Signals and Systems Homework 1
Solutions

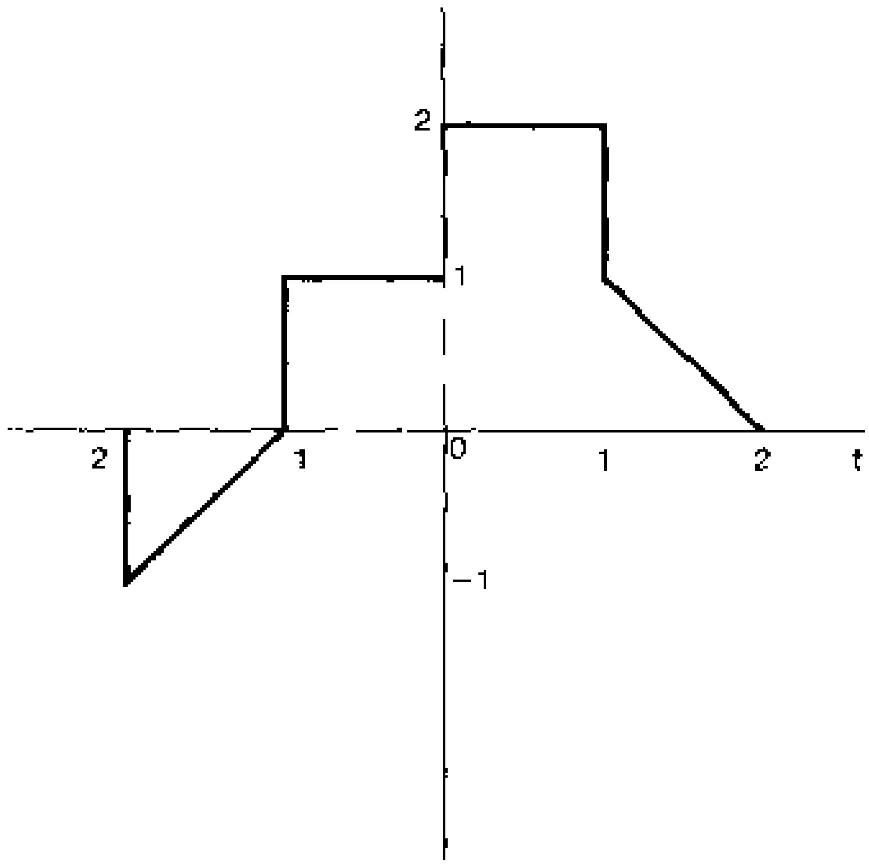
Exercises for Section 1: Basic Problems

1. A continuous-time signal $x(t)$ is shown in the following figure. Sketch and label carefully each of the following signals:

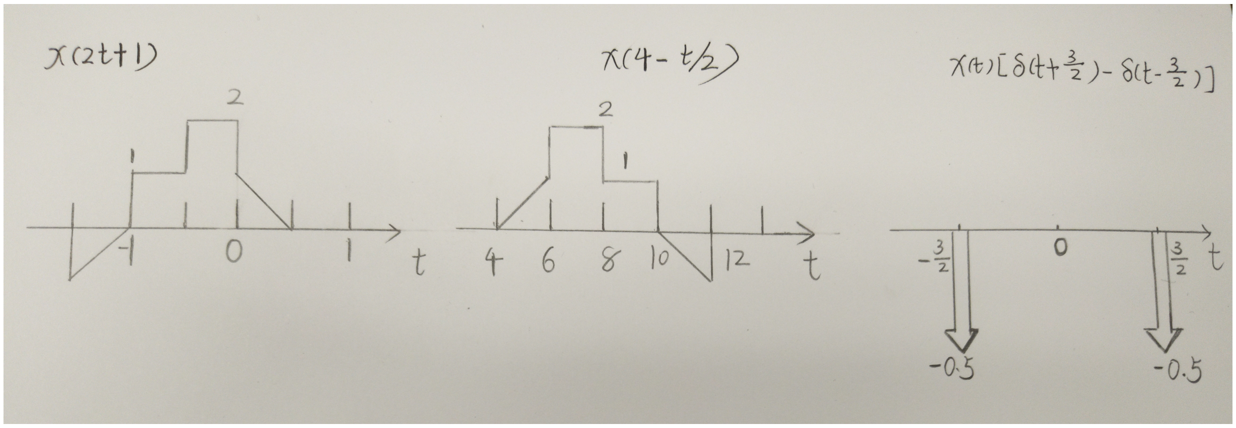
   (a) $x(2t+1)$

   (b) $x(4-\frac{t}{2})$

   (c) $x(t)[\delta(t+\frac{3}{2})-\delta(t-\frac{3}{2})]$



Solution



2. Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

   (a) $x(t) = 3\cos(4t + \frac{\pi}{3})$

   (b) $x(t) = e^{j(\pi t - 1)}$

   (c) $x(t) = \sum\limits_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t-n)$

   Solution

   (a) Periodic, period $= 2\pi/4 = \pi/2$.

   (b) Periodic, period $= 2\pi/\pi = 2$.

   (c) Periodic, period $= \frac{1}{2}$.

3. Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

   (a) $x[n] = \cos(\frac{\pi}{8}n^2)$

   (b) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$

   (c) $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

   Solution

   (a) Periodic, period $= 8$.

   (b) Periodic, period $= 8$, because $x[n] = \frac{1}{2}[\cos(\frac{3\pi}{4}n) + \cos(\frac{\pi}{4}n)]$.

   (c) Periodic, period $= 16$.

4. For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.

   (a) $y(t) = t^2 x(t-1)$
   (b) $y[n] = x^2[n-2]$
   (c) $y[n] = x[n+1] - x[n-1]$

   Solution

   (a) The system is linear, but not time-invariant.
   (b) The system is time-invariant, but not linear.
   (c) The system is both linear and time-invariant.

5. In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be

   (1) Memoryless
   (2) Invertible
   (3) Causal
   (4) Stable
   (5) Time invariant
   (6) Linear

   Determine which of the properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

   (a) $y(t) = [\cos(3t)]x(t)$
   (b)
   $$y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$$
   (c)
   $$y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$$

   Solution

   (a) The system is memoryless, causal, stable, linear.
   (b) The system is causal, stable, linear.
   (c) The system is causal, stable, time-invariant.

6. Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

   (a)
$$y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$$

   (b) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau) d\tau$

   (c) $y[n] = x[2n]$

   (d) $y(t) = \frac{dx(t)}{dt}$

   Solution

   (a) Invertible. Inverse system: $y[n] = x[n+1]$ for $n \geq 0$ and $y[n] = x[n]$ for $n < 0$.

   (b) Non invertible. If the value of $x(t)$ at some point changes, the value of $y(t)$ remains.

   (c) Non invertible. $x_1[n] = \delta[n] + \delta[n-1]$ and $x_2[n] = \delta[n]$ give $y[n] = \delta[n]$.

   (d) Non invertible. If $x(t)$ is any constant, then $y(t) = 0$.

Exercises for Section 2: Advanced Problems

1. Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \quad \text{and} \quad y_2[n] = \begin{cases} x[n/2], & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

   The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

   (1) If $x[n]$ is periodic, then $y_1[n]$ is periodic.

   (2) If $y_1[n]$ is periodic, then $x[n]$ is periodic.

   (3) If $x[n]$ is periodic, then $y_2[n]$ is periodic.

   (4) If $y_2[n]$ is periodic, then $x[n]$ is periodic.

   For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

   Solution

   (1) True. $x[n] = x[n+N]$; $y_1[n] = y[n+N_0]$. i.e. with period $N_0 = N/2$ if N is even, and with period $N_0 = N$ if N is odd.

   (2) False. $y_1[n]$ is periodic does not imply $x[n]$ is periodic. i.e. let $x[n] = g[n] + h[n]$ where $g[n] = \begin{cases} 1, & \text{n even} \\ 0, & \text{n odd} \end{cases}$ and $h[n] = \begin{cases} 0, & \text{n even} \\ (1/2)^n, & \text{n odd} \end{cases}$
   Then $y_1[n] = x[2n]$ is periodic but $x[n]$ is clearly not periodic.

   (3) True. $x[n+N] = x[n]$; $y_2[n+N_0] = y_2[n]$ where $N_0 = 2N$.

   (4) True. $y_2[n+N] = y_2[n]$; $x[n+N_0] = x[n]$ where $N_0 = N/2$.

2. (a) Is the following statement true or false?
The series interconnection of two linear time-invariant systems is itself a linear, time-invariant system.
Justify your answer.

(b) Is the following statement true or false?
The series interconnection of two nonlinear systems is itself nonlinear.
Justify your answer.

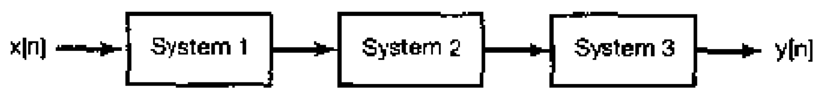
(c) Consider three systems with the following input-output relationships:

$$\text{System 1:} \qquad y[n] = \begin{cases} x[n/2], & \text{n even} \\ 0, & \text{n odd} \end{cases},$$

$$\text{System 2:} \qquad y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2],$$

$$\text{System 3:} \qquad y[n] = x[2n].$$

Suppose that these systems are connected in series as depicted in the following figure. Find the input-output relationship for the overall interconnected system. Is this system linear? Is it time invariant?



Solution

(a) True. Consider two systems S_1 and S_2 connected in series. Assume that if $x_1(t)$ and $x_2(t)$ are the inputs to S_1. Then $y_1(t)$ and $y_2(t)$ are the outputs respectively. Also, assume that if $y_1(t)$ and $y_2(t)$ are the inputs to S_2, then $z_1(t)$ and $z_2(t)$ are the outputs respectively. Since S_1 is linear, we may write

$$ax_1(t) + bx_2(t) \xrightarrow{S_1} ay_1(t) + by_2(t),$$

where a and b are constants. Since S_2 is also linear, we may write

$$ay_1(t) + by_2(t) \xrightarrow{S_2} az_1(t) + bz_2(t),$$

We may therefore conclude that

$$ax_1(t) + bx_2(t) \xrightarrow{S_1 S_2} az_1(t) + bz_2(t)$$

Therefore, the series combination of S_1 and S_2 is linear.
Since S_1 is time invariant, we may write

$$x_1(t - T_0) \xrightarrow{S_1} y_1(t - T_0),$$

and

$$y_1(t - T_0) \xrightarrow{S_2} z_1(t - T_0),$$

Therefore,

$$x_1(t - T_0) \xrightarrow{S_1 S_2} z_1(t - T_0)$$

Therefore, the series combination of S_1 and S_2 is time invariant.

(b) False. Let $y(t) = x(t) + 1$ and $z(t) = y(t) - 1$. These corresponds to two nonlinear systems. If these systems are connected in series, then $z(t) = x(t)$ which is a linear system.

(c) Let us name the output of system 1 as $w[n]$ and the output of system 2 as $z[n]$. Then,

$$y[n] = z[2n] = w[2n] + \frac{1}{2}w[2n-1] + \frac{1}{4}w[2n-2] = x[n] + \frac{1}{4}x[n-1]$$

The overall system is linear and time-invariant.