Discussion 4

- Capacitors and Inductors
- First-Order RL and RC Circuit

10/20/2016

OUTLINE

- Review & Extension
  - Linear Circuit Elements
    - ➢ Capacitor
    - ➢ Inductor
    - ✓ Mutual Inductance
  - First-Order Circuit
    - ➢ Natural Response
    - ➢ Step Response
    - ✓ Integrator
    - ✓ Differentiator
    - ✓ Application
- Q&A

Capacitor

- value:
  - 1.$C = \frac{q}{V}$ $C = \frac{\varepsilon A}{d}$

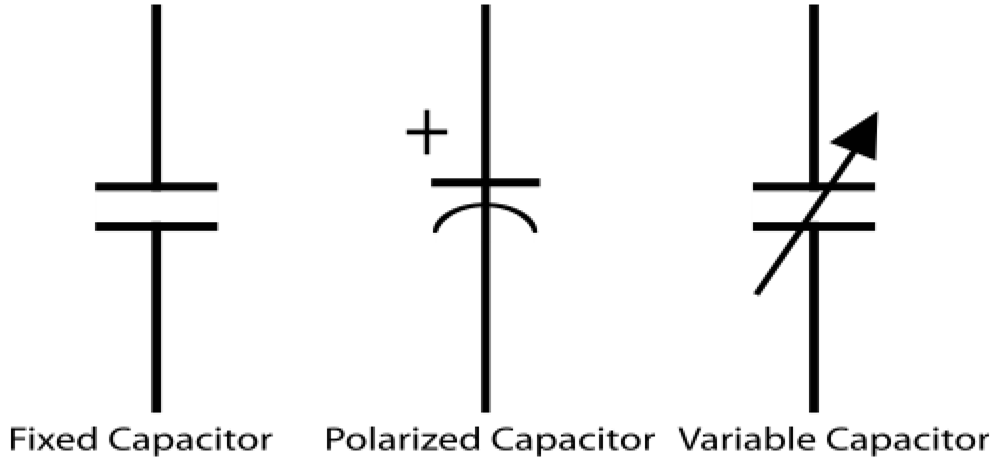
where A is the surface area of each plate, *d* is the distance between the plates, and ε is the permittivity of the dielectric material between the plates.

- Current and voltage:
  - 2.$i = C\frac{dv}{dt}$ $v = \frac{1}{C}\int_{t0}^{t} i(\tau)d\,\tau + \mathrm{v}(t_0)$
- energy:
  - 3.$E = \frac{1}{2}CV^2$
  - The voltage on the capacitor's plates can't change instantaneously, i.e., voltage must be continuous

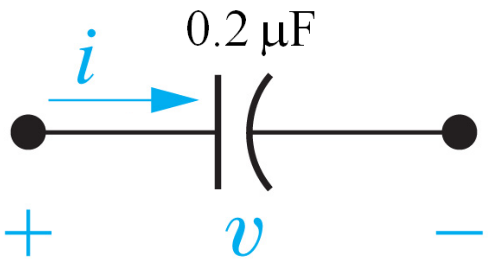
Capacitor

- Parallel and Series
  - $C_{eq} = C_1 + C_2 + C_3 + \cdots C_N$
  - $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$

- Symbols in circuit



- Example 1 – find the voltage, power, and energy



$$i(t) = \begin{cases} 0, & t \le 0; \\ 5000t \text{ A}, & 0 \le t \le 20\ \mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \le t \le 40\ \mu\text{s}; \\ 0, & t \ge 40\ \mu\text{s}. \end{cases}$$

For $t < 0$:

$$v(t) = 0\text{ V}; \qquad p(t) = 0\text{ W}; \qquad w(t) = 0\text{ J}$$

For $0 \le t \le 20\mu\text{s}$:

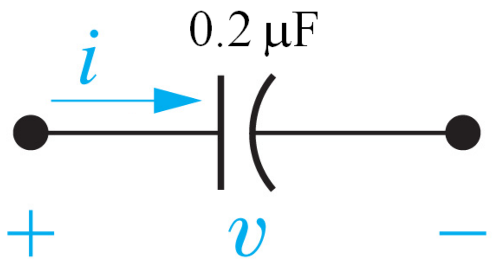
$$v(t) = \frac{1}{0.2\mu}\int_0^t 5000x\,dx + v(0) = \frac{1}{0.2\mu}\left.\frac{5000x^2}{2}\right|_0^t = 12.5\times10^9 t^2\text{ V}$$

$$p(t) = v(t)i(t) = (12.5\times10^9 t^2)(5000t) = 62.5\times10^{12} t^3\text{ W}$$

$$w(t) = \tfrac{1}{2}(0.2\mu)(12.5\times10^9 t^2)^2 = 15.625\times10^{12} t^4\text{ J}$$

At $t = 20\mu\text{s}$, $\qquad v(20\mu\text{s}) = 12.5\times10^9(20\mu)^2 = 5\text{ V}$

- Example 1, continued



$$i(t) = \begin{cases} 0, & t \le 0; \\ 5000t \text{ A}, & 0 \le t \le 20\ \mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \le t \le 40\ \mu\text{s}; \\ 0, & t \ge 40\ \mu\text{s}. \end{cases}$$

For $20\mu\text{s} \le t \le 40\mu\text{s}$:

$$v(t) = \frac{1}{0.2\mu}\int_{20\mu}^{t}(0.2 - 5000x)\,dx + v(20\mu)$$

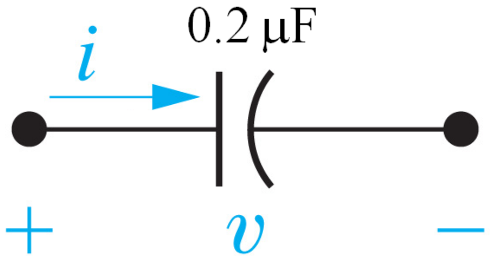
$$= \frac{1}{0.2\mu}\left[0.2x - \frac{5000x^2}{2}\right]\Bigg|_0^t + 5$$

$$= (10^6 t - 12.5\times10^9 t^2 - 10)\text{ V}$$

$$p(t) = v(t)i(t); \qquad w(t) = \tfrac{1}{2}Cv(t)^2$$

At $t = 40\mu\text{s}$, $\quad v(40\mu\text{s}) = [10^6(40\mu) - 12.5\times10^9(40\mu)^2 - 10) = 10\text{ V}$

- Example 1.continued



$$i(t) = \begin{cases} 0, & t \le 0; \\ 5000t \text{ A}, & 0 \le t \le 20\ \mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \le t \le 40\ \mu\text{s}; \\ 0, & t \ge 40\ \mu\text{s}. \end{cases}$$

For $t \ge 40\mu s$:

$$v(t) = \frac{1}{0.2\mu}\int_{40\mu}^{t} 0\, dx + v(40\mu) = 10\text{ V}$$

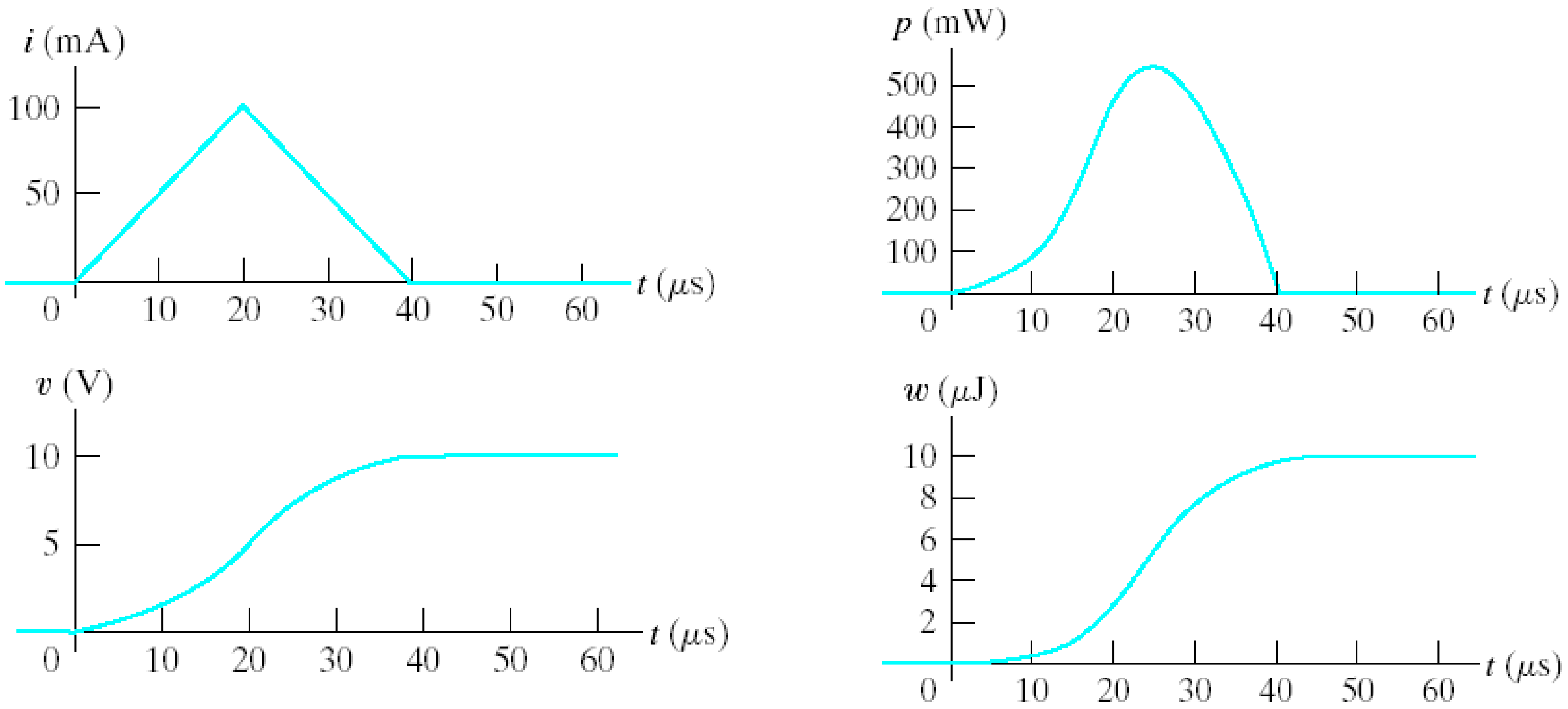
$$p(t) = v(t)i(t) = 0\text{ W}$$

$$w(t) = \tfrac{1}{2}(0.2\mu)(10)^2 = 10\mu\text{J!}$$

During the interval between 0 and 40μs, the power is positive (absorbed), energy is stored and “trapped” by the capacitor, so even when the current goes to 0, the voltage stays at 10 V and the energy is non-zero.

Capacitor

- Example 1 continued



Inductor

- value:
  - 1.$\mathrm{L} = \frac{N^2 \mu A}{\mathrm{l}}$

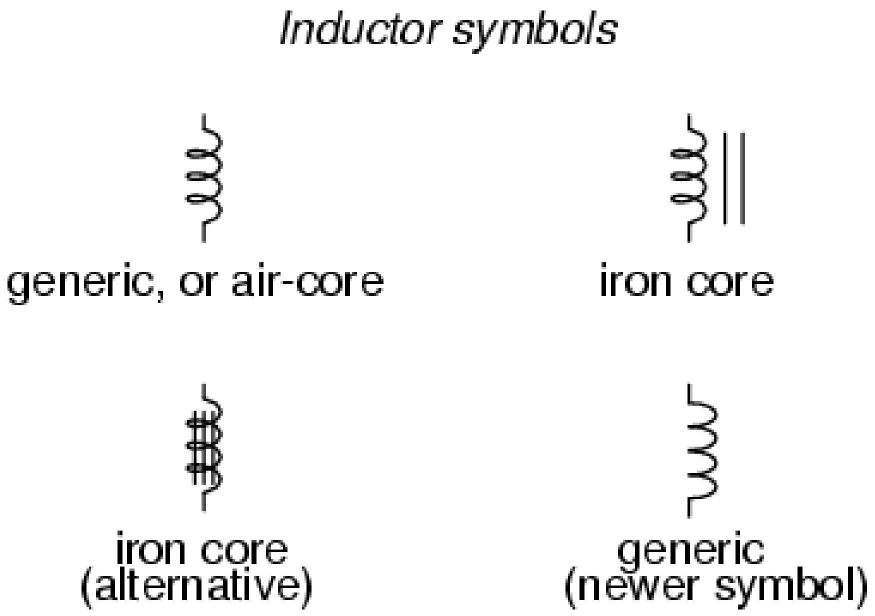
where N is the number of turns, l is the length, *A* is the cross-sectional area, μ is the permeability of the core.

- Current and voltage:
  - 2.$v = L\frac{di}{dt}$ $\quad \mathrm{i} = \frac{1}{L}\int_{t0}^{t} v(\tau)d\,\tau + \mathrm{i}(t_0)$
- energy:
  - 3.$E = \frac{1}{2}Li^2$
  - The current through an inductor can't change instantaneously, i.e., current must be continuous

Inductor

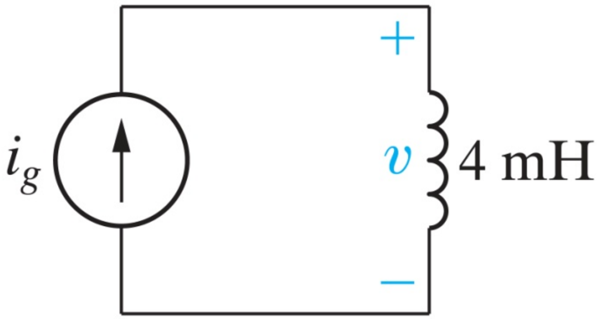
- Parallel and Series
  - $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$
  - $L_{eq} = L_1 + L_2 + L_3 + \cdots L_N$

- Symbols in circuit



Inductor

- Example – 2



$$i_g(t) = 0, \qquad t < 0,$$

$$i_g(t) = 8e^{-300t} - 8e^{-1200t}\ \text{A}, \qquad t \geq 0.$$

Is the current continuous? i(0) = 8 – 8 = 0: YES!

Find the voltage:

$$v(t) = 0, \qquad t < 0$$

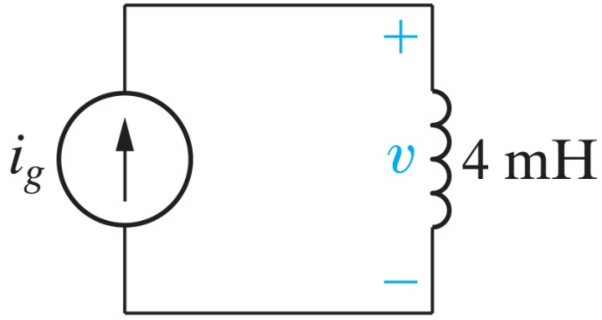
$$v(t) = L\frac{di(t)}{dt} = (0.004)\left[(-300)8e^{-300t} - (-1200)8e^{-1200t}\right]$$

$$= -9.6e^{-300t} + 38.4e^{-1200t}\ \text{V}, \qquad t \geq 0$$

Is the voltage continuous? v(0) = -9.6 + 38.4 = 28.8 V: NO!

Inductor

- Example – 2, continued



$$i_g(t) = 0, \qquad t < 0,$$

$$i_g(t) = 8e^{-300t} - 8e^{-1200t}\ \text{A}, \qquad t \geq 0.$$

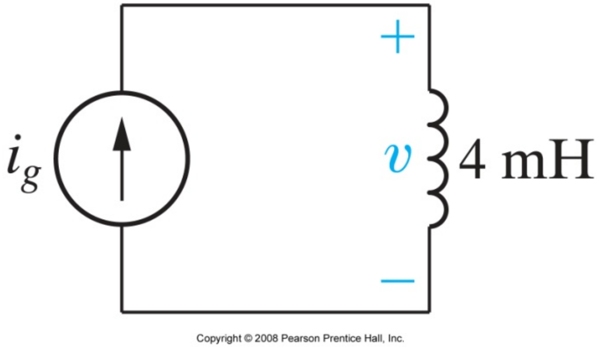
Find the power for the inductor:

$$\begin{aligned} p(t) &= v(t)i(t) \\ &= (-9.6e^{-300t} + 38.4e^{-1200t})(8e^{-300t} - 8e^{-1200t}) \\ &= -76.8e^{-600t} + 384e^{-1500t} - 307.2e^{-2400t}\ \text{W} \end{aligned}$$

To find the max power and the time at which the power is max, take the first derivative of p(t) and set it equal to 0.

Inductor

- Example – 2, continued



$$i_g(t) = 0, \qquad t < 0,$$

$$i_g(t) = 8e^{-300t} - 8e^{-1200t}\ \text{A}, \qquad t \geq 0.$$

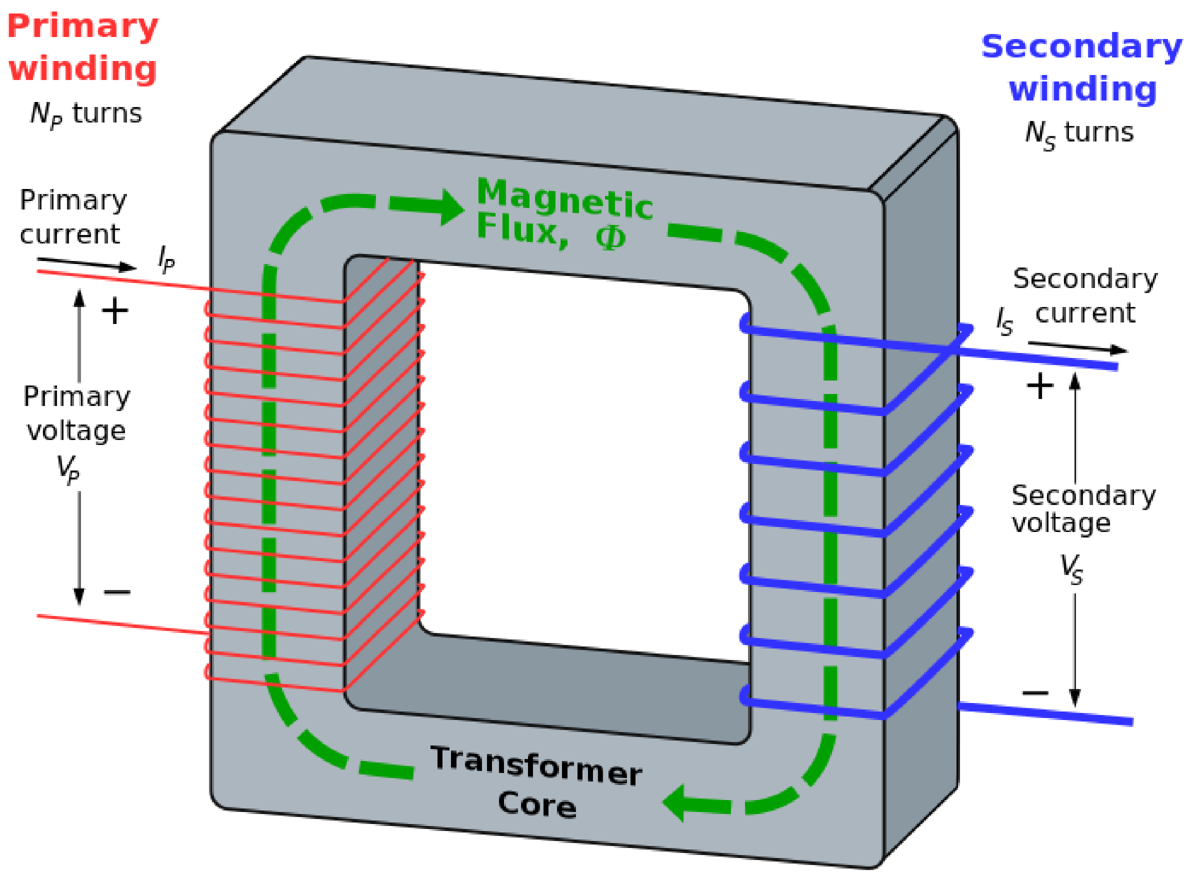
Find the energy for the inductor:

$$\begin{aligned} w(t) &= \tfrac{1}{2} Li(t)^2 \\ &= \tfrac{1}{2}(0.004)(8e^{-300t} - 8e^{-1200t})^2 \\ &= 128(e^{-600t} - 2e^{-1500t} + e^{-2400t})\ \text{mJ} \end{aligned}$$

To find the max energy and the time at which the energy is max, take the first derivative of w(t) and set it equal to 0. Or if you don't have w(t) yet, do the same for i(t)!

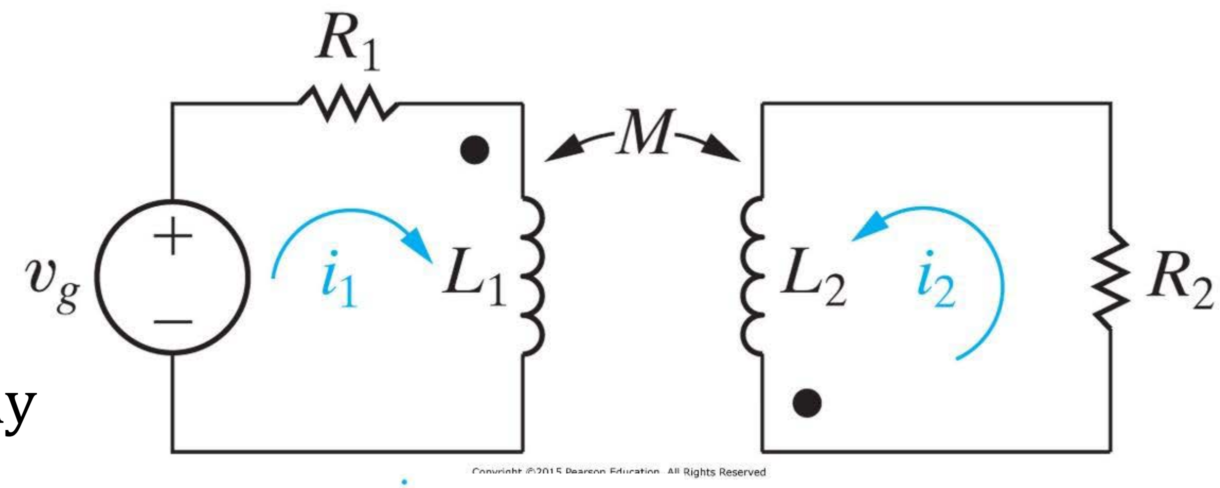
Mutual Inductance

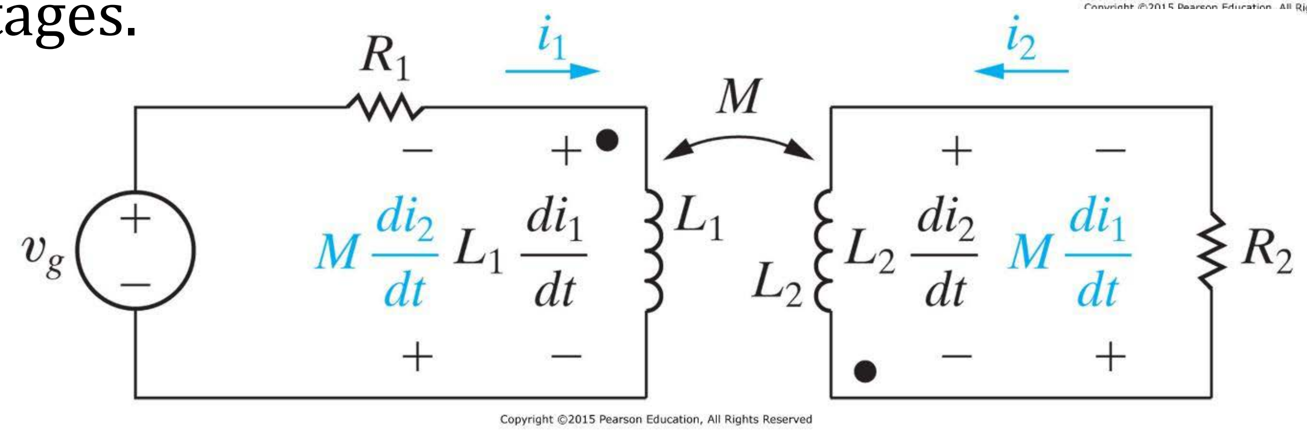
- Transformer



Mutual Inductance

- Two circuits lined by a magnetic field
  - ➢ L_1,L_2: self-inductances
  - ➢ M: mutual inductance
  - ➢ Dots: indicating polarity of mutually induced voltages.

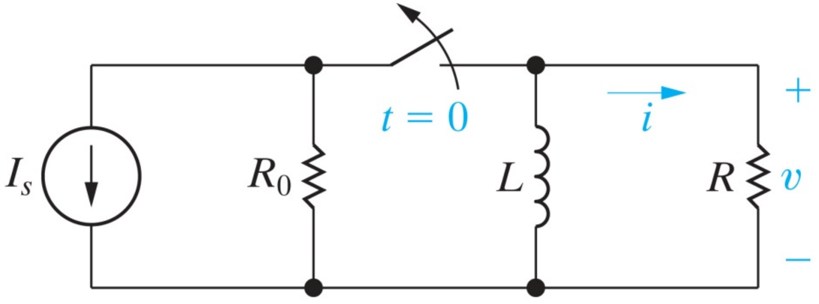


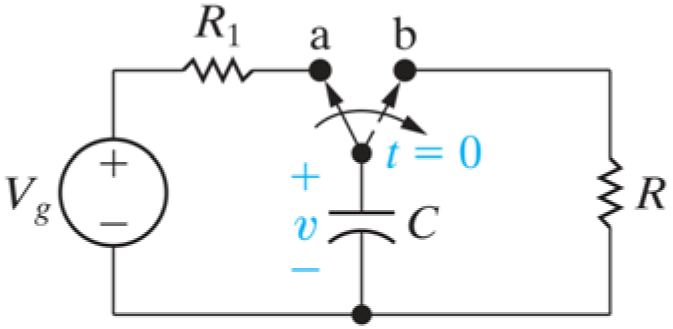


$$v_g = i_1 R_1 + L_1 \frac{di_1}{dt} \boxed{- M \frac{di_2}{dt}}$$

$$0 = i_2 R_2 + L_2 \frac{di_2}{dt} \boxed{- M \frac{di_1}{dt}}$$

First-Order Circuit -- Natural Response





KVL: $$L\frac{di(t)}{dt} + Ri(t) = 0$$

$$i(0) = I_s$$

$$i(t) = I_s e^{-(R/L)t}, \qquad t \geq 0$$

KCL: $$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = 0$$

$$v(0) = V_g$$

$$v(t) = V_g e^{-(1/RC)t}, \qquad t \geq 0$$

The form of the natural response is the same: $\mathrm{IC}e^{-t/\tau}$

IC is the initial condition and τ is the **time constant,** a measure of how quickly or slowly the exponential decays.

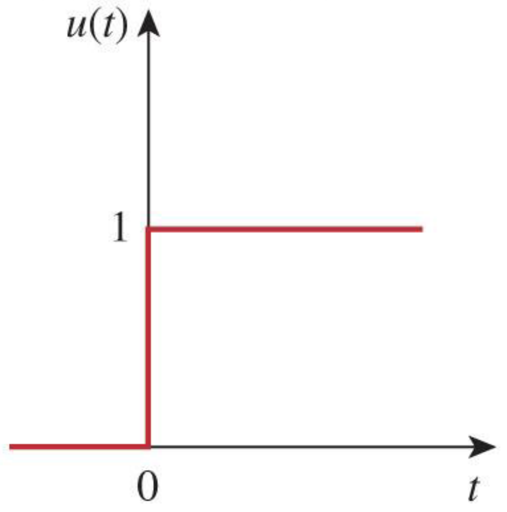
First-Order Circuit -- Natural Response

1. Identify the variable of interest.
   - For RL, $i(t)$ through L; For RC, $v(t)$ across C
2. Find the initial value of this variable, either $i(0) = I_o$ or $v(0) = V_o$. Usually, analyze the circuit for t < 0.
3. Find the time constant, τ
   - $\tau_{RL} = L/R_{eq}$ or $\tau_{RC} = R_{eq}C$
   - R_{eq} is the equivalent resistance seen by the inductor or capacitor.
4. Write the expression for the variable of interest:

   $$x(t) = X_o e^{-t/\tau}, \quad t \geq 0.$$
5. Use simple circuit analysis to calculate any other requested variables.

Singularity Functions -- Unit step

- A step function is one that maintains a constant value before a certain time and then changes to another constant afterwards.
- The prototypical form is 0 before $t = 0$ and 1 afterwards.



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

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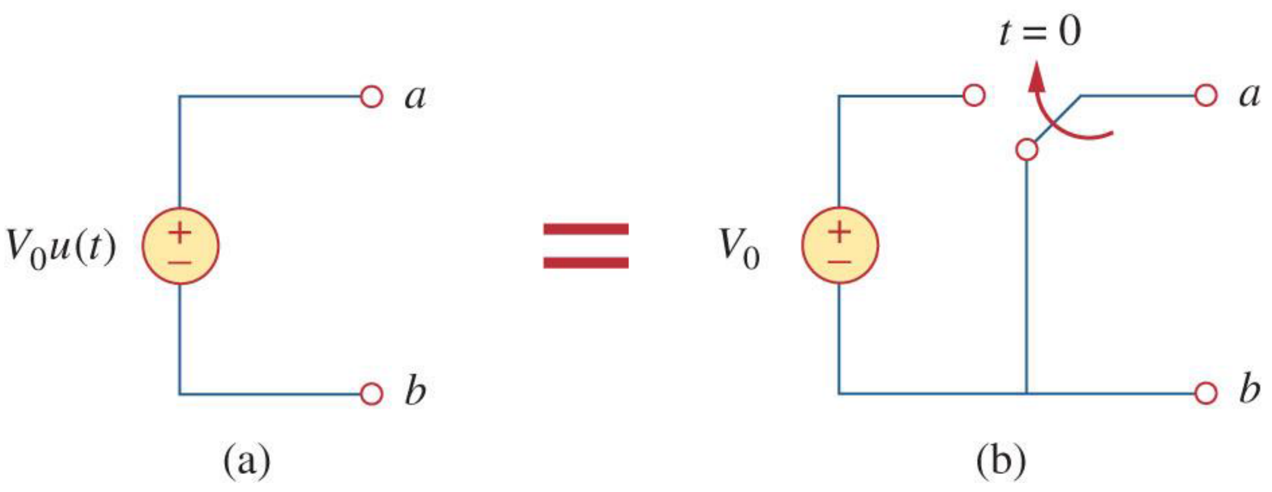
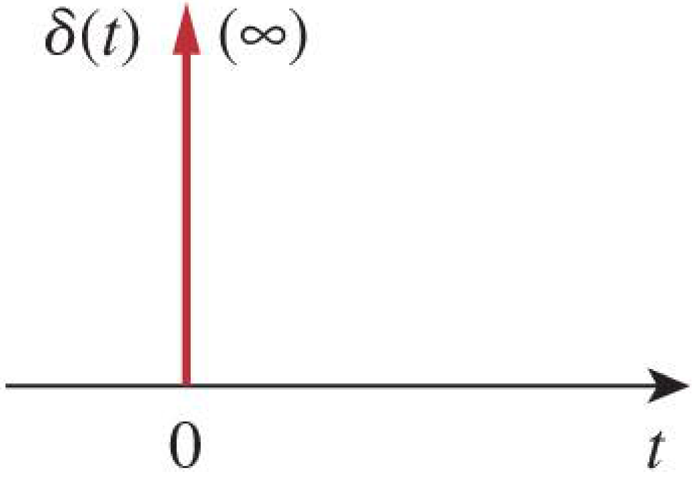


Singularity Functions -- Unit Impulse

- The derivative of the unit step function is the unit impulse function.
- This is expressed as:

$$u(t) = \begin{cases} 0, & t < 0 \\ Undefined & t = 0 \\ 0, & t > 0 \end{cases}$$

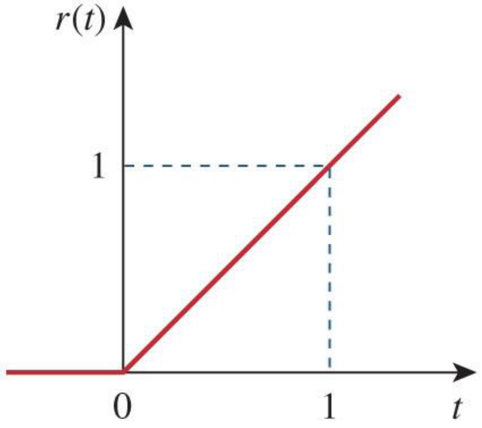


- Voltages of this form can occur during switching operations.

Singularity Functions -- Unit Ramp

- Integration of the unit step function results in the unit ramp function:

$$u(t)=\begin{cases}0, & t\le 0\\ t, & t\ge 0\end{cases}$$

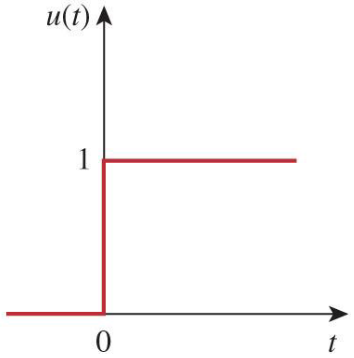


- Much like the other functions, the onset of the ramp may be adjusted.

First-Order Circuit -- Singularity Function

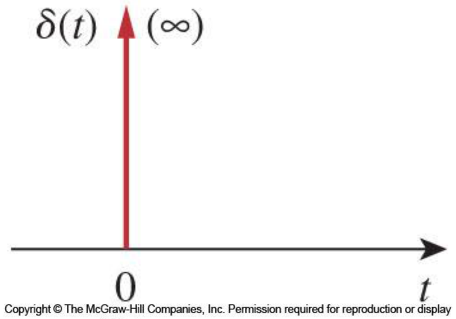
- Unit step

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$



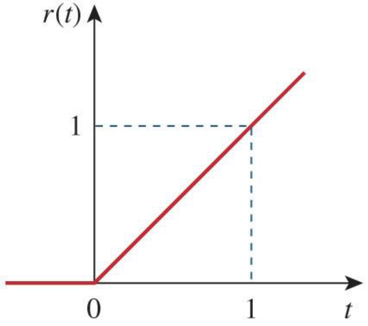
- Unit impulse

$$u(t)=\begin{cases}0, & t<0\\ Undefined & t=0\\ 0, & t>0\end{cases}$$

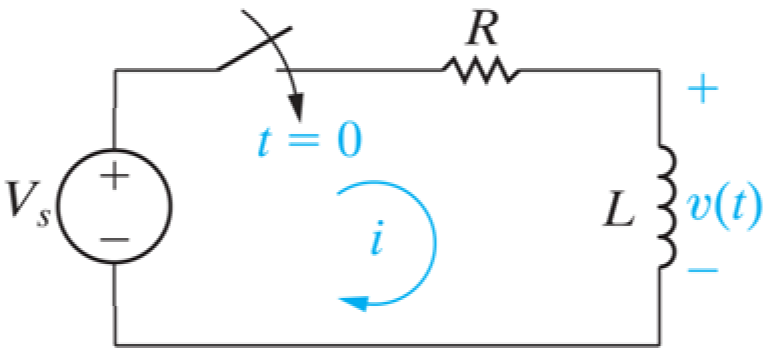


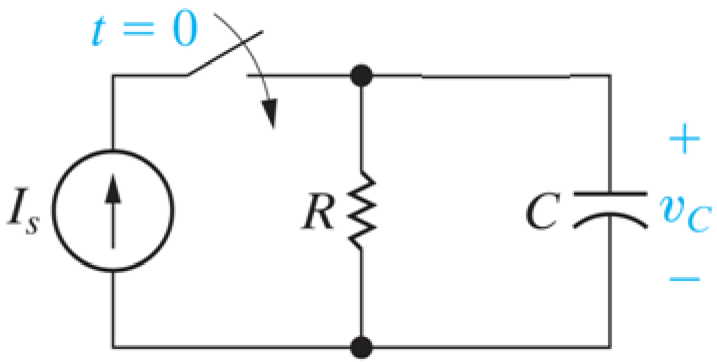
- Unit ramp

$$u(t)=\begin{cases}0, & t\le 0\\ t, & t\ge 0\end{cases}$$



First-Order Circuit -- Step Response





KVL for $t \geq 0$:

$$-V_S + L\frac{di(t)}{dt} + Ri(t) = 0$$

$$\Rightarrow \qquad L\frac{di(t)}{dt} + Ri(t) = V_S$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}$$

KCL for $t \geq 0$:

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = I_S$$

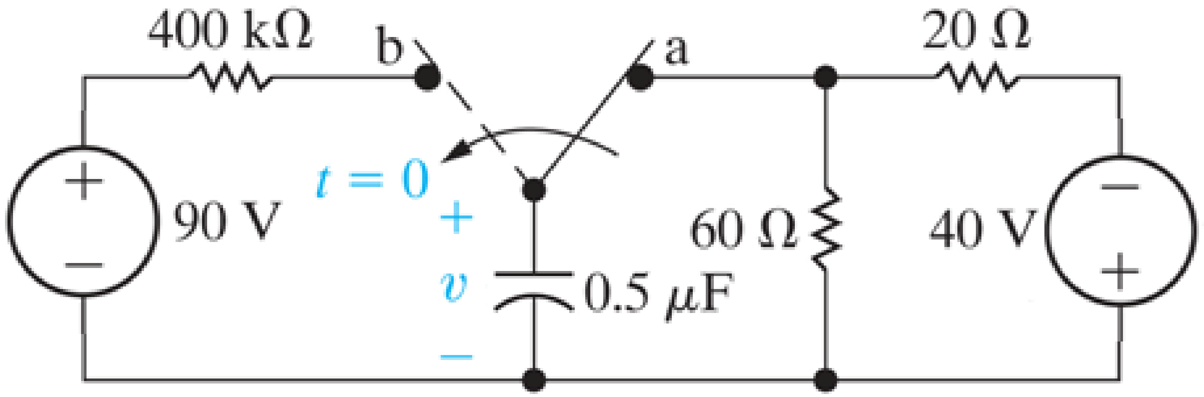
$$v_C(t) = I_s R + (V_0 - I_s R)e^{-t/RC}, \qquad t \geq 0$$

$$\boxed{x(t) = X_F + (X_0 - X_F)e^{-t/\tau}, \qquad t \geq 0.}$$

First-Order Circuit

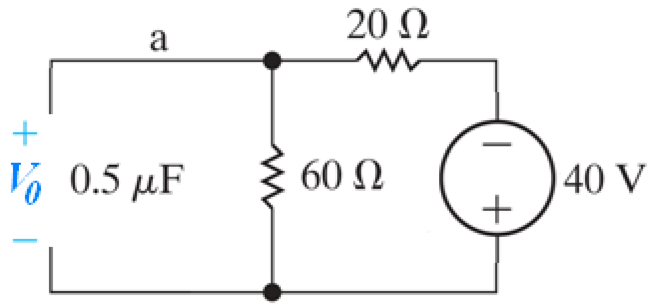
- Example – 3

Find $v(t)$ for $t \geq 0$



1. The variable of interest is the capacitor voltage drop, which is already defined in the circuit.
2. Find the initial voltage drop across the capacitor:

For t < 0



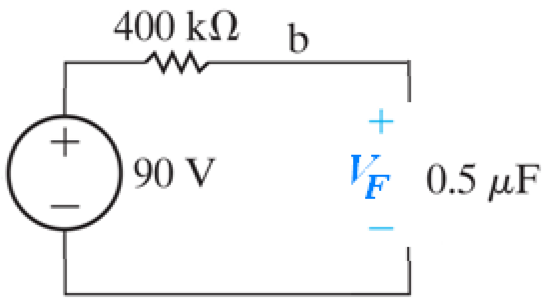
$$V_o = \frac{60}{60+20}(-40) = -30\text{ V}$$

First-Order Circuit

Example – 3, continued

3. Find the final voltage drop across the capacitor:

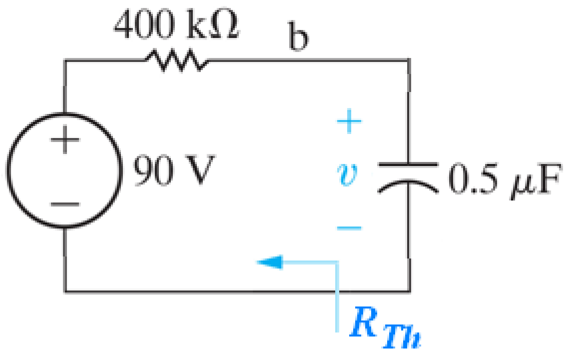
As $t \to \infty$



$$V_F = 90\ \text{V}$$

4. Find the time constant, $\tau = R_{eq}C$ by finding the equivalent resistance seen bv the capacitor for $t \geq 0$.

For $t \geq 0$



$$R_{Th} = 400\ \text{k}\Omega$$

$$\tau = (400{,}000)(0.5 \times 10^{-6}) = 0.2\ \text{s}$$

First-Order Circuit

Example – 3, continued

5. Write the expression for the inductor current:

$$v(t) = V_F + (V_0 - V_F)e^{-t/\tau} = 90 + [-30 - 90]e^{-t/0.2}$$

$$= 90 - 120e^{-5t} \text{ V}, \quad t \geq 0 \qquad (\text{check at } t = 0 \text{ and } t \to \infty)$$

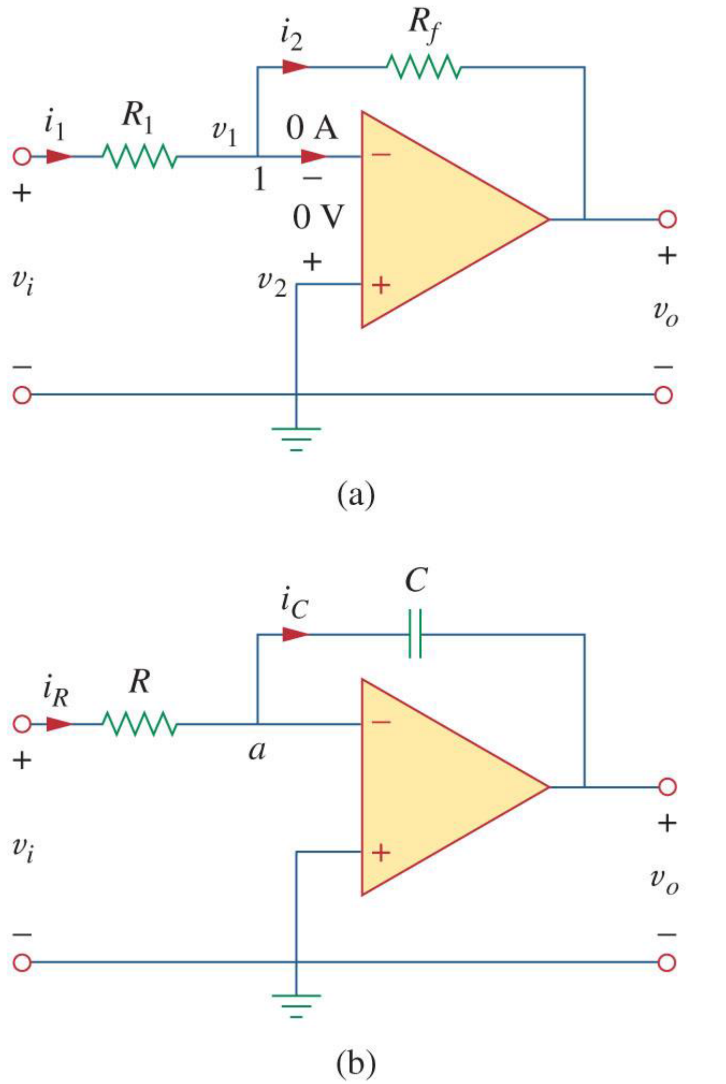
Integrator

- Review: Operational Amplifier(a)

$$v_1 = v_2 \qquad i_1 = i_2$$

- Integrator(b):
  - Capacitors, in combination with op-amps can be made to perform advanced mathematical functions.
  - By replacing the feedback resistor with a capacitor, the output voltage from the op-amp is:

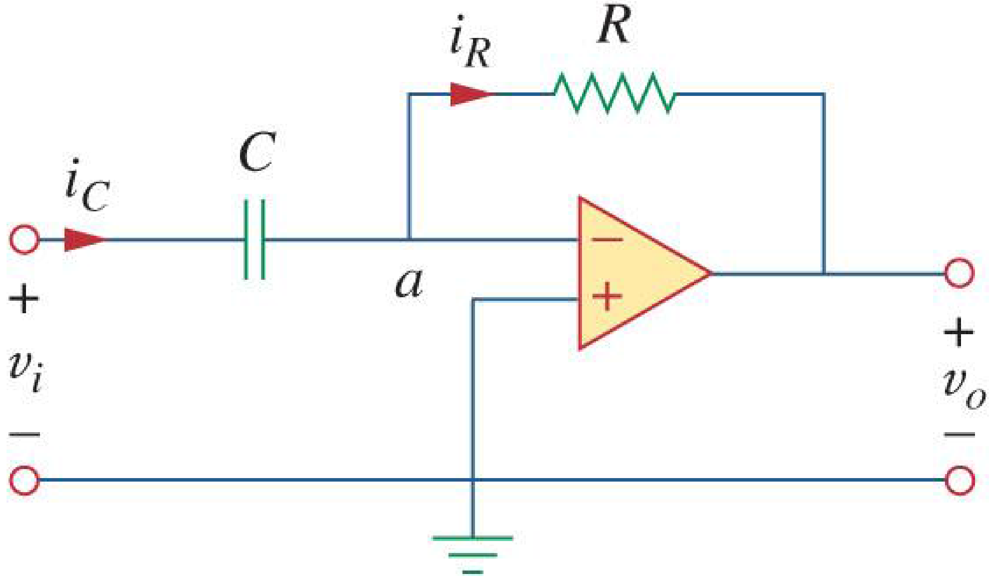
$$v_0 = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau$$



Differentiator

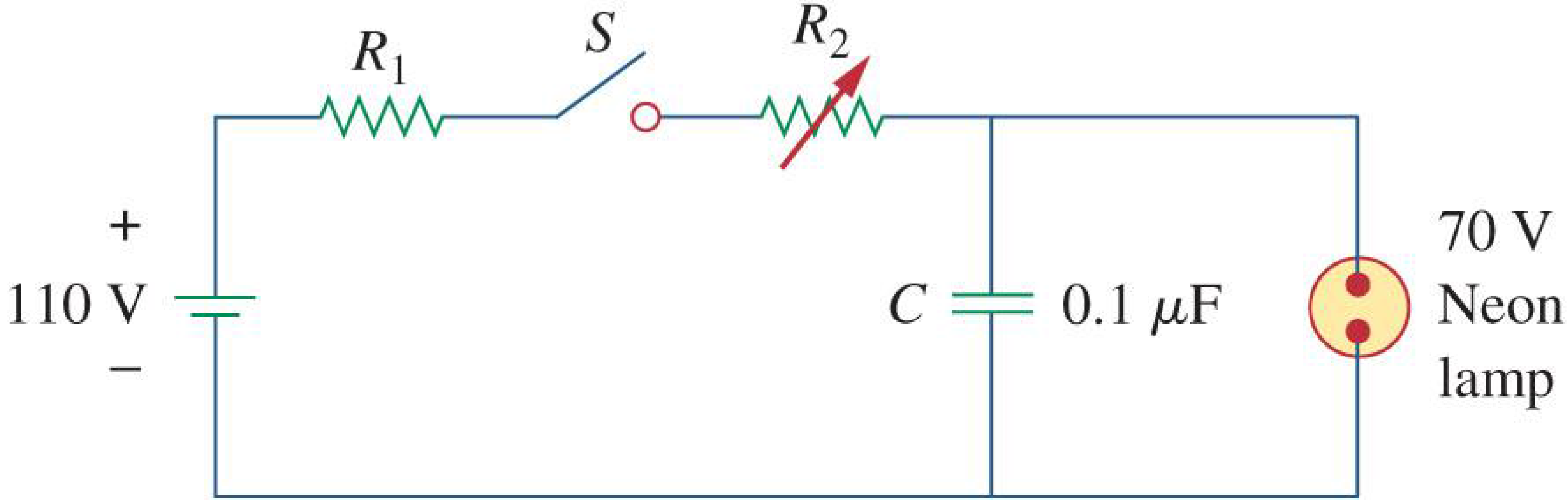
- The previous circuit functions as an integrator with time.
- If the capacitor is used in place of the input resistor instead of the feedback resistor, there will only be current flowing if the voltage is changing.
- The output voltage in this case will be:

$$v_0 = -RC\frac{dv_i}{dt}$$

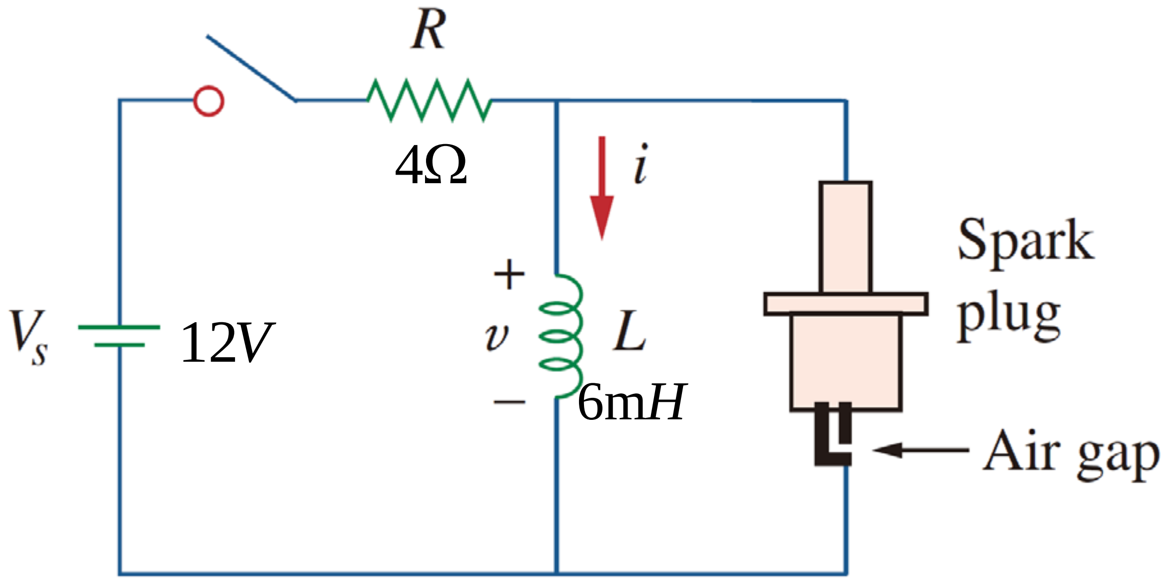


Application -- Delay Circuit

- The RC circuit can be used to delay the turn on of a connected device.
- For example, a neon lamp which only triggers when a voltage exceeds a specific value can be delayed using such a circuit.



Application -- Automobile Ignition Circuit



Assuming that the switch takes 1μs to open, determine:
the votage across the air gap.

Application -- Automobile Ignition Circuit

The final current through the coil is

$$I = \frac{V_s}{R} = \frac{12}{4} = 3\ \text{A}$$

The energy stored in the coil is

$$W = \frac{1}{2}L\,I^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 27\ \text{mJ}$$

The voltage across the gap is

$$V = L\frac{\Delta I}{\Delta t} = 6 \times 10^{-3} \times \frac{3}{1 \times 10^{-6}} = 18\ \text{kV}$$

Q&A