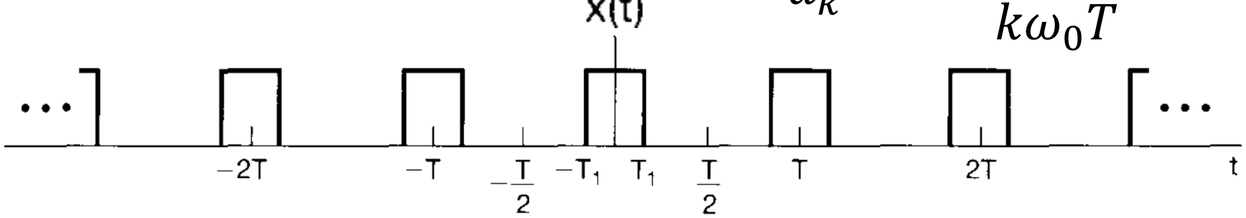
The Continuous-Time Fourier Transform (ch.4)

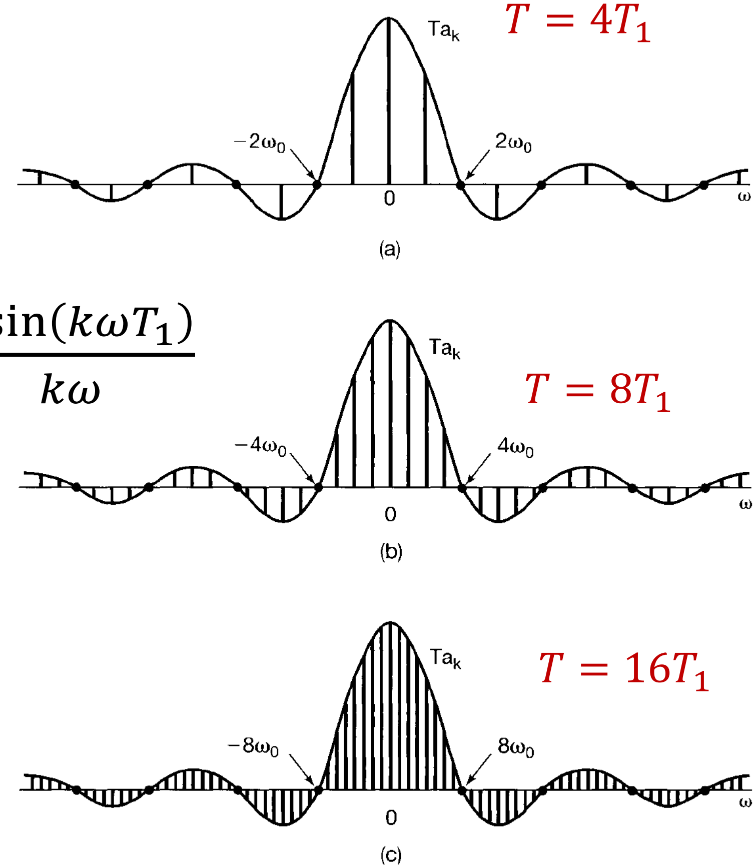
- Representation of aperiodic signals- Continuous Fourier Transform
- Fourier transform for periodic signals
- Properties of continuous-time Fourier Transform
- The convolution property
- The multiplication property
- System characterized by differential equations

Continuous Fourier Transform

Recall square wave

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$





- Ta_k: Samples of an envelope function $f(\omega) = \dfrac{2\sin(k\omega T_1)}{k\omega}$

$$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega=k\omega_0}$$

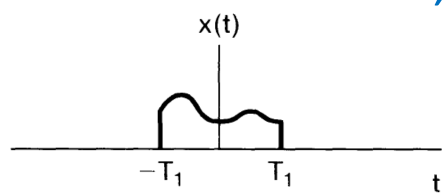
- $T\uparrow, \omega_0\downarrow \Rightarrow$ the envelope is sampled with closer spacing

- $T\to\infty, \Rightarrow Ta_k \to$ the envelope $\dfrac{2\sin(k\omega T_1)}{k\omega}$

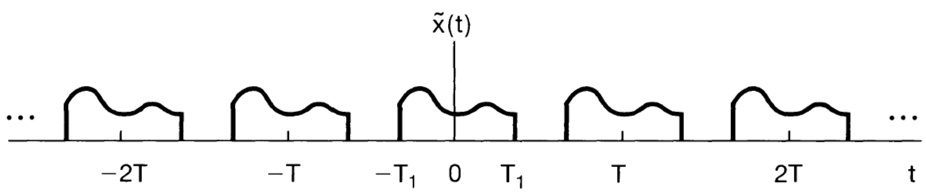
Continuous Fourier Transform

Development of FT

- Consider a signal of finite duration, $x(t) = 0 \; if \; |t| > T_1$



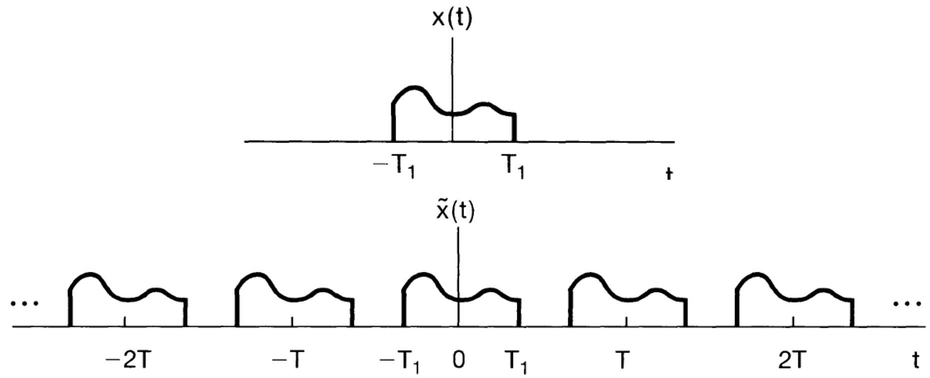
- Periodic extension of $x(t)$ with T



- FS representation of $\hat{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

Continuous Fourier Transform



Development of FT

- FS coefficients of $\hat{x}(t)$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\boxed{\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}\,dt} = \frac{1}{T}X(jk\omega_0)$$

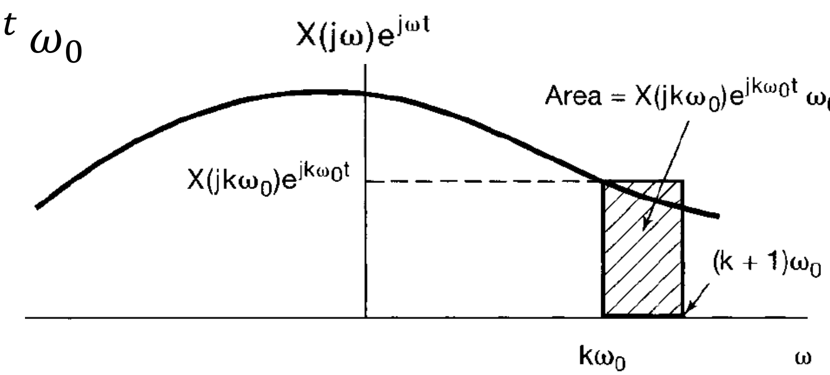
Define $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$

- FS of $\hat{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\,\omega_0$$

- $T \to \infty$, $\hat{x}(t) \to x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,e^{j\omega t}\,d\omega$$



Continuous Fourier Transform

FT pairs

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt$$ Fourier transform (FT)

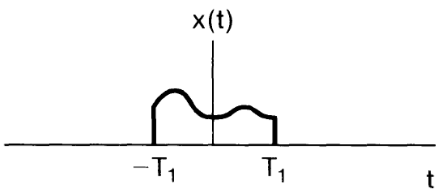
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$ Inverse Fourier transform

- $x(t)$ is a linear combination (specifically, an integral) of sinusoidal signals at different frequencies
- $X(j\omega)(d\omega/2\pi)$ is the weight for different frequencies
- $X(j\omega)$ is called the spectrum

Continuous Fourier Transform

FT vs. FS

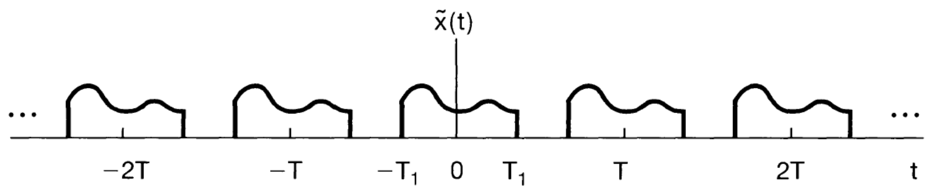
Fourier transform (FT)



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt$$

Fourier series (FS)



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

$a_k = \frac{1}{T} X(j\omega)$ with $\omega = k\omega_0$

Continuous Fourier Transform

Convergence of FT

- Condition 1: Finite energy condition

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

- Condition 2: Dirichlet condition

(1) Absolutely integrable $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

(2) Finite maxima and minima in one period with in any finite interval

(3) Finite number of finite discontinuities in any finite interval

Continuous Fourier Transform

Examples

Consider the signal $x(t) = e^{-at}u(t), a > 0$

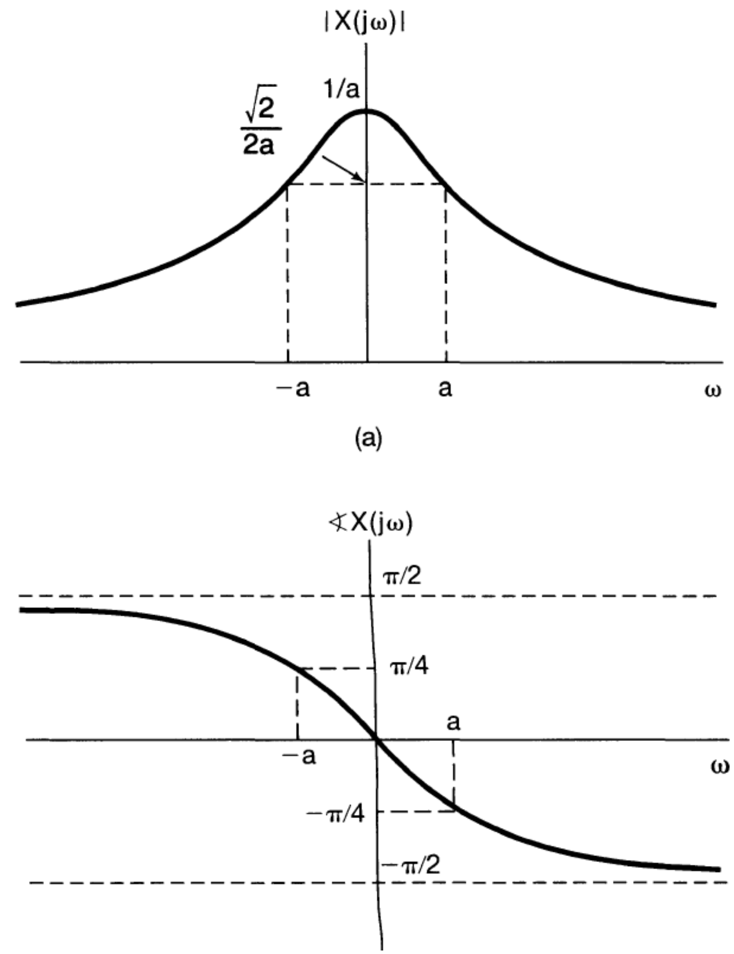
Determine its FT

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$= -\frac{1}{a + j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{1}{a + j\omega}, a > 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \qquad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



Continuous Fourier Transform

Examples

$$x(t) = e^{-a|t|}, a > 0 \qquad X(j\omega) = ?$$

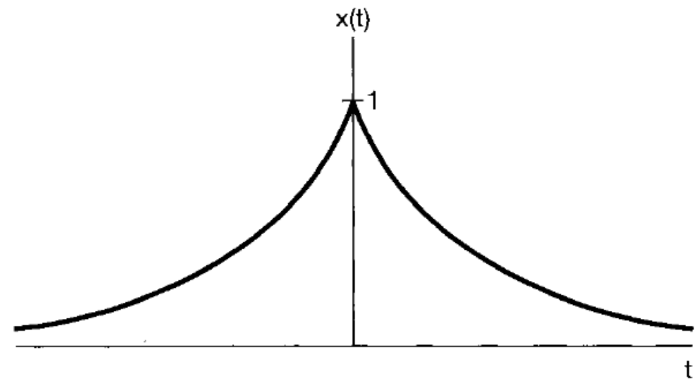
Solution

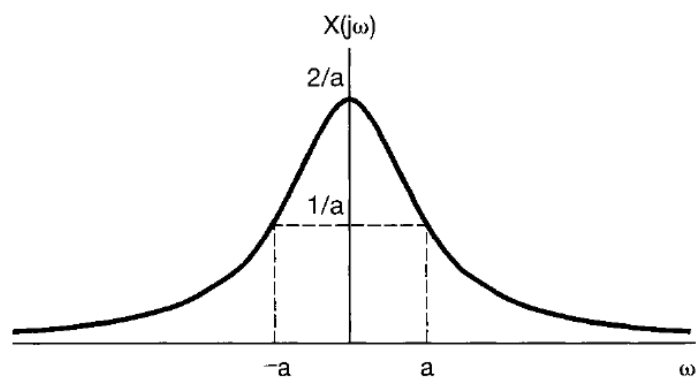
$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{+\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$





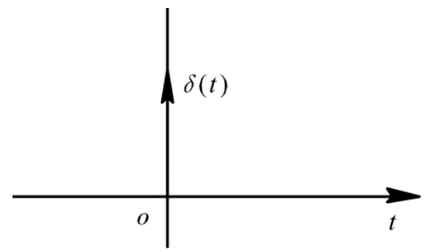
Continuous Fourier Transform

Examples

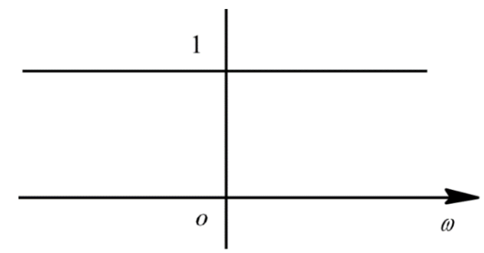
- $x_1(t) = \delta(t) \qquad X_1(j\omega) = ?$

$$X_1(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t}\, dt = 1$$

$x_1(t)$



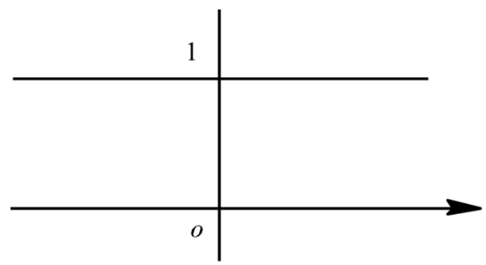
$X_1(j\omega)$



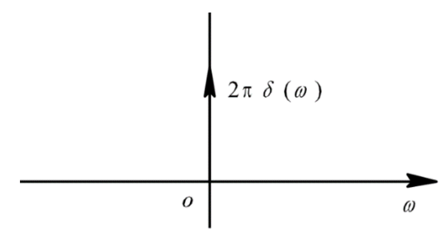
- $x_2(t) = 1 \qquad X_2(j\omega) = ?$

$$X_2(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t}\, dt = 2\pi\delta(\omega)$$

$x_2(t)$



$X_2(j\omega)$



Hints: $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1 \cdot e^{j\omega t}\, d\omega \Rightarrow \delta(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1 \cdot e^{-j\omega t}\, dt$

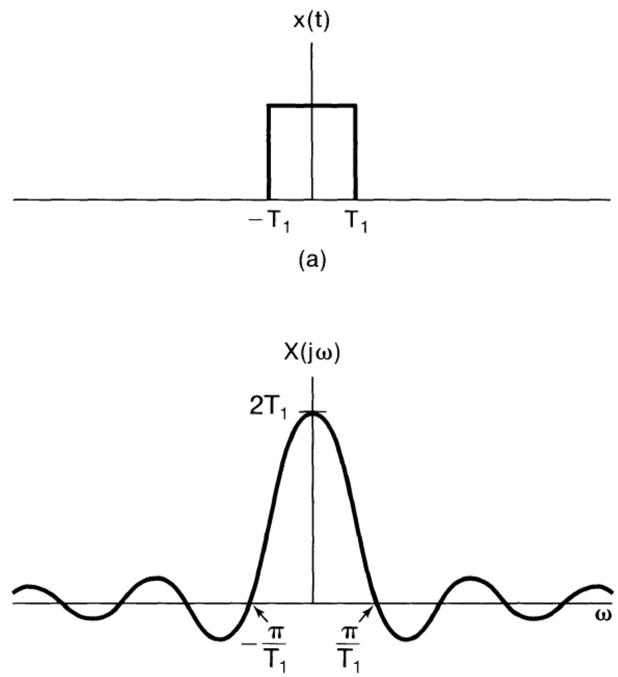
Continuous Fourier Transform

Examples

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases} \qquad X(j\omega) = ?$$

Solution

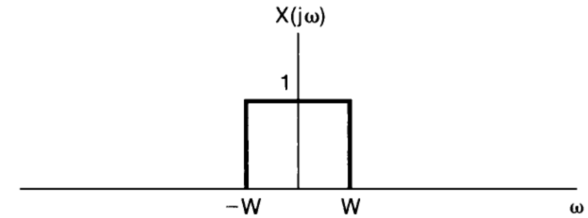
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega}$$



Continuous Fourier Transform

Examples

$$X(j\omega) = \begin{cases} 1, |\omega| < W \\ 0, |\omega| > W \end{cases} \qquad x(t) = ?$$

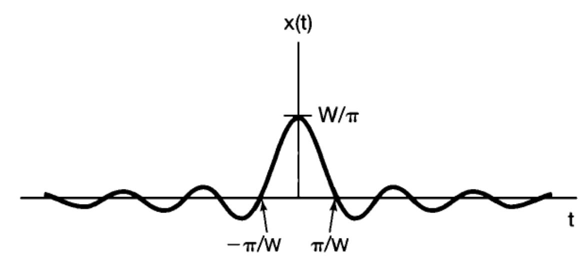


(a)

Solution

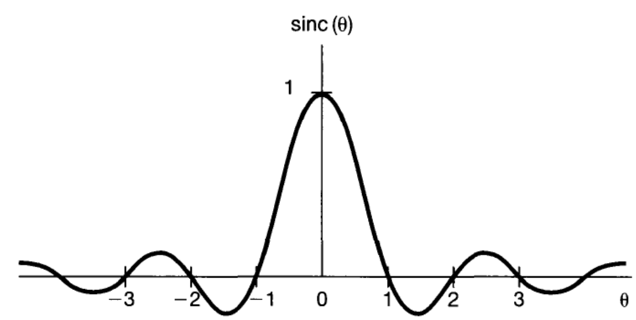
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t}$$



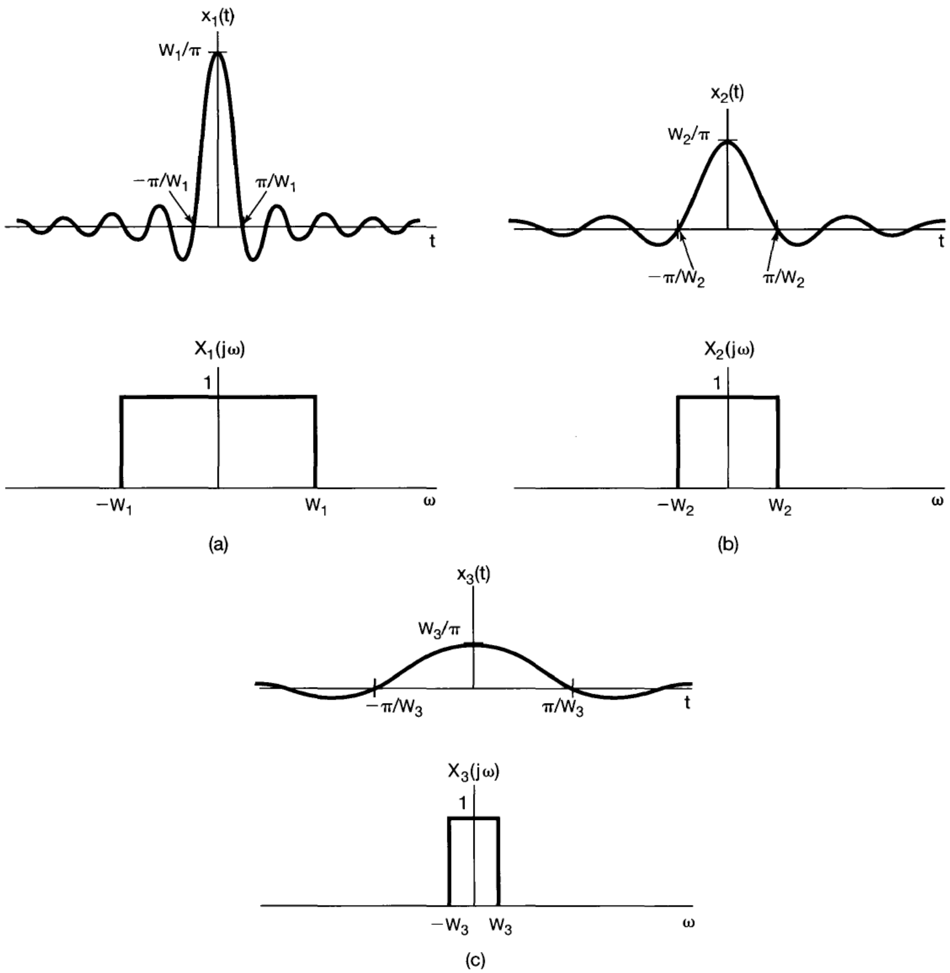
$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\frac{\sin Wt}{Wt} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$



Continuous Fourier Transform

Examples



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Fourier transform for periodic signals

- A period signal can be represented by a FS, but also a FT

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t}\, d\omega$$

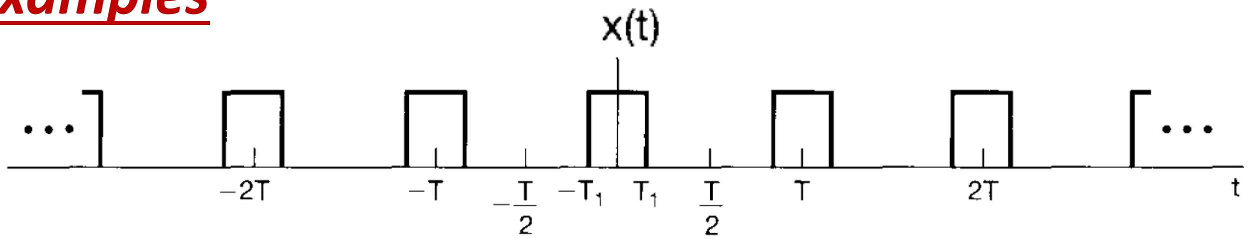
- The relationship between a_k and $X(j\omega)$?
  - Consider $x_1(t) = a_k e^{jk\omega_0 t}$, the FT of $x_1(t)$: $X_1(j\omega)$=?

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\omega) e^{j\omega t}\, d\omega = a_k e^{jk\omega_0 t} \Rightarrow X_1(j\omega) = 2\pi a_k \delta(\omega - k\omega_0)$$

  - For $x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\qquad X(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0)$

Fourier transform for periodic signals

Examples



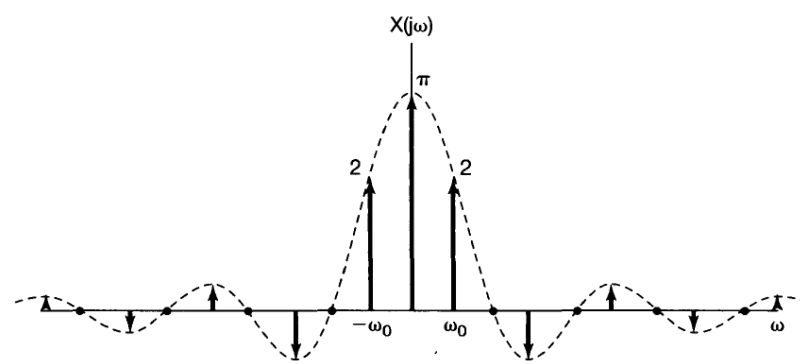
$X(j\omega)$?

Solution

$$x(t) = \sum_{K=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$= \sum_{K=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$

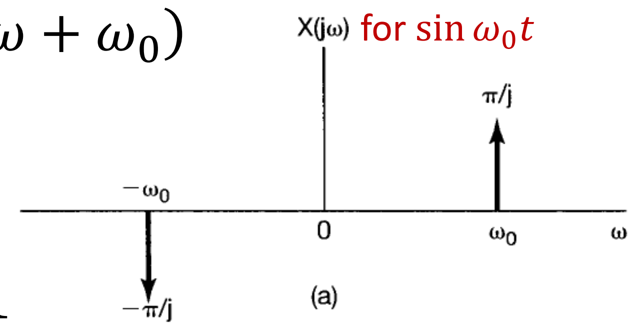


Fourier transform for periodic signals

Examples

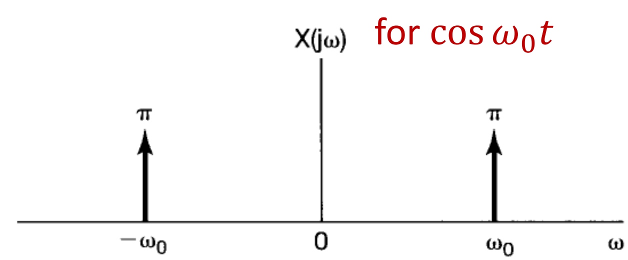
$$x_1(t) = \sin\omega_0 t \qquad a_1 = 1/2j \quad a_{-1} = -1/2j \quad a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \sum_{K=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



$$x_2(t) = \cos\omega_0 t \quad a_k = 1/2, k = \pm 1, a_k = 0, k \neq \pm 1$$

$$X_1(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier transform for periodic signals

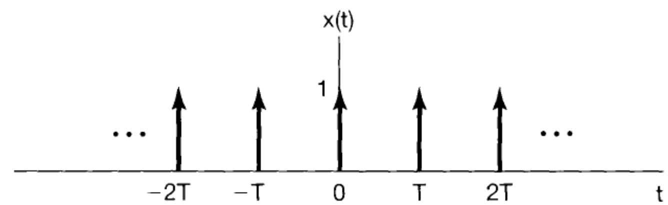
Examples

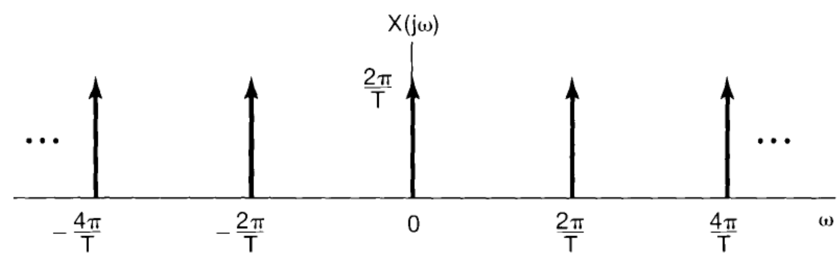
$$x(t) = \sum_{K=-\infty}^{\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$X(j\omega) = \frac{2\pi}{T}\sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T}\sum_{K=-\infty}^{\infty} \delta(\omega - \frac{2k\pi}{T})$$





The Continuous-Time Fourier Transform (ch.4)

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Properties of continuous-time Fourier Transform

Short notation for FT pairs

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}dt \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

Properties of continuous-time Fourier Transform

Linearity

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \qquad y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$$

$$\Downarrow$$

$$\boxed{ax(t) + bx(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bX(j\omega)}$$

Properties of continuous-time Fourier Transform

Time shifting

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \implies \boxed{x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)}$$

- proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}dt$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-j\omega t_0}X(j\omega)e^{j\omega t}dt$$

$$\mathcal{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]}$$

- A time shift on a signal introduces a phase shift into its FT, $-\omega t_0$, which is a linear function of ω.

Examples

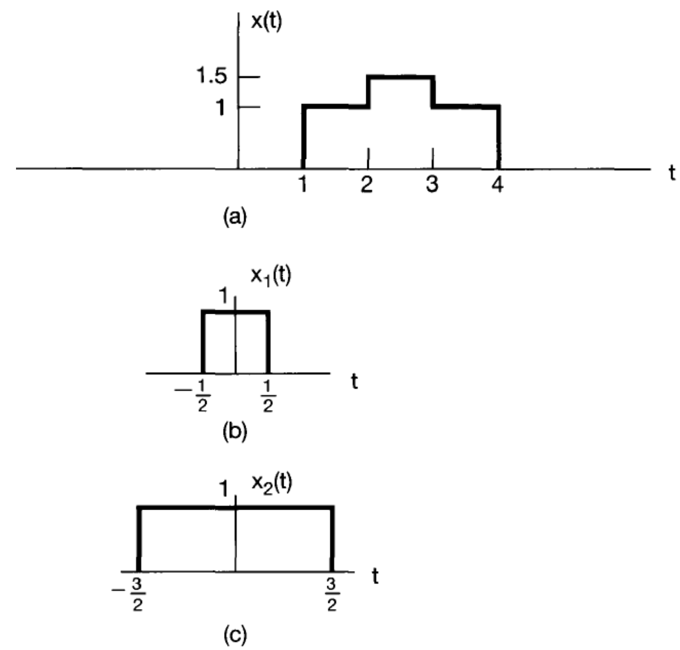
- $x(t)$ can be expressed as

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(j\omega) = 2\frac{\sin\omega T_1}{\omega} = 2\frac{\sin\omega/2}{\omega}$$

$$X_2(j\omega) = 2\frac{\sin 3\omega/2}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2}\left(\frac{\sin\omega/2 + 2\sin 3\omega/2}{\omega}\right)$$



Properties of continuous-time Fourier Transform

Conjugation and Conjugate Symmetry

- Conjugation property $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \implies \boxed{x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)}$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt = \mathcal{F}\{x^*(t)\}$$

- Conjugation Symmetry

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}]}$$

For a real-valued signal, the FT need only to be specified for positive frequencies.

Properties of continuous-time Fourier Transform

Time reversing $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Longrightarrow \boxed{x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)}$

- $x(t)$ even $\Longrightarrow X(j\omega) = X(-j\omega)$, $x(t)$ real $\Longrightarrow X(-j\omega) = X^*(j\omega)$

- $x(t)$ real and even $\Longrightarrow X(j\omega)$ real and even

- $x(t)$ real and odd $\Longrightarrow X(j\omega)$ purely imaginary and odd

- If $x(t)$ real

$$\left.\begin{array}{c} x(t) = x_e(t) + x_o(t) \\ \mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\} \end{array}\right\} \Leftrightarrow \begin{cases} E_v\{x(t)\} \overset{\mathcal{F}}{\leftrightarrow} R_e\{X(j\omega)\} \\ O_d\{x(t)\} \overset{\mathcal{F}}{\leftrightarrow} j \cdot I_m\{X(j\omega)\} \end{cases}$$

Properties of continuous-time Fourier Transform

Example

- For $a > 0$

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} 1/(a+j\omega)$$

$$e^{-a|t|} \overset{\mathfrak{F}}{\longleftrightarrow} 2a/(a^2+\omega^2)$$

- use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2E_v\{e^{-at}u(t)\}$$

$$E_v\{e^{-at}u(t)\} \overset{\mathcal{F}}{\leftrightarrow} R_e\left\{\frac{1}{a+j\omega}\right\}$$

$$\mathcal{F}\left\{e^{-a|t|}\right\} = 2R_e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

Properties of continuous-time Fourier Transform

Differential and integration

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad \Longrightarrow \quad \boxed{\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, X(j\omega)} \quad \boxed{\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)}$$

❑ Proof

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\frac{d(e^{j\omega t})}{dt}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\cdot j\omega \cdot e^{j\omega t}d\omega$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega\tau} d\omega\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) \int_{-\infty}^{t} e^{j\omega\tau} d\tau \quad d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{X(j\omega)}{j\omega} e^{j\omega t} d\omega \,, \omega \neq 0$$

$\pi X(0)\delta(\omega)$ DC components

Properties of continuous-time Fourier Transform

Example FT of unit sept $x(t) = u(t)$

$$g(t) = \delta(t) \overset{\mathcal{F}}{\leftrightarrow} G(j\omega) = 1 \qquad x(t) = u(t) \int_{-\infty}^{t} g(\tau)d\tau$$

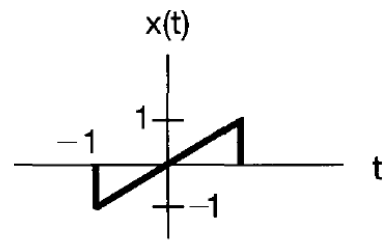
- use integration property

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi X(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

- Recover $G(j\omega)$ by differential property

$$\delta(t) = \frac{du(t)}{dt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1$$

Example Determine the FT of $x(t)$

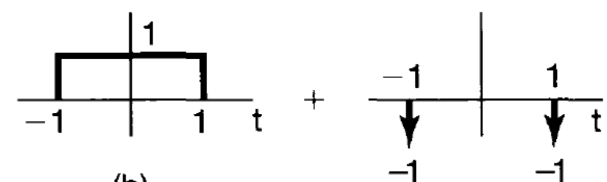


(a)

❑ Solution

$$g(t) = \frac{dx(t)}{dt}$$

$$g(t) = \frac{dx(t)}{dt} =$$



(b)

$$G(j\omega) = \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega}$$

use FT properties

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

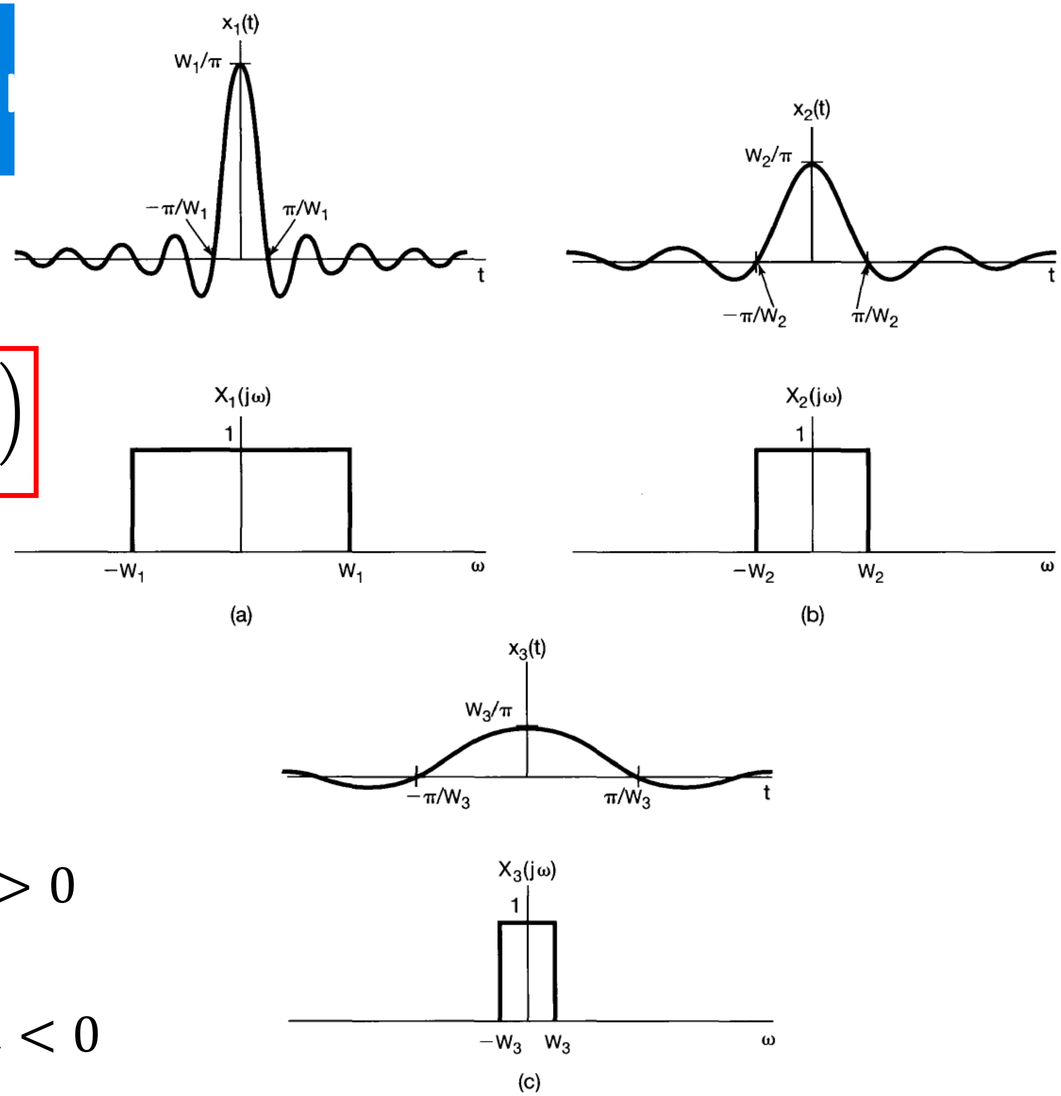
Properties of continuous-ti

Time and frequency scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \Longrightarrow \boxed{\begin{array}{c} x(at) \\ a \neq 0 \end{array} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)}$$

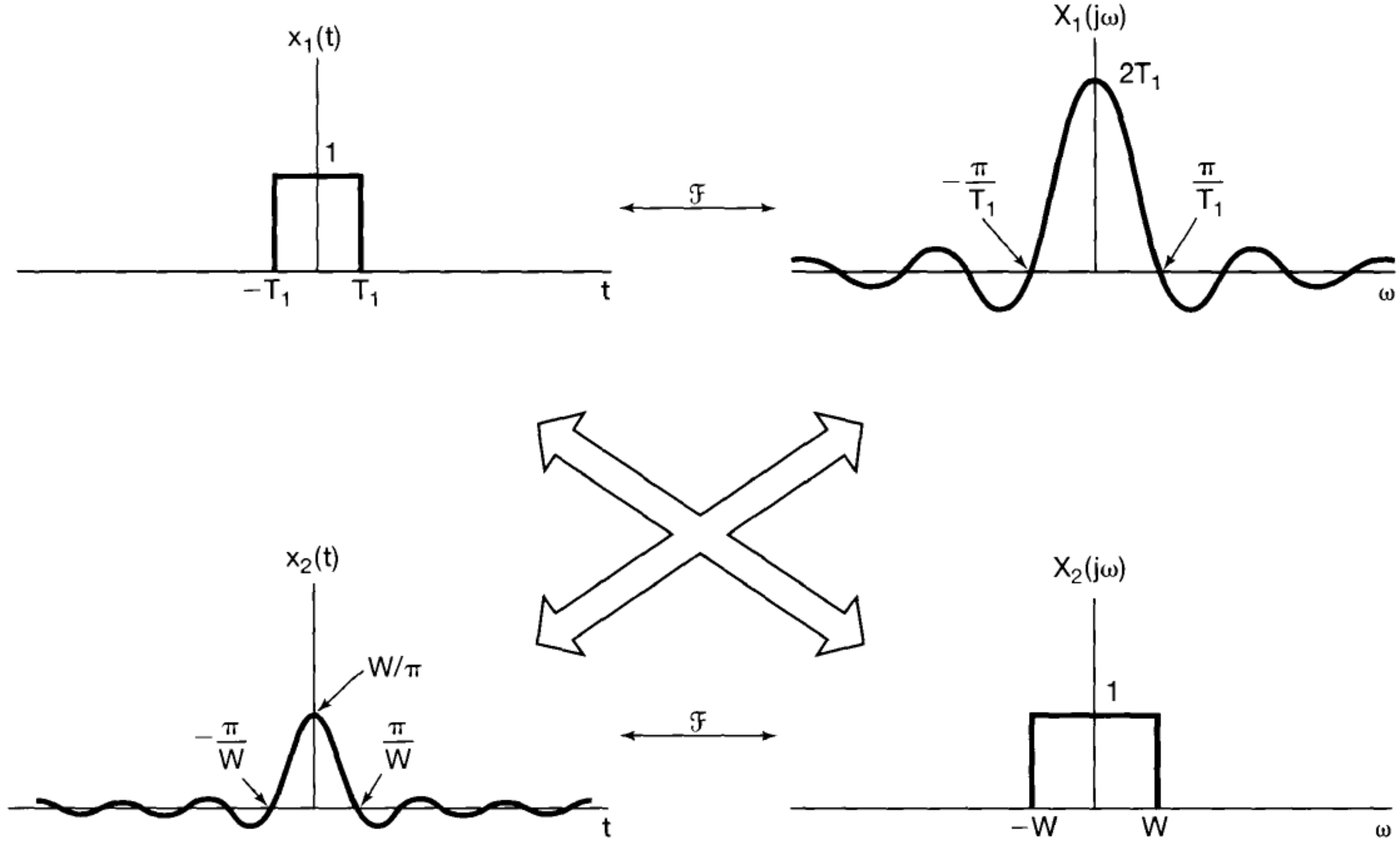
$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$\mathcal{F}\{x(at)\} = \begin{cases} \dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau \, , a > 0 \\ -\dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau \, , a < 0 \end{cases}$$



Properties of continuous-time Fourier Transform

Duality

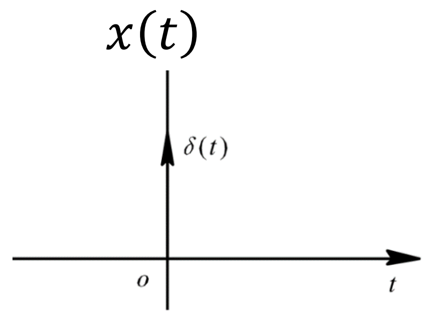


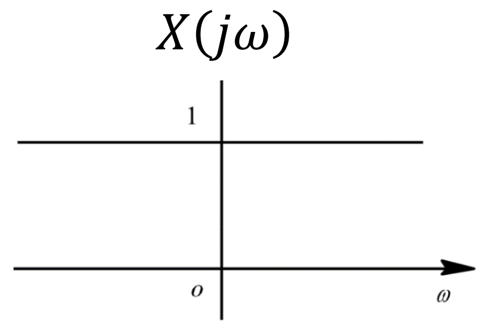
Properties of continuous-time Fourier Transform

Example

$$x(t) = \delta(t) \qquad X(j\omega) = 1$$

$$x(t) = 1 \qquad X(j\omega) = 2\pi\delta(\omega)$$



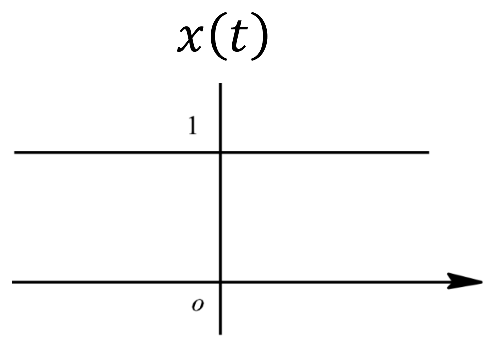


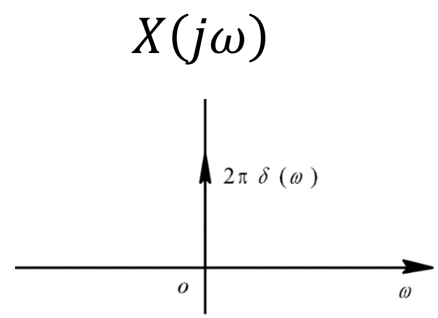
Principle

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\cdot e^{j\omega t}\, d\omega$$

$$x(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(t)\cdot e^{j\omega t}\, dt$$

$$x(-j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(t)\cdot e^{-j\omega t}\, dt$$





Properties of continuous-time Fourier Transform

Example $\quad g(t) = \dfrac{2}{1+t^2} \qquad G(j\omega) = ?$

Solution: calculate $G(j\omega)$ is difficult; use duality property

$$e^{-a|t|} \stackrel{\mathfrak{F}}{\longleftrightarrow} 2a/(a^2+\omega^2)$$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2}{1+\omega^2} \cdot e^{j\omega t}\, d\omega$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{+\infty} \frac{2}{1+t^2} \cdot e^{j\omega t}\, dt \qquad \therefore G(j\omega) = 2\pi e^{-|\omega|}$$

Properties of continuous-time Fourier Transform

Example

Duality property can determine or suggest other FT properties

$$\frac{dx(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega). \quad \Leftrightarrow \quad -jtx(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega). \quad \Leftrightarrow \quad -\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.$$

$$x(t - t_0) \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega). \quad \Leftrightarrow \quad e^{j\omega_0 t}x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Properties of continuous-time Fourier Transform

Parseval's relation

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |x(t)|^2\, dt &= \int_{-\infty}^{+\infty} x(t)x^*(t)\, dt \\
&= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}\, d\omega\right] dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt\right] d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega
\end{aligned}$$

The Continuous-Time Fourier Transform (ch.4)

- ❑ Representation of aperiodic signals- Continuous Fourier Transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of continuous-time Fourier Transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ System characterized by differential equations

The convolution property

$$\boxed{y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)}$$

- proof

$$Y(j\omega) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau) d\tau\right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} h(t-\tau)\, e^{-j\omega t} dt\right] d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} H(j\omega)\, d\tau = H(j\omega) \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau}\, d\tau = H(j\omega)X(j\omega)$$

- $H(j\omega)$: Frequency response; important for analyzing LTI systems
- Only stable continuous-time LTI systems have $H(j\omega)$
- Non-stable continuous-time LTI system: Laplace transform

The convolution property

Example

$$x(t) \longrightarrow \boxed{\mathrm{h(t)}} \longrightarrow y(t)$$

- Assume $h(t) = \delta(t - t_0), \mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$

- Solution 1

$$H(j\omega) = e^{-j\omega t_0} \qquad Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega)$$

- Solution 2

$$y(t) = x(t - t_0) \qquad Y(j\omega) = e^{-j\omega t_0}X(j\omega)$$

The convolution property

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

- ❑ Differentiation property $\Rightarrow\ Y(j\omega) = j\omega X(j\omega)$
- ❑ Convolution property $\Rightarrow\ Y(j\omega) = H(j\omega)X(j\omega)$
- ❑ Therefore, $H(j\omega) = j\omega$

The convolution property

Example

$$x(t) \longrightarrow \boxed{\mathrm{h(t)}} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad Y(j\omega) = ?$$

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \ = u(t)$$

- ❑ Frequency response $H(j\omega) = \dfrac{1}{j\omega} + \pi\delta(\omega)$

- ❑ Convolution property $Y(j\omega) = H(j\omega)X(j\omega)$

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

- ❑ Consistent with integration property

The convolution property

Example

$$x(t) \longrightarrow \boxed{\mathrm{h(t)}} \longrightarrow y(t)$$

$$h(t) = e^{-at}u(t), \mathrm{a} > 0 \qquad x(t) = e^{-bt}u(t), \mathrm{b} > 0 \qquad y(t) = ?$$

- Solution $b \neq a$

$$H(j\omega) = \frac{1}{b + j\omega} \qquad X(j\omega) = \frac{1}{a + j\omega} \qquad Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} \qquad A = \frac{1}{b - a} = -B$$

$$Y(j\omega) = \frac{1}{b - a}\left(\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right) \qquad y(t) = \frac{1}{b - a}[e^{-at} - e^{-bt}]u(t), b \neq a$$

The convolution property

Example

$$x(t) \longrightarrow \boxed{\mathrm{h(t)}} \longrightarrow y(t)$$

$$h(t) = e^{-at}u(t), \mathrm{a} > 0 \qquad x(t) = e^{-bt}u(t), \mathrm{b} > 0 \qquad y(t) = ?$$

- Solution $b = a$

$$Y(j\omega) = \frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right]$$

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} 1/(a + j\omega)$$

$$te^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right]$$

$$\therefore y(t) = te^{-at}u(t)$$

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The multiplication property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

- multiplication of two signals is often referred to as *amplitude modulation*

$$s(t)p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)e^{j\theta t}\,d\theta\,\frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\omega')e^{j\omega' t}\,d\omega'$$

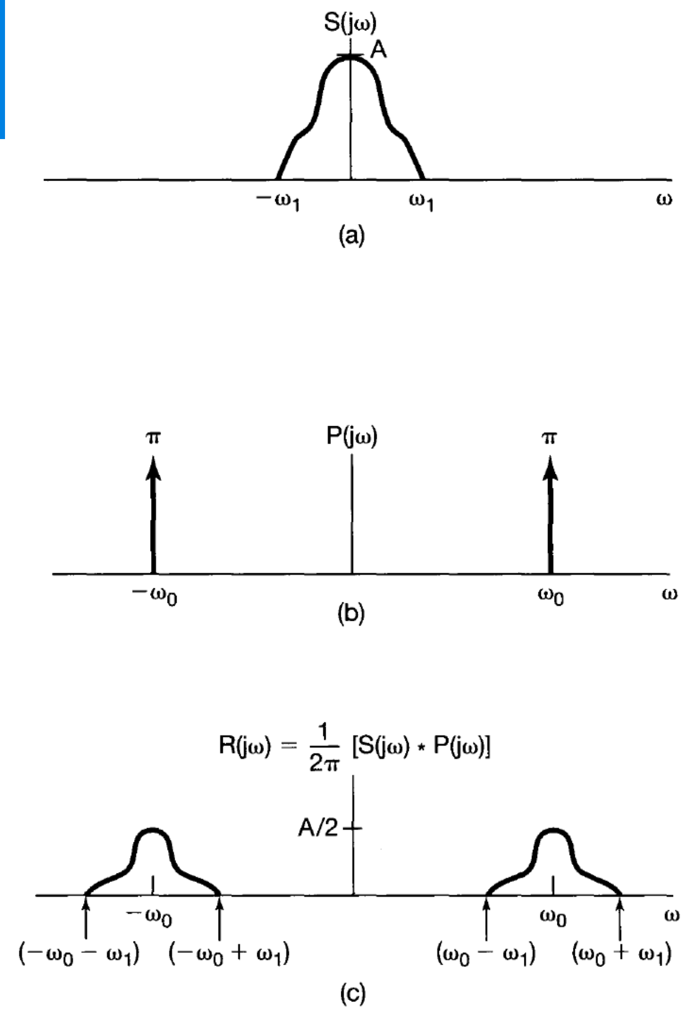
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\,P(j\omega')e^{j(\theta+\omega')t}\,d\theta d\omega'$$

$$\omega' = \omega - \theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} S[j(\theta)]\,P(j(\omega-\theta))e^{j\omega t}\,d\theta d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta}^{R(j\omega)}e^{j\omega t}\,d\omega$$

The multiplication property



Example

Consider a signal $p(t) = \cos\omega_0 t$ and a signal $s(t)$ *with spectrum* $S(j\omega)$, *determine the FT of* $r(t) = p(t)s(t)$

❑ Solution $\quad P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

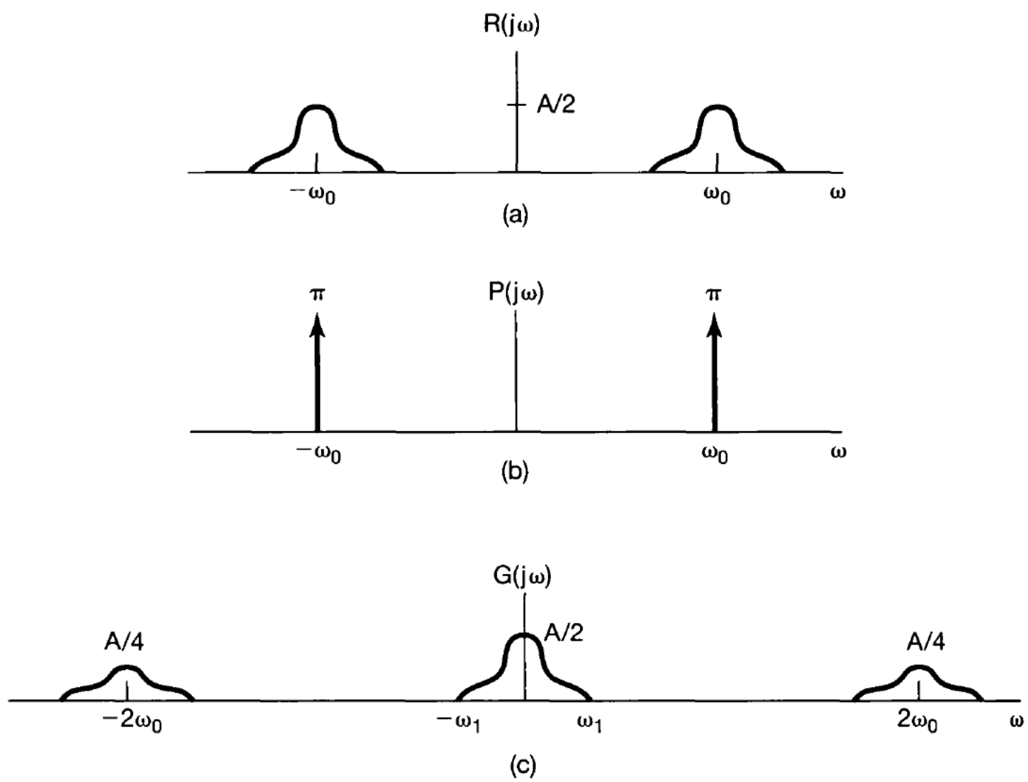
$$R(j\omega) = 1/2\pi \cdot S(j\omega) * P(j\omega)$$

$$= 1/2\pi \cdot S(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$= 1/2[S[j(\omega - \omega_0)] + S[j(\omega + \omega_0)]$$

The multiplication property

Example $\quad g(t) = r(t)p(t) \qquad G(j\omega) = ?$



The multiplication property

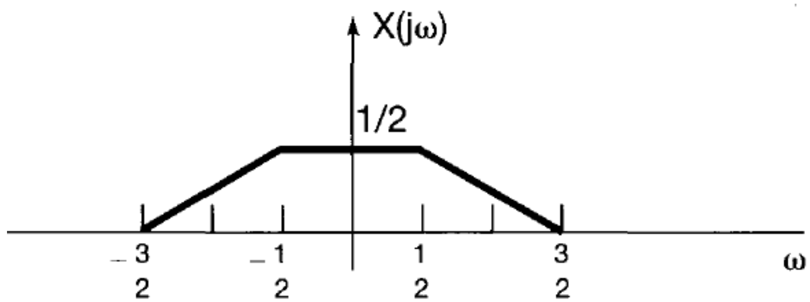
Example

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2} \qquad X(j\omega) = ?$$

- Solution

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$



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System characterized by differential equations

- Differential equation $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$Y(j\omega) = H(j\omega)X(j\omega) \quad \Longrightarrow \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \quad \Longrightarrow \quad \sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\Downarrow$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right] \quad \Longleftarrow \quad \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\searrow$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

System characterized by differential equations

Example

$$\frac{dy(t)}{dt} + ay(t) = x(t) \qquad a > 0$$

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \Longrightarrow h(t) = e^{-at}u(t)$$

System characterized by differential equations

Example

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

System characterized by differential equations

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x(t) = e^{-t}u(t) \qquad \frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t). \qquad y(t) = ?$$

❑ Solution

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right] = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}$$

$$= \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3} \qquad A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4}$$

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3} \quad \Longrightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$