1 20180411: Integration Property for Fourier Series

Let $x(t)$ be periodic with fundamental period T_0 (thus $w_0 = \frac{2\pi}{T_0}$), and a_k be its Fourier series. Recall that there is an integration property for Fourier Series:

$$\int_{t_0}^{t} x(s)ds \xrightarrow{FS} \frac{a_k}{jkw_0}.$$

The full version is actually: first the integral of $x(t)$ over one period should be zero (namely $a_0 = 0$), and then

$$\int_{t_0}^{t} x(s)ds \xrightarrow{FS} \begin{cases} \frac{a_k}{jkw_0}, & k \neq 0 \\ -\sum_{k\neq 0} \frac{a_k}{jkw_0} e^{jkw_0 t_0}, & k = 0 \end{cases}.$$

The case for $k = 0$ can also be calculated by definition (the average area over one period). The details are as follows:

1. In order to calculate the Fourier series for

$$y(t) := \int_{t_0}^{t} x(s)ds = \int_{t_0}^{t} \sum_k a_k e^{jkw_0 s} ds,$$

   it must be periodic.

2. For $k \neq 0$,

$$\int_{t_0}^{t} a_k e^{jkw_0 s} ds = \frac{a_k}{jkw_0}(e^{jkw_0 t} - e^{jkw_0 t_0})$$

   is periodic with period T_0.

3. For $k = 0$,

$$\int_{t_0}^{t} a_k e^{jkw_0 s} ds = a_0(t - t_0)$$

   is not periodic unless $a_0 = 0$.

4. A periodic signal plus a non-periodic signal cannot result into a periodic signal. Hence in order $y(t)$ to be periodic, a_0 must be zero.

5. Now

$$y(t) = \sum_{k\neq 0} \frac{a_k}{jkw_0}(e^{jkw_0 t} - e^{jkw_0 t_0}) = -\sum_{k\neq 0} \frac{a_k}{jkw_0} e^{jkw_0 t_0} + \sum_{k\neq 0} \frac{a_k}{jkw_0} e^{jkw_0 t},$$

   which means $b_k = a_k/(jkw_0)$ for $k \neq 0$, and $b_0 = -\sum_{k\neq 0} \frac{a_k}{jkw_0} e^{jkw_0 t_0}$.

Remark: In the book (Oppenheim), page 224, Table 4.2, it is not much precise to put the lower limit of the integral to be $-\infty$, since usually $\int_{-\infty}^{t} x(s)ds$ is not well defined for a periodic signal x.

2 20180402: Integration Property for Fourier Transform

Here we show that the Fourier Transform has the following integration property:

$$\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{FT} \frac{1}{jw}X(jw) + \pi X(0)\delta(w).$$

This is actually equivalent to

$$\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{FT} \frac{1}{jw}X(jw) + \pi X(jw)\delta(w).$$

The steps are as follows:

1. Convolution: $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$.
2. Convolution property: $x * u \xrightarrow{FT} X(jw)U(jw)$.
3. It remains to show that

$$u(t) \xrightarrow{FT} U(jw) = \frac{1}{jw} + \pi\delta(w).$$

Notice that we can decompose $u(t)$ into the even and odd parts:

$$u(t) = \frac{1}{2} + (u(t) - \frac{1}{2}).$$

Firstly, we already know that $F[\delta(t)] = 1$. From the duality we obtain $F[1] = 2\pi\delta(-w) = 2\pi\delta(w)$, which means the Fourier transform of $\frac{1}{2}$ is $\pi\delta(w)$.

Secondly, we need to show that $F[u(t) - \frac{1}{2}] = \frac{1}{jw}$. Let us do the inverse Fourier transform to show that $F^{-1}[\frac{1}{jw}] = u(t) - \frac{1}{2}$.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{jw}e^{jwt}dw &= \frac{1}{2\pi j}\Big(\int_{-\infty}^{0} \frac{e^{jwt}}{w}dw + \int_{0}^{+\infty} \frac{e^{jwt}}{w}dw\Big) \\
&= \frac{1}{2\pi j}\Big(\int_{+\infty}^{0} \frac{e^{j(-u)t}}{-u}d(-u) + \int_{0}^{+\infty} \frac{e^{jwt}}{w}dw\Big) \\
&= \frac{1}{2\pi j}\int_{0}^{+\infty} \frac{e^{jwt} - e^{-jwt}}{w}dw \qquad \text{// Cauchy principal values} \\
&= \frac{1}{\pi}\int_{0}^{+\infty} \frac{\sin wt}{w}dw \\
&= \begin{cases} +\frac{1}{\pi}\int_{0}^{+\infty} \frac{\sin u}{u}du, & t > 0, \\ -\frac{1}{\pi}\int_{0}^{+\infty} \frac{\sin u}{u}du, & t < 0. \end{cases}
\end{aligned}$$

Finally, one uses the result from Complex Analysis to show that $\int_{-\infty}^{+\infty} \frac{\sin u}{u}du = \pi$. Or, one can also notice that the Fourier transform of $rect(t)$ (with value 1 on [-1,1], and 0 otherwise) is $2\frac{\sin w}{w}$, one has the inverse Fourier transform

$$1 = rect(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\frac{\sin w}{w}e^{-jw0}dw.$$