

CS243: Introduction to Algorithmic Game Theory

Week 9.2 Cost Sharing and Public Goods (Dengji ZHAO)

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Recap: Coalitional/Cooperative Game

- A set of agents N .
- Each subset of agents (**coalition**) $S \subseteq N$ cooperate together can generate some value $v(S) \in \mathbb{R}$. Assume $v(\emptyset) = 0$. N is called **grand coalition**. $v : 2^N \rightarrow \mathbb{R}$ is called the **characteristic function** of the game.
- The possible outcomes of the game is defined by $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$.

Recap: Core and Shapley Value

Definition (Core)

The **core** of the coalitional game (N, v) is a set of vectors $x \in \mathbb{R}^N$ such that x is efficient and $\forall S \subseteq N \sum_{i \in S} x_i \geq v(S)$.

Definition (Shapley Value)

Given a coalitional game (N, v) , the **Shapley value** of each player i is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S))$$

Recap: Cost Sharing

Definition

A **cost sharing** game (N, c) is defined by

- a set of n agents N .
- a cost function $c : 2^N \rightarrow \mathbb{R}$ and assume $c(\emptyset) = 0$.

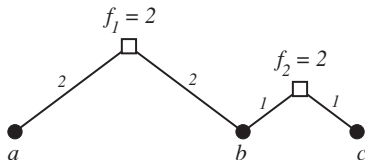


Figure 15.1. An example of the facility location game.

- $c(\{a\}) = 4, c(\{b\}) = 3, c(\{c\}) = 3$
- $c(\{a, b\}) = 6, c(\{b, c\}) = 4, c(\{a, c\}) = 7, c(\{a, b, c\}) = 8$

Public Goods

- A set of agents N .
- These agents have to choose to whether to produce some **indivisible, nonexcludable public good**. We denote the decision by $g \in \{0, 1\}$.
- There is a commonly known cost c to build the public good.
- Each agent i has a (**private**) valuation v_i for the public good.

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- How much will each agent share the cost?
 - Can we apply VCG here?

Application of VCG

Assume that the decision is to build the public good, i.e. $\sum_{i \in N} v_i > c$. What is the payment for i if

- $\sum_{j \neq i} v_j > c$
- $\sum_{j \neq i} v_j \leq c$

Application of VCG

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Assume that the decision is to **NOT** build the public good, i.e.

$\sum_{i \in N} v_i \leq c$. What is the payment for i if

- $\sum_{j \neq i} v_j \leq c$
 - Quiz: is the payment (1) > 0 , (2) $= 0$, (3) < 0 ?
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Question

Can their VCG payments cover the cost of the public good?

IC, IR, Budget Balanced Mechanisms

In the public good setting, a mechanism is **budget balanced** if

- the total payments of all agents is not less than the cost to build the public good, if the decision is to build the good.
- the total payments of all agents is non-negative, if the decision is to not build the good.

Theorem

*When there are only two agents, the only mechanisms that are IC, IR and budget balanced are **fixed-price mechanisms**. (It does not hold for more than three agents settings) [Tilman Borgers, 2015]*

For example: when the public good is built, agent 1 has to pay p_1 and agent 2 has to pay p_2 , where $p_1 + p_2 = c$ and p_1 and p_2 are independent of their valuations.

Cost Sharing of Excludable Good Production

- A set of agents N .
- There is a good can be produced with a cost c .
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- The good can be shared by a subset of agents.

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Question

How to design an IC, IR and budget balanced mechanism?

One Example [Moulin and Shenker, 2001]

- Find the largest k such that the highest k agents' valuation reports are at least c/k .
- Charge these k agents c/k and reject all others, i.e. the good is only shared by the k agents.

Other Deficit Issues of VCG

- Bilateral Trade (Double Auctions)

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Bilateral trade:

- One buyer and one seller trade one item.
- The seller's valuation is v_s and the buyer's valuation is v_b .
- The possible allocations are {Trade, NoTrade}.

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Question

What will happen if we apply VCG?

- If $v_b \leq v_s$, NoTrade
- if $v_b > v_s$, Trade, and [Quiz] their payments are?

Advanced Reading

- Tilman Borgers, *an introduction to the theory of mechanism design*, 2015.