Signals and Systems Homework 2
Solutions

1. (10')Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1]$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$

(b) $y_2[n] = x[n+2] * h[n]$

(c) $y_3[n] = x[n] * h[n+2]$

Solution:

(a) We have know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{1}$$

The signals $x[n]$ and $h[n]$ are as how in Figure 1 From this figure, we can easily see that the

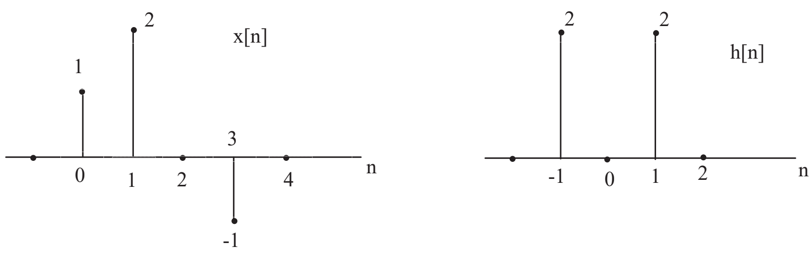

Figure 1:

above convolution sum reduces to

$$y_1[n] = h[-1]x[n+1] + h[1]x[n-1] = 2x[n+1] + 2x[n-1]$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

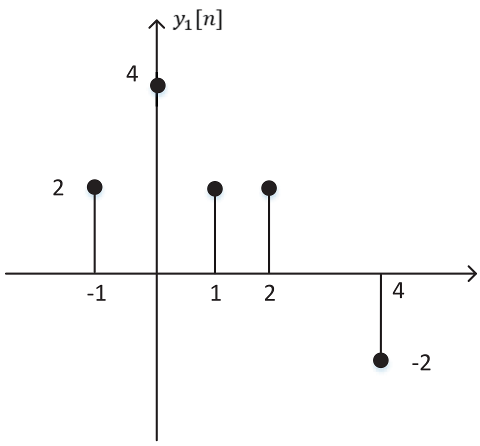

Figure 2:

(b) We know that

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$$

Comparing with eq.(1), we see that

$$y_2[n] = y_1[n+2]$$

(c) We may rewrite eq.(1) as

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similarly, we may write

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k]h[n+2-k]$$

Then, we see that

$$y_3[n] = y_1[n+2]$$

2. (15')For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $x(t)$ and $h(t)$ are as in Figure 3(a).

(b) $x(t)$ and $h(t)$ are as in Figure 3(b).

(c) $x(t)$ and $h(t)$ are as in Figure 3(c).

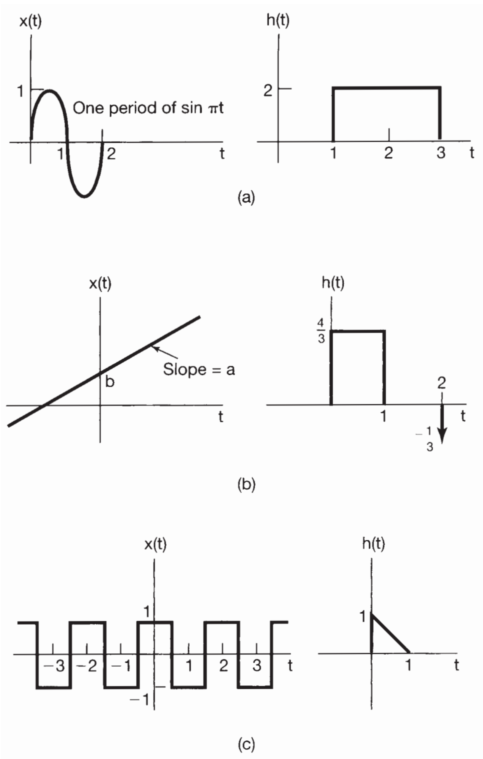

Figure 3:

Solution:

(a) The desire convolution is

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}2\sin(\pi\tau)[u(\tau)-u(\tau-2)][u(t-\tau-1)-u(t-\tau-3)]d\tau$$

This give us

$$y(t)=\frac{2}{\pi}[1-\cos\{\pi(t-1)\}][u(t-1)-u(t-3)]+\frac{2}{\pi}[\cos\{\pi(t-3)\}-1][u(t-3)-u(t-5)]$$

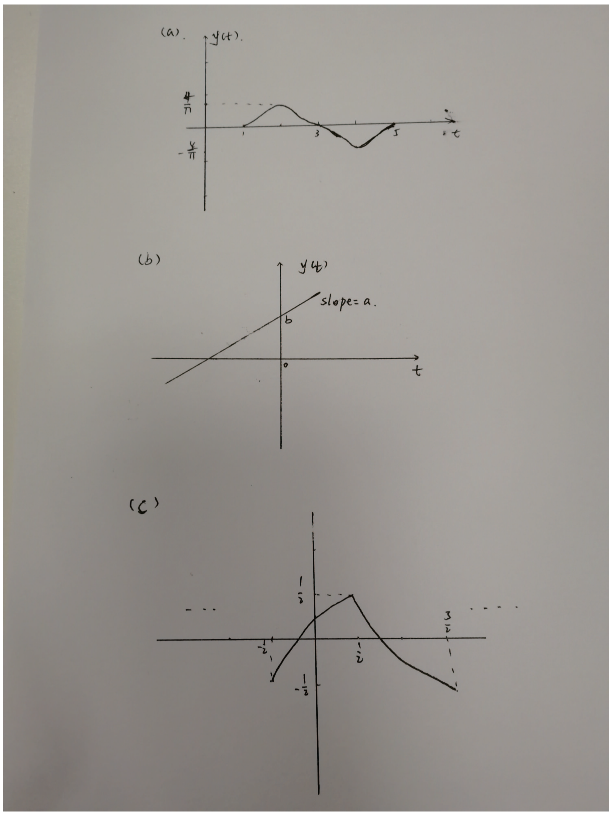

Figure 4:

(b) Let

$$h(t)=h_1(t)-\frac{1}{3}\delta(t-2)$$

where

$$h_1(t)=\frac{4}{3}[u(t)-u(t-1)]$$

Now

$$y(t)=x(t)*h(t)=x(t)*h_1(t)-\frac{1}{3}x(t-2)$$

We have

$$x(t)*h_1(t)=\int_{t-1}^{t}\frac{4}{3}(a\tau+b)d\tau=\frac{4}{3}at+\frac{4}{3}b-\frac{2}{3}a$$

Therefore,

$$y(t)=\frac{4}{3}at+\frac{4}{3}b-\frac{2}{3}a-\frac{1}{3}[a(t-2)+b]=at+b=x(t)$$

(c) $x(t)$ periodic implies $y(t)$ periodic, so determine 1 period only. we have

$$y(t)=\begin{cases}\int_{t-1}^{-\frac{1}{2}}(t-\tau-1)d\tau+\int_{-\frac{1}{2}}^{t}(1-t+\tau)d\tau=\frac{1}{4}+t-t^2, & -\frac{1}{2}<t<\frac{1}{2}\\ \int_{t-1}^{\frac{1}{2}}(1-t+\tau)d\tau+\int_{\frac{1}{2}}^{t}(t-\tau-1)d\tau=t^2-3t+\frac{7}{4}, & \frac{1}{2}<t<\frac{3}{2}\end{cases}$$

The period of $y(t)$ is 2.

3. (10')Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(t) = \delta(t+2) + 2\delta(t+1)$$

Solution:
Given that $h(t) = \delta(t+2) + 2\delta(t+1)$, the above integral reduces to

$$x(t) * h(t) = x(t+2) + 2x(t+1)$$

we can easily show that

$$y(t) = \begin{cases} t+3, & -2 < t \le -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

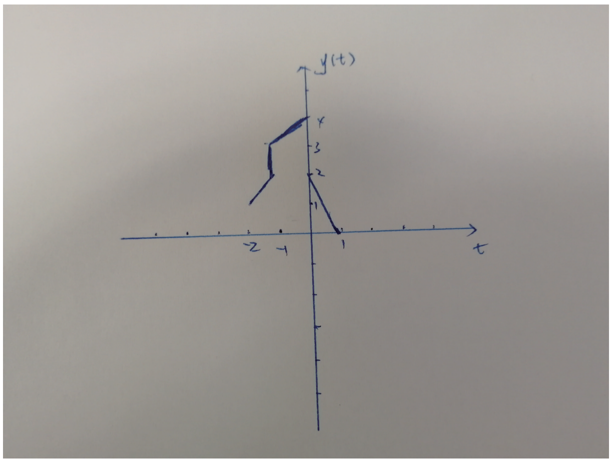

Figure 5:

4. (10')Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$.

(a) Determine and sketch $y(t) = x(t) * h(t)$.

(b) If $dy(t)/dt$ contains only three discontinuities, what is the value of α?

Solution:

(a) From the given information, we may sketch $x(t)$ and $h(t)$ as show,

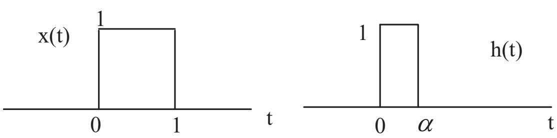

Figure 6:

Then, we can show that $y(t) = x(t) * h(t)$ is as shown in Figure 7
Thus,

$$y(t) = t[u(t) - u(t-\alpha)] + \alpha[u(t-\alpha) - u(t-1)] + (1+\alpha-t)[u(t-1) - u(t-1-\alpha)]$$

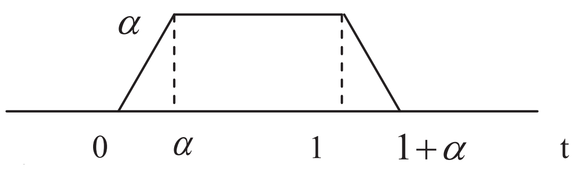

Figure 7:

(b) From the plot of $y(t)$, it is clear that $\frac{dy(t)}{dt}$ has discontinuities at 0, α , 1, and $1+\alpha$. If we want $\frac{dy(t)}{dt}$ to have only three discontinuities, then we need to ensure that $\alpha = 1$.

5. (10')Let

$$x(t) = u(t-3) - u(t-5) \quad \text{and} \quad h(t) = e^{-3t}u(t)$$

(a) Compute $y(t) = x(t) * h(t)$.

(b) Compute $g(t) = (dx(t)/dt) * h(t)$.

(c) How is $g(t)$ related to $y(t)$?

Solution:

(a) We have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} e^{-3\tau}[u(t-\tau-3) - u(t-\tau-5)]d\tau$$

Therefore, for $t \leq 3$, the above integral evaluates to zero. For $3 < t \leq 5$, the above integral is

$$y(t) = \int_{0}^{t-3} e^{-3\tau}d\tau = \frac{1-e^{-3(t-3)}}{3}$$

For $t > 5$, the integral is

$$y(t) = \int_{t-5}^{t-3} e^{-3\tau}d\tau = \frac{(1-e^{-6})e^{-3(t-5)}}{3}$$

Therefore, the result of this convolution may be expressed as

$$y(t) = \frac{1-e^{-3(t-3)}}{3}[u(t-3) - u(t-5)] + \frac{(1-e^{-6})e^{-3(t-5)}}{3}u(t-5)$$

(b) By differentiating $x(t)$ with respect to time we get

$$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$$

Therefore,

$$g(t) = \frac{dx(t)}{dt} * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

From the result of part (a),we may compute the derivative of $y(t)$ to be

$$\frac{dy(t)}{dt} = e^{-3(t-3)}[u(t-3) - u(t-5)] + (e^{-6}-1)e^{-3(t-5)}u(t-5)$$

This is exactly equal to $g(t)$. therefore, $g(t) = \frac{dy(t)}{dt}$.

6. (20')Let $h(t)$ be the triangular pulse shown in Figure 8(a), and let $x(t)$ be the impulse train depicted in Figure 8(b). That is

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT).$$

Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T:
(a). $T = 4$ (b). $T = 2$ (c). $T = 3/2$ (d). $T = 1$
Solution:

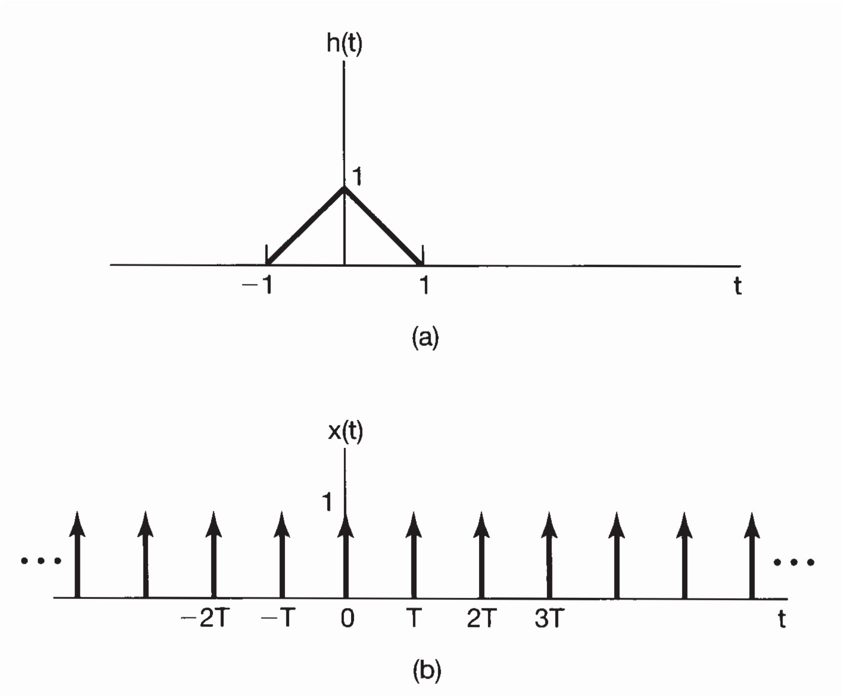

Figure 8:

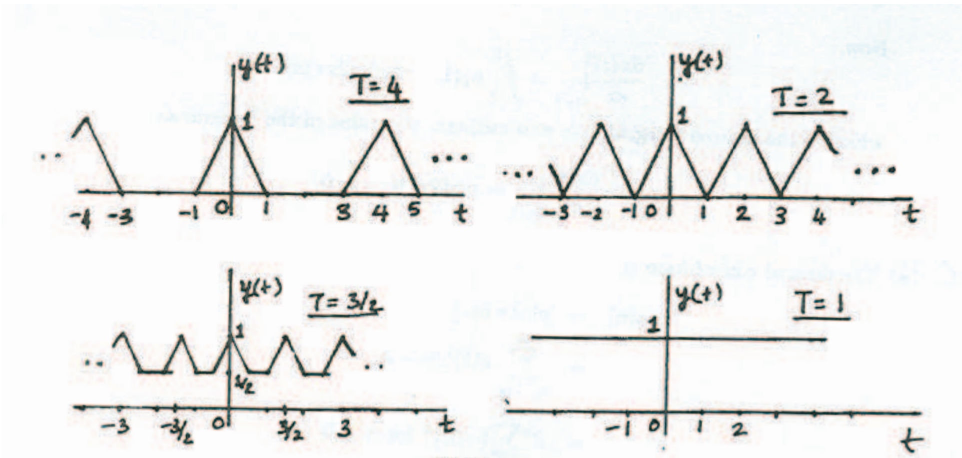

Figure 9:

7. (15')Let the signal

$$y[n] = x[n] * h[n],$$

where

$$x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$

and

$$h[n] = \left(\frac{1}{4}\right)^n u[n+3]$$

(a) Determine $y[n]$ without utilizing the distributive property of convolution.

(b) Determine $y[n]$ utilizing the distributive property of convolution.

Solution:

(a) we may write $x[n]$ as

$$x[n] = (\frac{1}{3})^{|n|}$$

Now the desire convolution is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{-1} (1/3)^{-k}(1/4)^{n-k}u[n-k+3] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3] \\ &= (1/12)\sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n+k}u[n+k+4] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3] \end{aligned}$$

By consider each summation in the above equation separately, we may show that

$$y[n] = \begin{cases} \frac{12^4}{11}3^n, & n \leq -4 \\ \frac{1}{11}4^{-n} - 3 \cdot 4^{-n} + 256 \cdot 3^{-(n+3)}, & n \geq -3 \end{cases}$$

(b) Now consider the convolution

$$y_1[n] = [(1/3)^n u[n]] * [(1/4)^n u[n+3]] = [-3 \cdot 4^{-n} + 256 \cdot 3^{-(n+3)}]u[u+3]$$

Also consider the convolution

$$y_2[n] = [3^n u[-n-1]] * [(1/4)^n u[n+3]] = \begin{cases} \frac{12^4}{11}3^n, & n \leq -4 \\ \frac{1}{11}4^{-n}, & n \geq -3 \end{cases}$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

8. (10')An analog system has the input-output relation

$$y(t) = \int_0^t e^{-(t-\tau)}x(\tau)d\tau \quad t \geq 0$$

and zero otherwise. The input is $x(t)$ and $y(t)$ is the output.

(a) Is this a linear time-invariant system? If so, can you determine without any computation the impulse response of the system? Explain.

(b) Is this system causal? Explain.

(c) Find the unit-step response $s(t)$ and from it find the impulse response $h(t)$. Is this a stable system? Explain.

(d) Find the response due to a pulse $x(t) = u(t) - u(t-1)$.

Solution:

(a) The system is LTI since the input $x(t)$ and the output $y(t)$ are related by a convolution integral with $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$ or $h(t) = e^{-t}u(t)$.

(b) Yes, this system is causal as the output $y(t)$ depends on present and past values of the input.

(c) Letting $x(t) = u(t)$, the unit-step response is

$$s(t) = \int_0^t e^{-t+\tau}u(\tau)d\tau = e^{-t}\int_0^t e^{\tau}d\tau = 1 - e^{-t}, \quad t \geq 0$$

and zero otherwise. The impulse response as indicated before is $h(t) = ds(t)/dt = e^{-t}u(t)$. The BIBO stability of the system is then determined by checking whether the impulse response is absolutely integrable or not,

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t}dt = 1$$

so yes, it is BIBO stable.

(d) Using superposition, the response to the pulse $x(t) = u(t) - u(t-1)$ would be

$$y(t) = s(t) - s(t-1) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$$