

The Continuous-Time Fourier Transform

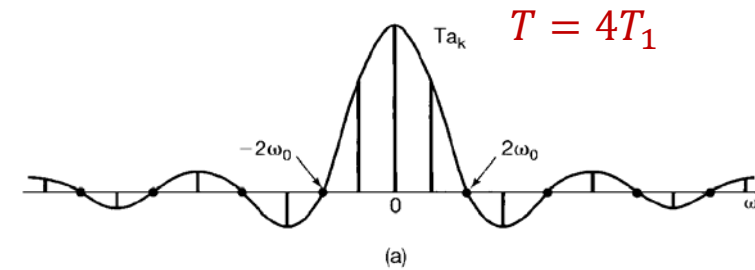
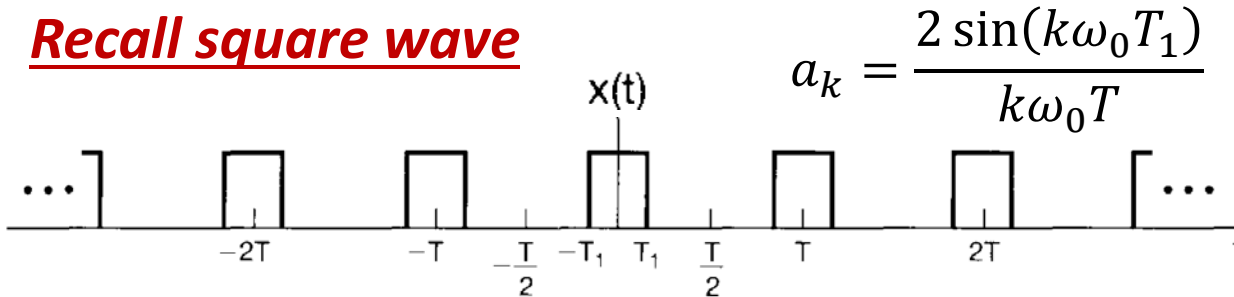
(ch.4)

- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations

Continuous Fourier Transform

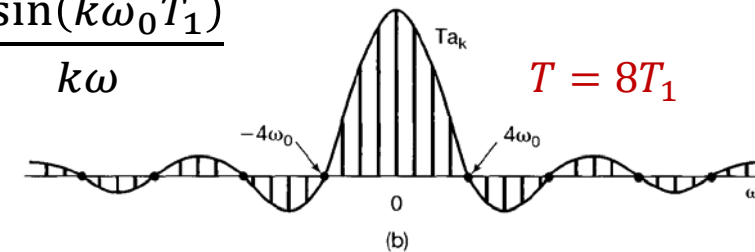


Recall square wave



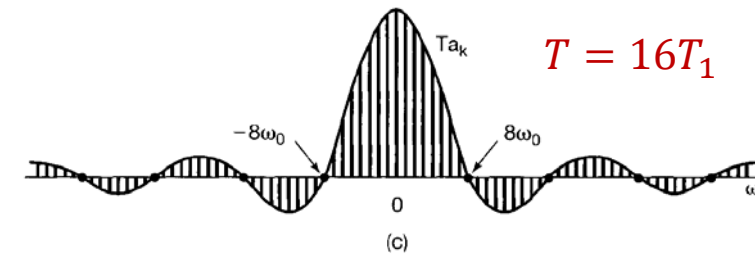
□ Ta_k : Samples of an envelope function $f(\omega) = \frac{2 \sin(k\omega_0 T_1)}{k\omega}$

$$Ta_k = \left. \frac{2 \sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$



□ $T \uparrow, \omega_0 \downarrow \Rightarrow$ the envelope is sampled with closer spacing

□ $T \rightarrow \infty, \Rightarrow Ta_k \rightarrow$ the envelope $\frac{2 \sin(k\omega_0 T_1)}{k\omega}$

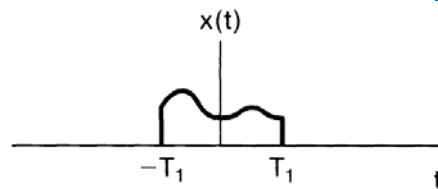


Continuous Fourier Transform

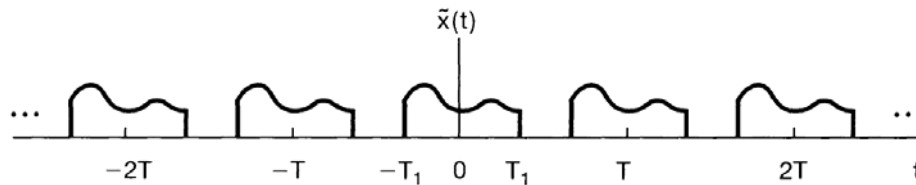


Development of FT

- Consider a signal of finite duration, $x(t) = 0$ if $|t| > T_1$



- Periodic extension of $x(t)$ with T



- FS representation of $\hat{x}(t)$
- $$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
- $$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

Continuous Fourier Transform

Development of FT

FS coefficients of $\hat{x}(t)$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} X(jk\omega_0)$$

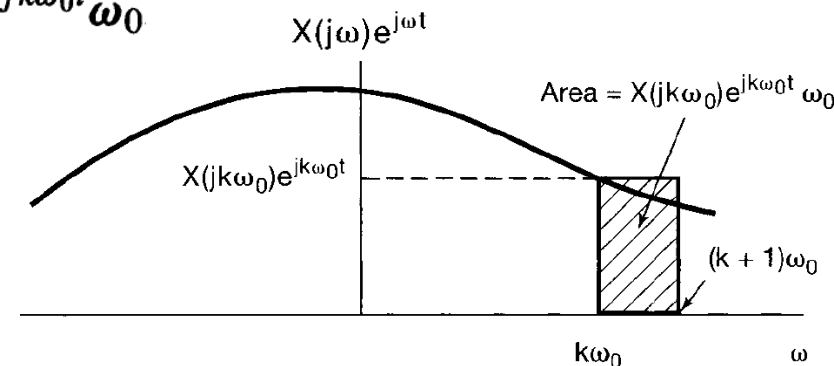
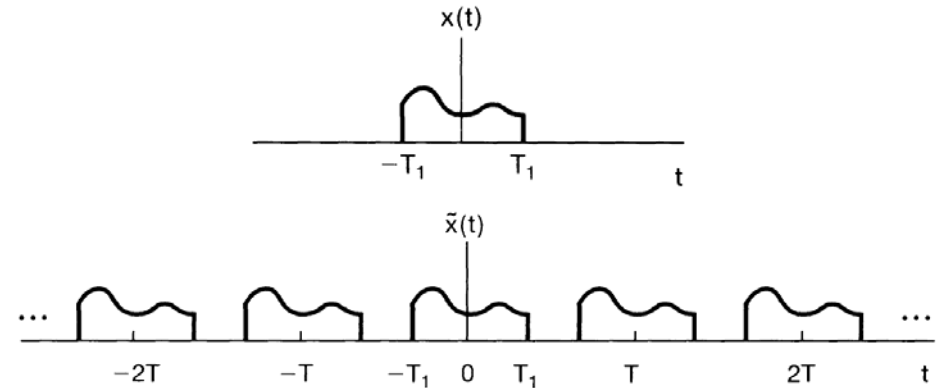
Define $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

FS of $\hat{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$T \rightarrow \infty, \hat{x}(t) \rightarrow x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$



Continuous Fourier Transform



FT pairs

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt \quad \text{Fourier transform (FT)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{Inverse Fourier transform}$$

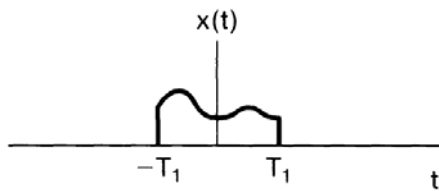
- $x(t)$ is a linear combination (specifically, an integral) of sinusoidal signals at different frequencies
- $X(j\omega)(d\omega/2\pi)$ is the weight for different frequencies
- $X(j\omega)$ is called the spectrum

Continuous Fourier Transform



FT vs. FS

Fourier transform (FT)

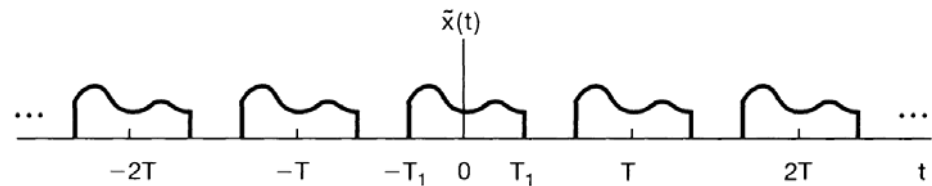


$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$a_k = \frac{1}{T} X(j\omega) \text{ with } \omega = k\omega_0$$

Fourier series (FS)



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

Continuous Fourier Transform



Convergence of FT

□ Condition 1: Finite energy condition

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

□ Condition 2: Dirichlet condition

(1) Absolutely integrable $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

(2) Finite maxima and minima in one period with in any finite interval

(3) Finite number of finite discontinuities in any finite interval

Continuous Fourier Transform



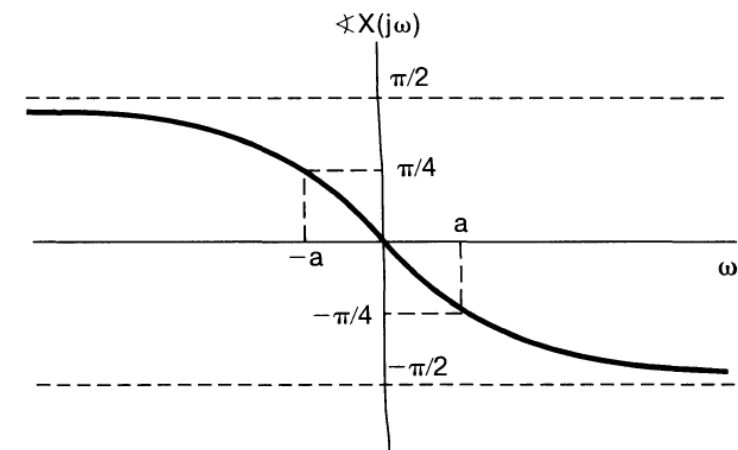
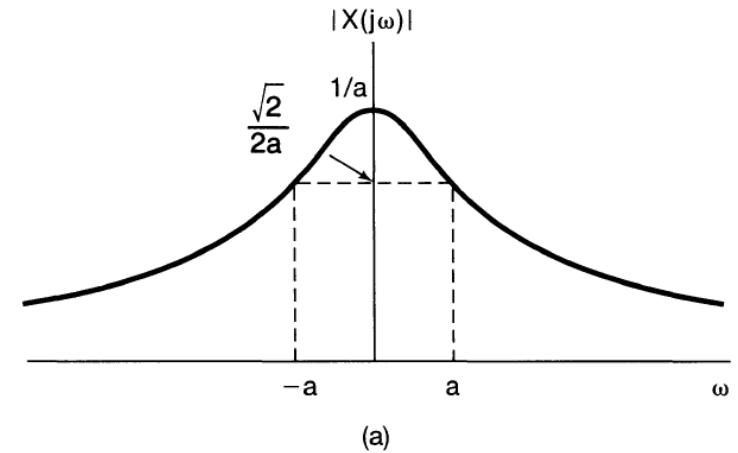
Examples

Consider the signal $x(t) = e^{-at}u(t), a > 0$

Determine its FT

$$\begin{aligned} X(j\omega) &= \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} \\ &= \frac{1}{a + j\omega}, a > 0 \end{aligned}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \quad \angle X(j\omega) = -\tan^{-1} \left(\frac{a}{\omega} \right)$$



Continuous Fourier Transform

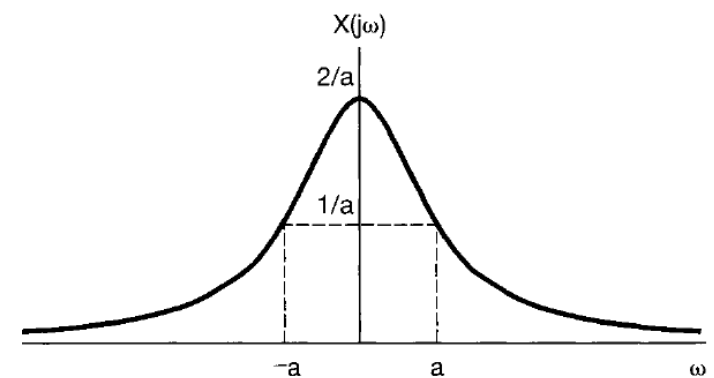
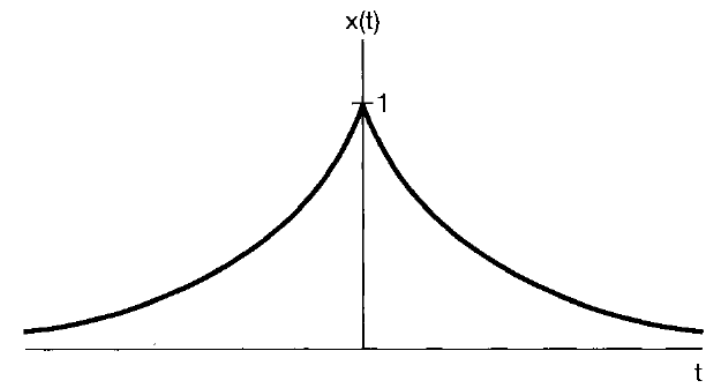


Examples

$$x(t) = e^{-a|t|}, a > 0 \quad X(j\omega) = ?$$

Solution

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$



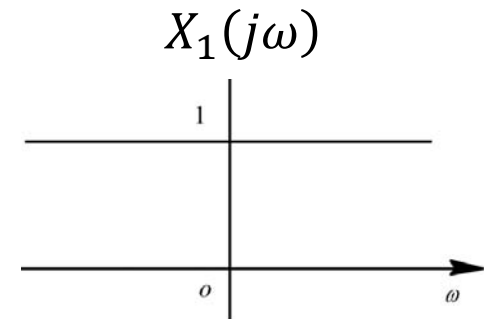
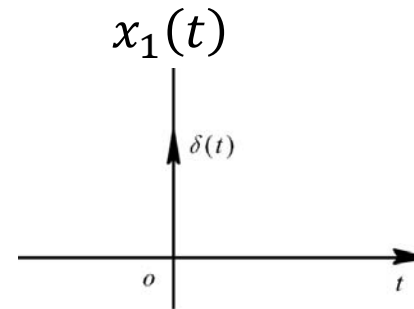
Continuous Fourier Transform



Examples

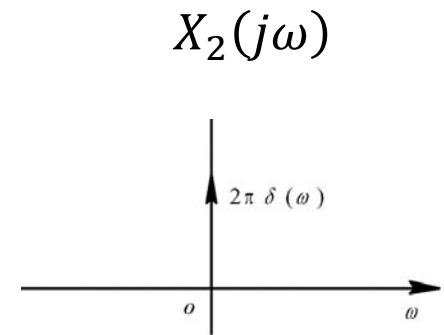
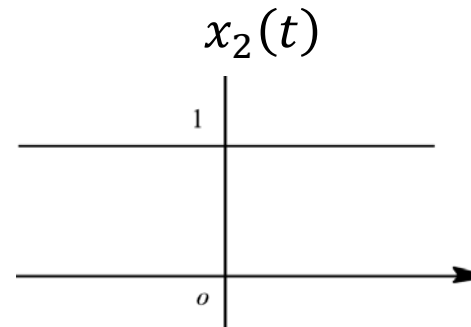
□ $x_1(t) = \delta(t) \quad X_1(j\omega) = ?$

$$X_1(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$



□ $x_2(t) = 1 \quad X_2(j\omega) = ?$

$$X_2(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(\omega)$$



Hints: $\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega \Rightarrow \delta(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t} dt$

Continuous Fourier Transform

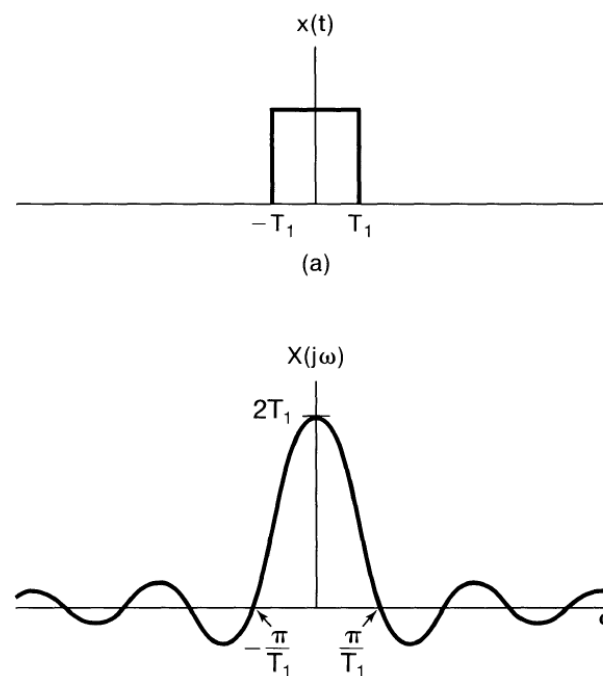


Examples

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad X(j\omega) = ?$$

Solution

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2 \frac{\sin \omega T_1}{\omega}$$



Continuous Fourier Transform



Examples

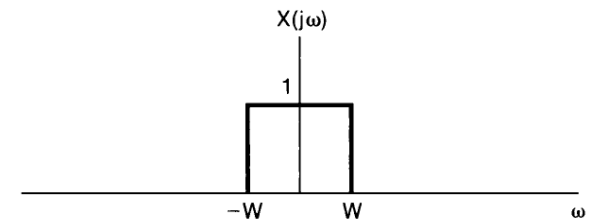
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \quad x(t) = ?$$

Solution

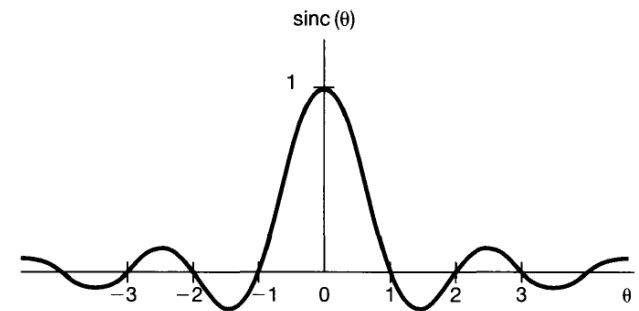
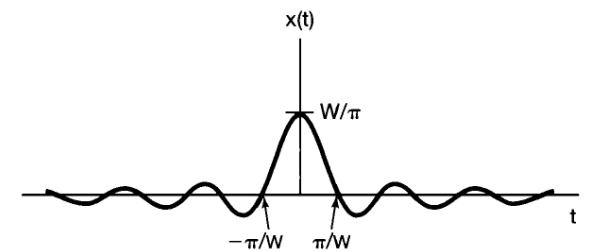
$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-W}^W e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t} \end{aligned}$$

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \frac{\sin Wt}{Wt} = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right)$$



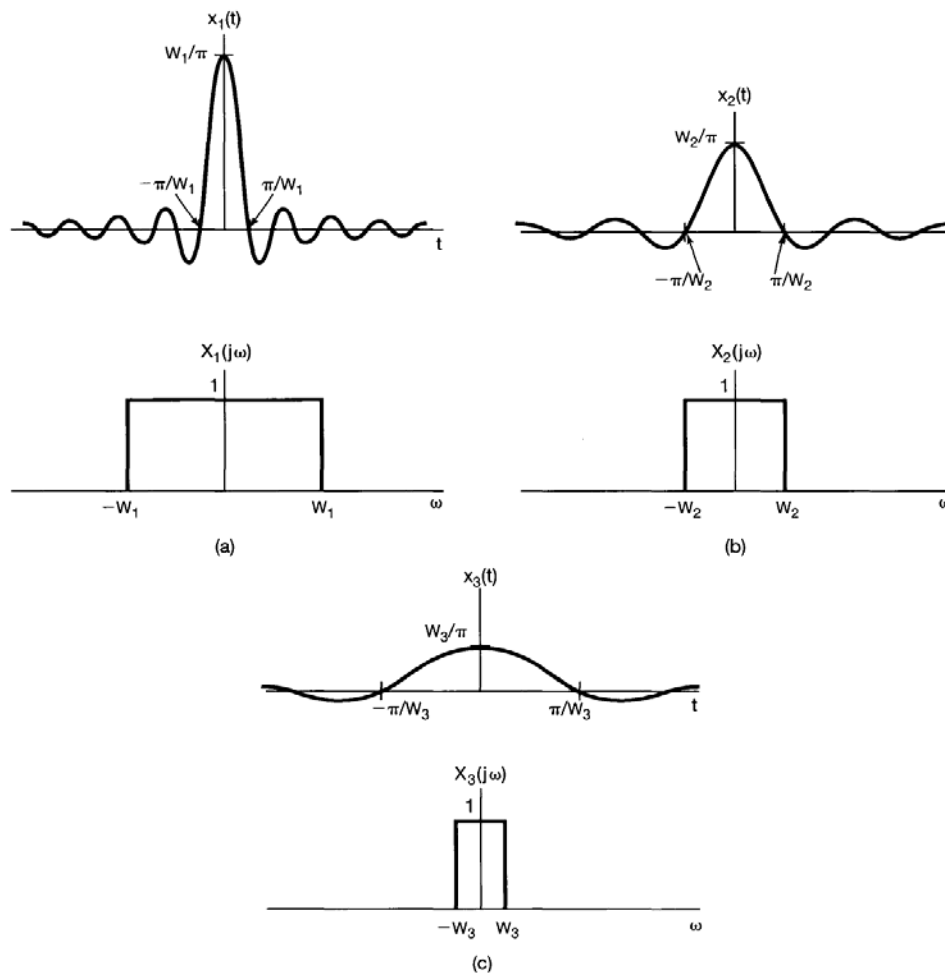
(a)



Continuous Fourier Transform



Examples



The Continuous-Time Fourier Transform (ch.4)

- ☐ Representation of aperiodic signals- Continuous Fourier Transform
- ☒ Fourier transform for periodic signals
- ☐ Properties of continuous-time Fourier Transform
- ☐ The convolution property
- ☐ The multiplication property
- ☐ System characterized by differential equations

Fourier transform for periodic signals



- A period signal can be represented by a FS, but also a FT

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

- The relationship between a_k and $X(j\omega)$?

➤ Consider $x_1(t) = a_k e^{jk\omega_0 t}$, whose FT is

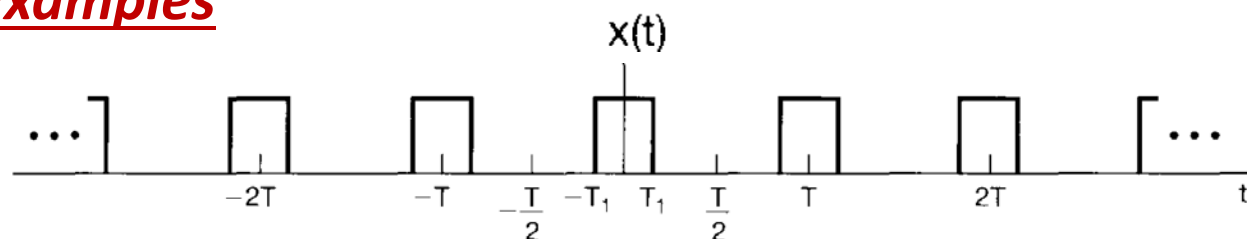
$$x_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\omega) e^{j\omega t} d\omega = a_k e^{jk\omega_0 t} \Rightarrow X_1(j\omega) = 2\pi a_k \delta(\omega - k\omega_0)$$

➤ For $x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $X(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0)$

Fourier transform for periodic signals



Examples

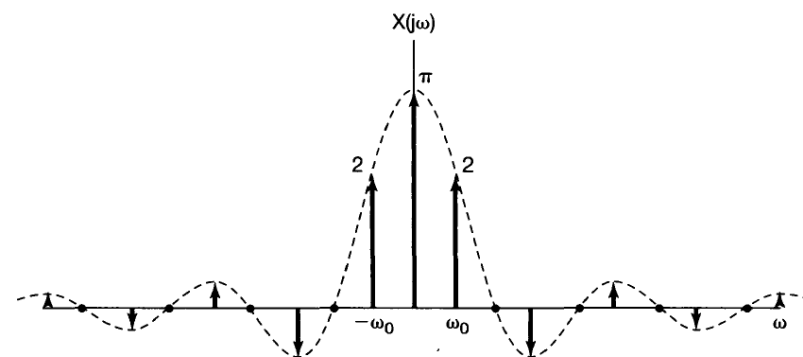


$X(j\omega)?$

Solution

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$\begin{aligned} X(j\omega) &= \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0) \\ &= \sum_{K=-\infty}^{\infty} \frac{2 \sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0) \end{aligned}$$



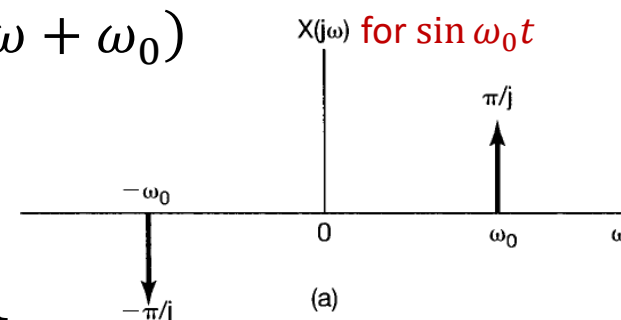
Fourier transform for periodic signals



Examples

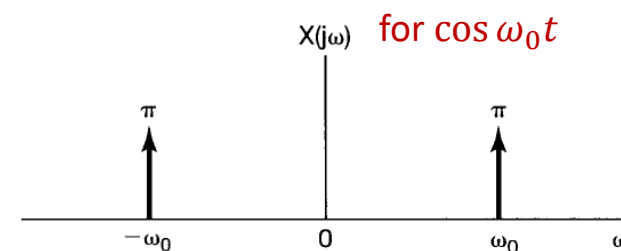
$$x_1(t) = \sin \omega_0 t \quad a_1 = 1/2j \quad a_{-1} = -1/2j \quad a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0) = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$



$$x_2(t) = \cos \omega_0 t \quad a_k = 1/2, k = \pm 1, a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$



Fourier transform for periodic signals

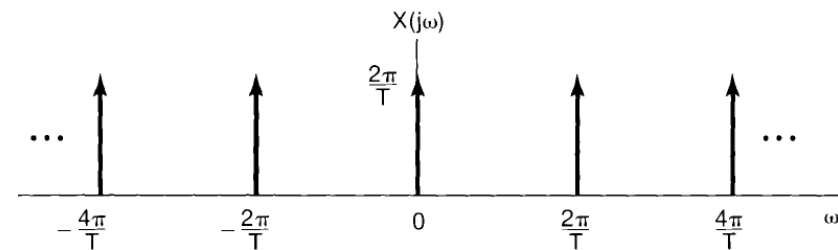
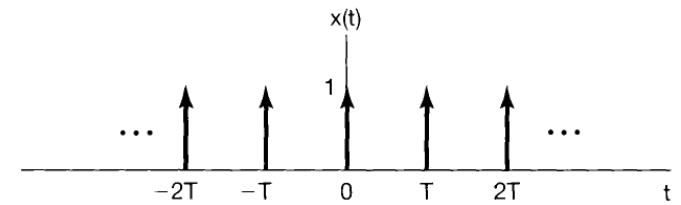


Examples

$$x(t) = \sum_{K=-\infty}^{\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$\begin{aligned} X(j\omega) &= \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_0) \\ &= \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2k\pi}{T}\right) \end{aligned}$$



The Continuous-Time Fourier Transform (ch.4)

- ☐ Representation of aperiodic signals- Continuous Fourier Transform
- ☐ Fourier transform for periodic signals
- ☒ **Properties of continuous-time Fourier Transform**
- ☐ The convolution property
- ☐ The multiplication property
- ☐ System characterized by differential equations



Properties of continuous-time Fourier Transform

Short notation for FT pairs

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt \quad X(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$



Properties of continuous-time Fourier Transform

Linearity

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega),$$



$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega).$$



Properties of continuous-time Fourier Transform

Time shifting

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow \boxed{x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega).}$$

□ proof

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(e^{-j\omega t_0} X(j\omega) \right) e^{j\omega t} d\omega$$

$$\mathcal{F}\{x(t)\} = X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\angle X(j\omega) - \omega t_0]}$$

□ A time shift on a signal introduces a phase shift into its FT, $-\omega t_0$, which is a linear function of ω .



Properties of continuous-time Fourier Transform

Examples

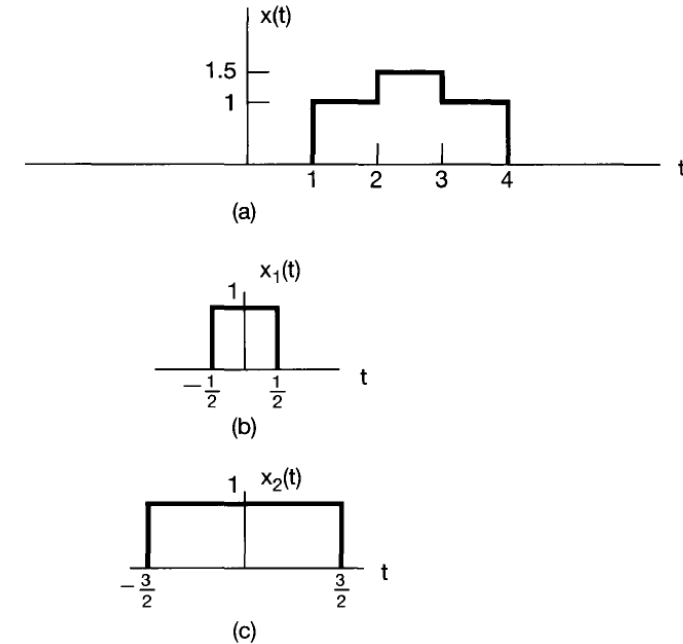
□ $x(t)$ can be expressed as

$$x(t) = \frac{1}{2}x_1(t - 2.5) + x_2(t - 2.5)$$

$$X_1(j\omega) = 2 \frac{\sin \omega T_1}{\omega} = 2 \frac{\sin \omega/2}{\omega}$$

$$X_2(j\omega) = 2 \frac{\sin 3\omega/2}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2} \left(\frac{\sin \omega/2 + 2 \sin 3\omega/2}{\omega} \right)$$





Properties of continuous-time Fourier Transform

Conjugation and Conjugate Symmetry

□ Conjugation property $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow \boxed{x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)}.$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right]^* = \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t) e^{-j\omega t} dt = \mathcal{F}\{x^*(t)\}$$

□ Conjugation Symmetry

$$\boxed{X(-j\omega) = X^*(j\omega) \quad [x(t) \text{ real}].} \Leftrightarrow \Re\{X(j\omega)\} = \Re\{X(-j\omega)\} \quad \Im\{X(j\omega)\} = -\Im\{X(-j\omega)\}$$

For a real-valued signal, the FT need only to be specified for positive frequencies



Properties of continuous-time Fourier Transform

Time reversing

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad \Rightarrow \quad \boxed{x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega).}$$

$$\square x(t) \text{ even} \Rightarrow X(j\omega) = X(-j\omega), \quad x(t) \text{ real} \Rightarrow X(-j\omega) = X^*(j\omega)$$

$$\square x(t) \text{ real and even} \Rightarrow X(j\omega) \text{ real and even}$$

$$\square x(t) \text{ real and odd} \Rightarrow X(j\omega) \text{ purely imaginary and odd}$$

$$\square \text{ If } x(t) \text{ real}$$

$$x(t) = x_e(t) + x_o(t).$$

$$\mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\},$$

$$\Leftrightarrow$$

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{\mathcal{F}} \Re\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \xleftrightarrow{\mathcal{F}} j\Im\{X(j\omega)\}.$$



Properties of continuous-time Fourier Transform

Example

□ For $a > 0$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} 1/(a + j\omega)$$

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} 2a/(a^2 + \omega^2)$$

□ use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2 \mathcal{E}_v\{e^{-at}u(t)\}$$

$$\mathcal{E}_v\{e^{-at}u(t)\} \xleftrightarrow{\mathcal{F}} \Re\left\{\frac{1}{a + j\omega}\right\}$$

$$\mathcal{F}\{e^{-a|t|}\} = 2\Re\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2}$$



Properties of continuous-time Fourier Transform

Differential and integration

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow$$

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega).$$

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega).$$

□ use FT properties

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \frac{d(e^{j\omega t})}{dt} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot j\omega \cdot e^{j\omega t} d\omega$$

$$\begin{aligned} \int_{-\infty}^t x(\tau) d\tau &= \int_{-\infty}^t \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega\tau} d\omega d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \int_{-\infty}^t e^{j\omega\tau} d\tau d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X(j\omega)}{j\omega} e^{j\omega t} d\omega, \omega \neq 0 \quad \pi X(0) \delta(\omega) \quad \text{DC components} \end{aligned}$$



Properties of continuous-time Fourier Transform

Example FT of unit step $x(t) = u(t)$

$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1 \quad x(t) = \int_{-\infty}^t g(\tau) d\tau$$

□ use integration property

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

□ Recover $G(j\omega)$ by differential property

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] = 1$$



Properties of continuous-time Fourier Transform

Example

Determine the FT of $x(t)$

□ Solution

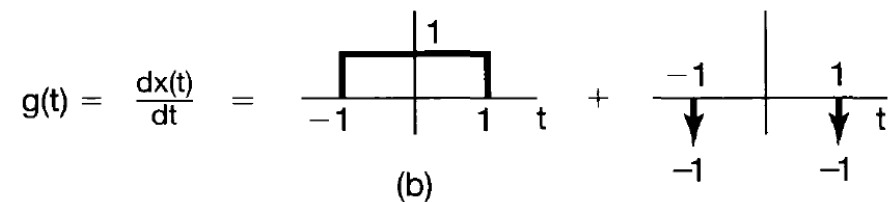
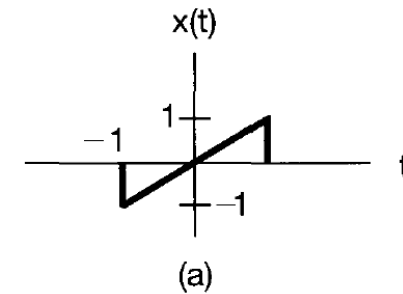
$$g(t) = \frac{d}{dt}x(t)$$

$$G(j\omega) = \left(\frac{2 \sin \omega}{\omega} \right) - e^{j\omega} - e^{-j\omega}.$$

□ use FT properties

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

$$X(j\omega) = \frac{2 \sin \omega}{j\omega^2} - \frac{2 \cos \omega}{j\omega}.$$



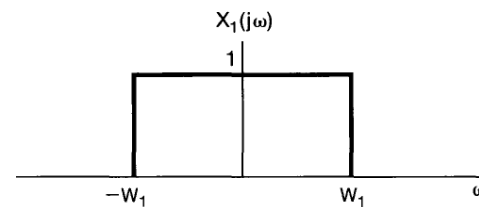
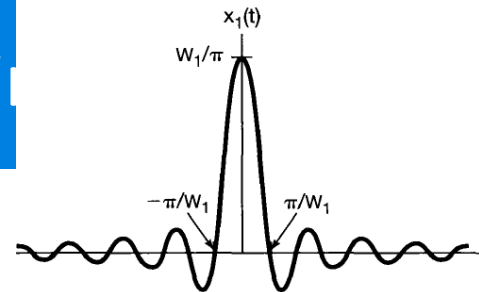
Properties of continuous-time

Time and frequency scaling

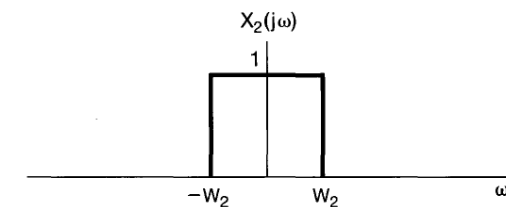
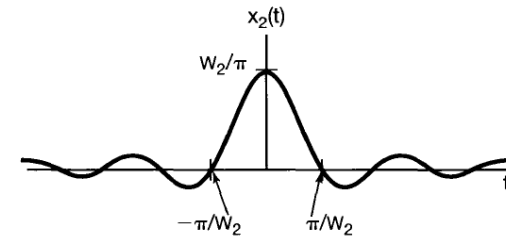
$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega), \quad \boxed{x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), \quad a \neq 0}$$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt$$

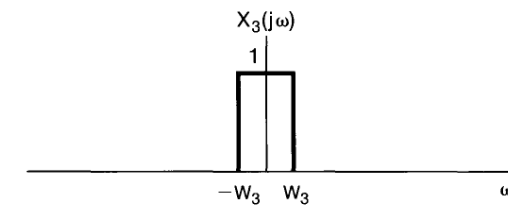
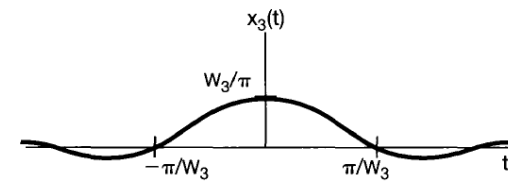
$$\mathcal{F}\{x(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases}$$



(a)



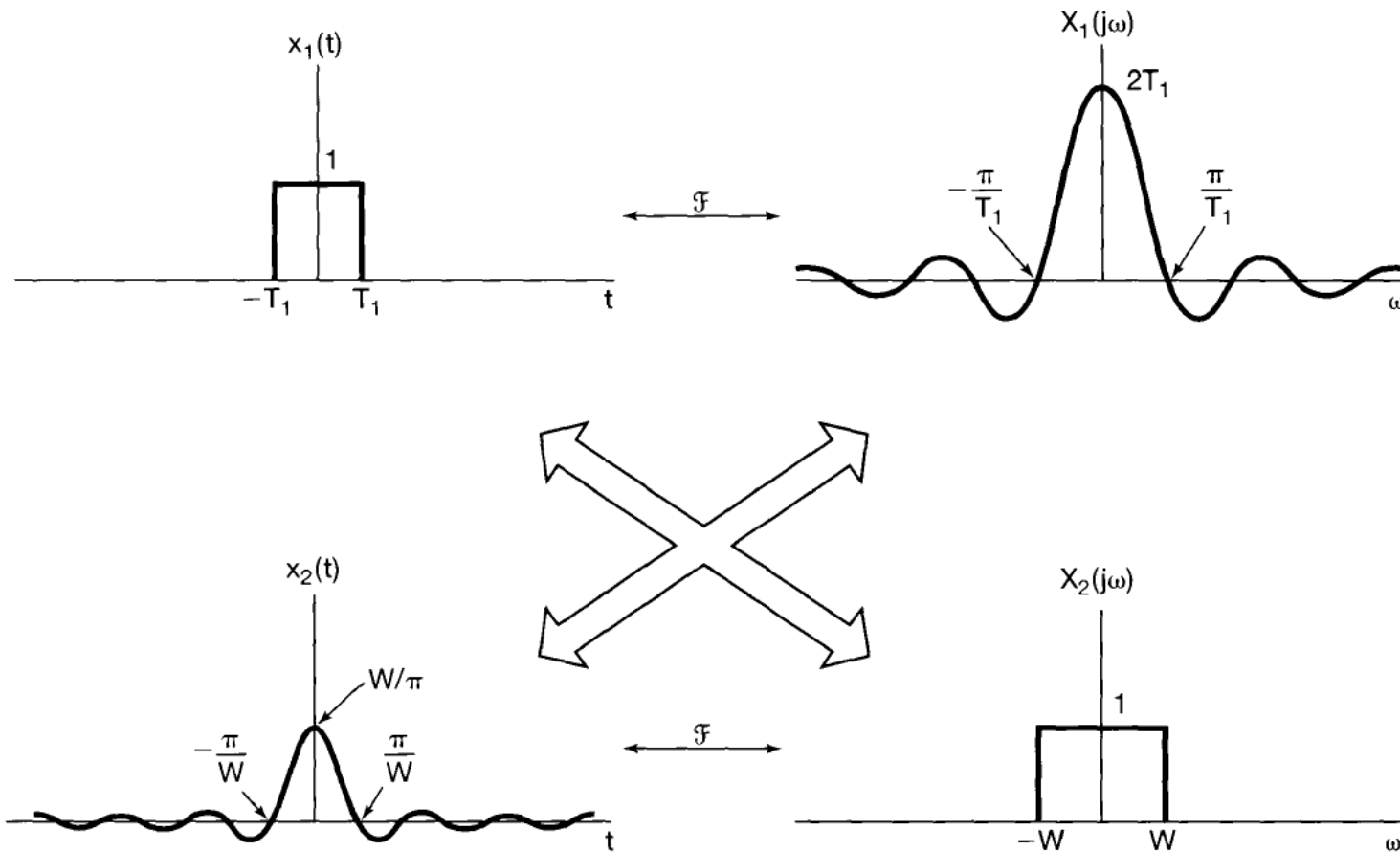
(b)



(c)

Properties of continuous-time Fourier Transform

Duality



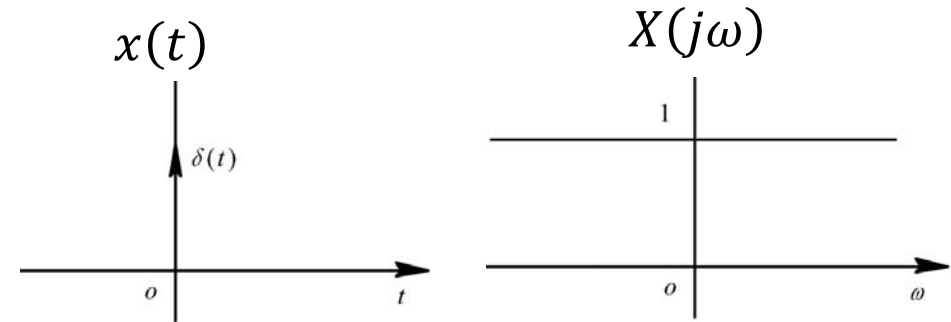


Properties of continuous-time Fourier Transform

Example

$$x(t) = \delta(t) \quad X(j\omega) = 1$$

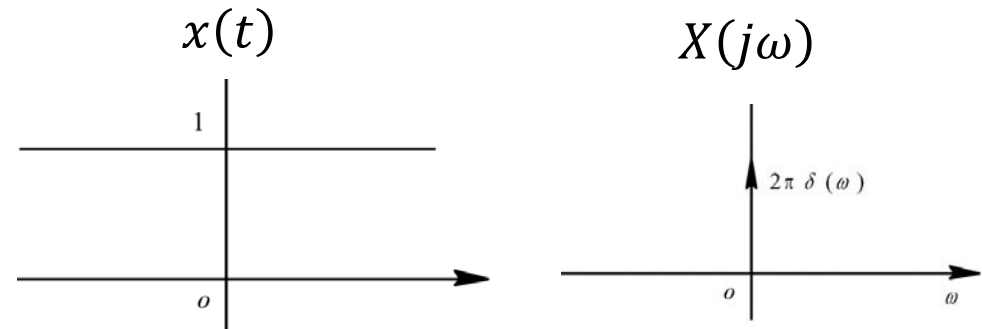
$$x(t) = 1 \quad X(j\omega) = 2\pi\delta(\omega)$$



Principle

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

$$x(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) \cdot e^{j\omega t} dt$$





Properties of continuous-time Fourier Transform

Example $g(t) = \frac{2}{1+t^2} \quad G(j\omega) = ?$

Solution: calculate $G(j\omega)$ is difficult; use duality property

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} 2a/(a^2 + \omega^2)$$

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} \cdot e^{j\omega t} d\omega$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} \cdot e^{j\omega t} dt \quad \therefore G(j\omega) = 2\pi e^{-|\omega|}$$



Properties of continuous-time Fourier Transform

Example

Duality property can determine or suggest other FT properties

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega).} \quad \Leftrightarrow \quad \boxed{-jtx(t) \xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega}.}$$

$$\boxed{\int_{-\infty}^t x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega).} \quad \Leftrightarrow \quad \boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\eta)d\eta.}$$

$$\boxed{x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega).} \quad \Leftrightarrow \quad \boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0))}$$



Properties of continuous-time Fourier Transform

Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

$$\begin{aligned} \int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} x(t) x^*(t) dt \\ &= \int_{-\infty}^{+\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega \end{aligned}$$

The Continuous-Time Fourier Transform (ch.4)

- ☐ Representation of aperiodic signals- Continuous Fourier Transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of continuous-time Fourier Transform
- ☒ The convolution property
- ☐ The multiplication property
- ☐ System characterized by differential equations



The convolution property

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega).$$

□ proof

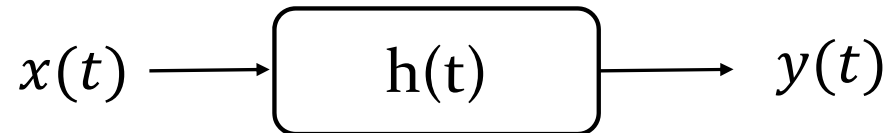
$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \right] e^{-j\omega t} dt \\ &= \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t} dt \right] d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau} H(j\omega) d\tau = H(j\omega) \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau} d\tau \\ &= H(j\omega)X(j\omega) \end{aligned}$$

- $H(j\omega)$: Frequency response; important for analyzing LTI systems
- Only stable continuous-time LTI systems have $H(j\omega)$
- Non-stable continuous-time LTI system: Laplace transform

The convolution property



Example



□ Assume $h(t) = \delta(t - t_0)$, $\mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$

□ Solution 1

$$H(j\omega) = e^{-j\omega t_0} \quad Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega)$$

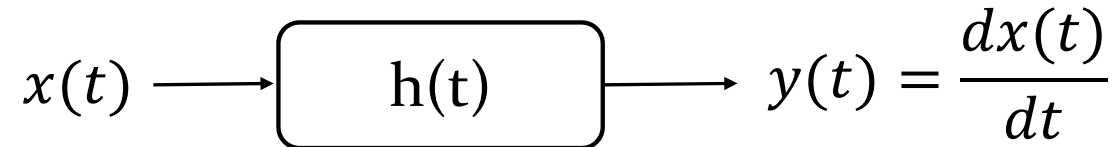
□ Solution 2

$$y(t) = x(t - t_0) \quad Y(j\omega) = e^{-j\omega t_0}X(j\omega)$$

The convolution property



Example



□ Differentiation property $\Rightarrow Y(j\omega) = j\omega X(j\omega)$

□ Convolution property $\Rightarrow Y(j\omega) = H(j\omega)X(j\omega)$

□ Therefore, $H(j\omega) = j\omega$

The convolution property



Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^t x(\tau) d\tau \quad Y(j\omega) = ?$$

$$h(t) = \int_{-\infty}^t \delta(\tau) d\tau = u(t)$$

□ Frequency response $H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

□ Convolution property $Y(j\omega) = H(j\omega)X(j\omega)$

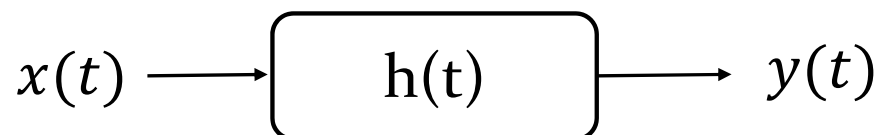
$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

□ Consistent with integration property

The convolution property



Example



$$h(t) = e^{-at}u(t), a > 0 \quad x(t) = e^{-bt}u(t), b > 0 \quad y(t) = ?$$

□ **Solution** $b \neq a$

$$H(j\omega) = \frac{1}{b + j\omega} \quad X(j\omega) = \frac{1}{a + j\omega} \quad Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

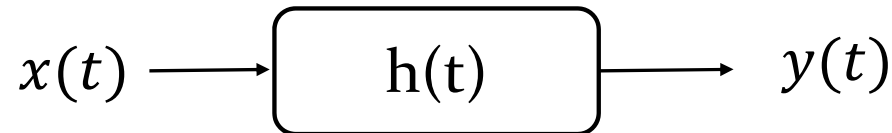
$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} \quad A = \frac{1}{b - a} = -B$$

$$Y(j\omega) = \frac{1}{b - a} \left(\frac{1}{a + j\omega} - \frac{1}{b + j\omega} \right) \quad y(t) = \frac{1}{b - a} [e^{-at} - e^{-bt}]u(t), b \neq a$$

The convolution property



Example



$$h(t) = e^{-at}u(t), a > 0 \quad x(t) = e^{-bt}u(t), b > 0 \quad y(t) = ?$$

□ Solution $b = a$

$$Y(j\omega) = \frac{1}{(a + j\omega)^2} = j \frac{d}{d\omega} \left[\frac{1}{a + j\omega} \right]$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} 1/(a + j\omega)$$

$$te^{-at}u(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} \left[\frac{1}{a + j\omega} \right]$$

$$\therefore y(t) = te^{-at}u(t)$$

The Continuous-Time Fourier Transform (ch.4)

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- ☒ **The multiplication property**
- ☐ System characterized by differential equations



The multiplication property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\theta) P(j(\omega - \theta)) d\theta$$

□ multiplication of two signals is often referred to as *amplitude modulation*

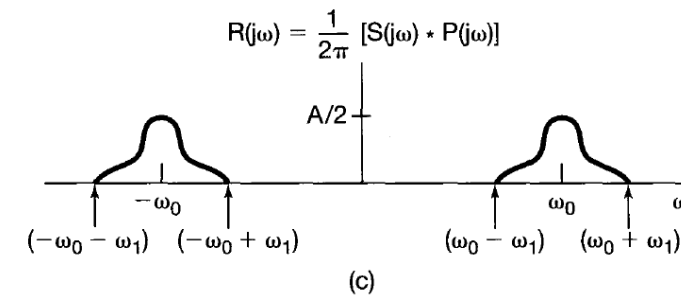
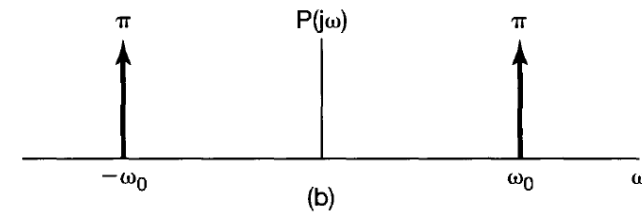
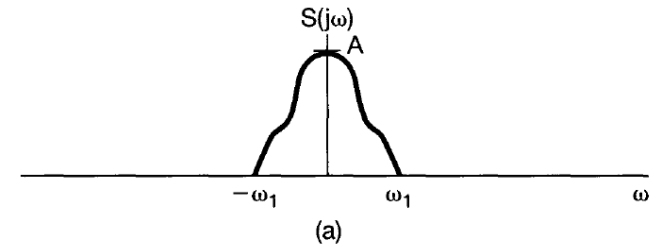
$$\begin{aligned} s(t)p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) e^{j\theta t} d\theta \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\omega') e^{j\omega' t} d\omega' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j\omega') e^{j(\theta + \omega')t} d\theta d\omega' \\ &\quad \omega' = \omega - \theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S[j(\theta)] P(j(\omega - \theta)) e^{j\omega t} d\theta d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j(\omega - \theta)) d\theta} e^{j\omega t} d\omega \\ &\quad \text{where } \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j(\omega - \theta)) d\theta = R(j\omega) \end{aligned}$$

The multiplication property

Example

Consider a signal $p(t) = \cos \omega_0 t$ and a signal $s(t)$ with spectrum $S(j\omega)$, determine the FT of $r(t) = p(t)s(t)$

□ **Solution** $P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$



$$R(j\omega) = 1/2\pi \cdot S(j\omega) * P(j\omega)$$

$$= 1/2\pi \cdot S(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

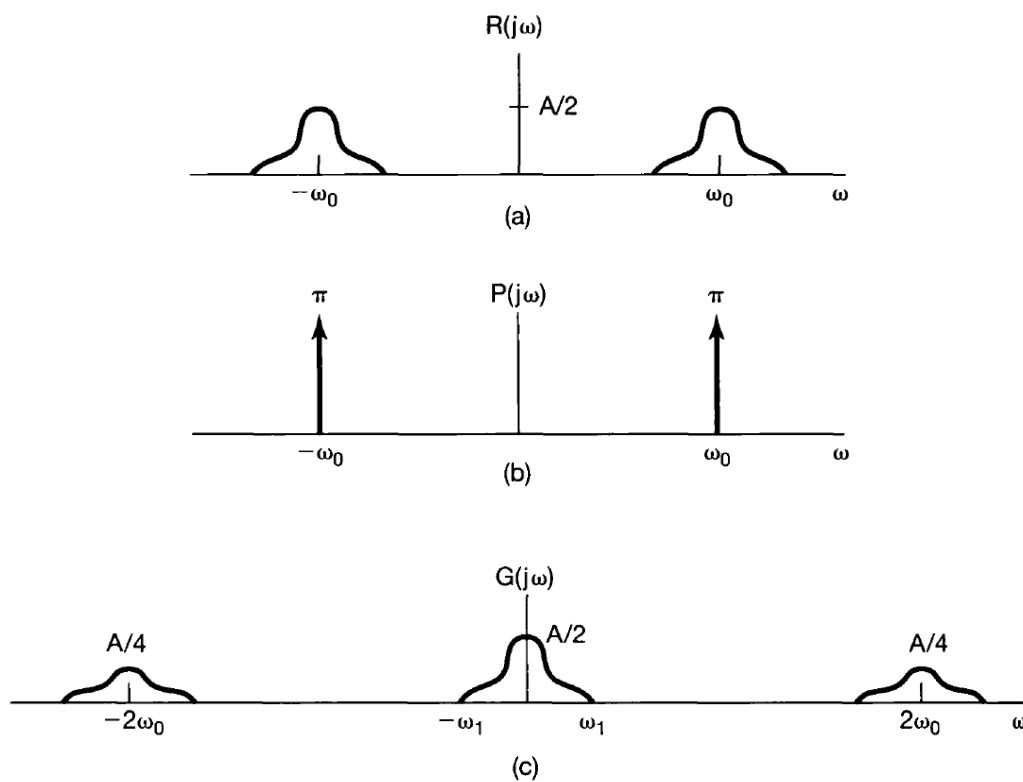
$$= 1/2[S[j(\omega - \omega_0)] + S[j(\omega + \omega_0)]]$$

The multiplication property



Example

$$g(t) = r(t)p(t) \quad G(j\omega) = ?$$





The multiplication property

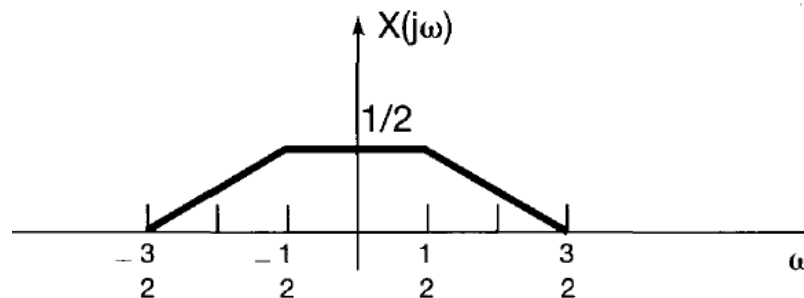
Example

$$x(t) = \frac{\sin(t) \sin(t/2)}{\pi t^2} \quad X(j\omega) = ?$$

□ Solution

$$x(t) = \pi \left(\frac{\sin(t)}{\pi t} \right) \left(\frac{\sin(t/2)}{\pi t} \right)$$

$$X(j\omega) = \frac{1}{2} \mathcal{F} \left\{ \frac{\sin(t)}{\pi t} \right\} * \mathcal{F} \left\{ \frac{\sin(t/2)}{\pi t} \right\}$$



The Continuous-Time Fourier Transform (ch.4)

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System characterized by differential equations

□ Differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

$$Y(j\omega) = H(j\omega)X(j\omega) \Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$\mathcal{F}\left\{\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}\right\} \Rightarrow \sum_{k=0}^N a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^M b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$Y(j\omega) \left[\sum_{k=0}^N a_k (j\omega)^k \right] = X(j\omega) \left[\sum_{k=0}^M b_k (j\omega)^k \right] \Leftrightarrow \sum_{k=0}^N a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^M b_k (j\omega)^k X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^M b_k (j\omega)^k}{\sum_{k=0}^N a_k (j\omega)^k}$$



System characterized by differential equations

Example

$$\frac{dy(t)}{dt} + ay(t) = x(t) \quad a > 0$$

$$\mathcal{F} \left\{ \frac{dy(t)}{dt} + ay(t) \right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \Rightarrow h(t) = e^{-at}u(t)$$



System characterized by differential equations

Example

$$\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

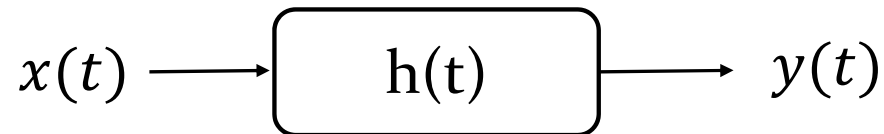
$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$



System characterized by differential equations

Example



$$x(t) = e^{-t}u(t) \quad \frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t), \quad y(t) = ?$$

□ Solution

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \right] \left[\frac{1}{j\omega + 1} \right] = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}$$

$$= \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3} \quad A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4}$$

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3} \quad \Rightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t} \right] u(t)$$