Properties of Series RLC Network - *v(t)*

- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - α determines the rate of damping

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

– $\alpha > \omega_0$, overdamped

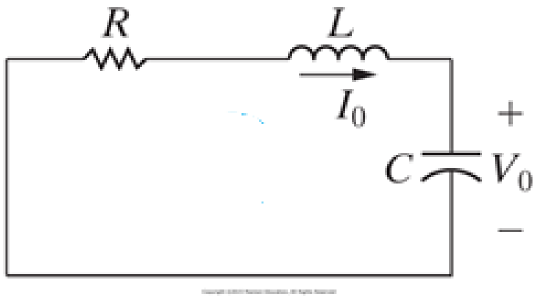
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

– $\alpha = \omega_0$, critically damped

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

– $\alpha < \omega_0$, underdamped

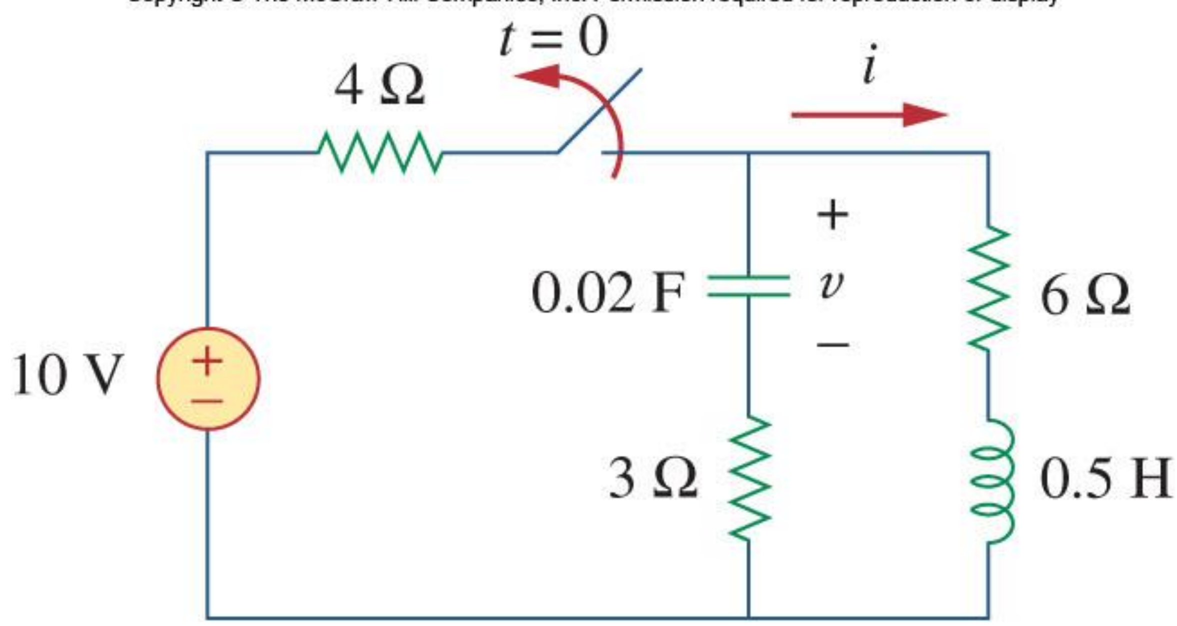
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$



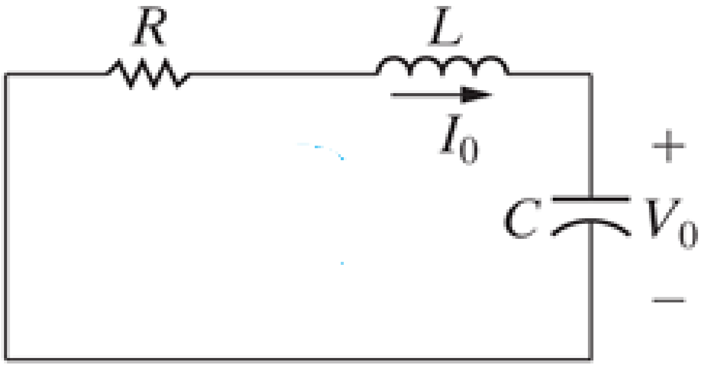
[Source: Berkeley]

Example

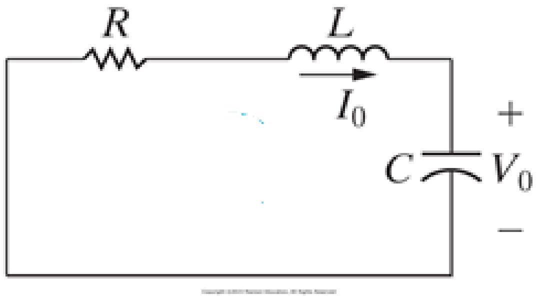
- Find $v(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.



Source-Free Series RLC Circuit



Properties of Series RLC Network - *i(t)*



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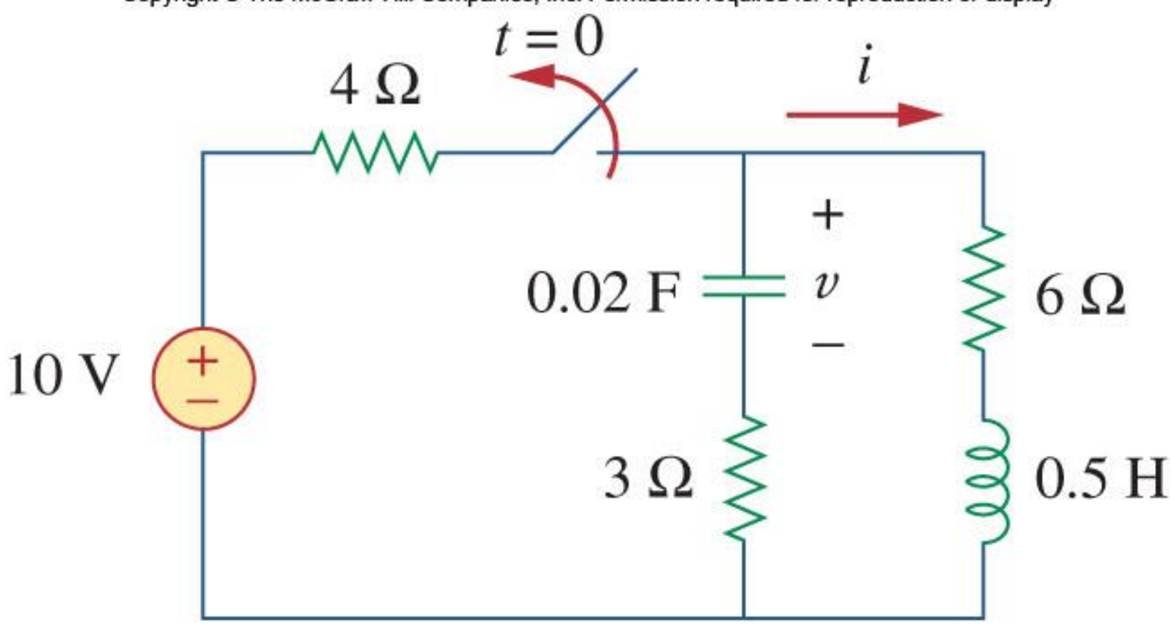
$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

[Source: Berkeley]

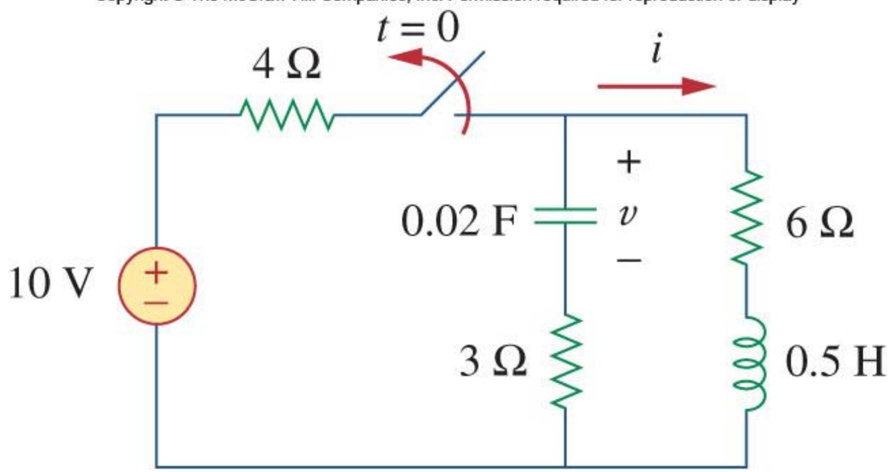
Example

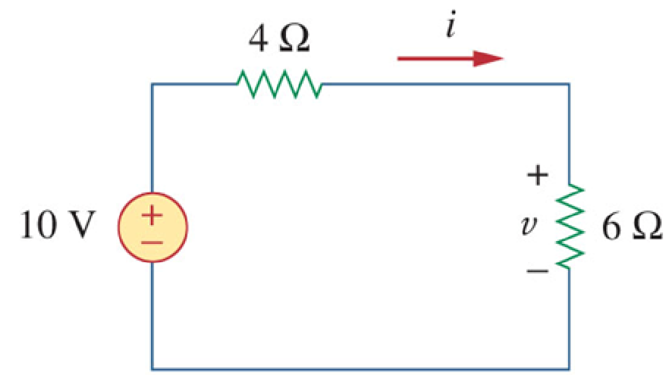
- Find $\boldsymbol{i(t)}$ in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.

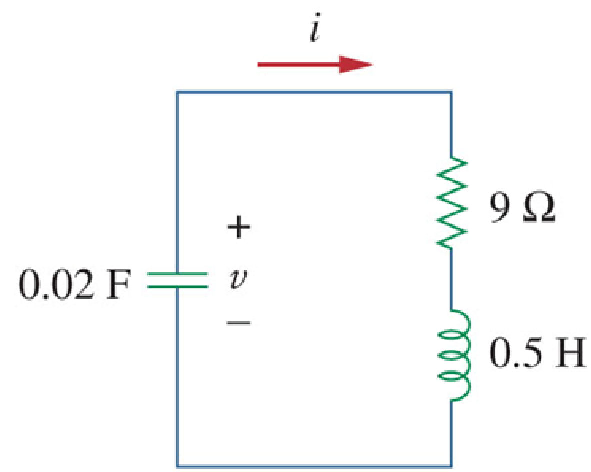


Example

- Find $i(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.



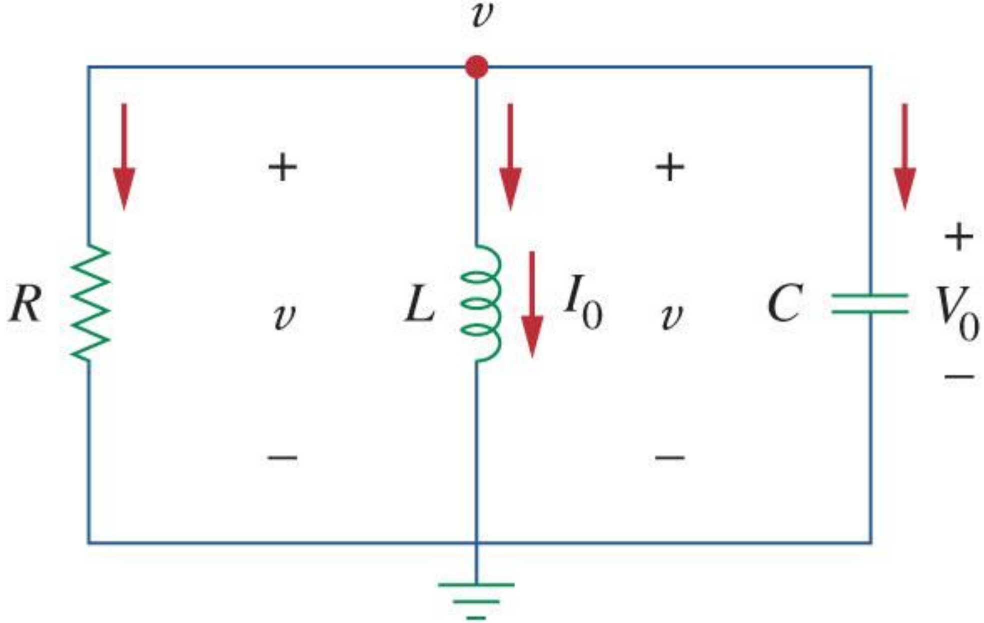




$$\alpha = \frac{R}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

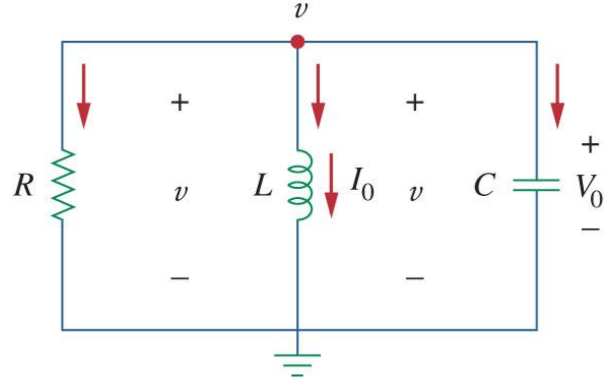
$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = -9 \mp j4.359$$

Source-Free Parallel RLC Network



Source-Free Parallel RLC Network - *v(t)*

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$



- The characteristic equation is:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

From this, we can find the roots of the characteristic equation to be:

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

As in last time, there are three scenarios to consider.

Three Damping Cases - *v(t)*

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

- For the overdamped case, the roots are real and negative,

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- For critically damped, the roots are real and equal

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$

- In the underdamped case, the roots are complex

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Three Damping Cases - *i(t)*

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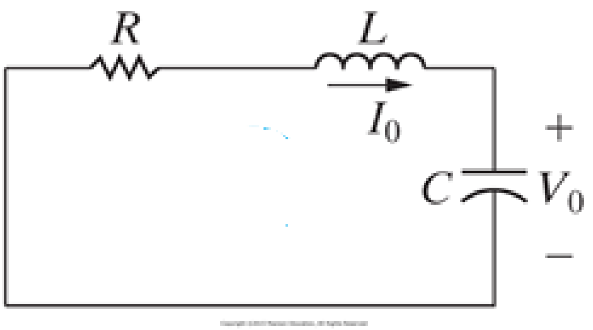
$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Series vs. Parallel (Source-Free RLC Circuit)

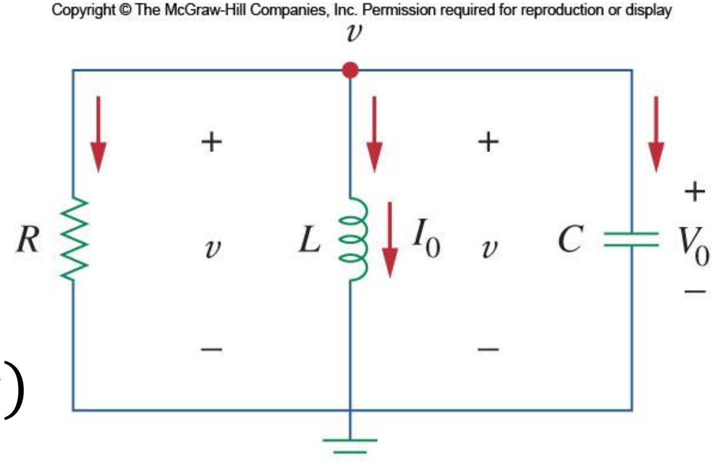
- Series $\alpha = \dfrac{R}{2L} \quad \omega_0 = \dfrac{1}{\sqrt{LC}}$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

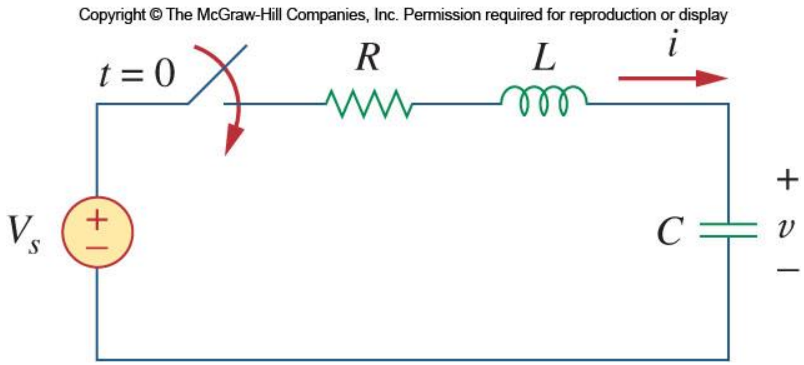


- Parallel $\alpha = \dfrac{1}{2RC} \quad \omega_0 = \dfrac{1}{\sqrt{LC}}$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$



Step Response of a Series RLC Circuit



$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

- The solution

$$v(t)=v_t(t)+v_{ss}(t)$$

- The complete solutions for the three conditions of damping are:

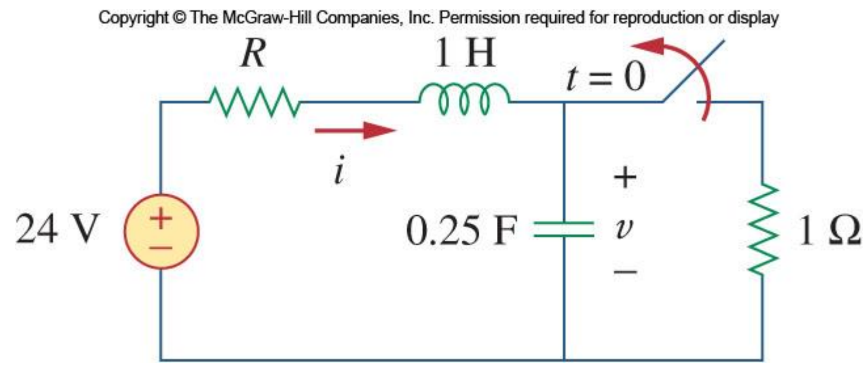
$$v(t)=V_s+(A_1e^{s_1t}+A_2e^{s_2t})\ \text{(Overdamped)}$$

$$v(t)=V_s+(A_1+A_2t)e^{-\alpha t}\ \text{(Critically Damped)}$$

$$v(t)=V_s+(B_1\cos\omega_d t+B_2\sin\omega_d t)e^{-\alpha t}\ \text{(Underdamped)}$$

Example

- Find $v(t)$ and $i(t)$ for $t > 0$.
  Consider three cases:
  - $R = 5\Omega$
  - $R = 4\Omega$
  - $R = 1\Omega$



When $R = 5\Omega$,

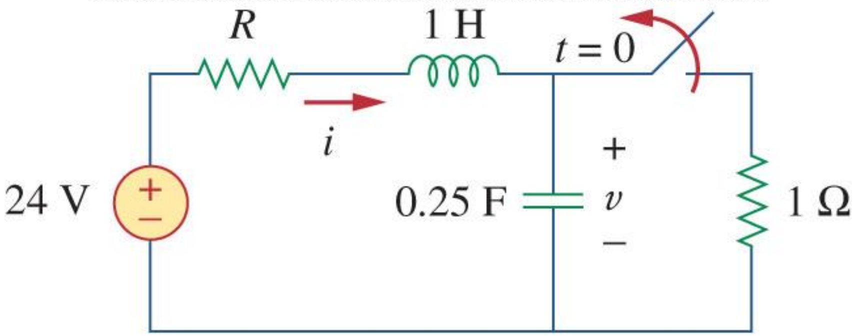
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4A = C\frac{dv(0)}{dt},\ \ v(0) = 4V,\ \ \frac{dv(0)}{dt} = 16$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2.5,\ \ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ \ s_{1,2} = -1, -4 \quad \text{Overdamped.}$$

$$v(t) = v_s + \left(A_1 e^{-t} + A_2 e^{-4t}\right)$$



When $R = 4\Omega$,

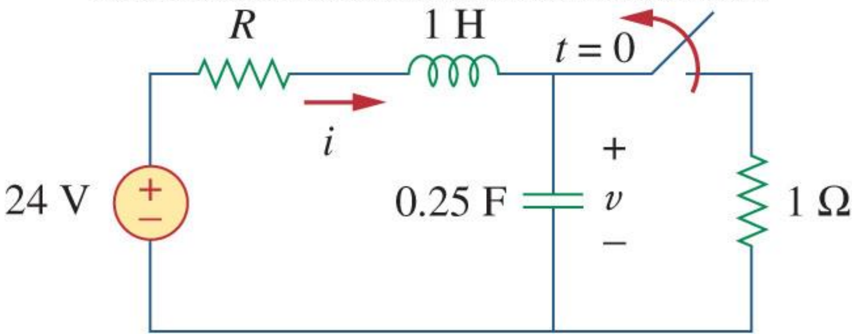
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4.8A = C\frac{dv(0)}{dt}, \;\; v(0) = 4.8V, \;\; \frac{dv(0)}{dt} = 19.2$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2, \;\; \omega_0 = \frac{1}{\sqrt{LC}} = 2, \;\; s_{1,2} = -2$$ Critically damped

$$v(t) = v_s + (A_1 + A_2 t)e^{-2t}$$



When $R = 1\Omega$,

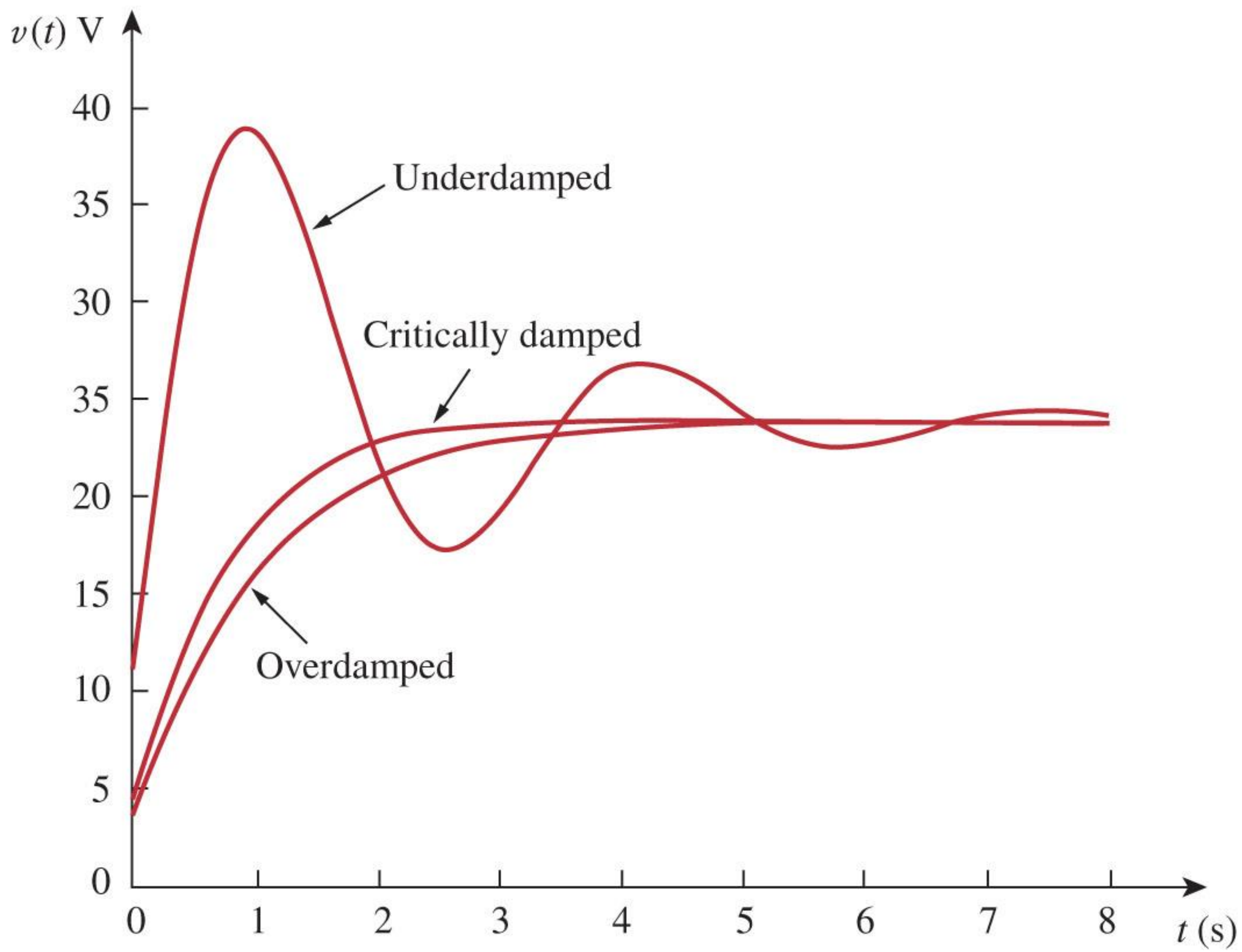
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 12A = C\frac{dv(0)}{dt}, \;\; v(0) = 12V, \;\; \frac{dv(0)}{dt} = 48$$

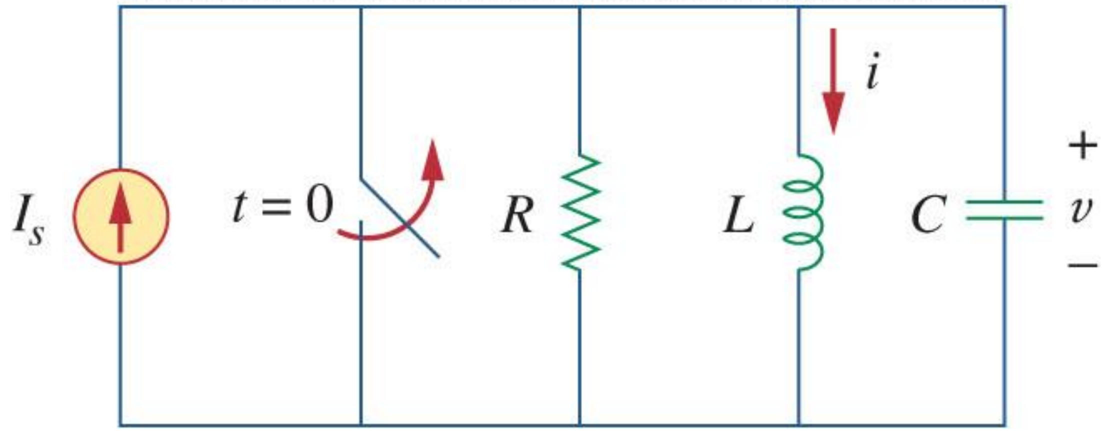
- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 0.5, \;\; \omega_0 = \frac{1}{\sqrt{LC}} = 2, \;\; s_{1,2} = -0.5 \pm j1.936 \quad \text{Underdamped}$$

$$v(t) = v_s + (B_1 \cos 1.936t + B_2 \sin 1.936t)e^{-0.5t}$$


v(t) V
40
35
30
35
20
15
10
5
0
Underdamped
Critically damped
Overdamped
0
1
2
3
4
5
6
7
8
t (s)

Step Response of a Parallel RLC Circuit



Apply KCL,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

& $\quad v = L\frac{di}{dt}$

So we get

$$\frac{d^2 i}{d^2 t} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

Step Response of a Parallel RLC Circuit

$$\frac{d^2i}{d^2t}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$$

- The total response is a combination of steady state responses and transient response:

$i(t)=I_s+(A_1e^{s_1t}+A_2e^{s_2t})$ (Overdamped)

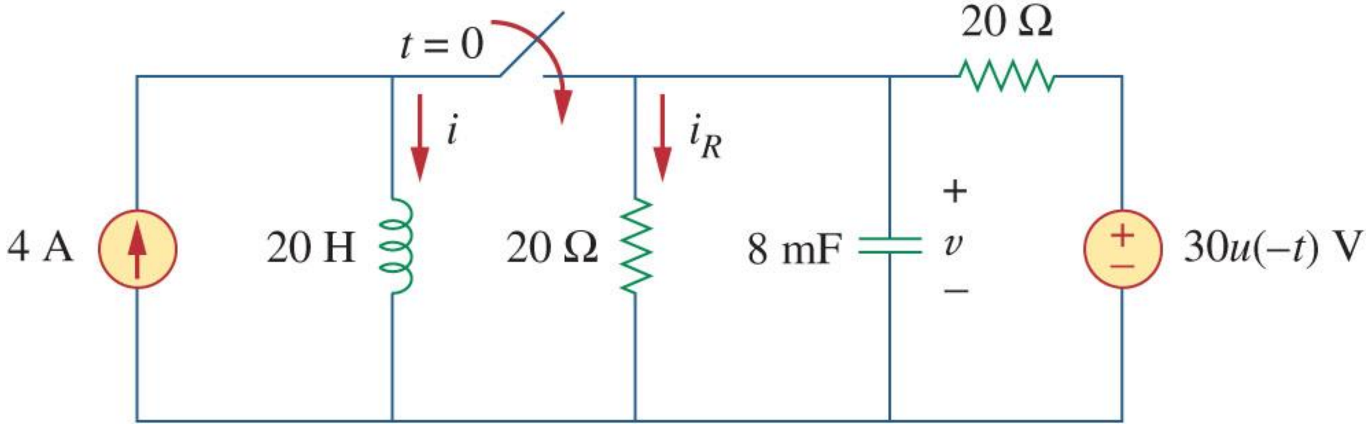
$i(t)=I_s+(A_1+A_2t)e^{-\alpha t}$ (Critically Damped)

i$(t)=I_s+(B_1\cos\omega_d t+B_2\sin\omega_d t)e^{-\alpha t}$ (Underdamped)

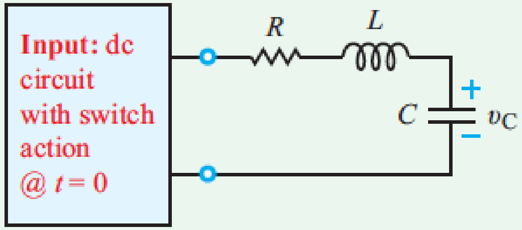
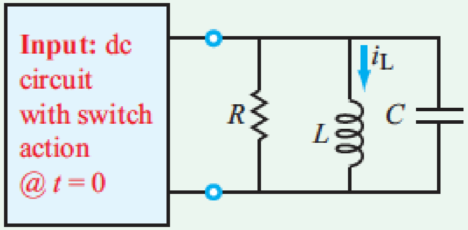
Here the variables A_1/A_2 B_1/B_2 are obtained from the initial conditions, *i(0)* and *di(0)/dt*.

Example

- Find $i(t)$ and $i_R(t)$ for $t > 0$.



Step response of RLC circuits for $t \geq 0$.

Series RLC	Parallel RLC
	

Total Response

Series RLC	Parallel RLC
Overdamped ($\alpha > \omega_0$)	**Overdamped** ($\alpha > \omega_0$)
$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty)$	$i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + i_L(\infty)$
$A_1 = \dfrac{\frac{1}{C}\, i_C(0) - s_2[v_C(0) - v_C(\infty)]}{s_1 - s_2}$	$A_1 = \dfrac{\frac{1}{L}\, v_L(0) - s_2[i_L(0) - i_L(\infty)]}{s_1 - s_2}$
$A_2 = \left[\dfrac{\frac{1}{C}\, i_C(0) - s_1[v_C(0) - v_C(\infty)]}{s_2 - s_1}\right]$	$A_2 = \left[\dfrac{\frac{1}{L}\, v_L(0) - s_1[i_L(0) - i_L(\infty)]}{s_2 - s_1}\right]$
Critically Damped ($\alpha = \omega_0$)	**Critically Damped** ($\alpha = \omega_0$)
$v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$	$i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$
$B_1 = v_C(0) - v_C(\infty)$	$B_1 = i_L(0) - i_L(\infty)$
$B_2 = \frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]$	$B_2 = \frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]$
Underdamped ($\alpha < \omega_0$)	**Underdamped** ($\alpha < \omega_0$)
$v_C(t) = e^{-\alpha t}(D_1 \cos \omega_d t + D_2 \sin \omega_d t) + v_C(\infty)$	$i_L(t) = e^{-\alpha t}(D_1 \cos \omega_d t + D_2 \sin \omega_d t) + i_L(\infty)$
$D_1 = v_C(0) - v_C(\infty)$	$D_1 = i_L(0) - i_L(\infty)$
$D_2 = \dfrac{\frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d}$	$D_2 = \dfrac{\frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]}{\omega_d}$

Auxiliary Relations

$$\alpha = \begin{cases} \dfrac{R}{2L} & \text{Series RLC} \\ \dfrac{1}{2RC} & \text{Parallel RLC} \end{cases} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$