Lab 6 Laplace Transform

Objective

- Laplace transform and inverse Laplace transform.
- Poles and zeros of a system and the effect.
- Application in system analysis.

Content

Laplace Transform and Inverse Laplace Transform

Laplace transform is a technique for solving differential equations. Here differential equation in the time domain form is first converted into an algebraic equation in the frequency domain form. After solving the algebraic equation in the frequency domain, the result is converted to the time domain form to achieve the final solution of the differential equation. The operation includes Laplace transform and inverse Laplace transform.

The definition of the Laplace Transform is:

a) Two-sided Laplace transform

$$F_b(\mathrm{s}) = \mathcal{L}[\mathrm{f(t)}] = \int_{-\infty}^{+\infty} f(t)e^{-st}dt\,, \quad \mathrm{s} = \sigma + \mathrm{j}\omega$$

b) One-sided Laplace transform

$$F(\mathrm{s}) = \mathcal{L}[\mathrm{f(t)}] = \int_{0}^{+\infty} f(t)e^{-st}dt\,, \quad \mathrm{s} = \sigma + \mathrm{j}\omega$$

And the inverse Laplace Transform is:

a) Inverse two-sided Laplace transform

$$\mathrm{f(t)} = \mathcal{L}^{-1}[F_b(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_b(s)e^{st}ds$$

b) Inverse one-sided Laplace transform

$$\mathrm{f(t)} = \mathcal{L}^{-1}[F(s)] = \left[\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds\right]u(t)$$

In MATLAB symbolic toolbox, function **laplace** and **ilaplace** are used to do Laplace transform and inverse Laplace transform, as shown in Table 1.

Table 1 laplace and ilaplace

Function	Syntax	Description
laplace	laplace(f)	Returns the Laplace transform of f using the default independent variable t and the default transformation variable s.
	laplace(f, transVar)	Uses the specified transformation variable transVar instead of s.

	laplace(f, var, transVar)	Uses the specified independent variable var and transformation variable transVar instead of t and s respectively.
ilaplace	ilaplace(F)	Returns the inverse Laplace transform of F using the default independent variable s for the default transformation variable t.
	ilaplace(F, transVar)	Uses the specified transformation variable transVar instead of t.
	ilaplace(F, var, transVar)	Uses the specified independent variable var and transformation variable transVar instead of s and t respectively.

Example 1: Find the Laplace transform of $\mathrm{f(t)} = e^{-t}\sin(at)\,u(t)$.

```
syms t a
f = exp(-t)*sin(a*t)*heaviside(t);
L = laplace(f);
```

Example 2: Find the inverse Laplace transform of $\mathrm{F(s)} = \frac{s^2}{s^2+1}$.

```
syms s
F = s^2/(s^2+1);
ft = ilaplace(F);
```

It is also able to calculate the inverse Laplace transform by expanding F(s). Function **residue** helps to find out a partial fraction expansion. That is

$$\mathrm{F}(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^{M} b_j s^j}{\sum_{i=0}^{N} a_i s^i} = \frac{\sum_{j=0}^{M} b_j s^j}{\prod_{i=0}^{N}(s-p_i)} = \sum_{i=0}^{N} \frac{k_i}{s-p_i} + \sum_{j=0}^{M-N} c_j s^j$$

$$\sum_{j=0}^{M-N} c_j s^j = 0, \text{when } \ \mathrm{M} < \mathrm{N}$$

The format of **residue** is:

$$[\mathrm{k}, \mathrm{p}, \mathrm{c}] = \mathrm{residue}(\mathrm{num}, \mathrm{den})$$

Vectors num and den specify the coefficients of the numerator and denominator polynomials in descending powers of s. The residues are returned in the column vector k, the pole locations in column vector p, and the direct terms in row vector c.

Example: $\mathrm{F(s)} = \frac{s+2}{s^3+4s^2+3s}$, find out the inverse Laplace transform of F(s).

```
format rat;       %Use a fractional representation
syms s
num = [1,2]; den = [1,4,3,0];
[k,p,c] = residue(num,den)
k =
     -1/6
     -1/2
      2/3
p =
     -3
```

```
     -1
      0
c =
    []
F = 2/(3*s)-1/2*(1/(1+s))-1/6*(1/(s+3));
ilaplace(F)
```

So $F(s) = \frac{2/3}{s} + \frac{-1/2}{s+1} + \frac{-1/6}{s+3}$, and $f(t) = (\frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t})u(t)$

If $p(j) = \cdots = p(j+m-1)$ is a pole of multiplicity m, then the expansion includes terms of the form:

$$\frac{k(j)}{s - p(j)} + \frac{k(j+1)}{(s - p(j))^2} + \cdots \frac{k(j+m-1)}{(s - p(j))^m}$$

Example: $F(s) = \frac{2s+1}{s^3+5s^2+8s+4}$, find out the inverse Laplace transform of F(s).

```
syms s
num = [2 1]; den = [1 5 8 4];
[k,p,c] = residue(num,den);
k =
    1.0000
    3.0000
   -1.0000
p =
   -2.0000
   -2.0000
   -1.0000
c =
    []
F = 1/(s+2)+3/(s+2)^2-1/(s+1);
ilaplace(F)
```

So $F(s) = \frac{1}{s+2} + \frac{3}{(s+2)^2} + \frac{-1}{s+1}$, and $f(t) = (e^{-2t} + 3te^{-2t} - e^{-t})u(t)$

$[num, den] = residue(k, p, c)$, with 3 input arguments and 2output arguments, converts the partial fraction expansion back to the polynomials with coefficient in num and den.

Example: $F(s) = \frac{5}{s-3} + \frac{6}{s+4} - \frac{7}{s+1/5}$

```
k = [5 6 -7];
p = [3 -4 -1/5];
[num, den] = residue(k,p,[])
num =
    4.0000   -2.8000   84.4000
den =
    1.0000    1.2000  -11.8000   -2.4000
```

This tells us that $F(s) = \frac{4s^2-2.8s+84.4}{s^3+1.2s^2-11.8s-2.4}$.

Poles and Zeros

The Symbolic toolbox of MATLAB provides function **pole** and **zero** to find out the poles and zeroes of linear systems. The format of the two functions is **pole**(sys) and **zero**(sys).
Function **pzmap** is used to draw the pole-zero distribution plot. The format is **pzmap**(sys).
Sys is the system transfer function which can be generated by function **tf**.

The distribution of zeroes and poles is of great importance. By analyzing zeroes and poles in s domain, it is possible to get the following characteristics:

1. The characteristics of the system impulse response (h(t)) in the time domain.
2. The stability of the system.
3. The frequency characteristics.

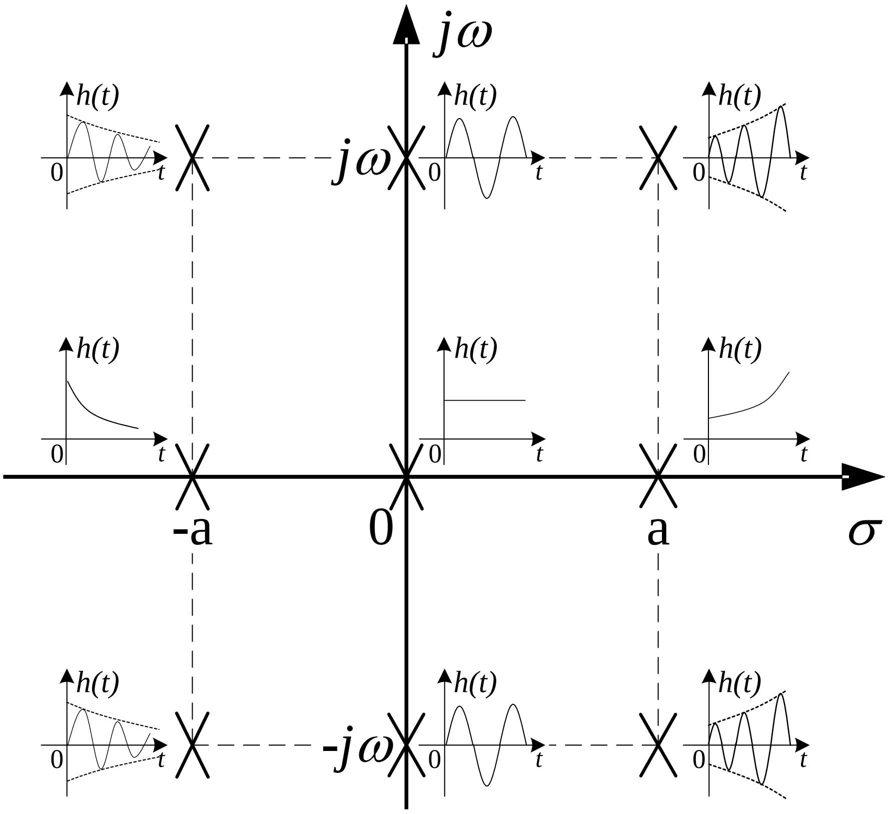
The relationship between poles and h(t) is displayed in Table 1.

Table 1 Relationship between Poles and h(t)

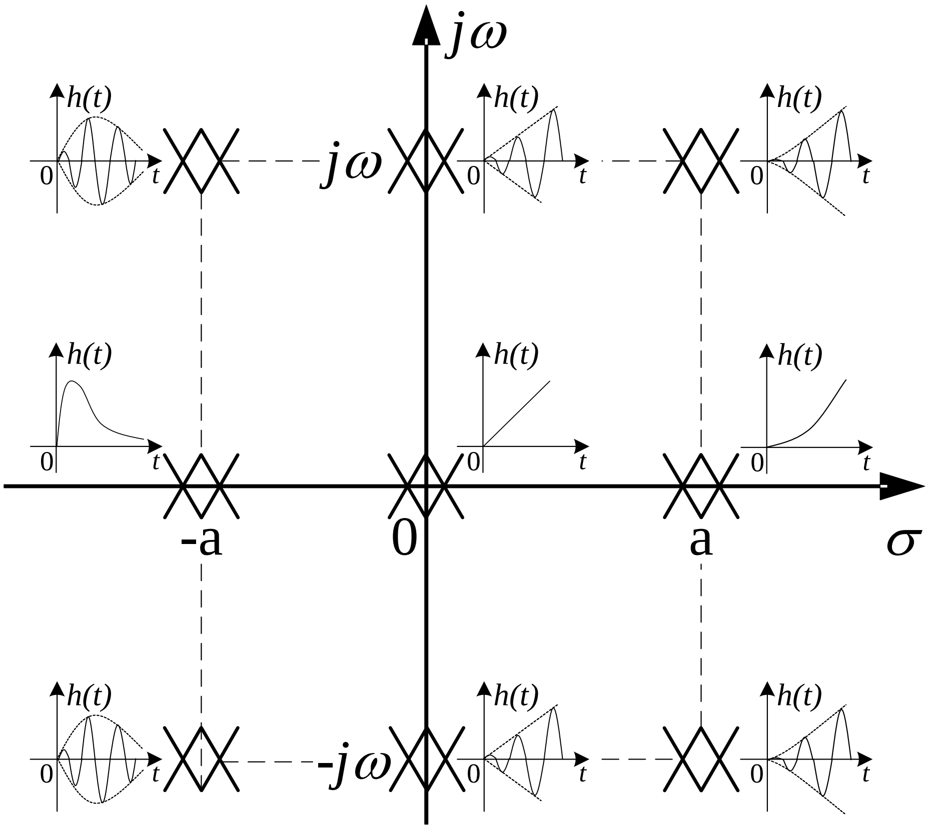
H(s)	Pole Position	h(t)
H(s)=1/s	p1=0	h(t)=u(t)
H(s)=1/(s+a)	p1=-a	
	a>0,left real axis	$h(t)=e^{-at}u(t)$,Exponential decay
	a<0,right real axis	$h(t)=e^{-at}u(t)$,-a>0,Exponential increase
$H(s)=w/(s^2+w^2)$	p1=±jw	h(t)=sinwt*u(t), continuous oscillation
$H(s)=w/((s+a)^2+w^2)$	p1=-a±jw	
	a>0,left half-plane	$h(t)=e^{-at}$sinwtu(t),Oscillation attenuation
	a<0,right half-plane	$h(t)=e^{-at}$sinwtu(t),-a>0,Oscillation increase
$H(s)=1/s^2$	p1=p2=0	h(t)=tu(t) t→∞,h(t)→∞
$H(s)=1/(s+a)^2$	p1=p2=-a	
	a>0,left real axis	$h(t)=te^{-at}u(t)$,a>0, t→∞,h(t)→0
	a<0,right real axis	$h(t)=te^{-at}u(t)$,a>0, t→∞,h(t)→∞
$H(s)=2ws/(s^2+w^2)^2$	p1=p2=±jw	h(t)=tsinwt*u(t),Oscillation increase
$H(s)=2w(s+a)/((s+a)^2+w^2)^2$	p1=p2=-a±jw	
	a>0,left half-plane	$h(t)=te^{-at}$sinwt*u(t) ,a>0, t→∞,h(t)→0
	a<0,right half-plane	$h(t)=te^{-at}$sinwt*u(t) , a>0, t→∞,h(t)→∞

The details of the table are shown in Figure 1. (a) is for first-order poles and (b) is for second-order poles.
If a continuous system is stable, all the poles of H(s) should be on the left half-plane of s plane.
The distribution of zeroes affects the amplitude and phase of h(t), which we will not discuss here in

detail.



(a) First-order Poles



(b) Second-order Poles

Figure 1 Relationship between H(s) Poles and h(t) Waveform Characteristics

Surface Plot for Laplace Transform

With Laplace transform, we have a complex-valued function of a complex variable (s = σ + jω). In order to examine the magnitude and phase, or real and imaginary parts of this function, it is necessary to import 3-dimensional surface plots of each component.

The Laplace transform of signal has the format as:

$$F(s) = |F(s)|e^{j\varphi(s)} = |F(s)|\cos\varphi(s) + j|F(s)|\sin\varphi(s)$$

To get the surface plot for Laplace transform, follow the steps below:

- Define vectors x (σ) and y (ω) as the real and imaginary axes of the complex s plane.
- Get 2-D grid coordinates based on the coordinates contained in vectors x and y with function **meshgrid**.
- Use function **abs**, **angle**, **real** or **imag** to calculate the magnitude, phase, real part or imaginary part of F(s) in complex s plane.
- Use function **mesh** or **surf** to get the surface plot of F(s).

The functions used here are listed in Table 2.

Table 2 Functions Used to Draw Surface Plot

Function	Format	Description
meshgrid	[X,Y]=meshgrid(x,y)	Returns 2-D grid coordinates based on the coordinates contained in vectors x and y. X is a matrix where each row is a copy of x, and Y is a matrix where each column is a copy of y. The grid represented by the coordinates X and Y has length(y) rows and length(x) columns.
mesh	mesh(X,Y,Z)	Draws a wireframe mesh with color determined by Z, so color is proportional to surface height. If X and Y are vectors, length(X) = n and length(Y) = m, where [m,n] = size(Z). In this case, (X(j), Y(i), Z(i,j)) are the intersections of the wireframe gridlines; X and Y correspond to the columns and rows of Z, respectively. If X and Y are matrices, (X(i,j), Y(i,j), Z(i,j)) are the intersections of the wireframe grid lines. The values in X, Y, or Z can be numeric, datetime, duration, or categorical values.
surf	surf(X,Y,Z)	Create a three-dimensional surface plot. The function plots the values in matrix Z as heights above a grid in the x-y plane defined by X and Y. The function also uses Z for the color data, so color is proportional to height.
colormap	colormap(map)	Sets the colormap for the current figure to the colormap specified by **map**.
rotate3d	rotate3d on	Enable mouse-base rotation on all axes within the current figure.

The parameter **map** of function **colormap** decides the color scale of the plot as shown in Figure 2.

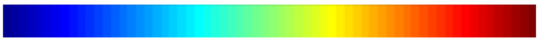
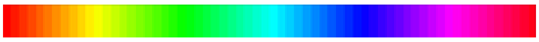
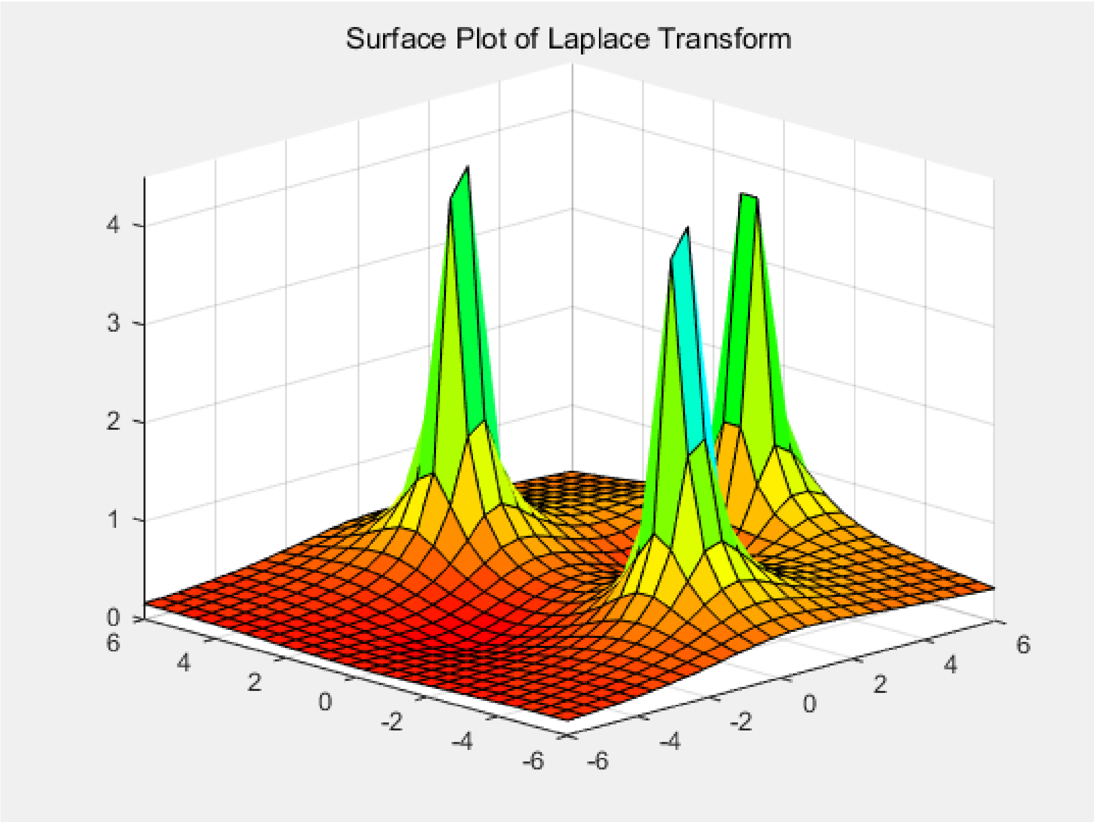
Colormap Name	Color Scale
parula	
jet	
hsv	
hot	
cool	
spring	
summer	
autumn	
winter	
gray	
bone	
copper	
pink	
lines	
colorcube	
prism	
flag	
white	

Figure 2 the Value of map

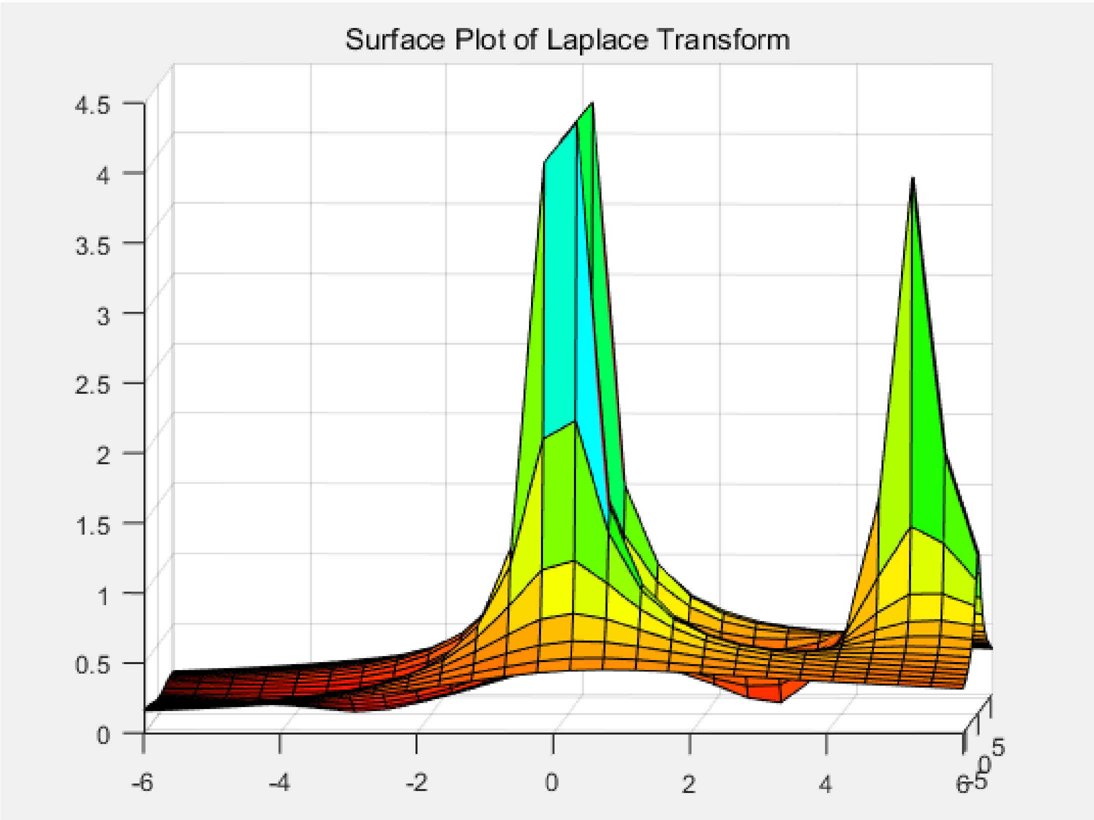
Eg: $F(s) = \frac{2(s-3)(s+3)}{(s-5)(s^2+10)}$, do the surface plot of F(s).

```
x = -6:0.48:6;y=x;
[x,y] = meshgrid(x,y);
s = x+1j*y;
Fs = (2*(s-3).*(s+3))./((s-5).*(s.*s+10));
Fsabs = abs(Fs);
surf(x,y,Fsabs);
axis([-6,6,-6,6,0,4.5]);
title('Surface Plot of Laplace Transform');
colormap(hsv);
rotate3d on;
```

The surface plot is show in Figure 3.



(a)



(b)

Figure 3 Surface Plot of Fs

From Figure 2, it is easy to identify that the three peaks (where $s = \pm j3.1623, s = 5$) on the surface plot correspond to the three poles of Fs, and the two valleys (where $s = \pm 3$ correspond to the zeros. In the later lab, we will discuss the relationship of Laplace transform and Fourier transform by their image.

Relationship between Laplace Transform and Fourier Transform

The Laplace transform of a function is just like the Fourier transform of the same function, except that the term in the exponential of a Laplace transform is a complex number instead of just an imaginary number. The exponential factor has the effect of forcing the signal to converge. That is why Laplace transform can be applied to a broader class of signals than Fourier transform, including exponentially growing signals. In a Fourier transform, both the signal in the time domain and its spectrum in the frequency domain are a one-dimensional, complex function. However, the Laplace transform of the 1D signal is a complex function defined over a two-dimensional complex plane (s plane) spanned by two variables, one for the horizontal real axis and one for the vertical imaginary axis. If this 2D function is evaluated along the imaginary axis, Laplace transform simply becomes the Fourier transform.

System Analysis with Laplace Transform

Laplace transform is often used to analyze the system. Here we will discuss two application of Laplace transform used in system analysis. They are the solution of difference equations and the analysis of circuits.

Solving Difference Equations with Laplace Transform

Take the example we talked about in Lab2:
$y''(t) + 3y'(t) + 2y(t) = f(t)$, $\mathrm{f(t)} = e^{-t}u(t), y(0_-) = 1, y'(0_-) = 2$, find out y(t)
To find out the y(t), we can

a) Do one-sided Laplace transform to both side of the differential equation

$$s^2Y(s) - sy(0_-) - y'(0_-) + 3 \cdot [\mathrm{sY(s)} - \mathrm{y}(0_-)] + 2\mathrm{Y(s)} = \mathrm{F(s)}$$

b) Solve the above algebraic equation

$$Y(s) = \underbrace{\frac{s+5}{s^2+3s+2}}_{Y_{zi}} + \underbrace{\overbrace{\frac{1}{s^2+3s+2}}^{H(s)} F(s)}_{Y_{zs}}$$

$$F(s) = \frac{1}{s+1}$$

Obviously the first part is only related to the characteristics of the system and the initial stat. This part is the zero-input response.
The second part is related to system characteristics and the inputs, and has no relation with the initial

state. This part is the zero-response response.

c) Do the inverse Laplace transform.

$$Y(s)$$
$$=\frac{s+5}{(s+2)(s+1)}+\frac{1}{(s+2)(s+1)}\bullet\frac{1}{(s+1)}$$
$$=[\frac{-3}{s+2}+\frac{4}{s+3}]+[\frac{1}{(s+1)^2}+\frac{-1}{s+1}+\frac{1}{s+2}]$$

$$\therefore y(t)=[-3e^{-2t}+4e^{-3t}]+[te^{-t}-e^{-t}+e^{-2t}]$$

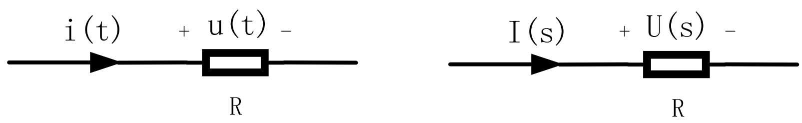
For the system with initial state of zero or a system only focus on zero-state response, the transmission model can be further simplified.

Analysis of Circuits with Laplace Transform

Based on the KCL and KVL theorems, we can equate devices such as resistors, capacitors and inductors with impedance, capacitive reactance, and inductive reactance in the complex frequency domain, thereby simplifying the analysis of the circuit.

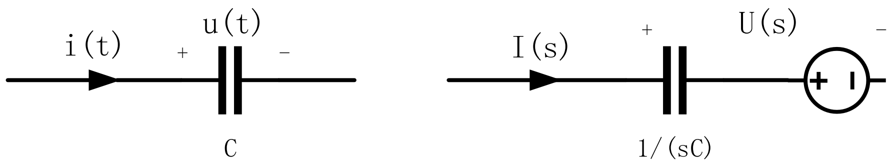
For resistance:

$$\mathrm{u}(t)=R\cdot i(t)$$
$$\mathcal{L}[\mathrm{u}(t)]=\mathcal{L}[R\cdot i(t)]$$
$$\frac{\mathrm{U}(s)}{I(s)}=R$$



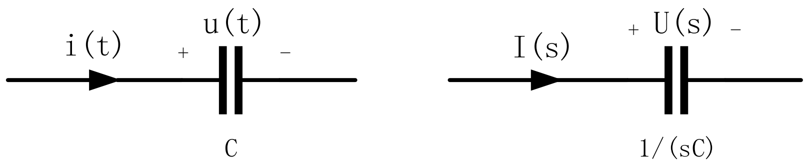
For capacitor:

$$i(t)=C\frac{du(t)}{dt}$$
$$\mathcal{L}[\mathrm{i(t)}]=\mathcal{L}[C\frac{du(t)}{dt}]$$
$$\mathrm{I}(s)=C[sU(s)-\mathrm{u}(0\_)]$$



If we are only interested in the steady-state response, we can ignore $\mathrm{u}(0\_)$, since it only effect transient response. For simplicity, we will take $\mathrm{u}(0\_)=0$. Thus we have:

$$\frac{U(s)}{I(s)}=\frac{1}{sC}$$

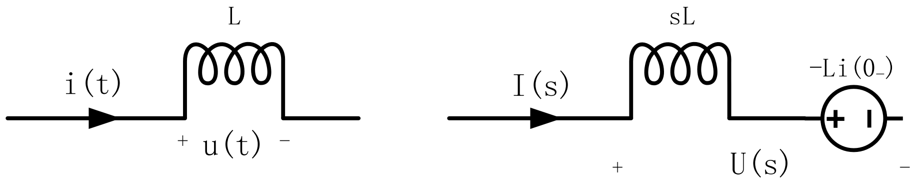


For inductor:

$$u(t) = L\frac{di(t)}{dt}$$

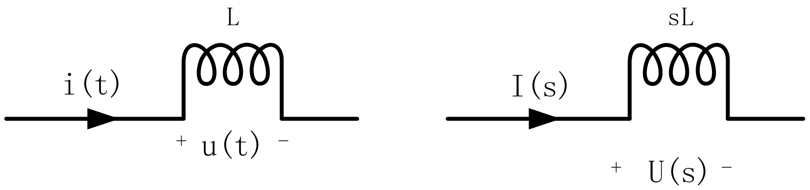
$$\mathcal{L}[u(t)] = \mathcal{L}[L\frac{di(t)}{dt}]$$

$$U(s) = L[sI(s) - i(0_-)]$$



Similarly, we take $i(0_-) = 0$ in this chapter also. Thus we have:

$$\frac{U(s)}{I(s)} = sL$$



When we equivalent the time domain circuit to the complex frequency domain circuit, we can use function **freqs** which we mentioned in Lab4 to analyze the frequency response of the circuit.