Nonlinear Control Systems

- Nonlinear Differential Equations
- Existence and uniqueness of solutions
- Taylor-Model Based Integrators
- Runge-Kutta Integrators
- Linear Approximation of Nonlinear Control Systems

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Problem Formulation

The focus of this lecture is on scalar ordinary differential equations (ODEs),

$$\forall t \in [0,T], \qquad \dot{x}(t) = f(t,x(t)) \quad \text{with} \quad x(0) = x_0 \ .$$

Here, $x : [0,T] \to \mathbb{R}$ is the state trajectory.

Assumptions:

- The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ may be nonlinear.
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Explicit solution

- In general: no explicit solution possible
- But it some special cases, we can solve the nonlinear differential equation by using the concept of separation of variables.

Seperation of variables:

- Assumption: f is separable, i.e.,

$$f(t,x) = f_1(x)f_2(t)\ .$$

- Strategy: integrate the equation

$$\frac{\dot{x}(t)}{f_1(x(t))} = f_2(t)\ ,$$

with respect to t on both sides and eliminate $x(t)$.

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Example: quadratic differential equation

Nonlinear ODE:

$$\dot{x}(t) = -x^2(t) \quad \text{with} \quad x(0) = 1 \; .$$

Separation of variables:

$$-\frac{\dot{x}(t)}{x(t)^2} = 1 \qquad \stackrel{\text{integrate}}{\Longrightarrow} \qquad \frac{1}{x(t)} - \frac{1}{x(0)} = t$$

Elimination of $x(t)$:

$$x(t) = \frac{1}{1+t} \quad \text{for all} \quad t \geq 0 \; .$$

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Example: Gauss' differential equation

ODE:

$$\dot{x}(t) = -t x(t) \quad \text{with} \quad x(0) = 1 \; .$$

Separation of variables:

$$\frac{\dot{x}(t)}{x(t)} = -t \qquad \Longrightarrow \qquad \log(x(t)) = -\frac{1}{2}t^2$$

Elimination of $x(t)$:

$$x(t) = e^{-\frac{t^2}{2}} \; ,$$

Example: Gauss' differential equation

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$$\dot{x}(t) = -tx(t) \quad \text{with} \quad x(0) = 1 \; .$$

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Integral Form

The ordinary differential equation (ODE)

$$\forall t \in [0, T], \qquad \dot{x}(t) = f(t, x(t)) \quad \text{with} \quad x(0) = x_0 \ .$$

can be equivalently be written in its integral form

$$\forall t \in [0, T], \qquad x(t) = x_0 + \int_0^t f(s, x(s)) \, \mathrm{d}s \ .$$

Lipschitz continuity

Definition:

- The function f is called (globally) Lipschitz continuous, if there exist a constant $L < \infty$ with

$$\forall x, y \in \mathbb{R}, \qquad |f(x) - f(y)| \leq L|x - y| \,.$$

Existence and Uniqueness

Theorem (Picard-Lindelöf):

- If f is globally Lipschitz continuous, the ODE has a unique solution.

Proof: (main idea, rough sketch only)

1) Start with <u>any</u> continuous function $y_1 : [0, T] \to \mathbb{R}$ and iterate

$$y_{i+1}(t) = x_0 + \int_0^t f(y_i(s))\,\mathrm{d}s \qquad \text{[Picard iteration]}$$

2) Show that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence, $y^* = \lim_{k\to\infty} y_i$.

3) Conclude that the (unique) limit point y^* satisfies the ODE.

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Some technical details

- Define $\Delta(t) = \max_{s\in[0,t]} |y_2(s) - y_1(s)|$.
- If $|y_{i+1}(t) - y_i(t)| \leq \frac{(tL)^{i-1}}{(i-1)!}\Delta(t)$, then

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\begin{aligned}
|y_{i+2}(t) - y_{i+1}(t)| &\leq L\left|\int_0^t [y_{i+1}(\tau) - y_i(\tau)]\,\mathrm{d}\tau\right| \\
&\leq \int_0^t L\frac{(\tau L)^{i-1}}{(i-1)!}\Delta(t)\,\mathrm{d}\tau = \frac{(tL)^i}{i!}\Delta(t)\ .
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- Thus, we have

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|y_n(t) - y_m(t)| &\leq \sum_{i=n}^{m-1} |y_{i+1}(t) - y_i(t)| \leq \sum_{i=n}^{m-1} \frac{(tL)^{i-1}}{(i-1)!}\Delta(t) \\
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Example: Linear ODEs

- Linear ODE: $\dot{x}(t) = ax(t)$, $a \in \mathbb{R}$, with $x(0) = x_0$.
- Picard iteration:

$$\begin{aligned} y_1(t) &= x_0 \\ y_2(t) &= x_0 + tax_0 \\ y_3(t) &= x_0 + tax_0 + \tfrac{t^2}{2}a^2x_0 \\ &\vdots \end{aligned}$$

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$$x(t) = e^{at}x_0 = \sum_{i=0}^{\infty} \frac{1}{i!}[ta]^i x_0 \ .$$

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Examples for nonlinear ODEs

- The ODE $\dot{x}(t) = x(t)^2$, with $x(0) = 1$ has the explicit solution

$$x(t) = \frac{1}{1-t} \quad \text{for} \quad t < 1$$

  Why does the solution not exist for $t \geq 1$?

- The ODE $\dot{x}(t) = 2\sqrt{x}$, with $x(0) = 0$ has more than one solution,

$$\text{for example} \quad x(t) = 0 \quad \text{and} \quad x(t) = t^2 \ .$$

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Taylor expansion of ODEs

A Taylor expansion of the solution $x(t)$ can be constructed recursively:

- $x(t_0) = x_0$
- $\dot{x}(t_0) = f(t_0, x_0)$
- $\ddot{x}(t_0) = \frac{\partial}{\partial t} f(t, x(t))\Big|_{t=t_0} = f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)$
- and so on ...
- Finally, $x(t) =$

  $x_0 + f(t_0, x_0)(t - t_0) + \frac{(t-t_0)^2}{2} \left[f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0) \right] + \ldots$

  for small t.

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Taylor expansion of ODEs

A general Taylor expansion can be computed by consecutive differentiation:

1. Set $\phi_0(t,x) = x$.
2. For $r = 0 : s-1$

   set $\phi_{r+1}(t,x) = \left(\frac{\partial}{\partial t}\phi_r(t,x)\right) + \left(\frac{\partial}{\partial x}\phi_r(t,x)\right) f(t,x)$.
3. Return the Taylor expansion

$$x(t) = \sum_{i=0}^{s} \frac{1}{i!}\phi_i(t_0,x_0)(t-t_0)^i + \mathbf{O}((t-t_0)^{s+1}) \ .$$

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Integration Algorithm (Constant Step-Size)

Input:

- The right-hand side function f and an initial value x_0.
- Order s and constant step-size $h = T/N$; set $i = 0$ and $y_0 = x_0$.

Repeat: (until $i = N$)

- Compute $y_{i+1} = \sum_{k=0}^{s} \frac{1}{k!} \phi_k(t_i, y_i) h^k$
- Compute $t_{i+1} = t_i + h$ and set $i \leftarrow i + 1$.

Theorem:

- If f is globally Lipschitz continuous and smooth, then

$$\forall i \in \{0, \ldots, N\}, \qquad y_i = x(t_i) + \mathbf{O}(h^s) \ .$$

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Proof (main idea)

1. Since f is globally Lipschitz, the solution x of the ODE exists.
2. Since f is smooth, the functions $\phi_0, \phi_1, \ldots, \phi_s$ are smooth, too.
3. We already know that $x(t) = \sum_{k=0}^{s} \frac{1}{k!} \phi_k(x_0) h^k + \mathbf{O}(h^{s+1})$.
4. Show by induction that

$$y_i = x(ih) + i \cdot \mathbf{O}(h^{s+1}) = x(ih) + \frac{T}{h} \cdot \mathbf{O}(h^{s+1}) = x(ih) + \mathbf{O}(h^s).$$

The integer s is called the convergence order of the integrator.

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3. We already know that $x(t) = \sum_{k=0}^{s} \frac{1}{k!} \phi_k(x_0) h^k + \mathbf{O}(h^{s+1})$.
4. Show by induction that

$$y_i = x(ih) + i \cdot \mathbf{O}(h^{s+1}) = x(ih) + \frac{T}{h} \cdot \mathbf{O}(h^{s+1}) = x(ih) + \mathbf{O}(h^s).$$

The integer s is called the convergence order of the integrator.

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Limitations of Taylor model based integrators

1. Taylor model based intgration is easy to implement, but
   - we need to evaluate derivatives of f
   - it is not the most efficient scheme for obtaining convergence order s.
2. Runge-Kutta integrators compute an approximation $y \approx x(h)$ by evaluating f at more than one point, but don't evaluate derivatives.

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Explicit Runge Kutta method (constant step-size)

Initialization:

- Set $h = T/N$, $t_0 = 0$, $i = 0$, and $y_0 = x_0$.

Repeat: (until $i = N$)

- Compute $t_{i+1} = t_i + h$.
- Compute $k_r = f(t_i + h\gamma_r, y_i + \sum_{j=1}^{r-1} h\alpha_{r,j}k_j)$ for $r = 1, \ldots, s$.
- Set $y_{i+1} = y_i + h\sum_{r=1}^{s} \beta_r k_r$ and then $i \leftarrow i + 1$.

Output:

- Time grid $[t_1, t_2, \ldots, t_N]$ and state trajectory $y_0, y_1, y_2, \ldots, y_N$.

Consistency conditions

Main idea:

- Choose the coefficients $\alpha_{r,j}$, β_r, and γ_r such that

$$\forall r \in \{1, \ldots, q\}, \quad \left. \frac{\partial^r y_{i+1}}{\partial h^r} \right|_{h=0} = \Phi_r(y_i) \; .$$

Example 1: Euler's method

- For $s = 1$, the Runge-Kutta method takes the form

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ y_{i+1} &= y_i + h\beta_1 k_1 = y_i + h\beta_1 f(t_i, y_i) \end{aligned} \tag{1}$$

- We have

$$\left.\frac{\partial y_{i+1}}{\partial h}\right|_{h=0} = \left|\frac{\partial}{\partial h}\left(y_i + h\beta_1 f(t_i, y_i)\right)\right|_{h=0} = \beta_1 f(t_i, y_i) \tag{2}$$

and

$$\phi_1(t, x) = f(t, x) \tag{3}$$

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- The equation

$$\left.\frac{\partial y_{i+1}}{\partial h}\right|_{h=0} = \phi_1(t_i, y_i) \qquad \stackrel{(2),(3)}{\Longleftrightarrow} \qquad \beta_1 f(t_i, y_i) = f(t_i, y_i)$$

is satisfied for $\beta_1 = 1$.

- Result: Euler's method

$$y_{i+1} = y_i + hf(t_i, y_i)\ .$$

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Example 2: Heun's method

- Heun's method is given by the coefficient scheme

$$
\begin{array}{c|cc}
\gamma_1 & 0 & \\
\gamma_2 & \alpha_{2,1} & 0 \\
\hline
 & \beta_1 & \beta_2
\end{array}
\qquad = \qquad
\begin{array}{c|cc}
0 & 0 & \\
1 & 1 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

- The corresponding method can be written as

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\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + h, y_i + hk_1) \\
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Example 3: RK 4

- A very elegant method of order 4 is given by the scheme

$$
\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right) \\
k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_2\right) \\
k_4 &= f(t_i + h, y_i + h k_3) \\
y_{i+1} &= y_i + h\left(\frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4\right) \; .
\end{aligned}
$$

This method is called the <u>classical</u> Runge Kutta method.

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Nonlinear Control Systems

A general nonlinear control system is a differential equation of the form

$$\forall t \in [0, T], \qquad \dot{x}(t) = f(x(t), u(t)) \quad \text{with} \quad x(0) = x_0 \ ,$$

where $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is a nonlinear function.

- If the control input function $u(t)$ is given, the differential equation can be computed by using the numerical integration techniques from the previous slides.
- BUT: there is often no explicit closed-form solution; nonlinear control systems are in general difficult to analyze.

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Steady States

A point $(x_\mathrm{s}, u_\mathrm{s}) \in \mathbb{R} \times \mathbb{R}$ is called a steady-state, if

$$f(x_\mathrm{s}, u_\mathrm{s}) = 0 \; .$$

- Sometimes steady-states can be found by simulation of

$$\dot{x}(t) = f(x(t), u_\mathrm{s}) \quad \text{with} \quad x(0) = x_0$$

  if $\lim_{t\to\infty} x(t) = x_\mathrm{s}$ (if the system is asymptotically stable).

- Otherwise, we need to solve the equation

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Linear Approximation at Steady States

If we have already a steady-state $(x_{\mathrm{s}}, u_{\mathrm{s}}) \in \mathbb{R} \times \mathbb{R}$ and if f is continously differentiable, we can compute the first order Taylor approximation

$$f(x,u) \approx a(x - x_{\mathrm{s}}) + b(u - u_{\mathrm{s}}) \ .$$

with

$$f(x_{\mathrm{s}}, u_{\mathrm{s}}) = 0 \ , \quad a = \frac{\partial}{\partial x} f(x_{\mathrm{s}}, u_{\mathrm{s}}) \ , \quad b = \frac{\partial}{\partial u} f(x_{\mathrm{s}}, u_{\mathrm{s}})$$

Linear Approximation at Steady States

The solution of the linear differential equation

$$\dot{z}(t) = az(t) + bv(t) \quad \text{with} \quad \begin{cases} z(0) = x(0) - x_{\mathrm{s}} \\ v(t) = u(t) - u_{\mathrm{s}} \end{cases}$$

approximates the solution trajectory $x(t)$ of the nonlinear system,

$z(t) \approx x(t) - x_{\mathrm{s}}$ for

- small $t \geq 0$ if $\|z(0)\|$ and $\|v(t)\|$ are small; and
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Summary of a “Practical Workflow”

1. Simulate the system for a suitable constant input u_{s} in order to find the corresponding steady-state x_{s}.
2. Linearize the system at the steaty state and store a and b.
3. Design a controller for the linear system, $\dot{z}(t) = az(t) + bv(t)$.
4. Test whether the controller happens to work reasonably well for the original nonlinear system (by simulation).