SI151A
Convex Optimization and its Applications in Information Science, Fall 2021
Homework 4

Due on Nov. 22, 2021, 23:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

1. Submit your homework at **Gradescope**. Homework 4 contains two parts: the theoretical part the and the programming part.
2. About the the theoretical part:
   (a) Submit your homework in **Homework 4** in gradescope. Make sure that you have correctly selected pages for each problem. If not, you probably will get no point.
   (b) Your homework should be uploaded in the **PDF** format, and the naming format of the file is not specified.
3. About the programming part:
   (a) Implement your program in Matlab (or Python), and make sure that your codes are executable and are consistent with your solutions.
   (b) When handing in your homework, package all your codes and results into your_student_id+hw4_code.zip and upload. In the package, you also need to include a file named README.txt/md to clearly identify the function of each file.
4. **No late submission is allowed.**

---

1. (*Linear Programming*) Consider the following compressive sensing problem via ℓ_1-minimization:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \|\boldsymbol{x}\|_1 \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{z}, \end{aligned} \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times d}$, $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{z} \in \mathbb{R}^m$.

(a) Equivalently reformulate (1) into a linear programming problem. (10 points)

(b) Write down the dual problem of the reformulated linear program in (a). (10 points)

(c) Figure 1 shows the phase transition phenomenon in compressed sensing [1]. This diagram shows the empirical probability that the ℓ_1-minimization method (1) successfully identifies a vector $\boldsymbol{x}_0 \in \mathbb{R}^d$ with s non-zero entries given a vector $\boldsymbol{z}_0$ consisting of m random measurements of the form $\boldsymbol{z}_0 = \boldsymbol{A}\boldsymbol{x}_0$ where A is an $m \times d$ matrix with independent standard normal entries. The brightness of each point reflects the observed probability of success, ranging from certain failure (black) to certain success (white).
Write a program to validate the phase transition phenomenon. Specifically, for a fixed $d = 100$, plot a two-dimensional image. The horizontal axis m is the number of random measurements, which ranges from 0 to 100. The vertical axis s is the number of nonzeros in $\boldsymbol{x}_0$, which ranges from 0 to 100. For each pair (s, m), generate 50 random problems. The intensity of the image shows the fraction of problems for which ℓ_1-minimization succeeds. (15 points)

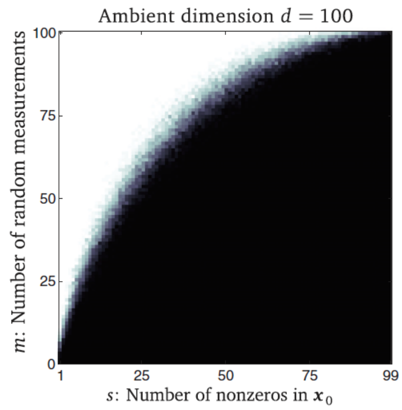

Figure 1: phase transition

2. (*Second-Order Cone Programming*) Consider the following coordinated beamforming design problem for transmit power minimization in wireless communication networks [2]

$$\begin{aligned} \mathscr{P} : \underset{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_K}{\text{minimize}} \quad & \sum_{k=1}^{K} \|\boldsymbol{w}_k\|^2 \\ \text{subject to} \quad & \text{SINR}_k(\boldsymbol{w}_1, \cdots, \boldsymbol{w}_K) \geq \gamma_k, k = 1, \cdots, K, \end{aligned} \tag{2}$$

where $\boldsymbol{w}_k \in \mathbb{C}^n$ is the transmit beamforming vector for user k, and $\gamma_k \geq 0$ is the threshold for quality-of-service (QoS) requirement. The signal-to-interference-plus-noise-ratio (SINR) for k -th user is given by

$$\text{SINR}_k(\boldsymbol{w}_1, \cdots, \boldsymbol{w}_K) = \frac{\left|\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{w}_k\right|^2}{\sum_{i \neq k} \left|\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{w}_i\right|^2 + \sigma^2}, \tag{3}$$

where $\boldsymbol{h}^k \in \mathbb{C}^n$ is the channel coefficient vector between the transmitter and the k -th user and $\sigma^2 \geq 0$ is noise power. Parameters $\boldsymbol{h}_k, \gamma_k, \sigma^2$ are known in this problem.

(a) Equivalently reformulate problem $\mathscr{P}$ into a second-order cone programming (SOCP) problem. (10 points)

(b) Find the global optimal solution to problem $\mathscr{P}$ using Lagrangian duality approach. (10 points)

(c) In (a), we have equivalently reformulated problem $\mathscr{P}$ into a second-order cone programming (SOCP) problem. Next, Consider the complex Gaussian channel channel, i.e., $\boldsymbol{h}_k \sim \mathcal{CN}\left(\boldsymbol{0}, s^2\boldsymbol{I}\right)$ with $s = 1/\sqrt{K}$.

And the noise power $\sigma^2 = 1$ without loss of generality. Each target SINR $\gamma_k \geq 0$ and it's often represented with dB, which is defined as $10 \log \gamma_k$. Consider the relationship between target SINR and the feasibility of $\mathscr{P}$. Please draw the *phase transition* figure where X-axis is target SINR in dB ($\gamma_1 = \cdots = \gamma_K = \gamma$), and Y-axis is the ratio when the problem is feasible over multiple realizations of channel, i.e.

$$R = \frac{\#\{\mathscr{P} \text{ is feasible}\}}{\# \text{ of tests(channel realizations)}}. \tag{4}$$

Assume $K = 50, n = 3$. You need to run 20 times and take average. (15 points)

3. (*Semidefinite Programming*) In many scenarios such as collaborative filtering, we wish to make predictions about all entries of an (approximately) low-rank matrix $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ (e.g., a matrix consisting of users' ratings about many movies), yet only a highly incomplete subset of the entries are revealed to us [3]. Consider the following matrix completion problem via nuclear norm minimization:

$$\begin{aligned} \underset{\mathbf{X} \in \mathbb{R}^{m \times n}}{\text{minimize}} \quad & \|\mathbf{X}\|_* \\ \text{subject to} \quad & X_{ij} = M_{ij}, \quad (i,j) \in \Omega \end{aligned} \tag{5}$$

where Ω denotes the set of locations corresponding to the observed entries ($(i,j) \in \Omega$ if M_{ij} is observed) and the nuclear norm $\|\cdot\|_*$ is defined as

$$\|\mathbf{X}\|_* = \sum_{k=1}^{p} \sigma_k(\mathbf{X}), \quad p = \min\{m, n\}$$

and $\sigma_k(\mathbf{X})$ denotes the kth largest singular value of $\mathbf{X}$.

(a) Solve problem (5) using **CVX**. The test data are generated as follows: (15 points)

```
%% Data Generation
seed = 97006855;
ss = RandStream('mt19937ar', 'Seed', seed);
RandStream.setGlobalStream(ss);
m = 100;
n = 20;
density = 0.2;
M = sprand(m, n, density);

save('data.mat', 'M');
```

(b) It has been proved that problem (5) can be equivalently reformulated into the following semidefinite programming (SDP) problem [4]:

$$\begin{aligned} \underset{\mathbf{X}, \mathbf{W_1}, \mathbf{W_2}}{\text{minimize}} \quad & \frac{1}{2}(\text{trace}(\mathbf{W}_1) + \text{trace}(\mathbf{W}_2)) \\ \text{subject to} \quad & X_{ij} = M_{ij}, \quad (i,j) \in \Omega \\ & \begin{bmatrix} \mathbf{W}_1 & \mathbf{X} \\ \mathbf{X}^\top & \mathbf{W}_2 \end{bmatrix} \succeq 0 \end{aligned} \tag{6}$$

with optimization variables $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{W}_1 \in \mathbb{R}^{m \times m}$ and $\mathbf{W}_2 \in \mathbb{R}^{n \times n}$. Solve problem (6) using **CVX** with the test data in (a). Verify that the optimal solutions of (5) and (6) are identical. (15 points)

Remarks: (Important!)

- The solution of (a) and (b) should be printed in files named "sol1.txt" and "sol2.txt" respectively.
- The optimizer's ouput of (a) and (b) should be printed in files named "out1.txt" and "out2.txt" respectively.

References

[1] D. Amelunxen, M. Lotz, M. B. McCoy, and J. A. Tropp, "Living on the edge: Phase transitions in convex programs with random data," *Information and Inference: A Journal of the IMA*, vol. 3, no. 3, pp. 224–294, 2014.

[2] E. Björnson, M. Bengtsson, and B. Ottersten, "Optimal multiuser transmit beamforming: A difficult problem with a simple solution structure [lecture notes]," *IEEE Signal Process. Mag.*, vol. 31, pp. 142–148, Jul 2014.

[3] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Foundations of Computational mathematics*, vol. 9, no. 6, pp. 717–772, 2009.

[4] M. Fazel, *Matrix rank minimization with applications.* PhD thesis, Stanford University, 2002.