SI 151, Spring 2020
The solution of quiz 15

1. **Solution:**
   A quadratic program can be expressed in the form

$$\begin{aligned} \text{minimize}_x \quad & \tfrac{1}{2}x^T Qx + r^T x + s \\ \text{subject to} \quad & Gx \preceq h, \\ & Ax = b, \end{aligned}$$

where $Q \in \mathbb{S}^n_+, G \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{p\times n}$. The original QP can be rewritten in epigraph form as the following QP in $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\begin{aligned} \text{minimize}_t \quad & t \\ \text{subject to} \quad & \tfrac{1}{2}x^T Qx + r^T x + s \le t, \\ & Gx \preceq h, \\ & Ax = b. \end{aligned}$$

Since Q is symmetric and positive semidefinite, there is some matrix P such that

$$Q = P^T P.$$

Using the Schur complement, the convex quadratic inequality constraint can be rewritten as the following LMI

$$\begin{bmatrix} -I & -Px \\ -x^T P^T & -t + s + r^T x \end{bmatrix} \preceq 0$$

and the linear inequality constraint can be written as the following LMI

$$\mathbf{diag}(Gx - h) \preceq 0.$$

Thus, the convex QP can be written as the SDP in $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\begin{aligned} \text{minimize}_{x,t} \quad & t \\ \text{subject to} \quad & \begin{bmatrix} -I & -Px & 0 \\ -x^T P^T & -t + s + r^T x & 0 \\ 0 & 0 & \mathbf{diag}(Gx - h) \end{bmatrix} \preceq 0 \end{aligned}$$

2. **Solution:**
   The Lagrangian is

$$L(x, z, \mu) = \sum_{i=1}^{n} x_i \log x_i + \lambda^T (Ax - b) + \mu^T (Cx - d).$$

Minimizing over x_i gives the conditions

$$1 + \log x_i + a_i^T \lambda + c_i^T \mu = 0, \quad i = 1, ..., n,$$

with solution

$$x_i = e^{-a_i^T \lambda - c_i^T \mu - 1},$$

where a_i and c_i are the ith column of A and C, respectively. Plugging this in in L gives the Lagrange dual function

$$g(\lambda, \mu) = -b^T \lambda - d^T \mu - \sum_{i=1}^{n} e^{-a_i^T \lambda - c_i^T \mu - 1}.$$