Linear Time-Varying Differential Equations

- Introduction
- Fundamental Solutions
- Periodic Orbits

Contents

Linear Time-Varying Differential Equations

Let $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $b : \mathbb{R} \to \mathbb{R}^{n_x}$ be integrable functions.

A differential equation of the form

$$\dot{x}(t) \ = \ A(t)x(t) + b(t) \quad \text{with} \quad x(0) = x_0$$

is called a linear time-varying system.

Important:

- The function x does not have to be differentiable!
- Solutions are understood in the weak sense.
- Example: $A(t) = 0$, $b(t) = \mathrm{sgn}(t)$, and $x(0) = 0$ implies $x(t) = |t|$.

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Integral Form

The so-called integral form is given by

$$x(t) \;=\; x_0 + \int_0^t \left[A(\tau)x(\tau) + b(\tau) \right] \, \mathrm{d}\tau \; .$$

- Any solution x of the differential equation is also a solution of the integral equation and vice versa.
- Advantage: no derivatives needed; mathematically "cleaner" syntax.

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Uniqueness of Solutions

We proceed as in the linear time-invariant case: if x_1 and x_2 are solutions, the difference function $y = x_1 - x_2$ satisfies

$$\dot{y}(t) = A(t)y(t) \quad \text{with} \quad y(0) = 0$$

Denote with σ and upper bound of A on compact interval $0 \in I \subseteq \mathbb{R}$, $\|A(t)\|_2 \leq \sigma$ for all $t \in I$. The auxiliary function

$$v(t) = e^{-2\sigma|t|}\|y(t)\|_2^2 \geq 0 \qquad \text{satisfies}$$

$$\begin{aligned} \forall t > 0, \quad \dot{v}(t) &= 2e^{-2\sigma|t|}y(t)^\intercal \left[A(t) - \sigma I\right] y(t) \leq 0 \\ \forall t < 0, \quad \dot{v}(t) &= 2e^{-2\sigma|t|}y(t)^\intercal \left[A(t) + \sigma I\right] y(t) \geq 0 \, ; \end{aligned}$$

only possible, if we have $v(t) = 0$ for all $t \in I$, so $x_1(t) = x_2(t)$.

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Existence of Solutions

The differential equation

$$\dot{y}(t) = A(t)y(t) + b(t) \quad \text{with} \quad y(0) = 0$$

has a solution if the functions A and b are integrable and bounded. This can be proven by using Picard iterations (Details: see lecture notes).

Scalar Case

If we have only one differential state, $n_x = 1$, the matrix $a(t) = A(t)$ is a 1×1 matrix. In the offset-free case:

$$\dot{x}(t) = a(t)x(t) \quad \text{with} \quad x(0) = x_0 \; .$$

- If we assume $x(t) \neq 0$ for all t,

$$\int_0^t a(\tau)\,\mathrm{d}\tau \;=\; \int_0^t \frac{\dot{x}(\tau)}{x(\tau)}\,\mathrm{d}\tau \;=\; \log(x(t)) - \log(x(0)) \; ,$$

- Solution can be found by eliminating $x(t)$,

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Differential equation of the Gaussian function

Example:

$$\dot{x}(t) = -tx(t) \quad \text{with} \quad x(0) = 1 \; .$$

The solution

$$\forall t \in \mathbb{R}, \qquad x(t) \;=\; e^{-\int_0^t \tau \, \mathrm{d}\tau} \;=\; e^{-\frac{1}{2}t^2} \; ,$$

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If we have a scalar time-varying system with offset,

$$\dot{x}(t) = a(t)x(t) + b(t) \quad \text{with} \quad x(0) = x_0 \ ,$$

the unique solution is given by

$$x(t) = e^{\int_0^t a(\tau)\,\mathrm{d}\tau} x_0 + \int_0^t e^{\int_\tau^t a(s)\,\mathrm{d}s}\, b(\tau)\,\mathrm{d}\tau \ .$$

Problem: For $n_x > 1$ the solution is in general NOT given by

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$$\dot{x}(t) \; = \; A(t)x(t) + b(t) \quad \text{with} \quad x(0) = x_0$$

Recall:

- In general no explicit solution possible
- in theory: we can construct solution by Picard iteration only

Question: Are there “tools” that help us to understand/discuss the behavior of solution trajectories?

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Fundamental solutions

Idea: consider the linear time-varying differential equations

$$\frac{\partial}{\partial t} G(t,\tau) = A(t) G(t,\tau) \quad \text{with} \quad G(\tau,\tau) = I$$

for all $t, \tau \in \mathbb{R}$.

- We don't have explicit expressions for $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$
- but at least G neither depends on b nor x_0
- for constant functions A we have $G(t,\tau) = e^{A(t-\tau)}$.

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Fundamental solutions

The solution of the original differential equation can be written as

$$x(t) = G(t,0)x_0 + \int_0^t G(t,\tau)b(\tau)\,\mathrm{d}\tau \ .$$

Proof:

$$\begin{aligned}\dot{x}(t) &= A(t)G(t,0)x_0 + \int_0^t A(t)G(t,\tau)b(\tau)\,\mathrm{d}\tau + b(t) \\ &= A(t)x(t) + b(t) \qquad \text{and} \qquad x(0) = G(0,0)x_0 = x_0 \ .\end{aligned}$$

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Properties of the fundamental solution

- If $A(t) = \bar{A}$ is constant, then $G(t, \tau) = e^{\bar{A}(t-\tau)}$.
- We have $G(t_3, t_2)G(t_2, t_1) = G(t_3, t_1)$ for all $t_1, t_2, t_3 \in \mathbb{R}$.
- The function G is invertible for all $t, \tau \in \mathbb{R}$, $G(t, \tau)^{-1} = G(\tau, t)$.

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Periodic Systems

Time-varying differential equations do typically not admit a steady state, as it is "unlikely" that we find one constant vector $x_{\mathsf{s}} \in \mathbb{R}^{n_x}$ with

$$A(t)x_{\mathsf{s}} + b(t) = 0$$

for all $t \in \mathbb{R}$. However, if A and b are periodic,

$$A(t+T) = A(t) \qquad \text{and} \qquad b(t+T) = b(t)$$

for all $t \in \mathbb{R}$, it often possible to find periodic solutions $x_{\mathrm{p}} : \mathbb{R} \to \mathbb{R}^{n_x}$ such that

$$x_{\mathrm{p}}(t+T) = G(t+T,t)x_{\mathrm{p}}(t) + \int_t^{t+T} G(t+T,\tau)b(\tau)\,\mathrm{d}\tau = x_{\mathrm{p}}(t)\,.$$

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Construction of periodic orbits

If we assume for a moment that $I - G(t+T, t)$ is invertible, then

$$x_{\mathrm{p}}(t) = [I - G(t+T,t)]^{-1} \left(\int_t^{t+T} G(t+T,\tau) b(\tau)\, \mathrm{d}\tau \right)$$

for all $t \in \mathbb{R}$. Notice that

$$\begin{aligned} I - G(t+T,t) &= I - G(t+T,T) G(T,0) G(0,t) \\ &= I - G(t,0) G(T,0) G(0,t) \\ &= G(t,0) \left[I - G(T,0) \right] G(t,0)^{-1} \ . \end{aligned}$$

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Monodromy matrix

The matrix $G(T,0)$ is called the *monodromy matrix*.

Equivalent statements:

- $I - G(t+T,t)$ is invertible
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Shifted state trajectory

The shifted state trajectory $y(t) = x(t) - x_{\mathrm{p}}(t)$ satisfies

$$\dot{y}(t) = A(t)y(t) \qquad \text{with} \qquad y(0) = y_0 = x_0 - x_{\mathrm{p}}(0)\ .$$

The function y can be written as

$$\begin{aligned} y(t+NT) &= G(t+NT,0)\, y_0 = G(t,0)\, G(NT,0)\, y_0 \\ &= G(t,0)\, G(T,0)^N\, y_0 \end{aligned}$$

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Limit behavior for $t \to \infty$

If the (potentially complex-valued) eigenvalues of the monodromy matrix $G(T,0)$ are all contained in the open unit disk in $\mathbb{C}$, we have

$$\lim_{N\to\infty} G(T,0)^N = 0 \; .$$

This implies $\lim_{t\to\infty} y(t) = 0$.

- If the eigenvalues of the monodromy matrix $G(T,0)$ are all different from 1, then there exist a unique periodic orbit x_{p}.
- If the eigenvalues of the monodromy matrix $G(T,0)$ are all contained in the open unit disk in $\mathbb{C}$, then

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