Quadratic Integral Forms

- Introduction
- Analyzing quadratic integral forms
- Quadratic integral forms with quadratic end term
- Reverse Lyapunov differential equations

Contents

Quadratic Integral Forms

We focus on homogeneous linear differential equations of the form

$$\dot{x}(t) = A(t)x(t) \qquad \text{with} \qquad x(0) = x_0 \; .$$

We are interested in analyzing quadratic integrals,

$$q_0 = \int_0^T x(\tau)^\intercal Q(\tau) x(\tau) \, \mathrm{d}\tau \; .$$

- $Q(t) : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ is a given matrix-valued function.
- In practice, $Q(t)$ is often symmetric, sometimes positive definite.

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Simple examples

- For $Q(t) = \frac{1}{T} I$ the term $q_0 = \frac{1}{T}\int_0^T x(\tau)^\mathsf{T} x(\tau)\,\mathrm{d}\tau$ is the average of the square of the Euclidean norm of the trajectory x.
- If $Q(t)$ is a positive definite function, the term

$$q_0 = \|x\|^2_{L_2[0,T]} = \int_0^T x(\tau)^T Q(\tau) x(\tau)\,\mathrm{d}\tau$$

can be interpreted as a weighted L_2-norm of the trajectory x.

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Energy consumption of a resistor in an RC-circuit

Assume $x(t) = I(t)$ is the current in an RC-circuit

$$\dot{I}(t) = -\frac{1}{RC} I(t) \qquad \text{with} \qquad I(0) = \frac{V_0}{R}\ .$$

The power that is consumed by the resistor at time t is given by $P(t) = RI(t)^2$. Quadratic performance measure

$$q_0 \ = \ \int_0^T x(\tau)^T Q(\tau) x(\tau)\, \mathrm{d}t \ = \ \int_0^T RI(\tau)^2\, \mathrm{d}t$$

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In this example, we can work out q explicitly:

$$q_0 \;=\; \int_0^T RI(\tau)^2 \,\mathrm{d}t \;=\; \int_0^T \frac{V_0^2}{R} e^{-\frac{2t}{RC}} \,\mathrm{d}t \;=\; \frac{CV_0^2}{2}\left[1 - e^{-\frac{2T}{RC}}\right] \;.$$

Overall energy consumption for $T \to \infty$ is given by

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Linear integral forms

Linear integral forms can be written as

$$\int_0^T g(\tau)^\mathsf{T} x(\tau)\,\mathrm{d}\tau$$

with $g : \mathbb{R} \to \mathbb{R}^{n_x}$ being a vector-valued function.

Important: Linear integral forms can be re-written as quadratic integral forms!

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Linear integral forms

Main idea: consider the augmented state $y(t) = [x(t)^\intercal, 1]^\intercal$,

$$\dot{y}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad y(0) = \begin{pmatrix} x_0 \\ 1 \end{pmatrix} .$$

With $Q(t) = \frac{1}{2} \begin{pmatrix} 0 & g(t) \\ g(t)^\intercal & 0 \end{pmatrix}$ we have

$$\int_0^T g(\tau)^\intercal x(\tau)\,\mathrm{d}\tau = \int_0^T y(\tau)^\intercal Q(\tau) y(\tau)\,\mathrm{d}\tau .$$

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Integrals over bi-linear terms

Integrals over products of components of x, e.g.,

$$q_0 = \int_0^T x_1(\tau)x_2(\tau)\,\mathrm{d}\tau$$

are called bi-linear forms. They can be written as quadratic forms, e.g.,

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(assume $n_x = 2$).

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Short Quiz

If you see a term like

$$q_0 = \int_0^T \left\{ x_1(\tau) x_2(\tau) + x_4(\tau)^2 + x_3(\tau) + \sin(\tau) + x_2(\tau)\cos(\tau) \right\} \, \mathrm{d}\tau \ ,$$

what can you say about it?

Contents

Analysis

Let us consider the auxiliary function

$$q(t) = \int_t^T x(\tau)^\mathsf{T} Q(\tau) x(\tau) \, \mathrm{d}\tau \ ,$$

- defined for all $t \leq T$ and satisfies $q(0) = q_0$
- If G denotes the fundamental solution,

$$\begin{aligned} q(t) &= \int_t^T x(\tau)^\mathsf{T} Q(\tau) x(\tau) \, \mathrm{d}\tau \\ &= \int_t^T x(t)^\mathsf{T} G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t) x(t) \, \mathrm{d}\tau \end{aligned}$$

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- $q(t)$ can be interpreted as a quadratic form in $x(t)$.
- the shape matrix $P(t)$ is given by the expression

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- If $Q(\tau)$ is symmetric [positive semi-definite] then $P(0)$ is symmetric [positive semi-definite].

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Quadratic integral form with end term

Notation:

$$q(t) = \int_t^T x(\tau)^\mathsf{T} Q(\tau) x(\tau)\,\mathrm{d}\tau + x(T)^\mathsf{T} Q_T x(T)\ ,$$

- $Q_T \in \mathbb{R}^{n_x \times n_x}$ is the weighting matrix of the end term.

Quadratic integral form with end term

The function q can be written in the form

$$q(t) = x(t)P(t)x(t)$$

with

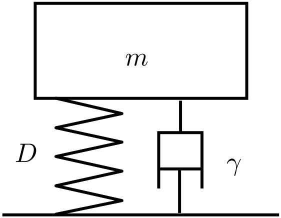
$$P(t) = \int_t^T G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t)\,\mathrm{d}\tau + G(T,t)^\mathsf{T} Q_T G(T,t)\ .$$

Proof (direct verification)

$$
\begin{aligned}
q(t) &= \int_t^T x(\tau)^\mathsf{T} Q(\tau) x(\tau)\,\mathrm{d}\tau + x(T)^\mathsf{T} Q_T x(T) \\
&= x(t)^\mathsf{T} \left(\int_t^T G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t)\,\mathrm{d}\tau + G(T,t)^\mathsf{T} Q_T G(T,t) \right) x(t)\ , \\
&= x(t)^\mathsf{T} P(t) x(t)\ .
\end{aligned}
$$

Kinetic energy of a spring

Consider a spring-damper system (no damping, $\gamma = 0$)



- States: elongation $s(t)$ and velocity $v(t)$.
- Differential equation: $\begin{cases} \dot{s}(t) &= v(t) \\ \dot{v}(t) &= -\frac{D}{m}s(t) \end{cases}$

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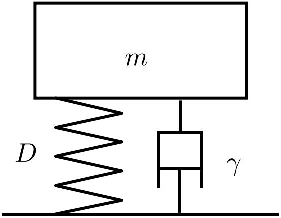

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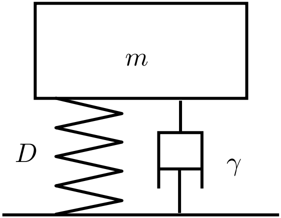

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Kinetic energy of a spring

Kinetic energy:

$$E_{\mathrm{kin}}(T) = \frac{1}{2} m \left[v(T)\right]^2$$

Can be written as

$$E_{\mathrm{kin}}(T) = x(T)^{\mathsf{T}} Q_T x(T) \quad \text{with} \quad Q_T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{m}{2} \end{pmatrix},$$

where $x(T) = (s(T), v(T))^{\mathsf{T}}$ is the state vector.

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Kinetic energy of a spring

We work out $P(0)$ explicitly:

$$\begin{aligned} P(0) &= G(T,0)^{\mathsf{T}} Q_T G(T,0) \\ &= \frac{m}{2} \begin{pmatrix} \omega^2 \sin(\omega T)^2 & -\omega \sin(\omega T)\cos(\omega T) \\ -\omega \sin(\omega T)\cos(\omega T) & \cos(\omega T)^2 \end{pmatrix} \end{aligned}$$

with $\omega = \sqrt{\frac{D}{m}}$.

Kinetic energy of a spring

- If we start at if we start $x(0) = (0, v_0)^\intercal$, the kinetic energy of the mass point is

$$E_{\text{kin}}(T) = x(T)^\intercal Q_T x(T) = x(0)^\intercal P(0) x(0) = \frac{1}{2} m v_0^2 \cos(\omega T)^2 \ .$$

Homework 5: details of the derivation & potential energy.

Contents

Reverse Lyapunov differential equations

The matrix-valued function

$$P(t) \;=\; \int_t^T G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t)\,\mathrm{d}\tau + G(T,t)^\mathsf{T} Q_T G(T,t)\;.$$

satisfies

$$\begin{aligned}
\dot{P}(t) \quad &= \quad -Q(t) - \int_t^T A(t)^\mathsf{T} G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t)\,\mathrm{d}\tau \\
&\qquad -A(t)^\mathsf{T} G(T,t)^\mathsf{T} Q_T G(T,t) \\
&\qquad -\int_t^T G(\tau,t)^\mathsf{T} Q(\tau) G(\tau,t) A(t)\,\mathrm{d}\tau - G(T,t)^\mathsf{T} Q_T G(T,t) A(t) \\
&= \quad -Q(t) - A(t)^\mathsf{T} P(t) - P(t) A(t)\;.
\end{aligned}$$

Reverse Lyapunov differential equations

The differential equation

$$-\dot{P}(t) = A(t)^{\mathsf{T}} P(t) + P(t)A(t) + Q(t) \quad \text{with} \quad P(T) = Q_T$$

is called a reverse (inhomogeneous) Lyapunov differential equation.

- We "start" with the end value $P(T) = Q_T$ and
- simulate "backwards" to find $P(0)$.
- We could also "mirror" the free variable $t \leftarrow T - t$. (Exercise).

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Summary

The reverse (inhomogeneous) Lyapunov differential equation

$$-\dot{P}(t) = A(t)^\mathsf{T} P(t) + P(t)A(t) + Q(t) \quad \text{with} \quad P(T) = Q_T$$

allows us to write $q(t)$ in the form

$$\int_t^T x(\tau)^\mathsf{T} Q(\tau) x(\tau)\,\mathrm{d}\tau + x(T)^\mathsf{T} Q_T x(T) = x(t)^\mathsf{T} P(t) x(t)\ .$$

- This result is a preparation for understanding optimal control concepts such as Dynamic Programming; we'll come back to this later.

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