Lecture 12

- Frequency Response

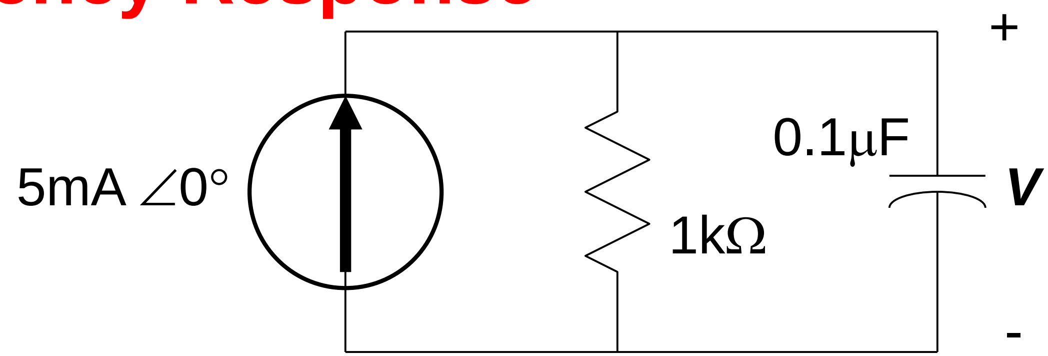
Outline

- Frequency response

-Transfer function

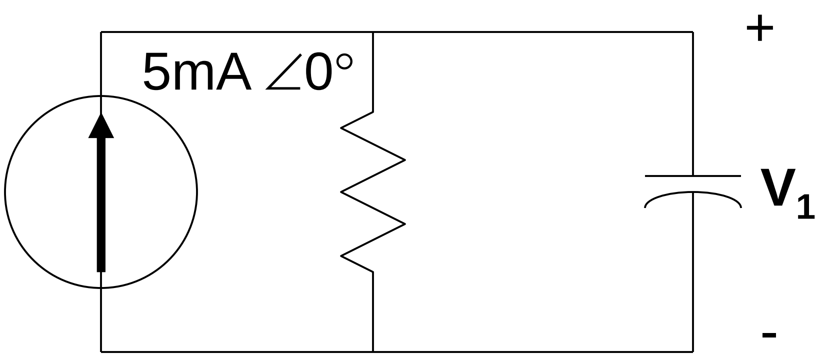
-Bode plots (or diagram)

Frequency Response



Find $v(t)$

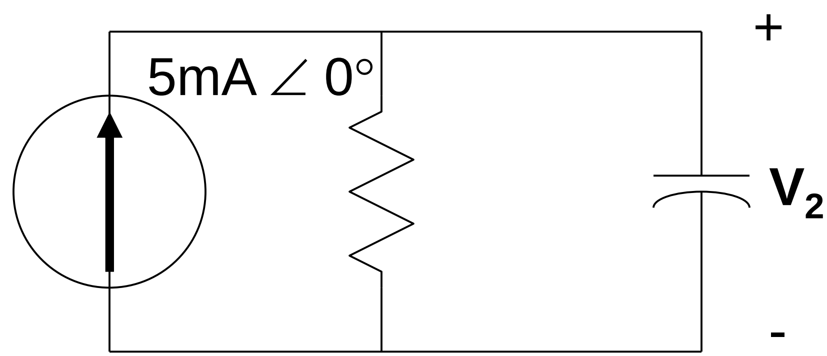
Case 1: $\omega = 2\pi \times 3000$



$$\mathbf{Z}_{eq} = 468.2\angle -62.1^\circ \Omega$$

$$\mathbf{V}_1 = 2.34\angle -62.1^\circ \mathrm{V}$$

Case 2: $\omega = 2\pi \times 455000$

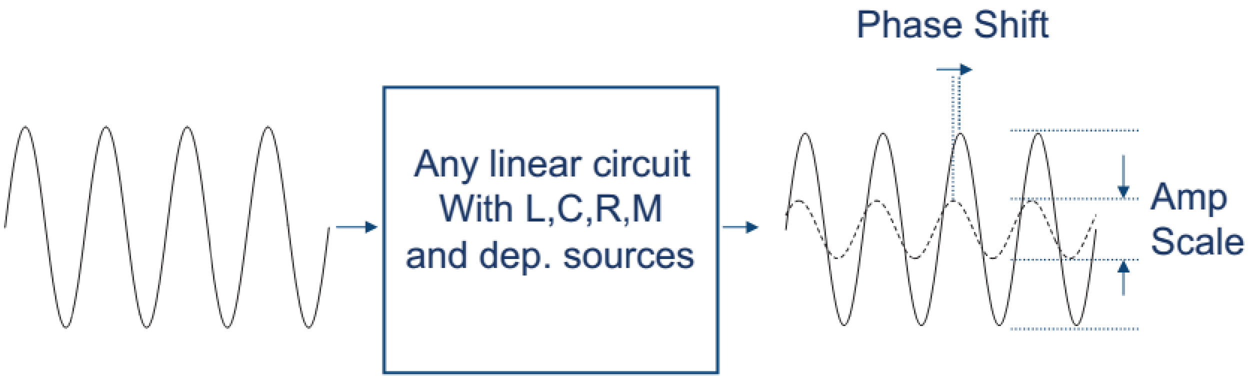


$$\mathbf{Z}_{eq} = 3.5\angle -89.8^\circ \Omega$$

$$\mathbf{V}_2 = 17.5\angle -89.8^\circ \mathrm{mV}$$

[Source: Berkeley]

Frequency Response

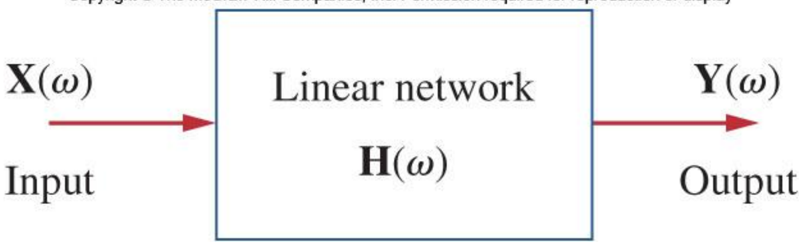


- The "Frequency Response" is a characterization of the input-output relation for sinusoidal inputs at **all** frequencies.
- Its output is also a sinusoid at the ***same*** frequency.
- Only the magnitude and phase of the output differ from the input.
- Significant for applications, esp. in communications and control systems.

[Source: Berkeley]

Transfer Function

- The transfer function *H(ω)* is the frequency-dependent ratio of a forced function *Y(ω)* to the forcing function *X(ω)*.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

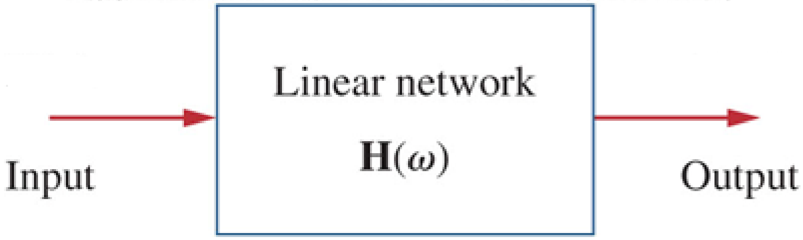
$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

Transfer Function

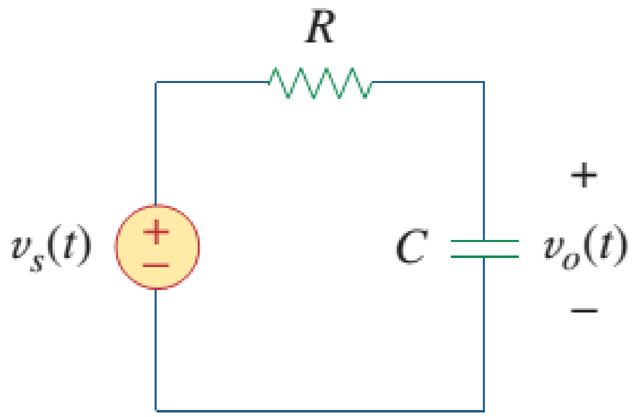
- Complex quantity
- Both *magnitude and phase of the output* are functions of frequency

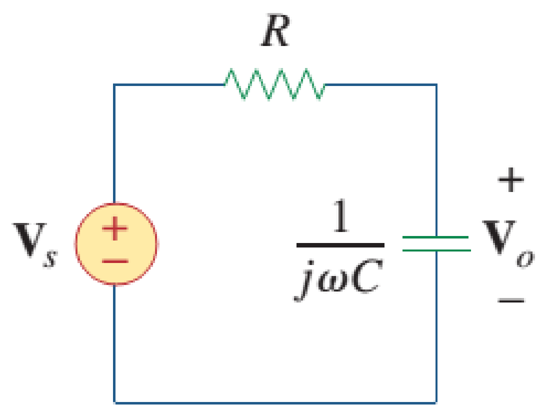


$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle \left(\theta_{out} - \theta_{in}\right)$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

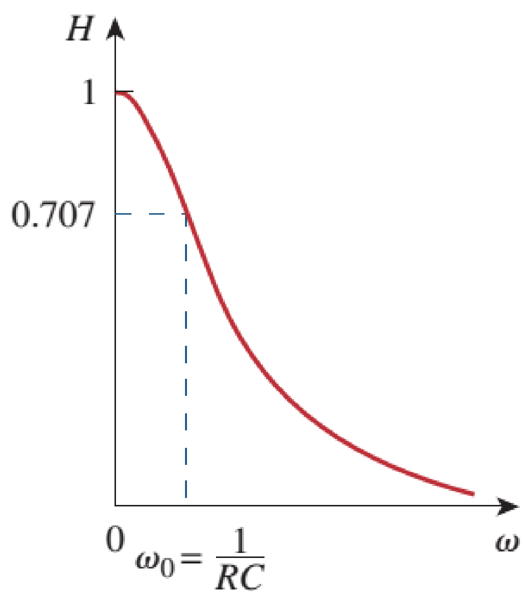
Example

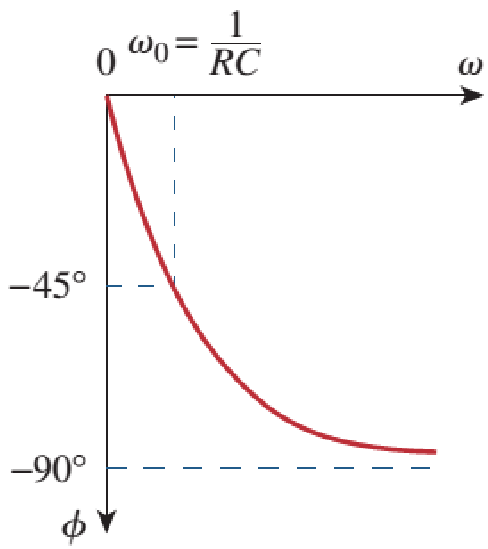




$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

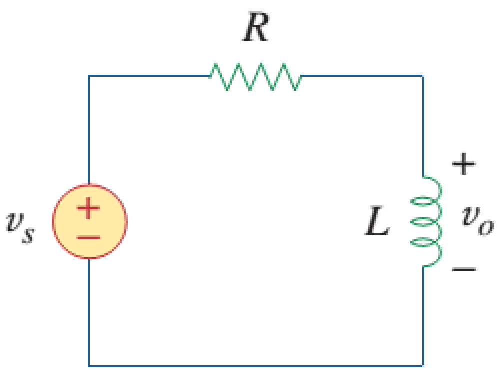
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

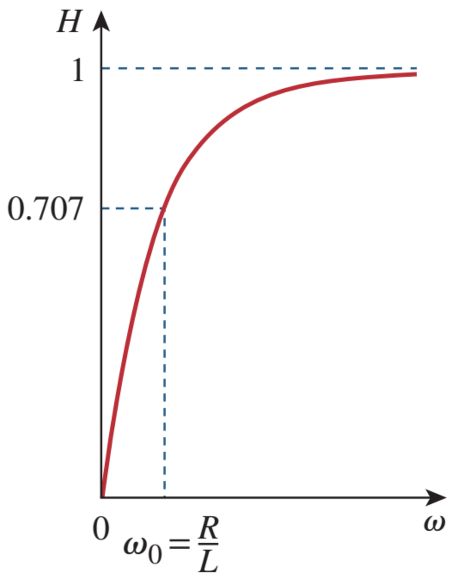


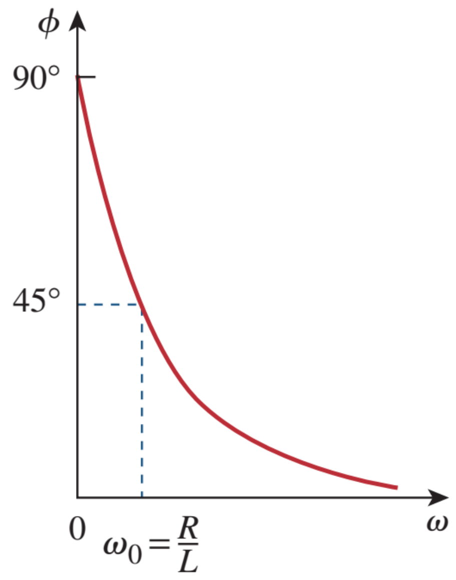


Exercise

- Obtain the transfer function $\mathbf{V_o}/\mathbf{V_s}$ of the RL circuit. Assuming $v_s = V_m \cos \omega t$.



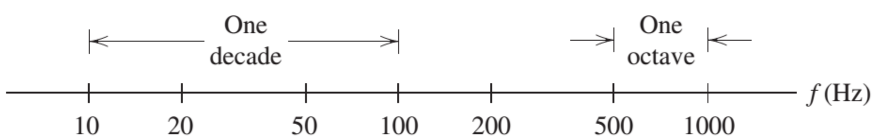




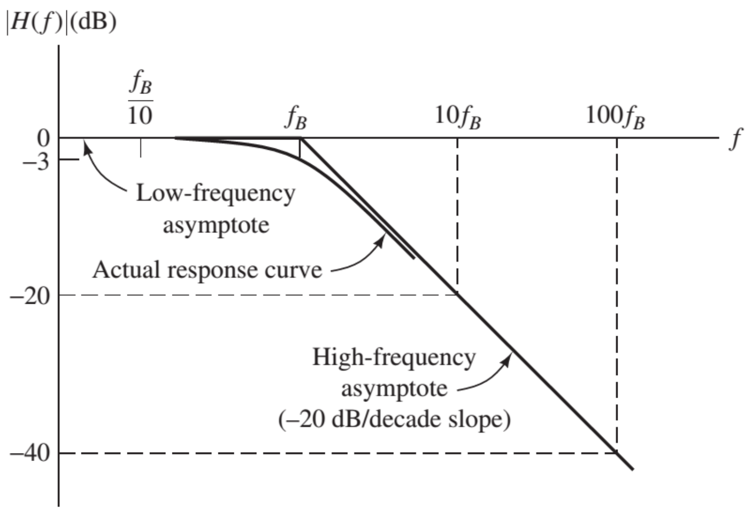
Bode Plots

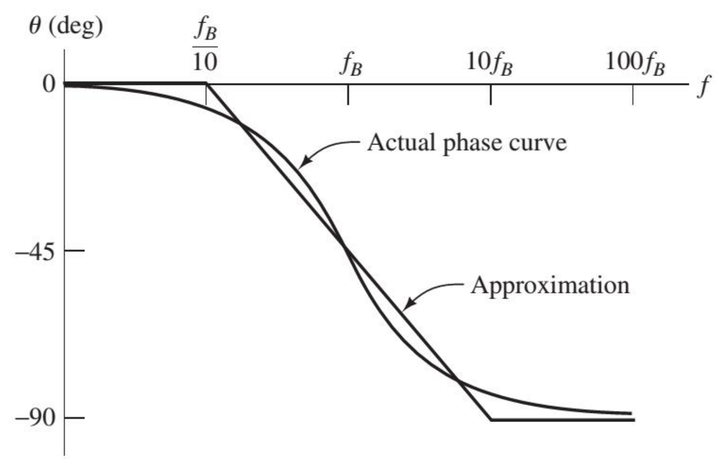
Plotting the frequency response, magnitude or phase, on plots with

- Frequency X in log scale



- Y scale in dB (for magnitude) or degree (for phase)





Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
  - Definition of bel:

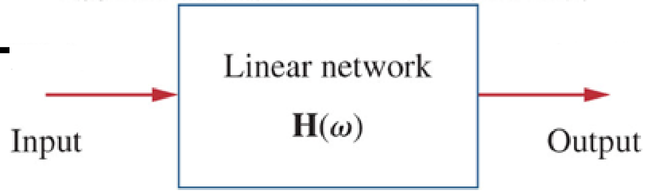
$$\text{Ratio with a unit of B} = \log_{10}(P_1/P_2)$$

  where P_1 and P_2 are power levels.

- One bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

$$\text{Ratio with a unit of dB} = 10\log_{10}(P_1/P_2)$$

  - used to measure electric power, gain or loss of amplifiers, etc.



dB for Voltage or Current

- We can similarly relate the reference voltage or current to the reference power, as

$$P = (V)^2/R \text{ or } P = (I)^2R$$

Hence,

$$\text{Voltage, V in decibels} = 20\log_{10}(V_1/V_2)$$
$$\text{Current, I, in decibels} = 20\log_{10}(I_1/I_2)$$

Question: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery?

Question: **The voltage gain** of an amplifier with input = 0.2 mV and output = 0.5 V is ?

[Source: Berkeley]

Summary

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

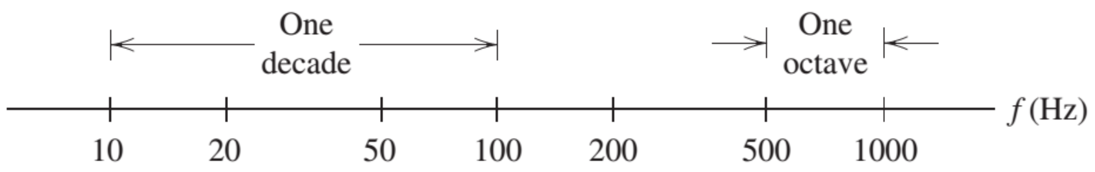
$$G\ [\mathrm{dB}] = 10\log G = 10\log\left(\frac{P}{P_0}\right) \qquad (\mathrm{dB}).$$

$$G\ [\mathrm{dB}] = 10\log\left(\frac{\frac{1}{2}|\mathbf{V}|^2/R}{\frac{1}{2}|\mathbf{V}_0|^2/R}\right) = 20\log\left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|}\right)$$

$\frac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

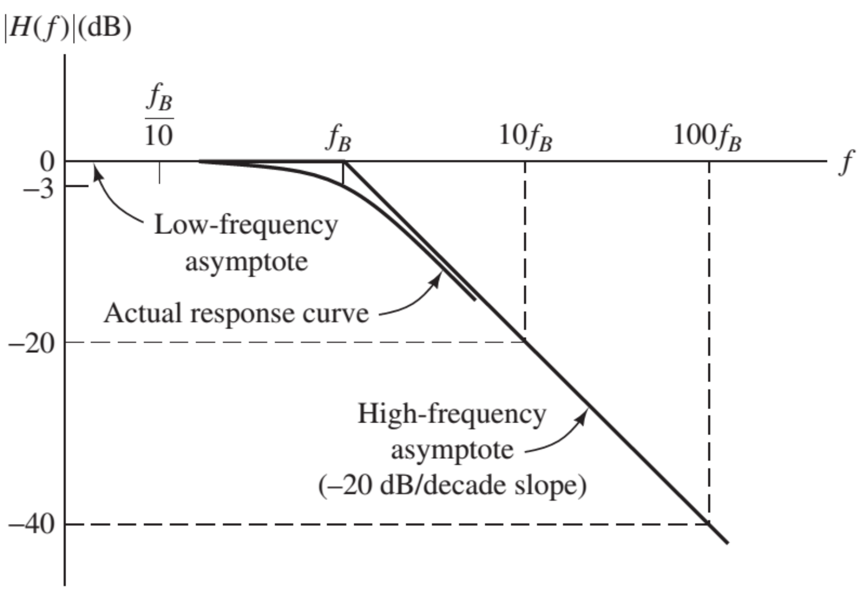
$\left\lvert\frac{\mathbf{V}}{\mathbf{V}_0}\right\rvert$ or $\left\lvert\frac{\mathbf{I}}{\mathbf{I}_0}\right\rvert$	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

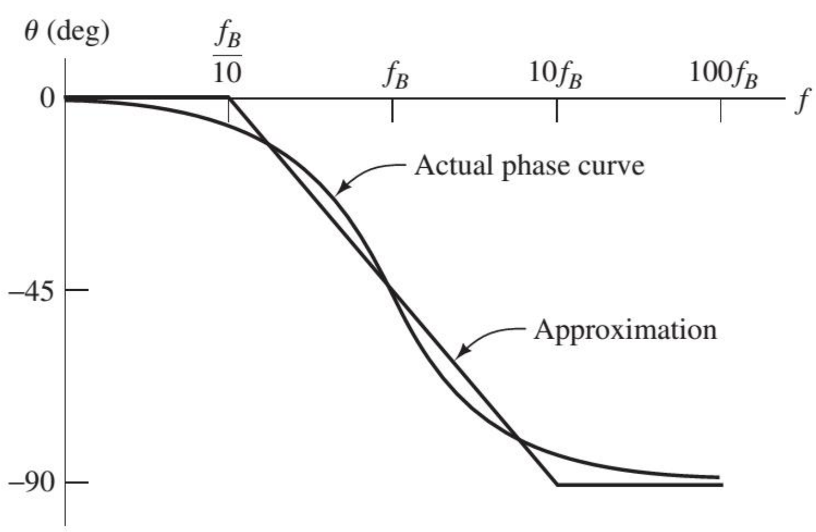
Bode Plots



Plotting the frequency response, magnitude or phase, on plots with

- Frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)





Bode Plots

- Bode plot is particularly useful for displaying transfer function-- a general form is displayed as:

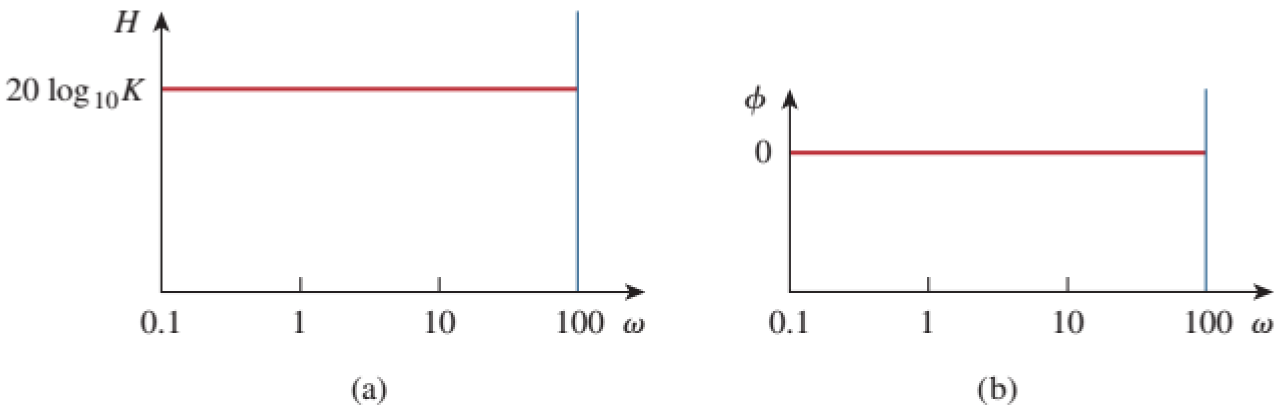
$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool.

Constant term K

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

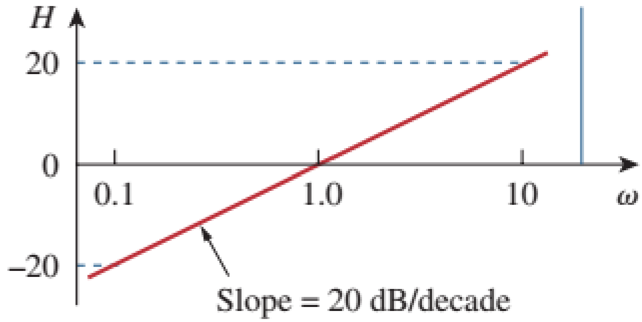
K>0



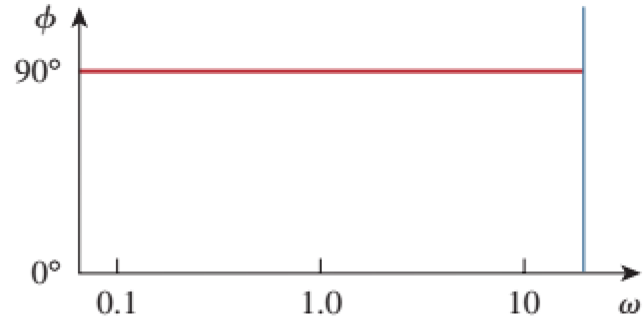
K<0

jω

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$



(a)

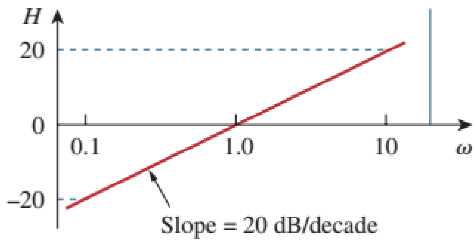


(b)

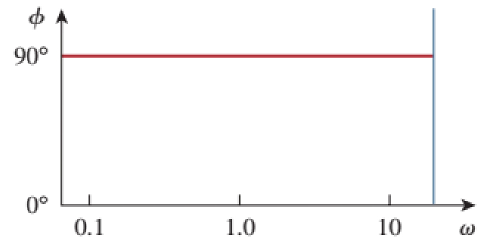
(jω)^−1

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

$j\omega$

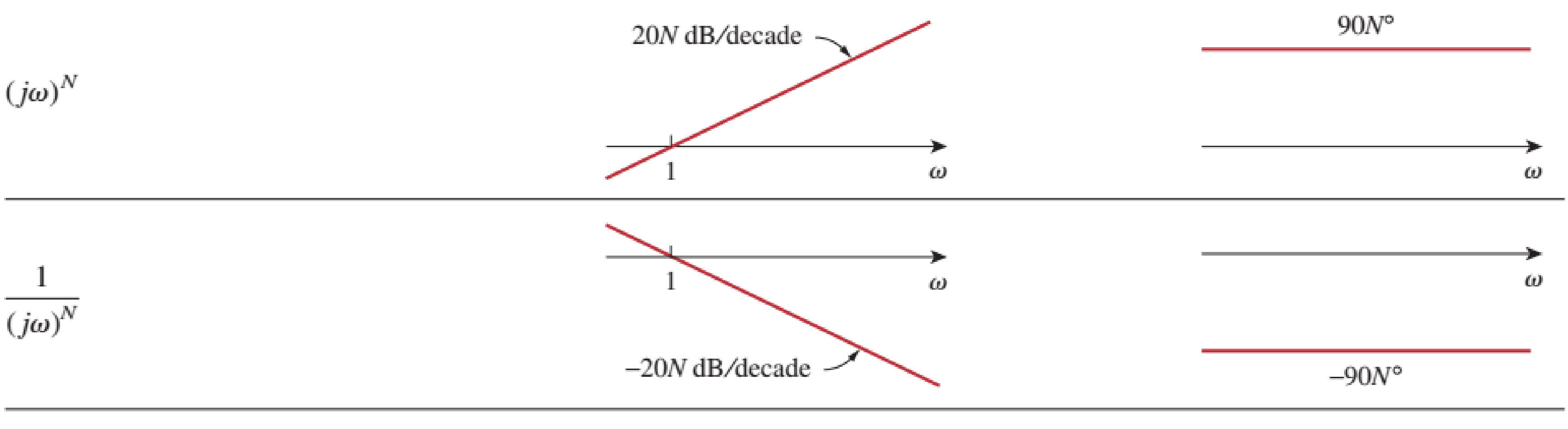


(a)



(b)

- In general：



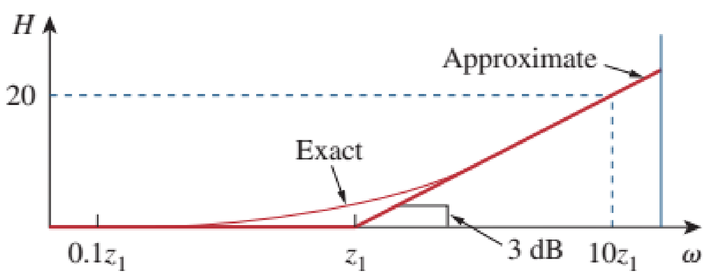
$1+j\omega/z_1$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

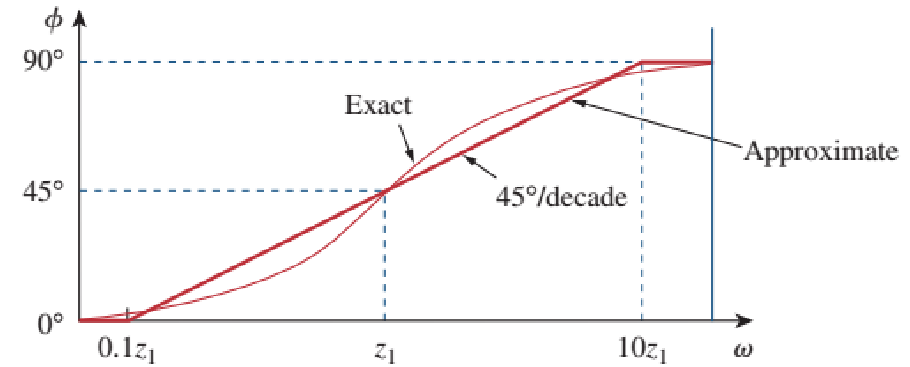
Simple pole/zero: For the simple zero $(1 + j\omega/z_1)$, the magnitude is $20\log_{10}|1 + j\omega/z_1|$ and the phase is $\tan^{-1}\omega/z_1$. We notice that

$$H_{\mathrm{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20\log_{10}1 = 0 \quad \text{as} \quad \omega \to 0$$

$$H_{\mathrm{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20\log_{10}\frac{\omega}{z_1} \quad \text{as} \quad \omega \to \infty$$



(a)



(b)

$1/(1+j\omega/p_1)$

$1/[1+2j\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]$

$$\mathbf{H}(\omega)=\frac{K(j\omega)^{\pm1}(1+j\omega/z_1)[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2]\cdots}{(1+j\omega/p_1)[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]\cdots}$$

Magnitude:

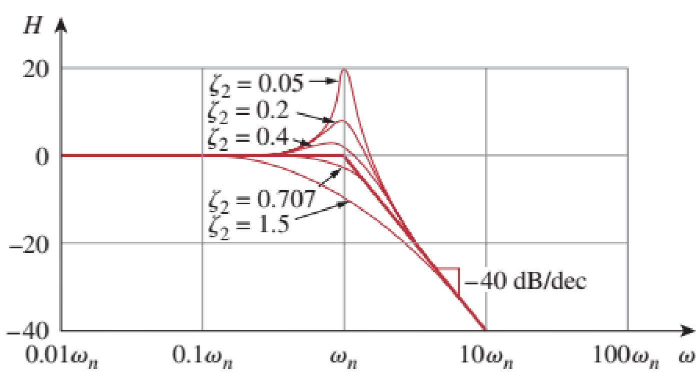
$$H_{\mathrm{dB}}=-20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad\Rightarrow\quad 0$$

and

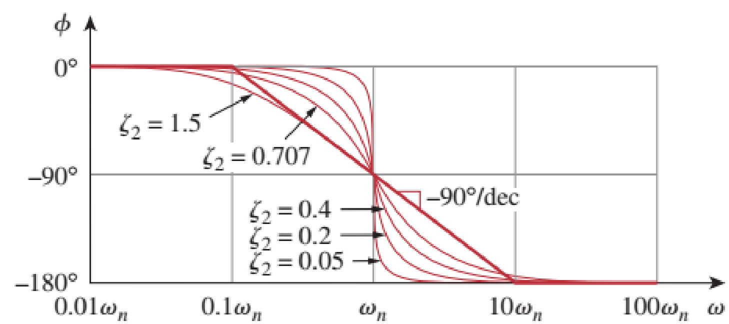
$$H_{\mathrm{dB}}=-20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad\Rightarrow\quad -40\log_{10}\frac{\omega}{\omega_n}$$

as $\omega\rightarrow\infty$

the phase is $-\tan^{-1}(2\zeta_2\omega/\omega_n)/(1-\omega^2/\omega_n^2)$.



(a)



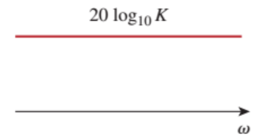
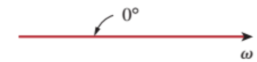
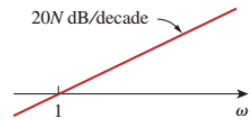
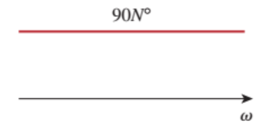
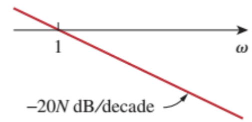
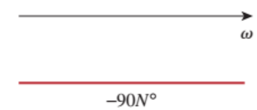
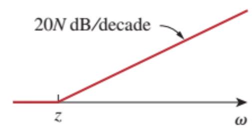
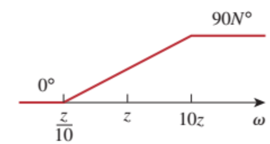
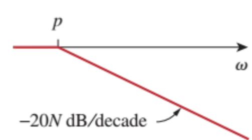
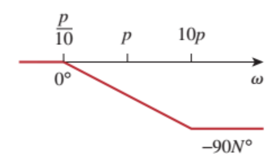
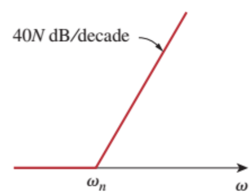
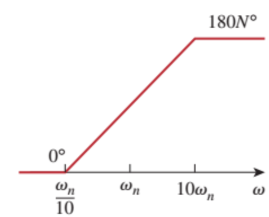
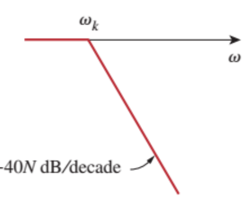
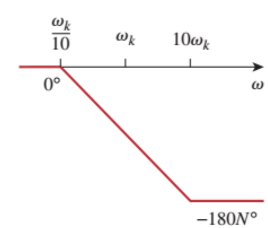
(b)

$1+2j\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

TABLE 14.3

Summary of Bode straight-line magnitude and phase plots.

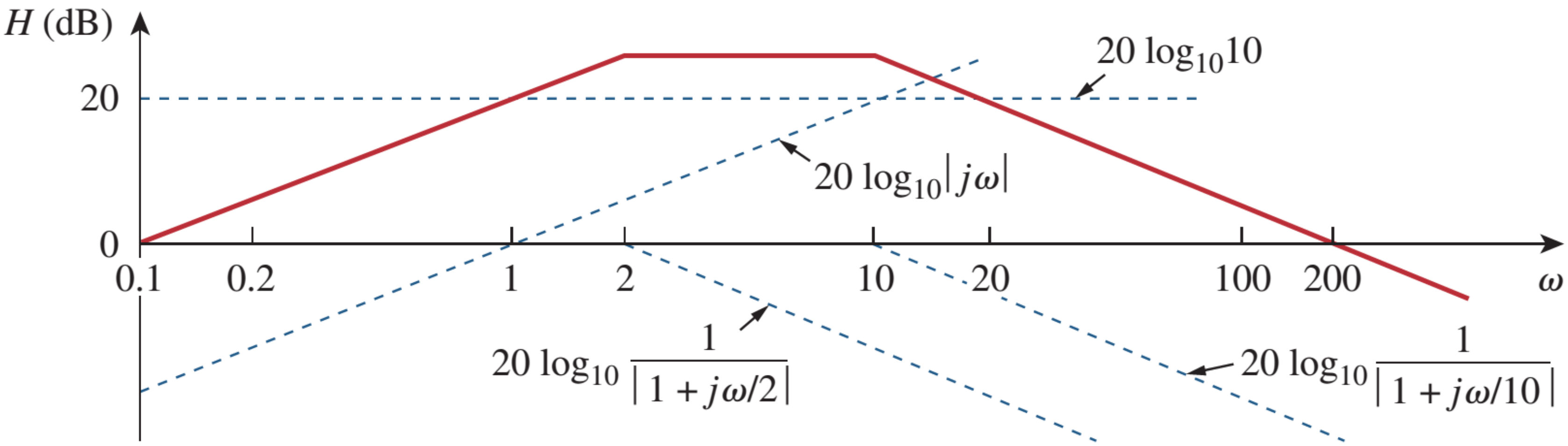
Factor	Magnitude	Phase
K		
$(j\omega)^N$		
$\frac{1}{(j\omega)^N}$		
$\left(1+\frac{j\omega}{z}\right)^N$		
$\frac{1}{(1+j\omega/p)^N}$		
$\left[1+\frac{2j\omega\zeta}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]^N$		
$\frac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$		

Example--Standard Form

$$\mathbf{H}(\omega) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)}$$

$$\mathbf{H}(\omega) = \frac{10\,j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

Example - Magnitude



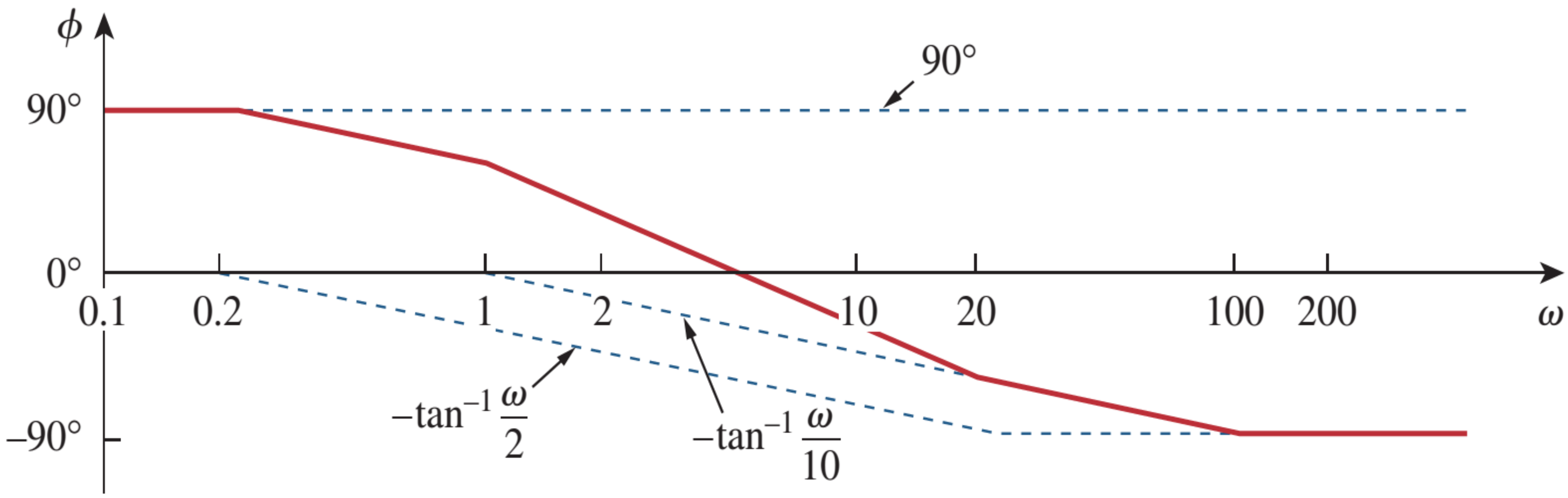
$$H_{\mathrm{dB}} = 20\log_{10}10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1+\frac{j\omega}{2}\right| - 20\log_{10}\left|1+\frac{j\omega}{10}\right|$$

Example - Phase

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \underline{/90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10}$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$



Exercises

- $\mathbf{H}(\omega) = \dfrac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$

- $\mathbf{H}(\omega) = \dfrac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}$

Obtain the transfer function