Lecture 2
Kirchhoff's Laws & Circuit Analysis

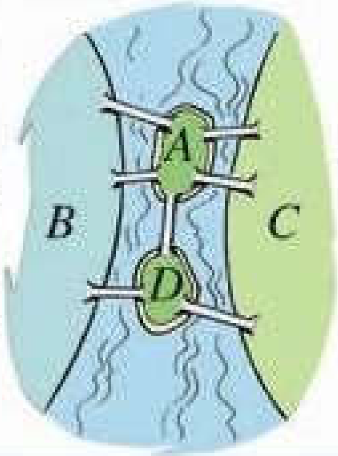
Outline

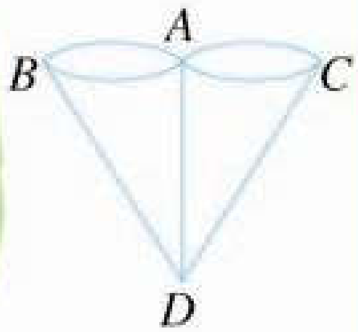
- Kirchhoff's Laws
  - KCL
  - KVL
- Circuit Analysis
  - Nodal Analysis
  - Mesh Analysis

你知道吗?

七桥问题

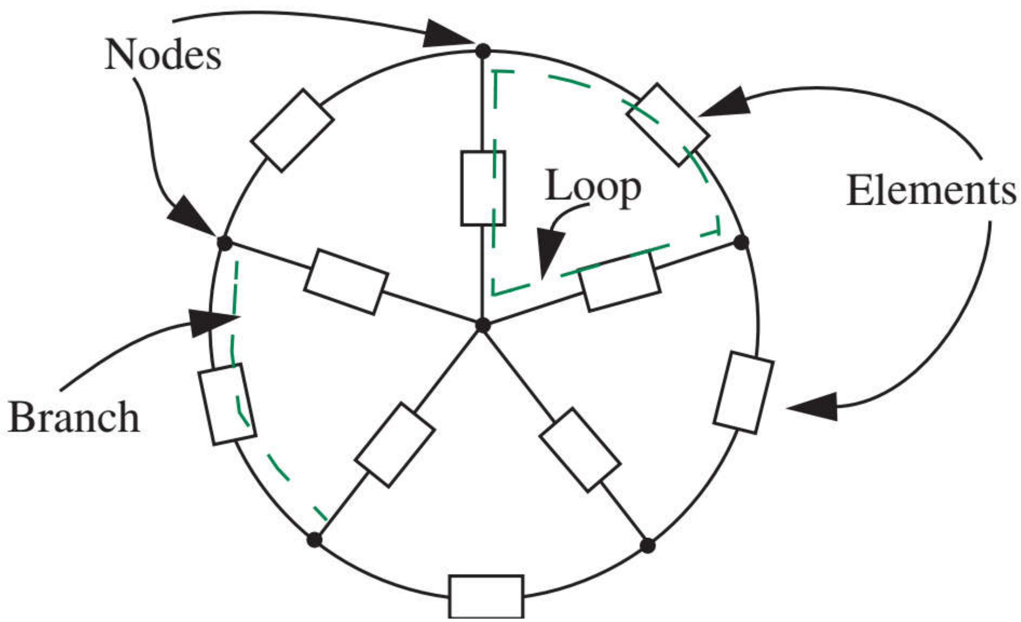
18 世纪东普鲁士的哥尼斯堡城，有一条河穿过，河上有两个小岛，有七座桥把两个岛与河岸联系起来（如下图）。有人提出一个问题：一个步行者怎样才能不重复、不遗漏地一次走完七座桥，最后回到出发点。后来大数学家欧拉把它转化成一个几何问题（如右图）——一笔画问题。





Terminology: Nodes, Branches and Loops

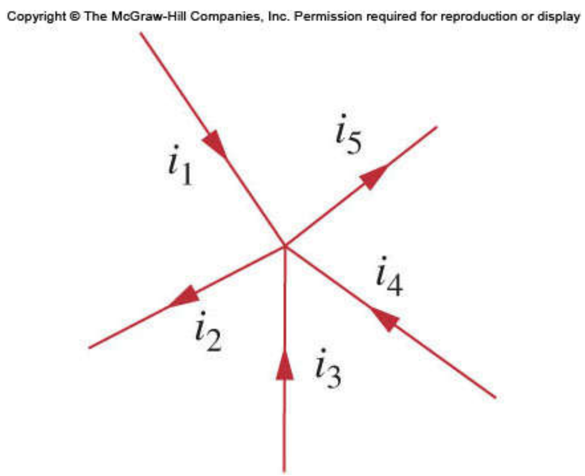
- **Node**: A point where two or more circuit elements are connected.
- **Branch**: A path that connects two nodes.
- **Loop**: Any closed path in a circuit.



[Source: MIT]

Kirchhoff's Laws

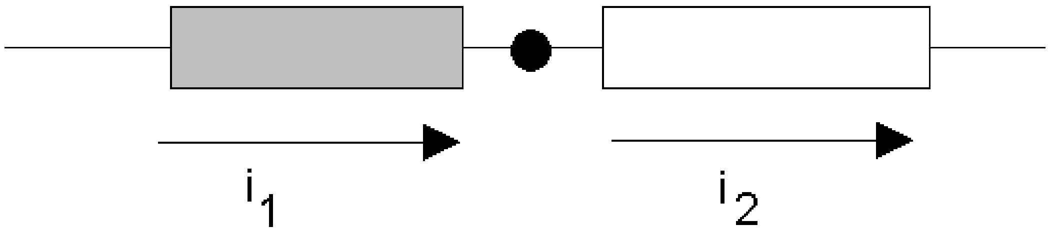
- Kirchhoff's Current Law (KCL):
  - The algebraic sum of all the **currents** entering any **node** in a circuit equals zero.
  - Why?



Gustav Robert Kirchhoff
1824-1887

A Major Implication of KCL

- KCL tells us that **all of the elements that are connected *in series* carry the same current.**

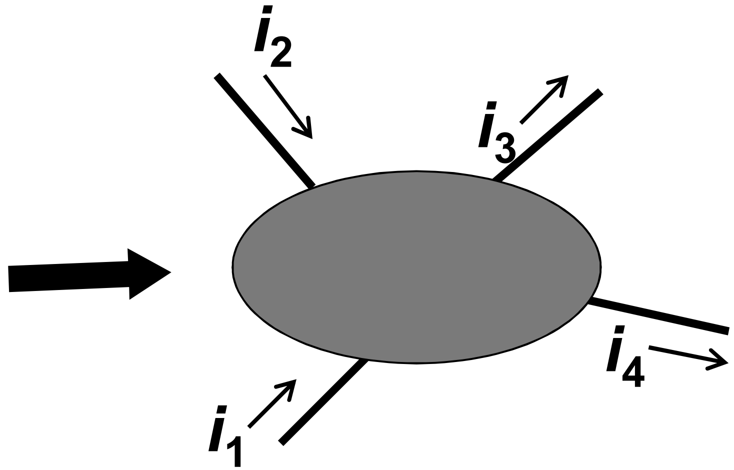


Current entering node = Current leaving node

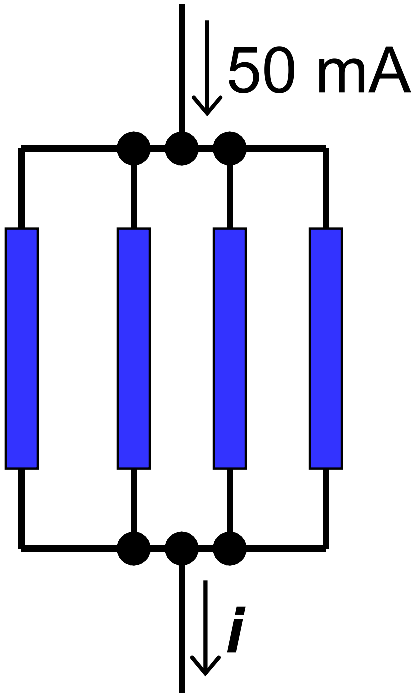
Generalization of KCL

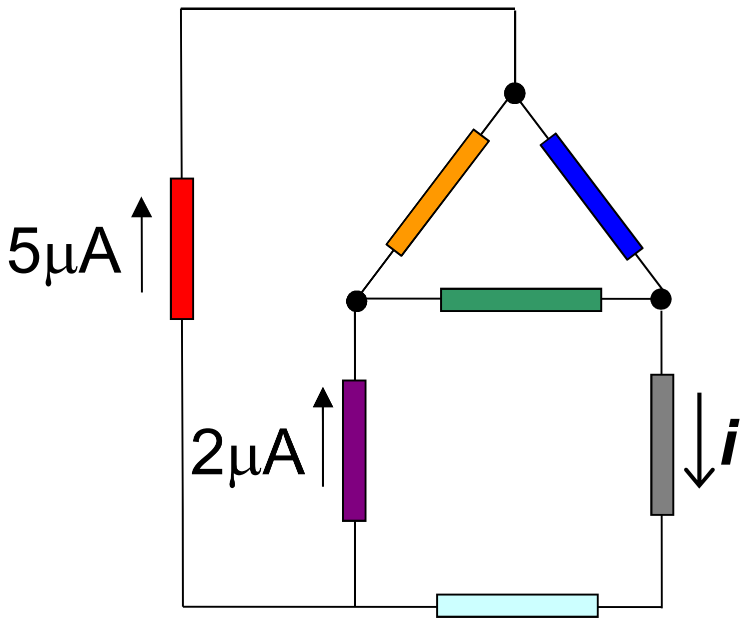
- The sum of currents entering/leaving a **closed surface** is zero.
  - Circuit branches can be inside this surface, *i.e.* the surface can enclose more than one node!

This could be a big chunk of a circuit, *e.g.* a “black box”



Generalized KCL Examples





When will KCL be Invalid?

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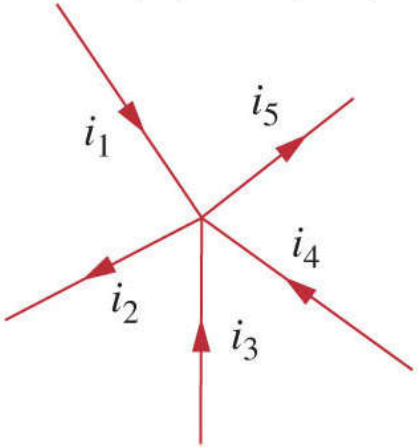
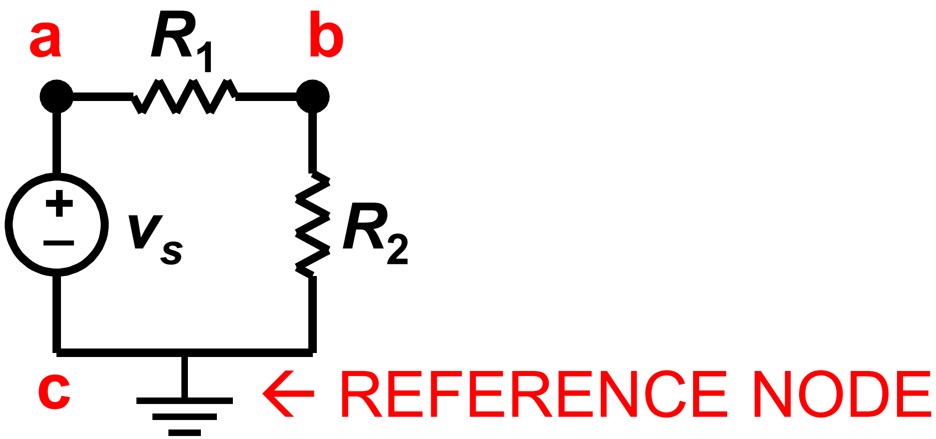


Notation: Node and Branch Voltages



- Use one node as the reference (the "common" or "ground" node) – label it with a symbol.
- The voltage drop from node **x** to the reference node is called the **node voltage v_x**.
- The voltage across a circuit element is defined as the difference between the node voltages at its terminals.

Kirchhoff's Voltage Law (KVL)

- The algebraic sum of all the **voltages** around any **loop** in a circuit equals zero.
- Why?

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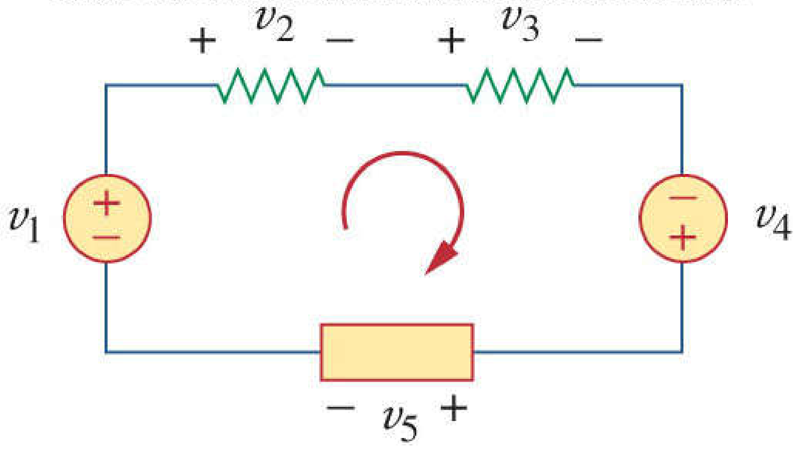
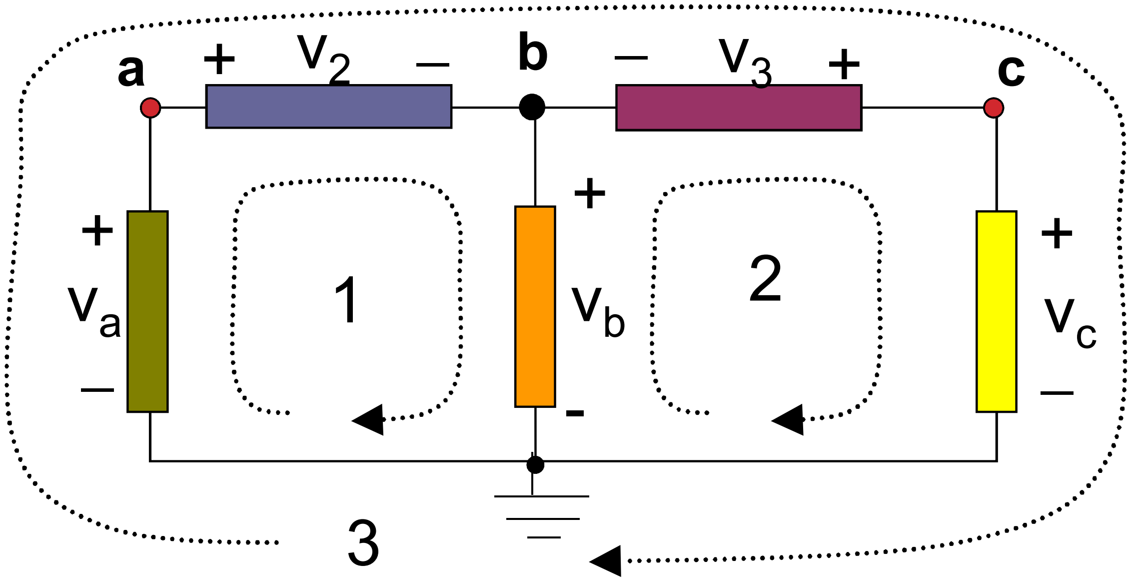


KVL Example

Three closed paths:



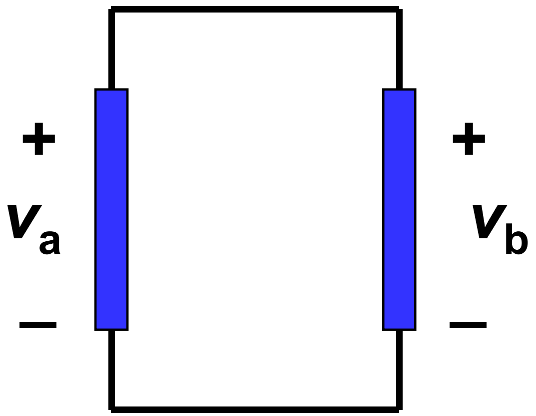
Path 1:

Path 2:

Path 3:

A Major Implication of KVL

- KVL tells us that **any set of elements which are connected at both ends carry the same voltage**.
- We say these elements are connected **in parallel**.



Series Resistors

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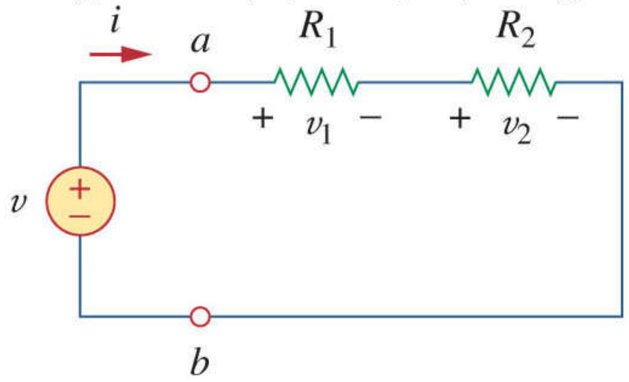

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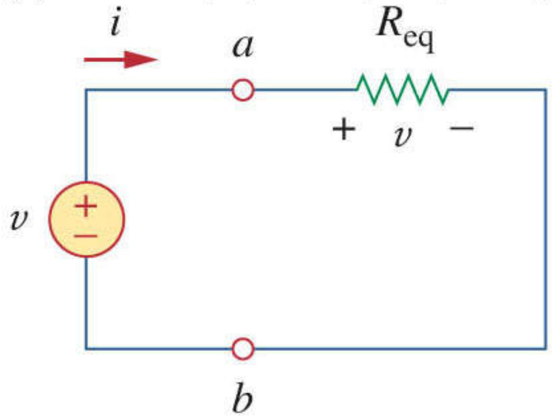

Voltage Division

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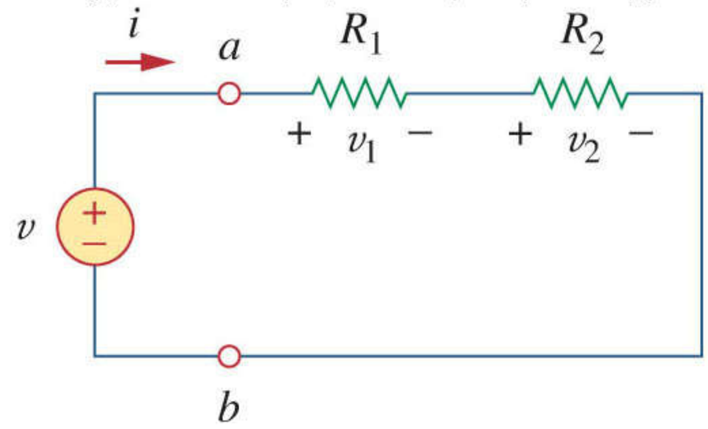
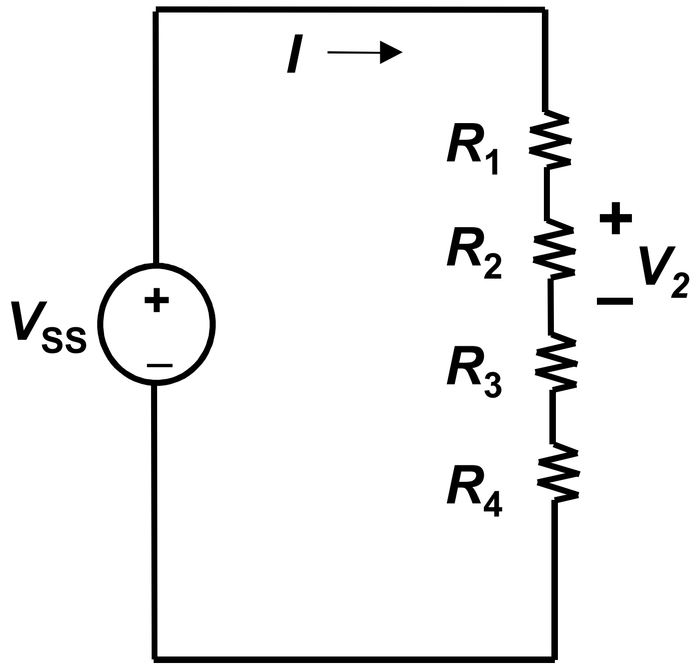
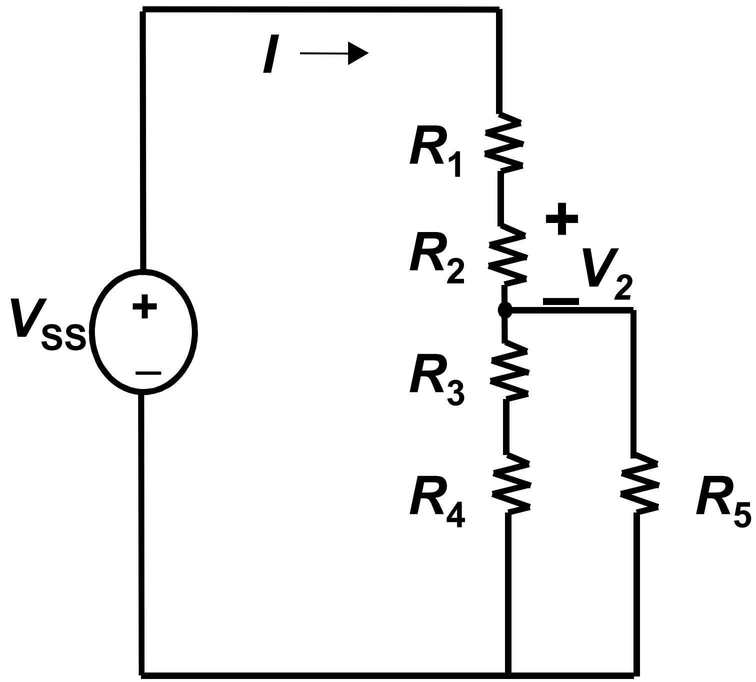


Three-terminal rheostat

When can the Voltage Divider Formula be Used?





Parallel Resistors

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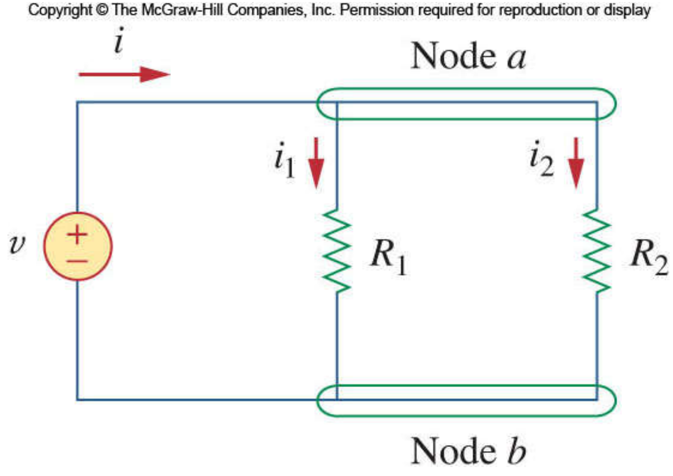

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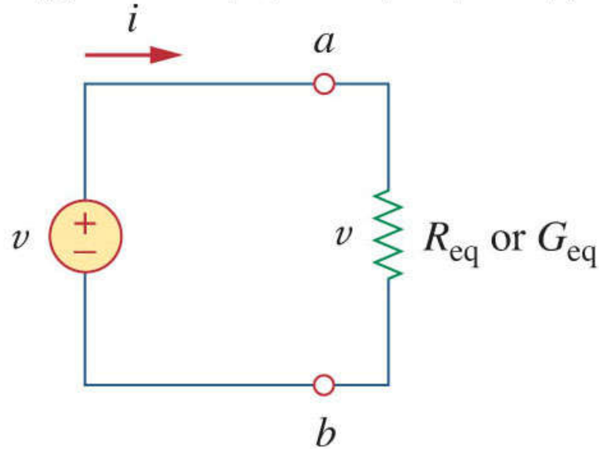

Current Division

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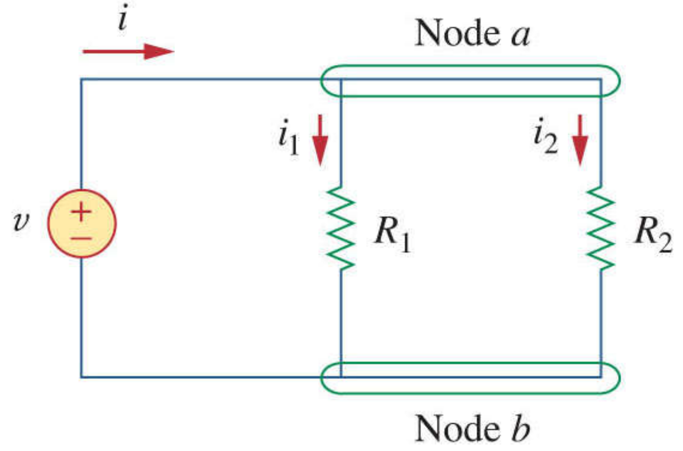
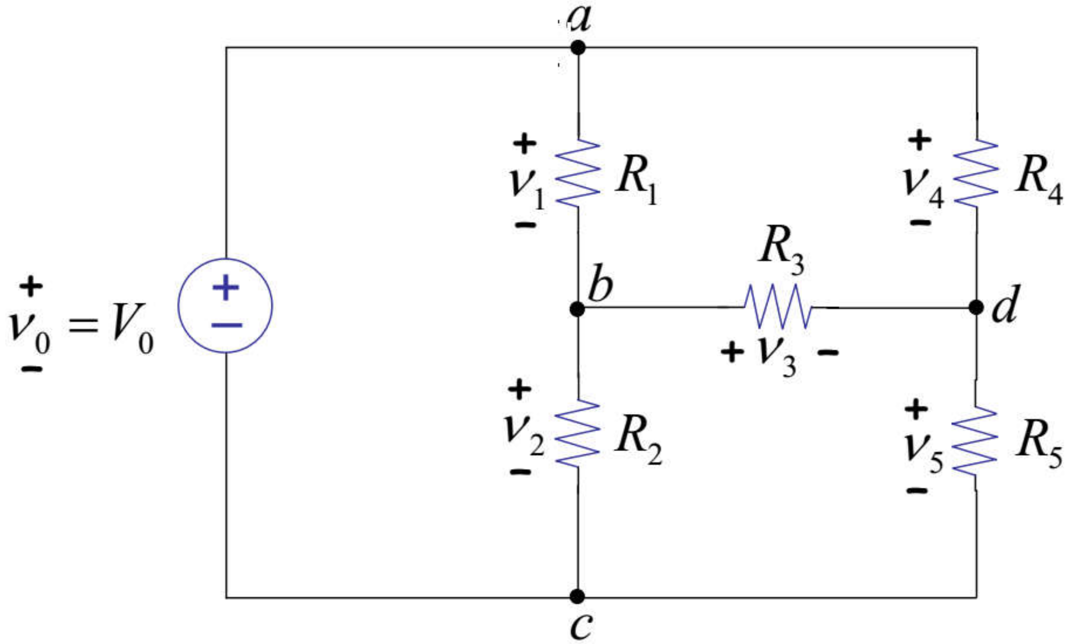


Exercise

- Find the voltage across each resistor



[Source: MIT]

Summary

- KCL and KVL

$$\sum_{n=1}^{N} i_n = 0 \qquad \sum_{m=1}^{M} v_m = 0$$

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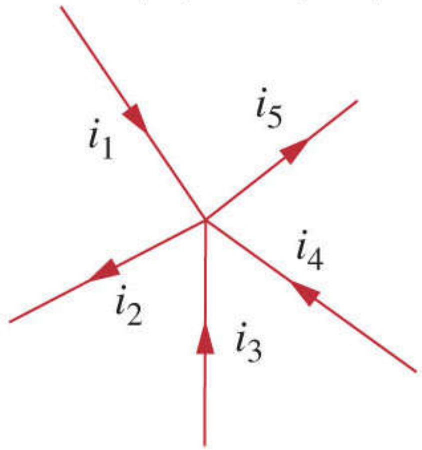

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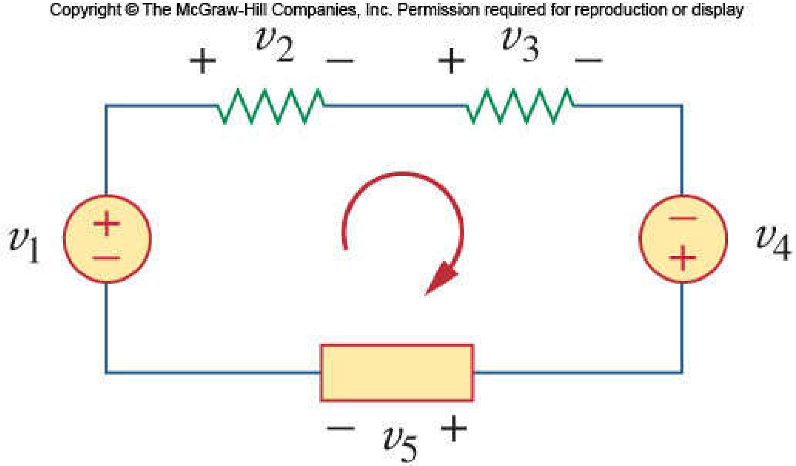


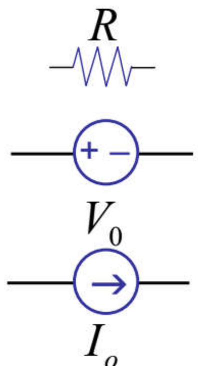
What you have learned

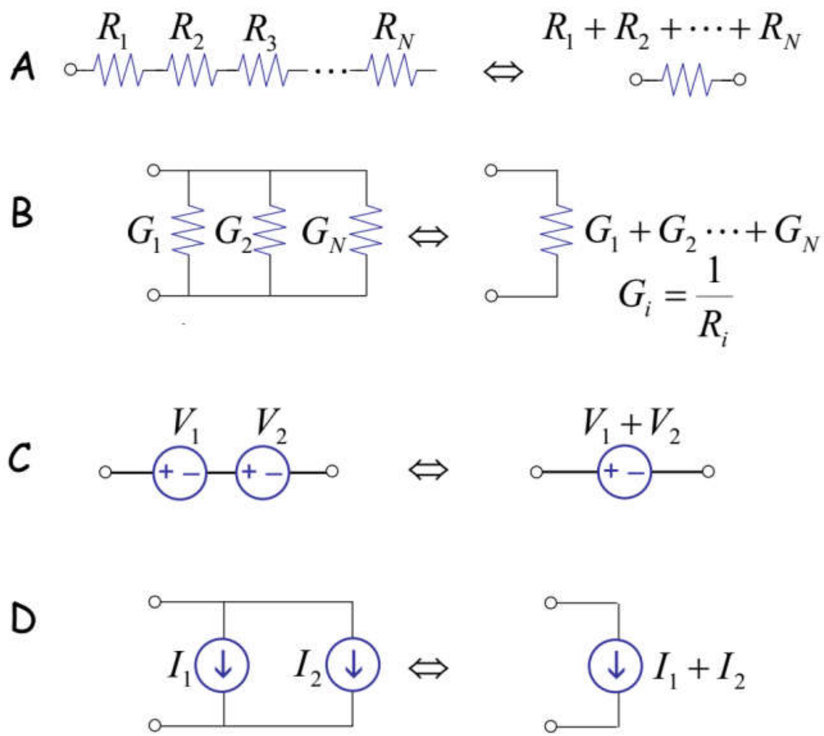
- KCL
- KVL
- Element relationships

For R, $V = IR$

For voltage source, $V = V_0$

For current source, $I = I_0$





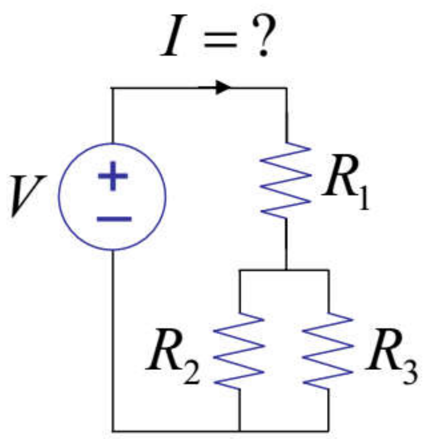
Circuit Analysis

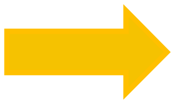
- Two techniques will be presented in this lecture:
  - Nodal analysis, which is based on KCL
    - Used in SPICE, the internal engine of circuit simulators.
  - Mesh analysis, which is based on KVL

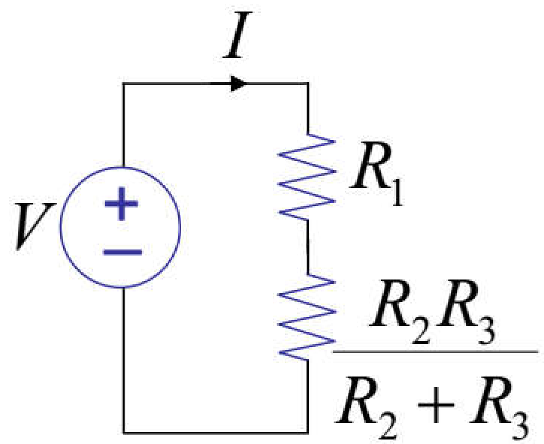
- The analysis will result in a set of simultaneous equations which may be solved by Cramer's rule or computationally (using MATLAB for example)

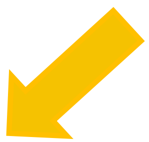
http://bwrcs.eecs.berkeley.edu/Classes/IcBook/SPICE/
http://www.ni.com/white-paper/5413/zhs/

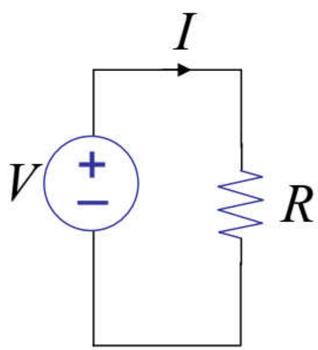
Example











$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$

[Source: MIT]

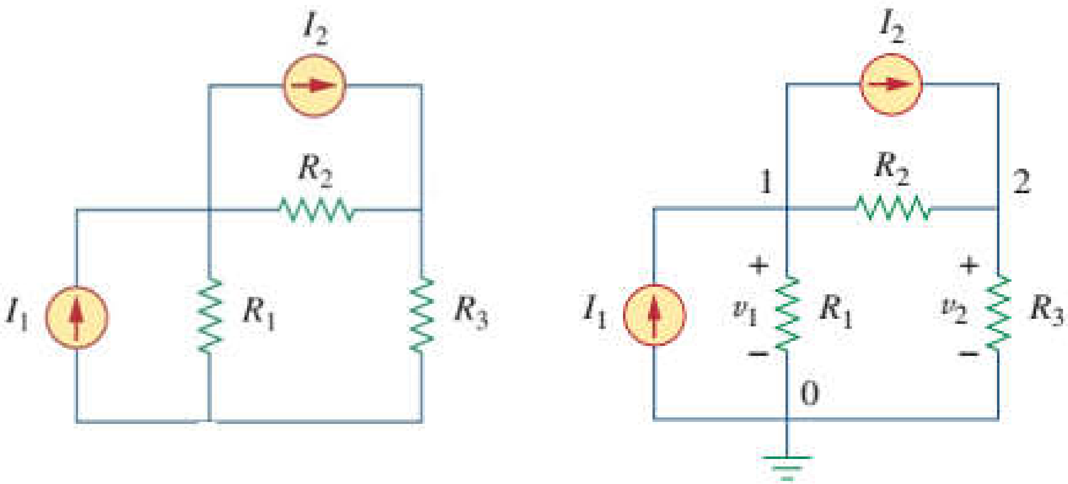
Nodal Analysis – Three Steps

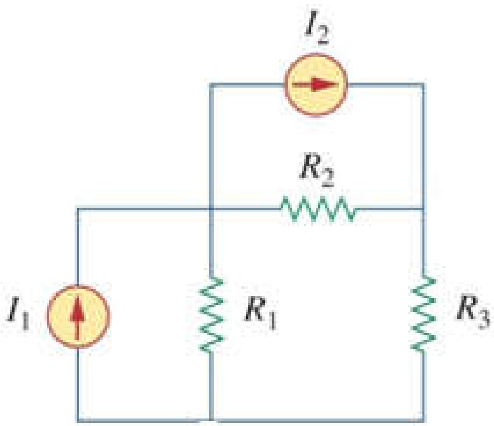
- Given a circuit with *n* nodes, the nodal analysis is accomplished via three steps:
  1. Select a node as the reference (i.e., ground) node. Define the node voltages (except reference node and the ones set by the voltage sources). Voltages are relative to the reference node.
  2. Apply KCL at nodes with unknown voltage, expressing current in terms of the node voltages (using the *I*-*V* relationships of branch elements).

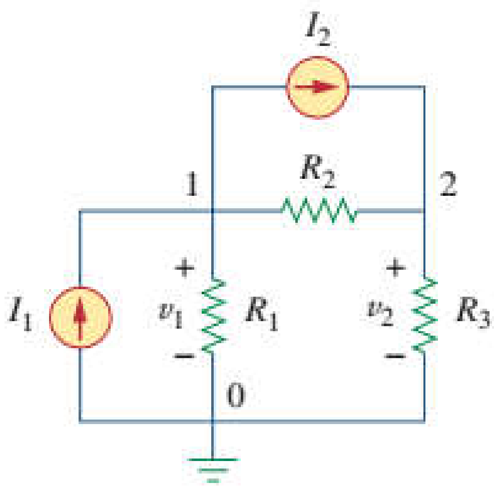
     Special cases: floating voltage sources.
  3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

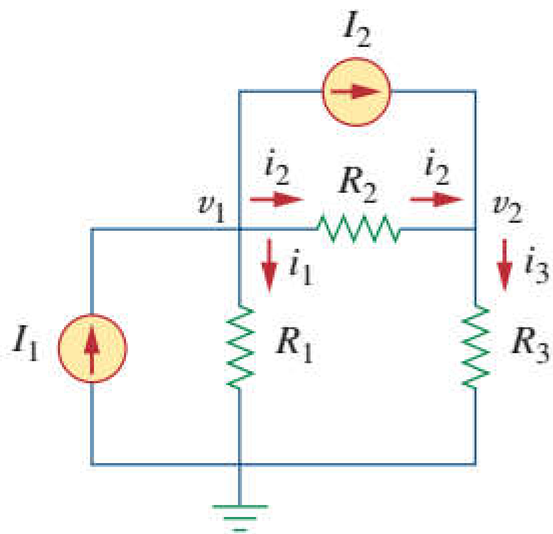
[Source: Berkeley]

Nodal Analysis Example #1




I_2
R_2
I_1
R_1
R_3


I_2
1
R_2
2
I_1
+
v_1
−
R_1
+
v_2
−
R_3
0


I_2
i_2
R_2
i_2
v_1
v_2
i_1
i_3
I_1
R_1
R_3

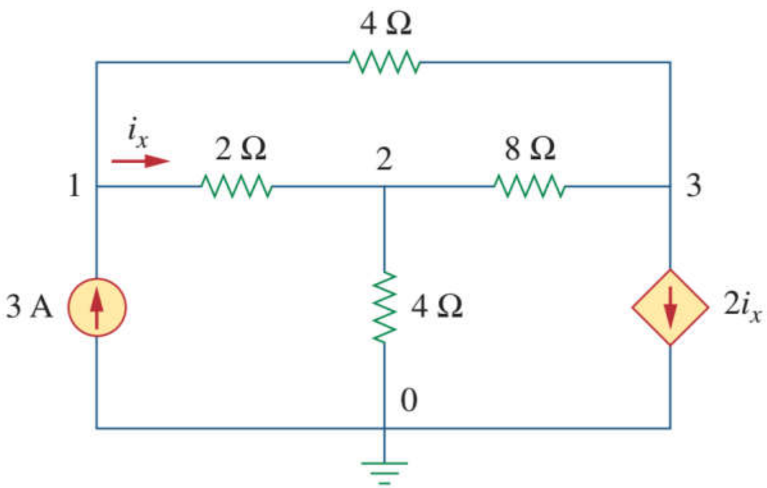
Cramer's Rule:
(optional)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

方程组有唯一解 $\Leftrightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

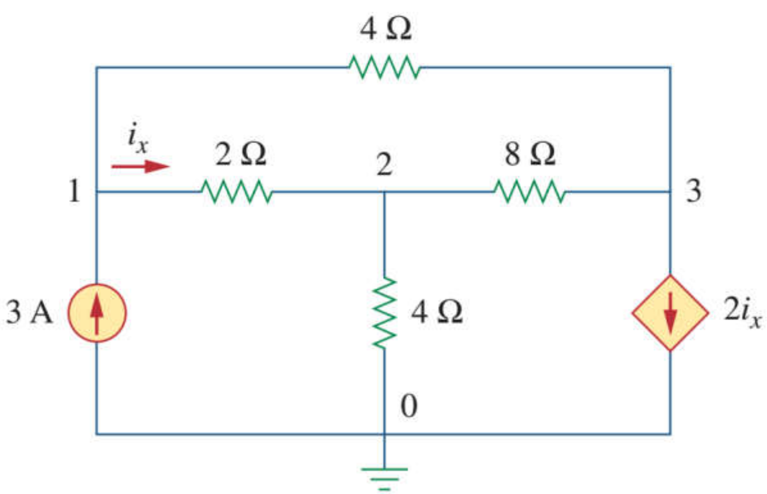
$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Nodal Analysis: Example #2

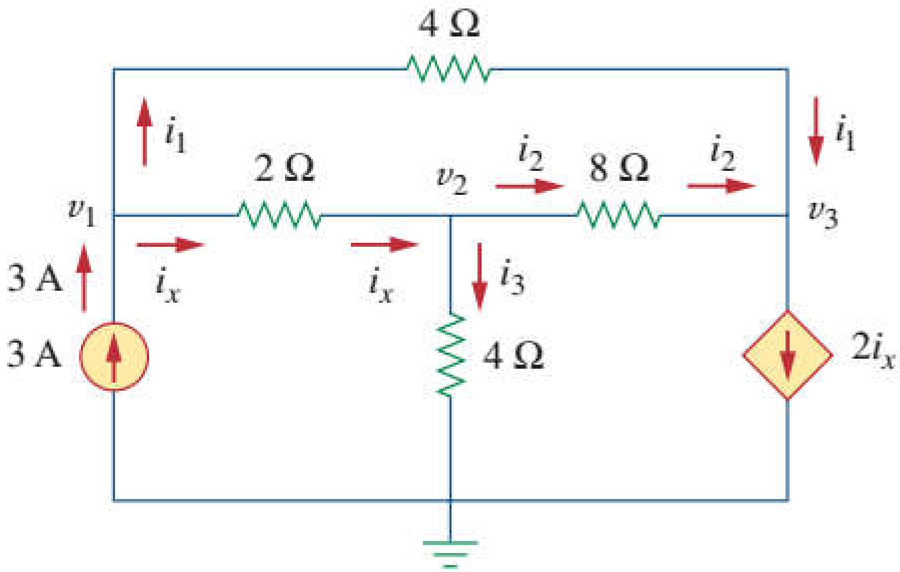


(a)

Nodal Analysis: Example #2



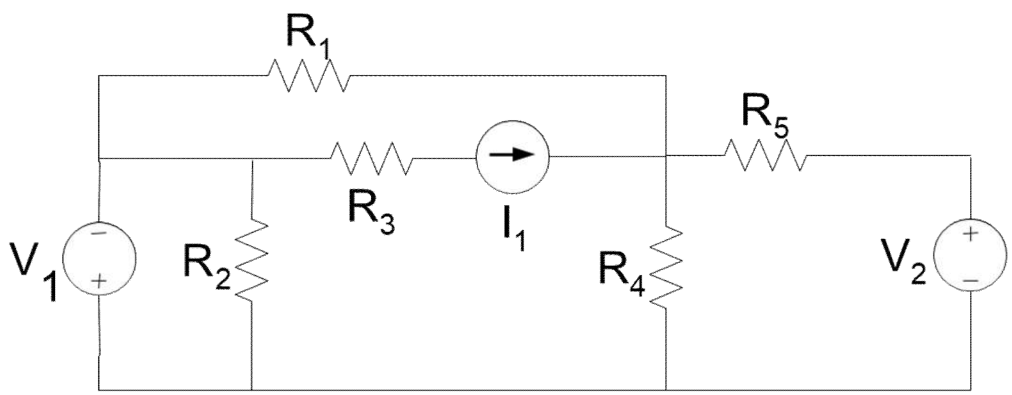
(a)



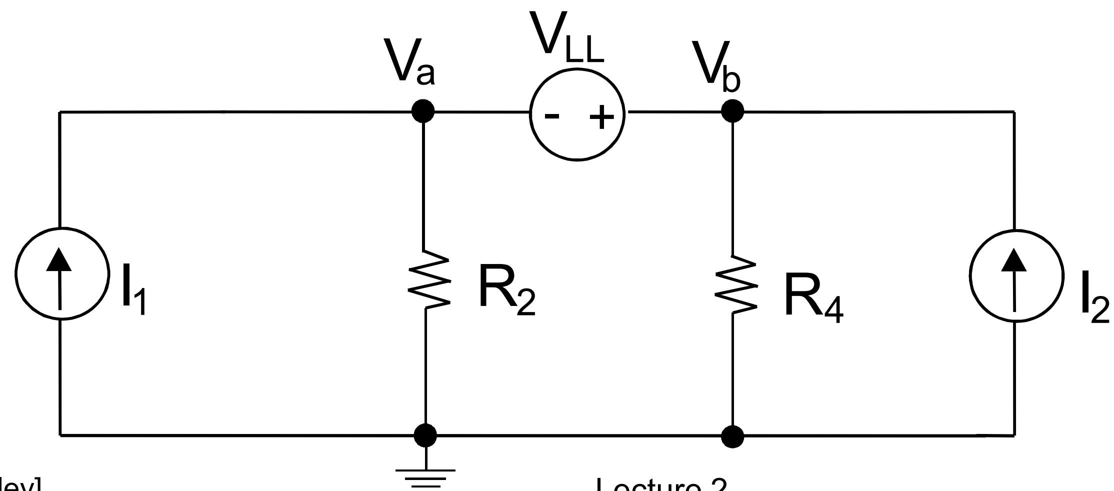
Nodal Analysis with Voltage Sources

Challenges:

- Determine node number
- Deal with different types of sources

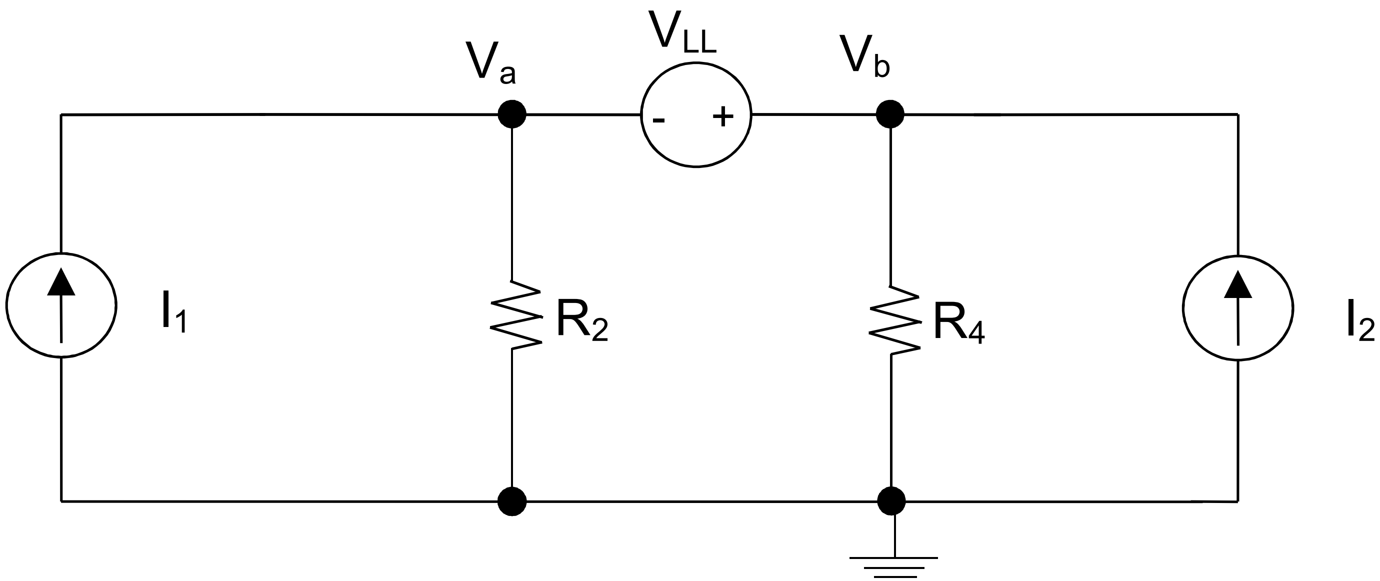


A "floating" voltage source is one for which neither side is connected to the reference node, e.g. V_{LL} in the circuit below:



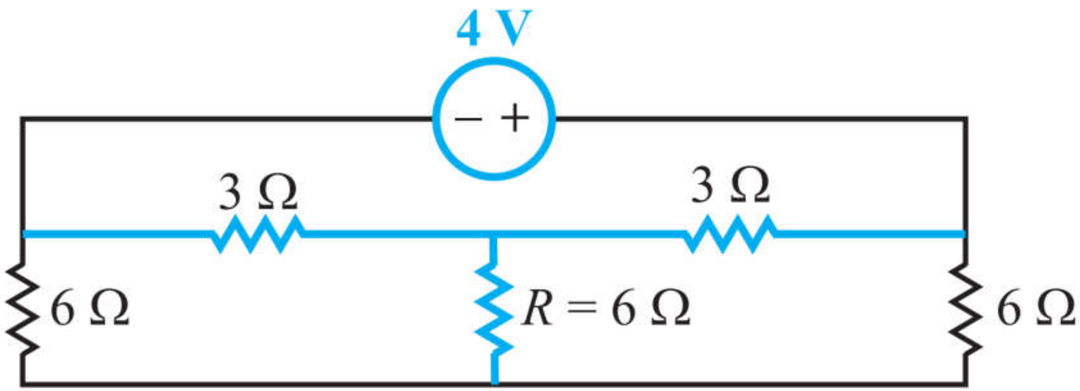
[Source: Berkeley]

Nodal Analysis: Supernode



Exercise

- Find the power supplied by the voltage source.



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Mesh Analysis

- Another general procedure for analyzing circuits is to use the mesh currents as the circuit variables.

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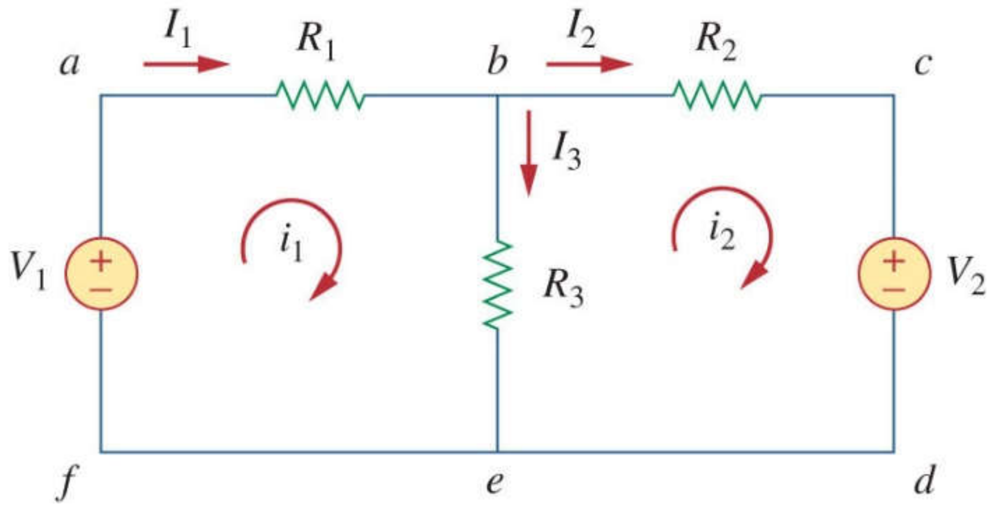

- Mesh analysis uses KVL to find unknown currents.

Loop, Independent Loop, Mesh

- A loop is a closed path with no node passed more than once.
- A loop is independent if it contains at least one branch which is not a part of any other independent loop.
- A mesh is a loop that does not contain any other loop within it.

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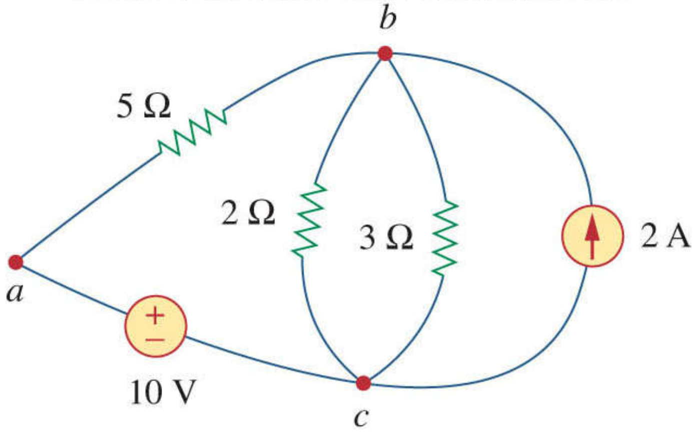


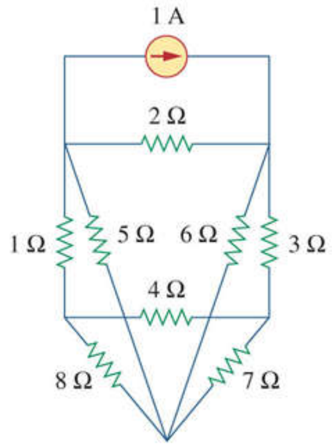
- b – number of branches
- n – number of nodes
- l_{ind} – number of ind. loops

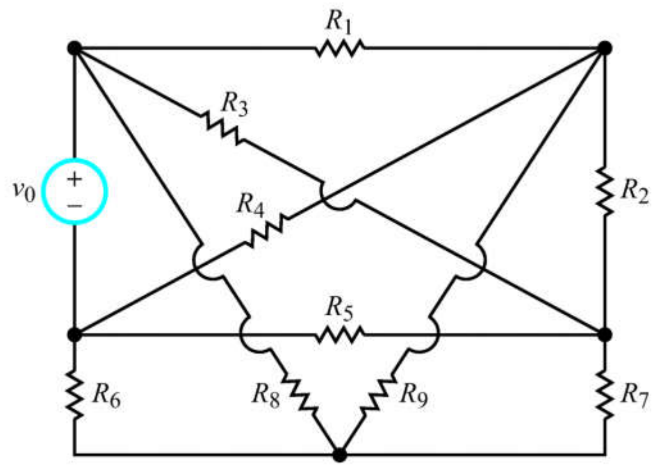
Mesh = Independent loop?

$$l_{ind} = b - (n - 1)$$

Planar vs Nonpalanar

- Mesh analysis is limited in one aspect: It can only apply to circuits that is planar.
  - A planar circuit can be drawn such that there are no crossing branches.





[Source: Berkeley]

Mesh Analysis Steps

- Mesh analysis follows these steps:
  1. Assign mesh currents $i_1, i_2, \ldots i_n$ to the n meshes
  2. Apply KVL to each of the n mesh currents.
  3. Solve the resulting n simultaneous equations to get the mesh currents.

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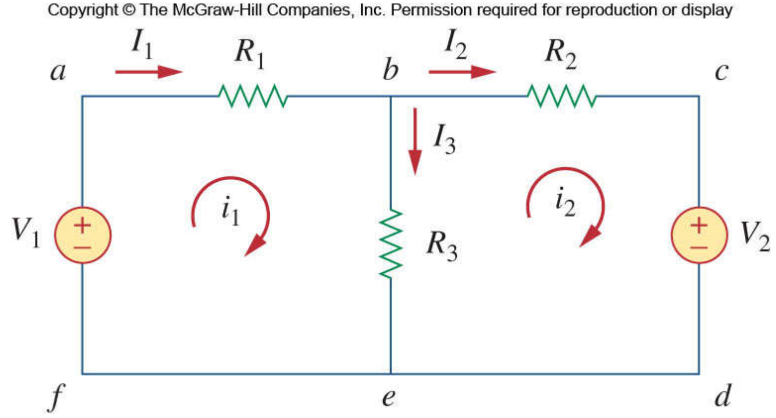

Mesh Analysis with Current Sources

- The presence of a current source makes the mesh analysis simpler in that it reduces the number of equations.
  - If the current source is located on only one mesh, the current for that mesh is defined by the source. For example:

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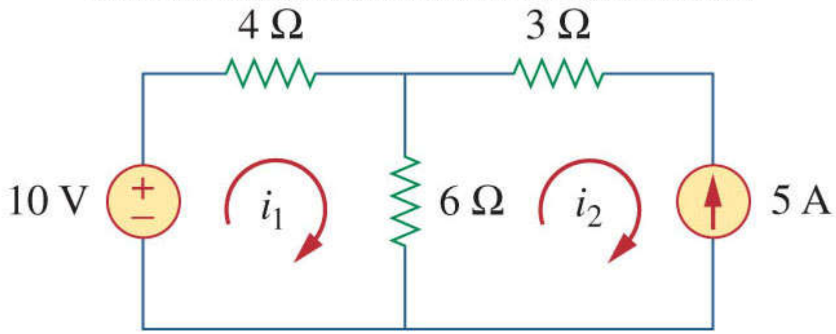
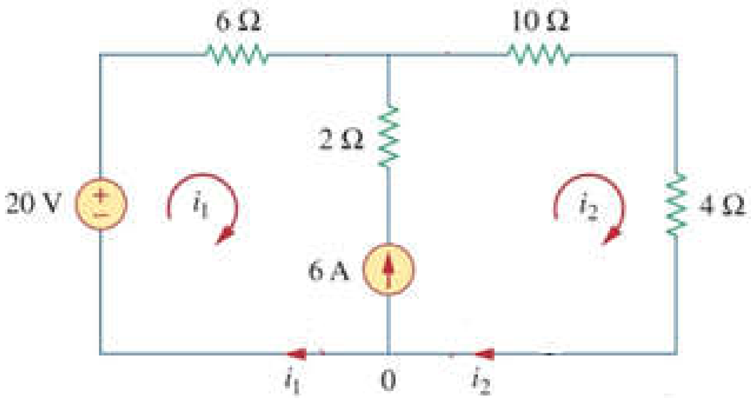
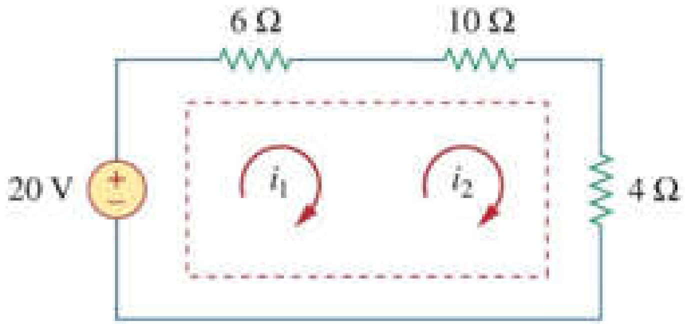


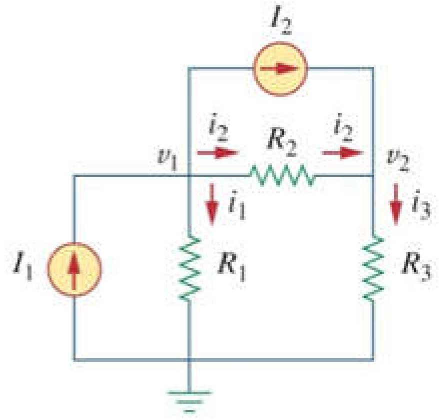
Supermesh





Summary

- Node Analysis
  - Node voltage is the unknown
  - Solve by KCL
  - Special case: Floating voltage source



- Mesh Analysis
  - Loop current is the unknown
  - Solve by KVL
  - Special case: Current source

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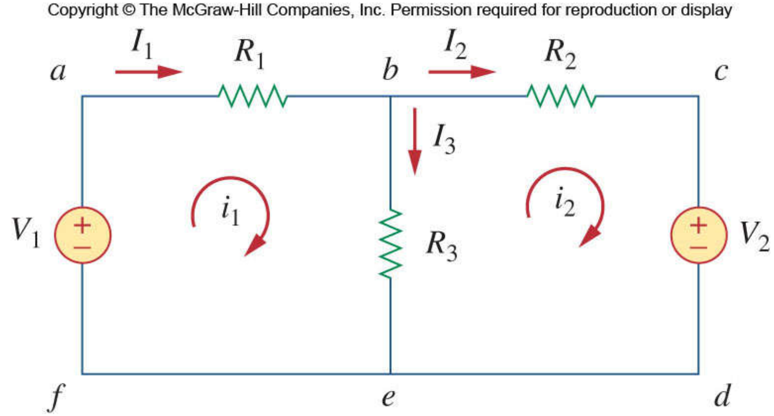
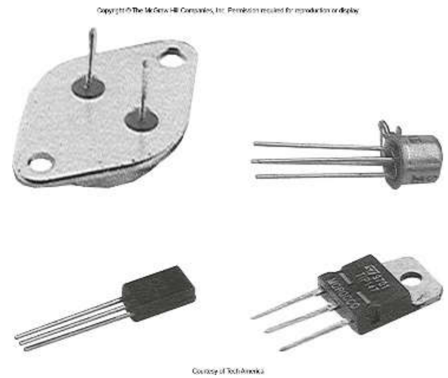
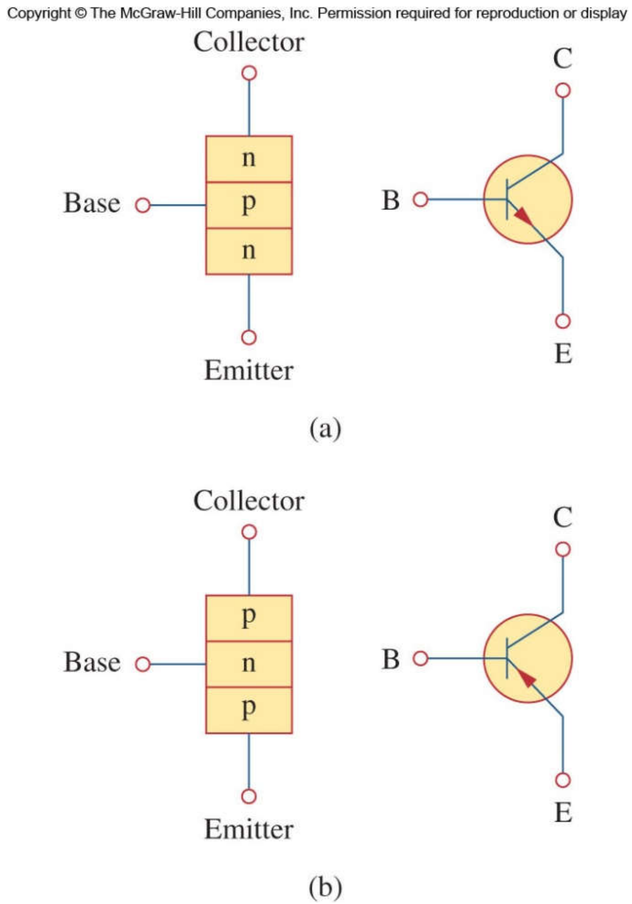


Application: DC Transistor Circuit

- In general, there are two types of transistors commonly used: Field Effect (FET) and Bipolar Junction (BJT). Here we will use the approaches learned in this lecture to analyze a BJT circuit.





- A BJT is a three terminal device, where
  - The input current into one terminal (the base) affects the current flowing out of a second terminal (the collector).
  - The third terminal (the emitter) is the common terminal for both currents.

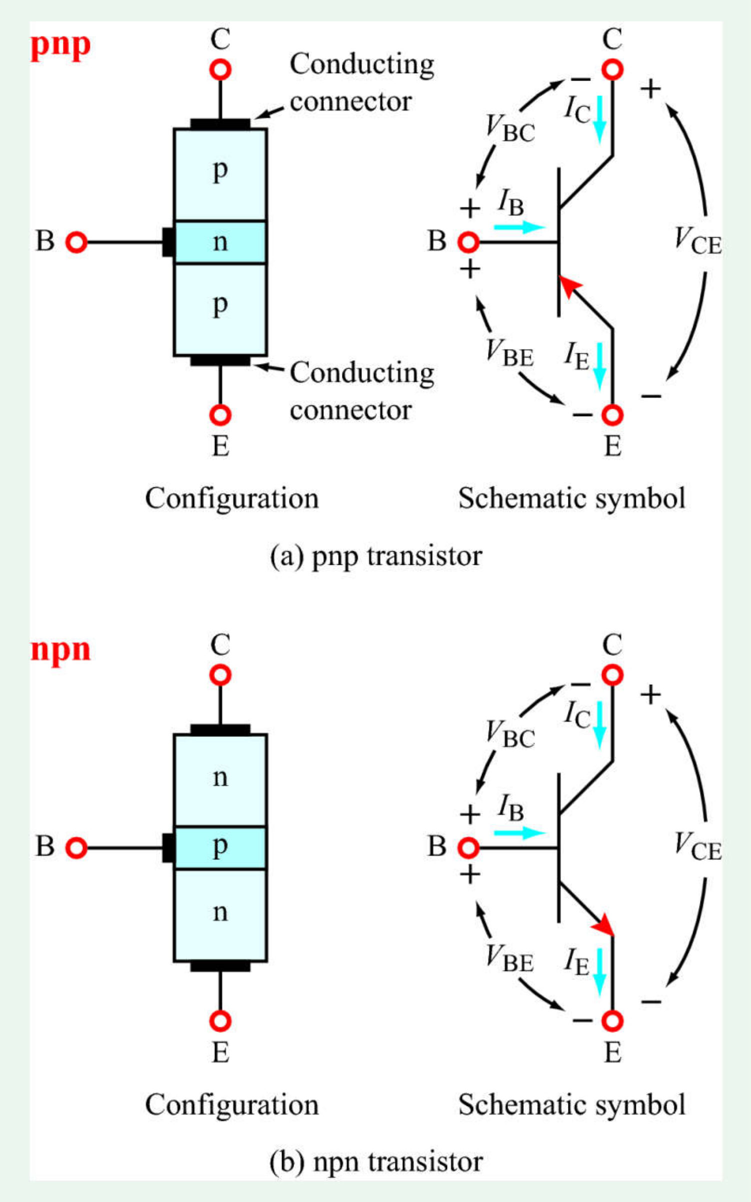
KCL and KVL for a BJT

- The currents from each terminal can be related to each other as follows:

- The base and collector current can be related to each other by the parameter β, which can range from 50-1000

$$I_C = \beta I_B$$

- Applying KVL to the BJT gives:



(a) pnp transistor

(b) npn transistor

[Source: Berkeley]

Analysis of a BJT Circuit

- A transistor has a few operating modes depending on the applied voltages/currents. In this problem, we will be interested in the operation in "active mode"
  - the mode used for amplifying signals.

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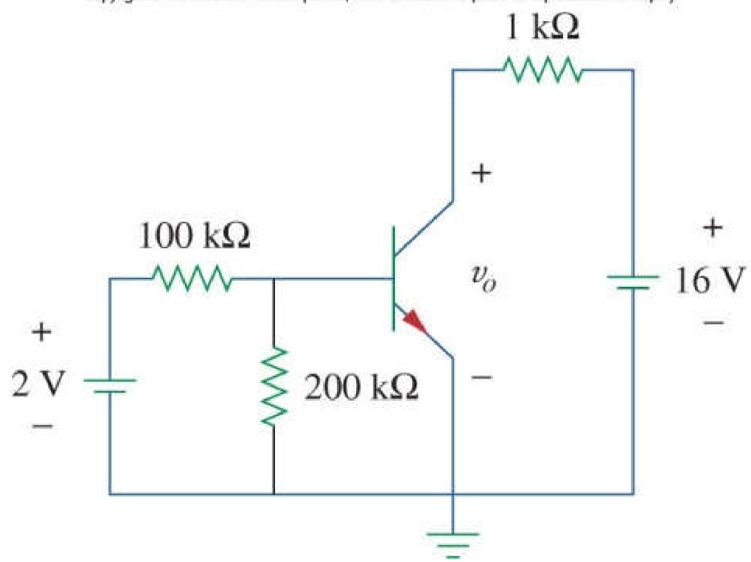

$\beta = 150$ $\quad v_{BE} = 0.7V$

Mesh Analysis？

$\beta = 150 \qquad v_{BE} = 0.7V$

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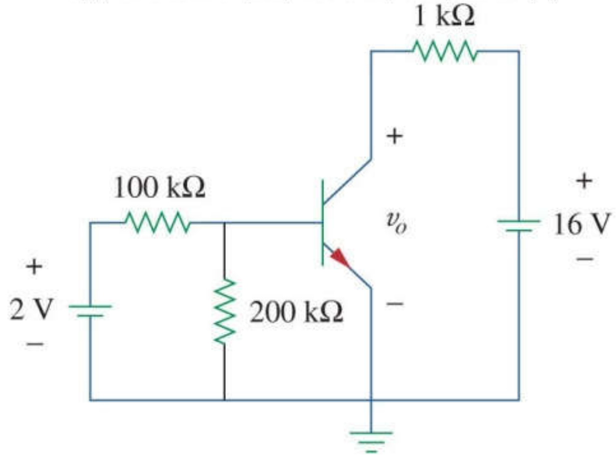
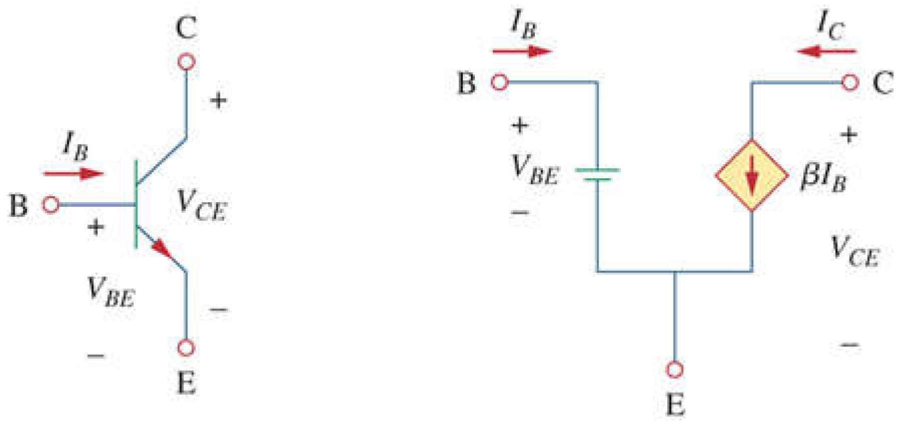


DC model of a BJT

- The figure below shows the equivalent DC model for a BJT in active mode



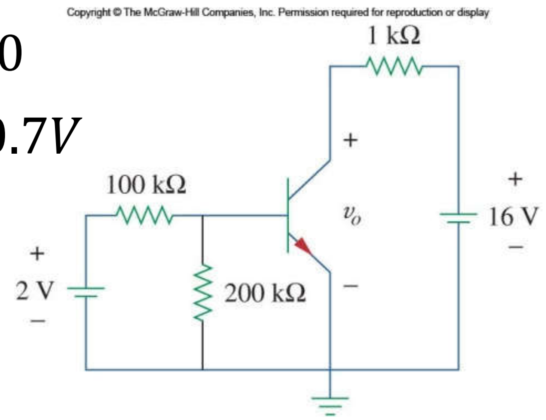
$$I_C = \beta I_B$$

Note that nodal analysis can be applied after using this model.

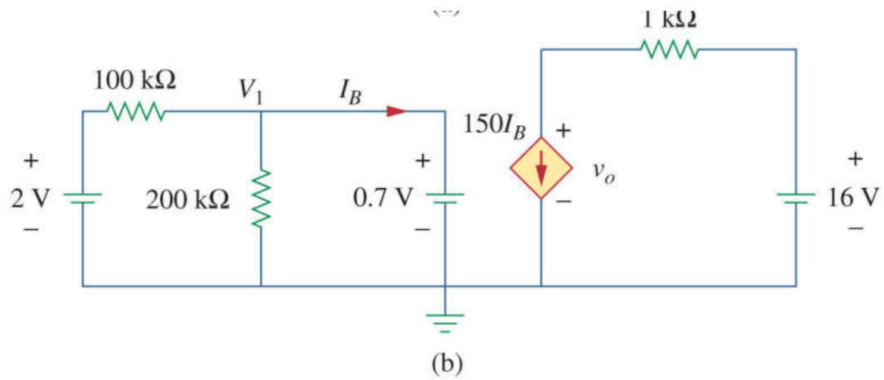
Setting up a BJT circuit

$\beta = 150$

$v_{BE} = 0.7V$



Original circuit



Circuit for nodal analysis