# Lab 4 Fourier Transform

## Objective

- Learn the Fourier transform of continuous-time signal with MATLAB.
- Analyze the signals with Fourier Transform.
- Analyze the LTI system with system models.

## Content

As for a periodic rectangular pulse signal, $\omega = \frac{2\pi}{T}$, as T increases, the density of the spectrum line increases, as shown in Figure 1.

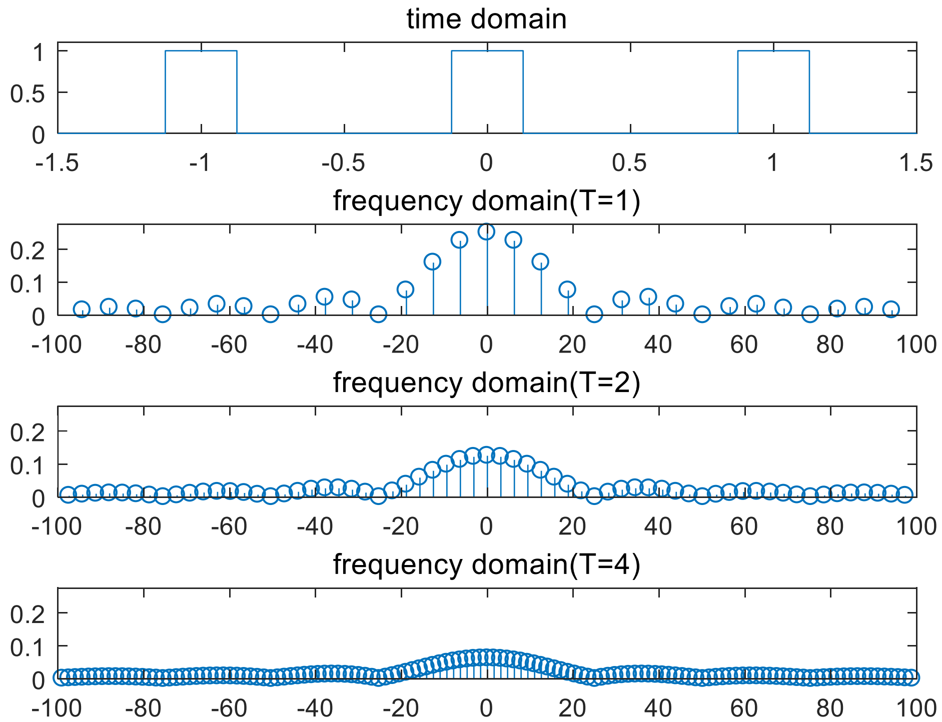



Figure 1 Influence of T

When $T \to \infty$, the periodic signal changes to aperiodic signal, and the spectrum line changes from discrete ones to continuous ones. However, the amplitude of the spectrum line also gets smaller, while the relative difference still exists. So for aperiodic signals, it is necessary to remove the effect of T in the frequency domain. As shown in Figure 2.

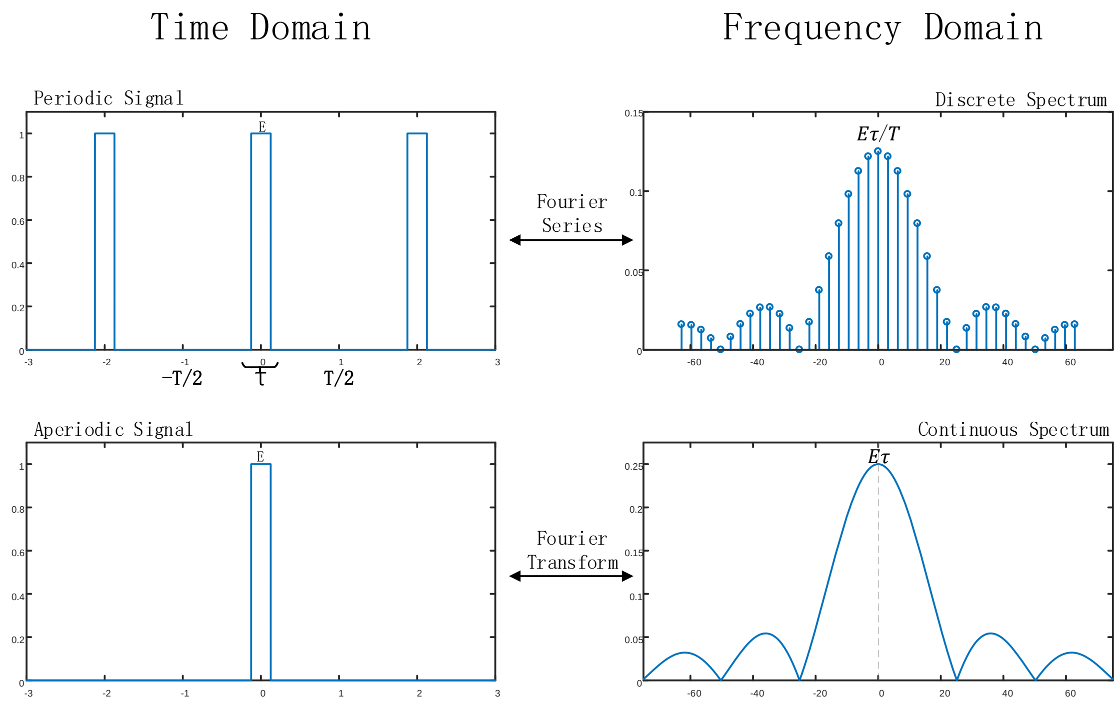



Figure 2 Fourier Series and Fourier Transform

A time signal f(t) is a function of time t, where $-\infty \leq t \leq \infty$. This signal may be represented in the frequency domain using the Fourier transform.

# Signal Analysis

## Fourier Transform and Inverse Fourier Transform with Symbolic Method

The Fourier transform of f(t) is defined as:

$$\mathrm{F}(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt$$

And the inverse Fourier transform is defined as:

$$\mathrm{f(t)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t}\, \mathrm{d\omega}$$

For signals that can be expressed by an expression, use the function **fourier** and **ifourier** to do the Fourier transfer and inverse Fourier transfer with MATLAB. Both **fourier** and **ifourier** are symbolic methods. The format of the two functions is listed in Table 1.

Table 1 Format of fourier and ifourier

| Format | Description |
|---|---|
| F=fourier(f) | Do Fourier transform of function f with variable x. The result F is the function with variable w. |
| F=fourier(f,v) | Do Fourier transform of function f with variable x. The result F is the function with variable v. |

| | |
|---|---|
| F=fourier(f,u,v) | Do Fourier transform of function f with variable u. The result F is the function with variable v. |
| f=ifourier(F) | Do inverse Fourier transform of function F with variable w. The result f is the function with variable x. |
| f = ifourier(F,u) | Do inverse Fourier transform of function F with variable w. The result f is the function with variable u. |
| f = ifourier(F,v,u) | Do inverse Fourier transform of function F with variable v. The result f is the function with variable u. |

Use function **abs** and **angle** to plot the amplitude-frequency and phase-frequency.
Example: Do the Fourier transfer of single-sided exponential signal.

$$f(t) = \begin{cases} e^{-2t}, t > 0 \\ 0, t < 0 \end{cases}$$

```
syms t
a = 2;
f = exp(-(a*t))*heaviside(t);
subplot(3,1,1);fplot(f);axis([-3,3,-0.1,1.1]);grid on;
F = fourier(f);
subplot(3,1,2);fplot(abs(F));axis([-10,10,0,.6]);grid on;
subplot(3,1,3);fplot(angle(F));axis([-10,10,-pi/2,pi/2]);grid on;
```

Example: Do the inverse Fourier transfer of $F(j\omega) = \frac{1}{1+\omega^2}$.

```
syms t w
ifourier(1/(1+w^2),t);
```

# Fourier Transform with Numeric Method

## Fourier Transform with Function fft

Function fft is used to do Fourier transform in the numeric method. Format fft(x) computes the discrete Fourier transform of x using a fast Fourier transform algorithm. Format fft(x,n) returns the n-point DFT.
Example: x=cos(2*pi*10.*t)+2sin(2*pi*15.*t)+3cos(2*pi*20.*t)). Observe the signal in frequency domain.

```
N = 256;
Fs = 100;
t = [0:N-1]./Fs;
x = 1*cos(2*pi*10.*t) + 2*sin(2*pi*15.*t) + 3*cos(2*pi*20.*t);
X = fft(x);
plot(abs(X))
```

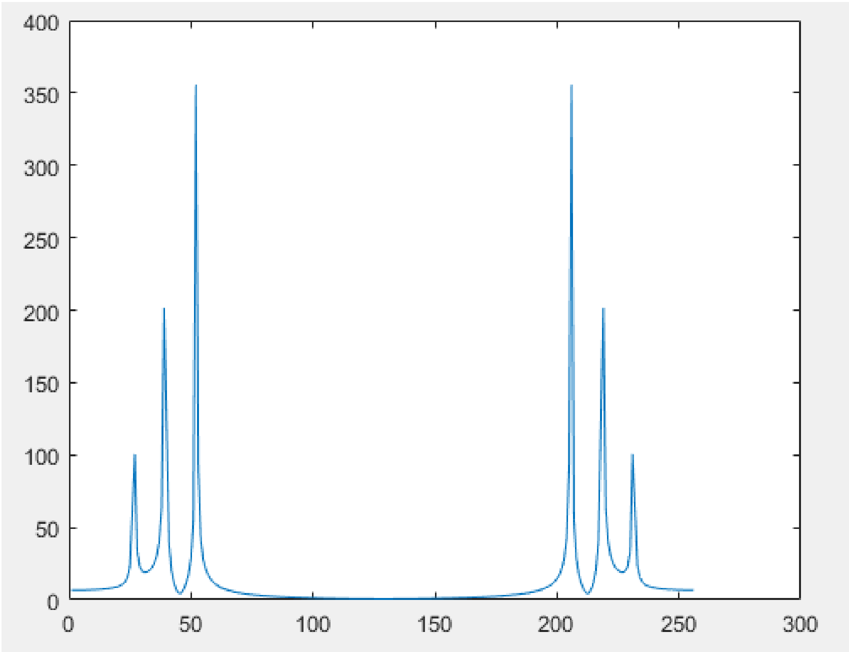
1. Figure 3 (a) shows the graphics of X and (b) shows the value. From (b), it is easy to find out that the $1^{th}$ and the $129^{th}$ items are real numbers, and the items at the sides of $129^{th}$ are conjugated. This is because MATLAB present frequency components from f=0 to f=Fs (from DC to the sampling frequency). However, we are only interested in the range of 0 to Fs/2. So only the $1^{th}$

to 128th items (N/2) are valuable.

2. The corresponding frequency to the X(n) is calculated by $f(n) = \frac{n-1}{N} Fs$, n $= 1,2, \dots,$ N, where $\frac{Fs}{N}$ is the frequency resolution.

3. As mentioned in 1 and 2, the effective observation items are from 1th to $\frac{N}{2}$ th, so the corresponding frequency is from 0 to $\frac{Fs}{2}$. So this is the basis for choosing a suitable Fs.

4. According to the characters of function fft, it is recommended that N prefers $2^n$.




(a)

```
1 至 5 列
0.0665 + 0.0000i   0.0665 + 0.0015i   0.0665 + 0.0029i   0.0667 + 0.0044i   0.0668 + 0.0060i
6 至 10 列
0.0670 + 0.0076i   0.0673 + 0.0092i   0.0676 + 0.0109i   0.0679 + 0.0128i   0.0683 + 0.0147i
...
121 至 125 列
-0.0067 - 0.0018i  -0.0066 - 0.0015i  -0.0066 - 0.0013i  -0.0065 - 0.0011i  -0.0065 - 0.0009i
126 至 130 列
-0.0065 - 0.0007i  -0.0064 - 0.0004i  -0.0064 - 0.0002i  -0.0064 + 0.0000i  -0.0064 + 0.0002i
131 至 135 列
-0.0064 + 0.0004i  -0.0065 + 0.0007i  -0.0065 + 0.0009i  -0.0065 + 0.0011i  -0.0066 + 0.0013i
...
251 至 255 列
0.0673 - 0.0092i   0.0670 - 0.0076i   0.0668 - 0.0060i   0.0667 - 0.0044i   0.0665 - 0.0029i
256 列
0.0665 - 0.0015i
```

(b)

Figure 3 fft Result

5. Amplitude can be got by using function abs(). To get amplitude, the 1st and $(\frac{N}{2} + 1)$th item

should be divided by N, and the others be divided by $\frac{N}{2}$.

The amplitude and frequency of x is 1@10Hz, 2@15Hz, 3@20Hz. Change the code to:

```
clear
N = 256;
Fs = 100;
df = Fs/N;
f = [0:N-1]*df;
t = [0:N-1]./Fs;
x=1*cos(2*pi*10.*t)+2*sin(2*pi*15.*t)+3*cos(2*pi*20.*t));
X = fft(x);
Xabs = abs(X);
Xabs = 2*Xabs/N;
Xabs(1)  = Xabs(1)/2;
Xabs = Xabs(1:N/2);
f = f(1:N/2);
plot(f,Xabs)
```

The result is show in Figure 4.

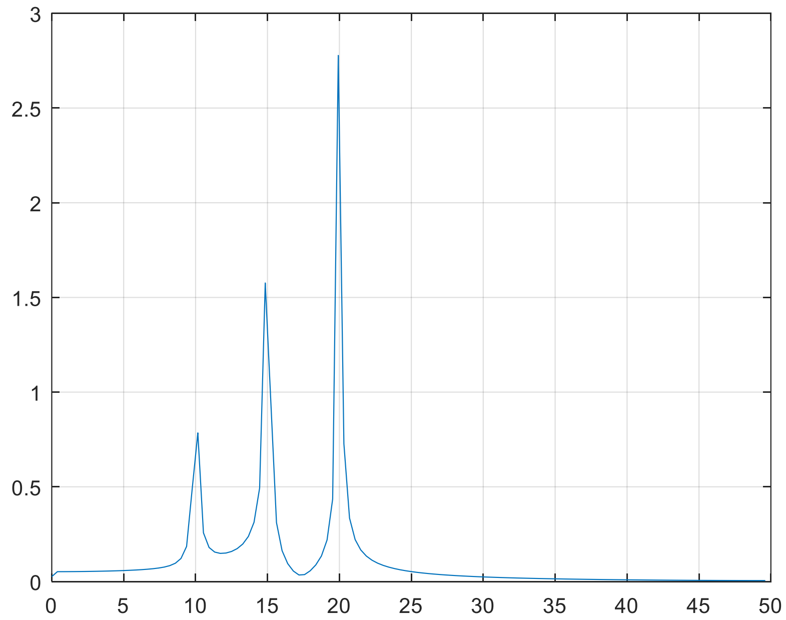



Figure 4 Adjustment

6. There is still some deviation of amplitude because of the spectrum leak. To reduce the spectrum leak, the frequency of signal should be an integer multiple of the frequency resolution. That is $f = \frac{Fs}{N} * k$. Adjust Fs and N to make the equation hold. Some adjustment of Fs and N is made in Table 2. Plot them to view the effect. The result is shown in Figure 5.

Table 2 Adjusting of Fs and N

| **k=f*N/Fs,N=256** | | | |
|---|---|---|---|
| | 1@10Hz | 2@15Hz | 3@20Hz |

| | | | |
|---|---|---|---|
| Fs=120 | 21.33 | 32 | 42.67 |
| Fs=160 | 16 | 24 | 32 |
| **k=f*N/Fs,Fs=100** | | | |
| | 1@10Hz | 2@15Hz | 3@20Hz |
| N=260 | 26 | 39 | 52 |
| N=250 | 25 | 37.5 | 50 |

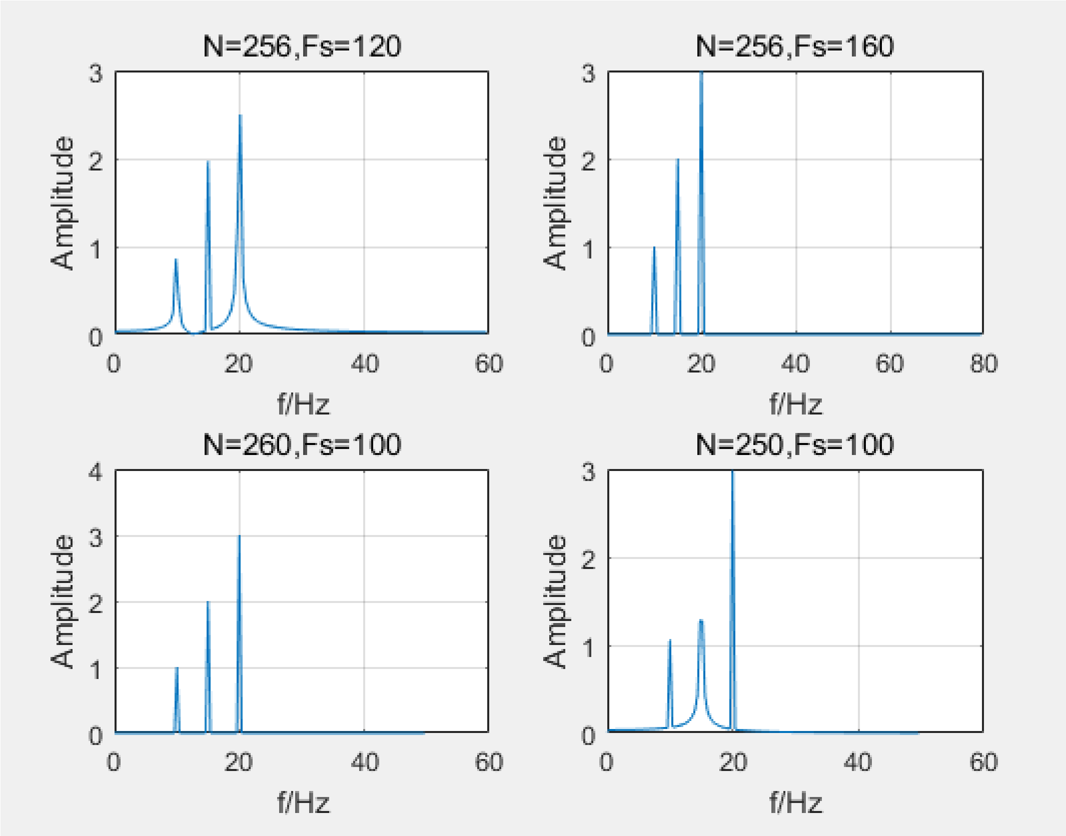



Figure 5 Adjust Fs and N

## Direct Fourier Transform

Except to do the Fourier transform with function fft, it is also possible to do it directly. For most signal,

$$\mathrm{F}(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt = \lim_{\tau\to 0}\sum_{n=-\infty}^{\infty} f(n\tau)e^{-jwn\tau}\tau$$

is true, if $\tau$ is small enough. Since the signal is time-limited, N can be taken as the border of n, we have

$$F(k) = \tau\sum_{n=-N}^{N} f(n\tau)e^{-j\omega_k n\tau}, -\mathrm{N} \le \mathrm{n} \le \mathrm{N}, -\mathrm{M} \le \mathrm{k} \le \mathrm{M}$$

$$f_s = \frac{1}{\tau}$$

$$\omega_k = \frac{2\pi f_s}{M} k = \frac{2\pi}{M\tau} k$$

Expressed in the matrix:

$$\mathrm{F(k)} = \tau \left[ f(-N \bullet \tau) \cdots \quad f(0 \bullet \tau) \quad \cdots \quad f(N \bullet \tau) \right] \left( \begin{bmatrix} e^{-jw_{-M}(-N\bullet\tau)} & \cdots & e^{-jw_0(-N\bullet\tau)} & \cdots & e^{-jw_M(-N\bullet\tau)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ e^{-jw_{-M}(0\bullet\tau)} & \cdots & e^{-jw_0(0\bullet\tau)} & \cdots & e^{-jw_M(0\bullet\tau)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ e^{-jw_{-M}(N\bullet\tau)} & \cdots & e^{-jw_0(N\bullet\tau)} & \cdots & e^{-jw_M(N\bullet\tau)} \end{bmatrix} \right)$$

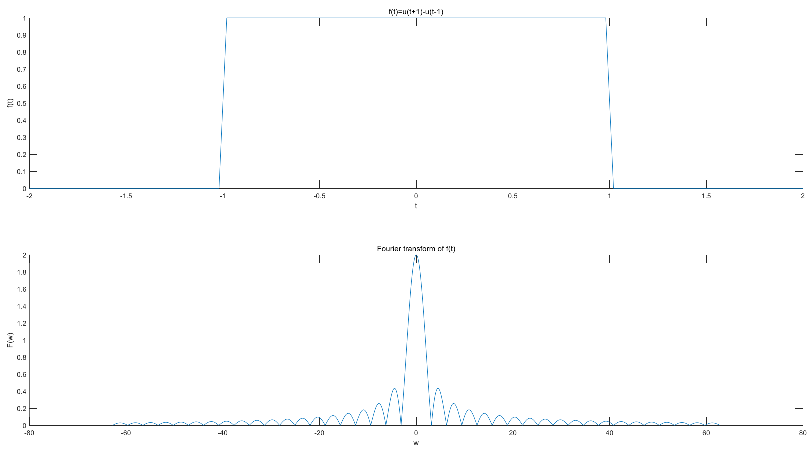
Example: Find out the Fourier transform of

$$f(t) = G(t) = \begin{cases} 1, |t| \le 1 \\ 0, |t| > 1 \end{cases}$$

```
tao = 0.02;
t = -2:tao:2;
f = heaviside(t+1)-heaviside(t-1);
M = 500; k=-M:M;
w1=2*pi*10;    % 10 indicates the range of the frequency to observe
W=k*w1/M;      % w1/M determines the resolution of the frequency

F = f*exp(-1i*t'*W)*tao;
F = abs(F);

subplot(2,1,1); plot(t,f);
xlabel('t'); ylabel('f(t)');
title('f(t)=u(t+1)-u(t-1)');
subplot(2,1,2); plot(W,F);
xlabel('w'); ylabel('F(w)');
title('Fourier transform of f(t)');
```




Tips:

1. You can use function fftshift to shift zero-frequency component to center of spectrum.
2. To find out the N that satisfy $N = 2^n$, use the function nextpow2().
   Example: N=7, to find out the closest integer to 7, which is also the power of 2.

```
1    N=7

     N = 7

2    N=2^nextpow2(N)

     N = 8
```

3. Functions used to read and play audio is listed in Table 3.
   Table 3 Audio Function

| format | description |
|---|---|
| [y,Fs]=audioread(FileName) | FileName:the audio to be read.<br>y: sampled data.<br>Fs: sample rate. |
| [y,Fs]=audioread(FileName,[Start,End]) | Only the samples from Start to End is returned. |
| soundsc(y, Fs) | Play vector as sound. |

## Signal Analyzer APP

The Signal Analyzer app is an interactive tool for visualizing, measuring, analyzing, and comparing signals, in the time domain, in the frequency domain, and in the time-frequency domain.
To launch the Signal Analyzer app, click the APP tab on the MATLAB desktop and select the Signal Analyzer under Signal Processing and Communication section. You can also add it to My Favorites by selecting the asterisk at the top right corner of the app's icon for easy viewing.
The Signal Analyzer interface is shown in Figure 4. The signals available in the MATLAB workspace are imported into the Workspace Browser. Select the items of interest and add it to the Signal Table by dragging it. Before doing analyzing, set the correct sample information for the signal by right-clicking the signal and selecting Time Values. Set the display items, styles, measure ways and etc. in the DISPLAY tab. And set the information of time and spectrum in TIME and SPECTRUM tab.

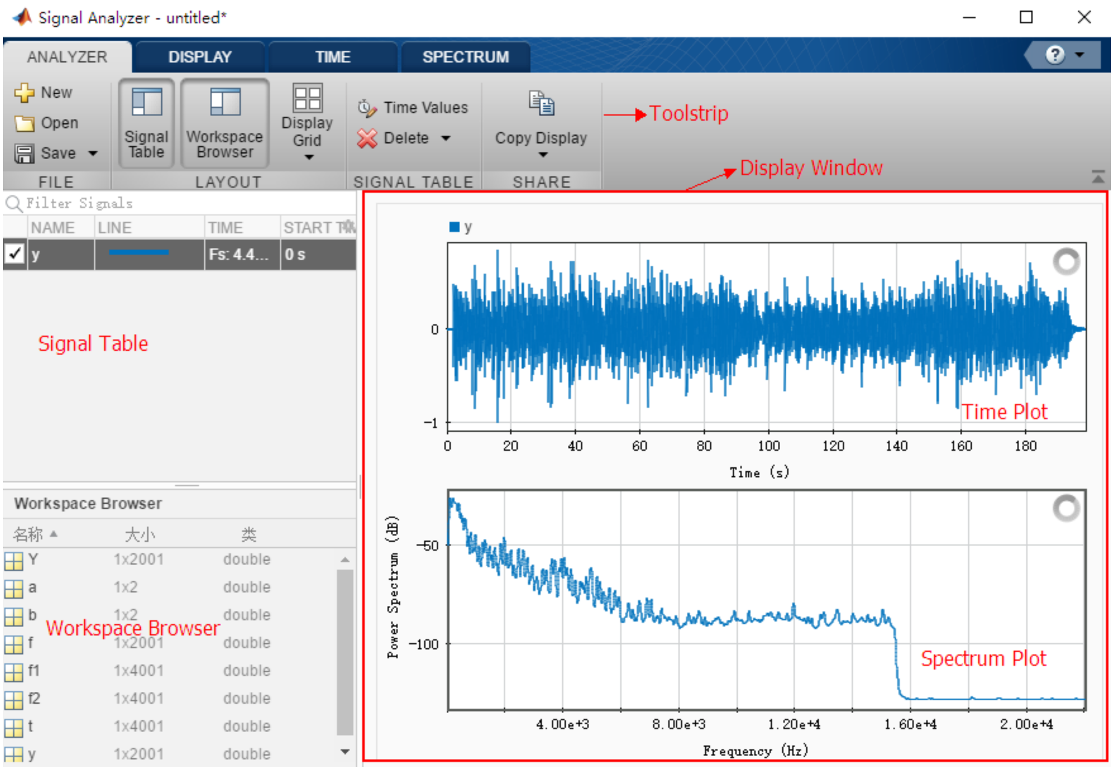



Figure 6 Signal Analyzer Interface

# System Analysis

As we mentioned in Lab 2, a dynamic system can be represented by a linear differential equation with constant coefficients, which is shown as below:

$$\sum_{i=0}^{N} a_i y^{(i)}(t) = \sum_{j=0}^{M} b_j x^{(j)}(t)$$

Do Fourier transform on both sides of the differential equation. So the frequency response of the system can be described as:

$$\mathrm{H(jw)} = \frac{Y(jw)}{X(jw)} = \frac{b_M(jw)^M + b_{M-1}(jw)^{M-1} + \cdots + b_1(jw) + b_0}{a_N(jw)^N + a_{N-1}(jw)^{N-1} + \cdots + a_1(jw) + a_0} = \frac{B(jw)}{A(jw)}$$

Function freqs is used to analyze H(jw), the format of which is:

$\mathrm{h = freqs(b, a, w)}$ or $\mathrm{[h, w] = freqs(b, a, n)}$

$\mathrm{b} = [b_M, b_{M-1}, \cdots, b_1, b_0], \mathrm{a} = [a_N, a_{N-1}, \cdots, a_1, a_0]$

w is the range of the frequency domain to observe, which is expressed as angular frequencies in rad/s.

n is the frequency point to compute the frequency response.

Example: the frequency response of an analog filter is like:

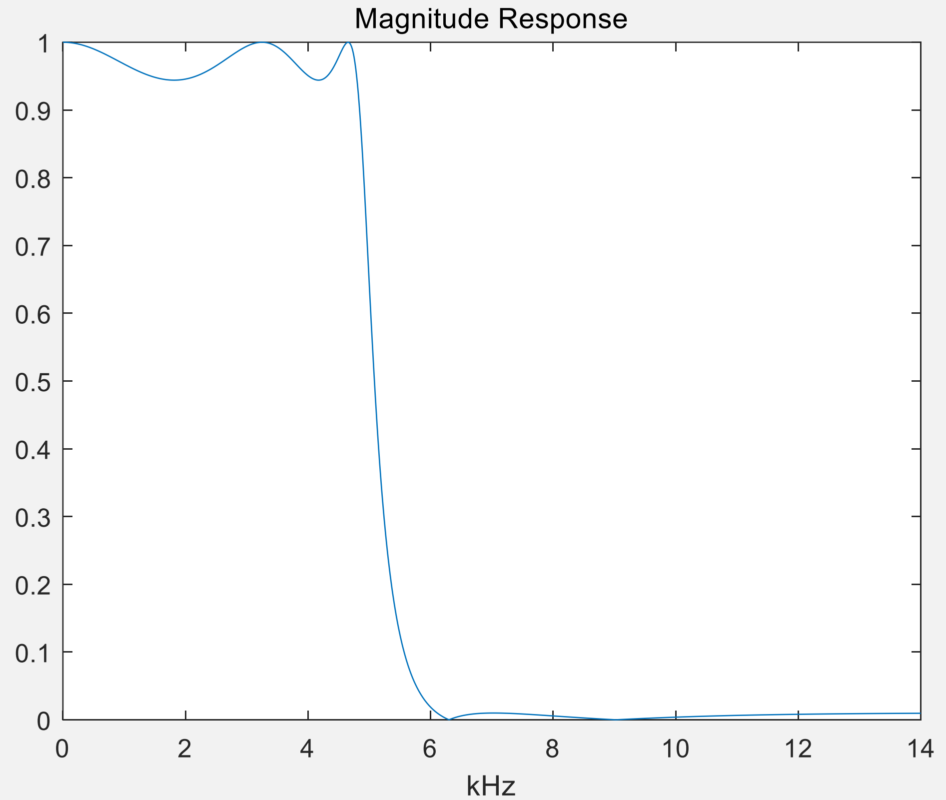
$$\mathrm{H(jw)} = \frac{b_4(jw)^4 + b_3(jw)^3 + b_2(jw)^2 + b_1(jw)^1 + b_0}{a_5(jw)^5 + a_4(jw)^4 + a_3(jw)^3 + a_2(jw)^2 + a_1(jw) + a_0}$$

$$\mathrm{b} = [b_4, b_3, b_2, b_1, b_0] = [1.53116389e + 03, -1.29990890e - 09, 7.32176217e + 12, -2.03715033e + 00, 7.71381999e + 21]$$

$$\mathrm{a} = [a_5, a_4, a_3, a_2, a_1, a_0] = [1, 3.47913978\mathrm{e} + 04, 1.87590501\mathrm{e} + 09, 4.03313474\mathrm{e} + 13, 7.97671668\mathrm{e} + 17, 7.71381999\mathrm{e} + 21]$$

So its frequency response can be calculated as:

```
b=[1.53116389e+03,-1.29990890e-09,7.32176217e+12,-
2.03715033e+00,7.71381999e+21];
a=[1,3.47913978e+04,1.87590501e+09,4.03313474e+13,7.97671668e+17,7.71
381999e+21];
w=linspace(1,14000*2*pi,1000);
H=freqs(b,a,w); plot(w/(2*pi)/1000, abs(H)); title('Magnitude
Response');xlabel('kHz');
```




Obviously it a low-pass filter.