# Lecture 7
# - Phasor

A beginning of AC circuits

AC usually refer to Sinusoidal signal

# Outline

- Sinusoidal signals
- Complex numbers
- Phasor

# Why Sinusoids?

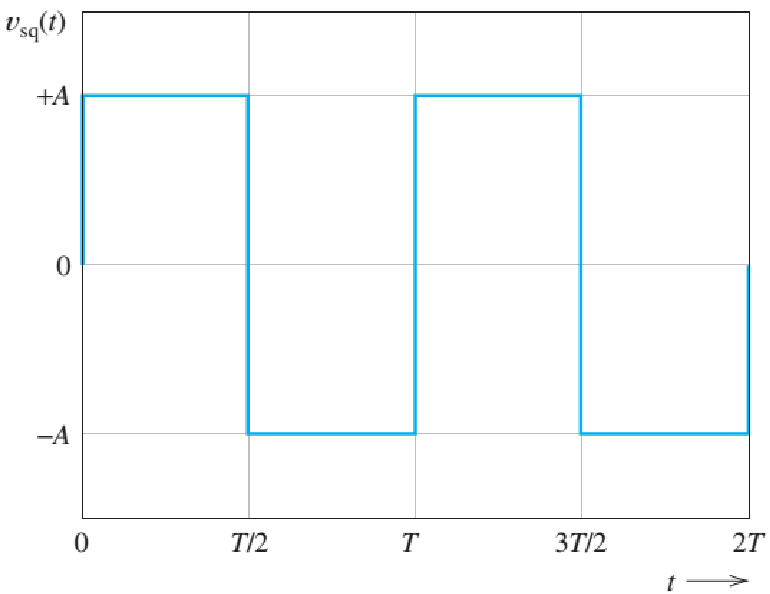
- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface

- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.

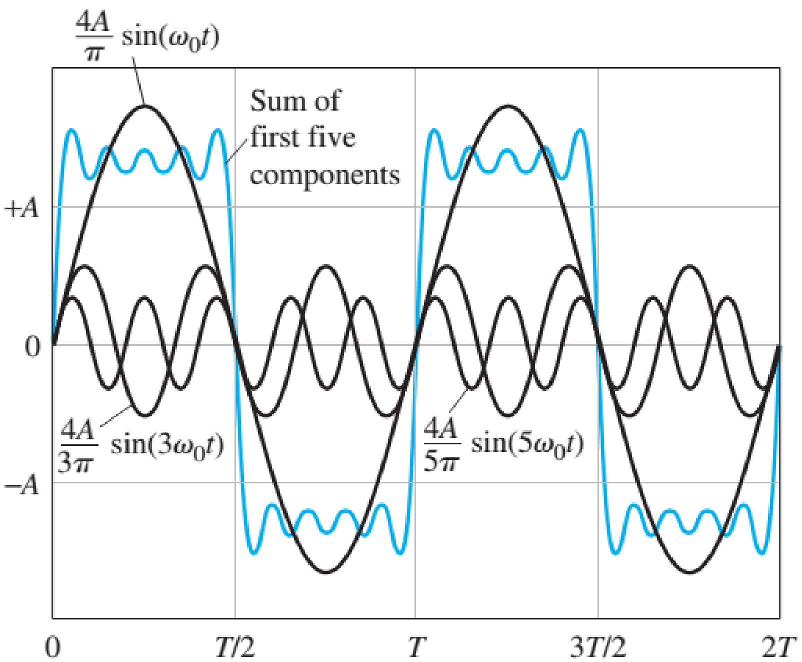
- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.

- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.

# Representing a Square Wave as a Sum of Sinusoids




(a) Periodic square wave

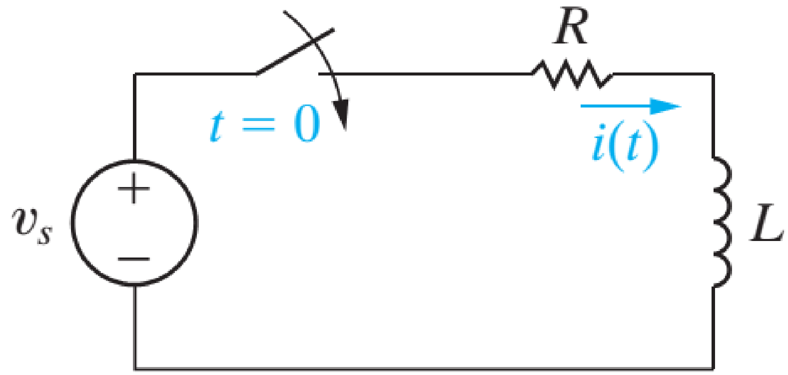



(b) Several of the sinusoidal components and the sum of the first five components

$$v_{sq}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \cdots$$

$$\omega_0 = 2\pi/T$$

# The Sinusoidal Response




$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.

Find $i(t)$, $t \geq 0$.

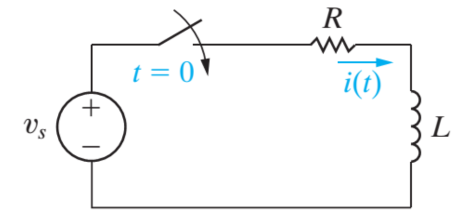
$$\boxed{L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)}$$

Ordinary differential equation

$$\frac{dx}{dt} + p(t)x = q(t) \Longleftrightarrow x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right) \qquad u(t) = e^{\int p(t)dt}$$

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

# Sinusoidal Steady-State Response




$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

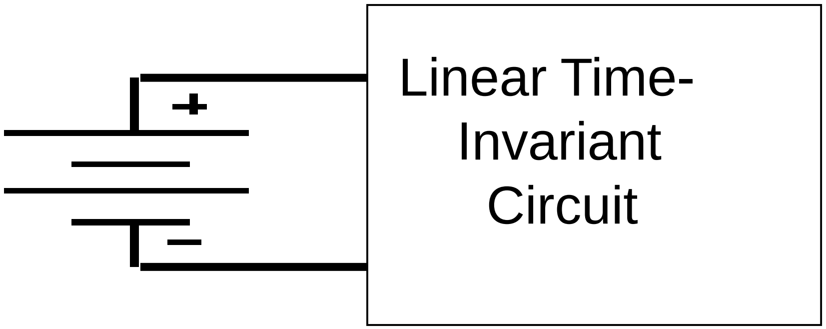
Transient response

Steady-state response

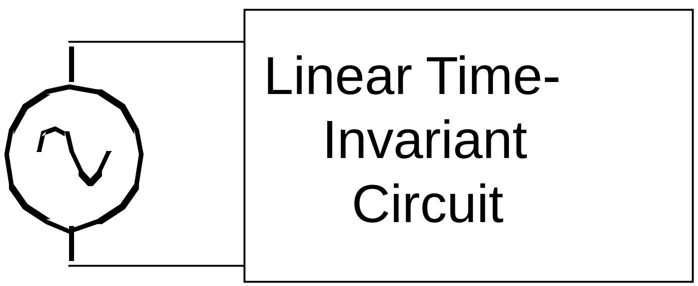
- Steady-state solution is sinusoidal
- Response frequency = source frequency
- Magnitude/phase of response differs from that of source

Sinusoidal sources create too much algebra!!!

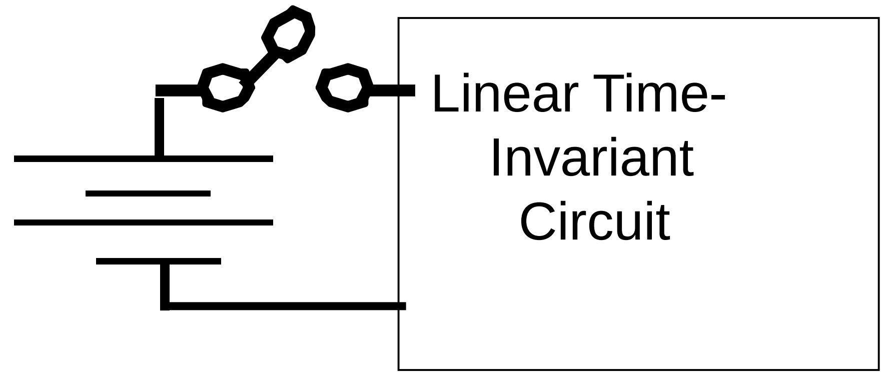
# Types of Circuit Excitation



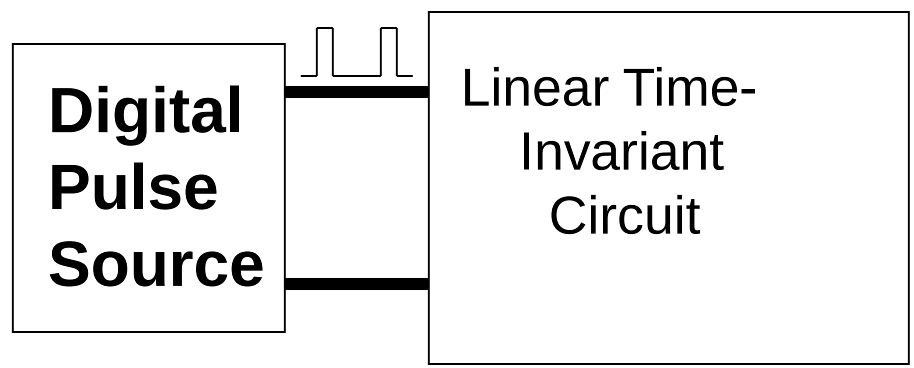

**Steady-State Excitation (DC Steady-State)**




**Sinusoidal (Single-Frequency) Excitation →AC Steady-State**




**OR**




**Transient Excitation**

[Source: Berkeley]

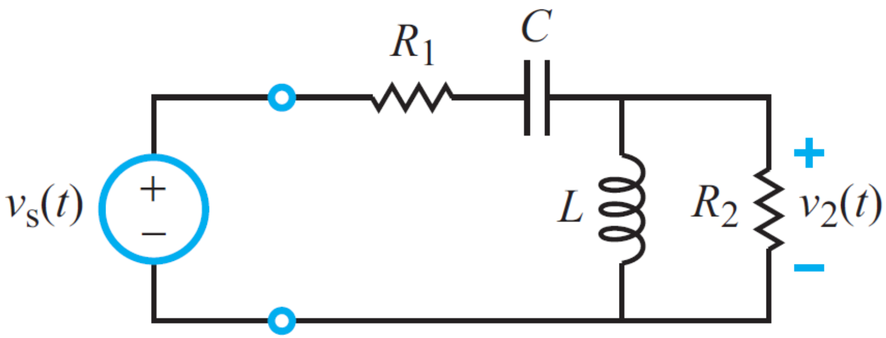
# AC Analysis of Linear Circuits

Objective: To determine the steady state response of a linear circuit to ac signals.

$$v_s(t) = V_0 \cos(\omega t + \phi)$$

***angular frequency*** $\omega$

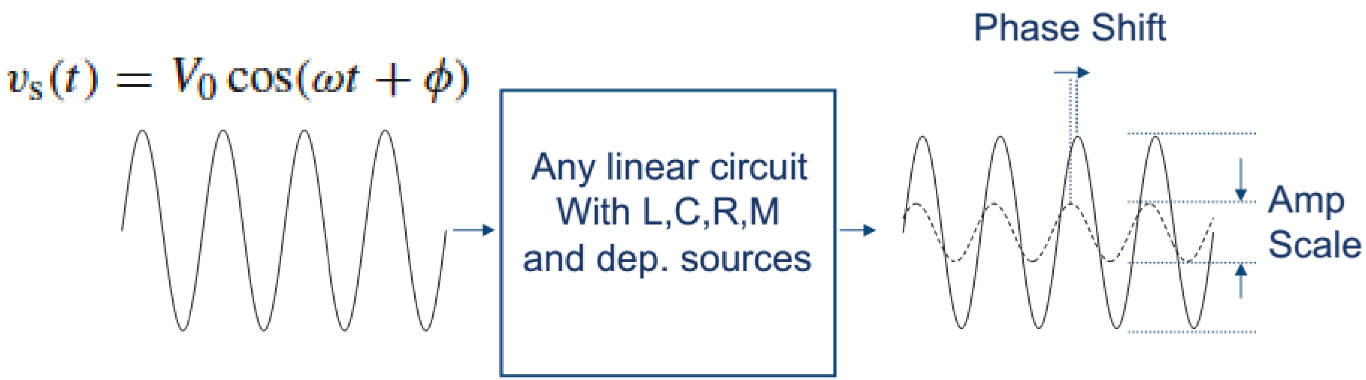
$\phi$ is called its ***phase angle***




- Sinusoidal input is common in electronic circuits.
- Any time-varying periodic signal can be represented by a series of sinusoids (Fourier Series).
- Time-domain solution method can be cumbersome.

[Source: Berkeley]

# The Magic of Sinusoids




- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it's output is a sinusoid at the *same* frequency.
  - Only the magnitude and phase of the output differ from the input.
- In order to find a steady-state voltage or current, all we need to know is its <u>magnitude</u> and its <u>phase</u> relative to the source.

# Sinusoidal Signal (Current or Voltage)

$$v(t) = V_m \cos(\omega t + \theta)$$

phase

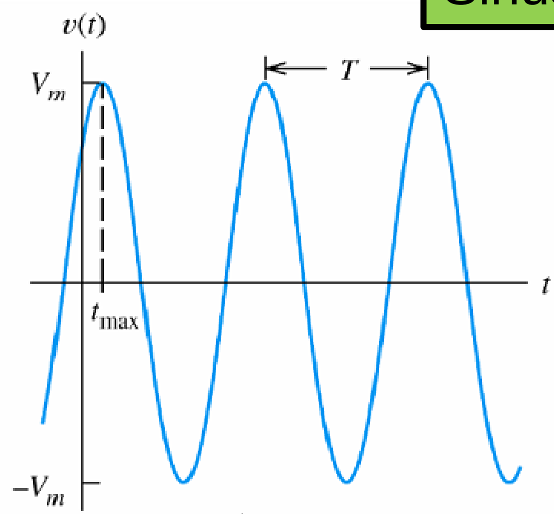
Sinusoidal can also be sine

$V_m$ is the **peak value**

$\omega$ is the **angular frequency** in radians per second

$\theta$ is the **phase angle**

$T$ is the **period**




$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

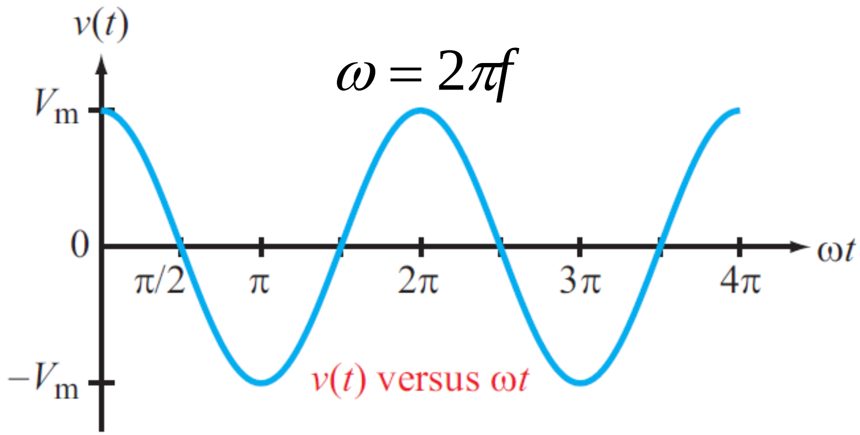
$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

AC usually refer to Sinusoidal current

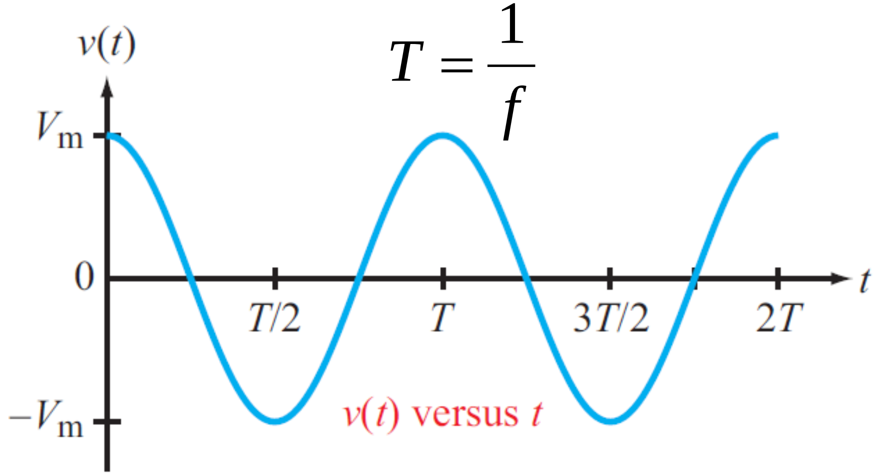
# Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\omega = 2\pi f$$




$$T = \frac{1}{f}$$




## Useful relations

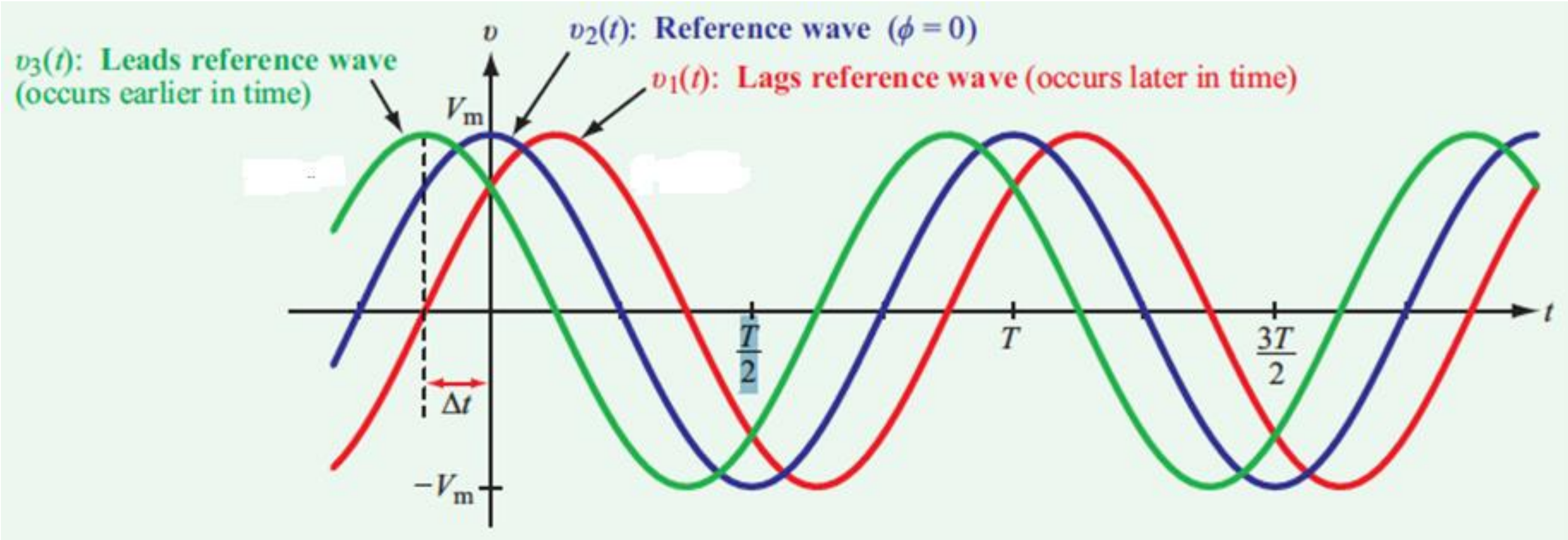
$$\sin x = \pm\cos(x \mp 90^\circ)$$
$$\cos x = \pm\sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

# Phase Lead/Lag

$$V_m \cos\frac{2\pi t}{T} \qquad V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right) \qquad V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right)$$




[Source: Berkeley]

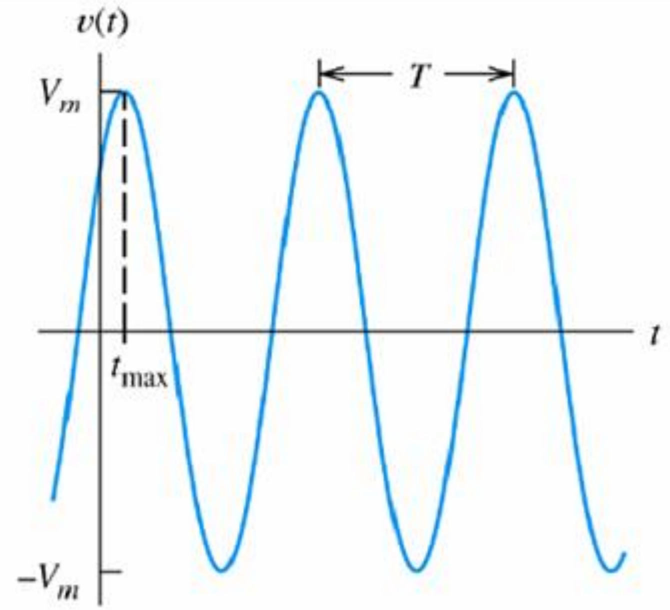
# Root-Mean-Square (RMS) Value

- The RMS value of $v(t) = V_m \cos(\omega t + \phi)$ is

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt}$$

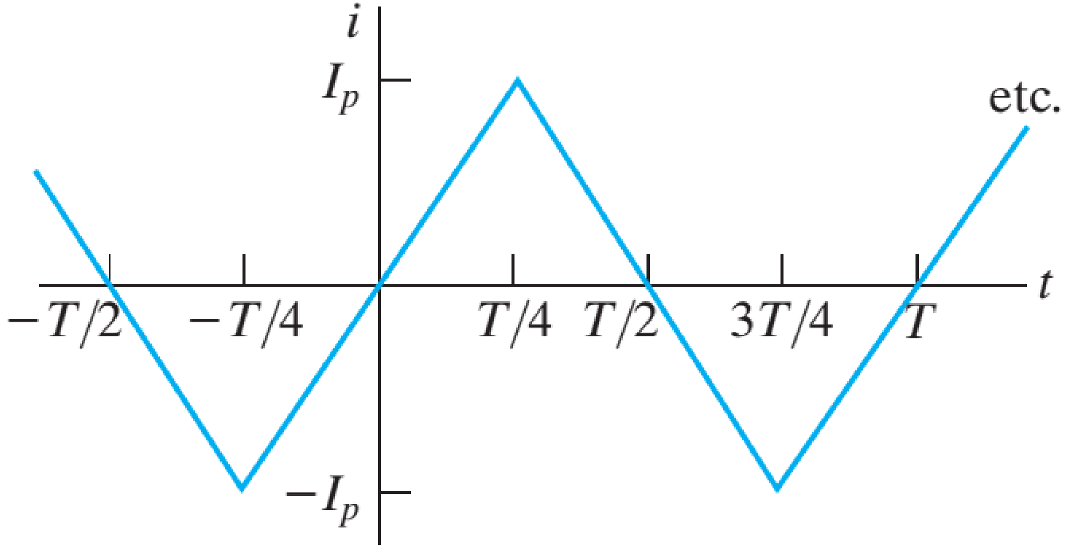
$$= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cdot \cos^2(\omega t + \phi)dt}$$

$$= \boldsymbol{\frac{V_m}{\sqrt{2}}}$$



# Exercise

- Find the rms value of the periodic triangular current shown below. Express your answer in terms of the peak current $I_p$.



# Outline

- Sinusoidal signals
- Complex numbers
- Phasor

# Complex Numbers

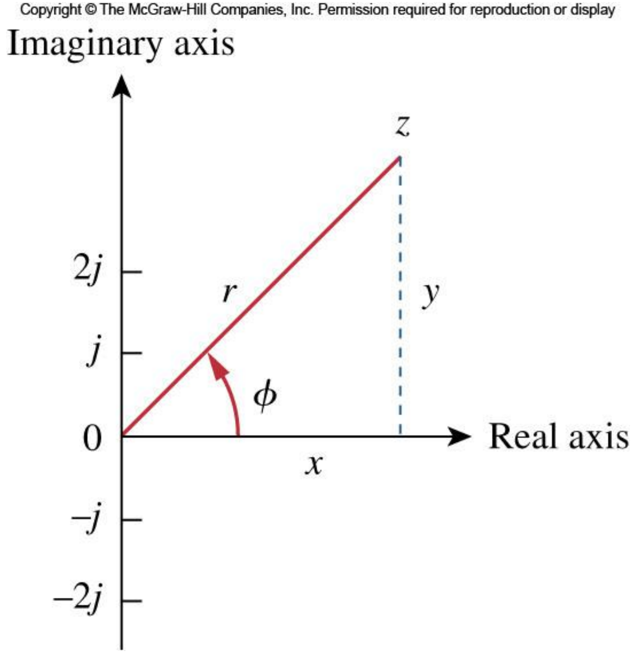
- A powerful method for representing sinusoids is the phasor.
- A complex number *z* can be represented in *rectangular form* as:

$$z = x + jy \qquad \mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y$$

- It can also be written in *polar* or *exponential* form as:

$$z = r\angle\phi = re^{j\phi}$$





# Euler's Formula

## Euler's Identities

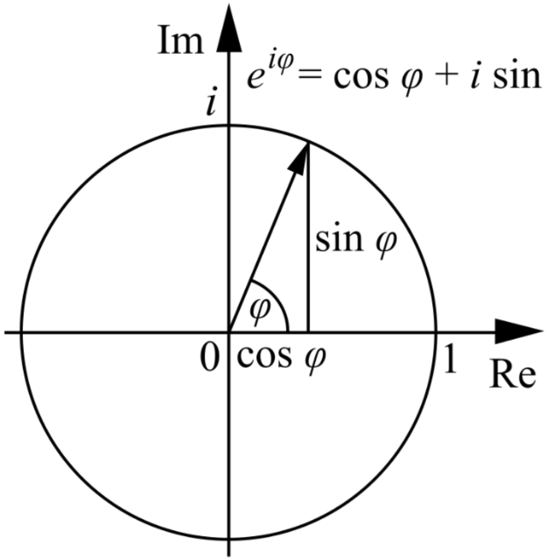
$$e^{j\pi} + 1 = 0$$

$$\boxed{e^{j\varphi} = \cos\varphi + j\sin\varphi}$$

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2}$$

$$|e^{j\varphi}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$




$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

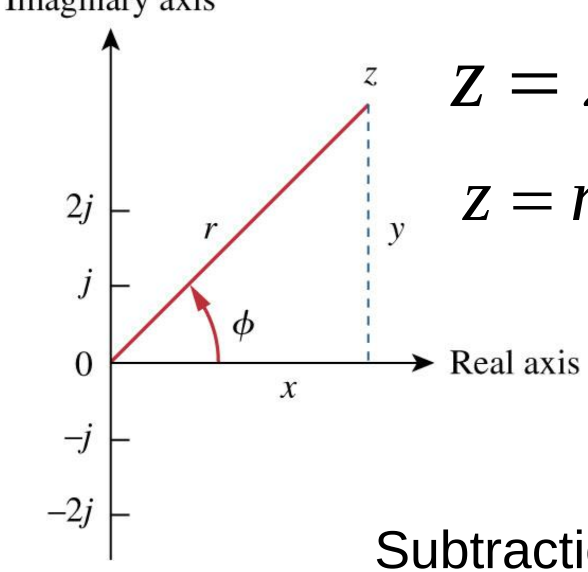
## Exponential Form of a complex number Z

$$\boldsymbol{Z} = |\boldsymbol{Z}|e^{j\varphi} = ze^{j\varphi} = z\angle\varphi$$

Source: https://en.wikipedia.org/wiki/Euler%27s_identity

# Arithmetic With Complex Numbers






$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

**Addition**

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

**Subtraction**

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

**Multiplication**

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

**Division**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

**Reciprocal**

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

**Square Root**

$$\sqrt{z} = \sqrt{r}\angle(\phi/2)$$

**Complex Conjugate**

$$z^* = x - jy = r\angle-\phi = re^{-j\phi}$$

# Example

- Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

# Exercise

- Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

# Relations for Complex Numbers

**Euler's Identity:** $e^{j\theta} = \cos\theta + j\sin\theta$

| | |
|---|---|
| $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$ | $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ |
| $\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$ | $\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$ |
| $x = \mathfrak{Re}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$ | $\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$ |
| $y = \mathfrak{Im}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$ | $\theta = \tan^{-1}(y/x)$ |
| $\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$ | $\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$ |
| $\mathbf{z}_1 = x_1 + jy_1$ | $\mathbf{z}_2 = x_2 + jy_2$ |
| $\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$ | $\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$ |
| $\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$ | $\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$ |
| $-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm180^\circ$ | |
| $j = e^{j\pi/2} = 1\angle 90^\circ$ | $-j = e^{-j\pi/2} = 1\angle -90^\circ$ |
| $\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$ | $\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$ |

[Source: Berkeley]

# Outline

- Sinusoidal signals
- Complex numbers
- Phasor

# Phasor

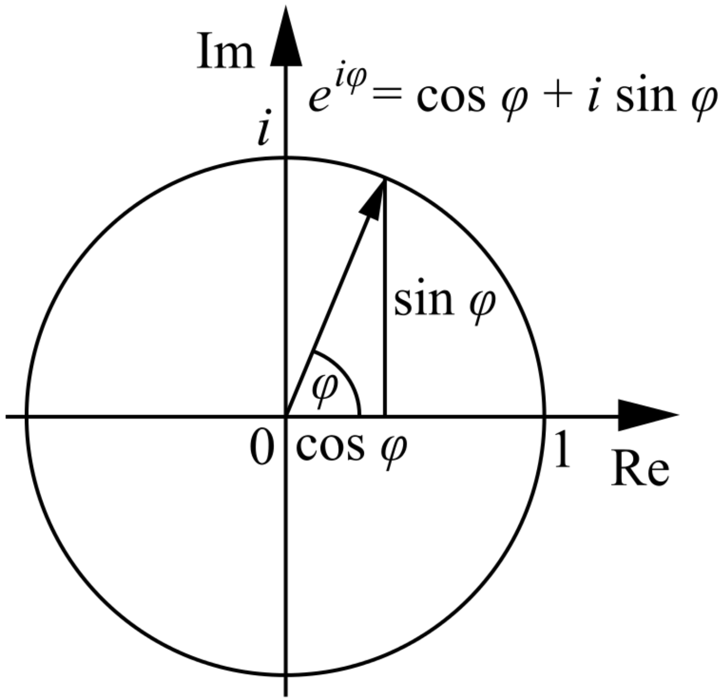
- The idea of phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the real component of a vector in the complex plane.

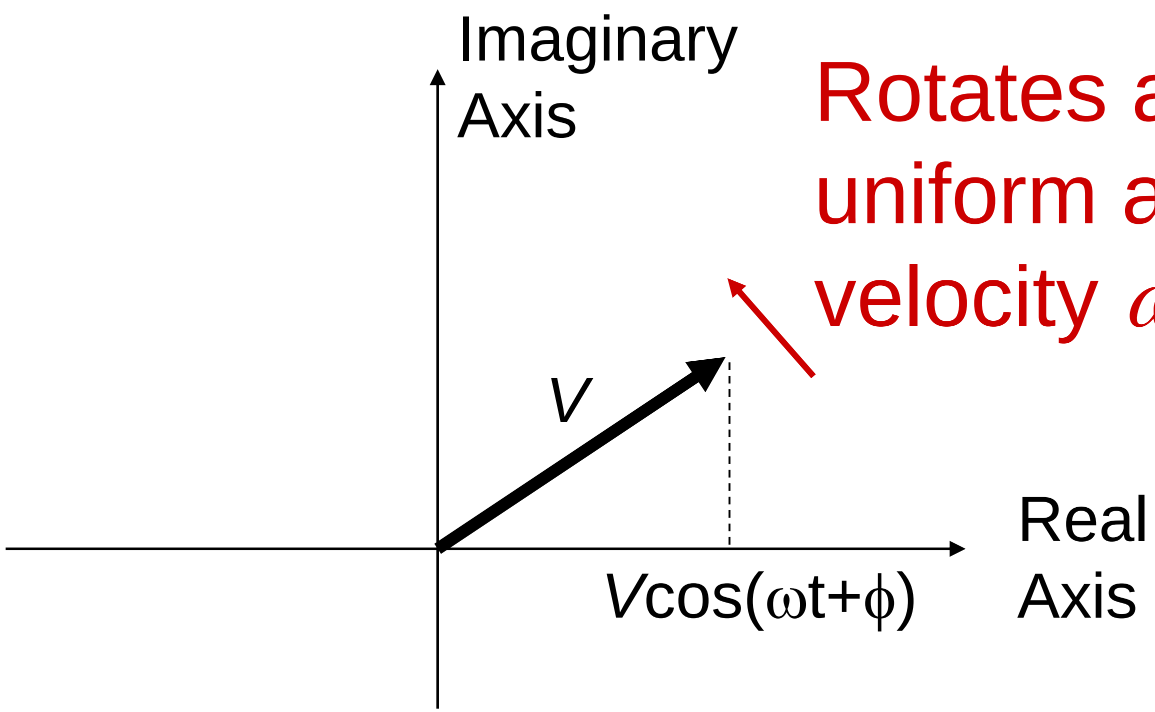
$$\cos\phi = \mathrm{Re}\left(e^{j\phi}\right)$$

$$\sin\phi = \mathrm{Im}\left(e^{j\phi}\right)$$



# Phasor: Rotating Complex Vector

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{j\omega t}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \qquad \boxed{\mathbf{V} = Ve^{j\phi}}$$



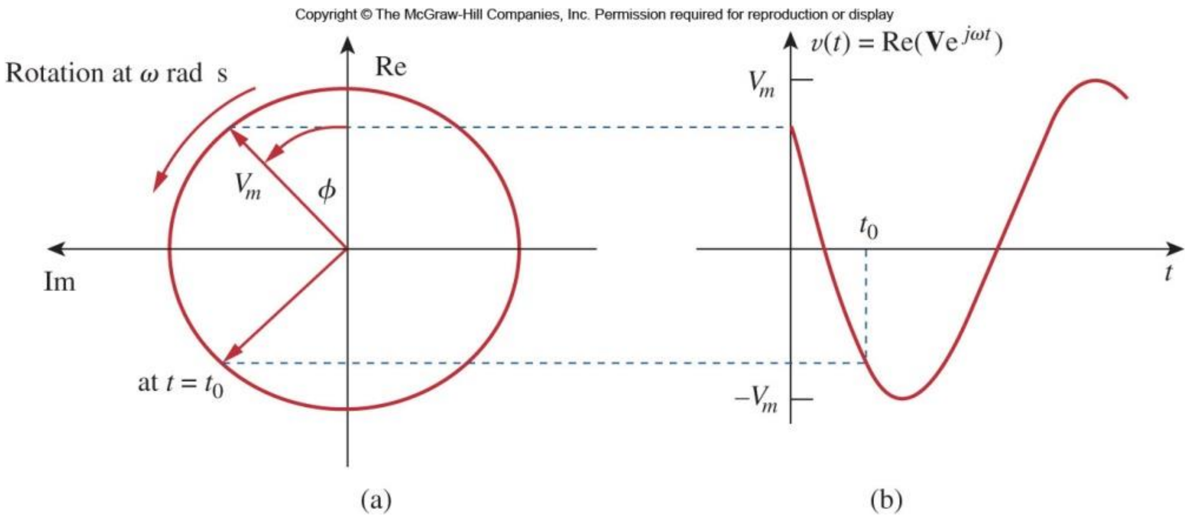

Rotates at uniform angular velocity $\omega t$

**The head start angle is $\phi$.**

# Phasors

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)




$V_m e^{j(\omega t+\phi)}$

- Phasors are typically represented at *t=0*.

# Example

- Transform these sinusoids to phasors

(a) $i = 6\cos(50t - 40^\circ)$ A

(b) $v = -4\sin(30t + 50^\circ)$ V

# Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

$$\mathbf{V} = -25\angle 40^\circ \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$

# Sinusoid-Phasor Transformation

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

- Applying a derivative to a phasor yields:

$$\frac{dv}{dt} \Leftrightarrow j\omega V$$

(Time domain) (Phasor domain)

- Applying an integral to a phasor yields:

$$\int v dt \Leftrightarrow \frac{V}{j\omega}$$

(Time domain) (Phasor domain)

# Time Domain - Phasor Domain Transformation

| $x(t)$ | | $\mathbf{X}$ |
|---|---|---|
| $A\cos\omega t$ | $\leftrightarrow$ | $A$ |
| $A\cos(\omega t+\phi)$ | $\leftrightarrow$ | $Ae^{j\phi}$ |
| $-A\cos(\omega t+\phi)$ | $\leftrightarrow$ | $Ae^{j(\phi\pm\pi)}$ |
| $A\sin\omega t$ | $\leftrightarrow$ | $Ae^{-j\pi/2}=-jA$ |
| $A\sin(\omega t+\phi)$ | $\leftrightarrow$ | $Ae^{j(\phi-\pi/2)}$ |
| $-A\sin(\omega t+\phi)$ | $\leftrightarrow$ | $Ae^{j(\phi+\pi/2)}$ |
| $\frac{d}{dt}(x(t))$ | $\leftrightarrow$ | $j\omega\mathbf{X}$ |
| $\frac{d}{dt}[A\cos(\omega t+\phi)]$ | $\leftrightarrow$ | $j\omega Ae^{j\phi}$ |
| $\int x(t)\,dt$ | $\leftrightarrow$ | $\frac{1}{j\omega}\mathbf{X}$ |
| $\int A\cos(\omega t+\phi)\,dt$ | $\leftrightarrow$ | $\frac{1}{j\omega}Ae^{j\phi}$ |

[Source: Berkeley]

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.

- You just need to track magnitude/phase, knowing that everything is at frequency $\omega$.

# Example

- Use the phasor approach, determine the current $i(t)$ in the circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

# Phasor Relationships for Resistors

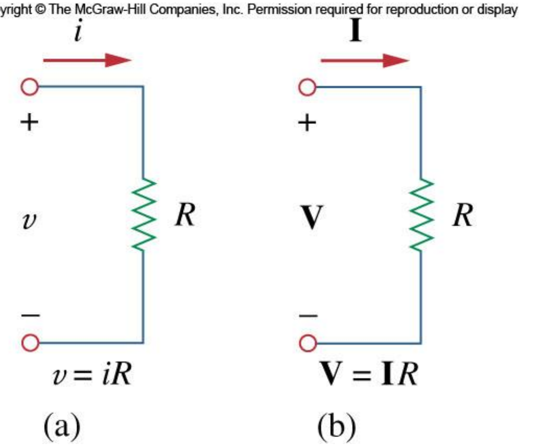
- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

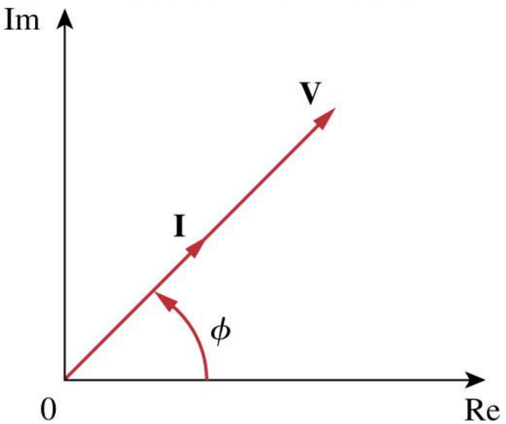
$$v = RI_m \cos(\omega t + \phi) \qquad \mathbf{V} = RI_m \angle \phi$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$





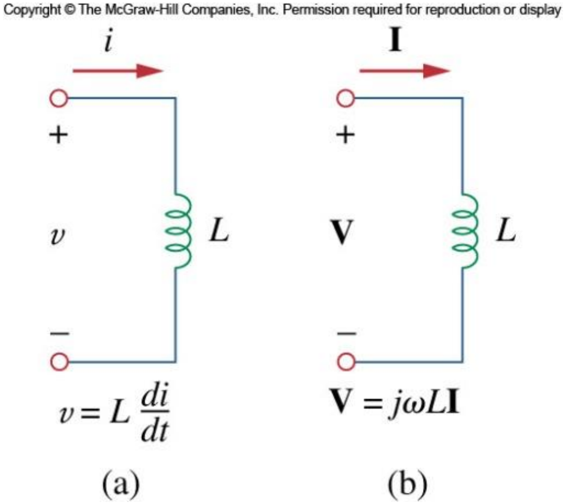


# Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

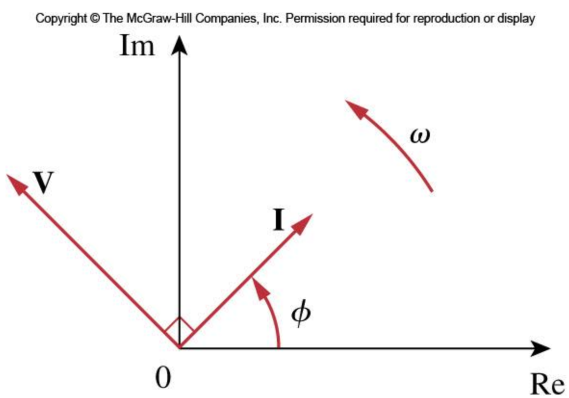
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$
$$= \omega L I_m \cos(\omega t + \phi + 90^\circ)$$




$$\mathbf{V} = \omega L I_m \angle \phi + 90^\circ = \omega L I_m e^{j(\phi+90^\circ)} = \omega L e^{j90^\circ} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\boxed{\mathbf{V} = j\omega L\mathbf{I}}$$

- The voltage leads the current by 90° (phase shift = 90°)

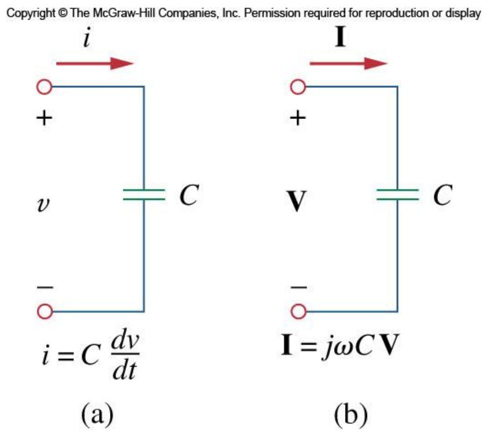


# Phasor Relationships for Capacitors

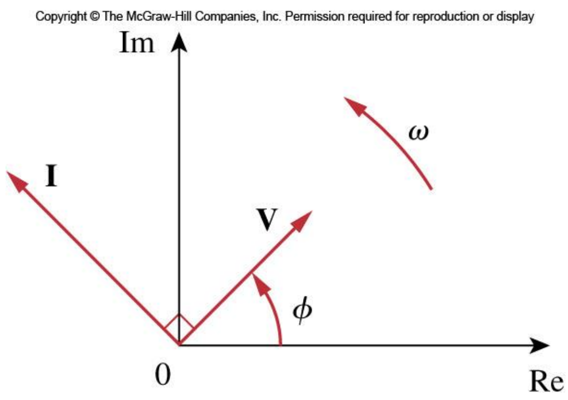
$$v = V_m \cos(\omega t + \phi) \qquad \mathbf{V} = V_m \angle \phi$$

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$
$$= \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90° = \cdots = j\omega C \cdot \mathbf{V}$$




$$\boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$




- The voltage lags the current by 90°.

# Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\boldsymbol{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - <u>Impedance is not a phasor</u>! But it is (often) a complex number.
  - Impedance depends on the frequency $\omega$.

# Admittance

Admittance is simply the inverse of impedance, unit: Simens.

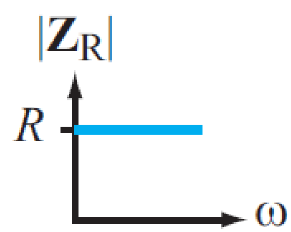
$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

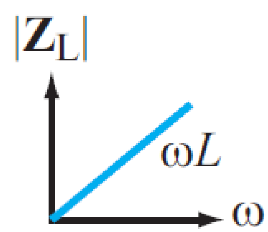
| Element | Impedance | Admittance |
|---|---|---|
| $R$ | $\mathbf{Z} = R$ | $\mathbf{Y} = \frac{1}{R}$ |
| $L$ | $\mathbf{Z} = j\omega L$ | $\mathbf{Y} = \frac{1}{j\omega L}$ |
| $C$ | $\mathbf{Z} = \frac{1}{j\omega C}$ | $\mathbf{Y} = j\omega C$ |

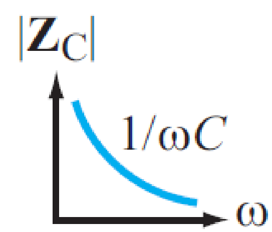
# Summary of *R, L, C*

| Property | $R$ | $L$ | $C$ |
|---|---|---|---|
| $v$–$i$ | $v = Ri$ | $v = L\dfrac{di}{dt}$ | $i = C\dfrac{dv}{dt}$ |
| **V**–**I** | $\mathbf{V} = R\mathbf{I}$ | $\mathbf{V} = j\omega L\mathbf{I}$ | $\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$ |
| **Z** | $R$ | $j\omega L$ | $\dfrac{1}{j\omega C}$ |
| dc equivalent | $R$ | Short circuit | Open circuit |
| High-frequency equivalent | $R$ | Open circuit | Short circuit |

Frequency response




[Source: Berkeley]