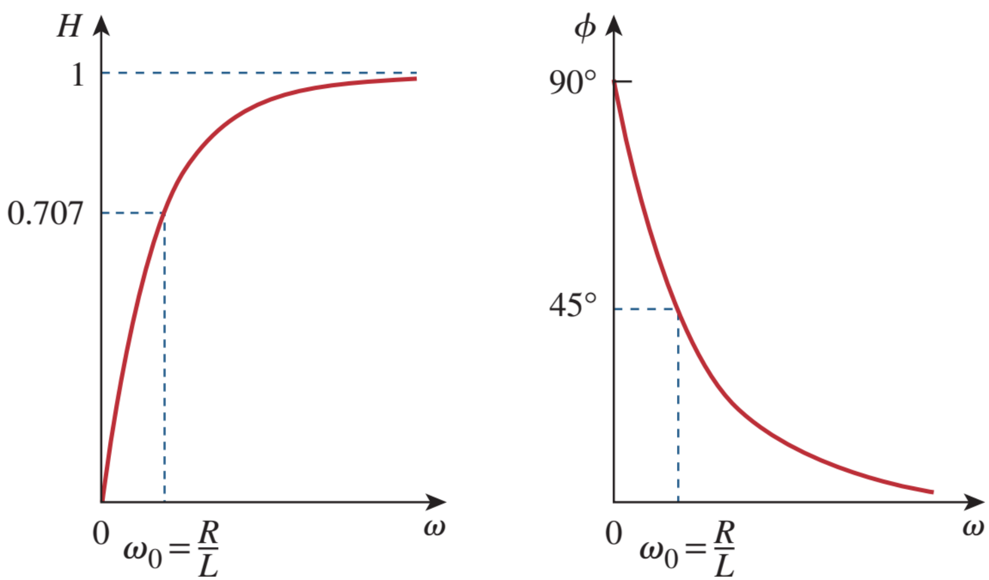




Exercise

- Obtain the transfer function $\mathbf{V_o}/\mathbf{V_s}$ of the RL circuit. Assuming $v_s = V_m \cos \omega t$.





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Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
  - Definition of bel:

$$\text{Ratio with a unit of B} = \log_{10}(P_1/P_2)$$

where P_1 and P_2 are power levels.

- One bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

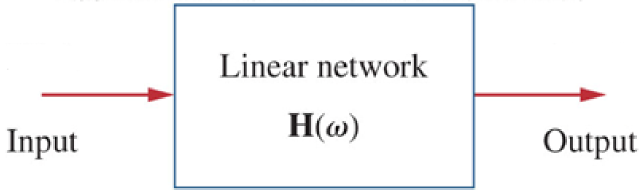
$$\text{Ratio with a unit of dB} = 10\log_{10}(P_1/P_2)$$

  - used to measure electric power, gain or loss of amplifiers, etc.

Decibel Scale

- The transfer function includes an expression of gain, which is typically expressed in log form.
  - in bels, or more commonly decibels

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1}$$



1. $\log P_1 P_2 = \log P_1 + \log P_2$
2. $\log P_1/P_2 = \log P_1 - \log P_2$
3. $\log P^n = n \log P$
4. $\log 1 = 0$

- We will soon discuss Bode plots, which are based on logarithmic scales.

dB for Power

- To express a power in terms of decibels, one starts by choosing a reference power, $P_{reference}$, and write

$$\text{Power P in decibels} = 10 \log_{10}(P/P_{reference})$$

- Exercise: Express a power of 50 mW in decibels relative to 1 watt and 1mW.

  P (dB) =

- dBm to express absolute values of power relative to a milliwatt.

$$\text{dBm} = 10 \log_{10} (\text{power in milliwatts} / 1 \text{ milliwatt})$$

  - 100 mW = dBm
  - 10 mW = dBm

dB for Voltage or Current

- We can similarly relate the reference voltage or current to the reference power, as

$$P_{\text{reference}} = (V_{\text{reference}})^2/R \text{ or } P_{\text{reference}} = (I_{\text{reference}})^2 R$$

Hence,

$$\text{Voltage, V in decibels} = 20\log_{10}(V/V_{\text{reference}})$$
$$\text{Current, I, in decibels} = 20\log_{10}(I/I_{\text{reference}})$$

Question: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery?

Question: The voltage gain of an amplifier with input = 0.2 mV and output = 0.5 V is ?

[Source: Berkeley]

Summary

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

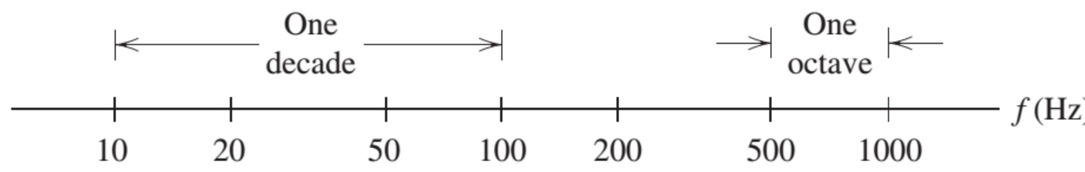
$$G\ [\text{dB}] = 10 \log G = 10 \log \left(\frac{P}{P_0} \right) \qquad (\text{dB}).$$

$$G\ [\text{dB}] = 10 \log \left(\frac{\frac{1}{2}|\mathbf{V}|^2/R}{\frac{1}{2}|\mathbf{V}_0|^2/R} \right) = 20 \log \left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|} \right)$$

$\dfrac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

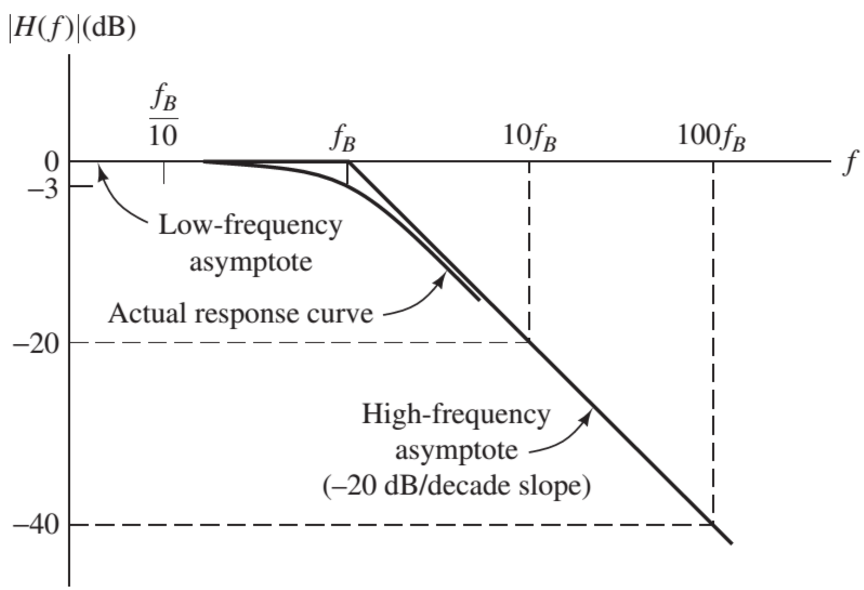
$\left\lvert\dfrac{\mathbf{V}}{\mathbf{V}_0}\right\rvert$ or $\left\lvert\dfrac{\mathbf{I}}{\mathbf{I}_0}\right\rvert$	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

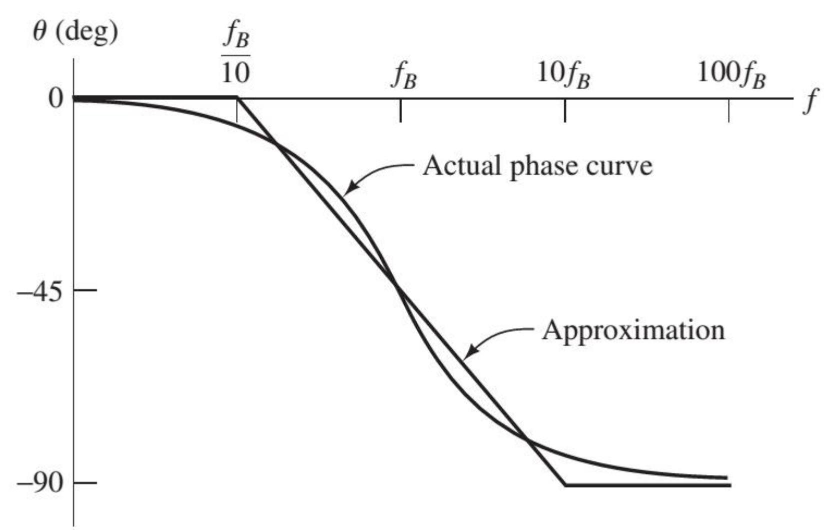
Bode Plots



Plotting the frequency response, magnitude or phase, on plots with

- frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)





Bode Plots

- Bode plot is particularly useful for displaying transfer function-- a general form is displayed as:

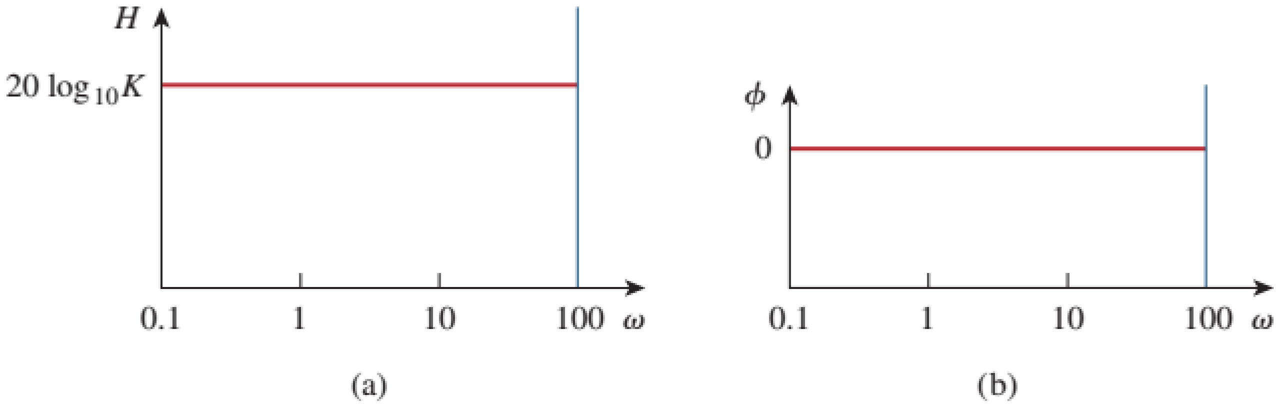
$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}\,(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool.

Constant term K

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

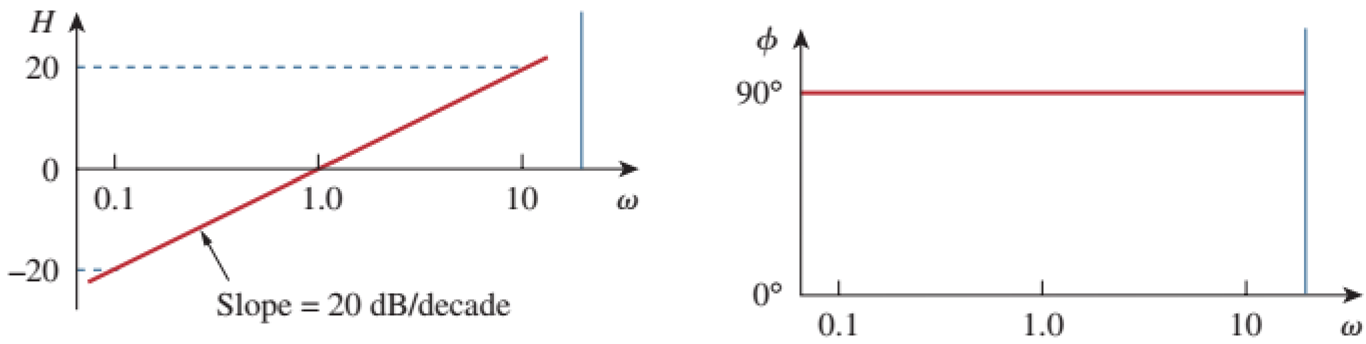
K>0



K<0

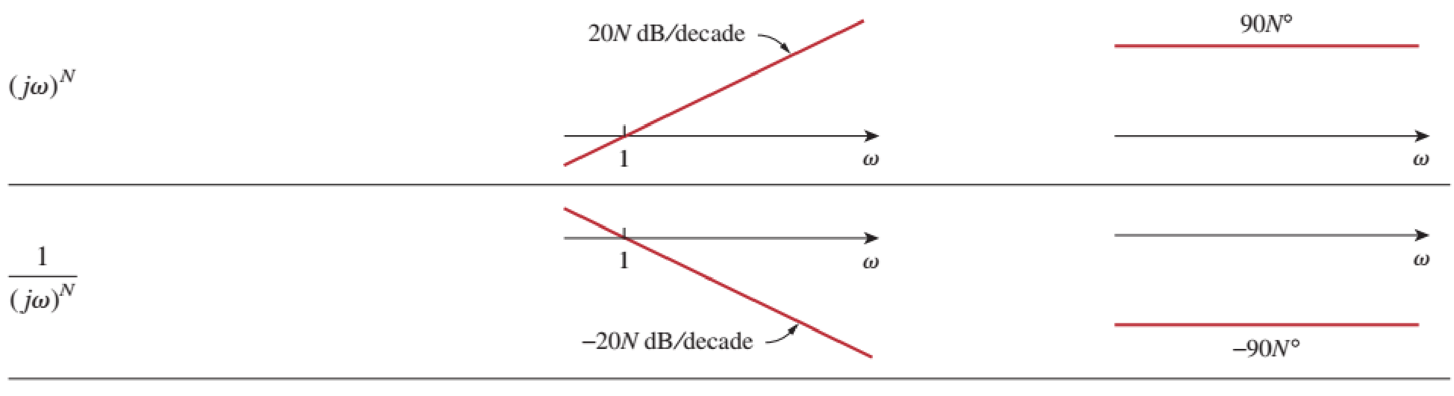
$j\omega$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}\,(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$



(a) (b)

- In general：



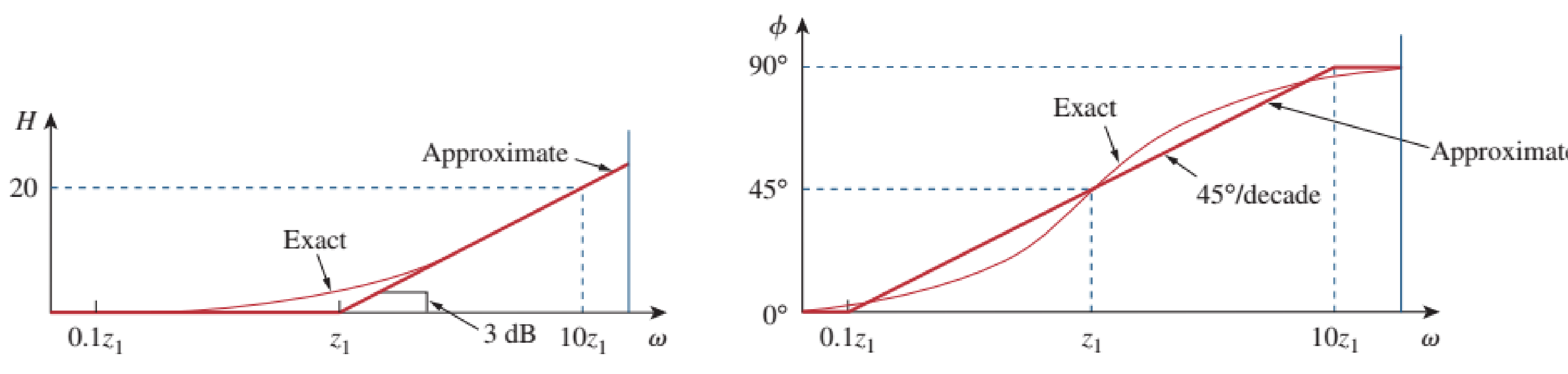
$1+j\omega/z_1$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

$(1 + j\omega/z_1)$, the magnitude is $20\log_{10}|1 + j\omega/z_1|$ and the phase is $\tan^{-1}\omega/z_1$. We notice that

$$H_{\mathrm{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20\log_{10}1 = 0 \quad \text{as} \quad \omega \to 0$$

$$H_{\mathrm{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20\log_{10}\frac{\omega}{z_1} \quad \text{as} \quad \omega \to \infty$$



(a) (b)

$1/(1+j\omega/p_1)$

$1/[1+2j\zeta_1\omega/\omega_n + (j\omega/\omega_n)^2]$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

Magnitude:

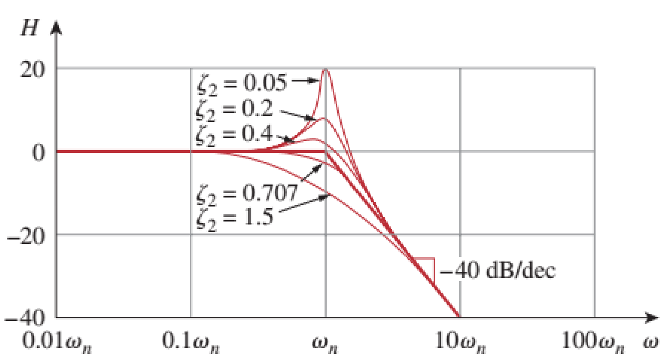
$$H_{\mathrm{dB}} = -20\log_{10}\left|1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad 0$$

and

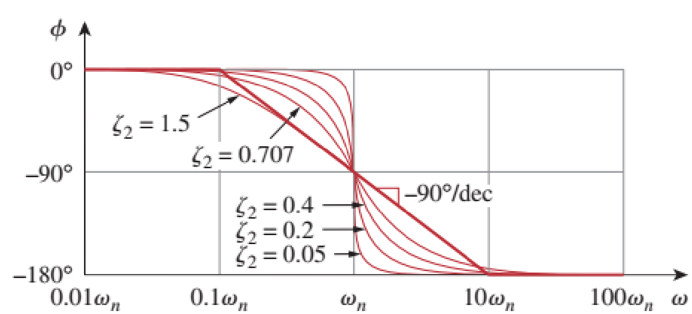
$$H_{\mathrm{dB}} = -20\log_{10}\left|1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad -40\log_{10}\frac{\omega}{\omega_n}$$

as $\omega \rightarrow \infty$

the phase is $-\tan^{-1}(2\zeta_2\omega/\omega_n)/(1 - \omega^2/\omega_n^2)$.



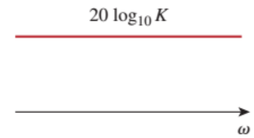
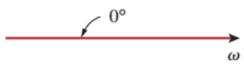
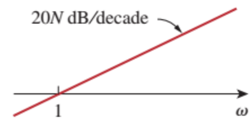
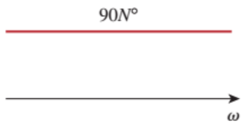
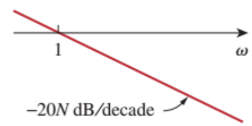
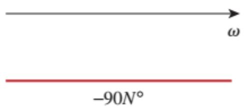
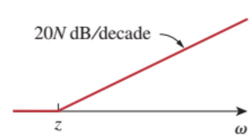
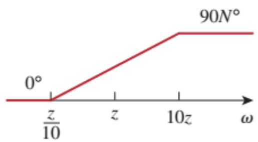
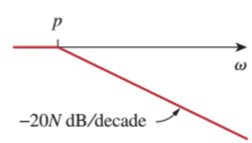
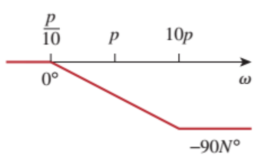
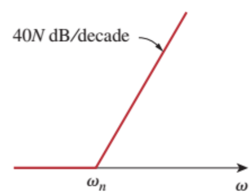
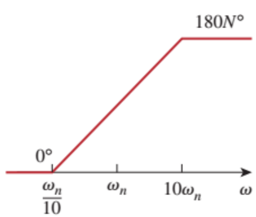
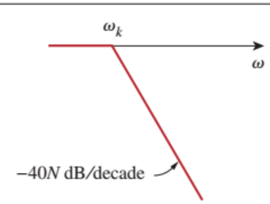
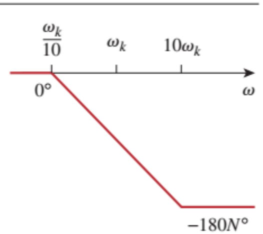
(a)



(b)

TABLE 14.3

Summary of Bode straight-line magnitude and phase plots.

Factor	Magnitude	Phase
K		
$(j\omega)^N$		
$\dfrac{1}{(j\omega)^N}$		
$\left(1+\dfrac{j\omega}{z}\right)^N$		
$\dfrac{1}{(1+j\omega/p)^N}$		
$\left[1+\dfrac{2j\omega\zeta}{\omega_n}+\left(\dfrac{j\omega}{\omega_n}\right)^2\right]^N$		
$\dfrac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$		

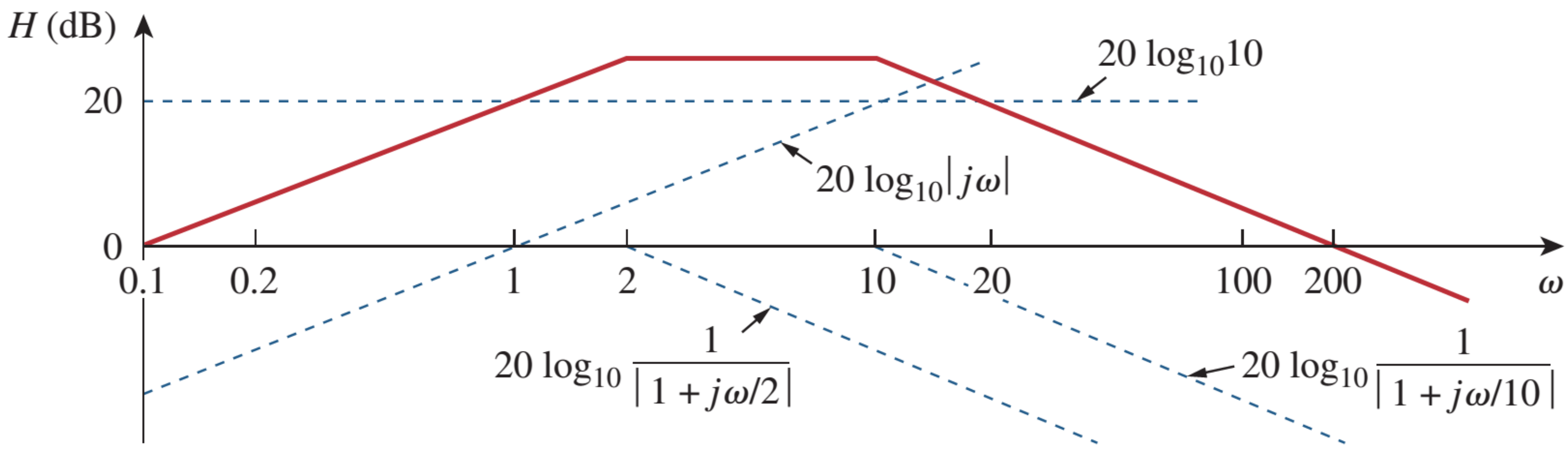
Example--Standard Form

$$\mathbf{H}(\omega) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)}$$

$$\mathbf{H}(\omega) = \frac{10\,j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

Example - Magnitude

$$H_{\mathrm{dB}} = 20\log_{10}10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1+\frac{j\omega}{2}\right| - 20\log_{10}\left|1+\frac{j\omega}{10}\right|$$

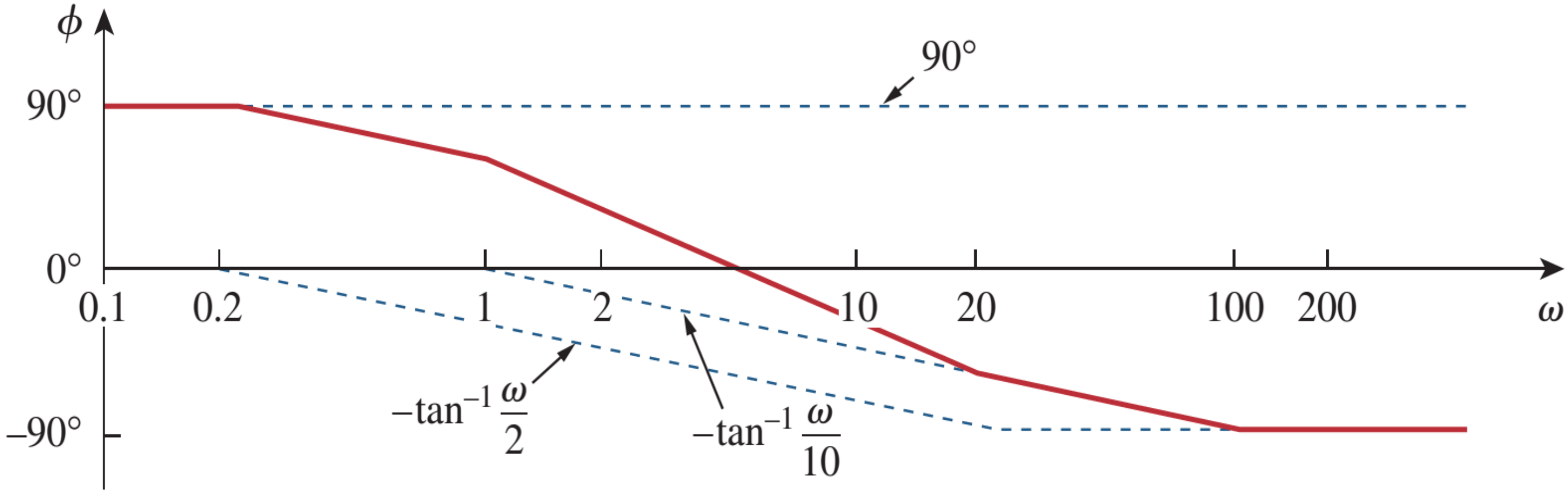


Example - Phase

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \underline{/90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10}$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$



Exercises

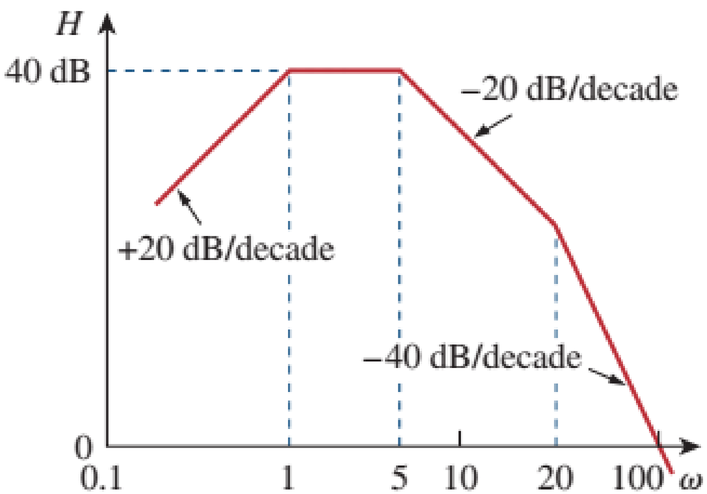
- $H(\omega) = K$
- $H(\omega) = (j\omega)^N$
- $H(\omega) = 1/(j\omega)^N$

Exercises

- $\mathbf{H}(\omega) = \dfrac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$

- $\mathbf{H}(\omega) = \dfrac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}$

Obtain the transfer function



Outline

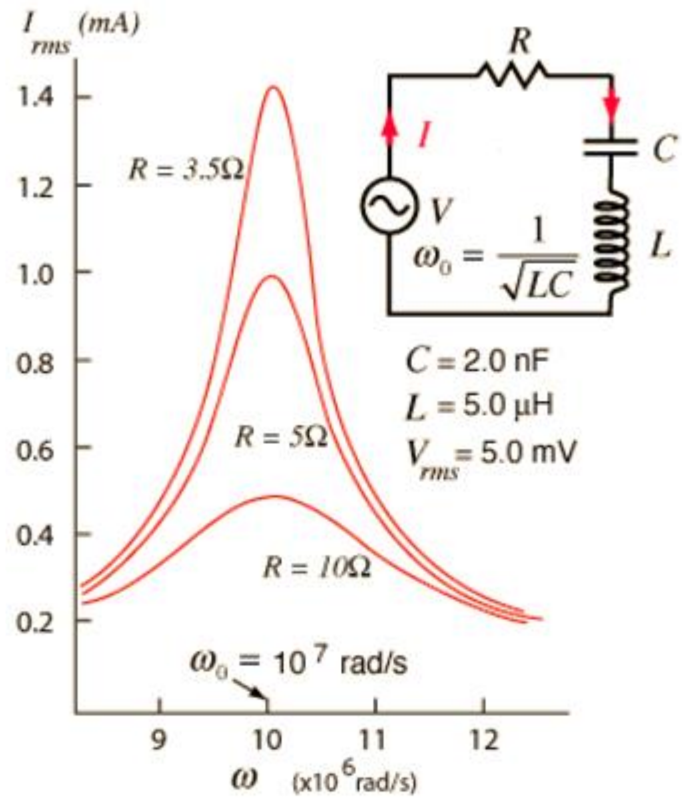
- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance

Series Resonance

- A series resonant circuit consists of an inductor and capacitor in series.

$$H(\omega) = \frac{I}{V} = \frac{1}{R + j\omega L + \dfrac{1}{j\omega C}}$$

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$



[Source: Georgia State U]

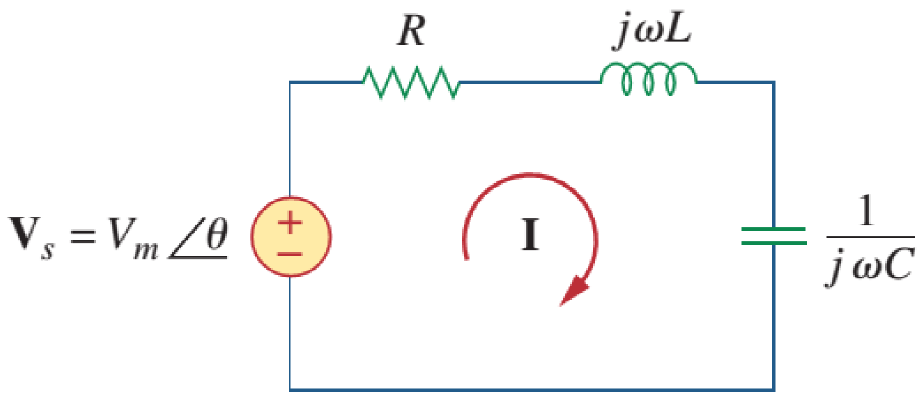
- Resonance occurs when the imaginary part of *Z* is zero.
- The value of ω that satisfies this is called the resonant frequency.

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

Series Resonance

- At resonance:
  - The impedance is purely resistive
  - The voltage V_s and the current I are in phase
  - The magnitude of the transfer function is maximum
  - The inductor and capacitor voltages can be much more than the source

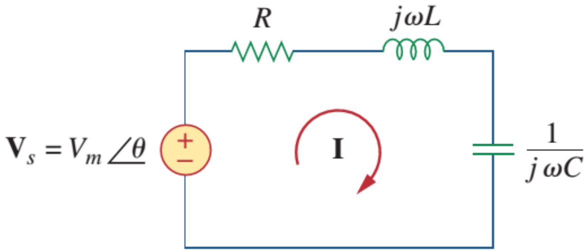


$$|\mathbf{V}_L| = \frac{V_m}{R}\omega_0 L$$

$$|\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C}$$

$$H(\omega) = \frac{I}{V} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$$

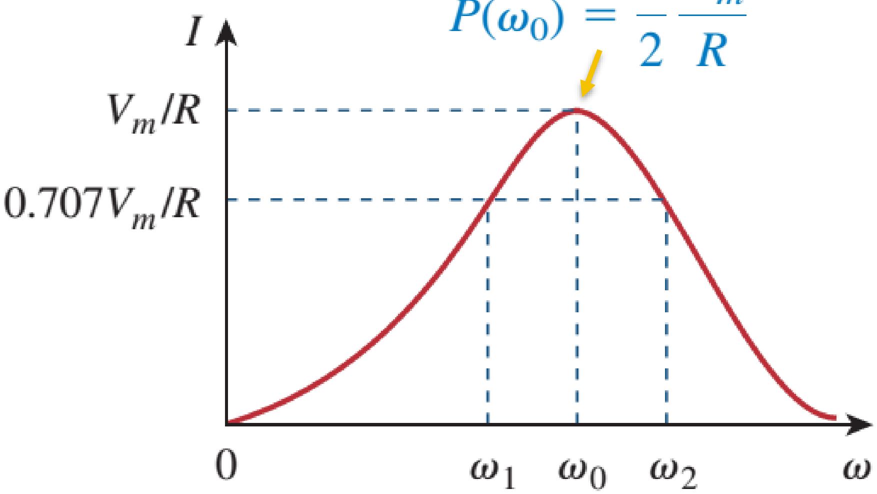
Half-Power Frequencies



- The response of the current magnitude:

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R}$$



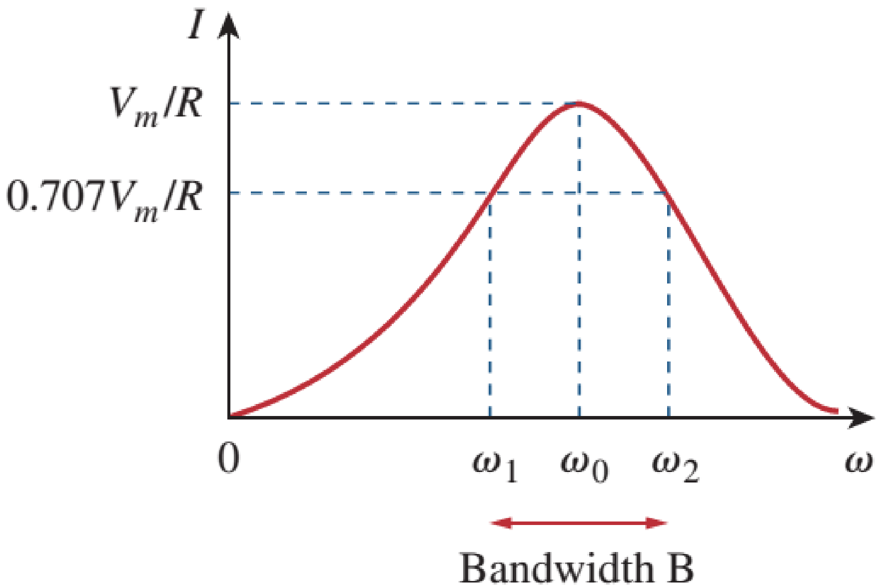
$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$\boxed{P(\omega_1) = P(\omega_2) = \frac{1}{2}P(\omega_0)}$$

$$\boxed{\begin{aligned} \omega_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{aligned}}$$

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

Bandwidth



$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

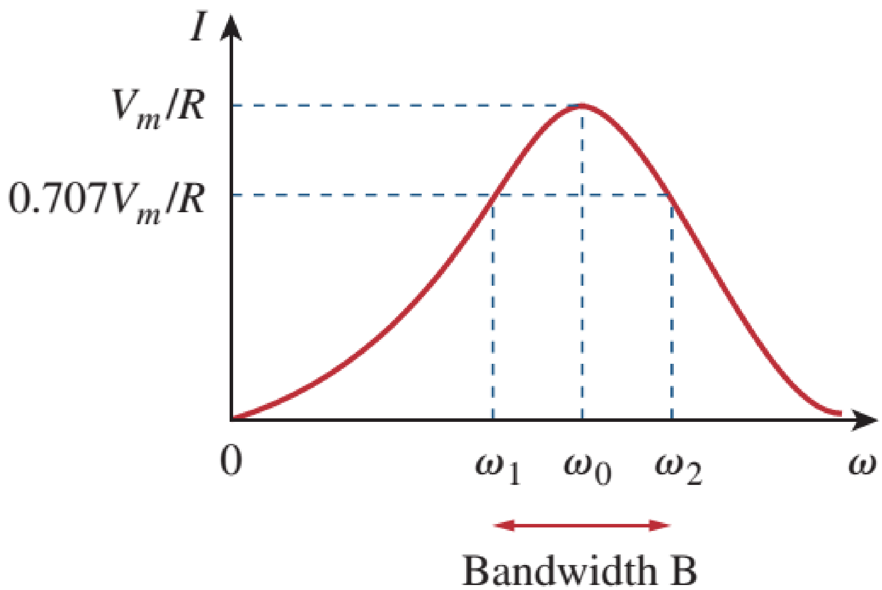
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

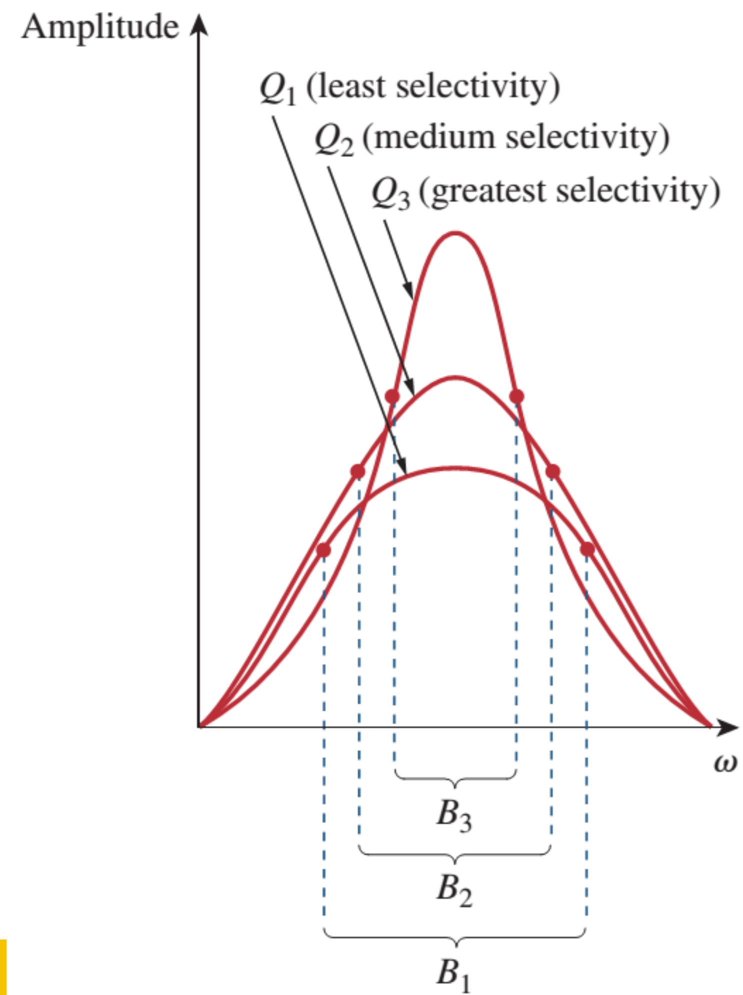
$$B = \omega_2 - \omega_1 = \frac{R}{L}$$

- Bandwidth: the difference between the two half-power frequencies

Quality Factor *Q*

- Quality factor *Q*: measure the "sharpness" of the resonance.



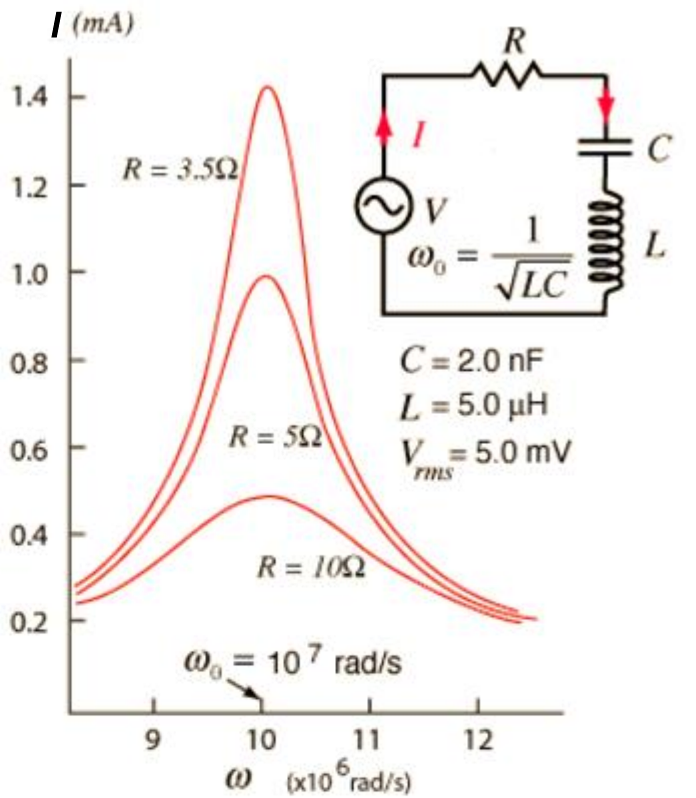


The smaller the *B*, the higher the *Q*.

$$Q = \frac{\omega_0}{B} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad B = \omega_2 - \omega_1 = \frac{R}{L}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Quality Factor Q – From Energy Perspective



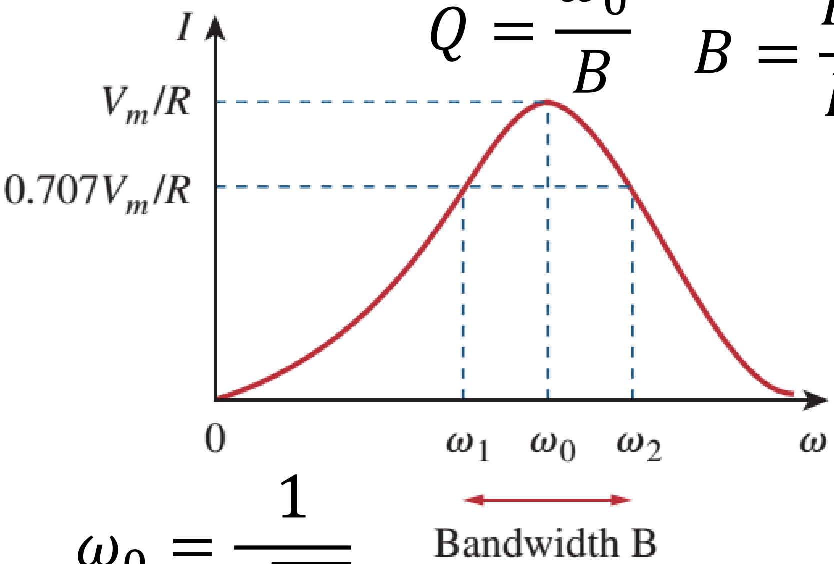
[Source: Georgia State U]

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit}}$$

in one period at resonance

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$$

Approximation of Half-Power Frequencies



$$Q = \frac{\omega_0}{B} \qquad B = \frac{R}{L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

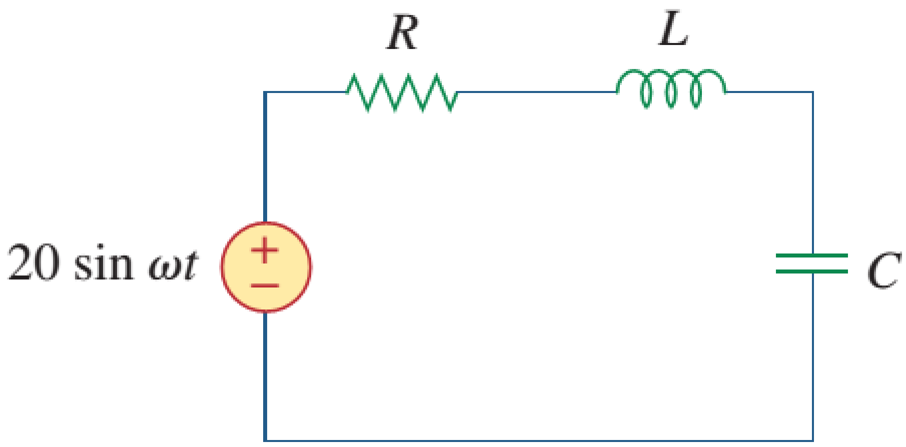
$$\frac{\omega_1}{\omega_0} = -\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}$$

$$\frac{\omega_2}{\omega_0} = \frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}$$

- For high-Q ($Q \geq 10$) circuits, half-power frequencies can be approximated as

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Example



In the circuit, $R = 2\Omega$, $L = 1\text{mH}$ and $C = 0.4\mu\text{F}$

- Find resonant frequency ω_0.
- Find half-power frequencies.
- Calculate *Q* and bandwidth *B.*
- Determine the amplitude of the current at ω_0, ω_1 and ω_2.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

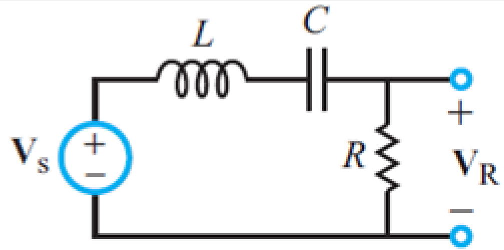
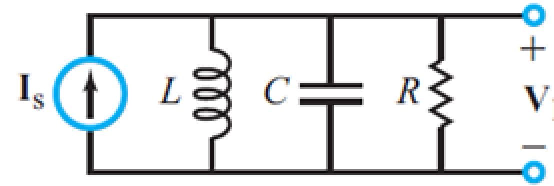
$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

At $\omega = \omega_0$,

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	$\dfrac{\omega_0}{B} = \dfrac{R}{\omega_0 L}$
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

Notes: (1) The expression for Q of the series RLC circuit is the inverse of that for Q of the parallel circuit. (2) For $Q \geq 10$, $\omega_1 \simeq \omega_0 - \frac{B}{2}$, and $\omega_2 \simeq \omega_0 + \frac{B}{2}$.

[Source: Berkeley]