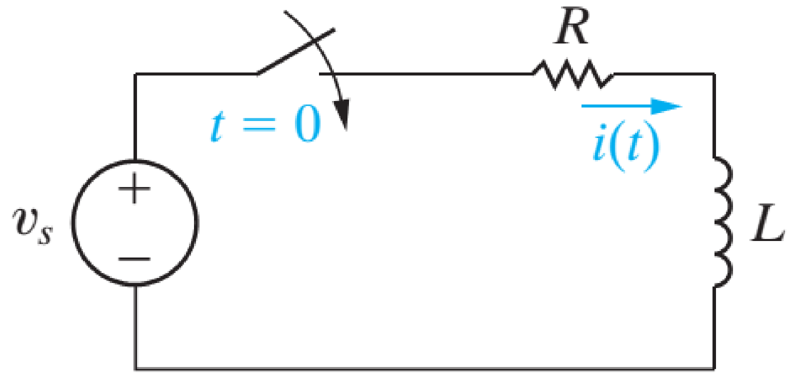
Lecture 8

- Sinusoidal Steady-State Analysis

AC Steady-State Analysis by Phasor Method

$$v_s = V_m \cos(\omega t + \phi)$$



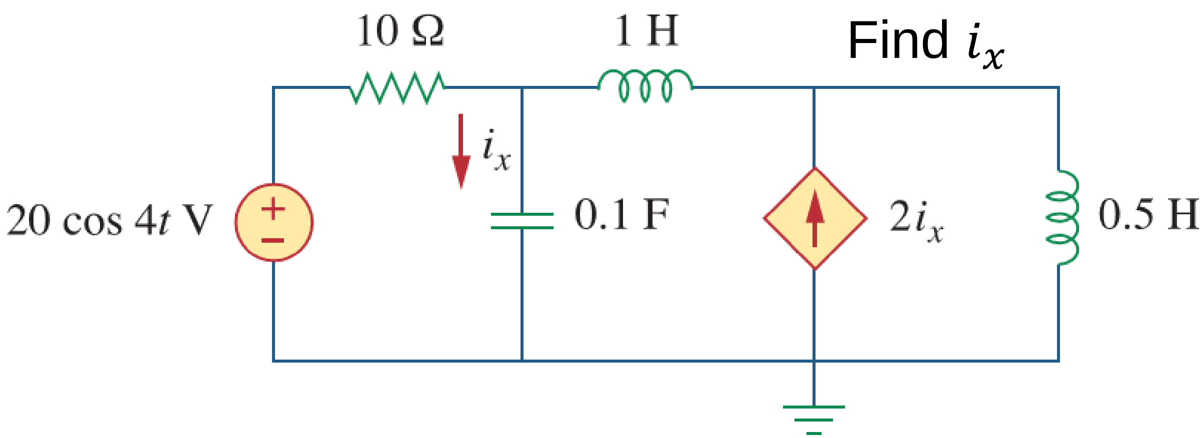
?

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

Transient response

Steady-state response

Circuit Analysis in Phasor Domain



- Phasor relationships for basic elements (R,L,C)
- Kirchhoff's laws in phasor domain
- Generalized impedance (series, parallel, Delta-to-Wye)
- Circuit analysis methods
  - Nodal/mesh analysis
  - Superposition
  - Source transformation/Thevenin/Norton

Outline

- Kirchhoff’s laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

Kirchhoff's Laws in the Frequency Domain

- Let $v_1, v_2, \cdots v_n$ be the voltages around a closed loop. Then according to KVL

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Similarly, KCL holds for phasors:

$$i_1 + i_2 + \cdots + i_n = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

Proof

If

$$v_1 + v_2 + \cdots + v_n = 0$$

where v_i are sinusoidal voltages <u>of the same frequency</u>, then

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\Updownarrow$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0$$

$$\Updownarrow$$

$$\mathrm{Re}\left(V_{m1}e^{j\theta_1} \cdot e^{j\omega t}\right) + \cdots + \mathrm{Re}\left(V_{mn}e^{j\theta_n} \cdot e^{j\omega t}\right) = 0$$

$$\Updownarrow$$

$$\mathrm{Re}\left((\mathbf{V}_1 + \cdots + \mathbf{V}_n) \cdot e^{j\omega t}\right) = 0 \qquad Where\ \mathbf{V}_k = V_{mk}e^{j\theta_k}$$

$$\Updownarrow$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Example

If $y_1 = 20\cos(\omega t - 30°)$ and $y_2 = 40\cos(\omega t + 60°)$, express $y = y_1 + y_2$ as a single sinusoidal function.

1. Use trigonometric identities
2. Use the phasor concept

$$y = (20\cos 30 + 40\cos 60)\cos\omega t$$
$$+ (20\sin 30 - 40\sin 60)\sin\omega t$$
$$= 37.32\cos\omega t - 24.64\sin\omega t.$$
$$y = 44.72\cos(\omega t + 33.43^\circ)$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$
$$= 20\angle -30^\circ + 40\angle 60^\circ$$
$$= (17.32 - j10) + (20 + j34.64)$$
$$= 37.32 + j24.64$$
$$= 44.72\angle 33.43^\circ.$$

Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

Review: Impedance and Admittance

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Impedance is voltage/current

$$\mathbf{Z} = R + jX$$

R = resistance = Re(Z)

X = reactance = Im(Z)

Admittance is current/voltage

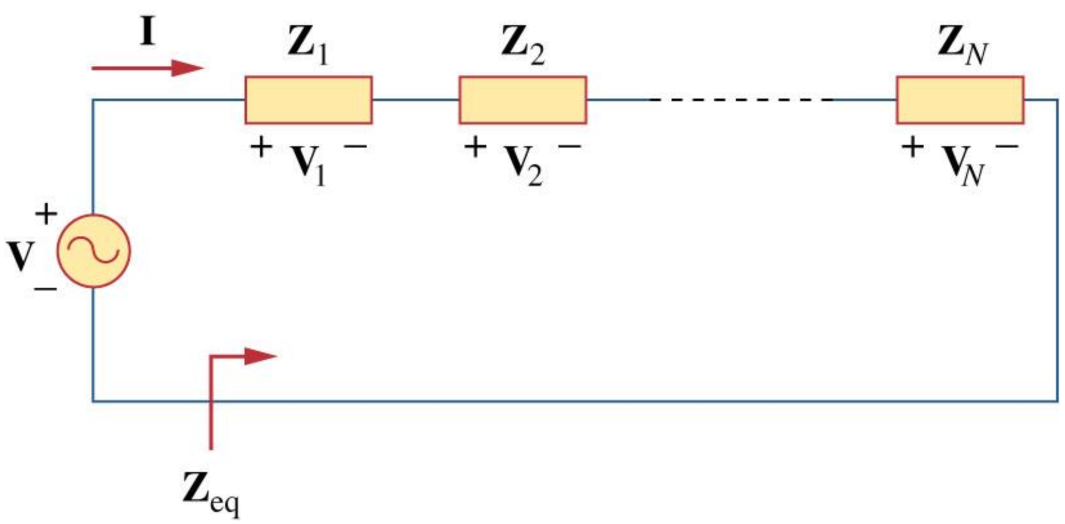
$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G + jB$$

G = conductance = Re(Y)

B = susceptance = Im(Y)

Series Impedance

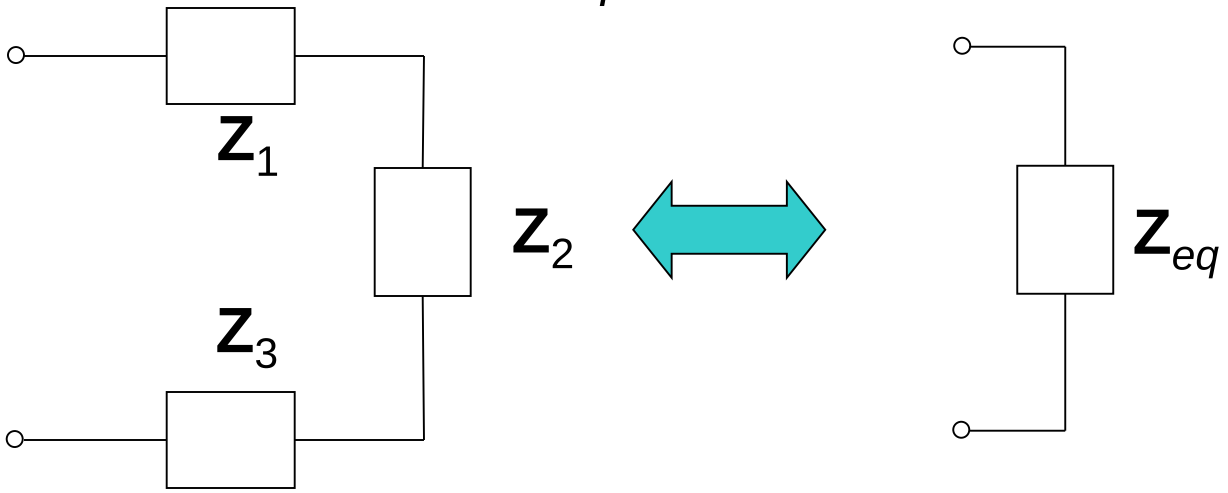
- Once in frequency domain, the impedance elements are generalized, combinations will follow the rules for resistors:



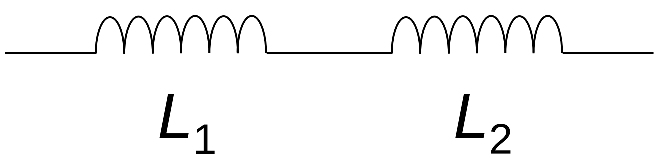
$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$

Series Impedance

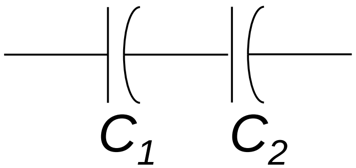
$$Z_{eq} = Z_1 + Z_2 + Z_3$$



For example:

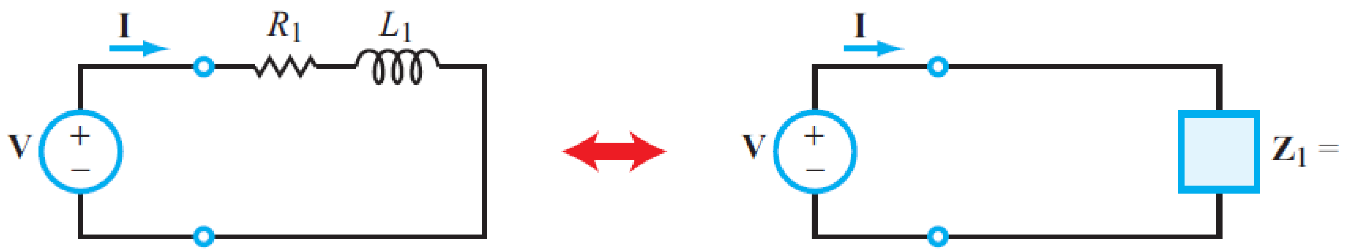


$Z_{eq} =$

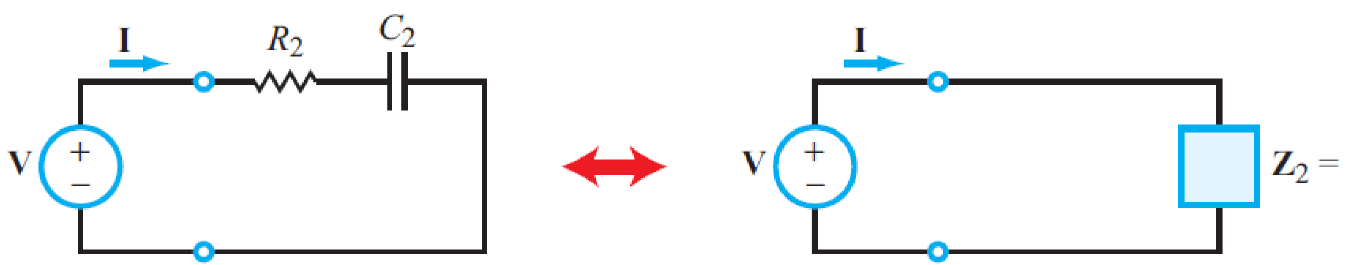


$Z_{eq} =$

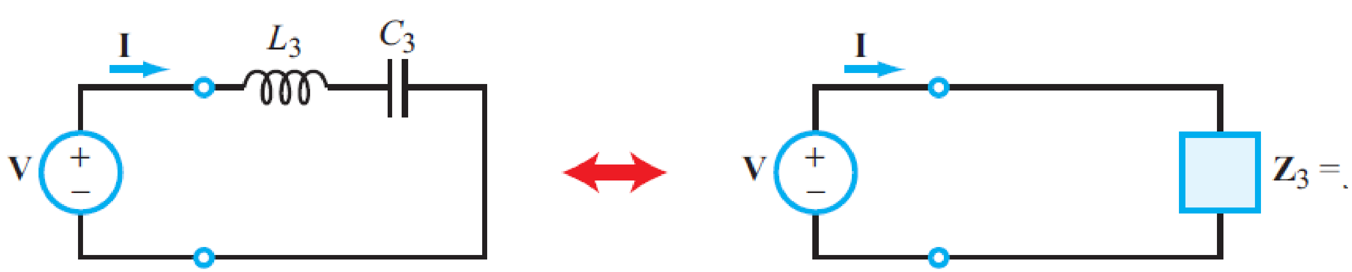
Impedance Transformation for RLC Circuit



(a) RL



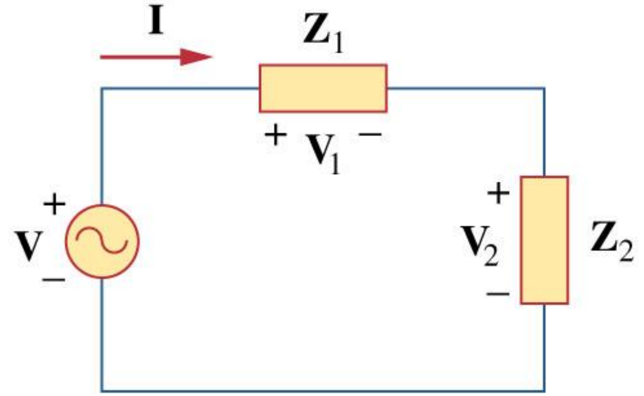
(b) RC



(c) LC

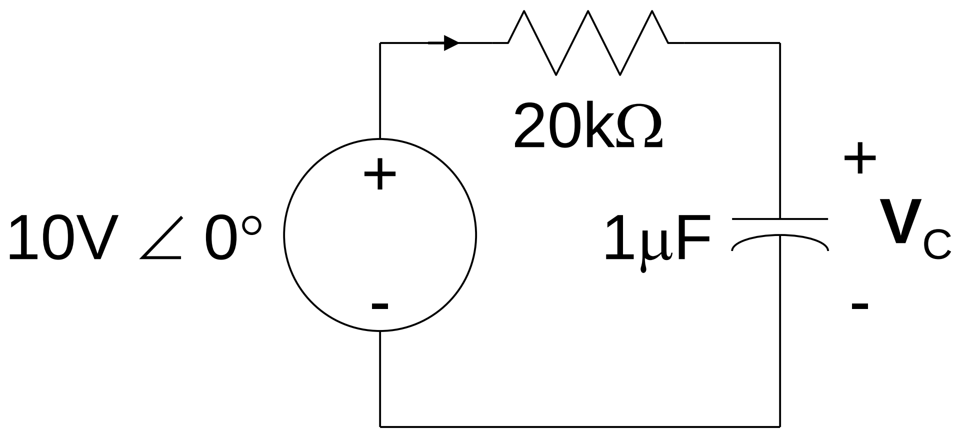
Voltage Divider

- Two elements in series can act like a voltage divider



$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \qquad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

Example

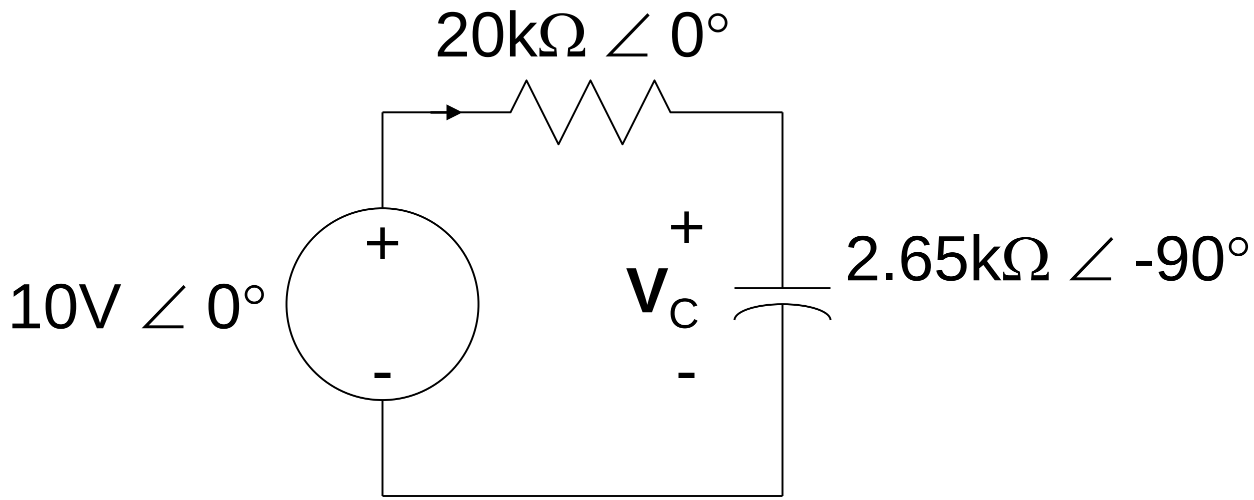


1. f=60 Hz, $\mathbf{V}_C$=?

First compute impedances for resistor and capacitor:

$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°

$\mathbf{Z}_C = 1/j\,(2\pi f \times 1\mu F) = 2.65k\Omega \angle -90^\circ$



Now use the voltage divider to find $\mathbf{V}_C$:

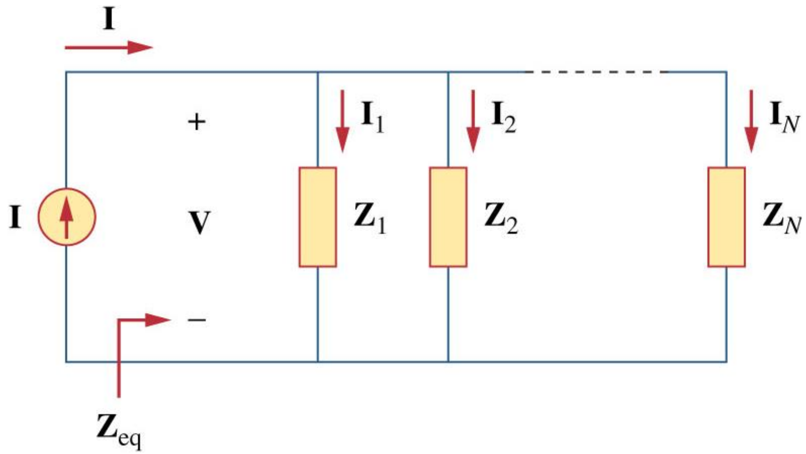
$$\mathbf{V}_C = 10\text{V} \angle 0^\circ \left(\frac{2.65\text{k}\Omega \angle -90^\circ}{2.65\text{k}\Omega \angle -90^\circ + 20\text{k}\Omega \angle 0^\circ} \right)$$

$$\mathbf{V}_C = 1.31\text{V} \angle -82.4^\circ$$

What if ω = 10, find $\mathbf{V}_C$

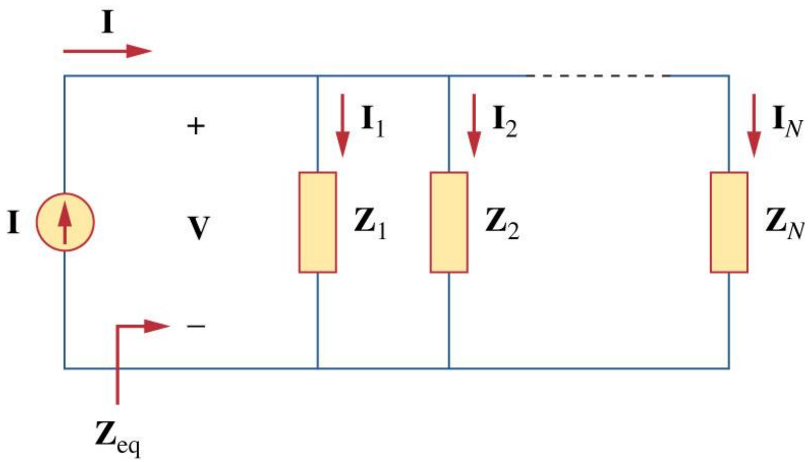
Parallel Combination

- Likewise, elements combined in parallel will combine in the same fashion as resistors in parallel:



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$

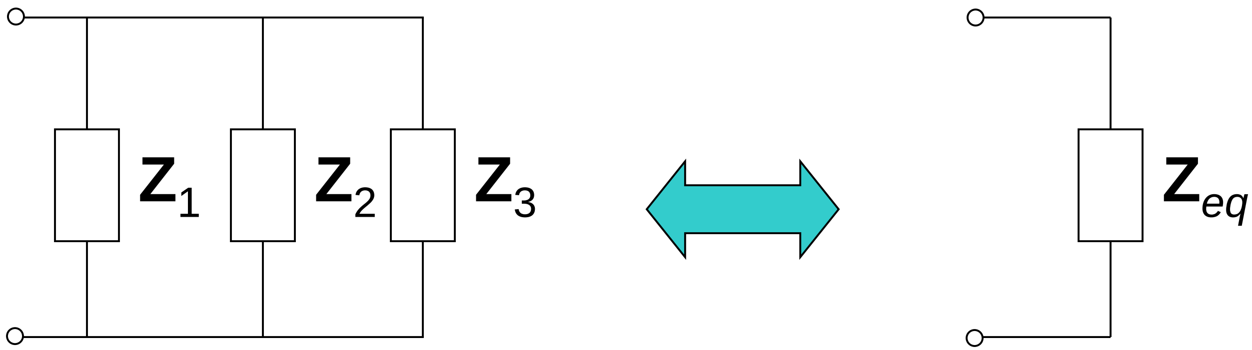
$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$



$$I_1 = \frac{Y_1}{Y_1 + \cdots + Y_N} I$$

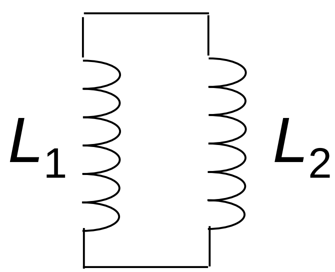
$$I_2 = \frac{Y_2}{Y_1 + \cdots + Y_N} I$$

Parallel Impedance

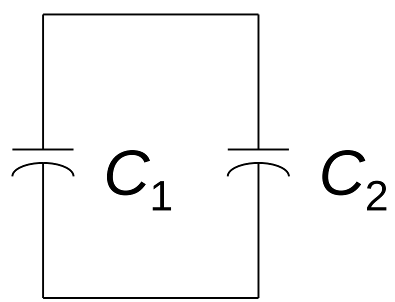


$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:

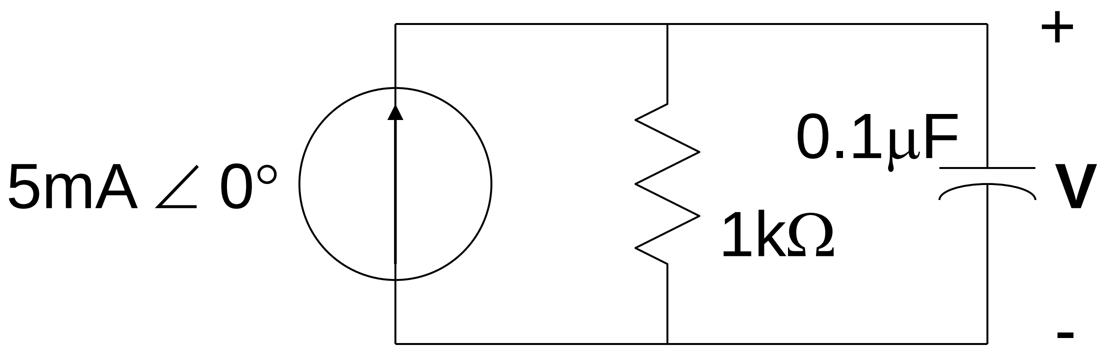


$$\mathbf{Z}_{eq} = j\omega \frac{L_1 L_2}{(L_1 + L_2)}$$

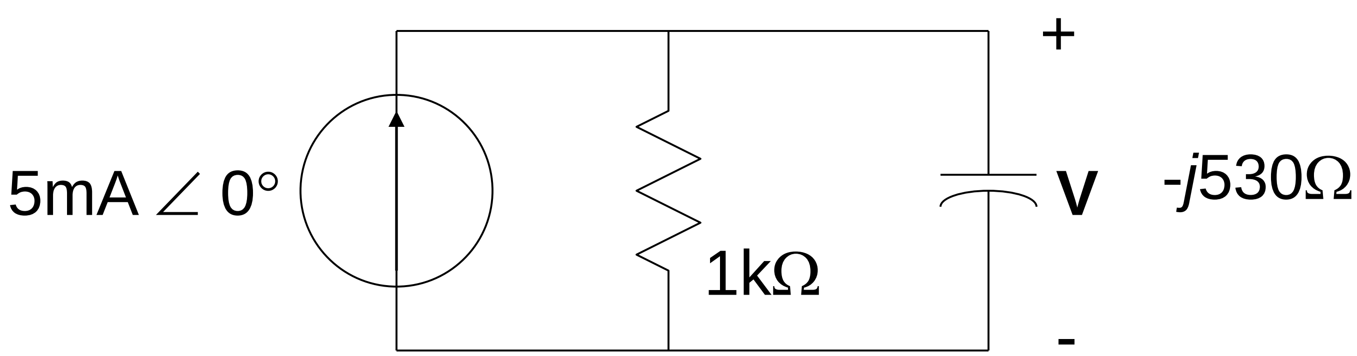


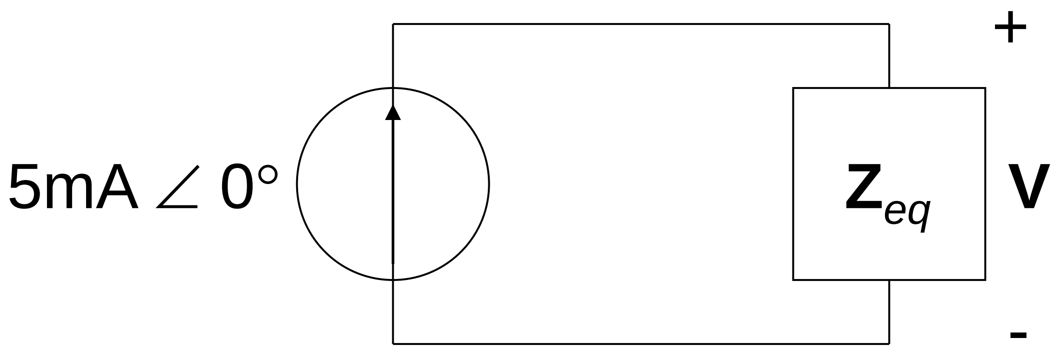
$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

Example



Find $v(t)$ for $\omega = 2\pi \times 3000$





$$\mathbf{Z}_{eq} = \frac{1000\times(-j530)}{1000-j530} = \frac{10^3\angle 0^\circ \times 530\angle -90^\circ}{1132\angle -27.9^\circ}$$

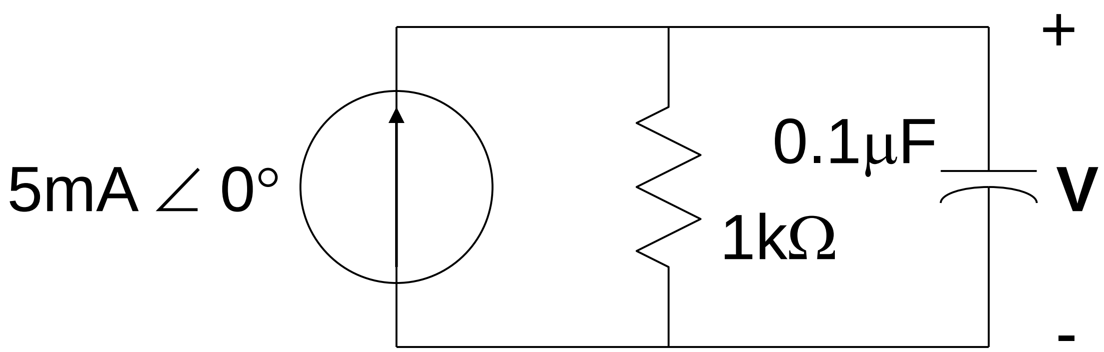
$$\mathbf{Z}_{eq} = 468.2\Omega\angle -62.1^\circ$$

$$\mathbf{V} = \mathbf{IZ}_{eq} = 5\text{mA}\angle 0^\circ \times 468.2\Omega\angle -62.1^\circ$$

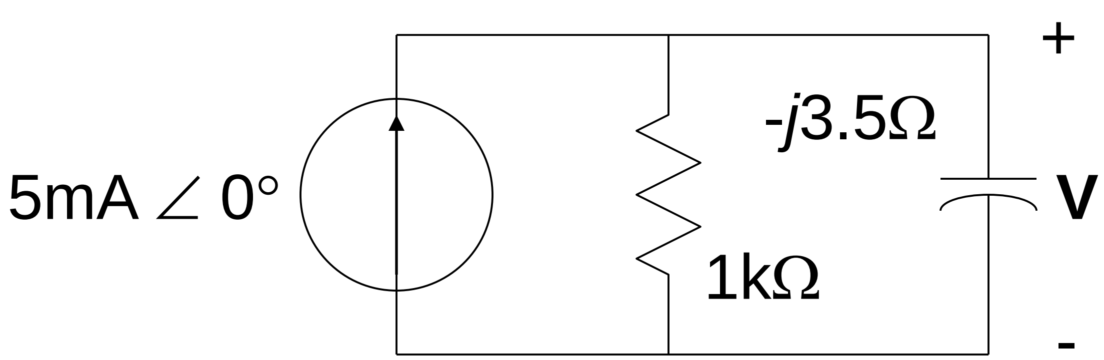
$$\mathbf{V} = 2.34\text{V}\angle -62.1^\circ$$

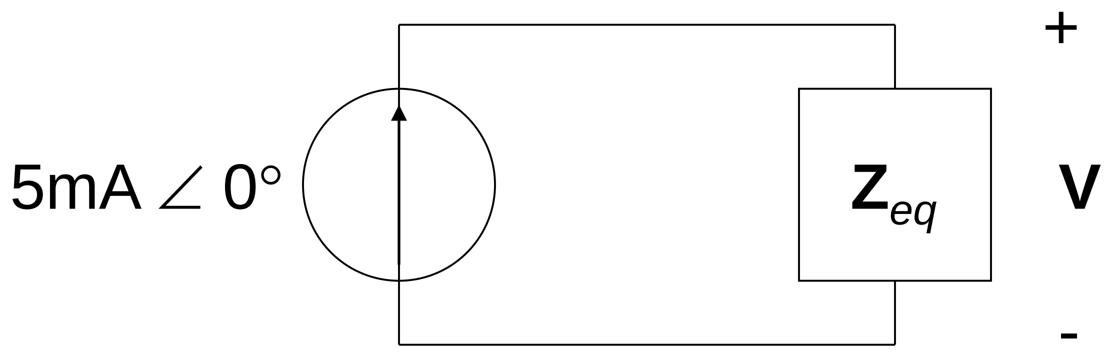
$$v(t) = 2.34\cos(2\pi 3000\text{t} - 62.1^\circ)\text{V}$$

Change the Frequency



Find $v(t)$ for $\omega=2\pi$ 455000





$$\mathbf{Z}_{eq} = \frac{1000 \times (-j3.5)}{1000 - j3.5} = \frac{10^3 \angle 0° \times 3.5 \angle -90°}{1000 \angle -0.2°}$$

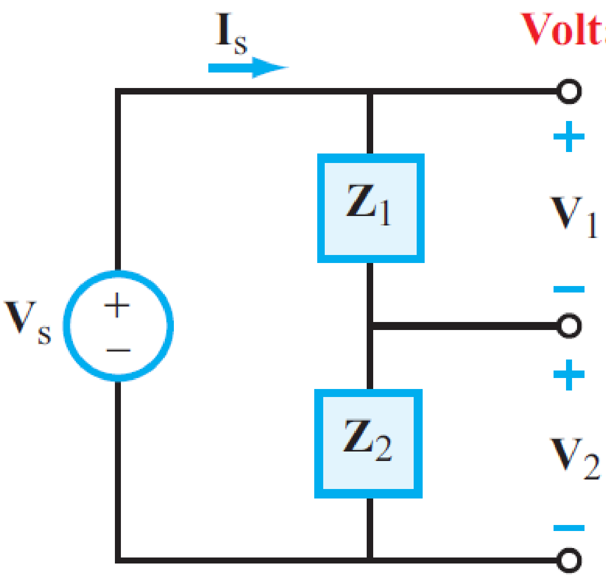
$$\mathbf{Z}_{eq} = 3.5 \angle -89.8° \Omega$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = 5 \angle 0° \text{mA} \times 3.5 \angle -89.8° \Omega \quad \mathbf{V} = 17.5 \angle -89.8° \text{mV}$$

$$v(t) = 17.5 \cos(2\pi 455000\text{t} - 89.8°)\text{mV}$$

Summary: Voltage & Current Division

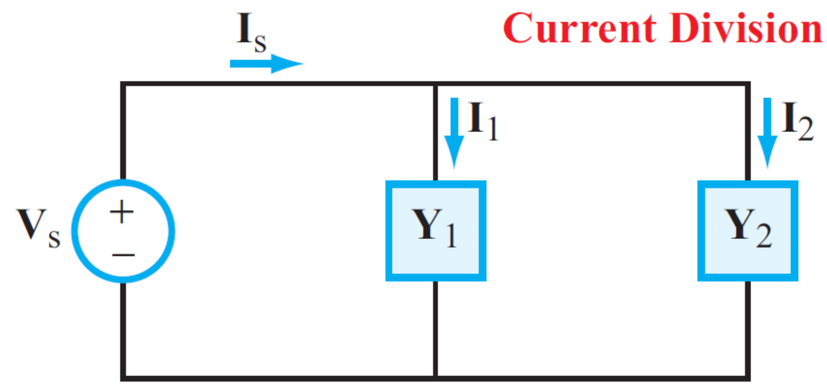
Voltage Division



$$V_1 = \left(\frac{Z_1}{Z_1 + Z_2}\right) V_s$$

$$V_2 = \left(\frac{Z_2}{Z_1 + Z_2}\right) V_s$$

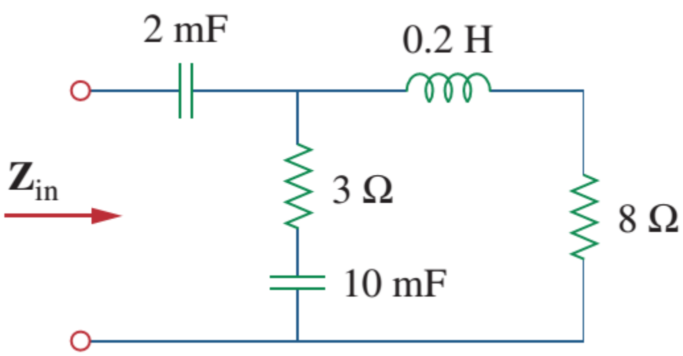
Current Division



$$I_1 = \left(\frac{Y_1}{Y_1 + Y_2}\right) I_s \qquad I_2 = \left(\frac{Y_2}{Y_1 + Y_2}\right) I_s$$

Exercise

- Find the input impedance of the circuit below. $\omega = 50$ rad/s.



Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
  - Nodal/mesh analysis
  - Superposition
  - Source transformation/Thevenin/Norton
- Phasor diagram

AC Phasor Analysis General Procedure

Step 1: Adopt cosine reference

$$v_s(t) = 12\sin(\omega t - 45^\circ)$$
$$= 12\cos(\omega t - 45^\circ - 90^\circ) = 12\cos(\omega t - 135^\circ)\ \text{V}.$$

$$\mathbf{V}_s = 12e^{-j135^\circ}\ \text{V}.$$

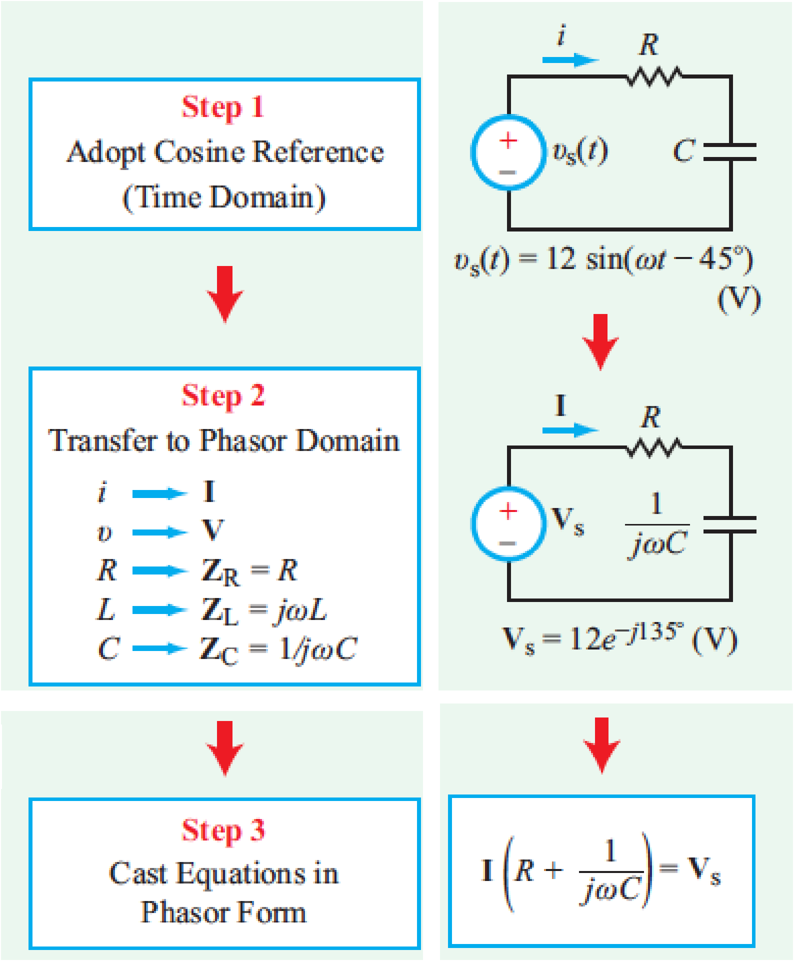
Step 2: Transform circuit to phasor domain

Step 3: Cast KCL and/or KVL equations in phasor domain

$$\mathbf{Z}_R\mathbf{I} + \mathbf{Z}_C\mathbf{I} = \mathbf{V}_s,$$

which is equivalent to

$$\left(R + \frac{1}{j\omega C}\right)\mathbf{I} = 12e^{-j135^\circ}.$$



AC Phasor Analysis General Procedure

Step 4: Solve for unknown variable

$$\mathbf{I} = \frac{12e^{-j135^\circ}}{R + \frac{1}{j\omega C}} = \frac{j12\omega C e^{-j135^\circ}}{1 + j\omega RC}.$$

Using the specified values, namely $R = \sqrt{3}$ kΩ, $C = 1$ μF, and $\omega = 10^3$ rad/s,

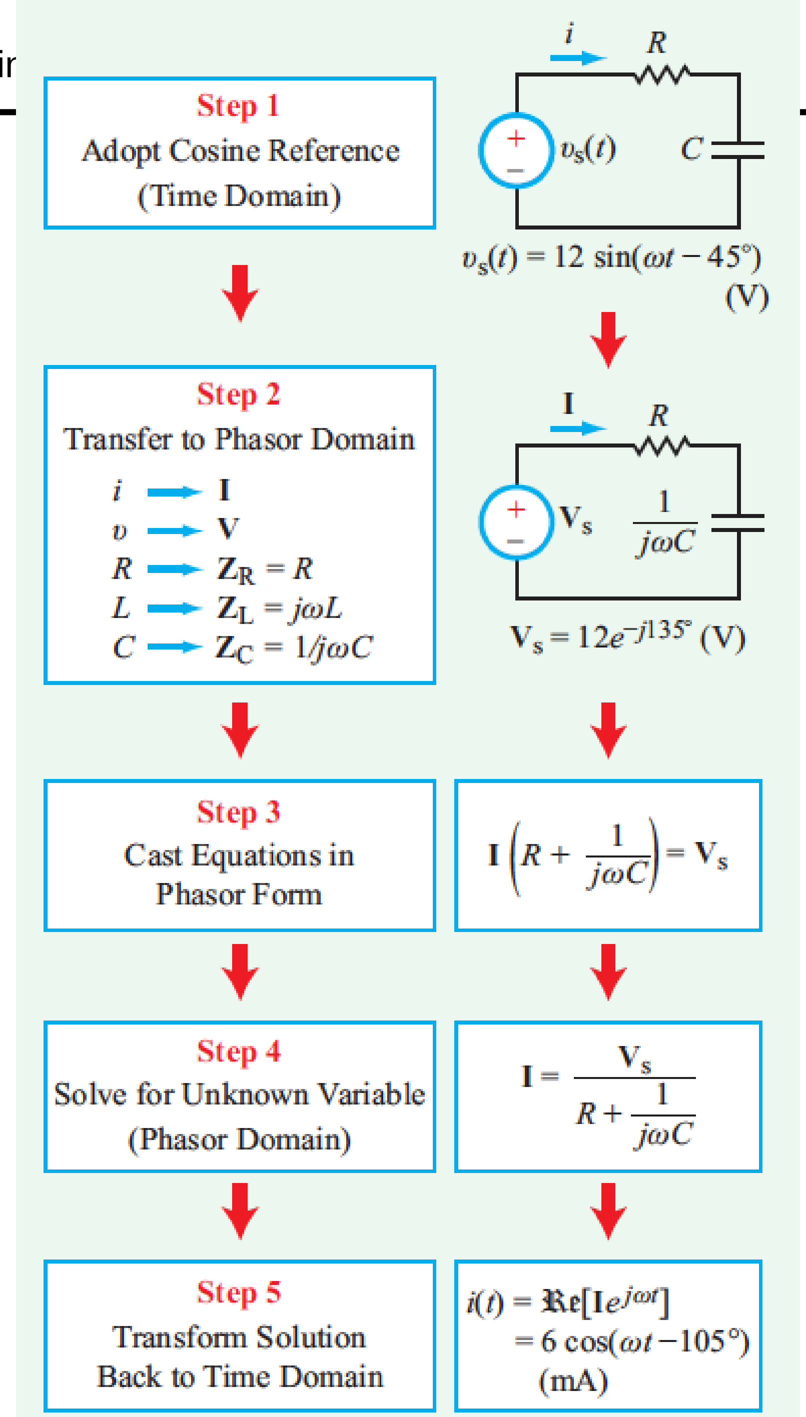
$$\mathbf{I} = \frac{j12 \times 10^3 \times 10^{-6} e^{-j135^\circ}}{1 + j10^3 \times \sqrt{3} \times 10^3 \times 10^{-6}} = \frac{j12e^{-j135^\circ}}{1 + j\sqrt{3}} \text{ mA}.$$

$$\mathbf{I} = \frac{12e^{-j135^\circ} \cdot e^{j90^\circ}}{2e^{j60^\circ}} = 6e^{j(-135^\circ + 90^\circ - 60^\circ)} = 6e^{-j105^\circ} \text{ mA}.$$

Step 5: Transform solution back to time domain

To return to the time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart, namely

$$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}] = \mathfrak{Re}[6e^{-j105^\circ} e^{j\omega t}] = 6\cos(\omega t - 105^\circ) \text{ mA}.$$



Example: RL Circuit

$$v_s(t) = 15\sin(4 \times 10^4 t - 30°) \text{ V}.$$

Also, $R = 3\ \Omega$ and $L = 0.1$ mH. Obtain an expression for the voltage across the inductor.

Solution:

Step 1: Convert $\boldsymbol{v_s(t)}$ to the cosine reference

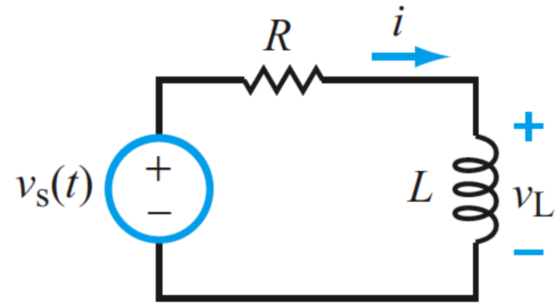
$$v_s(t) = 15\sin(4 \times 10^4 t - 30°) \;= 15\cos(4 \times 10^4 t - 120°) \text{ V},$$

$$\mathbf{V}_s = 15e^{-j120°} \text{ V}.$$

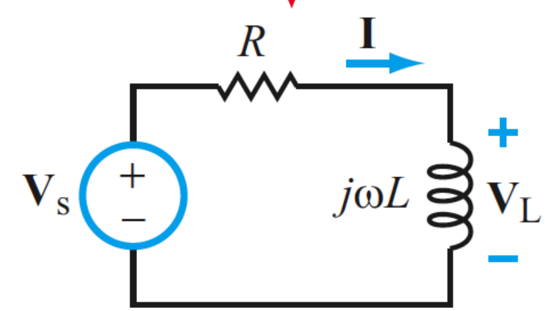
Step 2: Transform circuit to the phasor domain

Step 3: Cast KVL in phasor domain

$$R\mathbf{I} + j\omega L\mathbf{I} = \mathbf{V}_s.$$



(a) Time domain



(b) Phasor domain

Step 4: Solve for unknown variable

$$\mathbf{I} = \frac{\mathbf{V}_\mathrm{s}}{R + j\omega L} = \frac{15e^{-j120^\circ}}{3 + j4 \times 10^4 \times 10^{-4}}$$

$$= \frac{15e^{-j120^\circ}}{3 + j4} = \frac{15e^{-j120^\circ}}{5e^{j53.1^\circ}} = 3e^{-j173.1^\circ} \text{ A.}$$

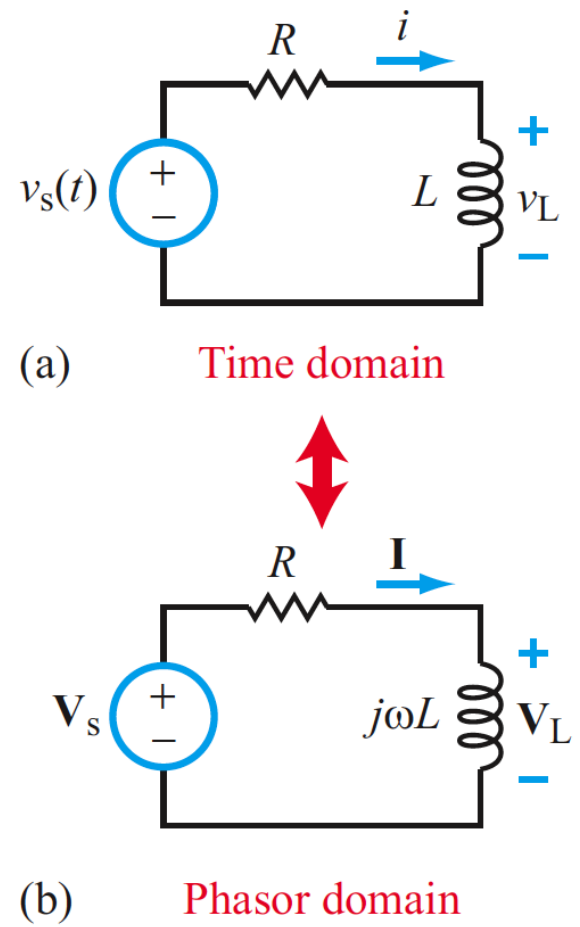
The phasor voltage across the inductor is related to **I** by

$$\begin{aligned} \mathbf{V}_\mathrm{L} &= j\omega L\mathbf{I} \\ &= j4 \times 10^4 \times 10^{-4} \times 3e^{-j173.1^\circ} \\ &= j12e^{-j173.1^\circ} \\ &= 12e^{-j173.1^\circ} \cdot e^{j90^\circ} = 12e^{-j83.1^\circ} \text{ V,} \end{aligned}$$

where we replaced j with e^{j90°.

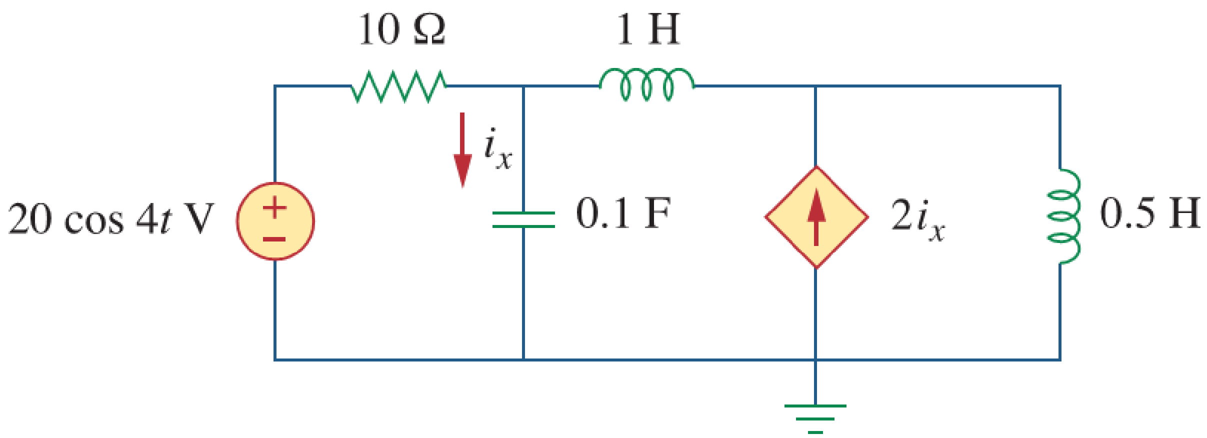
Step 5: Transform solution to the time domain

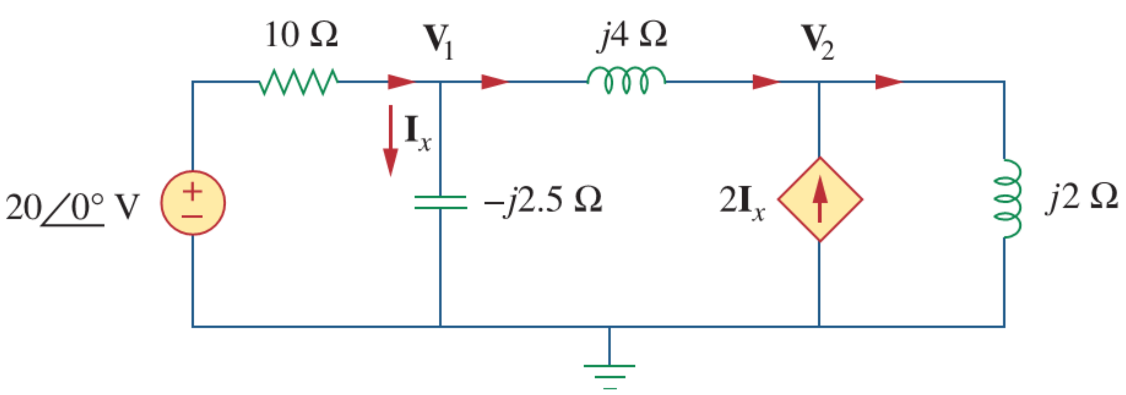
$$\begin{aligned} v_\mathrm{L}(t) &= \mathfrak{Re}[\mathbf{V}_\mathrm{L} e^{j\omega t}] \\ &= \mathfrak{Re}[12e^{-j83.1^\circ} e^{j4\times 10^4 t}] \\ &= 12\cos(4 \times 10^4 t - 83.1^\circ) \text{ V.} \end{aligned}$$



Nodal Analysis

- Example---Find i_x



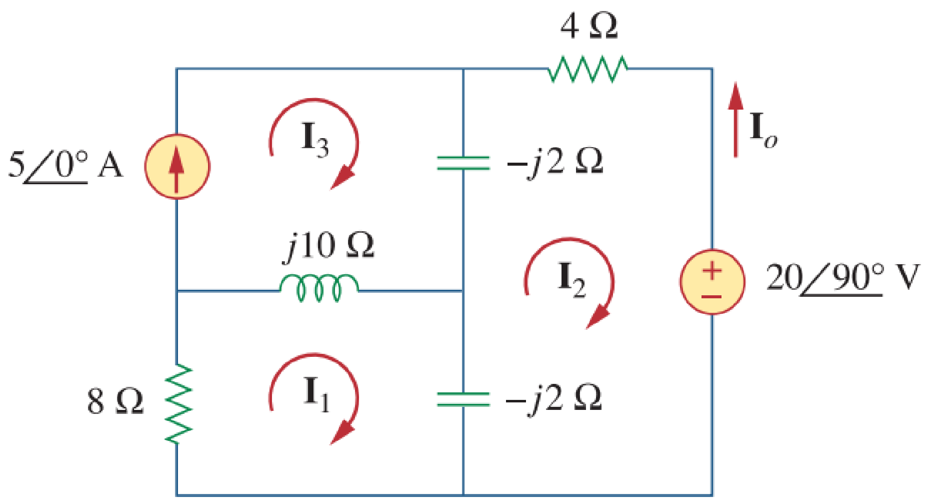


$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

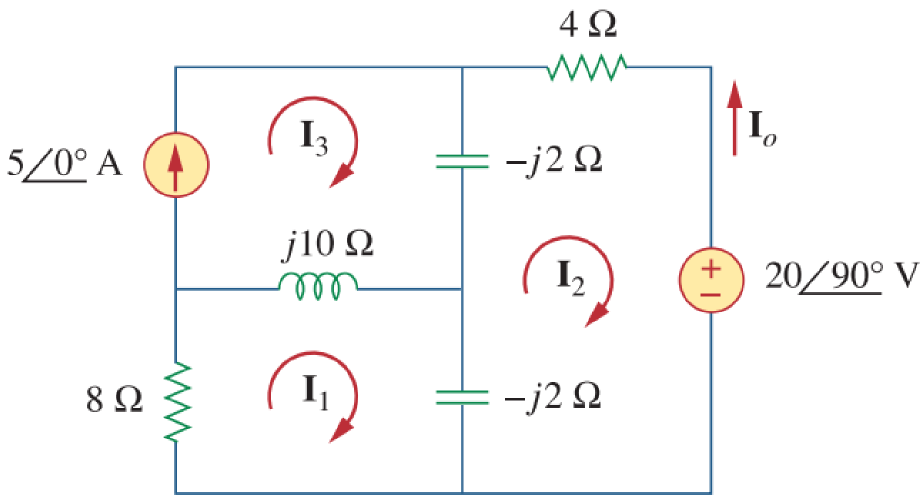
$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

$$i_x = 7.59 \cos(4t + 108.4°) \text{ A}$$

Mesh Analysis



Mesh Analysis



Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

For mesh 2,

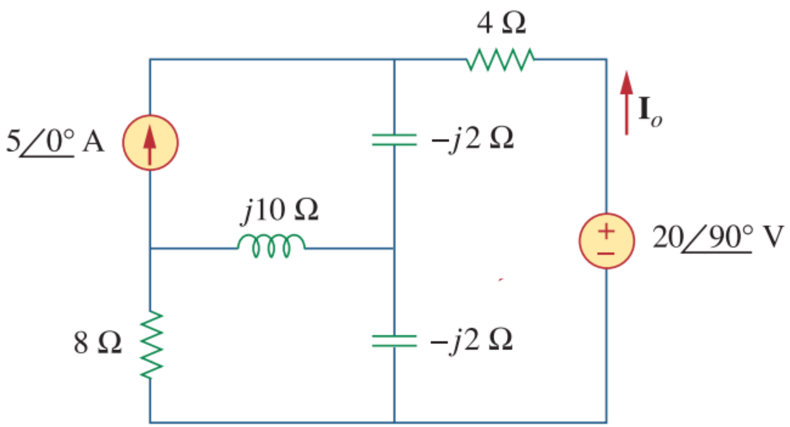
$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0 \qquad \textbf{(10.3.2)}$$

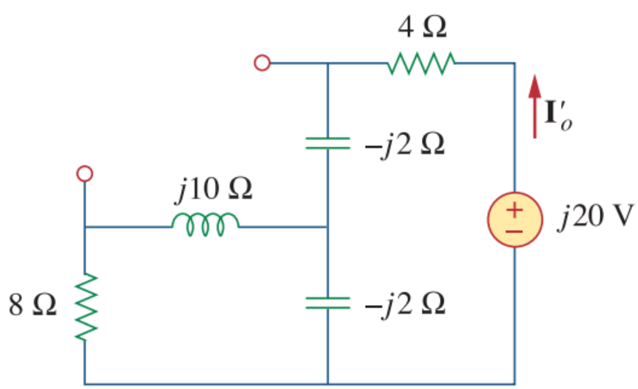
For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

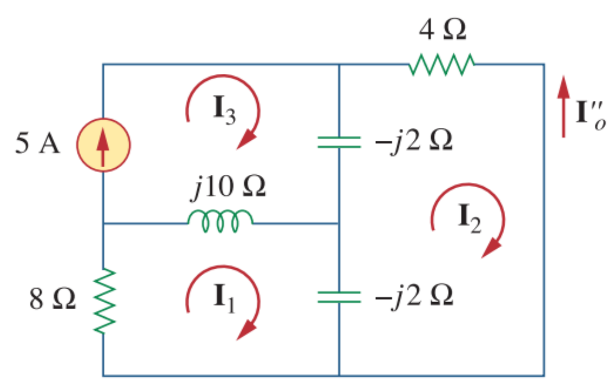
$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

Superposition-Example



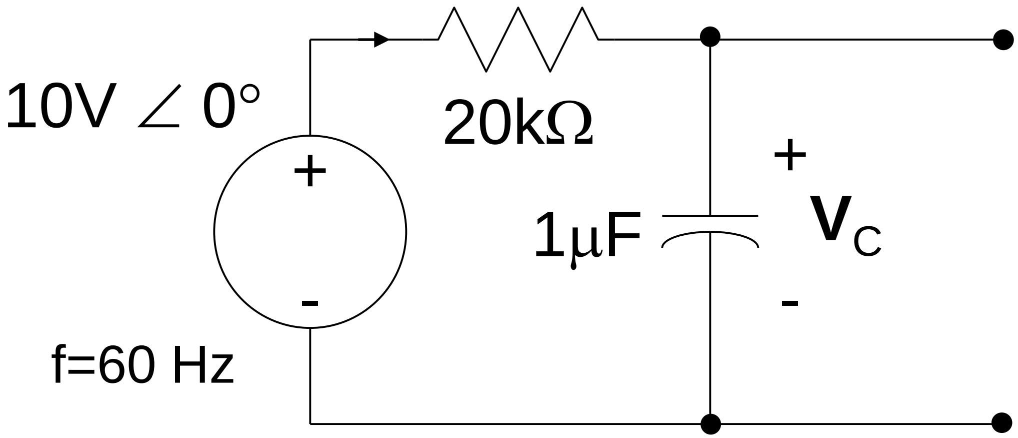


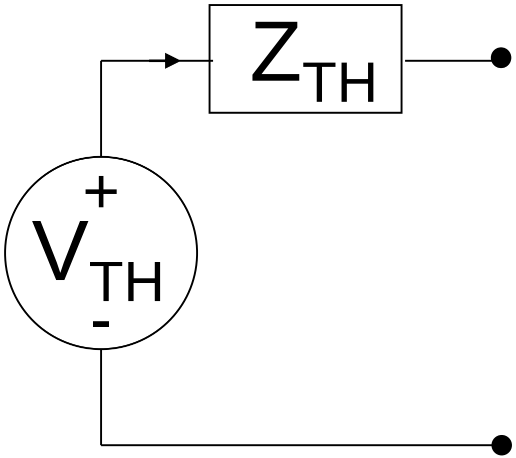
(a)



(b)

Thevenin Equivalent



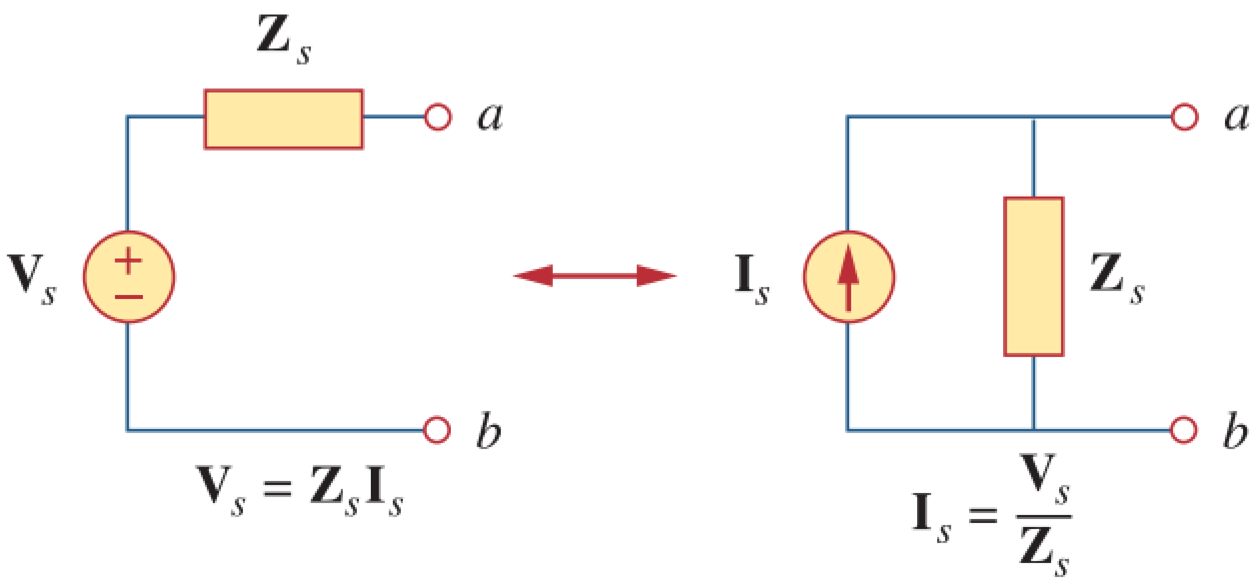


$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°
$\mathbf{Z}_C$ = 1/*j* (2πf x 1μF) = 2.65kΩ ∠ -90°

$$\mathbf{V}_{TH} = \mathbf{V}_{OC} = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right) = 1.31 \angle -82.4$$

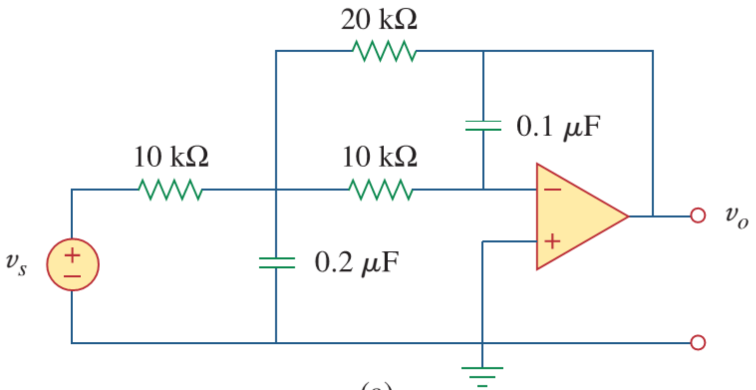
$$\mathbf{Z}_{TH} = \mathbf{Z}_R \,||\, \mathbf{Z}_C = \left(\frac{20\text{k}\Omega \angle 0° \cdot 2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right) = 2.62 \angle -82.4$$

Source transformation/Norton



$$\boxed{\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}}$$

AC Op Amp Circuits



(a)

Question 1: Are op amps used in ac circuits?

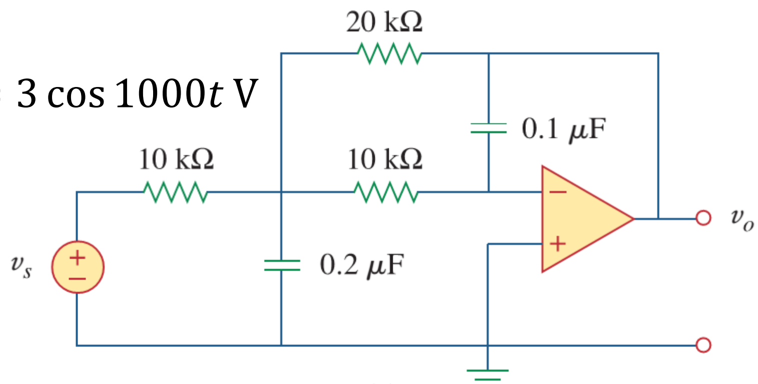
Answer 1: Yes.

Question 2: Is the ideal op-amp model applicable to ac circuits?

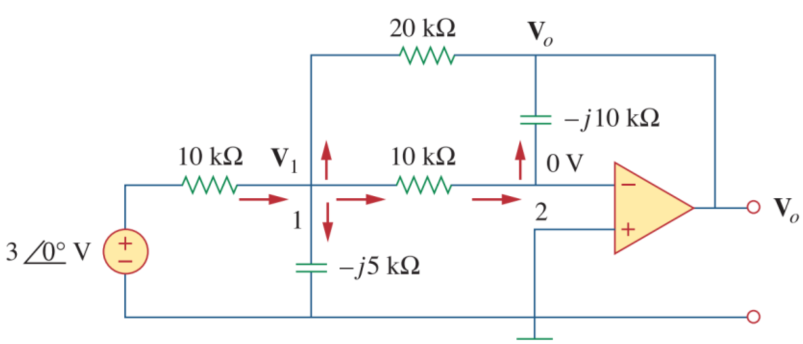
Answer 2: The ideal op-amp model is based on the assumption that the open-loop gain A is very large ($> 10^4$), which is true at dc and low frequencies, but not necessarily so at high frequencies. The range of frequencies over which A is large depends on the specific op-amp design. .

Example

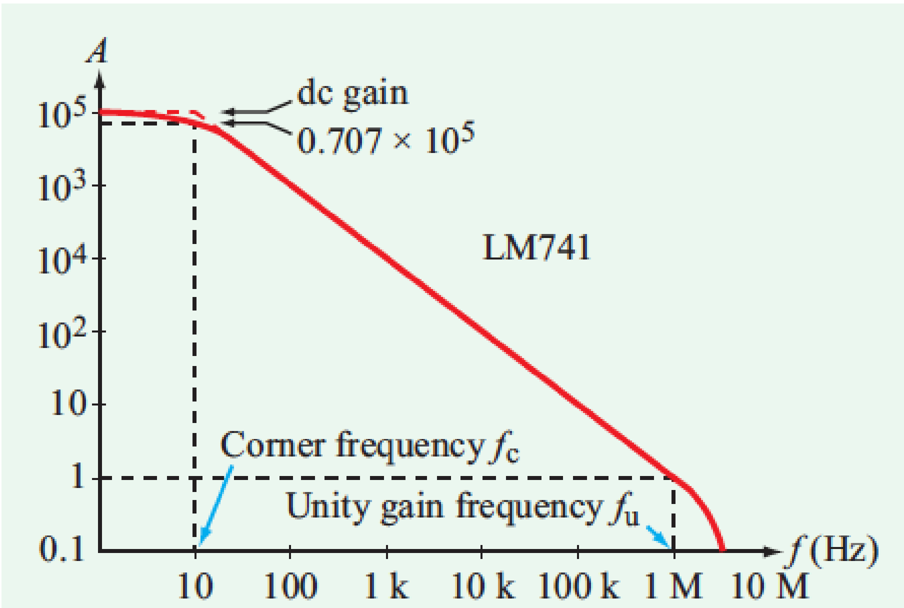
$v_s = 3\cos 1000t$ V



(a)



(b)



Open-loop gain A versus frequency for the LM741 op amp.

f (Hz)	A	G	Error
0 (dc)	10^5	−4.997	0.06%
100	10^4	−4.970	0.6%
1 k	10^3	−4.714	5.7%
10 k	10^2	−3.111	37.8%
100 k	10	−0.707	85.9%
1 M	1	−0.081	98.4%

The error is defined as $G_{\text{ideal}} = -5$

$$\%\ \text{error} = \left(\frac{G_{\text{ideal}} - G}{G_{\text{ideal}}}\right) \times 100.$$

Audio: dc to 1 kHz: Minor distortion

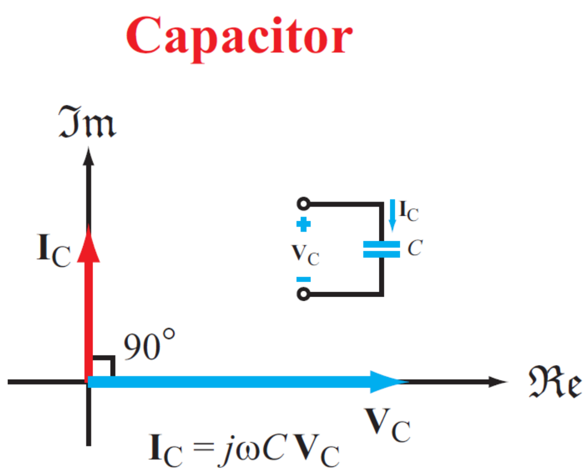
Video: Up to 1 MHz: Serious distortion

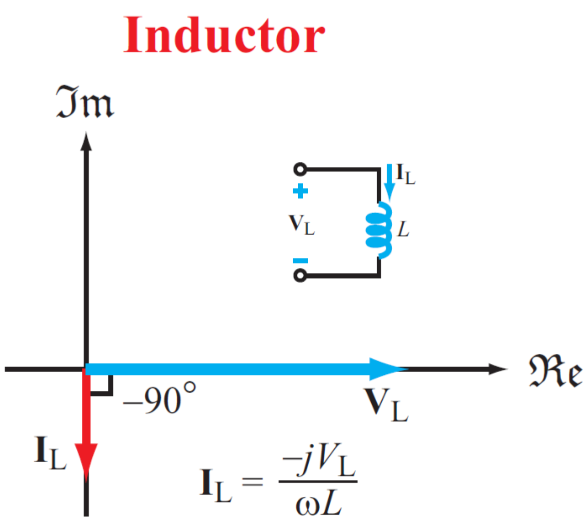
Conclusion: The LM741 model is not suitable for video signals; it is necessary to use an Op-amp model with a higher corner frequency.

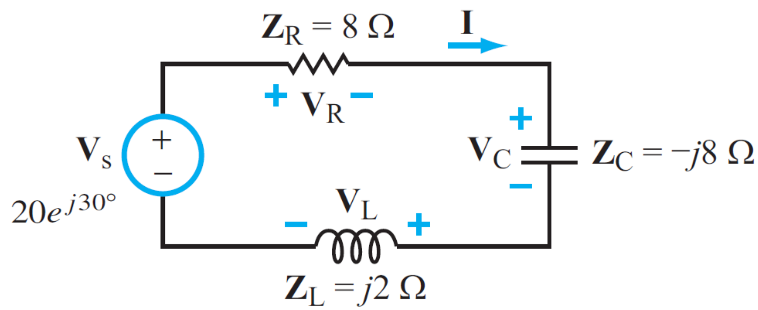
Outline

- Kirchhoff's laws in phasor domain
- Generalized impedance
- General AC phasor analysis
- Phasor diagram

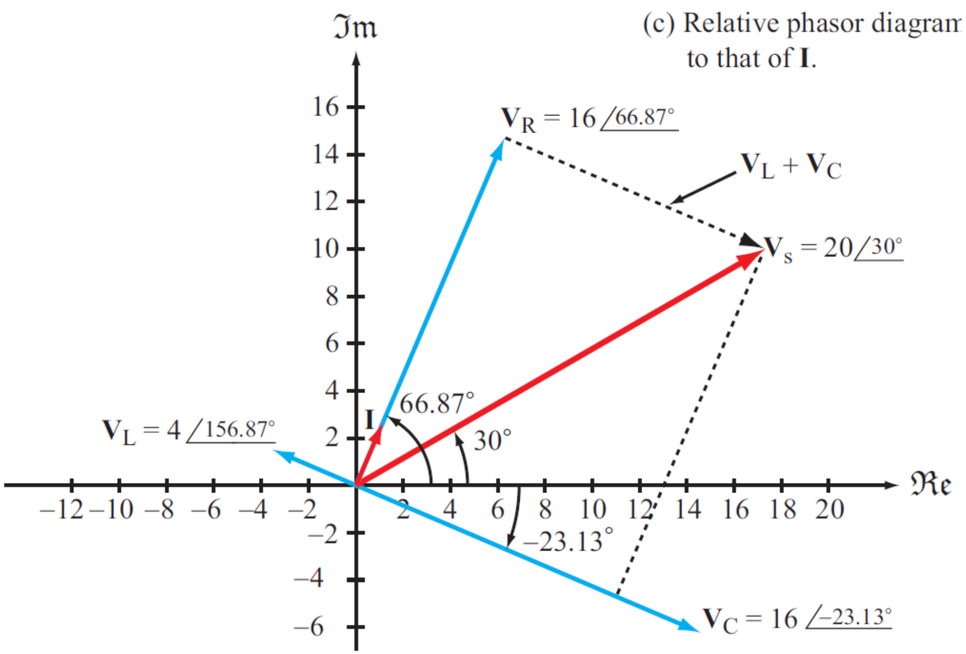
Phasor Diagrams







$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j\omega L - \dfrac{j}{\omega C}} = \frac{20e^{j30^\circ}}{8 + j2 - j8} = \frac{20e^{j30^\circ}}{8 - j6} = \frac{20e^{j30^\circ}}{10e^{-j36.87^\circ}} = 2e^{j66.87^\circ}\ \text{A}$$



(c) Relative phasor diagram to that of **I**.