1 VCG Auction with Redistribution

Consider a seller who wants to sell 2 homogeneous items, and each buyer requires at most one item. There are $n = 5$ buyers $\{a_1, a_2, a_3, a_4, a_5\}$, and the i-th buyer a_i bids v'_i (w.l.o.g., $v'_1 \geq \cdots \geq v'_5$). Suppose their bids are $\{10, 9, 5, 5, 2\}$ respectively. The mechanism firstly determines the allocation and payments using VCG, and then redistributes the payments received back to buyers as follows. Let r_i be the money that is returned to buyer a_i.

$$r_i = \begin{cases} 2v'_4/n & \text{for } i = a_1, a_2, \\ 2v'_3/n & \text{for } i = a_3, \ldots, a_n. \end{cases}$$

1.1 (1pt)

Compute all agents' payments after redistribution. Will this mechanism be budget-balanced (the amount of the money that cannot be redistributed among the agents is 0) in all cases?

1.2 (1pt)

Is this mechanism incentive-compatible? If so, give the proof. If not, state which agent can get higher utility by misreporting.

2 Facility Location

Consider the following facility location problem on a line $[0, l]$. We want to build a facility at the location $x \in [0, l]$. There are n agents and each agent i has a position $p_i \in [0, l]$ (w.l.o.g., assuming $p_1 \leq p_2 \leq \cdots \leq p_n$), which is the place she prefers to build the facility, and her cost if the location x is chosen is the distance to the facility, i.e., $c_i(x) = |x - p_i|$.

2.1 (1pt)

Suppose the objective is to minimize the maximum cost $max_i\{c_i(x)\}$. Give a truthful mechanism which guarantees a 2-approximation to the objective (the maximum cost of your proposed mechanism is at most twice than that of the optimal location).

2.2 (2pt)

Prove that the mechanism you propose in 2.1 is truthful and has a 2-approximation ratio.

3 Ranked Pairs Method

The ranked pairs method is a voting method that returns a total order of candidates. Following steps show the process of ranked pairs method.

1. Count all pairwise voter preferences (e.g. Consider the pair of candidates a and b. If 3 of 10 voters agree with $a \succ b$, then the count of $a \succ b$ is 3 and the count of $b \succ a$ is 7[1]).
2. Sort (rank) all pairwise preferences by the count calculated in 1 with descending order[2].
3. Initialize a directed graph where each node represents a candidate, and no edges.
4. Enumerate the sorted pairwise preferences in 2. For each pairwise preference $a \succ b$, add an edge (a, b) to the graph if it will not create a cycle.
5. Finally, the topological sort of the graph will give the output as the total order.

[1] We always assume that all voters give a total order of candidates.

[2] If there are more than one possible orders, we always assume a random tie-breaking

3.1 (1pt)

Given the following voters and their preferences, calculate the output of the ranked pairs method. (Show your process of calculation)

$$\begin{aligned}
\texttt{V1} &: a \succ_1 b \succ_1 c \\
\texttt{V2} &: b \succ_2 a \succ_2 c \\
\texttt{V3} &: a \succ_3 c \succ_3 b \\
\texttt{V4} &: a \succ_4 b \succ_4 c \\
\texttt{V5} &: a \succ_5 c \succ_5 b
\end{aligned}$$

3.2 (2pt)

Does the ranked pairs method satisfy the properties of unanimity and independence of irrelevant alternatives? For each property, if it is satisfied, give a proof; otherwise, give a counterexample.