1 VCG Auction with Redistribution

Consider a seller who wants to sell 2 homogeneous items, and each buyer requires at most one item. There are $n = 5$ buyers, and their values are $\{10, 9, 5, 5, 2\}$ respectively. Suppose agent a_i bids v'_i ($v'_1 \geq \cdots \geq v'_5$), the mechanism first uses VCG determines the allocation and payment and then returns the surplus to agents as follows:

$$r_i = \begin{cases} 2v'_4/n & \text{for } i = a_1, a_2, \\ 2v'_3/n & \text{for } i = a_3, \ldots, a_n. \end{cases}$$

1.1 (1pt)

Compute all agents' payments after redistribution. Will this mechanism be budget-balanced in all cases?

$$p_{a_1} = p_{a_2} = 5 - 2 = 3$$
$$p_{a_3} = p_{a_4} = p_{a_5} = -2$$

It will not be budget balanced when $v'_3 \neq v'_4$.

1.2 (1pt)

Is this mechanism incentive-compatible? If so, give the proof. If not, state which agent can get higher utility by misreporting.

No, it is not incentive compatible. a_3 could increase her utility by reporting v'_3 between $(5, 9)$.

2 Facility Location

Consider the following facility location problem on a line $[0, l]$. We want to build a facility at the location $x \in [0, l]$. There are n agents have an ideal (private) position $p_i \in [0, l]$ (w.l.o.g., assuming $p_1 \leq p_2 \leq \cdots \leq p_n$), where they would like the facility to be built, and their costs if the location x is chosen are the distances to the facility $c_i(x) = |x - p|$.

2.1 (1pt)

Suppose the objective is to minimize the maximum cost $max_i\{c_i(x)\}$. Give a truthful mechanism which guarantees a 2-approximation to the objective.

Locate the facility at p_1.

2.2 (2pt)

Prove that the mechanism you propose in 2.1 is truthful and has a 2-approximation ratio.

Truthfulness: agent 1 is a dictator.
2-approximation: $p_n - p_1 = 2\frac{p_n - p_1}{2}$.

3 Ranked Pairs Method

The ranked pairs method is a social welfare function for voting, and its calculation is done in three steps. The first step is to count pairwise voter preferences (e.g. Consider the pair of candidates a and b. If 3 of 10 voters agree with $a \succ b$, then the count of $a \succ b$ is 3). The second step is to sort (rank) each pair, by the largest strength of victory first to smallest last. The third step is to lock in each pair. Start with the one with the largest number of winning votes, and add one in turn to a graph as long as they do not create a cycle (which would create an ambiguity). The completed graph shows the winner (if there are more than one possible outcomes, we always assume a random tie-breaking).

3.1 (1pt)

Given the following voters and their preferences, calculate the output of the ranked pairs method. (Show your process of calculation)

$$\texttt{V1} : a \succ_1 b \succ_1 c$$
$$\texttt{V2} : b \succ_2 a \succ_2 c$$
$$\texttt{V3} : a \succ_3 c \succ_3 b$$
$$\texttt{V4} : a \succ_3 b \succ_3 c$$
$$\texttt{V5} : a \succ_3 c \succ_3 b$$

Pair	Count
$a \succ c$	5
$a \succ b$	4
$b \succ c$	3
$c \succ b$	2
$b \succ a$	1
$c \succ a$	0

Thus the result is $a \succ b \succ c$.

3.2 (2pt)

Does the ranked pairs method satisfy the properties of unanimity and independence of irrelevant alternatives? For each property, if it is satisfied, give a proof; otherwise, give a counterexample.

1) The Ranked pairs method satisfies the property of unanimity since the preference the voters hold will receive the highest score.
2) The Ranked pairs method does not satisfy the property of independence of irrelevant alternatives. An counterexample can be:

# of voters	preference
3	$A \succ B \succ C$
2	$C \succ A \succ B$
2	$B \succ C \succ A$
1	$C \succ B \succ A$

Given a tie-breaking rule $A \succ B \succ C$. If turn two of $A \succ B \succ C$ to $A \succ C \succ B$, where the order between A and C does not change. However, the result is turned into $C \succ A \succ B$ from $A \succ B \succ C$ where the order between A and C reverses.
Every other valid counterexample is OK.