Lecture 7
- Phasor

A beginning of AC circuits

AC usually refers to Sinusoidal signal

Outline

- Sinusoidal signals

- Phasor

Sinusoidal Signal (Current or Voltage)

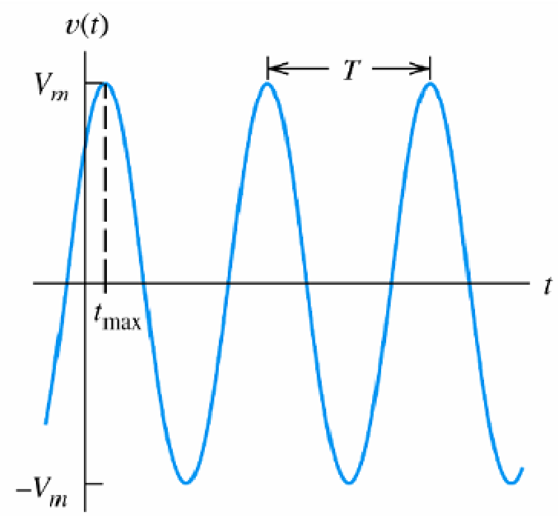
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

$(\omega t + \theta)$ is the **phase angle**

T is the **period**

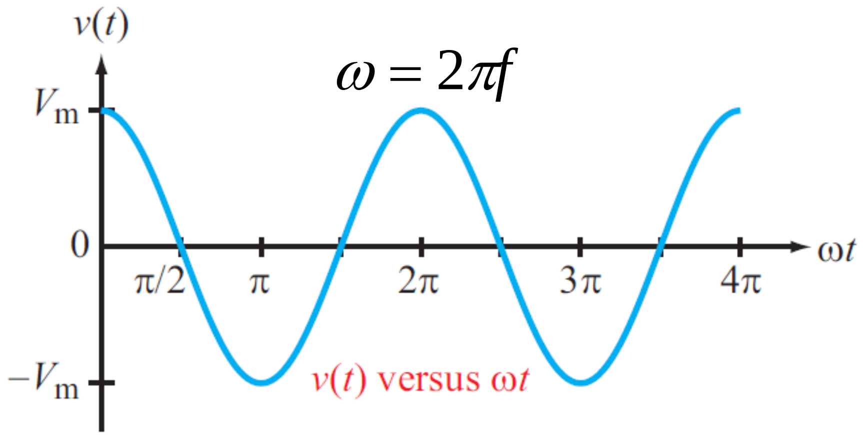


$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

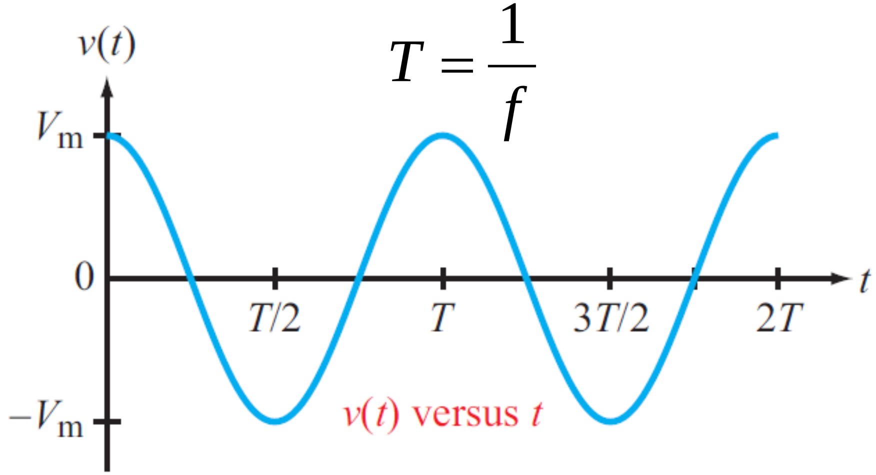
Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\omega = 2\pi f$$



$$T = \frac{1}{f}$$



Useful relations

$$\sin x = \pm \cos(x \mp 90^\circ)$$
$$\cos x = \pm \sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

Why Sinusoids?

- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface

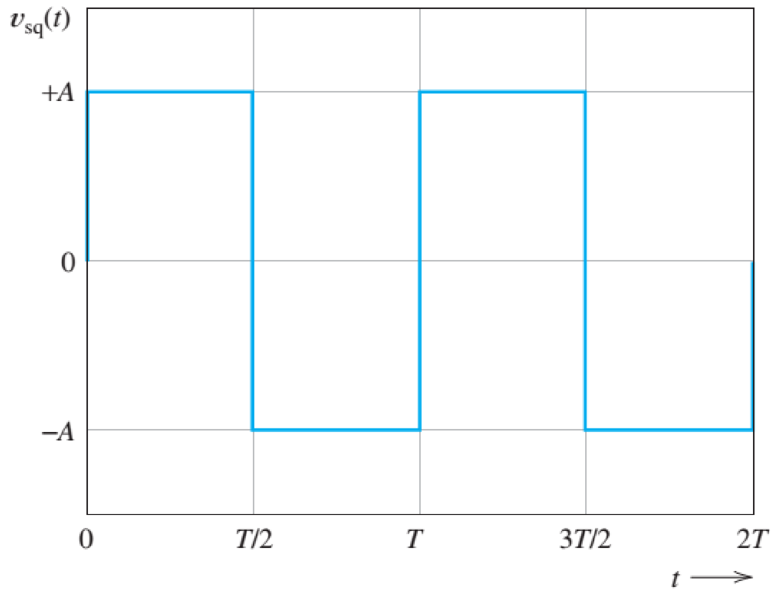
- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.

- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.

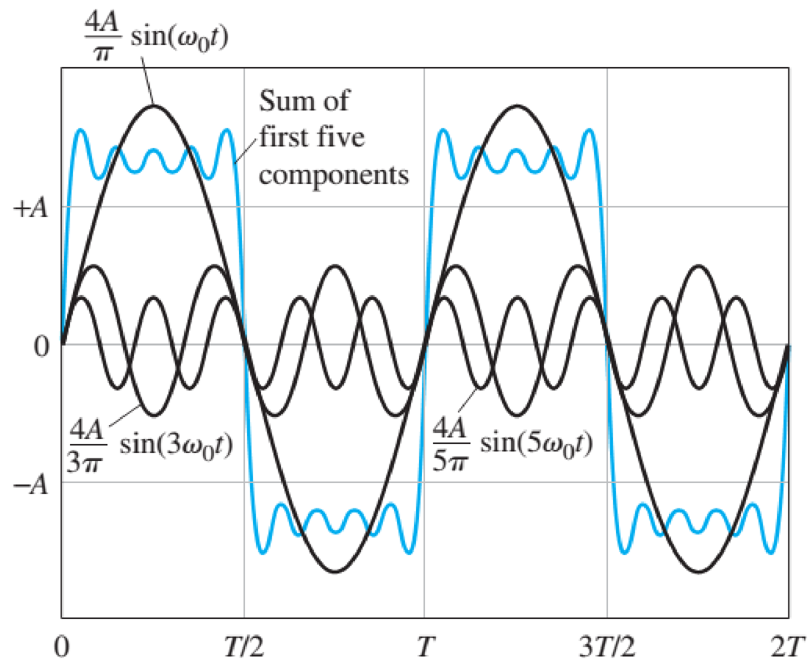
- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.

[Source: Berkeley]

Representing a Square Wave as a Sum of Sinusoids



(a) Periodic square wave



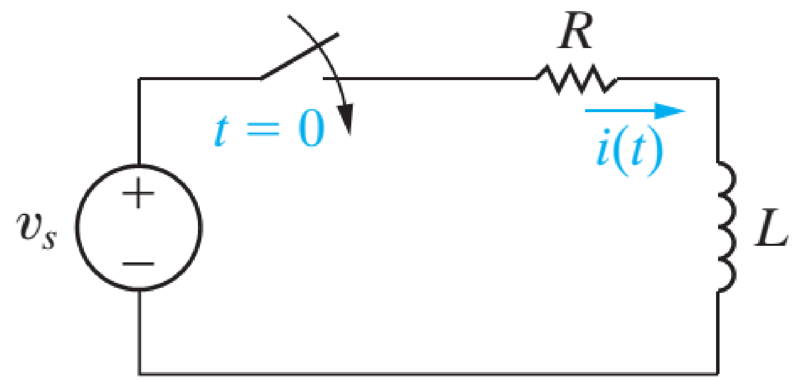
(b) Several of the sinusoidal components and the sum of the first five components

$$v_{sq}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \cdots$$

$$\omega_0 = 2\pi/T$$

The Sinusoidal Response

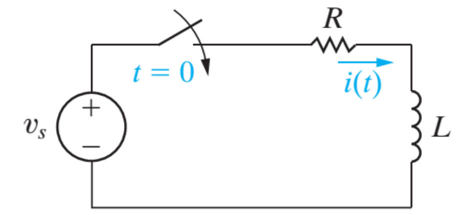
$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.
Find $i(t)$, $t \geq 0$.



$$\boxed{L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)}$$

Ordinary differential equation

Sinusoidal Steady-State Response



$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

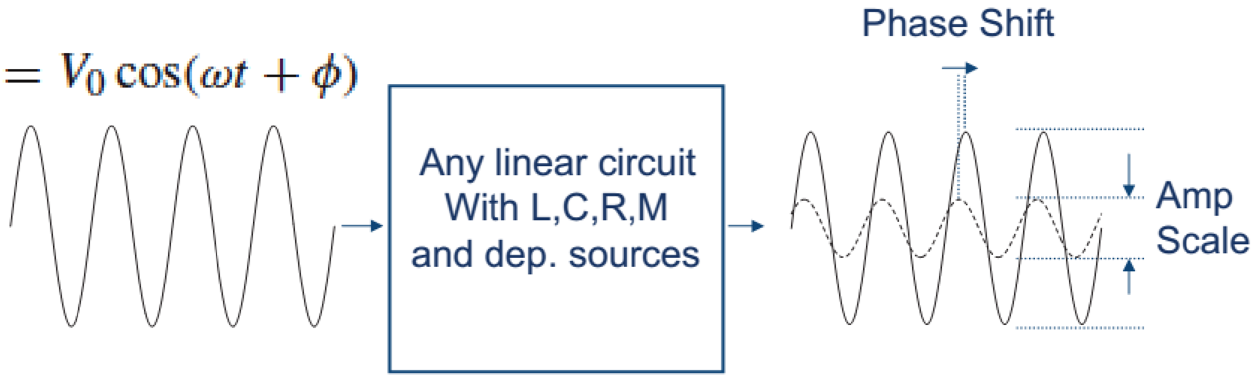
Transient response

Steady-state response

- Steady-state solution is sinusoidal
- Response frequency = source frequency
- Magnitude & phase of response differs from that of source

The Magic of Sinusoids

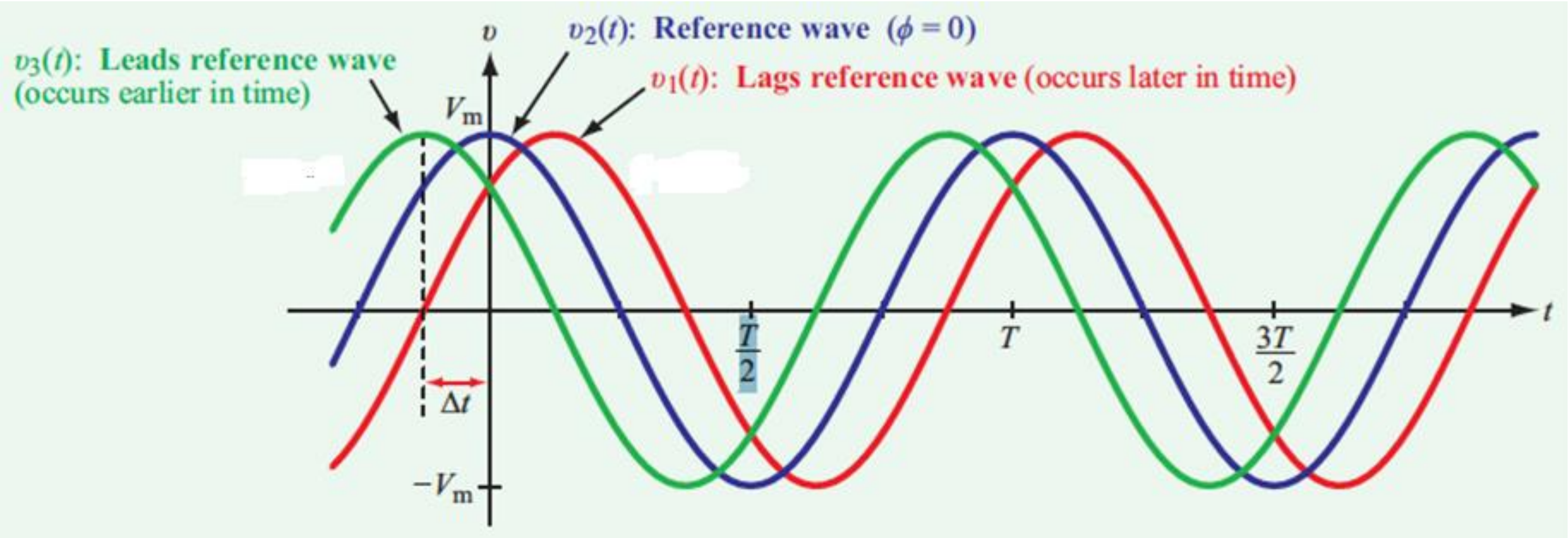
$$v_s(t) = V_0 \cos(\omega t + \phi)$$



- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it's output is a sinusoid at the *same* frequency.
  - Only the magnitude and phase (<u>initial phase angle</u>) of the output differ from the input.
- In order to find a steady-state voltage or current, all we need to know is its <u>magnitude</u> and its <u>initial phase angle</u> relative to the source.

Phase Lead/Lag

$$V_m \cos \frac{2\pi t}{T} \qquad V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right) \qquad V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right)$$



[Source: Berkeley]

Outline

- Sinusoidal signals

- Phasor

Phasor

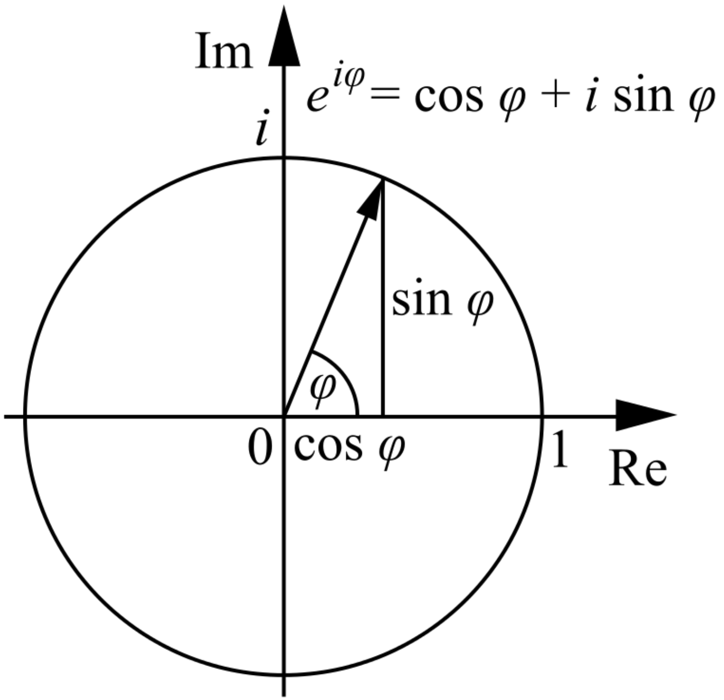
- The idea of phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the real component of a vector in the complex plane.

$$\cos\phi = \mathrm{Re}\left(e^{j\phi}\right)$$

$$\sin\phi = \mathrm{Im}\left(e^{j\phi}\right)$$

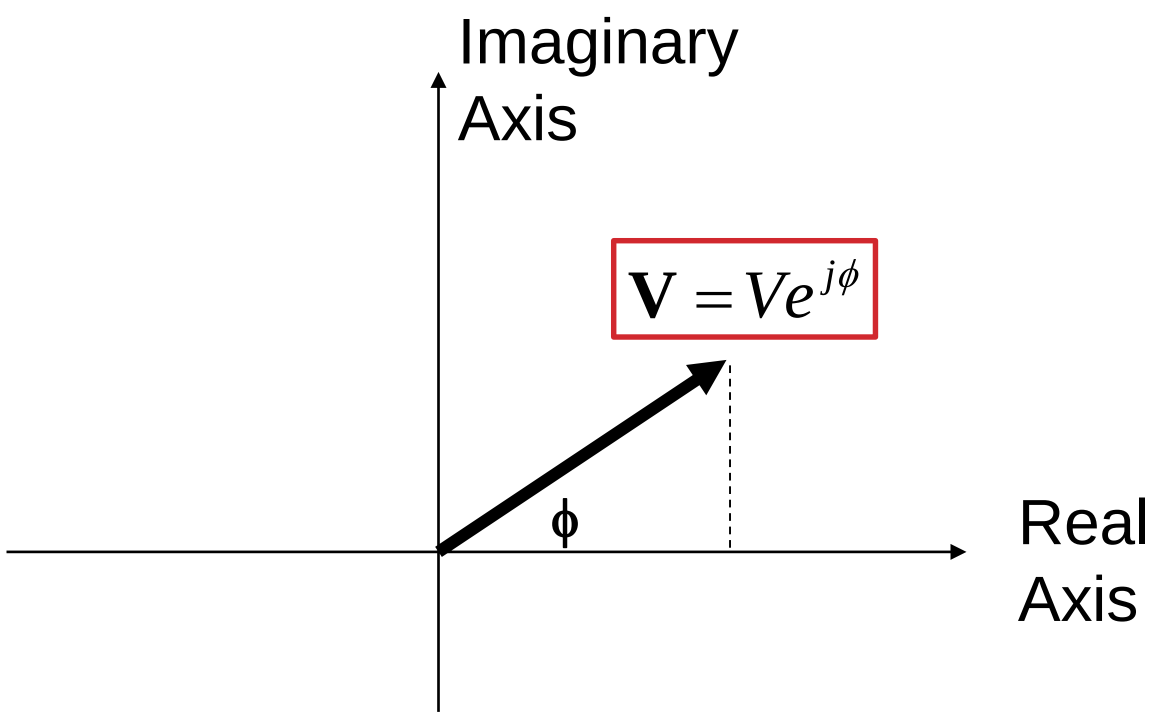


Similarly, $v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\{Ve^{j\phi}e^{j\omega t}\}$

Phasor:

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{j\omega t}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \quad \boxed{\mathbf{V} = Ve^{j\phi}}$$

Complex representation of the magnitude and phase of a sinusoid



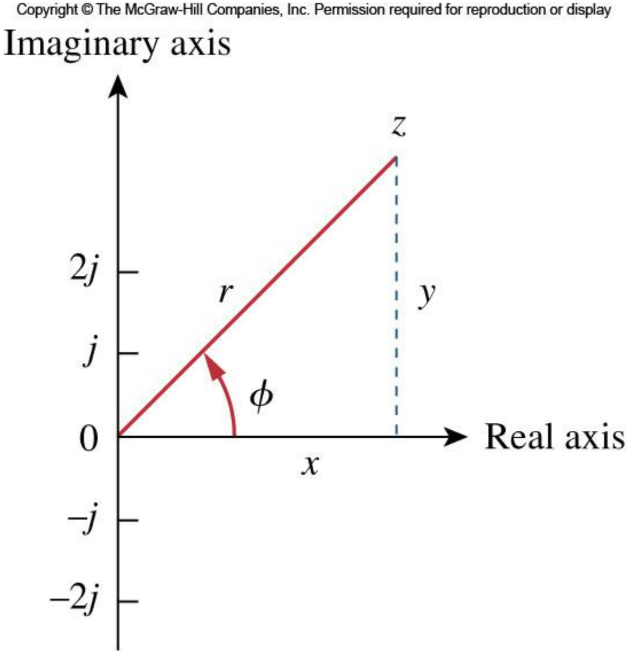
Complex Numbers

- A powerful method for representing sinusoids is the phasor.
- A complex number z can be represented in *rectangular form* as:

$$z = x + jy \qquad \begin{aligned} \mathrm{Re}(z) &= x \\ \mathrm{Im}(z) &= y \end{aligned}$$

- It can also be written in *polar* or *exponential* form as:

$$z = r\angle\phi = re^{j\phi}$$

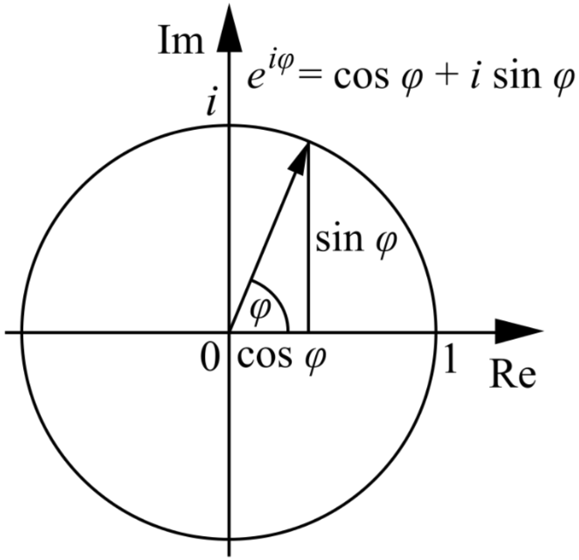


Euler's Formula

Euler's Identities

$$e^{j\pi} + 1 = 0$$

$$\boxed{e^{j\varphi} = \cos\varphi + j\sin\varphi}$$



$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2}$$

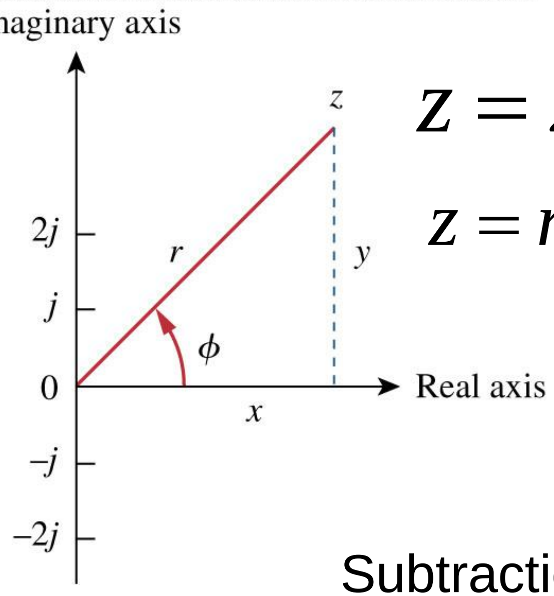
$$\left|e^{j\varphi}\right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

Exponential Form of a complex number Z

$$\boldsymbol{Z} = |\boldsymbol{Z}|e^{j\varphi} = ze^{j\varphi} = z\angle\varphi$$

Source: https://en.wikipedia.org/wiki/Euler%27s_identity

Arithmetic With Complex Numbers



$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

Square Root

$$\sqrt{z} = \sqrt{r}\angle(\phi/2)$$

Complex Conjugate

$$z^* = x - jy = r\angle -\phi = re^{-j\phi}$$

Example

- Evaluate these complex numbers

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Exercise

- Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Relations for Complex Numbers

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

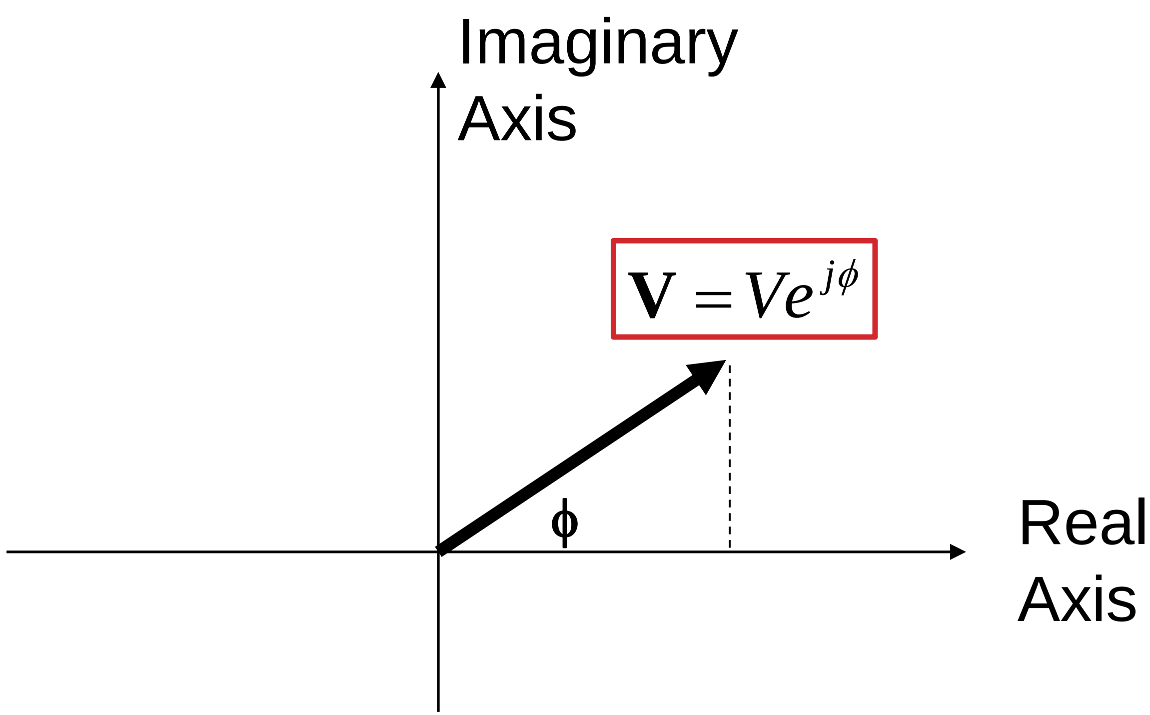
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \mathfrak{Re}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \mathfrak{Im}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm 180^\circ$	
$j = e^{j\pi/2} = 1\angle 90^\circ$	$-j = e^{-j\pi/2} = 1\angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

[Source: Berkeley]

Back to Phasor:

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\{Ve^{j\phi}e^{j\omega t}\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) \qquad \boxed{\mathbf{V} = Ve^{j\phi}}$$

Complex representation of the magnitude and phase of a sinusoid



Example

- Transform these sinusoids to phasors

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

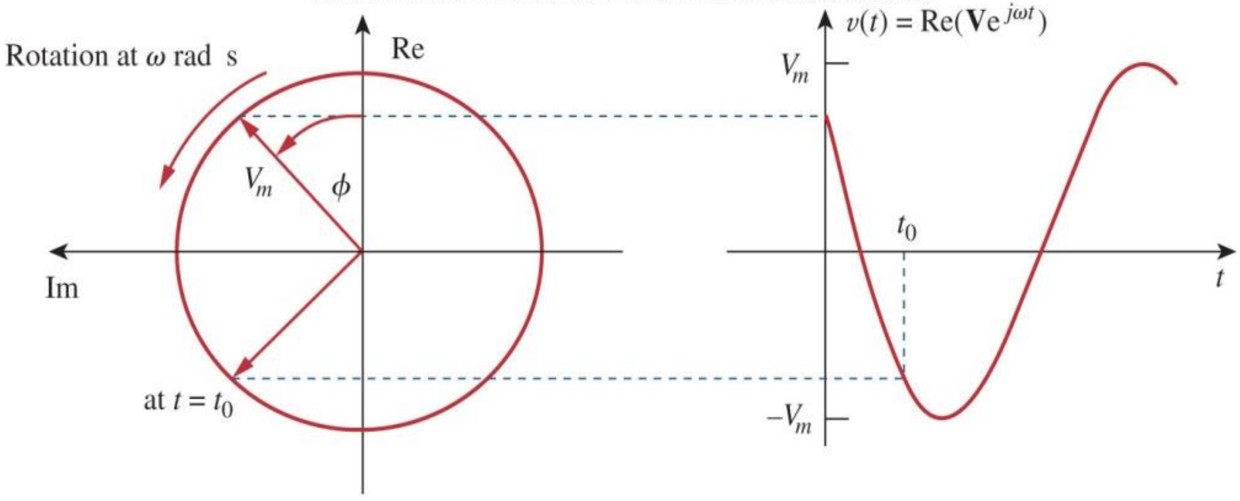
$$\mathbf{V} = -25\angle 40^\circ \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$

Phasors

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)



(a) (b)

$V_m e^{j(\omega t+\phi)}$

Sinusoid-Phasor Transformation

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

- Applying a derivative to a phasor yields:

$$\frac{dv}{dt} \Leftrightarrow j\omega V$$

(Time domain) (Phasor domain)

- Applying an integral to a phasor yields:

$$\int v\,dt \Leftrightarrow \frac{V}{j\omega}$$

(Time domain) (Phasor domain)

Time Domain - Phasor Domain Transformation

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\Longleftrightarrow$	A
$A\cos(\omega t+\phi)$	$\Longleftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\Longleftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\Longleftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\Longleftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\Longleftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}(x(t))$	$\Longleftrightarrow$	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\Longleftrightarrow$	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	$\Longleftrightarrow$	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t+\phi)\,dt$	$\Longleftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

[Source: Berkeley]

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.

- You just need to track magnitude/phase, knowing that everything is at frequency ω.

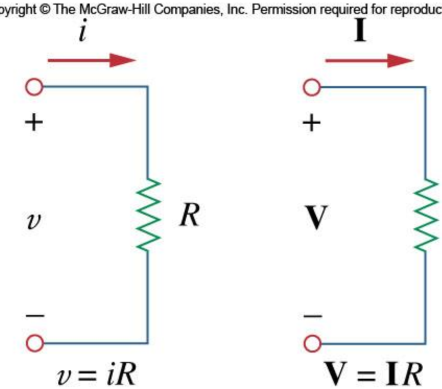
Phasor Relationships for Resistors

- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

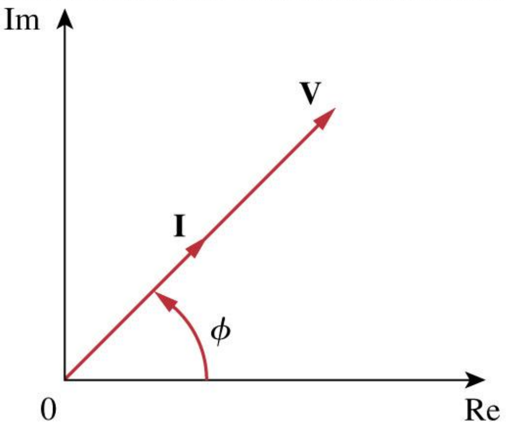
$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = RI_m \cos(\omega t + \phi) \qquad \mathbf{V} = RI_m \angle \phi$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$



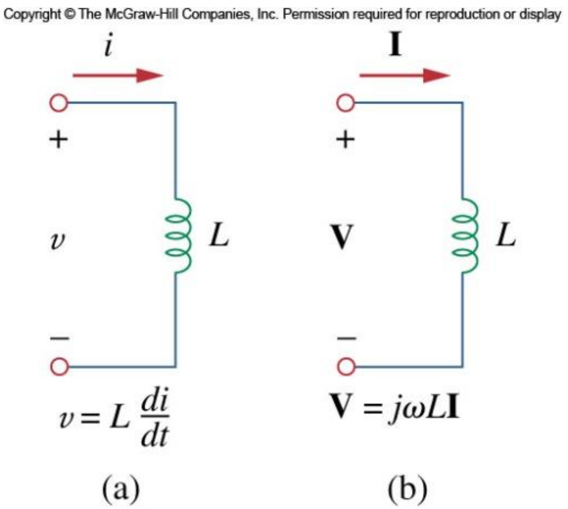
(a) (b)



Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

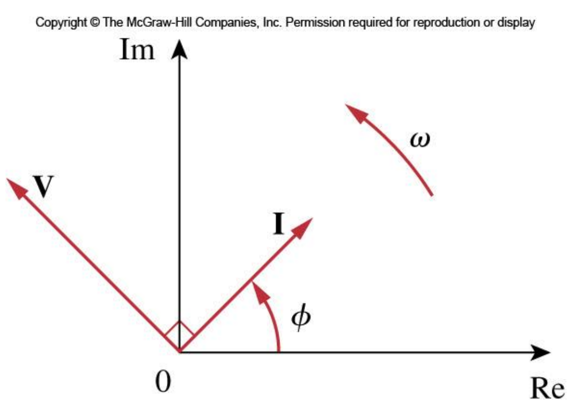
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$
$$= \omega L I_m \cos(\omega t + \phi + 90°)$$



$$\mathbf{V} = \omega L I_m \angle \phi + 90° = \omega L I_m e^{j(\phi + 90°)} = \omega L e^{j90°} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\boxed{\mathbf{V} = j\omega L\mathbf{I}}$$

- The voltage leads the current by 90° (phase shift = 90°)

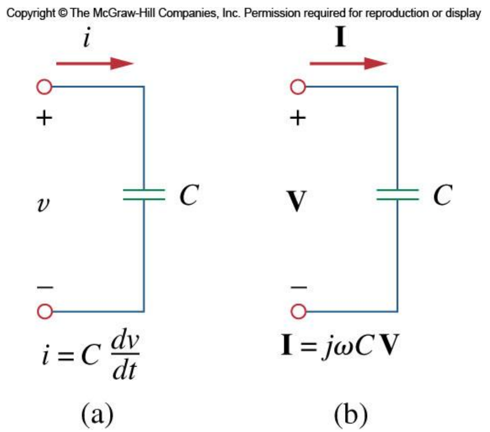


Phasor Relationships for Capacitors

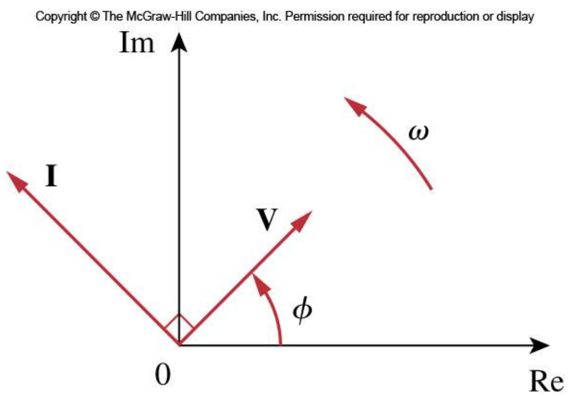
$$v = V_m \cos(\omega t + \phi) \qquad \mathbf{V} = V_m \angle \phi$$

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$
$$= \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90° = \cdots = j\omega C \cdot \mathbf{V}$$



$$\boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$



- The voltage lags the current by 90°.

Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\mathbf{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - Impedance is not a phasor! But it is (often) a complex number.
  - Impedance depends on the frequency ω.

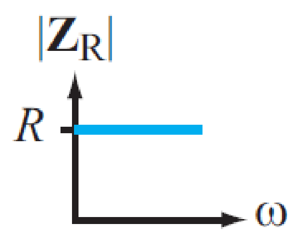
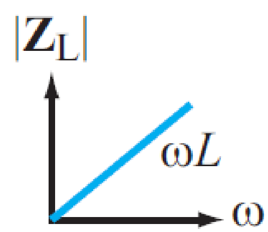
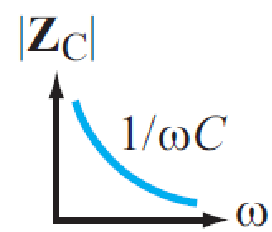
Admittance

Admittance is simply the inverse of impedance, unit: Simens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Summary of *R, L, C*

Property	R	L	C
v–i	$v = Ri$	$v = L\dfrac{di}{dt}$	$i = C\dfrac{dv}{dt}$
$\mathbf{V}$–$\mathbf{I}$	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
$\mathbf{Z}$	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

[Source: Berkeley]

Example

- Use the phasor approach, determine the current $i(t)$ in the circuit described by the integrodifferential equation

$$4i + 8\int i\,dt + 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$