Problem 1

(20 points)

(a) Determine the Fourier series coefficients a_k for $x_1(t)$ shown below.

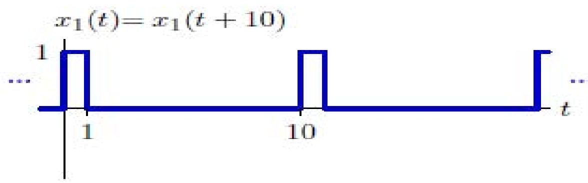

Figure 1: Problem 1(a)

(b) Determine the Fourier series coefficients b_k for $x_2(t)$ shown below.

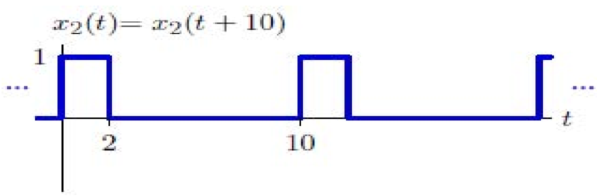

Figure 2: Problem 1(b)

(c) Determine the Fourier series coefficients c_k for $x_3(t)$ shown below.

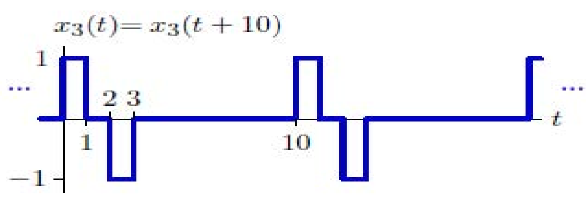

Figure 3: Problem 1(c)

Solution

(a)

$$a_k = \frac{1}{T}\int_T x_1(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{10}\int_0^1 1e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k}\Big|_0^1 = \frac{1}{j2\pi k}(1-e^{-j\pi k/5}) \tag{1}$$

Notice that this expression is badly formed at $k = 0$. We could use L'Hospital's rule to evaluate this expression, but an easier method (which is also more robust against errors) is to simply evaluate the average value of $x_1(t)$ to find that $a_0 = 1/10$.
This solution could also be written in terms of sinusoids as

$$a_k = \begin{cases} \frac{1}{10} & k = 0 \\ \frac{1}{\pi k}e^{-j\pi k/10}sin(\pi k/10) & k \neq 0 \end{cases} \tag{2}$$

(b)

$$b_k = \frac{1}{T}\int_T x_2(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{10}\int_0^2 1e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k}\Bigg|_0^2 = \frac{1}{j2\pi k}(1-e^{-j2\pi k/5}) \tag{3}$$

As with the previous part, this expression is badly formed for $k=0$. Therefore, we obtain $b_0 = 1/5$ by calculating the average value of $x_2(t)$.
This solution could also be written in terms of sinusoids as

$$b_k = \begin{cases} \frac{1}{5} & k=0 \\ \frac{1}{\pi k}e^{-j\pi k/5}sin(\pi k/5) & k \neq 0 \end{cases} \tag{4}$$

(c)

$$\begin{aligned} & x_3(t) = x_1(t) - x_1(t-2) \\ & \int_T x_1(t-2)e^{-j\frac{2\pi}{T}kt}dt = \int_T x_1(t)e^{-j\frac{2\pi}{T}k(t+2)}dt = e^{-j\frac{2\pi}{T}k2}\int_T x_1(t)e^{-j\frac{2\pi}{T}kt}dt = e^{-j\frac{2\pi}{T}k2}a_k \\ & c_k = a_k - e^{-j\frac{2\pi}{T}k2}a_k = (1-e^{-j2\pi k/5})\frac{1}{j2\pi k}(1-e^{-j\pi k/5}) \end{aligned} \tag{5}$$

The average value of $x_3(t)$ is zero, so $c_0 = 0$.
This solution could also be written in terms of sinusoids as

$$c_k = \begin{cases} 0 & k=0 \\ \frac{j2}{\pi k}e^{-j3\pi k/10}sin(\pi k/5)sin(\pi k/10) & k \neq 0 \end{cases} \tag{6}$$

Problem 2

(20 points) Suppose that we are given the following information about a signal $x[n]$

1. $x[n]$ is a real and even signal.

2. $x[n]$ has a period $N = 10$ and Fourier coefficients a_k.

3. $a_{11} = 5$.

4. $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$.

Show that $x[n] = Acos(Bn + C)$, and specify numerical values for the constants A, B and C.

Solution

Since the Fourier series coefficients of $x[n]$ has the same period with $x[n]$, we can get

$$a_1 = a_{11} = 5$$

Furthermore,since $x[n]$ is a real and even signal, then a_k is also real and even, so

$$a_{-1} = a_1 = 5$$

And it also implies

$$a_9 = a_{-1} = 5$$

We are also given

$$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$$

With $Parseval's\ theorem$,

$$\sum_{n=0}^{9}|a_k|^2 = 50$$

For

$$a_1^2 + a_9^2 = 50$$

Then $a_k = 0$, for $k = 0, 2, 3, ..., 8$ (in a period 0,...,9), so

$$x[n] = \sum_{k\in N} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{9} a_k e^{j\frac{2\pi}{10}kn} = 5e^{j\frac{2\pi}{10}n} + 5e^{j\frac{18\pi}{10}n} = 10cos(\frac{\pi}{5}n)$$

$$A = 10, B = \frac{\pi}{5}, C = 0.$$

Problem 3

(20 points) Consider the following three continuous-time signals with a fundamental period of $T = \frac{1}{2}$:

$$\begin{aligned} x(t) &= cos(4\pi t) \\ y(t) &= sin(4\pi t) \\ z(t) &= x(t)y(t) \end{aligned} \tag{7}$$

(a) Determine the Fourier series coefficients of $x(t)$.

(b) Determine the Fourier series coefficients of $y(t)$.

(c) Use the result of part(a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.

(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part(c).

Solution

(a)

$$x(t) = cos(4\pi t) = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} \tag{8}$$

So that the nonzero FS coefficients of $x(t)$ are $a_{-1} = a_1 = \frac{1}{2}$

(b)

$$y(t) = sin(4\pi t) = \frac{1}{2j}e^{j4\pi t} - \frac{1}{2j}e^{-j4\pi t} \tag{9}$$

So that the nonzero FS coefficients of $y(t)$ are $b_{-1} = -\frac{1}{2j}$, $b_1 = \frac{1}{2j}$

(c) Using the *multiplication property*, we know that

$$z(t) = x(t)y(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} \tag{10}$$

Therefore,

$$c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2] \tag{11}$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$

(d) We have

$$z(t) = sin(4\pi t)cos(4\pi t) = \frac{1}{2}sin(8\pi t) = \frac{1}{4j}e^{j8\pi t} - \frac{1}{4j}e^{-j8\pi t} \tag{12}$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$, it is the same as part(c).

Problem 4

(20 points)

(a) Draw the Fourier series coefficients of $x_1(t)$ and give explanation.

$$x_1(t) = 2 - 2cos(\frac{2\pi}{3}t) \tag{13}$$

(b) Draw the Fourier series coefficients of $x_2(t)$ and give explanation.

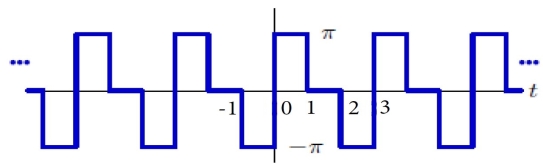

Figure 4: $x_2(t)$

Hint: When you graph, just draw the case where $k \in [-6, 6]$. And make sure to write their Fourier series coefficients' expressions.

Solution

(a) From the constant 2, it is clear that the zero coefficient is 2. Since $cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$, the coefficients for $k = \pm 1$ are -1. And the Fourier series coefficients of $x_1(t)$ could be drawn as Figure 5.

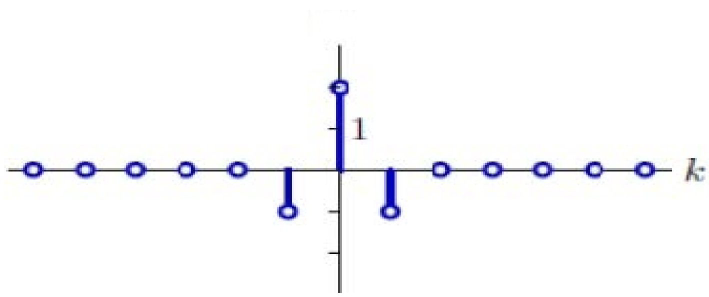

Figure 5: Problem 4(a)

(b) The signal is real and odd, so its FS coefficients must be purely imaginary and odd. Thus the only candidate is b_k. Solving

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x_2(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{3}\int_{-1}^{0} -\pi e^{-j\frac{2\pi}{3}kt}dt + \frac{1}{3}\int_0^1 \pi e^{-j\frac{2\pi}{3}kt}dt \\ &= \frac{1}{j2k}(e^{-j2\pi kt/3}\Big|_{-1}^{0} - e^{-j2\pi kt/3}\Big|_0^1) = \frac{1}{j2k}(2 - e^{j2\pi k/3} - e^{-j2\pi k/3}) \\ &= \frac{1}{jk}(1 - cos(2\pi k/3)) = \begin{cases} 0, & \text{if k is evenly divisible by 3} \\ 3/j2k, & \text{otherwise} \end{cases} \end{aligned} \tag{14}$$

And the Fourier series coefficients of $x_2(t)$ could be drawn as Figure 6.

$$b_k = \begin{cases} \frac{3}{j2k} & k = \pm 1, \pm 2, \pm 4, \pm 5, \pm 7, \ldots \\ 0 & k = 0, \pm 3, \pm 6, \cdots \end{cases}$$

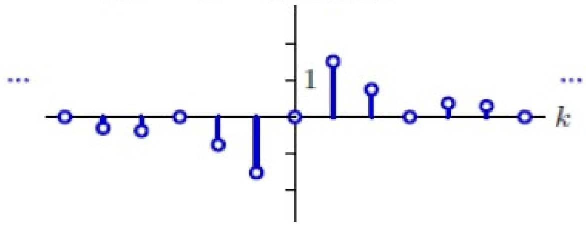

Figure 6: Problem 4(b)

Problem 5

(20 points)

(1) Consider a continuous-time ideal lowpass filter $h(t)$ whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq 100 \\ 0, & |\omega| > 100 \end{cases}$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

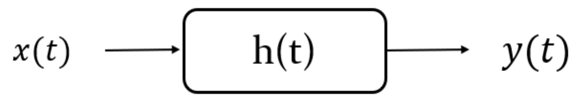

Figure 7: y(t)

Where $y(t) = x(t)$, and for what values of k is it guaranteed that $a_k = 0$?

(2) Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} 1, & 0 \leq n \leq 2 \\ -1, & -2 \leq n \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

Given that the input to this system is

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k],$$

determine the Fourier series coefficients of the output $y[n]$.

Solution

(1) $\omega_0 = 2\pi/T = 12$, and

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{H(j\omega)} y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

since $y(t) = x(t)$, there must be:

$$\forall a_k \neq 0, k \in Z, |k\omega_0| \leq 100$$

This implies that $|k| \leq 8$. Therefore, for $|k| > 8$, a_k is guaranteed to be 0.

(2) The frequency response of the system may be evaluated as

$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

For $x[n]$, $N = 4$ and $\omega_0 = \pi/2$. The FS coefficients of the input $x[n]$ are

$$a_k = \frac{1}{4}, \text{for all n.}$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4}[1 - e^{jk\pi/2} + e^{-jk\pi/2}].$$