Lecture 12-1 Image reconstruction

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Course piazza link： piazza.com/shanghaitech.edu.cn/spring2021/cs270spring2021

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Outline

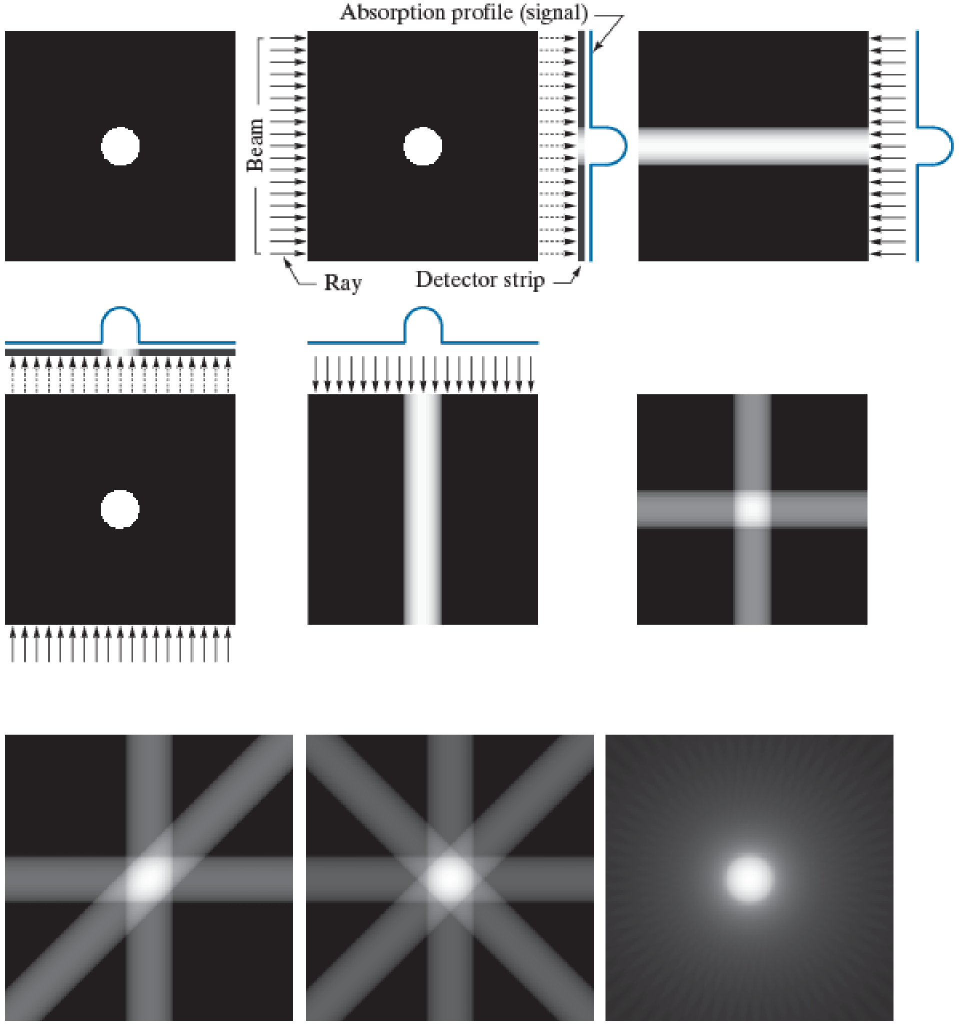
– Projection and back-projection

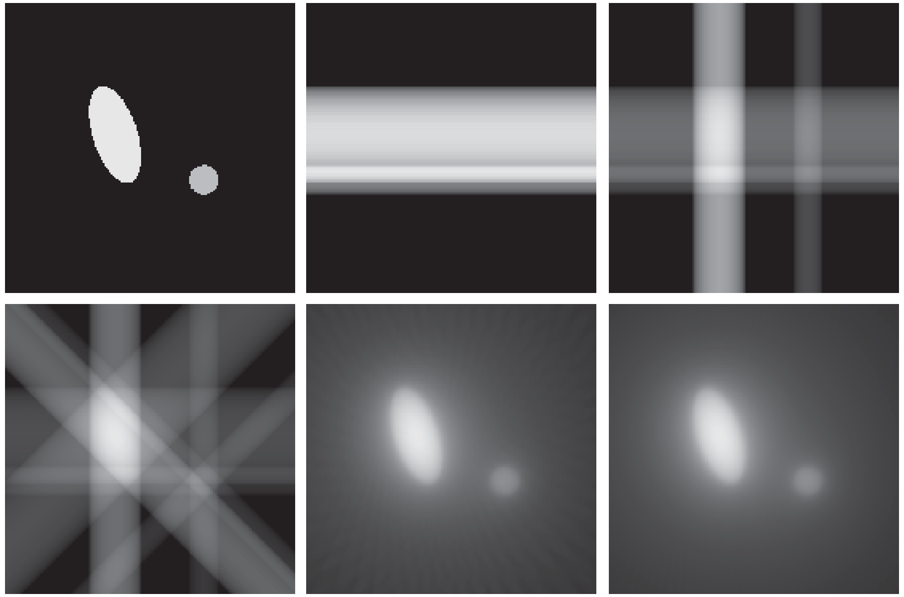
– Radon transform

– Fourier-Slice Theorem

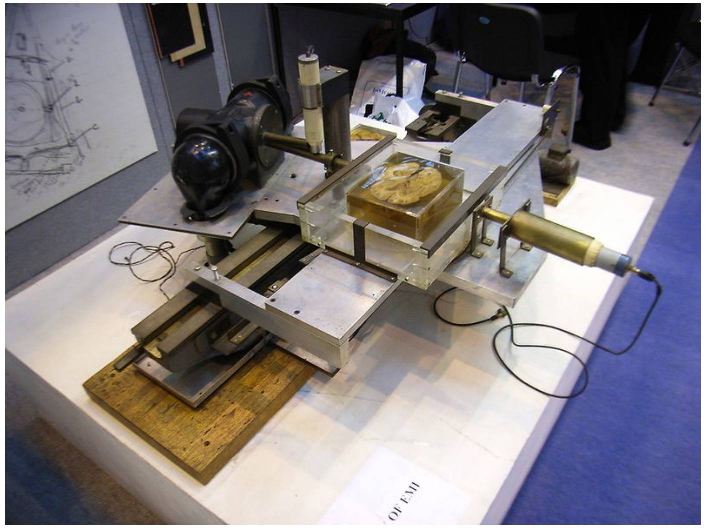
– Filtered back-projection

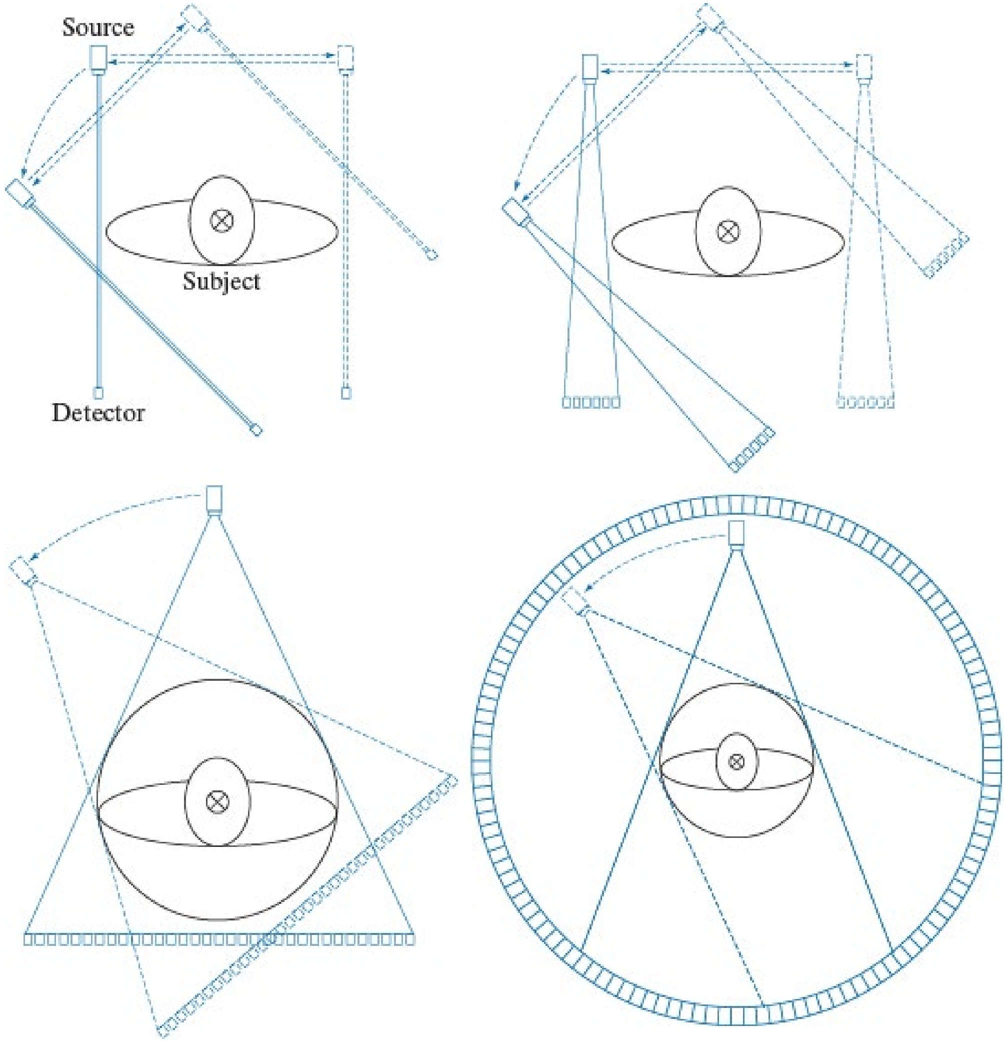
Projection and back projection



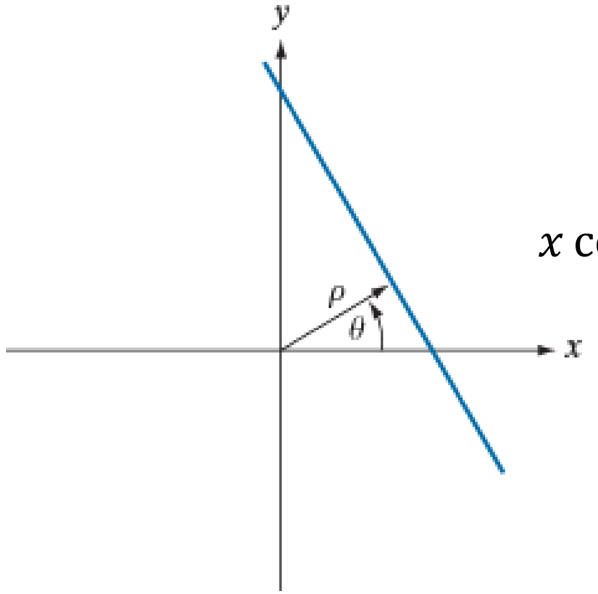


CT scanning methods





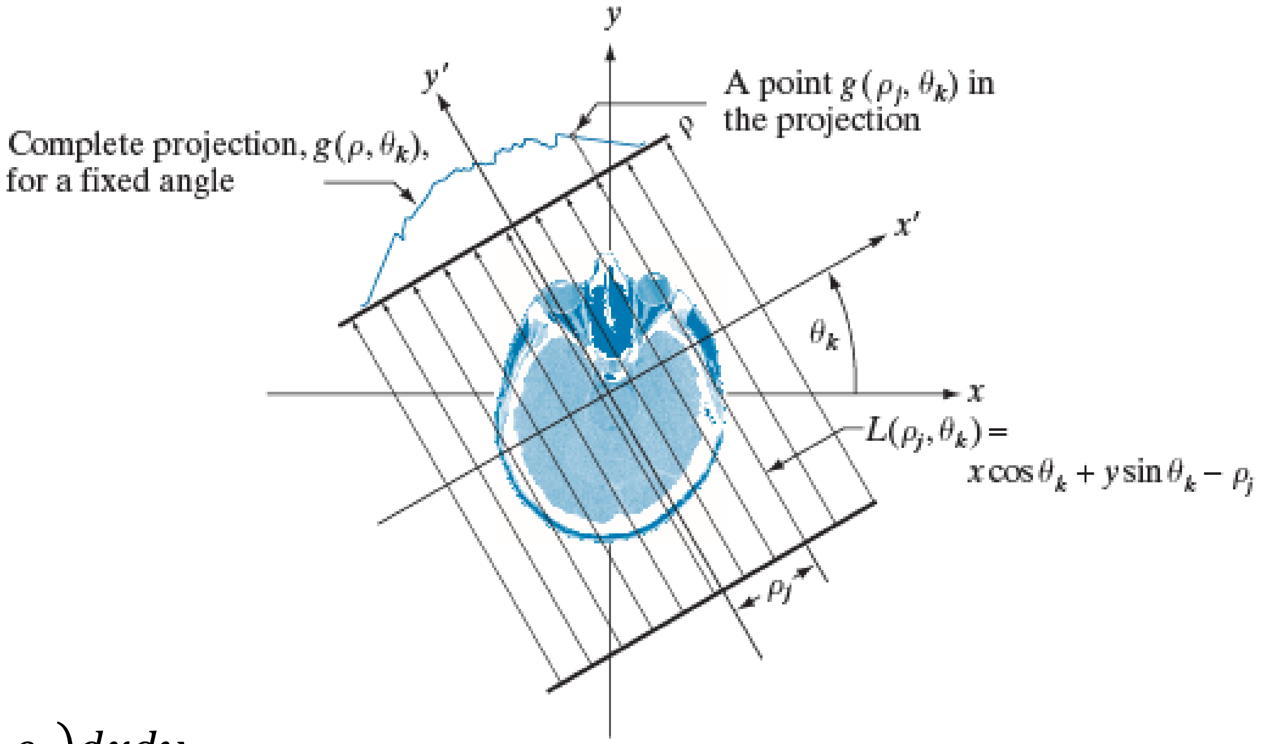
Radon transform



$$y = ax + b$$

$$\downarrow$$

$$x\cos\theta + y\sin\theta = \rho$$



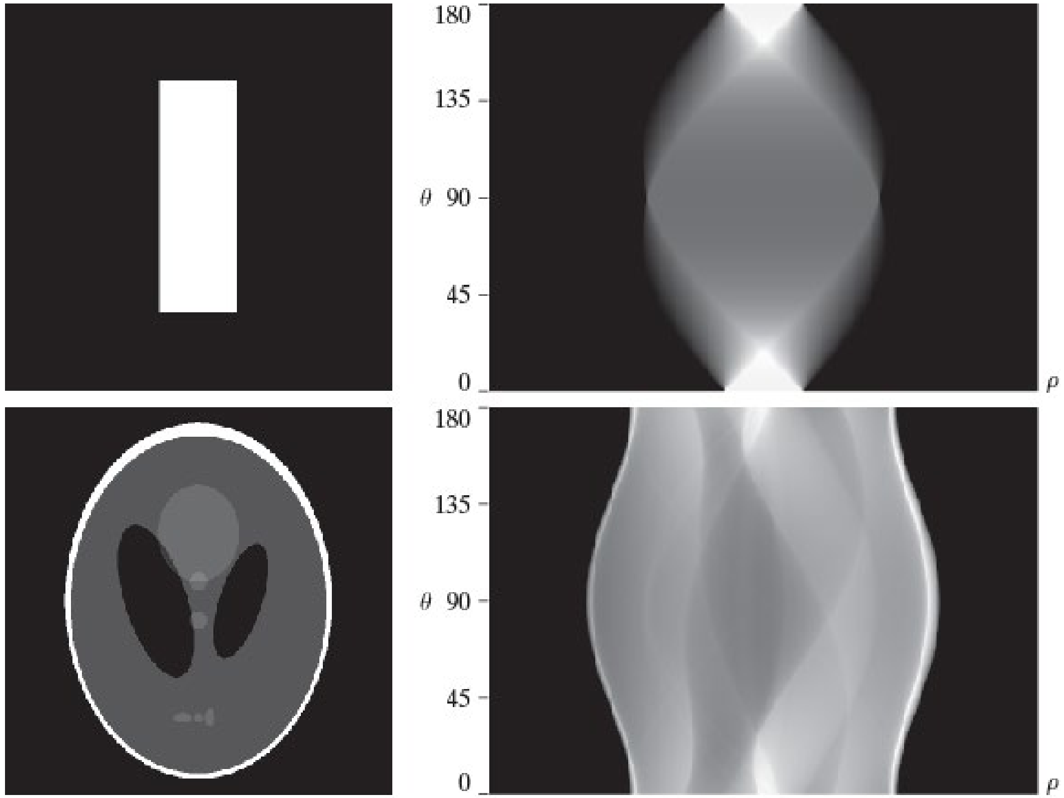
$$g(\rho_j, \theta_k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\delta(x\cos\theta_k + y\sin\theta_k - \rho_j)dxdy$$

$$g(\rho, \theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\delta(x\cos\theta + y\sin\theta)dxdy$$

$$g(\rho, \theta) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)$$

Sinogram

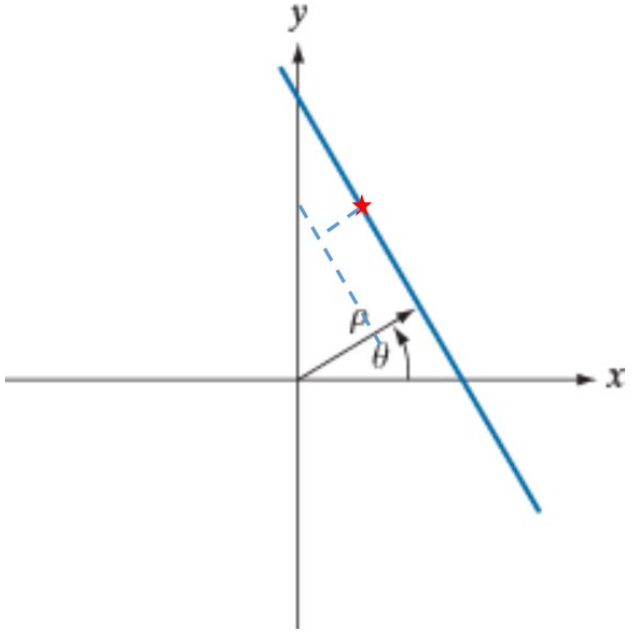
- Radon transform $g(\rho, \theta)$ is displayed as an image with ρ and θ as rectilinear coordinates



Back-projection from Sinogram

- For a fixed value of rotation θ_k:

$$f_{\theta_k}(x, y) = g(\rho, \theta_k) = g(x\cos\theta_k + y\sin\theta_k\,, \theta_k)$$



- Then a single back-projection image obtained at an angle θ is :

$$f_\theta(x, y) = g(x\cos\theta + y\sin\theta\,, \theta)$$

- The reconstructed image is obtained by summing over all the back-projected images:

$$f(x, y) = \sum_{\theta=0}^{\pi} f_\theta(x, y)$$

Fourier-Slice Theorem

- The 1D FT of a projection with respect to ρ is:

$$G(\omega,\theta)=\int_{-\infty}^{+\infty} g(\rho,\theta)\mathrm{e}^{-j2\pi\omega\rho}d\rho$$

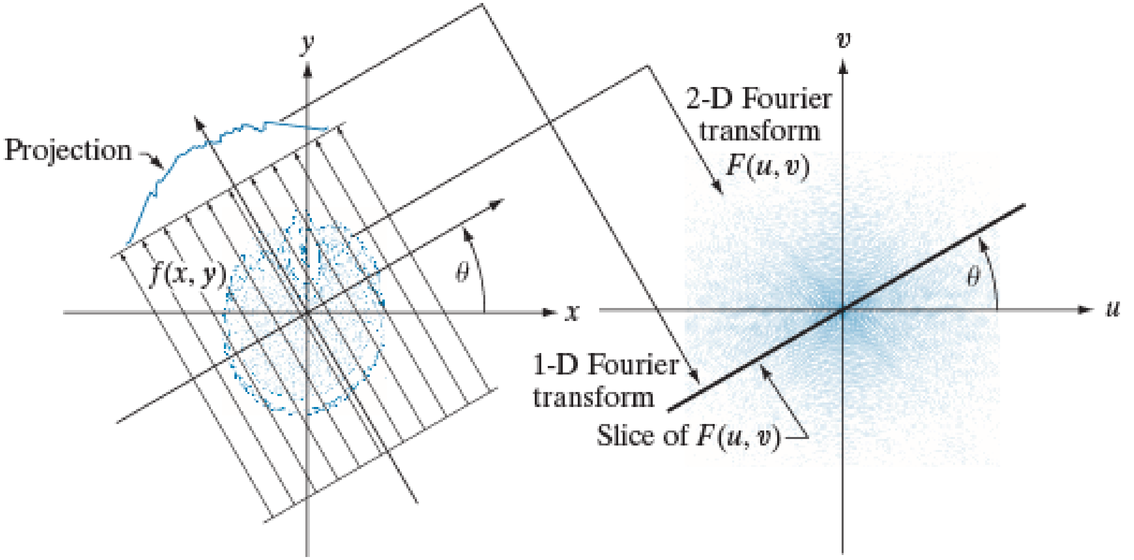
- Then

$$G(\omega,\theta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{e}^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dxdy$$

$$=\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{e}^{-j2\pi(ux+vy)}dxdy\right]_{u=\omega\cos\theta;v=\omega\sin\theta}$$

- Therefore

$$G(\omega,\theta)=[F(u,v)]_{u=\omega\cos\theta;v=\omega\sin\theta}=F(\omega\cos\theta,\omega\sin\theta)$$



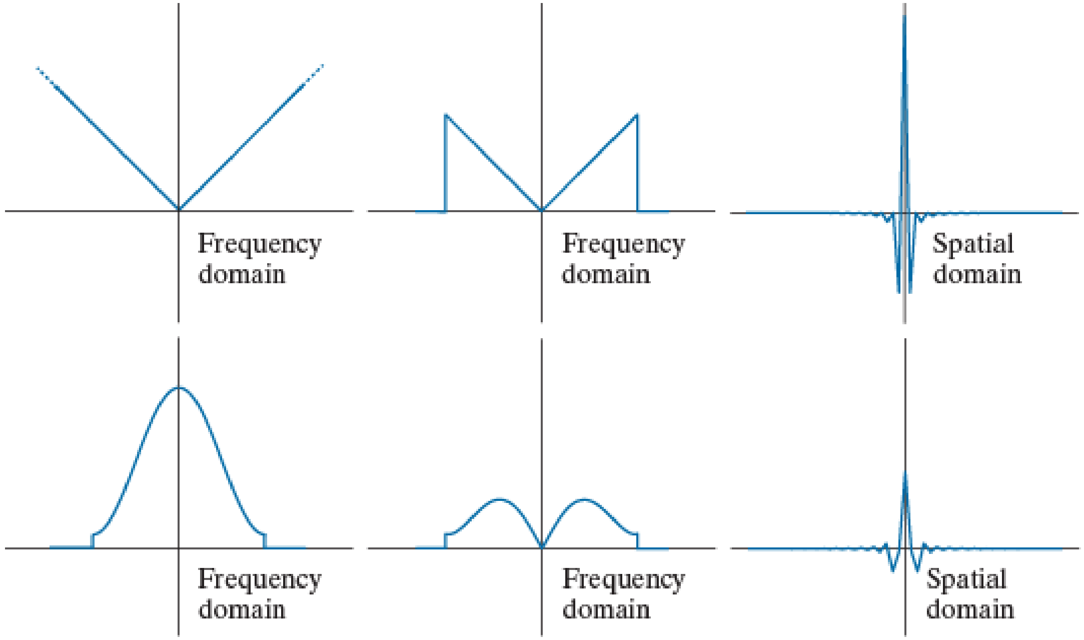
Filtered back-projection

- The 2D IFT of $\mathrm{F}(\mathrm{u},\mathrm{v})$ is:

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(u,v)\mathrm{e}^{j2\pi(ux+vy)}dudv$$

$$=\int_{0}^{2\pi}\int_{0}^{\infty}G(\omega,\theta)\mathrm{e}^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta$$

$$=\int_{0}^{\pi}\left[\int_{-\infty}^{+\infty}|\omega|G(\omega,\theta)\mathrm{e}^{j2\pi\omega\rho}d\omega\right]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

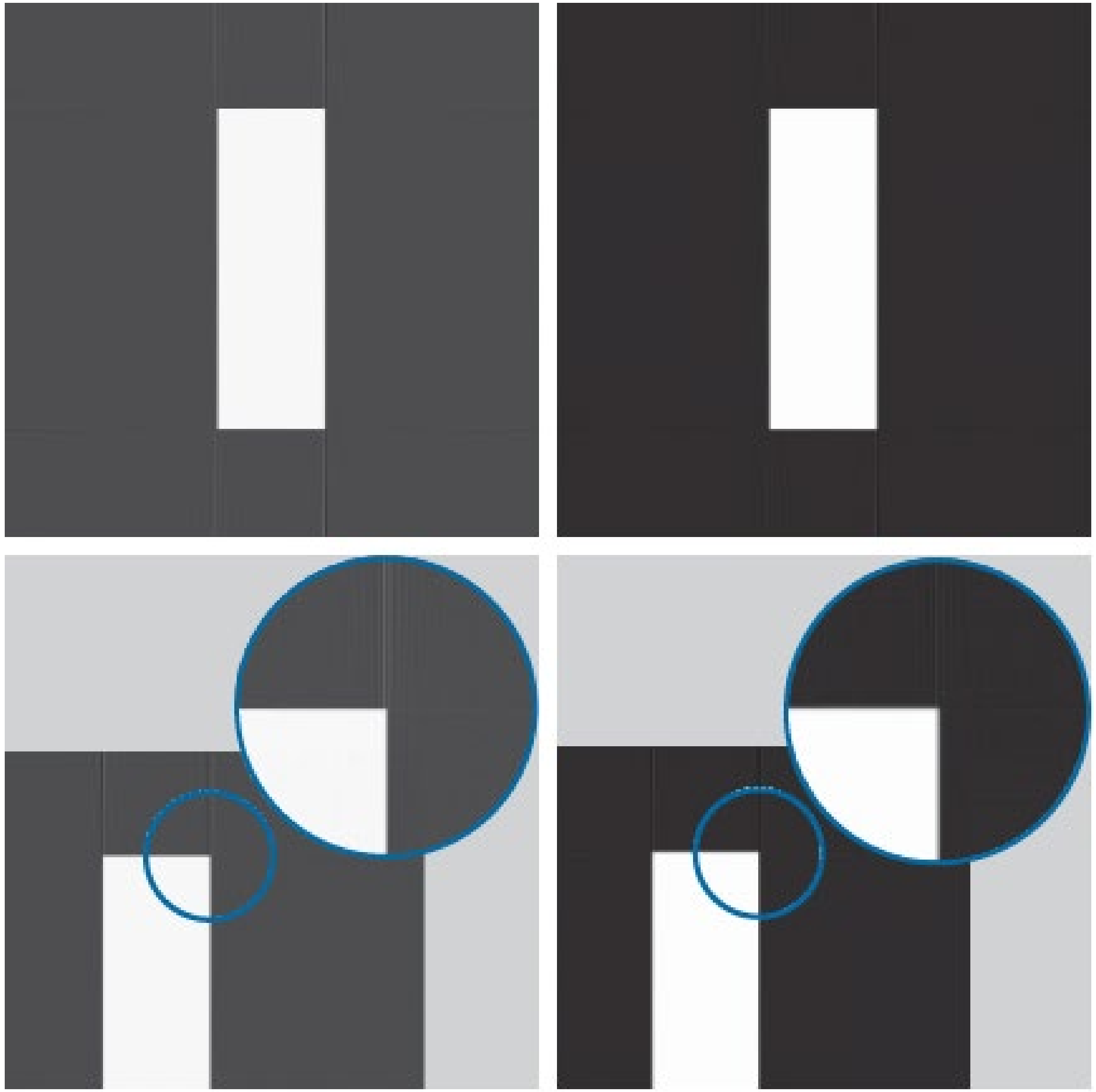


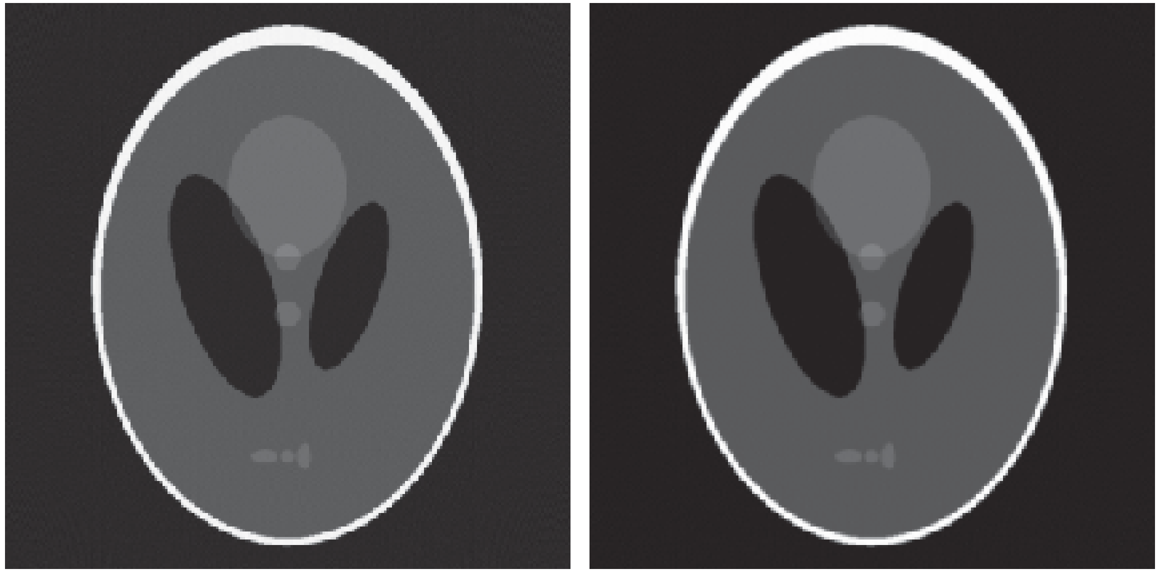
- Back-projection with convolution

$$f(x,y)=\int_{0}^{\pi}[s(\rho)\otimes g(\rho,\theta)]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

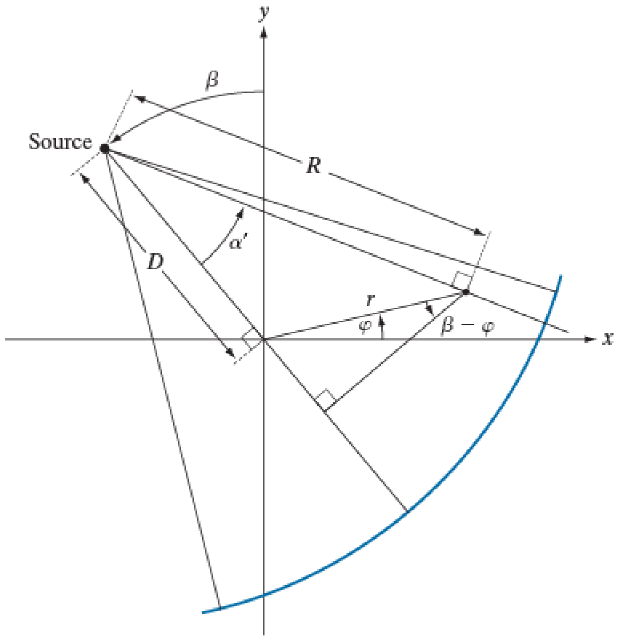
Where $s(\rho)=IFT(|\omega|)$, $g(\rho,\theta)=IFT(G(\omega,\theta))$

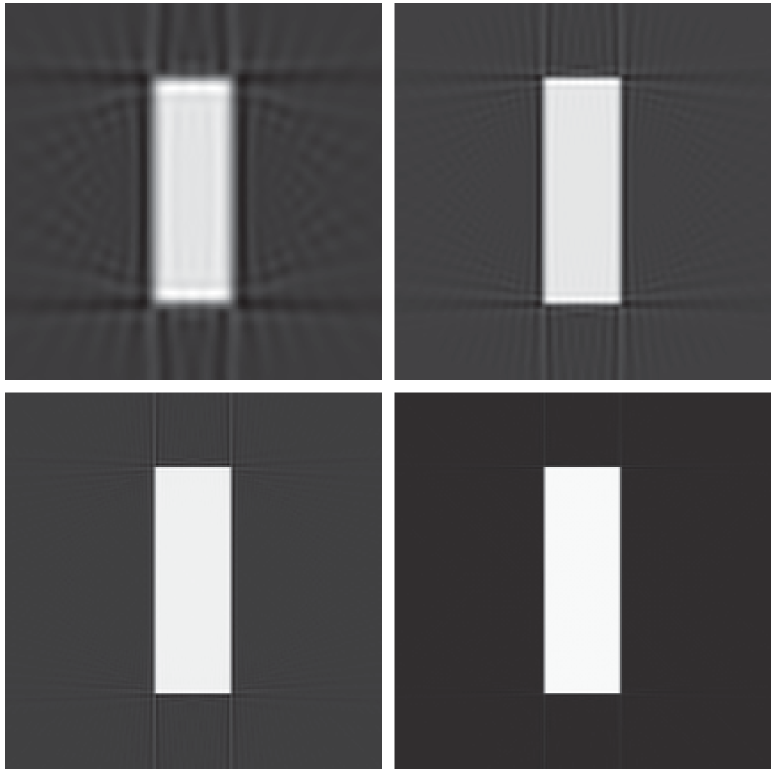
Filtered back-projection

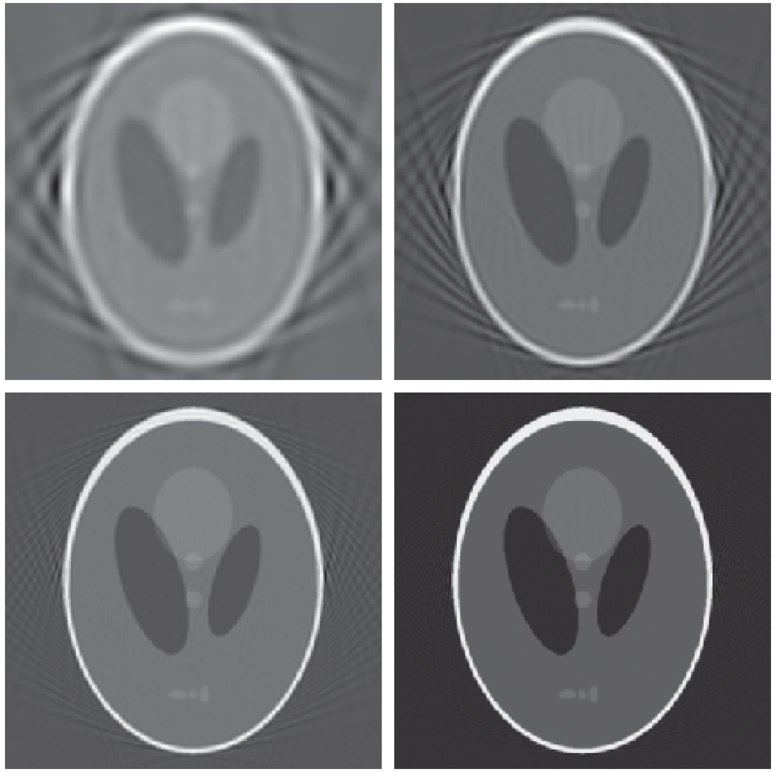




Fan-Beam based Filtered back-projection







Take home message

- CT imaging reconstruction is based on accumulating back-projections data directive, while the reconstructed images are blurred.
- The 2D Fourier transform of an image for reconstruction can be obtained by accumulating the Fourier transform of the projections at different angles (Fourier-Slice Theorem).
- Filtered back-projection can mitigate the blurring effects.