Solutions for HW6

Problem 1 Note that

$$H(j\omega) = \begin{cases} \dfrac{j\omega}{3\pi}, & -3\pi \le \omega \le 3\pi \\ 0, & \text{otherwise} \end{cases}$$

(a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 2\pi) + e^{-j\theta}\pi\delta(\omega + 2\pi)$. This is zero outside the region $-3\pi < \omega < 3\pi$. Thus, $Y(j\omega) = H(j\omega)X(j\omega) = (j\omega/3\pi)X(j\omega)$. This implies that $y(t) = (1/3\pi)dx(t)/dt = (-2/3)\sin(2\pi t + \theta)$.

(b) Since $x(t) = \cos(4\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 4\pi) + e^{-j\theta}\pi\delta(\omega + 4\pi)$. Therefore, the nonzero portions of $X(j\omega)$ lie outside the range $-3\pi < \omega < 3\pi$. This implies that $Y(j\omega) = X(j\omega)H(j\omega) = 0$. Therefore, $y(t) = 0$.

(c) The Fourier series coe icients of the signal $x(t)$ are given by

$$a_k = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t},$$

where $T_0 = 1$ and $\omega_0 = 2\pi/T_0 = 2\pi$. Also,

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k\delta(\omega - k\omega_0)$$

The only impulses of $X(j\omega)$ which lie in the region $-3\pi < \omega < 3\pi$ are at $\omega = 0, -2\pi$, and 2π. Defining the signal $x_{tp}(t) = a_0 + a_1 e^{j2\pi t} + a_{-1}e^{-j2\pi t}$, we note that $y(t) = (1/3\pi)dx_{tp}(t)/dt$. We can also easily show that $a_0 = 1/\pi, a_1 = 1/(4j), a_{-1} = -1/(4j)$. Putting these into the expression for $x_{tp}(t)$ we obtain $x_{tp}(t) = (1/\pi) + (1/2)\sin(2\pi t)$.
Finally, $y(t) = (1/3\pi)dx_{tp}(t)/dt = (1/3)\cos(2\pi t)$.

Problem 2 (a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2 + j\omega}$$

The Bode plot is as shown in **Figure 2**.

First-order systems

Bold Plots (Continuous time)

$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

- $\mathbf{20log_{10}|H(j\omega)|} = -10\log_{10}[(\omega\tau)^2 + 1]$

$$\simeq \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}$$

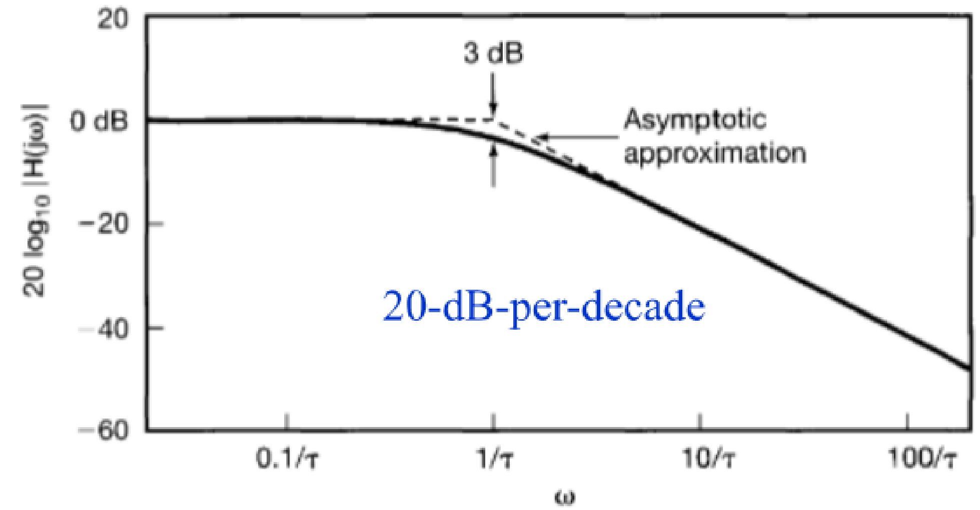
$$\omega = 1/\tau, 20\log_{10}|H(j\omega)| = -10\log_{10}(2) \simeq -3dB$$

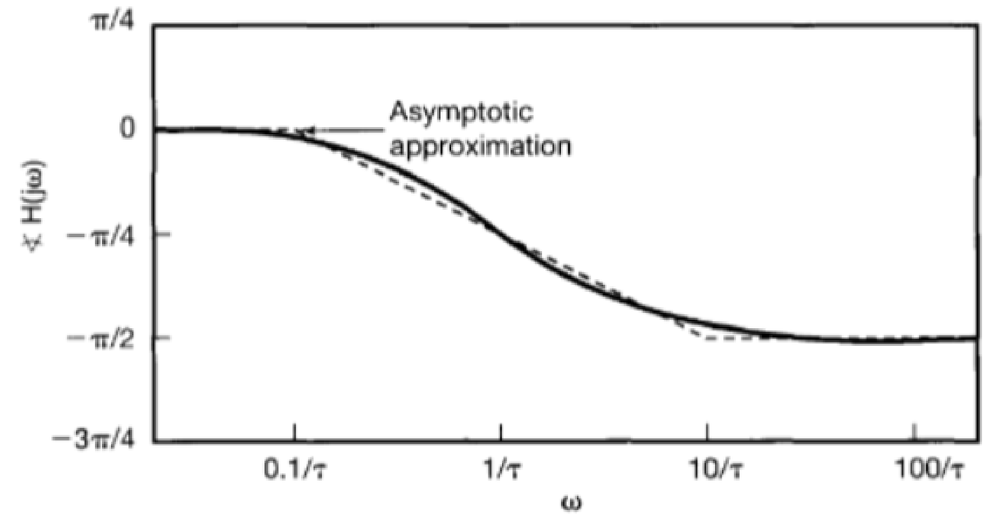
$\omega = 1/\tau$: break frequency

- $\angle \boldsymbol{H(j\omega)} = -\tan^{-1}(\omega\tau)$

$$\simeq \begin{cases} 0, & \omega \le 0.1/\tau \\ -\frac{\pi}{4}[log_{10}(\omega\tau) + 1], & 0.1/\tau \le \omega \le 10/\tau \\ -\pi/2, & \omega \ge 10/\tau \end{cases}$$

$$\omega = 1/\tau, \angle H(j\omega) = -\pi/4$$





$\tau\downarrow$, $h(t)$ and $s(t)$ more sharply, break frequency $\uparrow$.

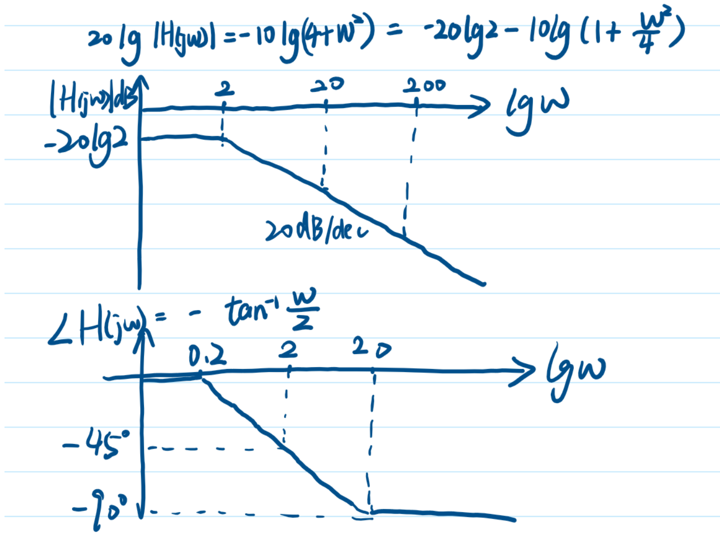


Figure 2

(b) From the expression for $H(j\omega)$ we obtain

$$\angle H(j\omega) = -\tan^{-1}(\omega/2).$$

Therefore,

$$\tau(\omega) = \frac{d\angle H(j\omega)}{d\omega} = \frac{2}{4+\omega^2}$$

.

(c) (i) Here,

$$Y(j\omega) = \frac{1+j\omega}{(2+jw)^2}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-2t}u(t) - te^{-2t}u(t).$$

(ii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)}$$

Taking the inverse Fourier transform of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t)$$

.

(iii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)^2}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t) - \ e^{-2t}u(t) - te^{-2t}u(t).$$

Problem 3

(10 points) Consider the discrete-time sequence $x[n] = cos[n\pi/5]$, find two different continuous-time signals that would produce this sequence when sampled at a frequency of $f_s = 500$Hz.

Solution

A continuous-time sinusoid

$$x_a(t) = cos(w_0 t) = cos(2\pi f_0 t) \tag{8}$$

that is sampled with a sampling frequency of f_s results in the discrete-time sequence

$$x[n] = x_a(nT_s) = cos(2\pi \frac{f_0}{f_s} n) \tag{9}$$

However, note that for any integer k,

$$cos(2\pi \frac{f_0}{f_s} n) = cos(2\pi \frac{f_0 + kf_s}{f_s} n) \tag{10}$$

Therefore, any sinusoid with a frequency

$$f = f_0 + kf_s \tag{11}$$

will produce the same sequence of samples $x[n]$ when sampled with a sampling frequency f_s. With $x[n] = cos(n\pi/5)$, we want

$$2\pi \frac{f_0}{f_s} = \frac{\pi}{5} \tag{12}$$

or

$$f_0 = \frac{1}{10}\ f_s = 50Hz \tag{13}$$

Therefore, two signals that produce the given sequence are

$$x_1(t) = cos(100\pi t) \tag{14}$$

and

$$x_2(t) = cos(1100\pi t) \tag{15}$$

Problem4

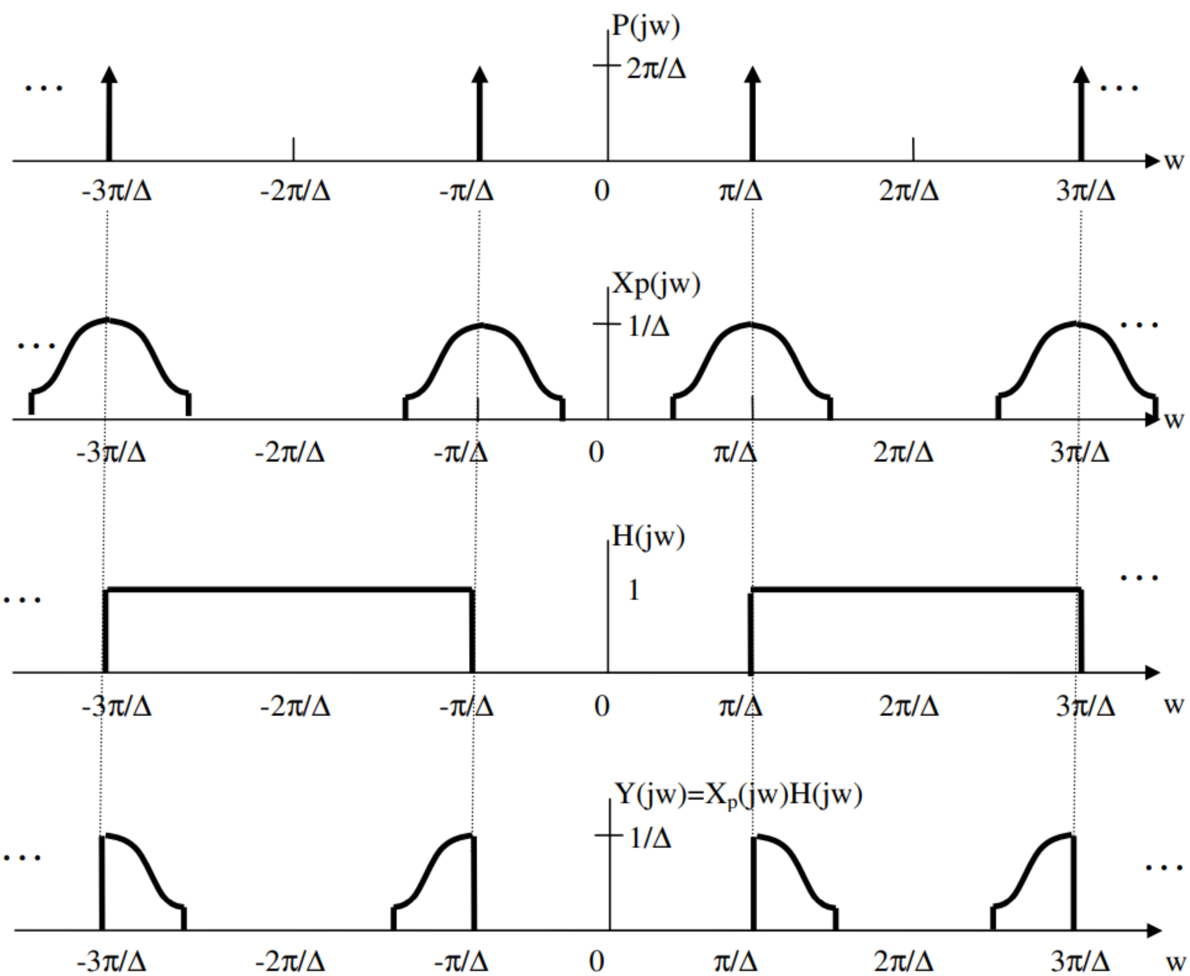
a) We can write p(t) = $p_1(t) - p_1(t-\Delta)$ where $p_1(t) = \sum_{k=-\infty}^{\infty}\delta(t-k2\Delta) \Rightarrow P_1(jw) = \frac{\pi}{\Delta}\sum_{k=-\infty}^{\infty}\delta(w-(k\pi)/\Delta)$

using the above information and the time shifting property we can write P(jw) as:

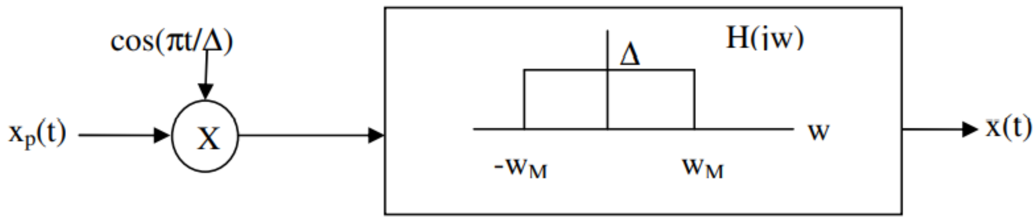
$P(jw) = P_1(jw) - e^{-jw\Delta}P_1(jw)$

$$P(jw) = \frac{\pi}{\Delta}\{\sum_{k=-\infty}^{\infty}\delta(w-\frac{\pi k}{\Delta}) - \sum_{k=-\infty}^{\infty}\delta(w-\frac{\pi k}{\Delta})e^{-jw\Delta}\} \quad where \quad e^{-jw\Delta} = e^{-j\pi k} = (-1)^k$$ where $w=\pi k/\Delta$

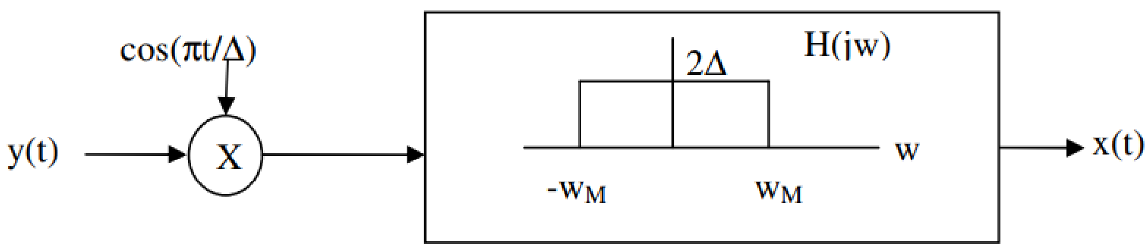
$$x_p(t) = x(t)p(t) \xrightarrow{FT} X_p(jw) = \frac{1}{2\pi}[X(jw) * P(jw)]$$



b) recovering x(t) from $x_p(t)$
Let's use:
1) $FT\{\cos(w_0t)\} = \pi[\delta(w-w_0) + \delta(w-w_0)$
2) Convolutions $FT\{x_p(t)\cos(\pi t/\Delta)\} = (1/2\pi)X_p(jw)*\{\ \pi[\delta(w-\pi/\Delta) + \delta(w-\pi/\Delta)\}$

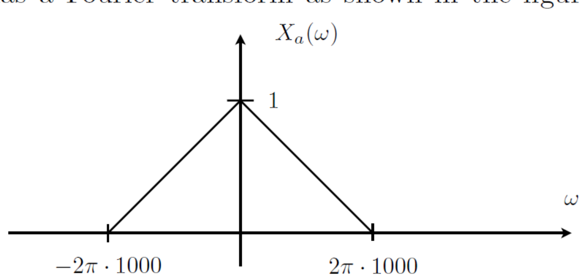


c) recovering x(t) from y(t) – use similar process as b.



Problem5

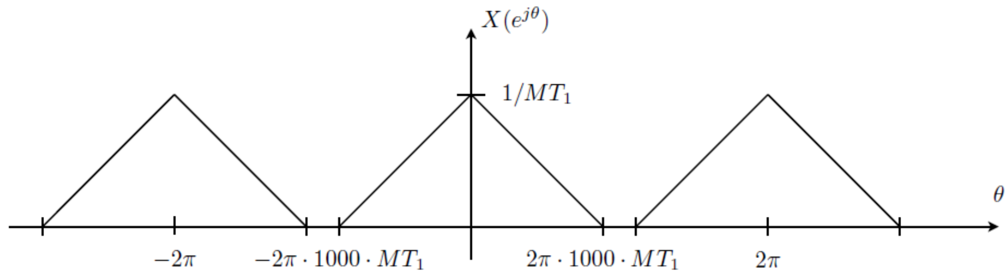
(a) Suppose that $x_a(t)$ has a Fourier transform as shown in the figure below. Because $y(n) =$



$x(Mn) = x_a(nMT_1)$, in order to prevent $x(n)$ from being aliased, it is necessary that

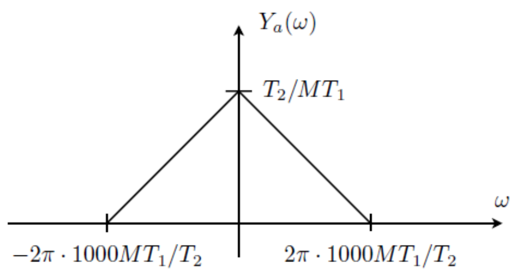
$$MT_1 < \frac{1}{2000}$$

If this constraint is satisfied, the output of the down-sampler has a FTD as shown below.



Going through the D/C converter produces signal $y_a(t)$, which has the Fourier transform shown below.

Therefore, in order for $y_a(t)$ to be equal to $x_a(t)$, we require that



1. $MT_1 \leq 1/2000$ in order to avoid aliasing.
2. $T_2 = MT_1$ to prevent frequency scaling.

(b) With $T_1 = T_2 = 1/20000$ and $M = 4$, note that

$$MT_1 = \frac{1}{5000} < \frac{1}{2000}$$

Therefore, there is no aliasing. Thus, as we see from the figure above,

$$Y_a(\omega) = \frac{1}{4} X_a\left(\frac{\omega}{4}\right)$$

or

$$y_a(t) = x_a(4t)$$