Lecture 8-1 Frequency Domain Filtering

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Course piazza link： piazza.com/shanghaitech.edu.cn/spring2021/cs270spring2021

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Outline

– 2D Discrete Fourier Transform （傅里叶变换）

– Frequency Domain Filtering（频率域滤波）

- Lowpass Filtering（低通滤波器）
- Highpass Filtering （高通滤波器）
- Selective Filtering（选择性滤波）

A recall of 1-D Discrete Fourier Transform

- Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(nt) + b_n\sin(nt)]$$

- Fourier transform (continuous-time)

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

- Discrete-time Fourier Transform (DTFT)

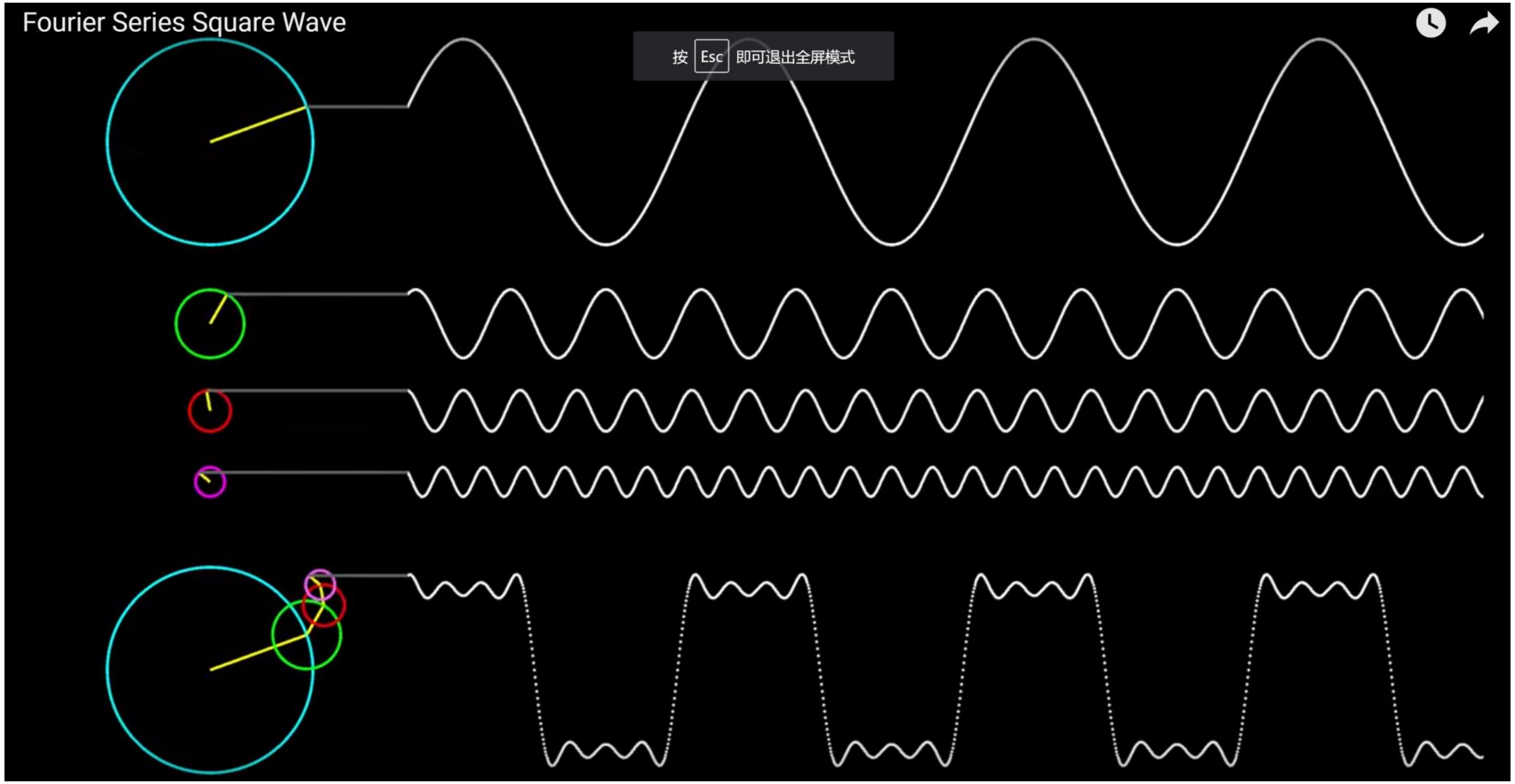
$$F(\omega) = \sum_{n=1}^{\infty} x[n]\; e^{-j\omega n},\ \omega \in [0,2\pi], \omega = 2\pi/T$$

- Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}\ ,\ (k = 1,2,\cdots,N)$$

An inside view of 1-D DFT

- An inside view: https://www.youtube.com/watch?v=cUD1gMAl6W4



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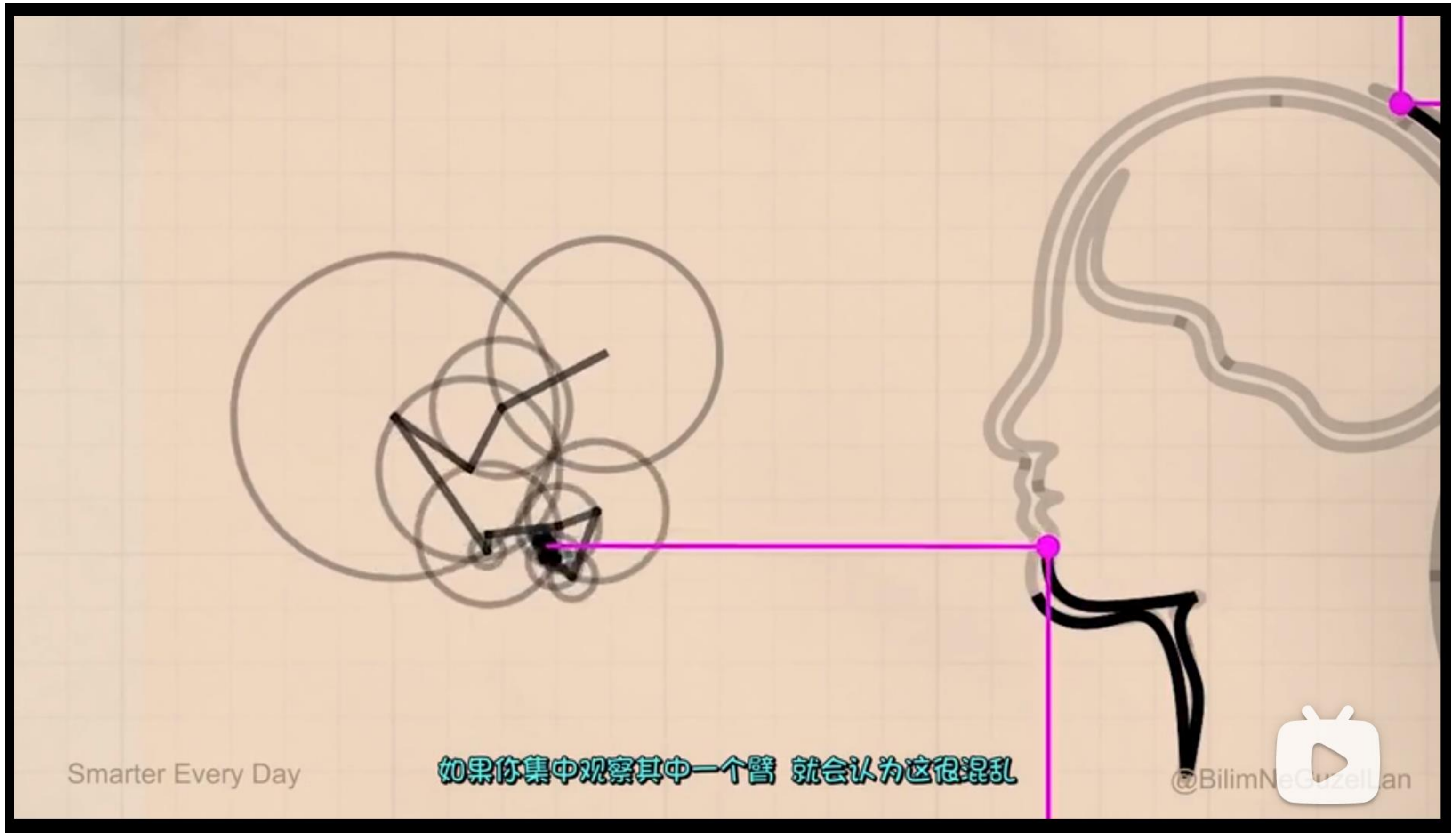

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Discrete Fourier Transform （离散傅里叶变换）

➢ Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}, (k = 1,2,\cdots,N)$$

➢ **2D Discrete Fourier Transform (DFT)**

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

➢ **2D Inverse Discrete Fourier Transform (IDFT)**

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

- $f(x,y)$: M*N input image
- (x,y): spatial variables, $(x = 0,2,\cdots,N-1; y = 0,2,\cdots,M-1)$
- (u,v): frequency variables, defines the continuous frequency domain $(u = 0,2,\cdots,N-1; v = 0,2,\cdots,M-1)$.
- **Examples of basis function.**

Separability（可分性）

2D DFT to 1D DFT

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}=\sum_{x=0}^{M-1}e^{-j2\pi\frac{ux}{M}}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi\frac{vy}{N}}=\mathcal{F}_x\{\mathcal{F}_y\{f(x,y)\}\}$$

Calculate IDFT by DFT

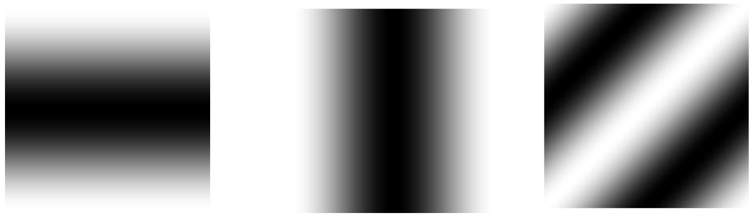
$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F^*(u,v)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

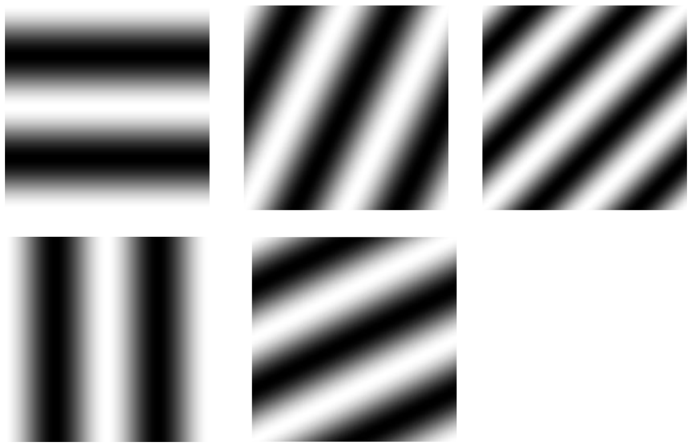
Examples of Basis Functions in 2D DFT

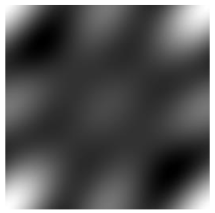
- [0,0]: Constant

- [0,1] [1,0] [1,1]



- [0,2] [2,0] [1,2] [2,1] [2,2]



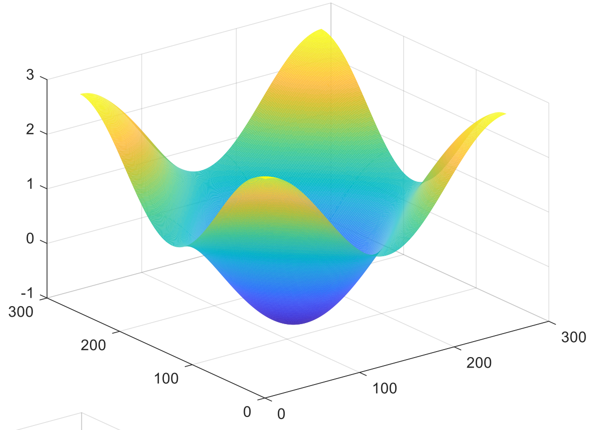


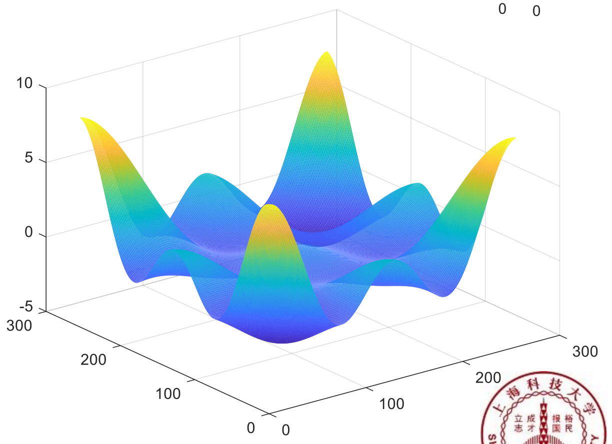
```
function im = bf(m,n)

N = 256;
[x,y] = meshgrid(0:(N-1),0:(N-1));
im = real(exp(-j*2*pi*(m*x/N +n*y/N)));

if (m==0) && (n==0)
    im = round(im);
end

figure; imshow(im,[]);
```





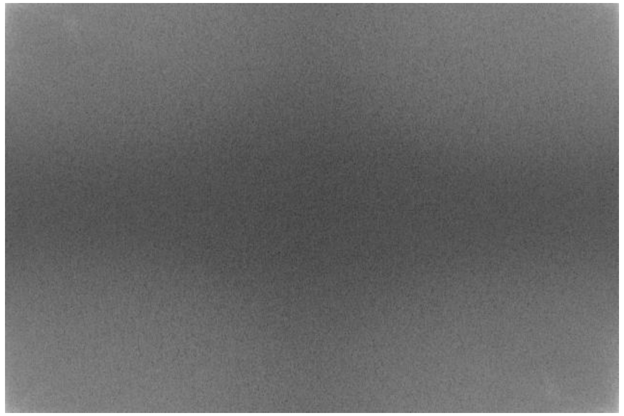
Coding task 1

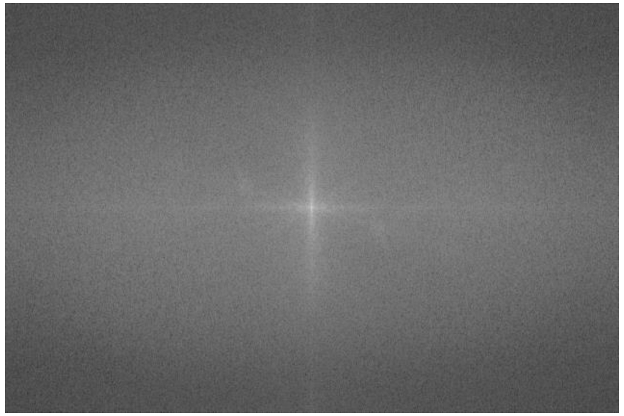
- 1. Load image vallay-house2.jpeg.
- 2. fft_im = fft2(im);
- 3. Try to show the magnitude of DFT.

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

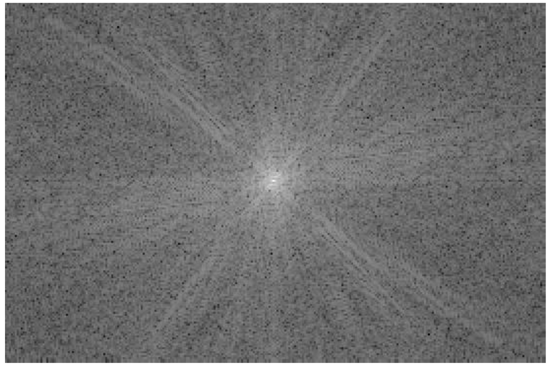
Coding task 1

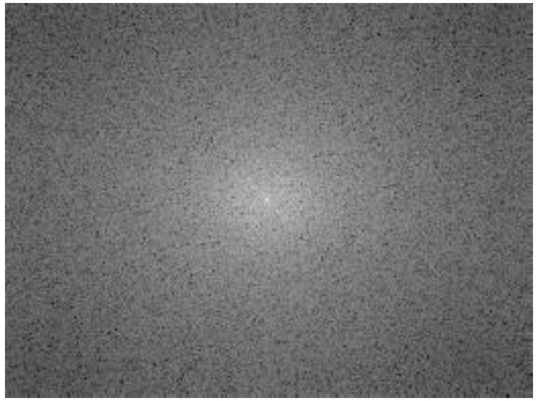
- 1. Load image vallay-house2.jpeg.
- 2. fft_im = fft2(im);
- 3. Try to show the magnitude of DFT.
- 4. fft_im_shifted = fftshift(fft_im);
- 5. im_recover = ifft(fft_im);
- 6. Try to show the inversed DFT image.

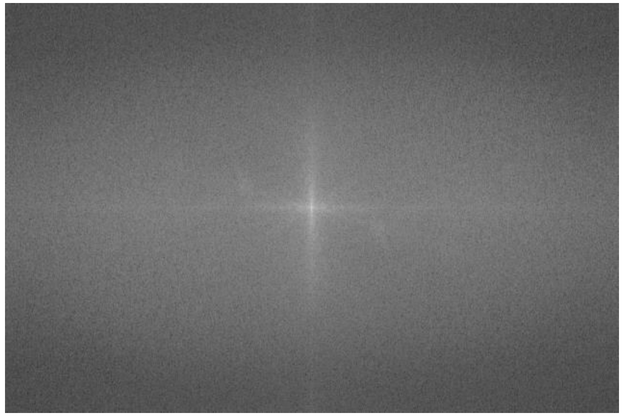




Visualization of FFT

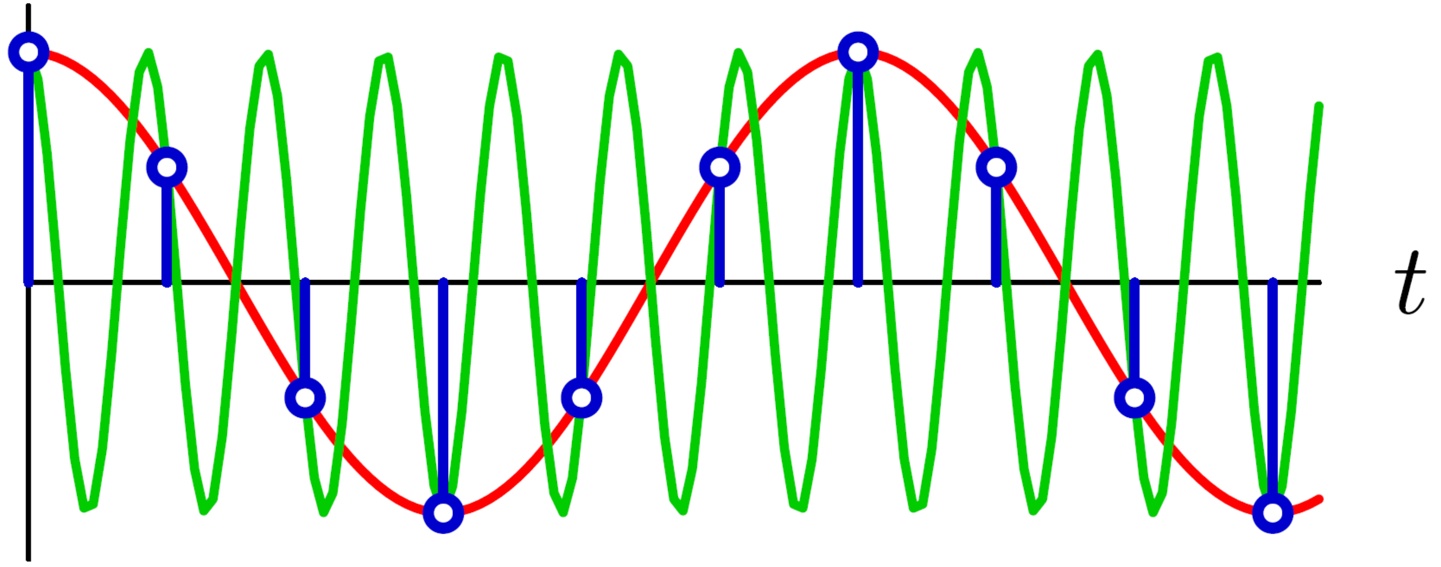






Aliasing

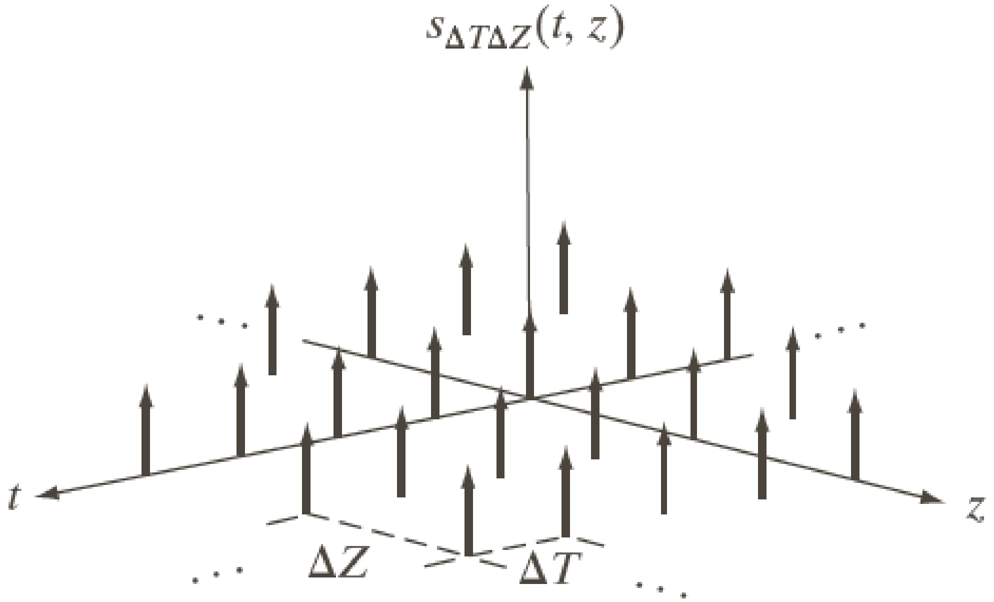
$\cos \frac{7\pi}{3} n?$ $\cos \frac{\pi}{3} n?$



2D Sampling

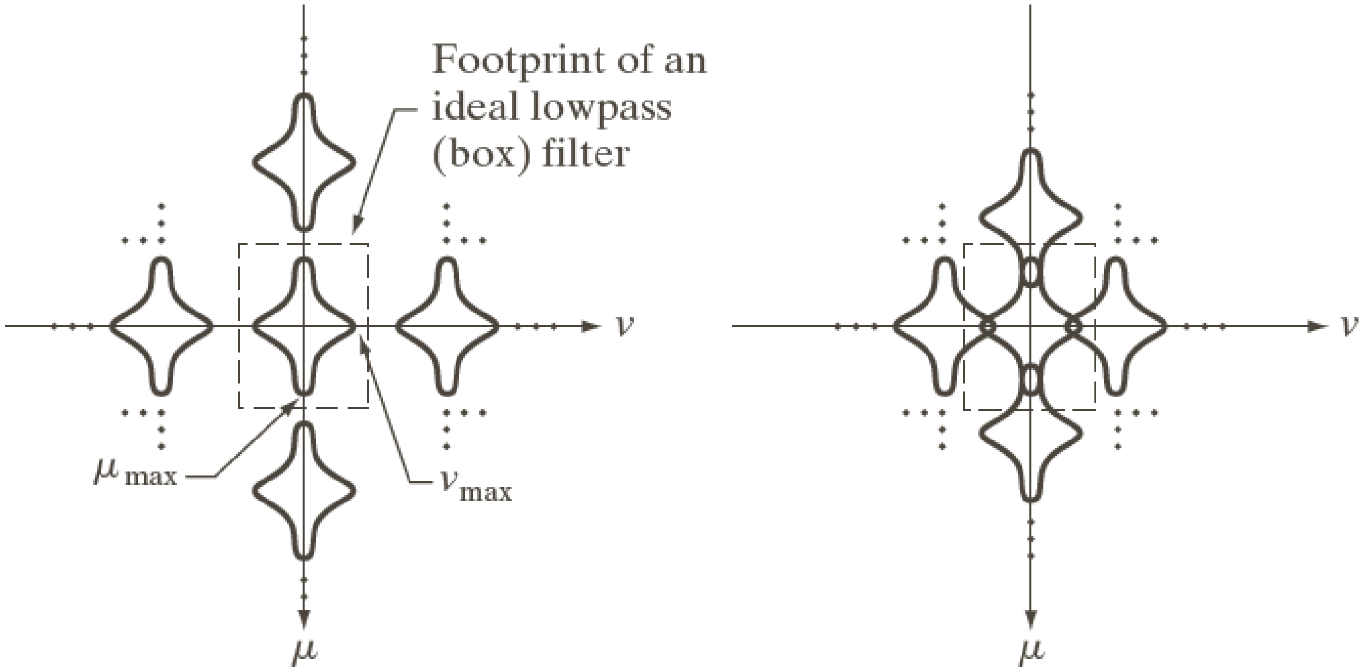
2D Sampling function (二维取样函数)

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



2D Sampling Theorem (二维取样定理)

- $f(t,z)$ is band-limited (带限函数) if $F(\mu,\nu)=0,\ |\mu|\geq\mu_{\max}$ and $|\nu|\geq\nu_{\max}$
- The sampling rate: $\frac{1}{\Delta T}>2\mu_{\max},\quad \frac{1}{\Delta Z}>2\nu_{\max}$



Spatial Aliasing（空间混淆）

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Properties of 2D DFT

- Spatial and frequency intervals （空间和频率间隔）
- Translation （平移）
- Periodicity（周期性）
- Rotation（旋转）
- Separability (可分性）
- Symmetry（对称性）
- Spectrum and Phase angle（频谱和相角）
- 2D Convolution theorem（卷积定理）

Translation（平移）

Translation

$$f(x,y)e^{j2\pi(\frac{u_0x}{M}+\frac{v_0y}{N})} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})}$$

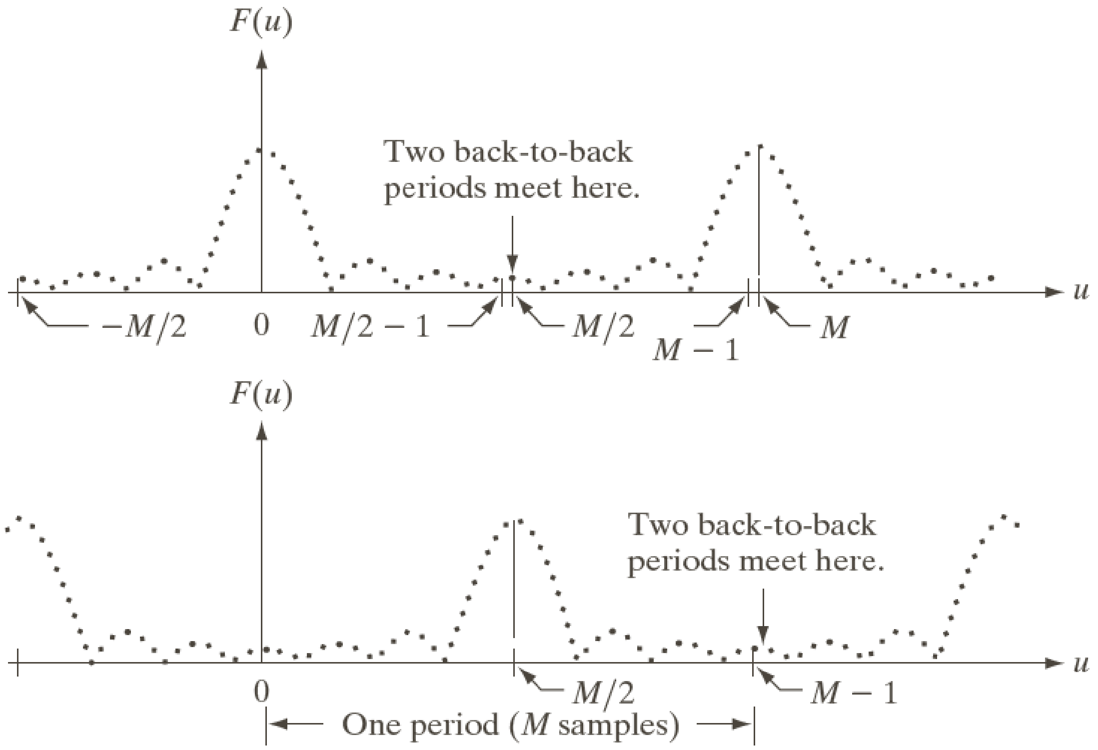
When $u_0 = \frac{M}{2}, v_0 = \frac{N}{2}$

$$F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) \Leftrightarrow f(x,y)e^{j\pi(x+y)} = f(x,y)(-1)^{(x+y)}$$

Periodicity（周期性）

- $f(x,y) = f(x + k_1 M, y) = f(x, y + k_2 N) = f(x + k_1 M, y + k_2 N)$
- $F(u,v) = F(u + k_1 M, v) = F(u, v + k_2 N) = F(u + k_1 M, v + k_2 N)$

Where k_1 and k_2 are integers

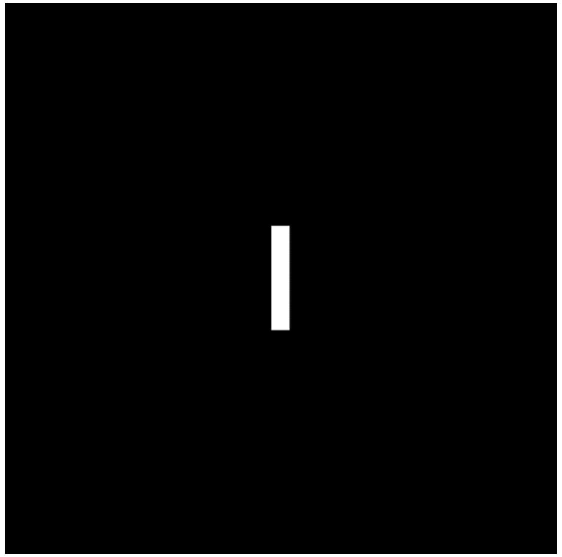
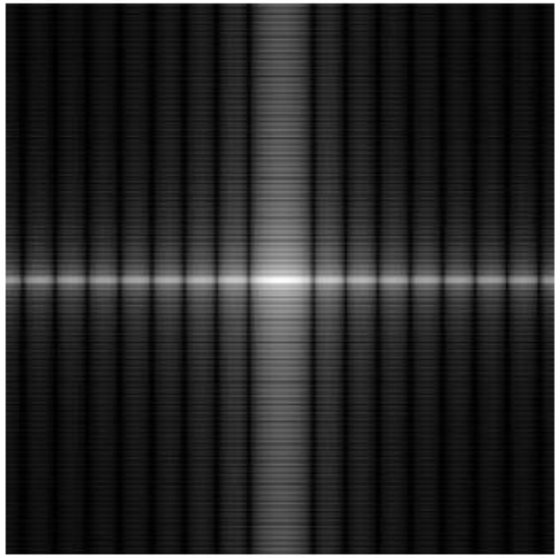
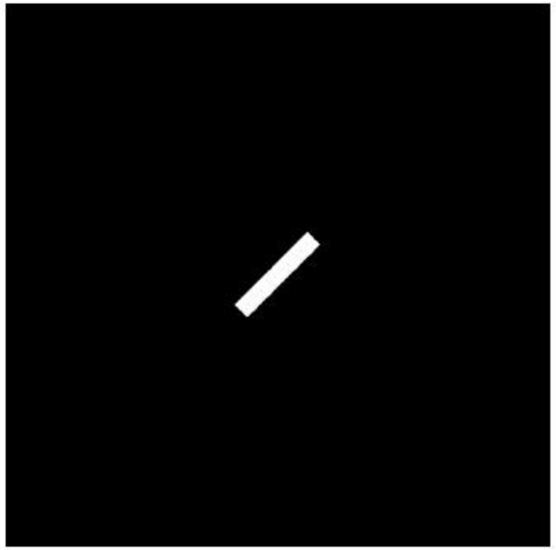
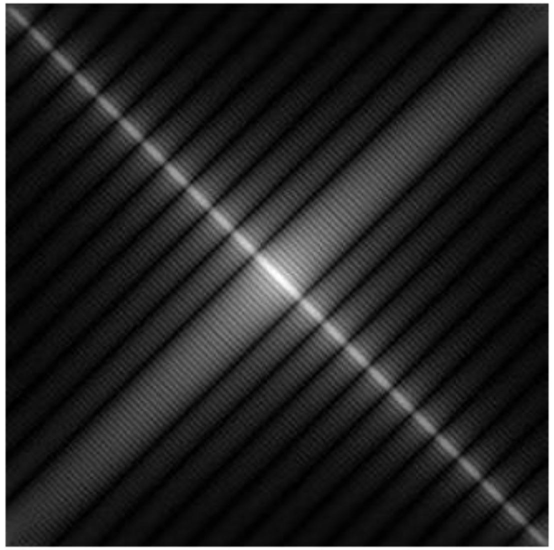


Rotation（旋转）

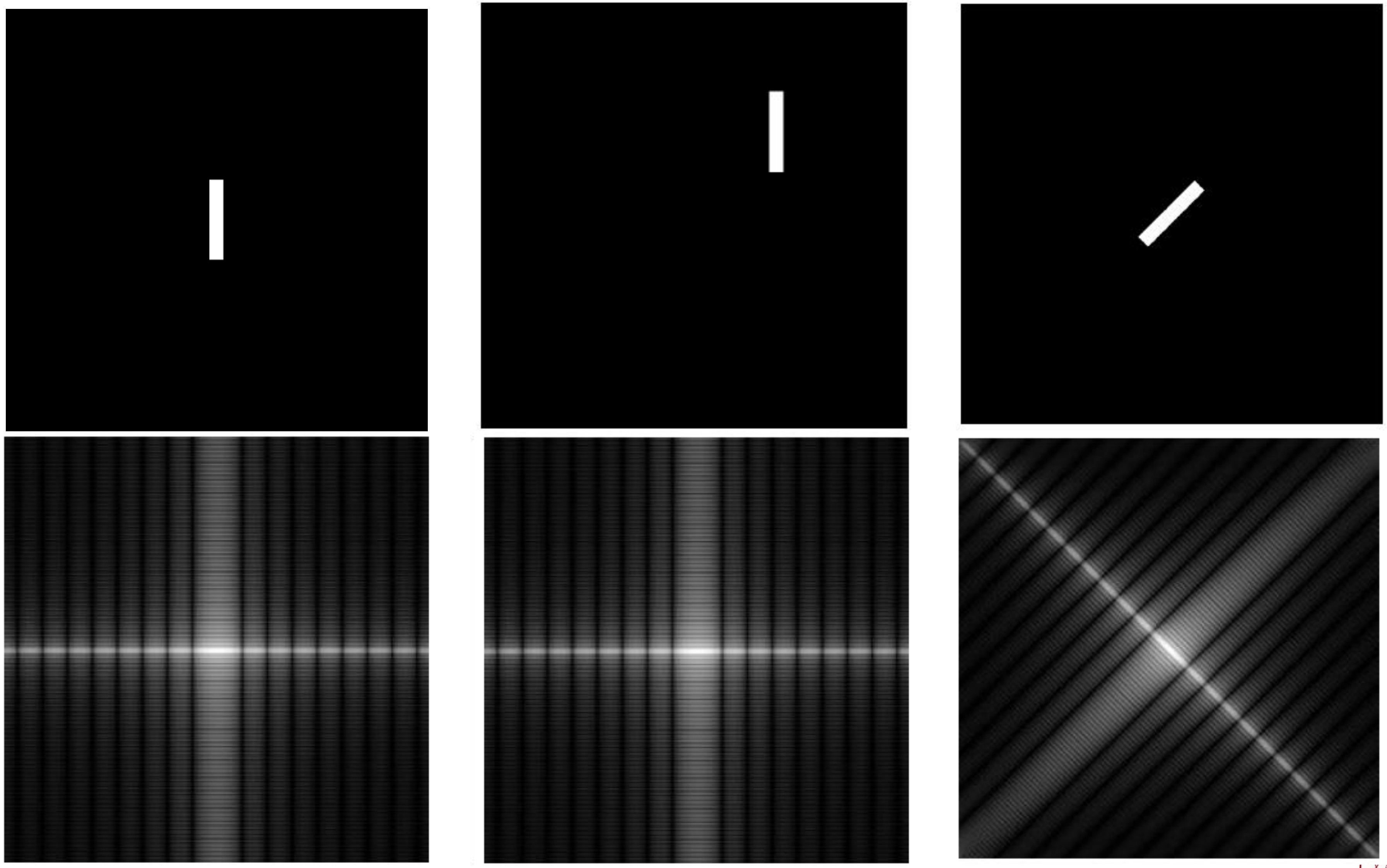
- **Rotation**

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

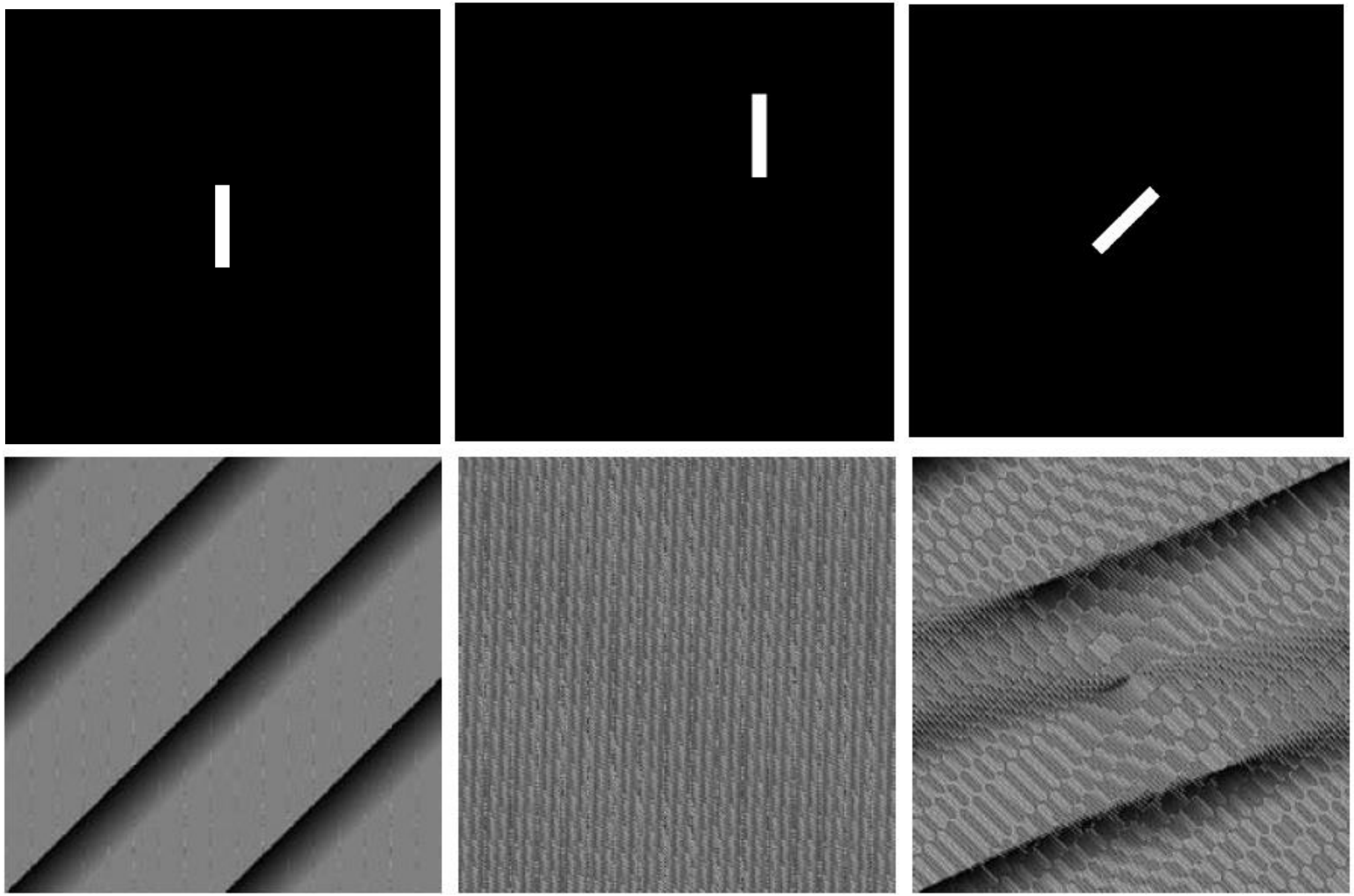
Where $x = r\cos\theta,\ \ y = r\sin\theta,\ \ u = \omega\cos\varphi,\ \ v = \omega\sin\varphi$

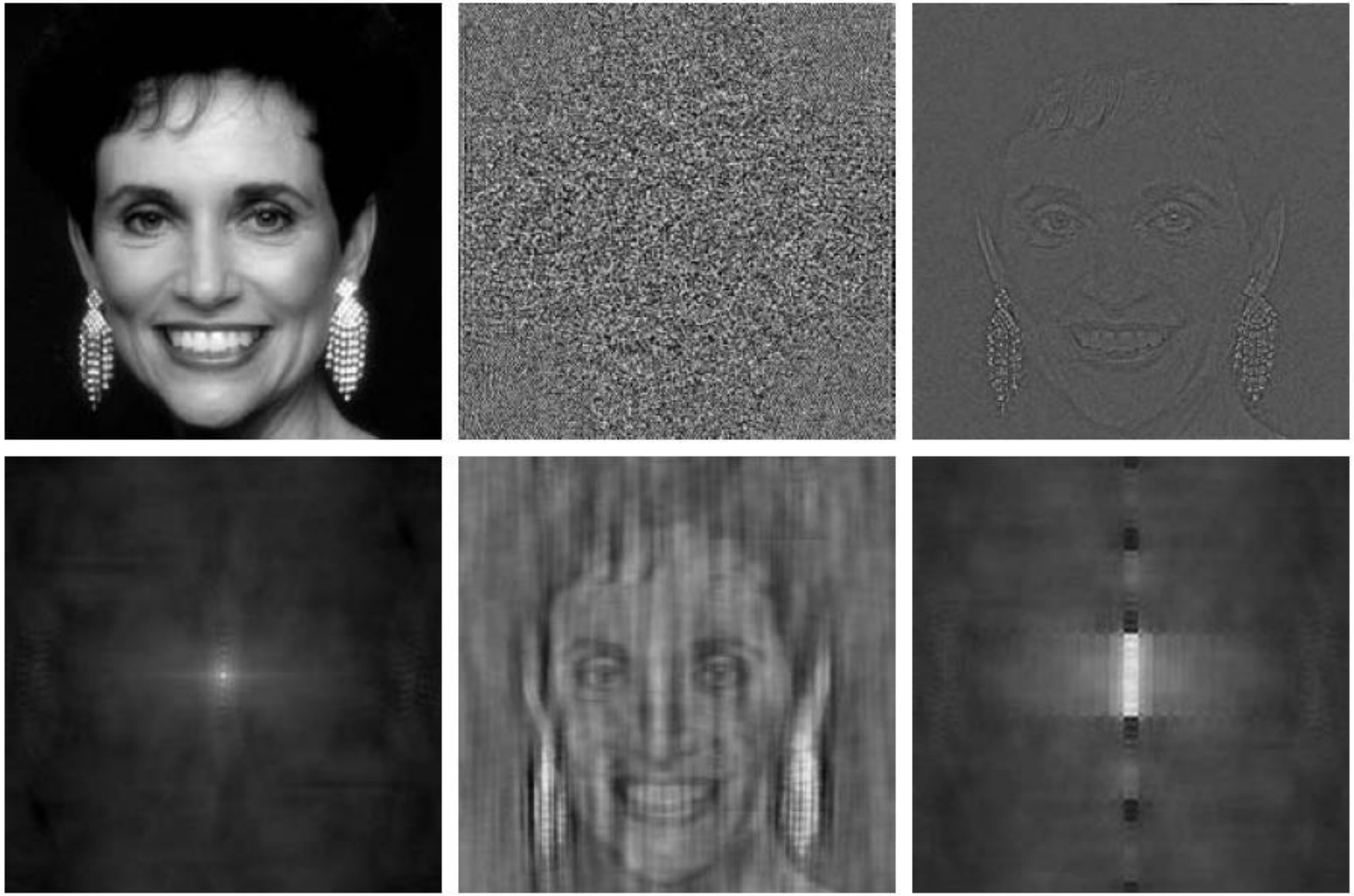
Fourier Spectrum（频谱）



Phase angle（相角）



Spectrum and Phase angle （频谱和相角）



Take home message

- 2D-DFT is a useful tool for estimating the frequency domain property about an image, however, it is not as powerful as the 1-D DFT for time domain sequence.
- Nearly all the property about 1D-DFT can be extended to 2D-DFT.

Symmetry（对称性）

- **Even Function (偶函数)**

$$w_e(x,y) = w_e(-x,-y) \quad w_e(x,y) = w_e(M-x, N-y)$$

- **Odd Function (奇函数)**

$$w_o(x,y) = -w_o(-x,-y) \quad w_o(x,y) = -w_o(M-x, N-y)$$

- **Conjugate symmetric (共轭对称)**

$$F^*(u,v) = F(-u,-v) \quad F^*(u,v) = F(M-u, N-v)$$

- **Conjugate antisymmetric (共轭反对称)**

$$F^*(u,v) = -F(-u,-v) \quad F^*(u,v) = -F(M-u, N-v)$$

Symmetry （对称性）

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

Spectrum and Phase angle（频谱和相角）

2D DFT in polar form: $F(u,v) = |F(u,v)|e^{-j\Phi(u,v)}$ **, then**

- **Fourier spectrum（频谱）:** $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{\frac{1}{2}}$
- **Phase angle（相角）:** $\Phi(u,v) = \arctan\frac{I(u,v)}{R(u,v)}$
- **Power spectrum(功率谱):** $P(u,v) = |F(u,v)|^2$
- **DC component(直流分量):** $F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\overline{f(x,y)}$

2D Convolution theorem（卷积定理）

- **Convolution theorem**

$$f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v) \text{ or } f(x,y)\, h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$$

- **Zero padding (零填充)**

$$f_p(x,y) = \begin{cases} f(x,y), & 0 \le x \le A-1, 0 \le y \le B-1 \\ 0, & A \le x \le P, B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y), & 0 \le x \le C-1, 0 \le y \le D-1 \\ 0, & C \le x \le P, D \le y \le Q \end{cases}$$

Where $f(x,y)$: $A \times B$ image; $h(x,y)$: $C \times D$ image; $P \ge A + C - 1$; $Q \ge B + D - 1$

Summary of DFT

Name	DFT Pairs
7) Correlation theorem†	$f(x,y) \star h(x,y) \Leftrightarrow F^*(u,v)\,H(u,v)$ $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$
8) Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$
9) Rectangle	$\text{rect}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\frac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) - \delta(u-Mu_0, v-Nv_0)\right]$
11) Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\frac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) + \delta(u-Mu_0, v-Nv_0)\right]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\frac{\partial}{\partial t}\right)^m\left(\frac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu);\ \dfrac{\partial^n f(t,z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13) *Gaussian*	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

Properties of DFT

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M, v) = F(u, v+k_2N)$ $= F(u+k_1M, v+k_2N)$ $f(x,y) = f(x+k_1M, y) = f(x, y+k_2N)$ $= f(x+k_1M, y+k_2N)$
9) Convolution	$f(x,y) \star h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$
10) Correlation	$f(x,y) \,☆\, h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m, y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Properties of DFT

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

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