## Problem 1 (20 points)

You must show your detailed work to get full credit.

For the Wheatstone bridge circuit shown in Fig.1, solve the following problems:

(a) Express $I_a$, the reading on the ammeter, as a function of all the circuit elements $R_1$, $R_2$, $R_3$, $R_x$, $R_a$ and $V_0$.

(b) If $R_1 = 1\Omega$, $R_2 = 2\Omega$, and $R_x = 3\Omega$, to what value should $R_3$ be adjusted so as to achieve a balanced condition, that is, $I_a = 0$?

(c) Further, if $V_0 = 6\text{V}$, $R_a = 0.1\Omega$, and $R_x$ were then to deviate by a small amount to $R_x = 3.01\Omega$, what would be the reading on the ammeter?

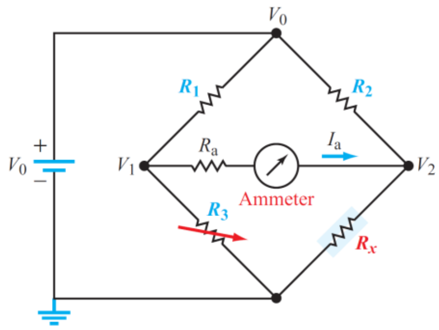
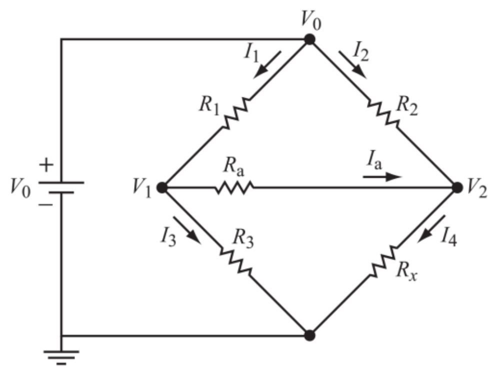



Fig. 1 for Problem 1.

**Solution**

**(a)**




KCL equations at nodes $V_1$ and $V_2$ are

$$I_1 = I_3 + I_a$$
$$I_4 = I_2 + I_a$$

KVL for the left loop and outside perimeter loop are

$$-V_0 + I_1R_1 + I_3R_3 = 0,$$
$$-V_0 + I_2R_2 + I_4R_x = 0.$$

Also, for the upper triangle of the bridge,

$$I_1R_1 + I_aR_a - I_2R_2 = 0.$$

Simultaneous solution of the five equations leads to

$$I_a = \frac{(R_3R_2 - R_xR_1)V_0}{R_2R_x(R_1+R_3) + (R_2+R_x)[R_1R_3 + R_a(R_1+R_3)]}.$$

(b)

$$\frac{R_3}{R_1} = \frac{R_x}{R_2}, \qquad R_3 = \frac{R_1R_x}{R_2} = \frac{1\times 3}{2} = 1.5\ \Omega.$$

(c)

For $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 1.5\ \Omega$, $R_x = 3.01\ \Omega$, and $V_0 = 6$ V,

$$I_a = -2.5\text{ mA}.$$

## Problem 2 (15 points)

You must show your detailed work to get full credit.

The circuit shown in Fig. 2 contains a variable load $R_L$.

(a) Choose $R_s$ so that $I_L$ never exceeds 4mA, regardless of the value of $R_L$.

(b) Given that choice, what is the maximum power that $R_L$ can extract from the circuit?

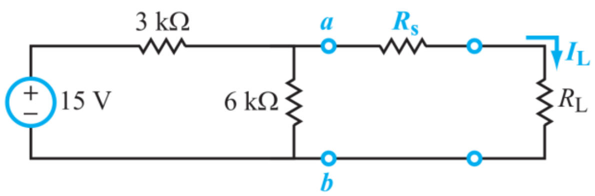
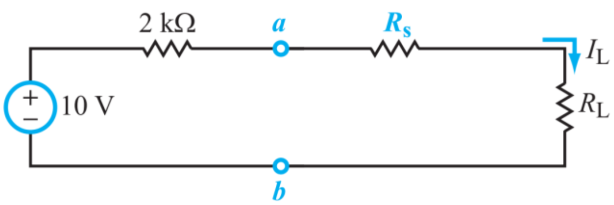



Fig. 2 for Problem 2.

**Solution:** We should start by finding the Thévenin equivalent of the circuit to the left of $(a, b)$. Simple source-transformation steps lead to:




To satisfy the stated condition, we need to choose $R_s$ such that $I_L = 4$ mA when $R_L = 0$. That is

$$I_L = 4 \text{ mA} = \frac{10}{2\text{k} + R_s},$$

which leads to $R_s = 0.5$ kΩ.

For maximum power transfer by $R_L$, it should be equal to:

$$R_L = 2 \text{ k}\Omega + R_s = 2.5 \text{ k}\Omega$$

$$I_L = \frac{10}{5\text{k}} = 2 \text{ mA}$$

$$P_{\max} = I_L^2 R_L = (2 \times 10^{-3})^2 \times 2.5 \times 10^3 = 10 \qquad \text{(mW)}.$$

## Problem 3 (20 points)

You must show your detailed work to get full credit.

For the circuit shown in Fig. 3, assume the op amp is ideal. Given $v_{in} = A\,u(t)$, $A = 6\text{V}$, $R_1 = 10\ \text{k}\Omega$, $R_2 = 5\ \text{k}\Omega$, $R_f = 50\ \text{k}\Omega$, and $C_1 = C_2 = 1\ \mu F$, determine $v_{out}(t)$ for $t \geq 0$.
Hint: There is no energy stored in the two capacitors at time $t = 0$.

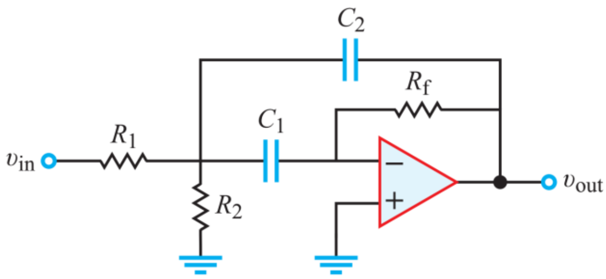



Fig. 3 for Problem 3.

**Solution:** Prior to $t = 0$, $v_{in}$ was equal to zero. Hence,

$$v_{C_1} = v_{C_1}(0^-) = 0, \qquad \text{[short-circuit equivalent]}$$
$$v_{C_2} = v_{C_2}(0^-) = 0. \qquad \text{[short-circuit equivalent]}$$

At $t = 0$ (Fig. (a)):

The capacitors have been replaced with short circuits. Also, the input circuit has been modified by applying source transformation. The new input-circuit parameters are

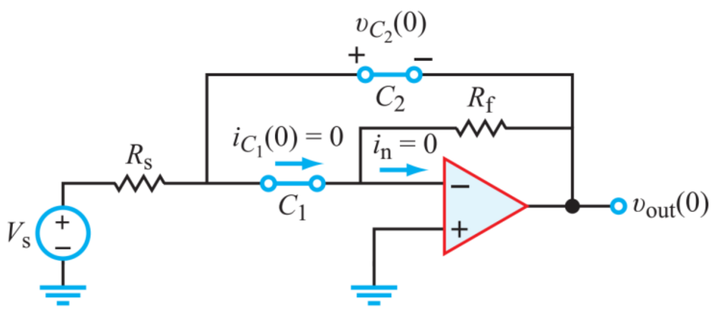
$$V_s = \frac{AR_2}{R_1 + R_2} = \frac{6 \times 5 \times 10^3}{10 \times 10^3 + 5 \times 10^3} = 2\ \text{V},$$
$$R_s = \frac{R_1 R_2}{R_1 + R_2} = \frac{5 \times 10}{5 + 10} = 3.33\ \text{k}\Omega.$$

In view of $i_n = 0$, if a current is generated by $V_s$, it will flow to $v_{out}$ entirely through the short-circuit branch containing $C_2$. Hence,

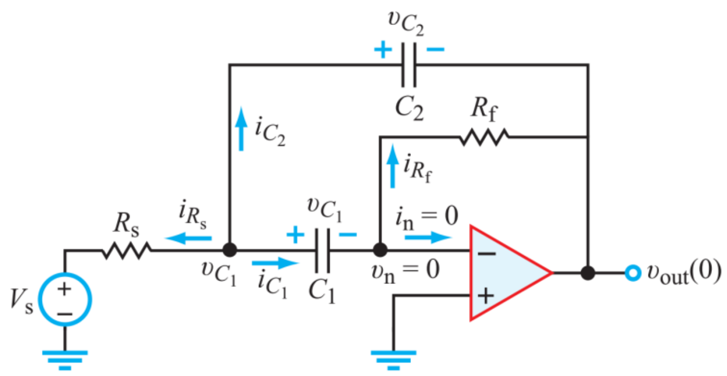
$$i_{C_1}(0) = 0,$$

which implies that

$$v'_{C_1}(0) = \frac{i_{C_1}(0)}{C} = 0.$$




(a) At $t = 0$




(b) At $t \geq 0$

At $t \geq 0$ (Fig. (b)):

At node $v_{C_1}$, KCL gives:

$$i_{R_S} + i_{C_2} + i_{C_1} = 0,$$

or

$$\frac{v_{C_1} - V_s}{R_s} + C_2(v'_{C_1} - v'_{out}) + C_1 v'_{C_1} = 0. \quad (1)$$

At node $v_n$:

$$i_{C_1} = i_{R_f},$$

or

$$C_1 v'_{C_1} = -\frac{v_{out}}{R_f}. \quad (2)$$

From (2)

$$v'_{out} = -R_f C_1 v''_{C_1}. \quad (3)$$

Upon using this expression for $v'_{out}$ in Eq. (1) and rearranging terms, we have

$$v''_{C_1} + a v'_{C_1} + b v_C = c,$$

where

$$a = \frac{C_1 + C_2}{C_1 C_2 R_f} = \frac{2 \times 10^{-6}}{10^{-12} \times 5 \times 10^4} = 40,$$

$$b = \frac{1}{C_1 C_2 R_s R_f} = \frac{1}{10^{-12} \times 3.33 \times 10^3 \times 5 \times 10^4} = 6000,$$

$$c = \frac{V_s}{C_1 C_2 R_s R_f} = 12000.$$

$$\alpha = \frac{a}{2} = 20 \text{ Np/s},$$

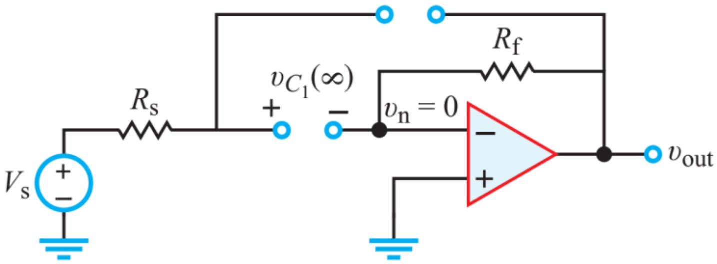
$$\omega_0 = \sqrt{b} = \sqrt{6000} = 77.46 \text{ rad/s}.$$

Since $\alpha < \omega_0$, the response will be underdamped:

$$v_{C_1}(t) = v_{C_1}(\infty) + [D_1 \cos \omega_d t + D_2 \sin \omega_d t] e^{-\alpha t},$$

with

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{6000 - 400} = 74.83 \text{ rad/s}.$$




(c) At $t = \infty$

At $t = \infty$ (Fig. (c)), both capacitors behave like open circuits. Consequently, no current flows through the circuit and

$$v_{C_1}(\infty) = V_s = 2\text{ V}.$$

Given that $v_{C_1}(0) = 0$, $v'_{C_1}(0) = 0$, and $v_{C_1}(\infty) = 2$ V,

$$D_1 = v_{C_1}(0) - v_{C_1}(\infty) = -2\text{ V},$$

$$D_2 = \frac{v'_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d} = \frac{0 + 20[0-2]}{74.83} = -0.53\text{ V}.$$

Hence,

$$v_{C_1}(t) = 2 - [2\cos 74.83t + 0.53\sin 74.83t]e^{-20t} \qquad \text{(V)}.$$

Returning to Eq. (2),

$$\begin{aligned} v_{out} &= -R_f C_1 v'_{C_1} \\ &= -5 \times 10^4 \times 10^{-6} v'_{C_1} \\ &= -5 \times 10^{-2}\left\{[2 \times 74.83 \sin 74.83t - 0.53 \times 74.83 \cos 74.83t]e^{-20t}\right. \\ &\qquad \left. + 20[2\cos 74.83t + 0.53 \sin 74.83t]e^{-20t}\right\} \\ &= -8e^{-20t}\sin 74.83t \qquad \text{(V)}. \end{aligned}$$

## Problem 4 (15 points)

You must show your detailed work to get full credit.

Determine the amount of <u>average power</u> delivered to $R_L$ in the circuit shown in Fig. 4. Assume that the op amp is ideal and $v_{\text{in}}(t) = 0.5\cos 2000t$ V, $R_1 = 1$ kΩ, $R_2 = 10$ kΩ, C = 0.1 μF, $R_L = 1$ kΩ and $L = 0.2$ H.

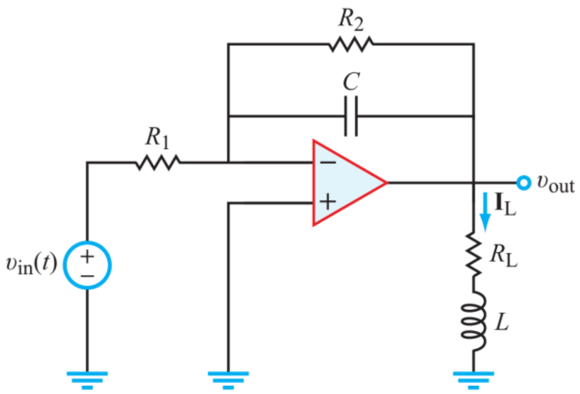
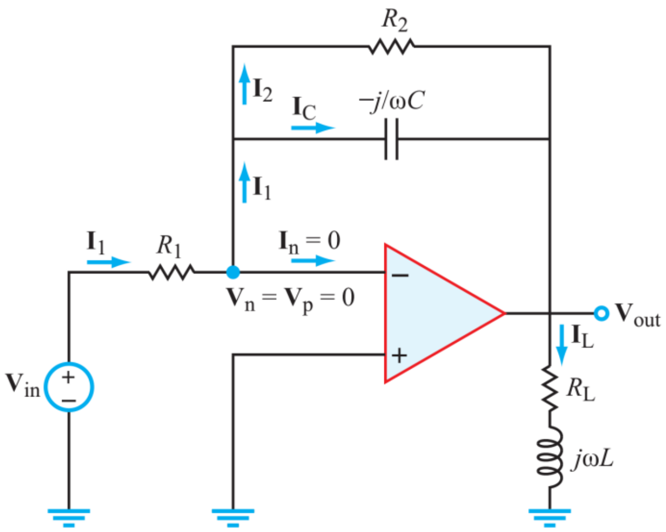



Fig. 4 for Problem 4.

**Solution:**




$\mathbf{V}_{\text{in}} = 0.5\angle 0°$ V.

Since $\mathbf{V}_n = 0$,

$$\mathbf{I}_1 = \frac{\mathbf{V}_{\text{in}}}{R_1}.$$

Also,

$$\begin{aligned}\mathbf{I}_1 &= \mathbf{I}_2 + \mathbf{I}_C \\ &= \frac{\mathbf{V}_n - \mathbf{V}_{\text{out}}}{R_2} + \frac{\mathbf{V}_n - \mathbf{V}_{\text{out}}}{-j/\omega C} \\ &= -\mathbf{V}_{\text{out}}\left(\frac{1}{R_2} + j\omega C\right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{V}_{\text{out}} &= -\left(\frac{R_2}{R_1}\right)\left(\frac{1}{1+j\omega R_2 C}\right)\mathbf{V}_{\text{in}}\\ &= -\left(\frac{R_2}{R_1}\right)\frac{1-j\omega R_2 C}{1+\omega^2 R_2^2 C^2}\,\mathbf{V}_{\text{in}}.\end{aligned}$$

For $\mathbf{V}_{\text{in}} = 0.5$ V, $R_1 = 1$ k$\Omega$, $R_2 = 10$ k$\Omega$, $\omega = 2000$ rad/s, and $C = 0.1\ \mu$F,

$$\begin{aligned}\mathbf{V}_{\text{out}} &= -(1-j2)\\ &= 2.24e^{j116.6^\circ}\ \text{V}.\\ \mathbf{I}_{\text{L}} &= \frac{1}{R_{\text{L}}+j\omega L}\,\mathbf{V}_{\text{out}}\\ &= \frac{1}{10^3+j400}\times 2.24e^{j116.6^\circ} = 2.08e^{j94.8^\circ} \qquad \text{(mA)}.\end{aligned}$$

Power delivered to $R_{\text{L}}$ alone is:

$$P_{\text{av}} = \frac{1}{2}\,|\mathbf{I}_{\text{L}}|^2 R_{\text{L}} = \frac{1}{2}\times(2.08)^2\times 10^{-6}\times 10^3 = 2.16\ \text{mW}.$$

## Problem 5 (15 points)

You must show your detailed work to get full credit.

For the circuit shown in Fig. 5,

a) Find the steady-state expressions for the currents $i_g$ and $i_L$ when $v_g = 168\cos 800t$ V.

b) Find the coefficient of coupling $k = \frac{M}{\sqrt{L_1 L_2}}$.

c) Find the total energy stored in the magnetically coupled coils at time $t = 1250\pi$ μs.

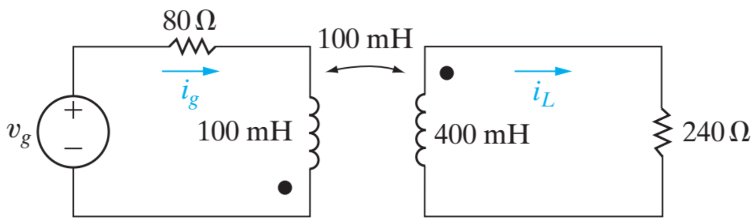


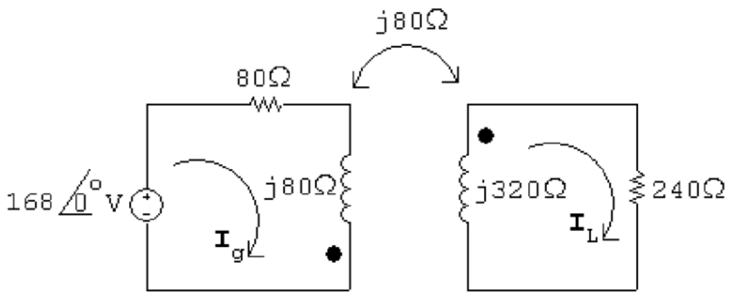

Fig. 5 for Problem 5.

**Solution**

[a] $j\omega L_1 = j(800)(100 \times 10^{-3}) = j80\,\Omega$

$$j\omega L_2 = j(800)(400 \times 10^{-3}) = j320\,\Omega$$

$$j\omega M = j80\,\Omega$$




$$168 = (80 + j80)\mathbf{I}_g + j80\mathbf{I}_L$$

$$0 = j80\mathbf{I}_g + (240 + j320)\mathbf{I}_L$$

Solving,

$$\mathbf{I}_g = 1.2 - j0.9\,\text{A}; \qquad \mathbf{I}_L = -0.3\,\text{A}$$

$$i_g = 1.5\cos(800t - 36.87^\circ)\,\text{A}$$

$$i_L = 0.3\cos(5000t - 180^\circ)\,\text{A}$$

**[b]** $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{0.1}{\sqrt{(0.1)(0.4)}} = 0.5$

**[c]**

When $t = 1250\pi\ \mu\text{s}$,

$$800t = \pi\,\text{rad} = 180^\circ$$

$$i_g(1250\pi\mu s) = 1.5\cos(180 - 36.87) = -1.2\,\text{A}$$

$$i_L(1250\pi\mu s) = 0.3\cos(180 - 180) = 0.3\,\text{A}$$

$$w = \frac{1}{2}(100 \times 10^{-3})(1.44) + \frac{1}{2}(400 \times 10^{-3})(0.09)$$
$$+100 \times 10^{-3}(-1.2)(0.3) = 54\,\text{mJ}$$

## Problem 6 (15 points)

You must show your detailed work to get full credit.

For the series RLC circuit of Fig. 6, $R = 5\ \Omega$, $L = 20$ mH, $C = 0.5$ μF.

(a) Obtain an expression for the transfer function $H(\omega) = V_R/V_s$.

(b) What are the values of the resonant frequency $\omega_0$ and quality factor $Q$?

(c) What are the values of half-power frequencies $\omega_{c1}$ and $\omega_{c2}$?

(d) Is it possible to double the magnitude of $Q$ by changing the values of $L$ and/or $C$, while keeping $\omega_0$ and $R$ unchanged? If yes, propose such values, and if no, explain why.

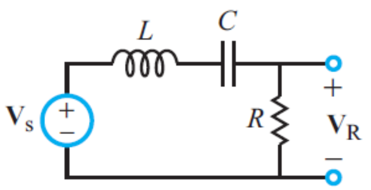



Fig. 6 for Problem 6.

**Solution:**

(a)

$$\frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{R\mathbf{I}}{\mathbf{V}_s} = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC}$$

(b)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 10^{-3} \times 0.5 \times 10^{-6}}} = 10^4 \text{ rad/s},$$

$$Q = \frac{\omega_0 L}{R} = \frac{10^4 \times 20 \times 10^{-3}}{5} = 40,$$

(c) Approximately

$$\omega_{c_1} = \omega_0 - \frac{B}{2} = 10^4 - \frac{250}{2} = 9875 \text{ rad/s},$$

$$\omega_{c_2} = \omega_0 + \frac{B}{2} = 10^4 + \frac{250}{2} = 10125 \text{ rad/s}.$$

(d)

$$Q = \frac{\omega_0 L}{R} \quad \Longrightarrow \quad \frac{\omega_0}{R} = \frac{Q}{L}.$$

Since $\omega_0$ and $R$ are constants, doubling $Q$ requires that $L$ be doubled, but to keep $\omega_0$ constant would require $C$ to be reduced to one half. Thus, the new set of element values are:

$$R = 5\ \Omega, \qquad L = 40 \text{ mH}, \qquad \text{and } C = 0.25\ \mu\text{F}.$$

The corresponding values of $\omega_0$ and $Q$ are:

$$\omega_0 = 10^4 \text{ rad/s (unchanged)}$$

$$Q = \frac{\omega_0 L}{R} = 80.$$