Solutions to Homework

1 (a) The desired convolution is

$$
\begin{aligned}
y[n] =& x[n] * h[n] \\
=& \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
=& \beta^n \sum_{k=0}^{n} (\alpha/\beta)^k \text{ for } n \geq 0 \\
=& \{\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\} u[n] \text{ for } \alpha \neq \beta
\end{aligned}
$$

(b) From (a),

$$y[n] = \alpha^n \left[\sum_{k=0}^{n} 1 \right] u[n] = (n+1)\alpha^n u[n].$$

(c) For $n \leq 6$,

$$y[n] = 4^n \left\{ \sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{3} (-\frac{1}{8})^k \right\}$$

For $n > 6$,

$$y[n] = 4^n \left\{ \sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{n-3} (-\frac{1}{8})^k \right\}$$

Therefore,

$$y[n] = \begin{cases} (8/9)(-1/8)^4 4^n, & n \leq 6 \\ (8^3/9)(-1/2)^n, & n > 6 \end{cases}$$

(d) We know that

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The signals $x[n]$ and $y[n]$ are as shown in Figure S2.4. From this figure, we see that the above summation reduces to

$$y[n] = \sum_{k=3}^{8} x[k]h[n-k]$$

This gives

$$y[n] = \begin{cases} n-6, & 7 \le n \le 11 \\ 6, & 12 \le n \le 18 \\ 24-n, & 19 \le n \le 23 \\ 0, & \text{otherwise} \end{cases}$$

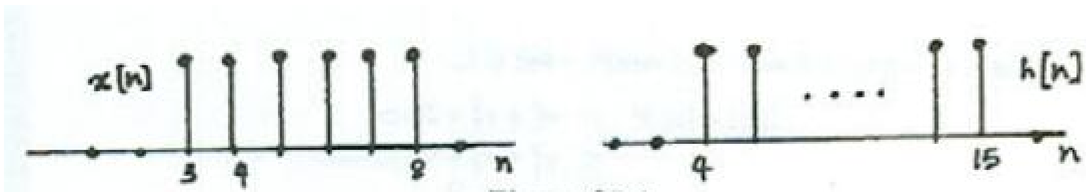


Figure S2.4

2 The desired convolution is

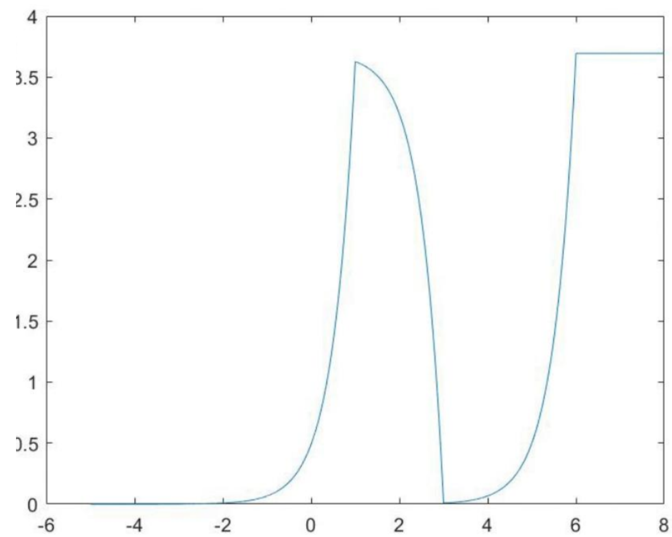
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^2 h(t-\tau)d\tau + \int_5^{\infty} h(t-\tau)d\tau. \end{aligned}$$

This may be written as

$$y(t) = \begin{cases} \int_0^2 e^{2(t-\tau)}d\tau + \int_5^{\infty} e^{2(t-\tau)}d\tau, & t \le 1 \\ \int_{t-1}^2 e^{2(t-\tau)}d\tau + \int_5^{\infty} e^{2(t-\tau)}d\tau, & 1 \le t \le 3 \\ \int_5^{\infty} e^{2(t-\tau)}d\tau, & 3 \le t \le 6 \\ \int_{t-1}^{\infty} e^{2(t-\tau)}d\tau, & 6 < t \end{cases}$$

Therefore,

$$y(t) = \begin{cases} (1/2)[e^{2t} - e^{2(t-2)} + e^{2(t-5)}], & t \le 1 \\ (1/2)[e^2 + e^{2(t-5)} - e^{2(t-2)}], & 1 \le t \le 3 \\ (1/2)[e^{2(t-5)}], & 3 \le t \le 6 \\ (1/2)e^2, & 6 < t \end{cases}$$



3 (a) We are given that $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

Since

$$h[n] = h_1[n] * [h_2[n] * h_2[n]]$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$$

Therefore,

$$\begin{aligned}
h[0] = h_1[0] \quad &\Rightarrow \quad h_1[0] = 1 \\
h[1] = h_1[1] + 2h_1[0] \quad &\Rightarrow \quad h_1[1] = 3 \\
h[2] = h_1[2] + 2h_1[1] + h_1[0] \quad &\Rightarrow \quad h_1[2] = 3 \\
h[3] = h_1[3] + 2h_1[2] + h_1[1] \quad &\Rightarrow \quad h_1[3] = 2 \\
h[4] = h_1[4] + 2h_1[3] + h_1[2] \quad &\Rightarrow \quad h_1[4] = 1 \\
h[5] = h_1[5] + 2h_1[4] + h_1[3] \quad &\Rightarrow \quad h_1[5] = 0
\end{aligned}$$

$h_1[n] = 0$ for $n < 0$ and $n > 5$.

(b) In this case,

$$y[n] = x[n] * h[n] = h[n] - h[n-1]$$

$$h[n] = \begin{cases} 1, & n = 0, 6 \\ 5, & n = 1 \\ 10, & n = 2 \\ 11, & n = 3 \\ 8, & n = 4 \\ 4, & n = 5 \\ 0, & Otherwise \end{cases}$$

$$y[n] = \begin{cases} 1, & n = 0, 3 \\ 4, & n = 1 \\ 5, & n = 2 \\ -3, & n = 4, 6 \\ -4, & n = 5 \\ -1, & n = 7 \\ 0, & Otherwise \end{cases}$$

4 (a) True. If $h(t)$ periodic and nonzero, then

$$\int_{-\infty}^{\infty} |h(t)| dt = \infty$$

Therefore, $h(t)$ is unstable.

(b) False. For example, inverse of $h[n] = \delta[n-k]$ is $g[n] = \delta[n+k]$ which is noncausal.

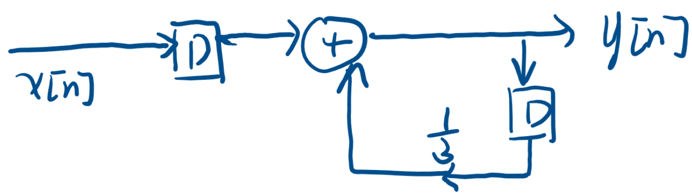
(c) True. Assuming that $h[n]$ is bounded and nonzero in the range $n_1 \leq n \leq n_2$,

$$\sum_{k=n_1}^{n_2} |h[k]| < \infty$$

This implies that the system is stable.

(d) False. For example, $h(t) = e^t u(t)$ is causal but not stable.

5 (a) $y[n] = \frac{1}{3}y[n-1] + x[n-1]$
$x_1[n] = K\delta[n]$
$y_1[0] = 0$
$y_1[1] = x[0] + \frac{1}{3}y_1[0] \Rightarrow y_1[1] = K$
$y_1[2] = x_1[1] + \frac{1}{3}y[1] = \frac{1}{3}K$
$y_1[3] = x_1[2] + \frac{1}{3}y[2] = (\frac{1}{3})^2 K$
...
$y_1[n] = (\frac{1}{3})^{n-1}K, n \geq 1$
$y_1[n] = 0, n \leq 0$



(b) First assume that $y_p(t)$ is of the form Ke^{2t} for $t > 0$. Then we get for $t > 0$

$$2Ke^{2t} + 2Ke^{2t} = e^{2t} \;\Rightarrow\; K = \frac{1}{4}$$

We now know that $y_p(t) = \frac{1}{4}e^{2t}$ for $t > 0$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}$$

Therefore,

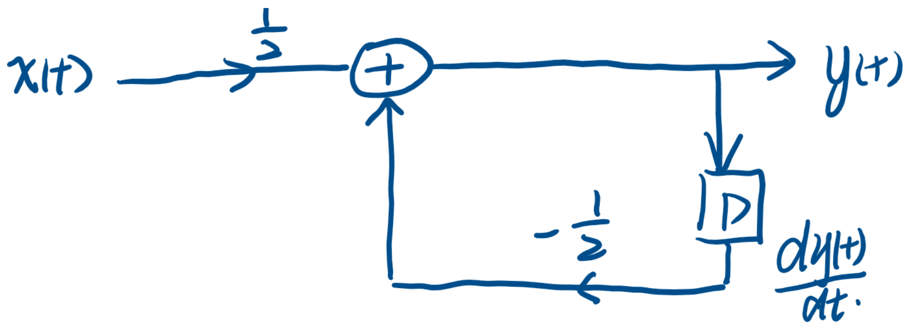
$$y_2(t) = Ae^{-2t} + \frac{1}{4}e^{2t}, \quad \text{for } t > 0$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq 0$. Therefore,

$$y_2(0) = 0 = A + \frac{1}{4} \;\Rightarrow\; A = -\frac{1}{4}$$

Then,

$$y_2(t) = \left[\frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}\right]u(t)$$



or