Time and frequency characterization of signals and systems (ch.6)

- The magnitude-phase representation of Fourier Transform
- The magnitude-phase representation of the frequency response of LTI systems
- Time-domain properties of ideal frequency-selective filters
- Time-domain and frequency-domain aspects of non-ideal filters
- First-order system

The magnitude-phase representation of FT

Magnitude and phase spectrum

- Continuous FT $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \qquad X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$

- Discrete FT $\quad x[n] \longleftrightarrow X(e^{j\omega}) \qquad X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

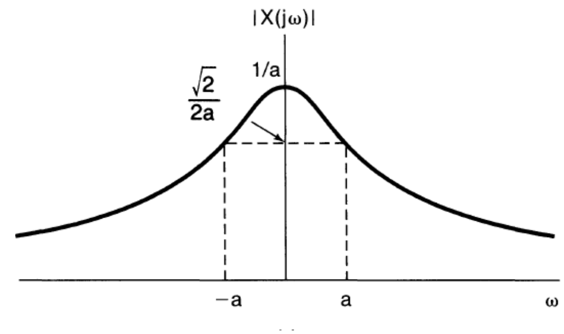
- Amplitude spectrum: $|X(j\omega)|$ and $|X(e^{j\omega})|$

- Phase spectrum (angle): $\angle X(j\omega)$and $\angle X(e^{j\omega})$

The magnitude-phase representation of FT

Magnitude spectrum

Continuous time as an example



IFT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

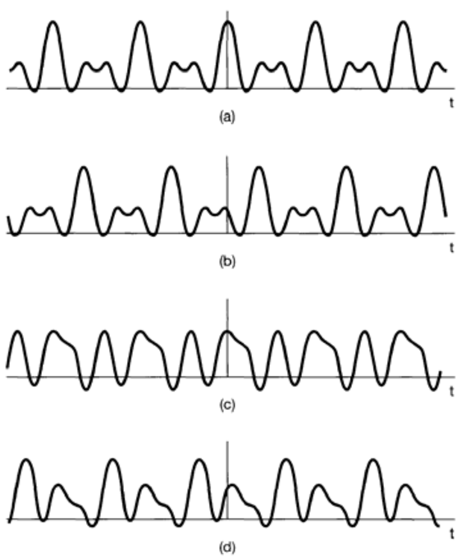
- IFT: decomposition of the signal $x(t)$ into a "sum" of complex exponentials at different frequencies
- $|X(e^{j\omega})|$: describes the basic frequency content of a signal, and the relative magnitude of the each frequency (complex exponential)
- $|X(j\omega)|^2$: energy-density spectrum of $x(t)$
- $|X(j\omega)|^2 d\omega/2\pi$: energy in the signal between ω and $\omega + d\omega$

The magnitude-phase representation of FT

Phase spectrum

- $\angle X(j\omega)$
  - relative phase of the each complex exponential
  - significant effect on the nature of the signal
  - changes in $\angle X(j\omega)$ lead to phase distortion

- Example 1: $x(t) = 1 + \frac{1}{2}cos(2\pi t + \varphi_1) + \frac{1}{2}cos(4\pi t + \varphi_2) + \frac{1}{2}cos(6\pi t + \varphi_3)$



$\varphi_1 = \varphi_2 = \varphi_3 = 0$

$\varphi_1 = 4rad, \varphi_2 = 8rad, \varphi_3 = 12rad$

$\varphi_1 = 6rad, \varphi_2 = -2.7rad, \varphi_3 = 0.93rad$

$\varphi_1 = 1.2rad, \varphi_2 = 4.1rad, \varphi_3 = -7.02rad$

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The magnitude-phase representation of LTI

Gain and phase shift

- For LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

- The frequency response $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$

- $|H(j\omega)|$: Gain of the LTI system; $\angle H(j\omega)$: phase shift of the LTI system

$$Y(j\omega) = H(j\omega)X(j\omega) = |H(j\omega)||X(j\omega)|e^{j(\angle H(j\omega)+\angle X(j\omega))}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \quad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

The magnitude-phase representation of LTI

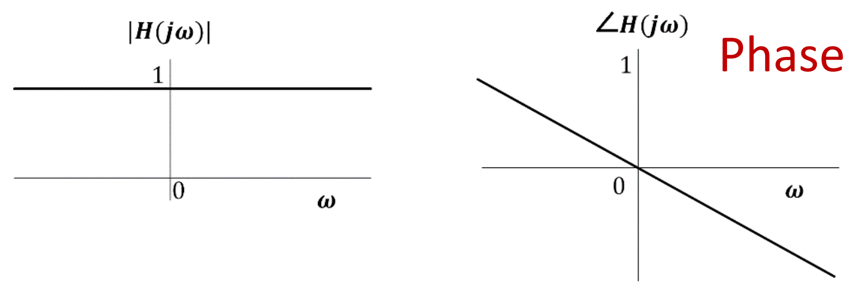
Linear phase system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$H(j\omega) = e^{-j\omega t_0}$$

$$|H(j\omega)| = 1 \qquad \angle H(j\omega) = -\omega t_0$$

All-pass system



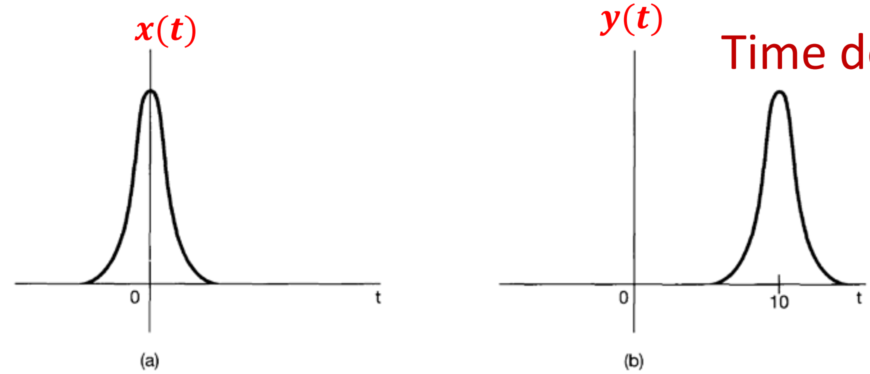
Phase shit

Output of system:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= X(j\omega)e^{-j\omega t_0}$$

$$y(t) = x(t - t_0)$$



Time delay

The magnitude-phase representation of LTI

Non-linear phase system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$H(j\omega) = H_1(j\omega)H_2(j\omega)$$

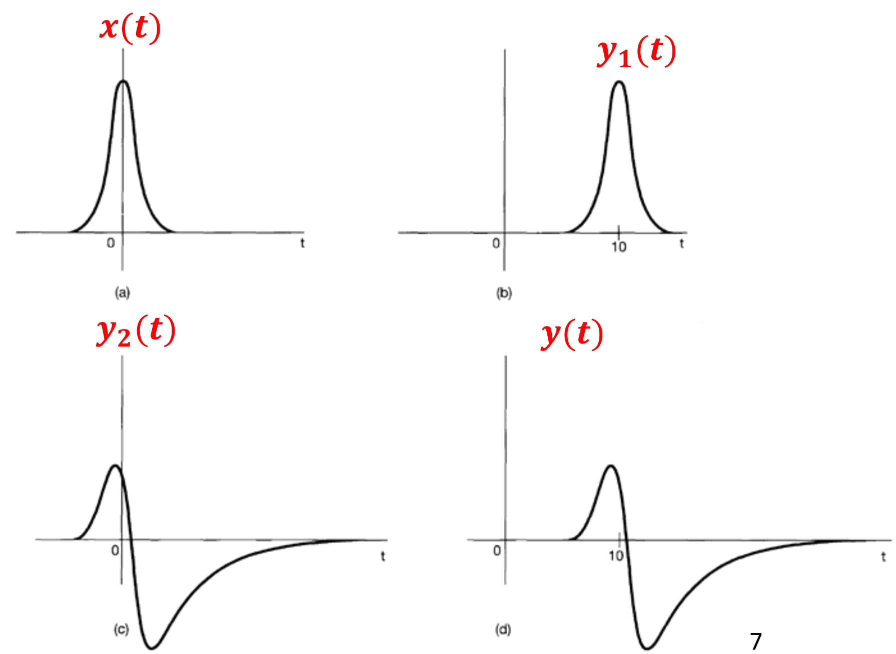
$$H_1(j\omega) = e^{-j\omega t_0}$$

$$H_2(j\omega) = e^{\angle H_2(j\omega)}$$

$\angle H_2(j\omega)$ is a nonlinear function of ω

$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0 + \angle H_2\,(j\omega)$$



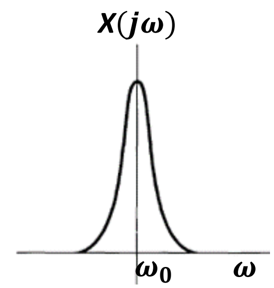
The magnitude-phase representation of LTI

Group delay

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Consider a system with $\angle H(j\omega)$ a nonlinear function of ω

- For a narrow band input $x(t)$, $\angle H\,(j\omega) \simeq -\phi - \alpha\omega$



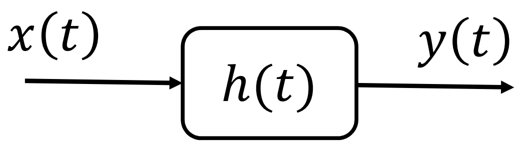
$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\alpha\omega}$$

- The time delay α is referred to as the group delay at $\omega = \omega_0$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H\,(j\omega)\}$$

The magnitude-phase representation of LTI

Group delay: example



- Consider

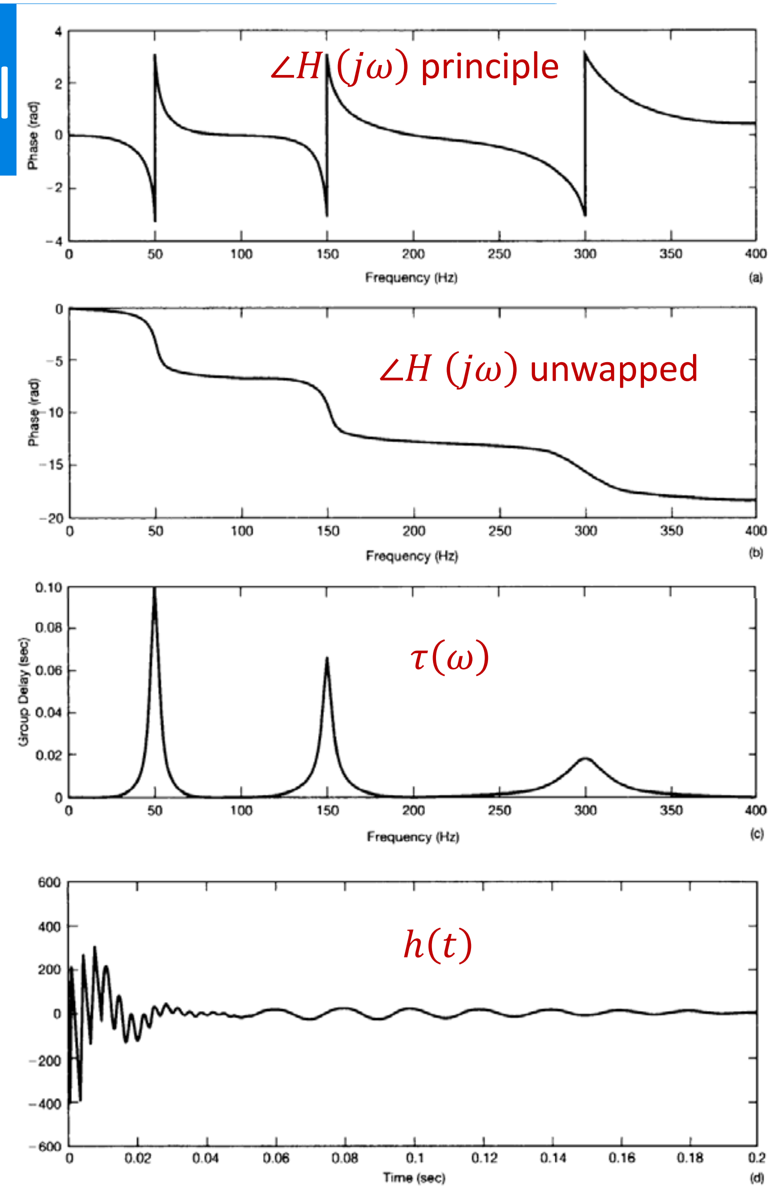
$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega) \qquad H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec and } \zeta_1 = 0.066, \\ \omega_2 = 943 \text{ rad/sec and } \zeta_2 = 0.033, \\ \omega_3 = 1888 \text{ rad/sec and } \zeta_3 = 0.058. \end{cases}$$

$$|H_i(j\omega)| = 1 \Rightarrow |H(j\omega)| = 1$$

$$\angle H_i(j\omega) = -2\arctan\left[\frac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

$$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i(j\omega) \qquad \tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\}$$



The magnitude-phase representation of LTI

Log-Magnitude and Bode Plots

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Time domain: $y(t) = x(t) * h(t)$ Convolution

Frequency domain: $Y(j\omega) = H(j\omega)X(j\omega)$ Multiplication

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Logarithmic amplitude: $\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$ Summation

Logarithmic amplitude scale: $20\log_{10}$, referred to as *decibels* (dB).

Bode plots: Plots of $20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ versus $\log_{10}(\omega)$

The magnitude-phase representation of LTI

Log-Magnitude and Bode Plots

Plot of $20\log_{10}|H(j\omega)|$ vs $\log_{10}(\omega)$

Plot of $\angle H(j\omega)$ vs. $\log_{10}(\omega)$

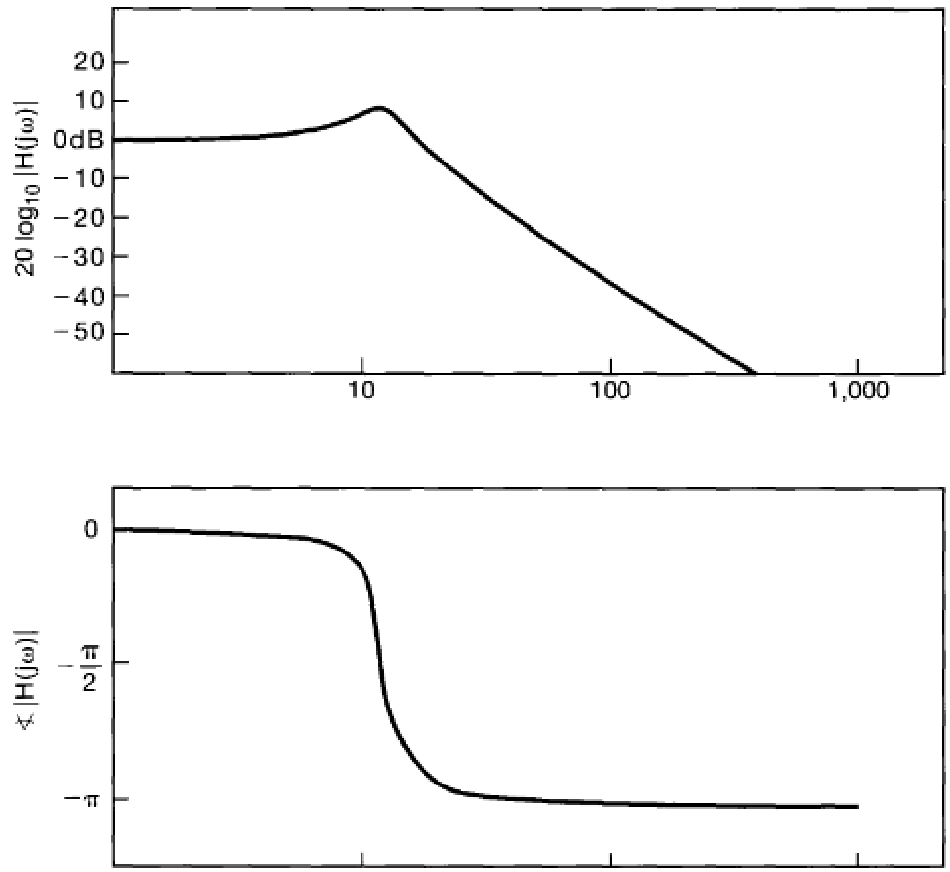


Figure 6.8 A typical Bode plot. (Note that ω is plotted using a logarithmic scale.)

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Time-domain properties of ideal frequency-selective filters

Frequency-selective filters

Low-pass filter

High-pass filter

Band-pass filter

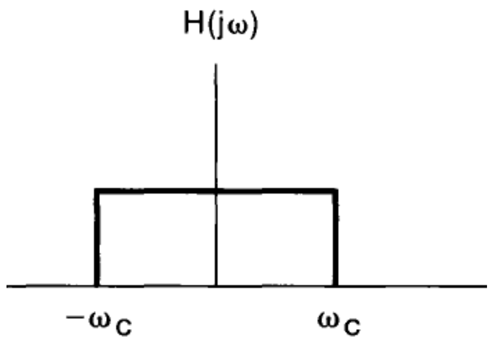
We focus on low-pass filter, similar concepts and results hold for high-pass and band pass filter.

Time-domain properties of ideal frequency-selective filters

Ideal low-pass filters: zero phase

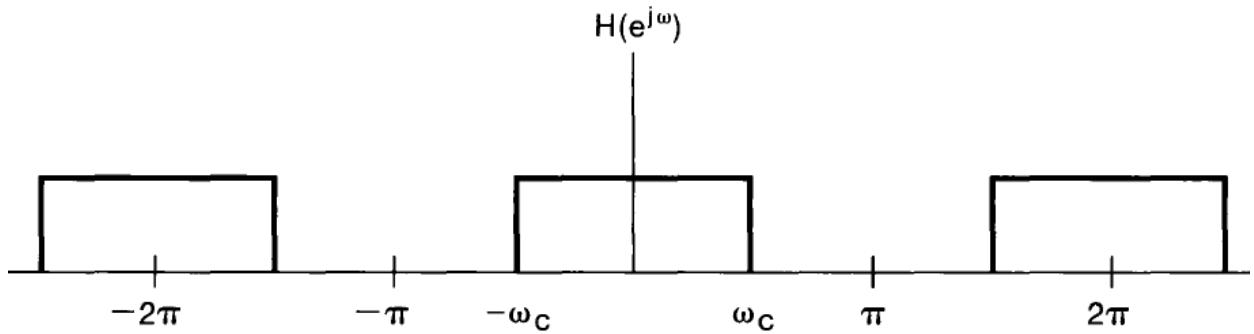
CT

$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$



DT

$$H(e^{j\omega})=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \omega_c<|\omega|<\pi\end{cases}$$



Time-domain properties of ideal frequency-selective filters

Ideal low-pass filters: zero phase

- Impulse response:

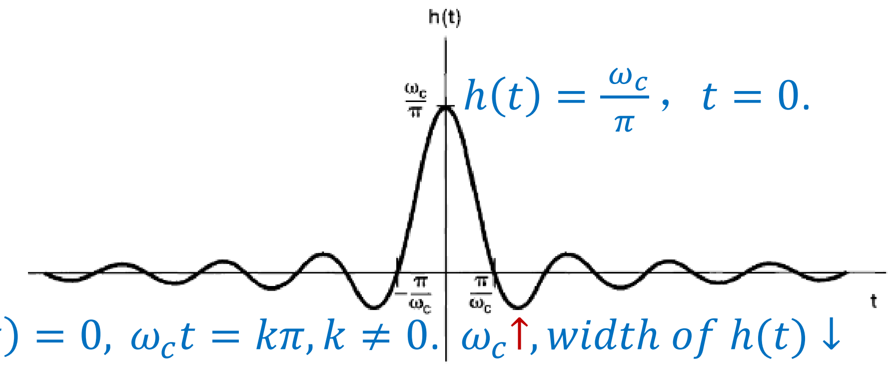
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{H}(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\cdot 2j\sin(\omega_c t) = \frac{\sin\omega_c t}{\pi t}$$

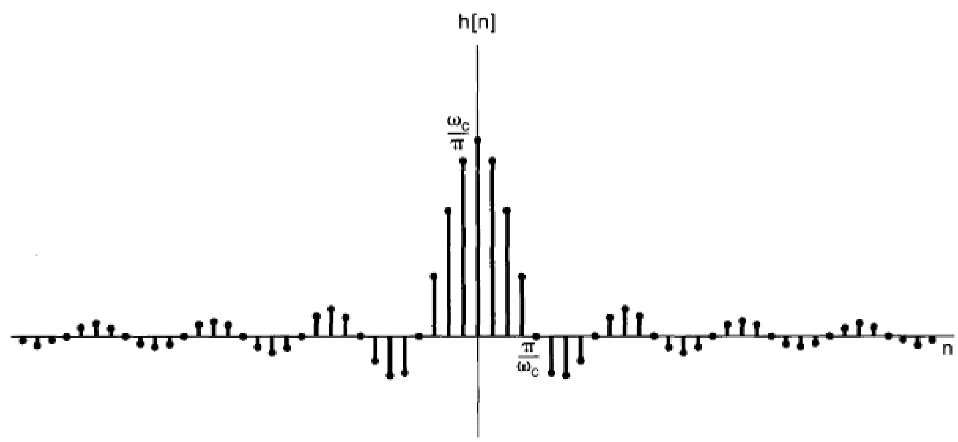
$$h(n) = \frac{\sin\omega_c n}{\pi n}$$



$h(t) = \frac{\omega_c}{\pi}, \quad t = 0.$

$h(t) = 0, \ \omega_c t = k\pi, k \neq 0.$ $\omega_c\uparrow, width\ of\ h(t)\downarrow$

(a)



(b)

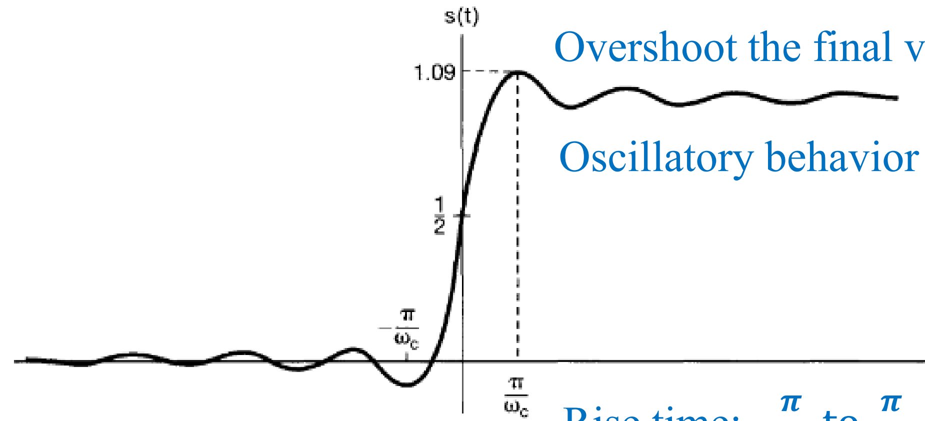
Time-domain properties of ideal frequency-selective filters

Ideal low-pass filters: zero phase

- Step response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(n) = \sum_{m=-\infty}^{n} h(m)$$

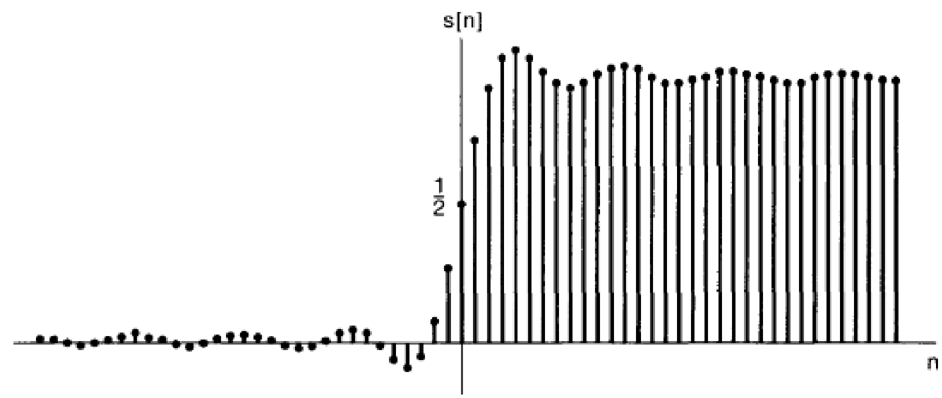


(a)

Overshoot the final value

Oscillatory behavior

Rise time: $-\frac{\pi}{\omega_c}$ to $\frac{\pi}{\omega_c}$

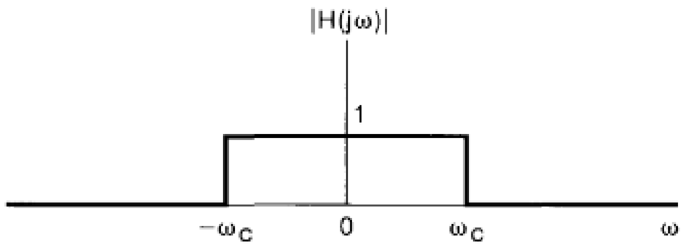


(b)

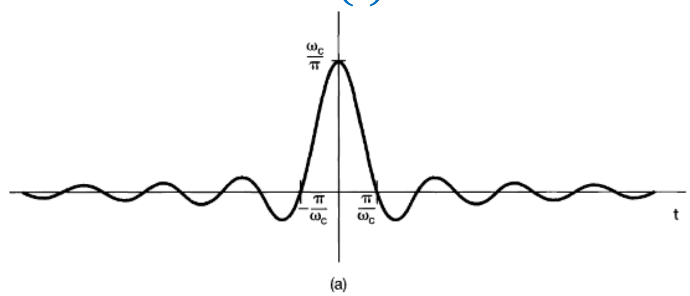
Time-domain properties of ideal frequency-selective filters

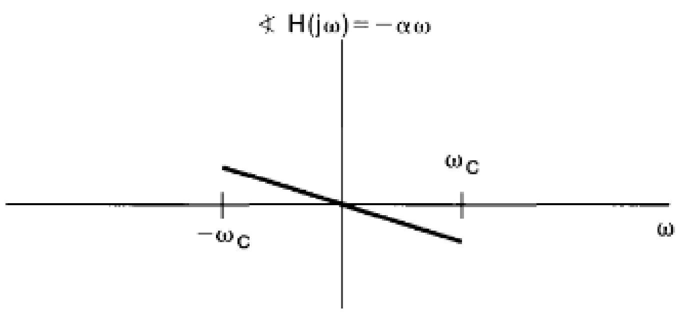
Ideal low-pass filters: linear phase

- Impulse response:

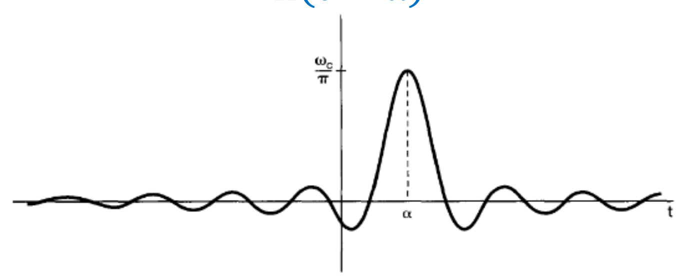


h(t)





$h(t-\alpha)$



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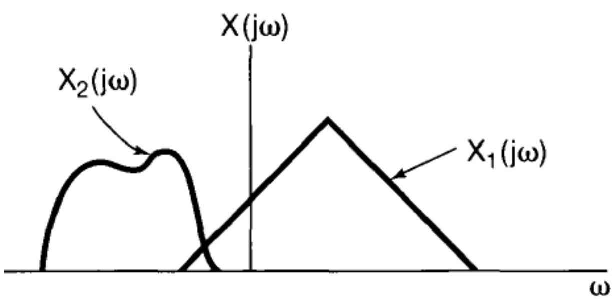
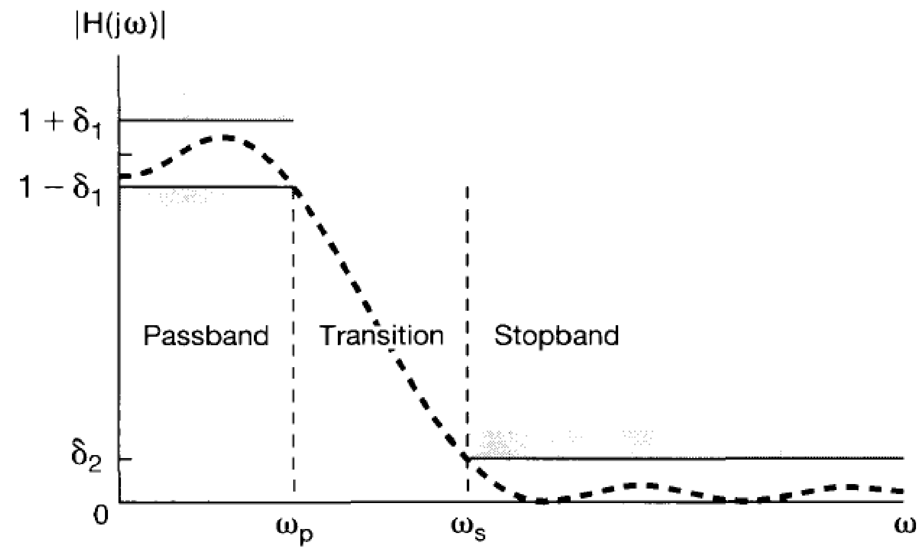
Non-ideal filters

Frequency domain (low-pass)

- Idea Low-pass filter is not implementable
- Gradual transition band is sometimes preferable

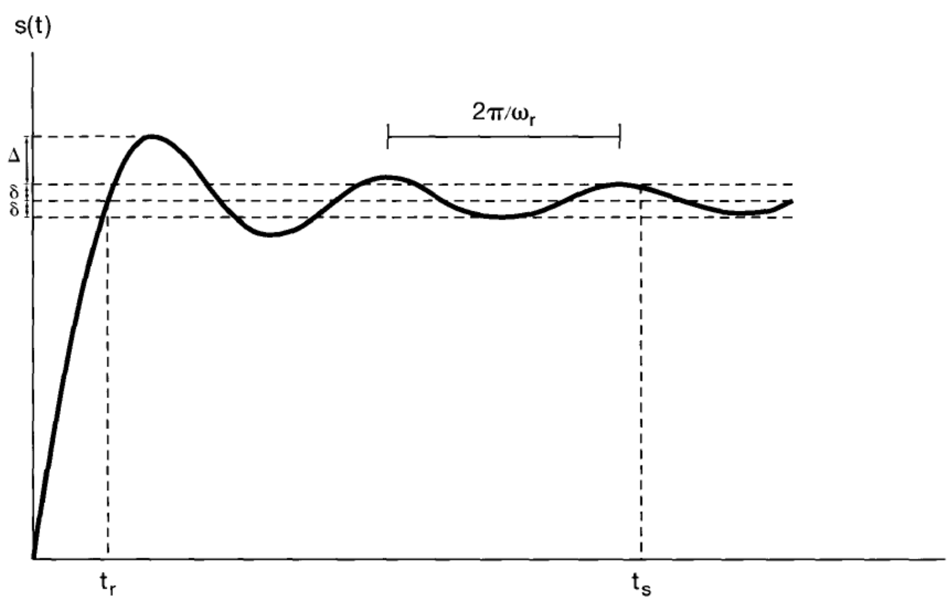




- Pass band: 0 - ω_p, stop band: ω>ω_s, transition: ω_s - ω_p
- Pass-band ripple: δ_1, stop-band ripple: δ_2
- Linear (nearly) linear phase over the passband is desirable.

Non-ideal filters

Time domain (low-pass)

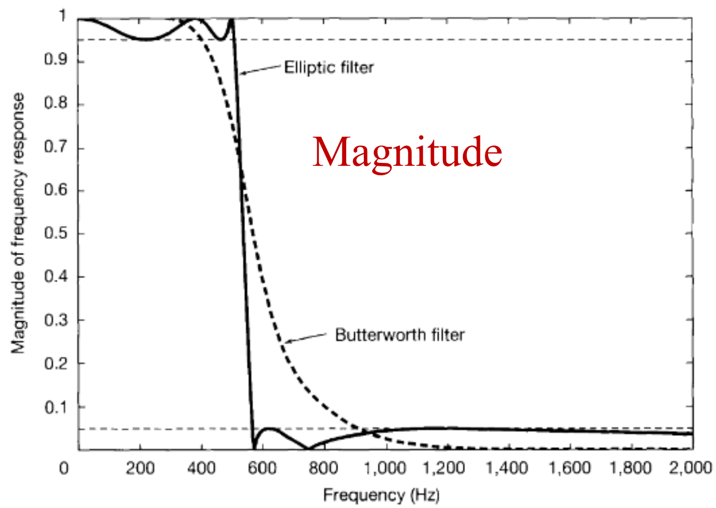


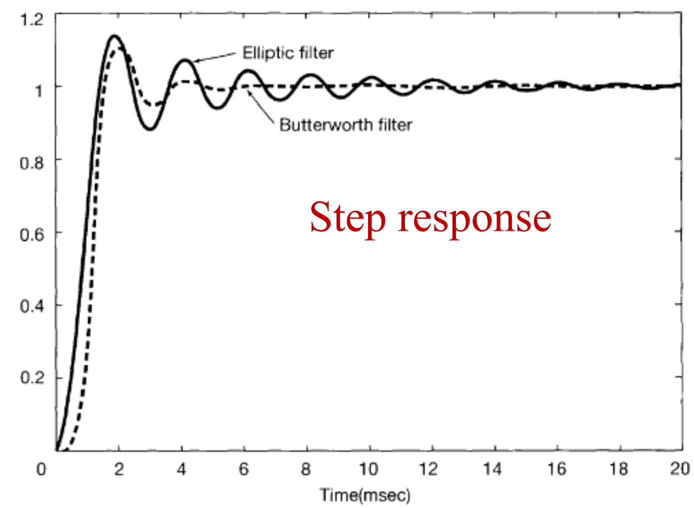
Step response of a CT low-pass filter

- Rise time: t_r
- Overshoot: Δ
- Ringing frequency: ω_r
- Settling time: t_s

Non-ideal filters

An example





- Fifth-order Butterworth filter and a fifth-order elliptic filter
- Same cutoff frequency
- Same passband and stopband ripple

Trade-off between time-domain (t_s) and frequency-domain (ω_s - ω_p).

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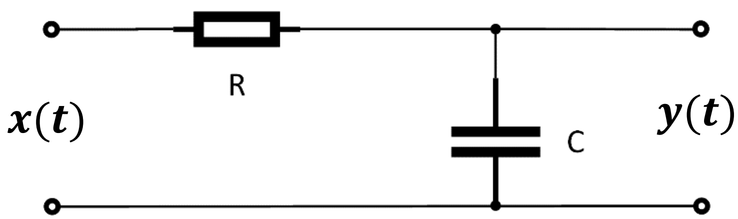
First-order systems

First-order system (Continuous time)



- Differential equation:

$$C\frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}$$

$$\tau\frac{dy(t)}{dt} + y(t) = x(t), \tau = RC$$

- Frequency response:

$$\tau j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega\tau + 1}$$

First-order systems

First-order system (Continuous time)

❑ Impulse response $$H(j\omega) = \frac{1}{j\omega\tau + 1} = \frac{1/\tau}{j\omega + 1/\tau}$$

$$e^{-at}u(t), a > 0 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega + a}$$

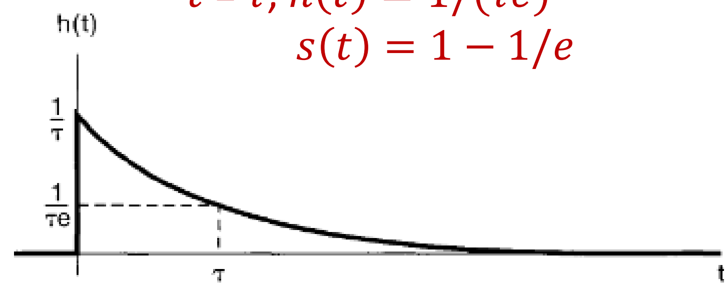
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

❑ Step response

$$s(t) = \int_{-\infty}^{t} h(t)\, dt = \frac{1}{\tau}\int_{0}^{t} e^{-t/\tau}\, dt = \begin{cases} 0, t < 0 \\ 1 - e^{-t/\tau}), t \geq 0 \end{cases}$$

$$s(t) = (1 - e^{-t/\tau})\, u(t)$$

- τ: time constant
- $t = \tau$, $h(t) = 1/(\tau e)$
  $s(t) = 1 - 1/e$



(a)

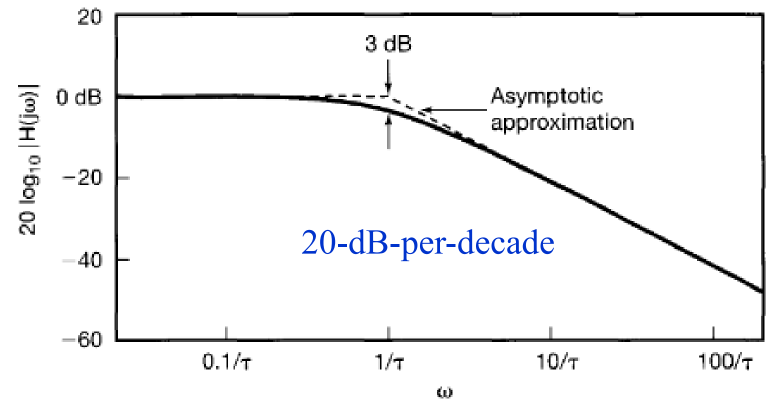
- $\tau \downarrow$, $h(t)$ decays more sharply
  $s(t)$ rises more sharply

(b)

First-order systems

Bold Plots (Continuous time)

$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

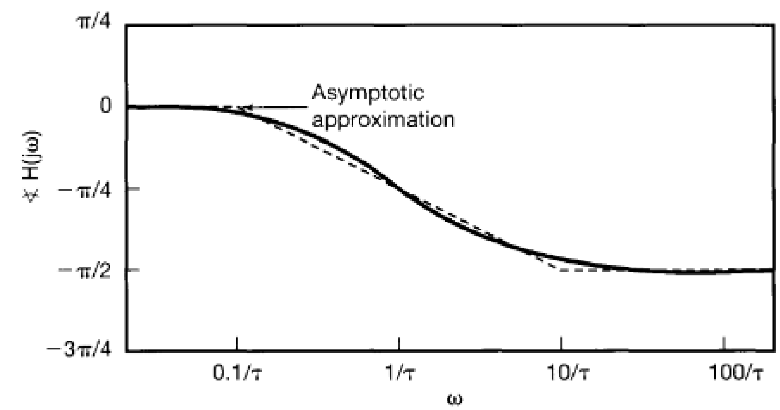


- **$20log_{10}|H(j\omega)|$** $= -10\log_{10}[(\omega\tau)^2 + 1]$

$$\simeq \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}$$

$$\omega = 1/\tau, 20\log_{10}|H(j\omega)| = -10\log_{10}(2) \simeq -3dB$$

$\omega = 1/\tau$： break frequency



- **$\angle H(j\omega)$** $= -\tan^{-1}(\omega\tau)$

$$\simeq \begin{cases} 0, & \omega \le 0.1/\tau \\ -\frac{\pi}{4}[log_{10}(\omega\tau) + 1], & 0.1/\tau \le \omega \le 10/\tau \\ -\pi/2, & \omega \ge 10/\tau \end{cases}$$

$$\omega = 1/\tau, \angle H(j\omega) = -\pi/4$$

$\tau \downarrow$, $h(t)$ and $s(t)$ more sharply, break frequency $\uparrow$.