CS101 Algorithms and Data Structures

Shortest Path: Floyd-Warshall

Textbook Ch 24, 25

Outline

- Dijkstra’s algorithm
- Floyd-Warshall algorithm

Dijkstra's algorithm

We will iterate $|V|$ times:

- Find the unvisited vertex v that has a minimum distance to it
- Mark it as visited
- Consider its every adjacent vertex w that is unvisited:
  - Is the distance to v plus the weight of the edge (v, w) less than our currently known shortest distance to w ?
  - If so, update the shortest distance to w and record v as the previous pointer

Continue iterating until all vertices are visited or all remaining vertices have a distance of infinity

Outline

- Dijkstra’s algorithm
- Floyd-Warshall algorithm

Background

Dijkstra's algorithm finds the shortest path between two nodes

- Run time: $\mathrm{O}(|E| \ln(|V|))$

If we wanted to find the shortest path between all pairs of nodes, we could apply Dijkstra's algorithm to each vertex:

- Run time: $\mathrm{O}(|V| |E| \ln(|V|))$

Background

Now, Dijkstra's algorithm has the following run times:

- Best case:
  If $|E| = \Theta(|V|)$, running Dijkstra for each vertex is $\mathrm{O}(|V|^2 \ln(|V|))$

- Worst case:
  If $|E| = \Theta(|V|^2)$, running Dijkstra for each vertex is $\mathrm{O}(|V|^3 \ln(|V|))$

Problem

Question: for the worst case, can we find a $o\left(|V|^3 \ln|V|\right)$ algorithm?

We will look at the Floyd-Warshall algorithm

- It works with positive or negative weights with no negative cycle

Strategy

First, let's consider only edges that connect vertices directly:

$$d_{i,j}^{(0)} = \begin{cases} 0 & \text{If } i = j \\ w_{i,j} & \text{If there is an edge from } i \text{ to } j \\ \infty & \text{Otherwise} \end{cases}$$

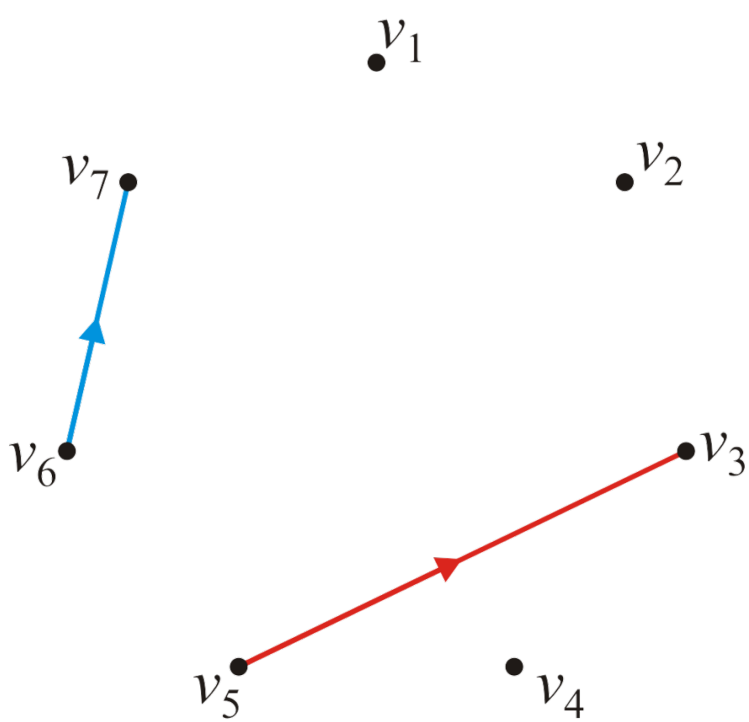
Here, $w_{i,j}$ is the weight of the edge connecting vertices i and j

- Note, this can be a directed graph; *i.e.*, it may be that $d_{i,j}^{(0)} \neq d_{j,i}^{(0)}$

Strategy

Consider this graph of seven vertices

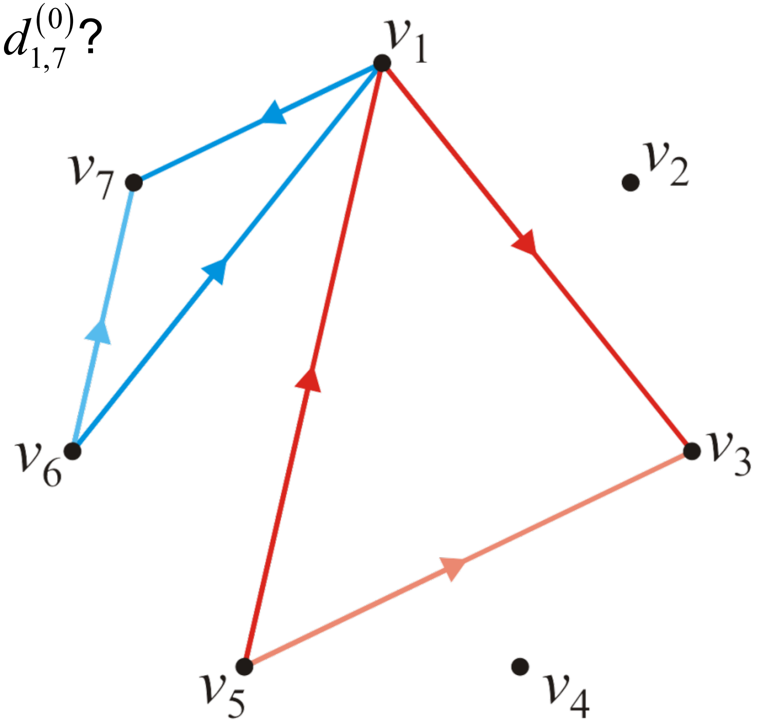
– The edges defining the values $d_{5,3}^{(0)}$ and $d_{6,7}^{(0)}$ are highlighted



Strategy

Suppose now, we want to see whether or not the path going through vertex v_1 is shorter than a direct edge?

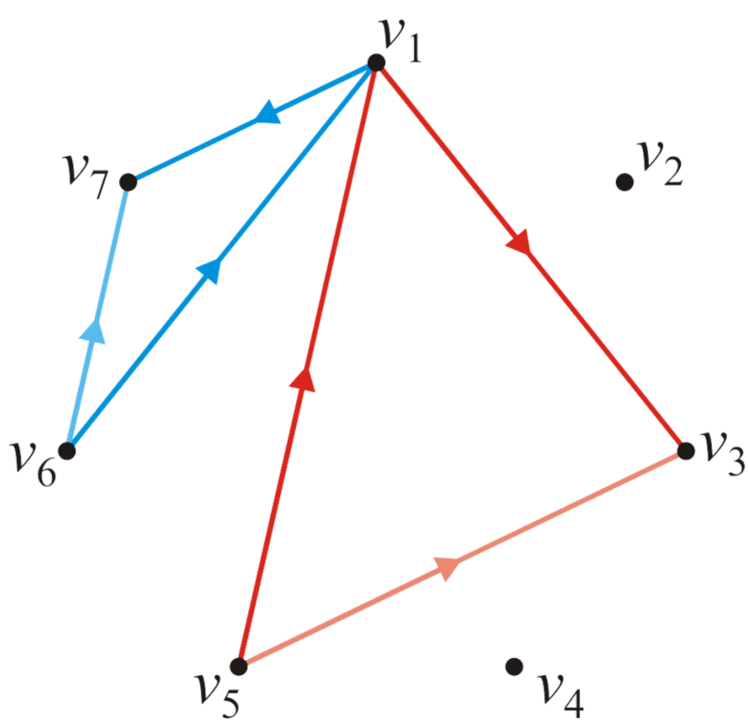
- Is $d_{5,3}^{(0)} > d_{5,1}^{(0)} + d_{1,3}^{(0)}$?
- Is $d_{6,7}^{(0)} > d_{6,1}^{(0)} + d_{1,7}^{(0)}$?



Strategy

Thus, for each pair of edges, we will define $d_{i,j}^{(1)}$ by calculating:

$$d_{i,j}^{(1)} = \min\left\{d_{i,j}^{(0)}, d_{i,1}^{(0)} + d_{1,j}^{(0)}\right\}$$



Strategy

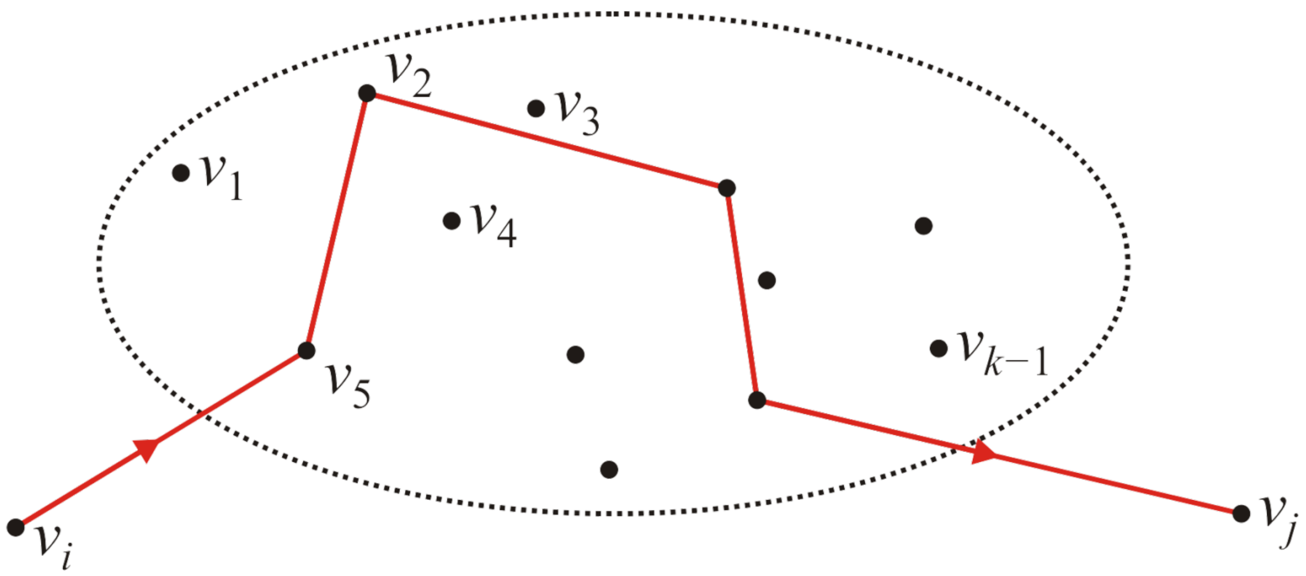
We need just run the algorithm for each pair of vertices:

```
for ( int i = 0; i < num_vertices; ++i ) {
    for ( int j = 0; j < num_vertices; ++j ) {
        d[i][j] = std::min( d[i][j], d[i][0] + d[0][j] );
    }
}
```

The General Step

Define $d_{i,j}^{(k-1)}$ as the shortest distance, but only allowing intermediate visits to vertices $v_1, v_2, \ldots, v_{k-1}$

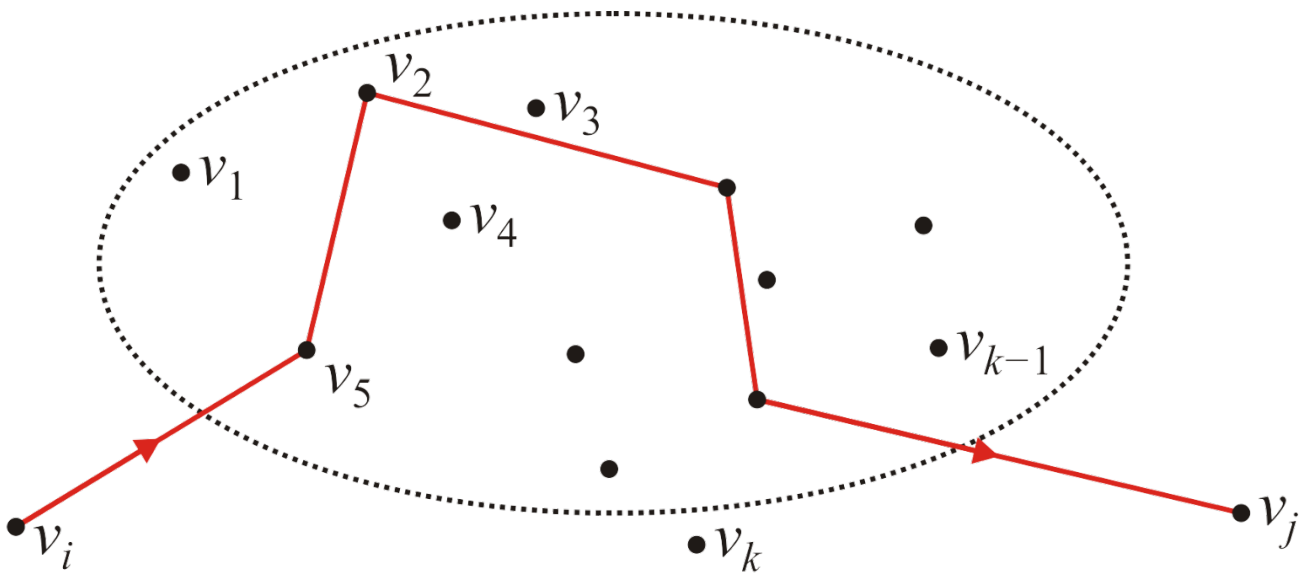
– Suppose we have an algorithm that has found these values for all pairs



The General Step

How could we find $d_{i,j}^{(k)}$; that is, the shortest path allowing intermediate visits to vertices v_1, v_2, …, v_{k-1} , v_k?

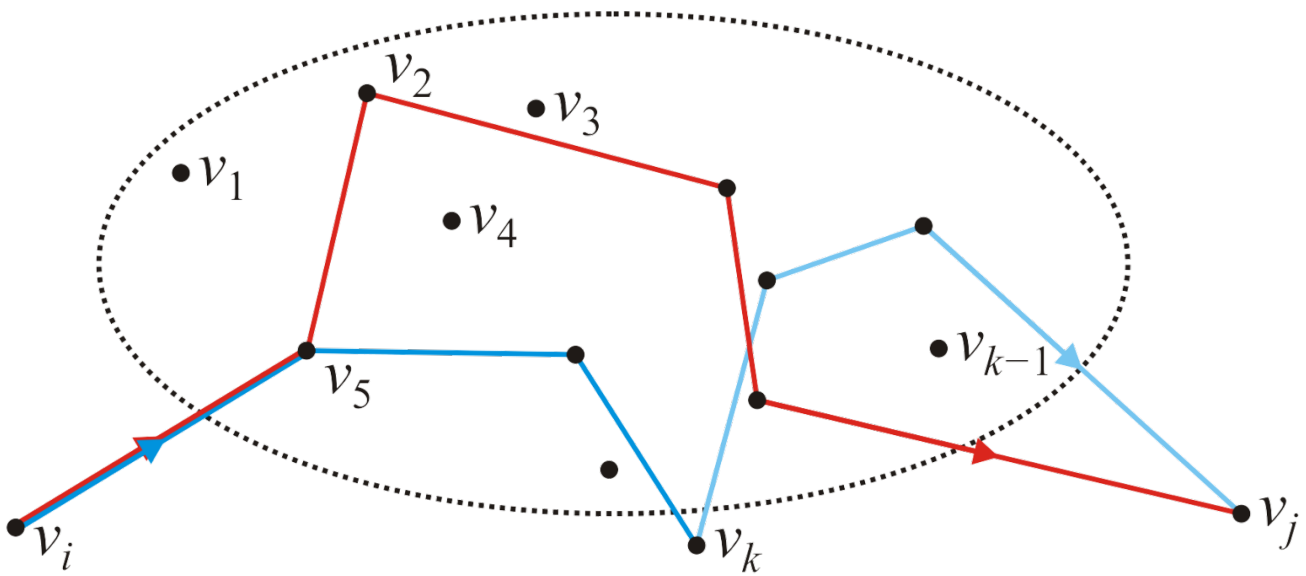
– Two possibilities: the shortest path includes or does not include v_k



The General Step

If the shortest path includes v_k, then it must consist of:

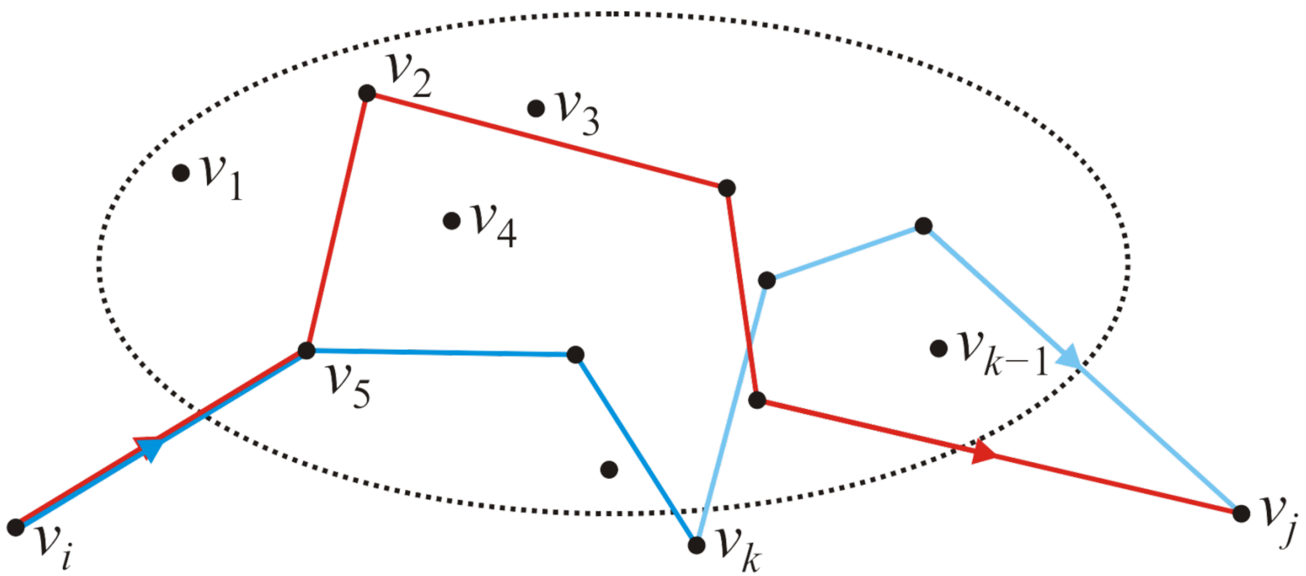
- the shortest path from v_i to v_k
- and then the shortest path from v_k to v_j
- both only allowing intermediate visits to vertices $v_1, v_2, \ldots, v_{k-1}$



The General Step

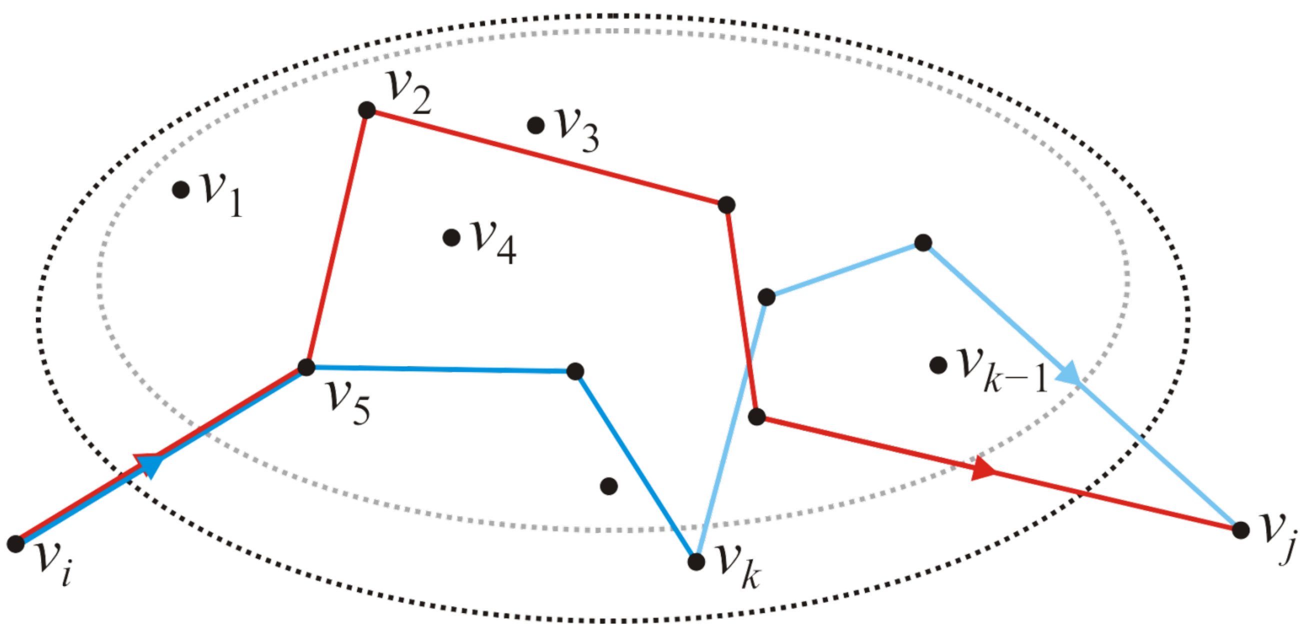
With v_1, v_2, …, v_{k-1} as intermediates, we already know the shortest paths from v_i to v_j, v_i to v_k and v_k to v_j

Thus, we calculate $d_{i,j}^{(k)} = \min\left\{d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}\right\}$



The General Step

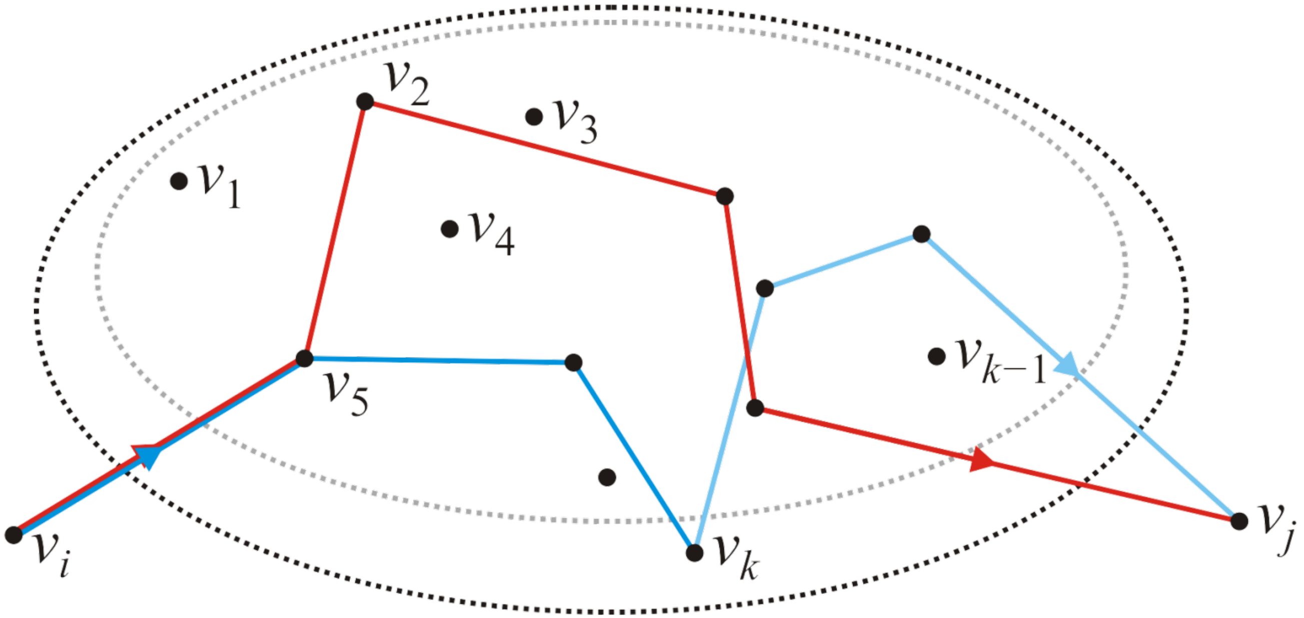
Finding $d_{i,j}^{(k)}$ for all pairs of vertices gives us all shortest paths from v_i to v_j possibly going through vertices $v_1, v_2, \ldots, v_k$



The General Step

The calculation is straight forward:

```
for ( int i = 0; i < num_vertices; ++i ) {
    for ( int j = 0; j < num_vertices; ++j ) {
        d[i][j] = std::min( d[i][j], d[i][k-1] + d[k-1][j] );
    }
}
```



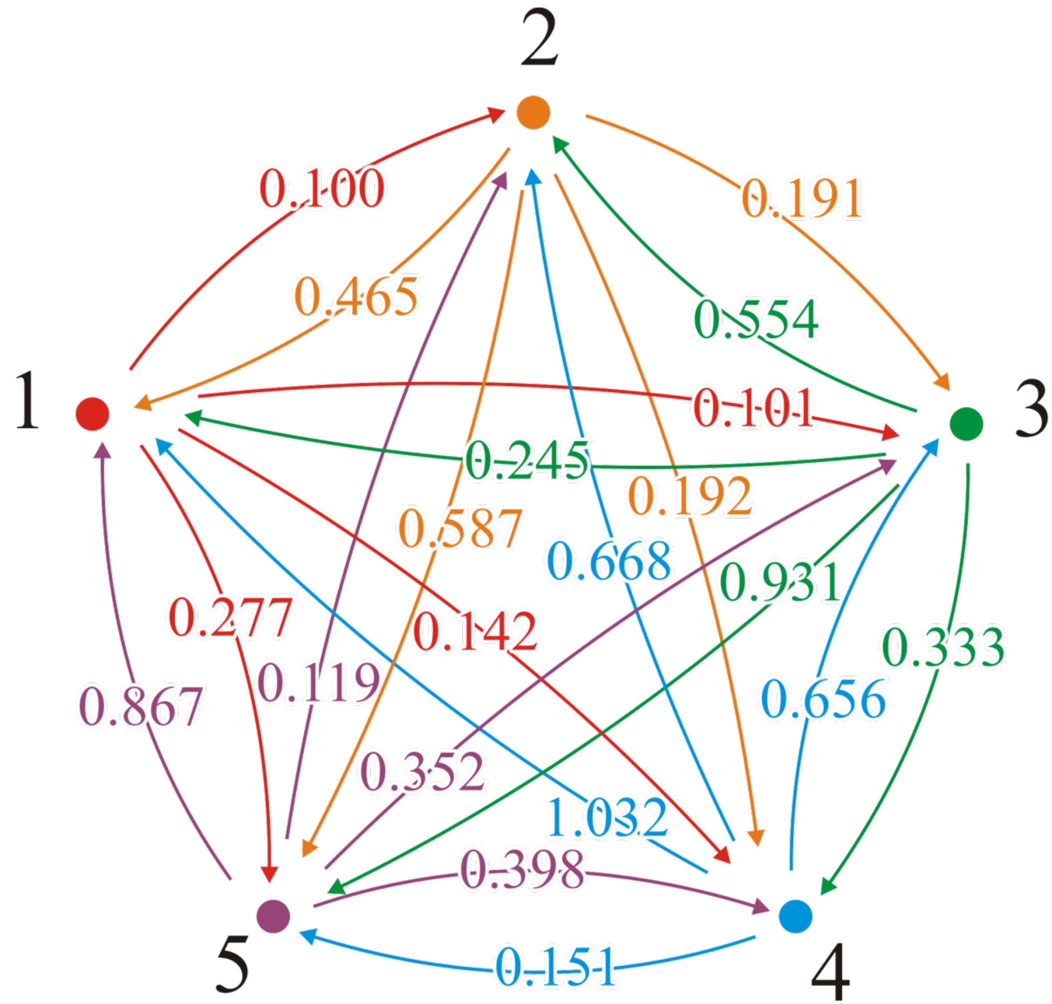
The Floyd-Warshall Algorithm

```
// Initialize the matrix d
//    ...

for ( int k = 0; k < num_vertices; ++k ) {
    for ( int i = 0; i < num_vertices; ++i ) {
        for ( int j = 0; j < num_vertices; ++j ) {
            d[i][j] = std::min( d[i][j], d[i][k] + d[k][j] );
        }
    }
}
```

Run time? $\Theta\left(|V|^3\right)$

Example

Consider this graph

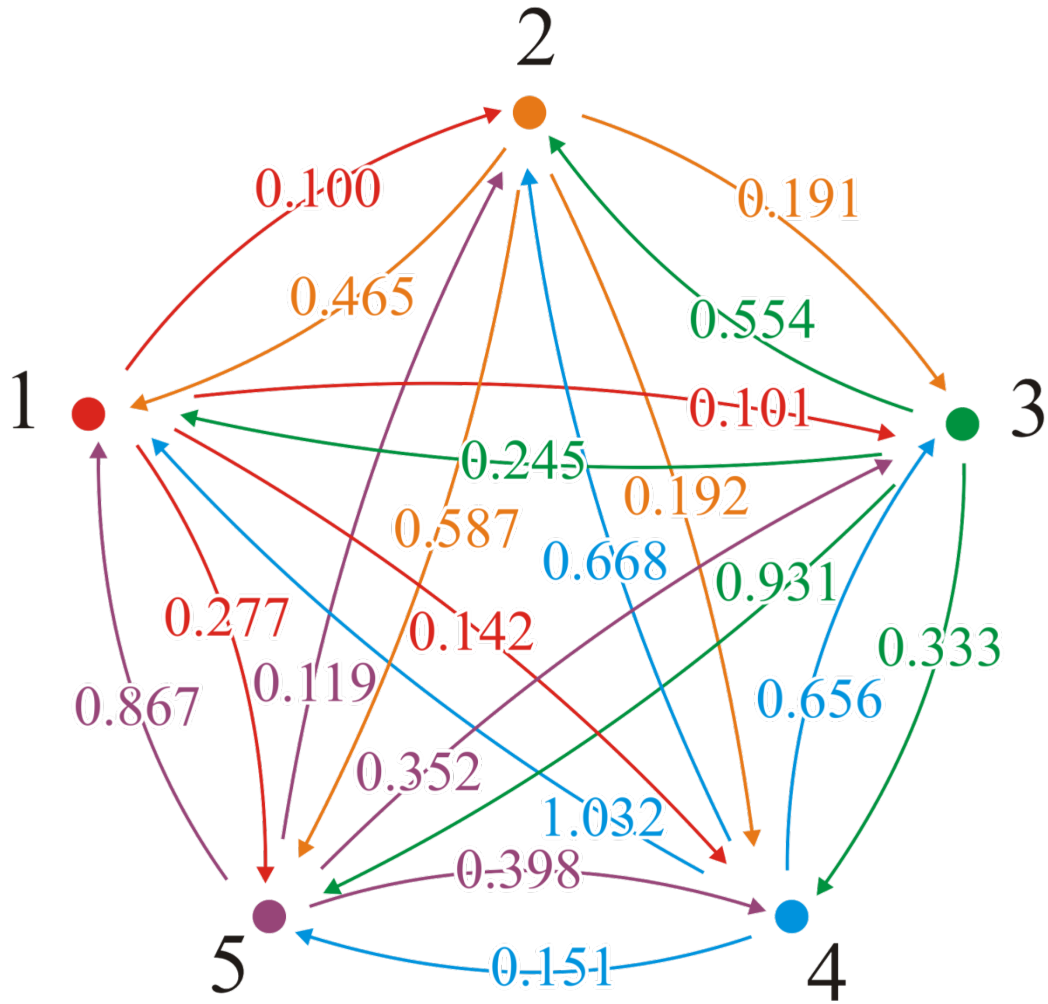


Example

The adjacency matrix is

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$

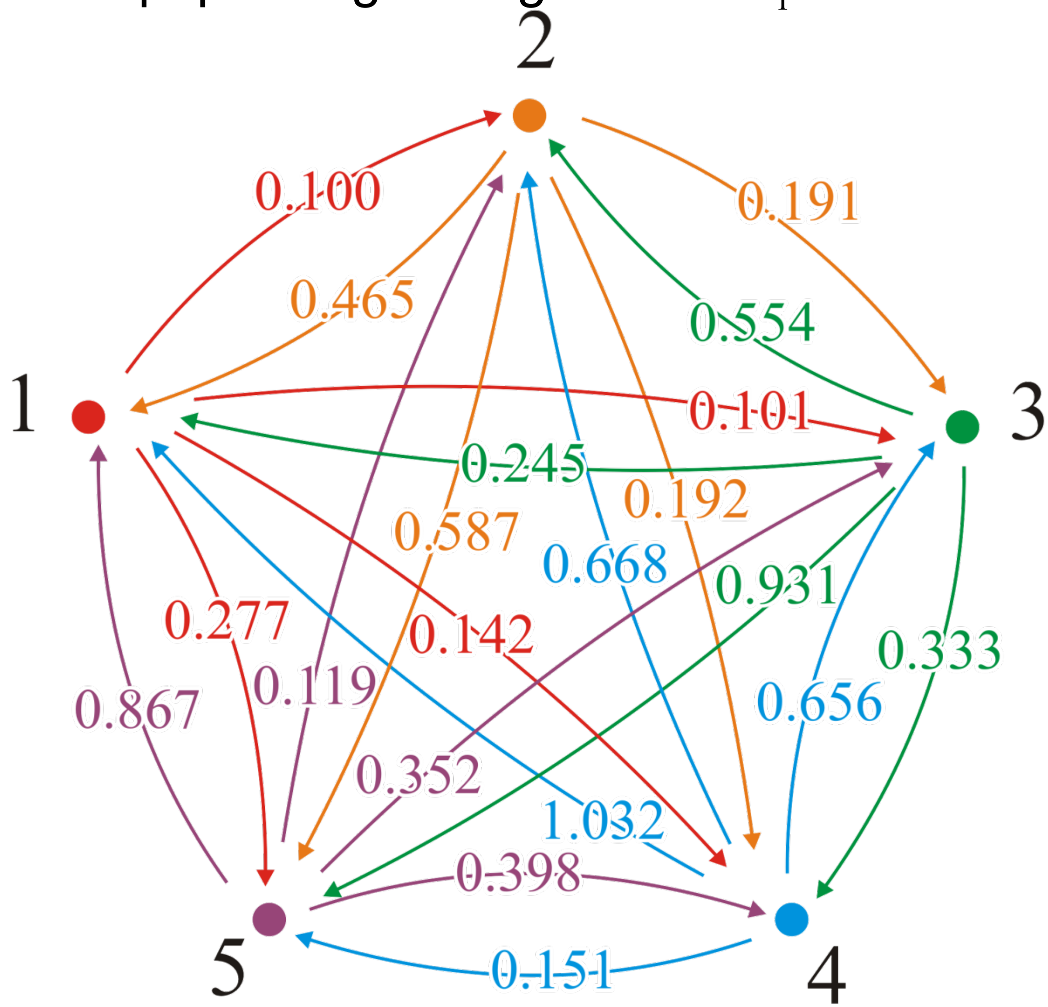
This would define our matrix $\mathbf{D} = (d_{ij})$



Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

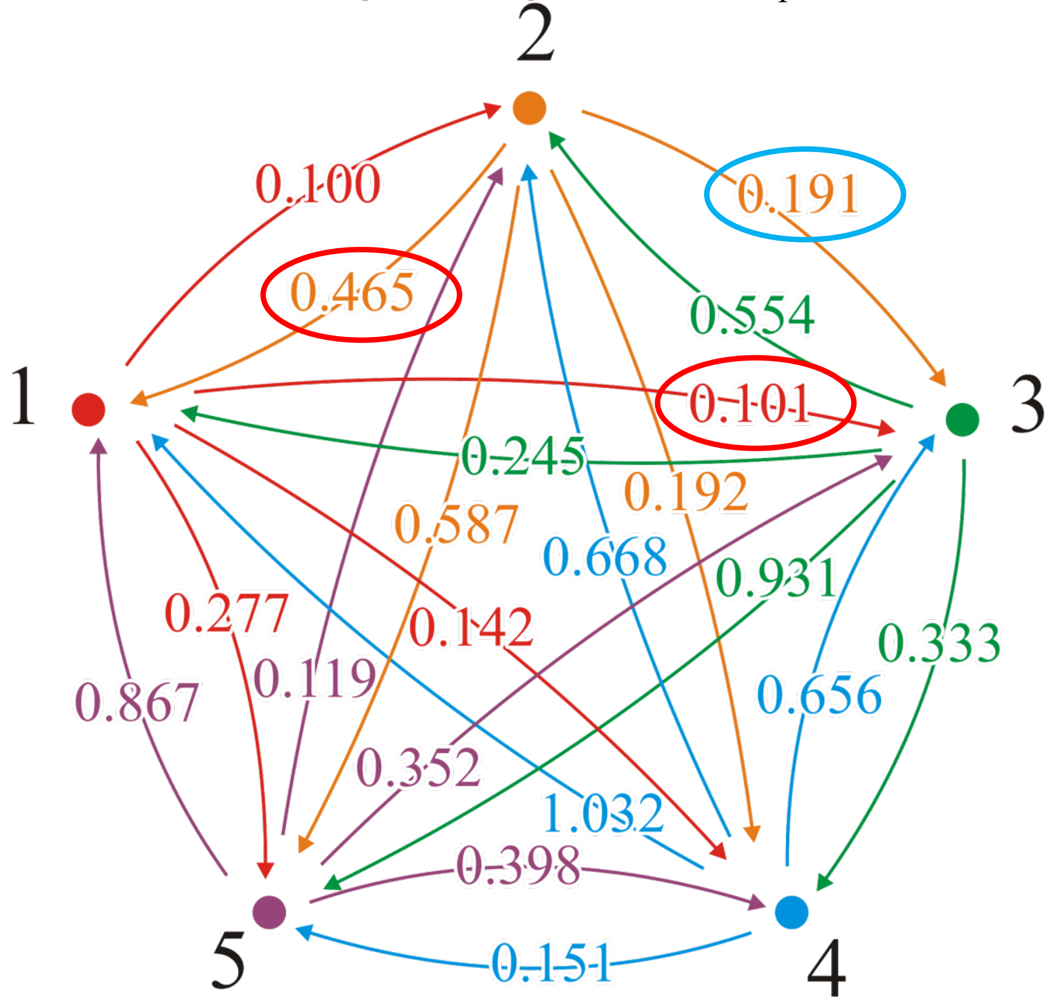
$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



We would start:

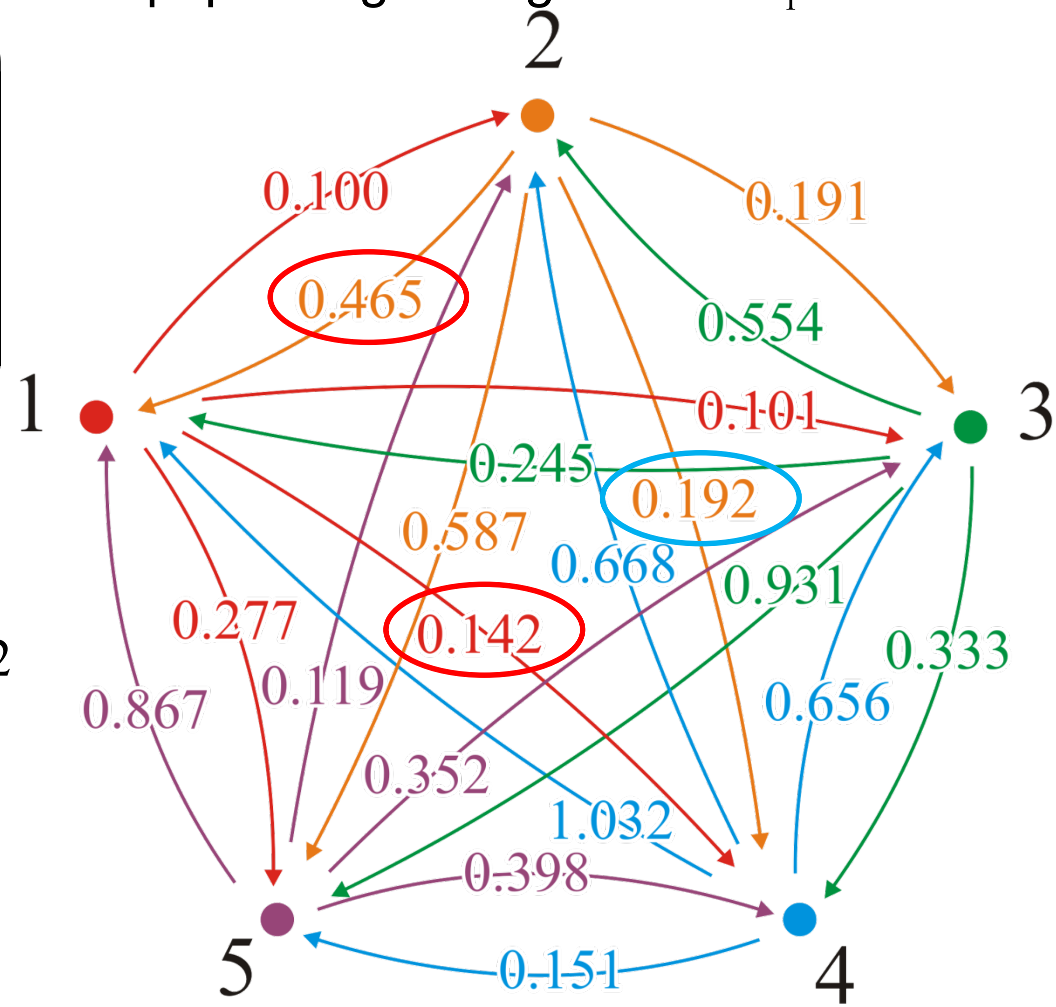
$(2, 3) \rightarrow (2, 1, 3)$

$$0.191 \not> 0.465 + 0.101$$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



We would start:

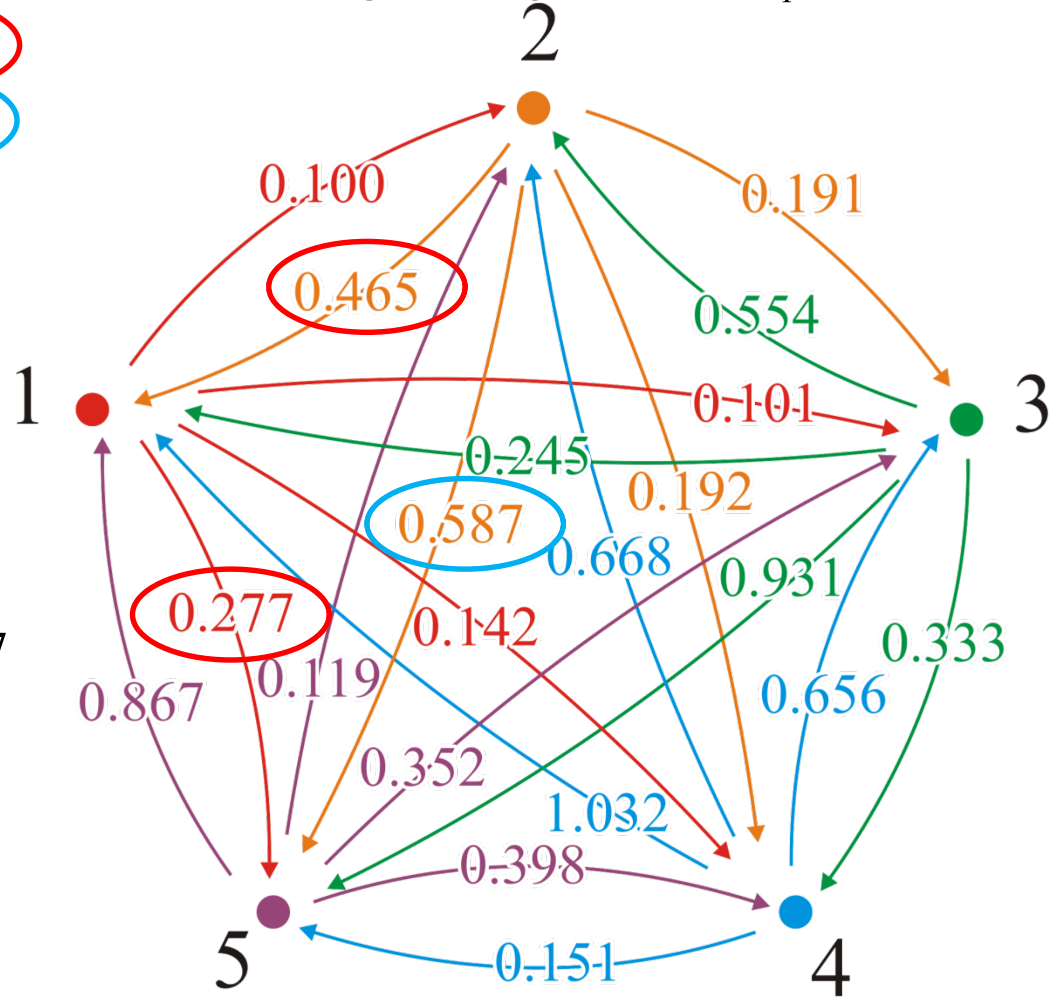
$(2, 4) \rightarrow (2, 1, 4)$

$$0.192 \not> 0.465 + 0.142$$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



We would start:

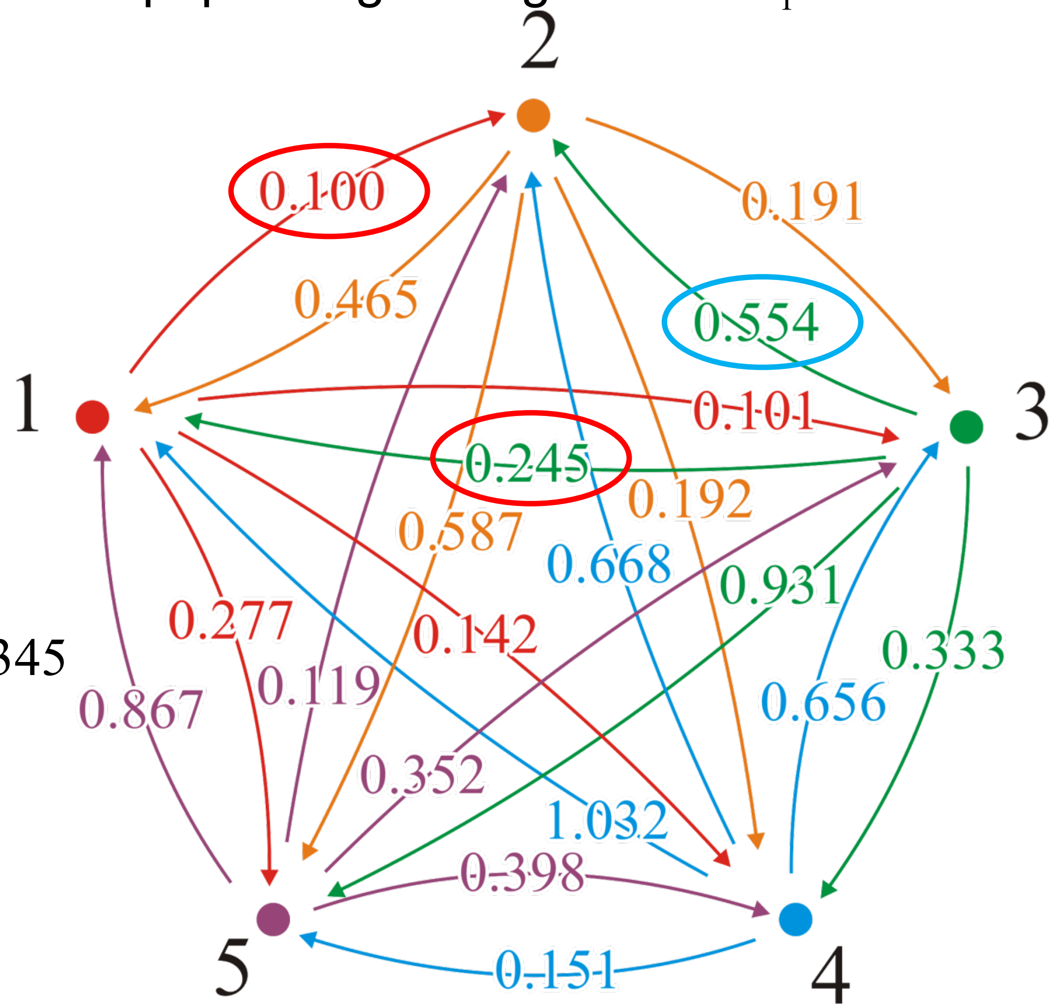
$(2, 5) \to (2, 1, 5)$

$$0.587 \not> 0.465 + 0.277$$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.554 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



Here is a shorter path:

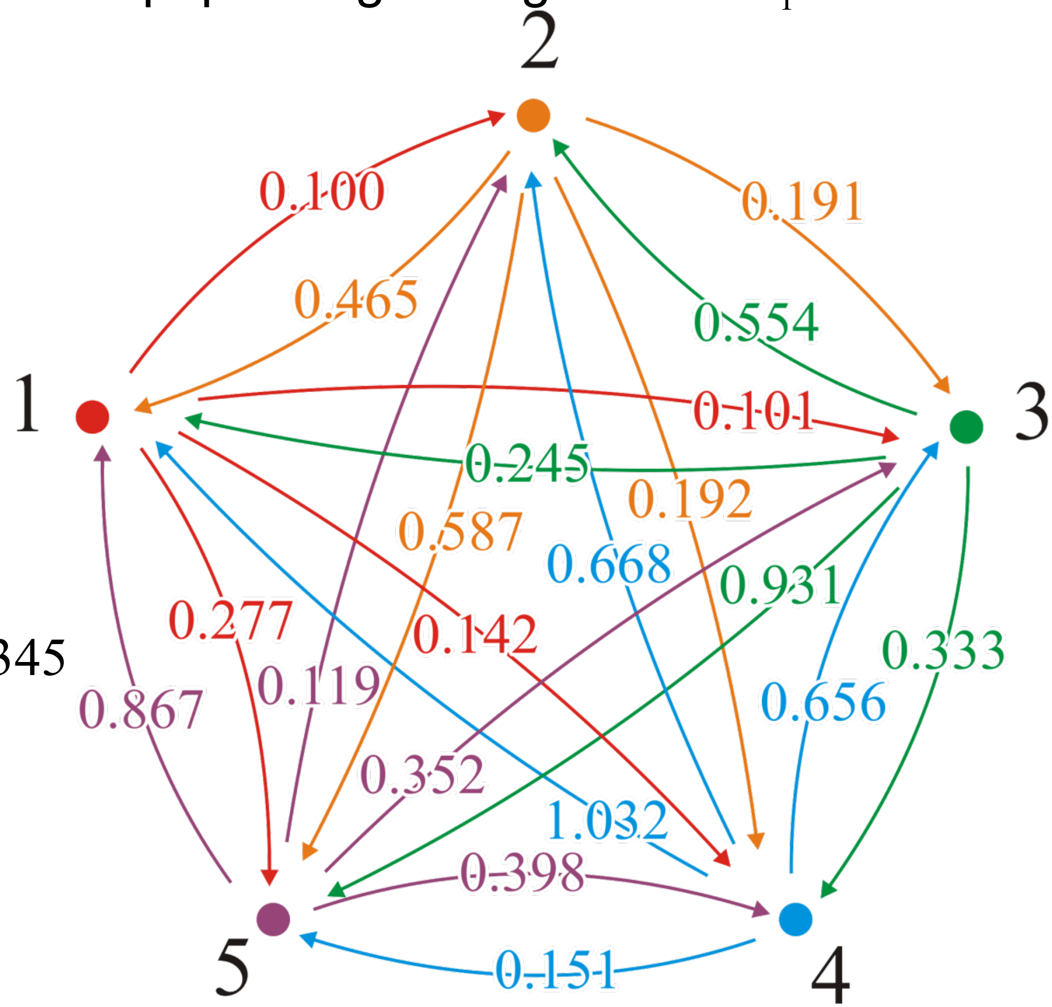
$(3, 2) \to (3, 1, 2)$

$0.554 > 0.245 + 0.100 = 0.345$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



We update the table

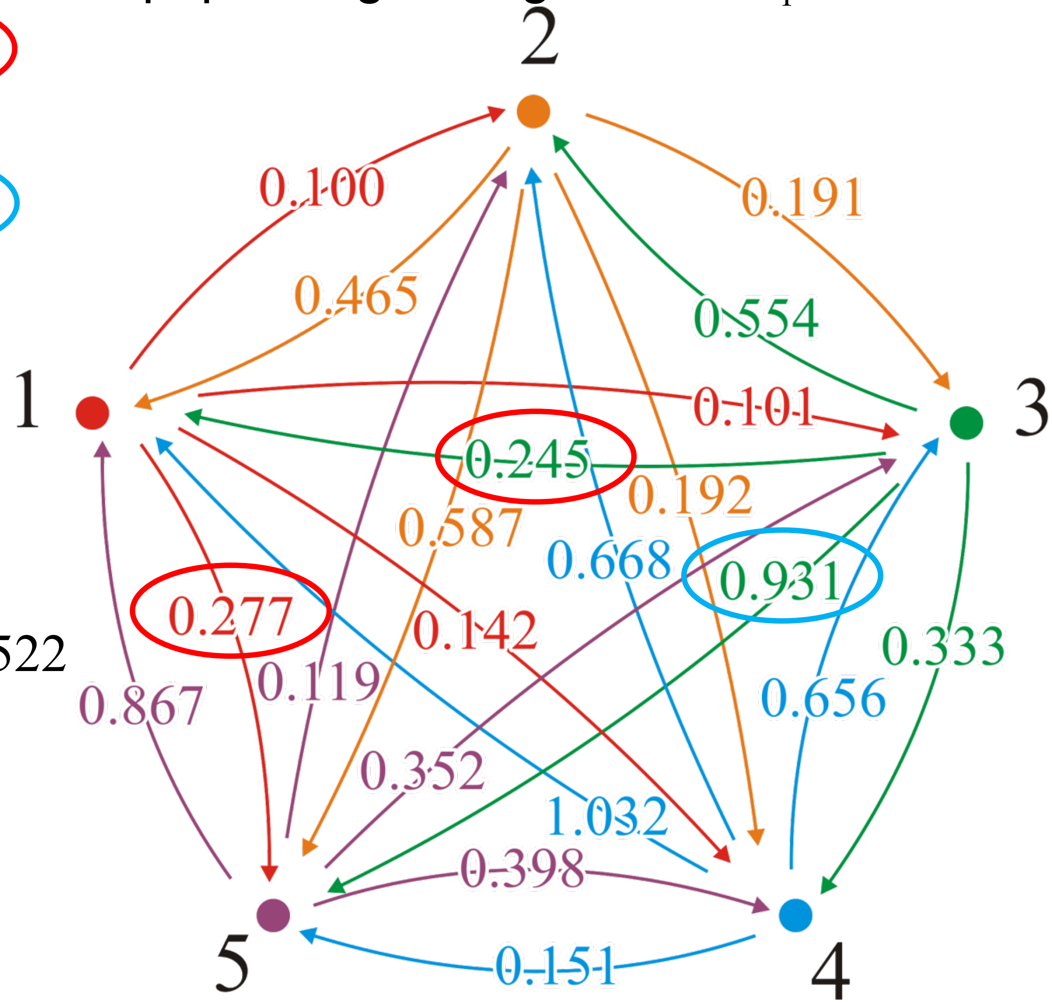
$(3, 2) \rightarrow (3, 1, 2)$

$0.554 > 0.245 + 0.100 = 0.345$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.931 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



And a second shorter path:

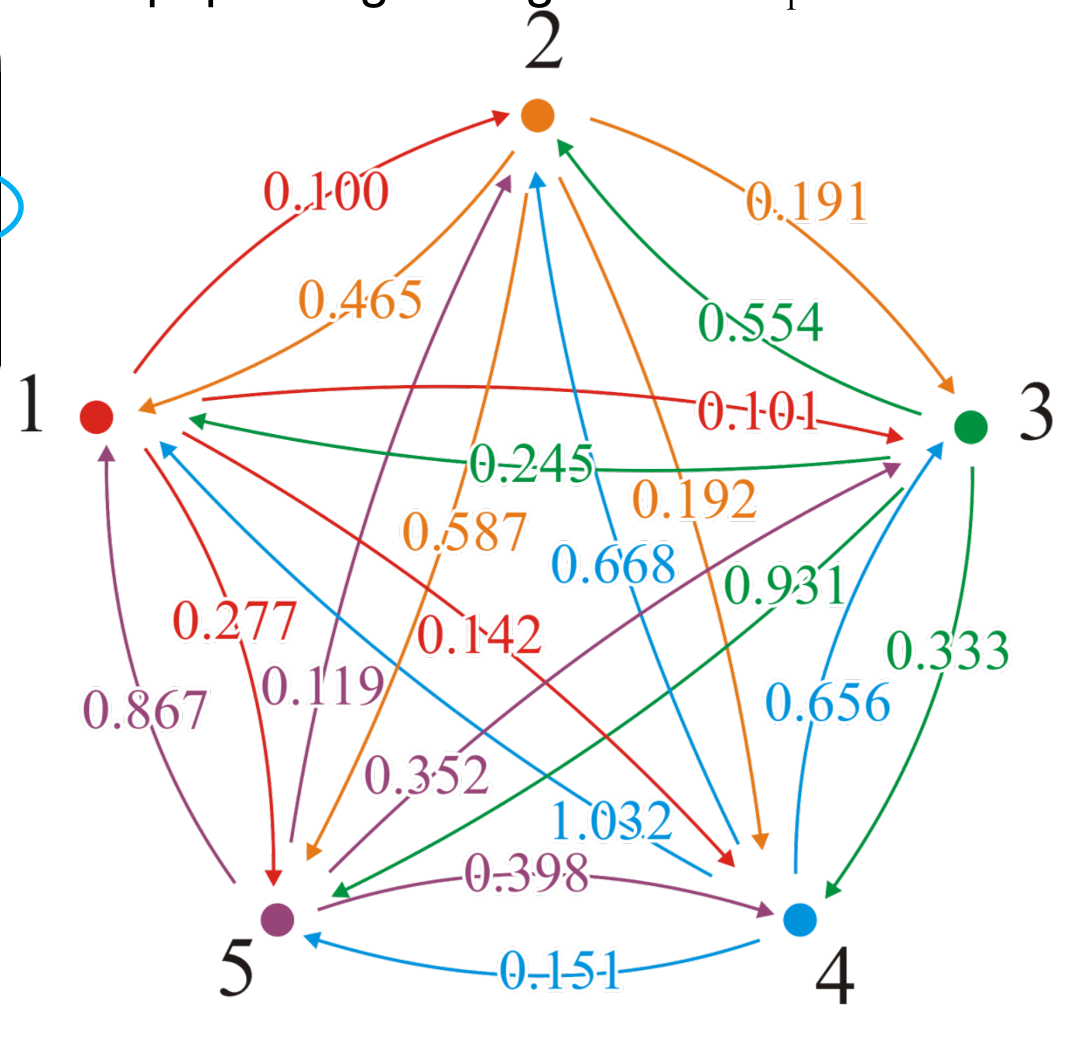
$(3, 5) \to (3, 1, 5)$

$$0.931 > 0.245 + 0.277 = 0.522$$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$

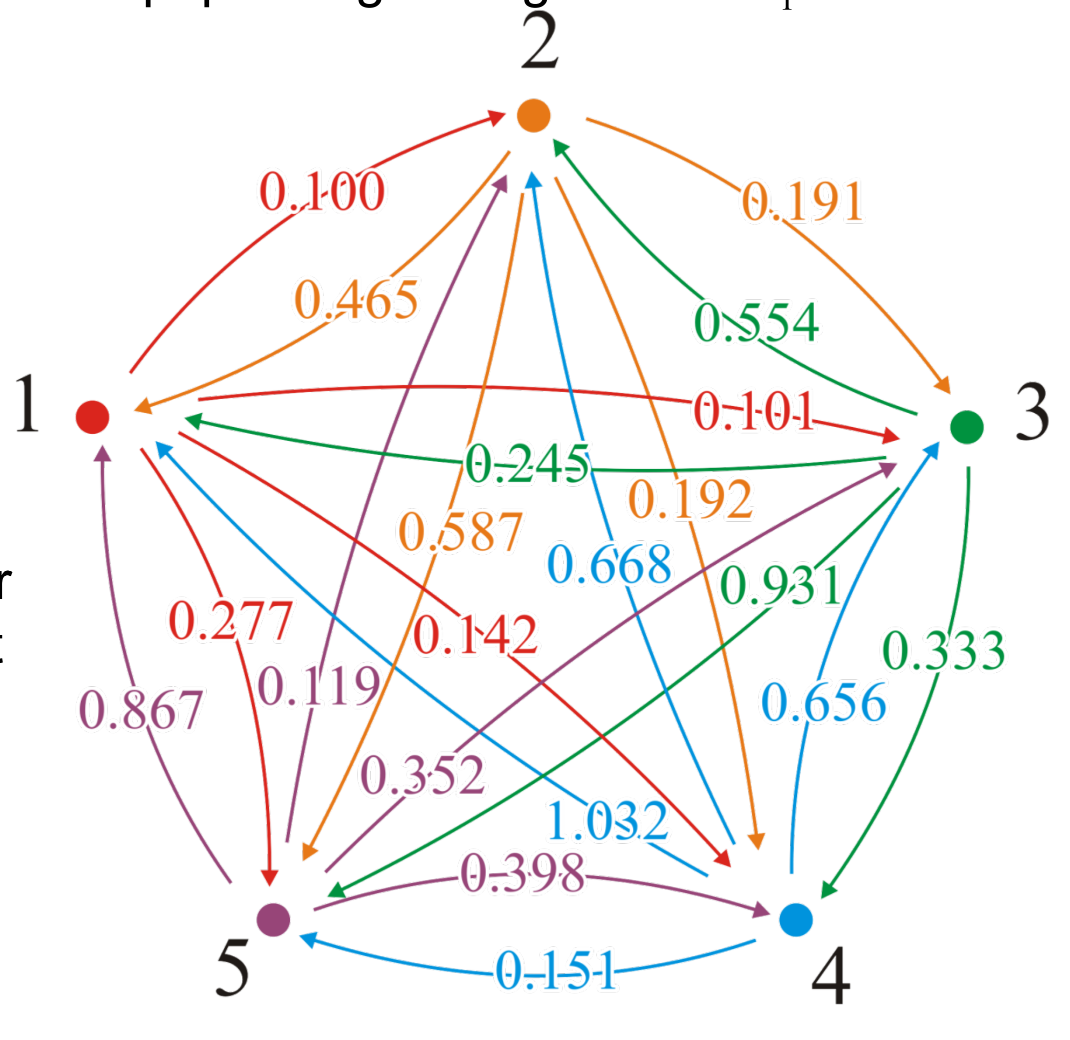


We update the table

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$



Continuing…
We find that no other shorter paths through vertex v_1 exist

Example

With the next pass, $k = 2$, we attempt passing through vertex v_2

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.867 & 0.119 & 0.352 & 0.398 & 0 \end{pmatrix}$$

There are three shorter paths:

$(5, 1) \rightarrow (5, 2, 1)$

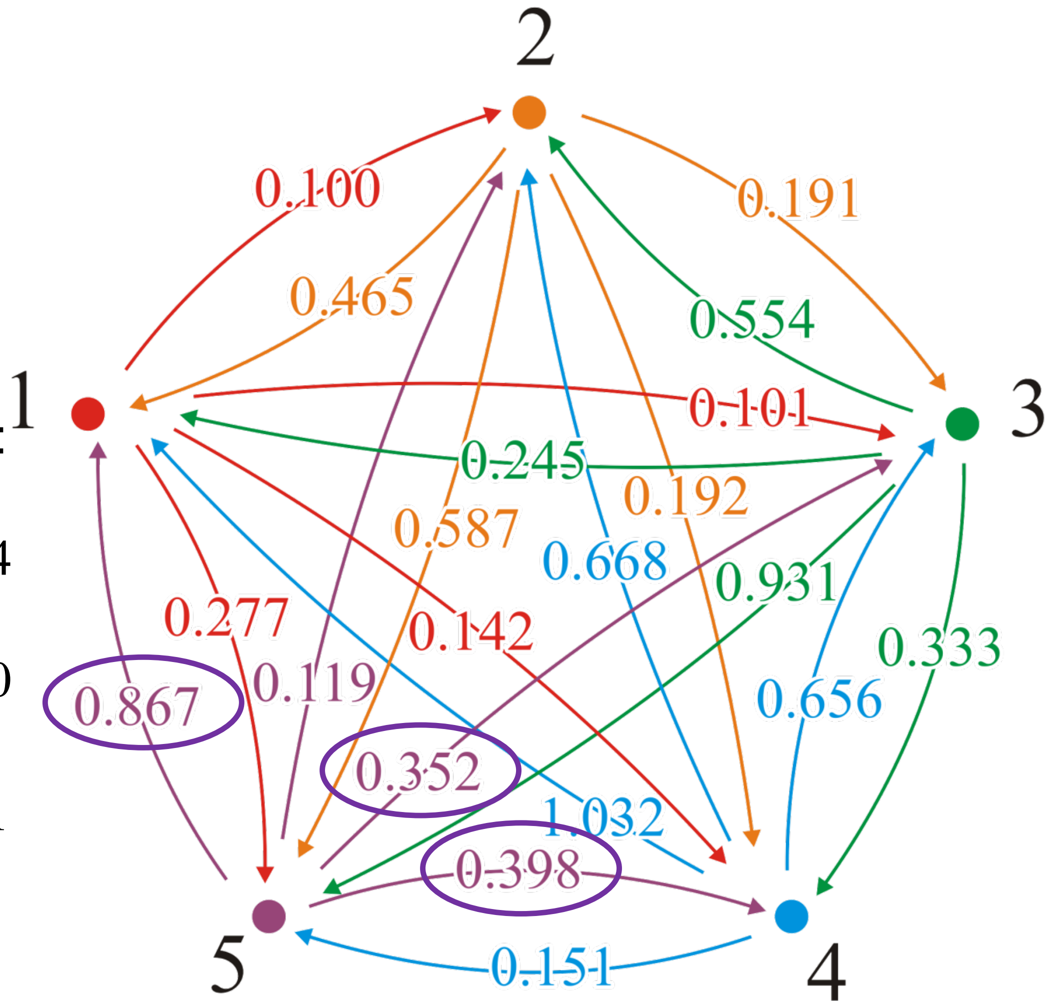
$0.867 > 0.119 + 0.465 = 0.584$

$(5, 3) \rightarrow (5, 2, 3)$

$0.352 > 0.119 + 0.191 = 0.310$

$(5, 4) \rightarrow (5, 2, 4)$

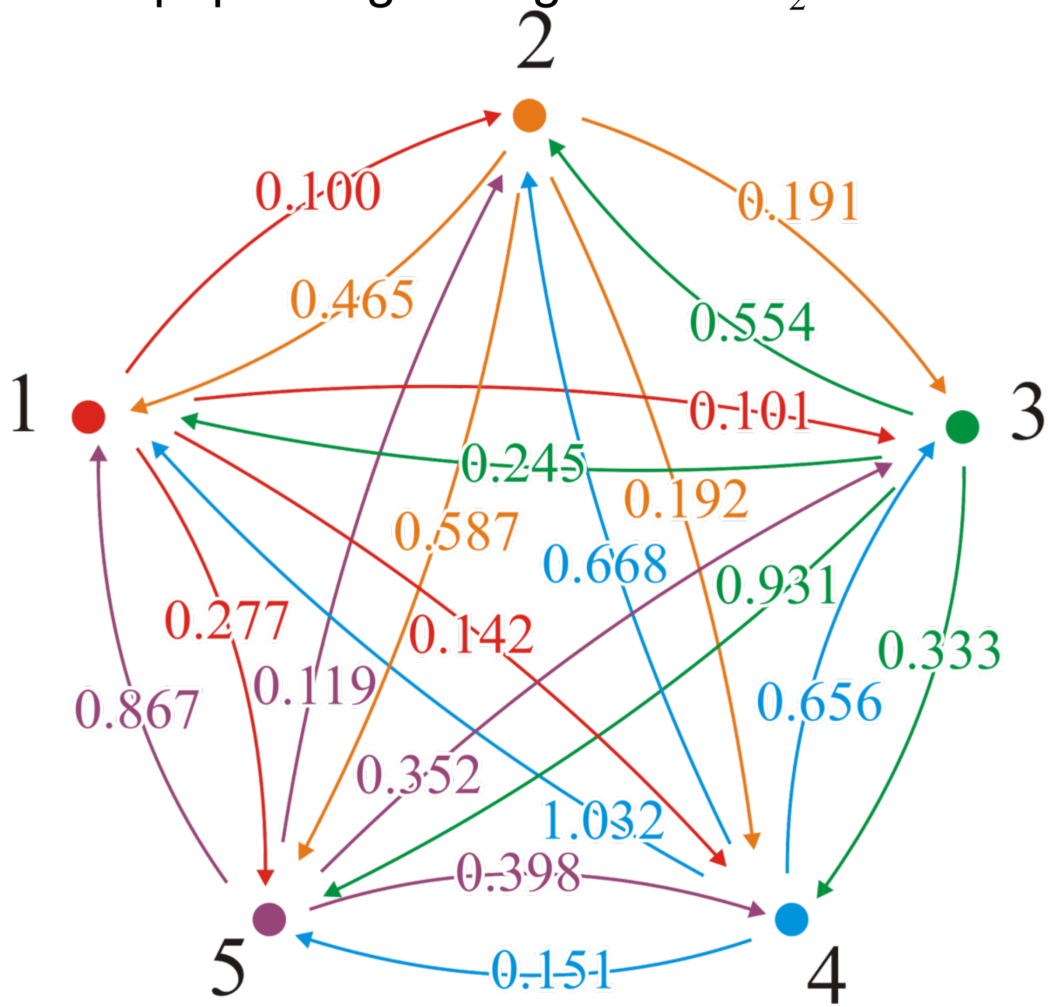
$0.398 > 0.119 + 0.192 = 0.311$



Example

With the next pass, $k = 2$, we attempt passing through vertex v_2

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.584 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$



We update the table

Example

With the next pass, $k = 3$, we attempt passing through vertex v_3

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.465 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 1.032 & 0.668 & 0.656 & 0 & 0.151 \\ 0.584 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$

There are three shorter paths:

$(2, 1) \rightarrow (2, 3, 1)$

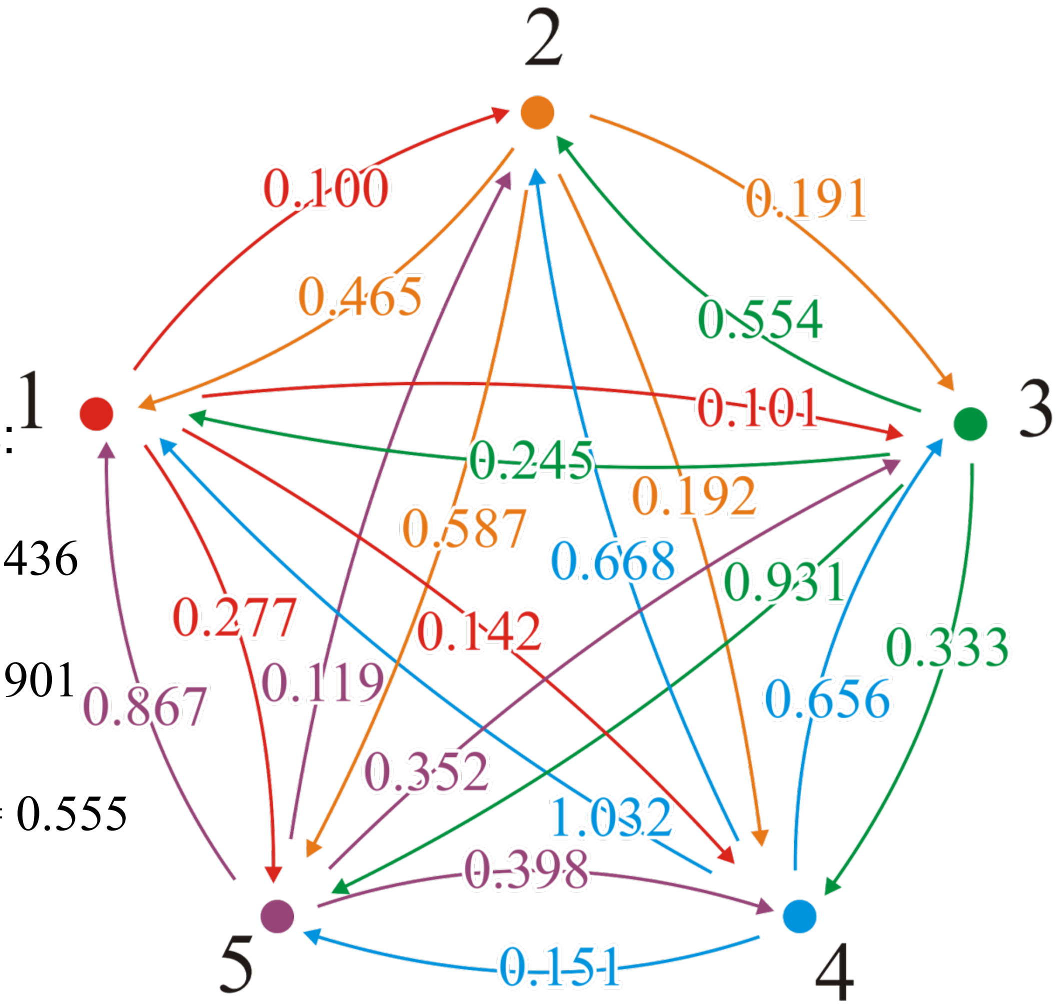
$0.465 > 0.191 + 0.245 = 0.436$

$(4, 1) \rightarrow (4, 3, 1)$

$1.032 > 0.656 + 0.245 = 0.901$

$(5, 1) \rightarrow (5, 3, 1)$

$0.584 > 0.310 + 0.245 = 0.555$

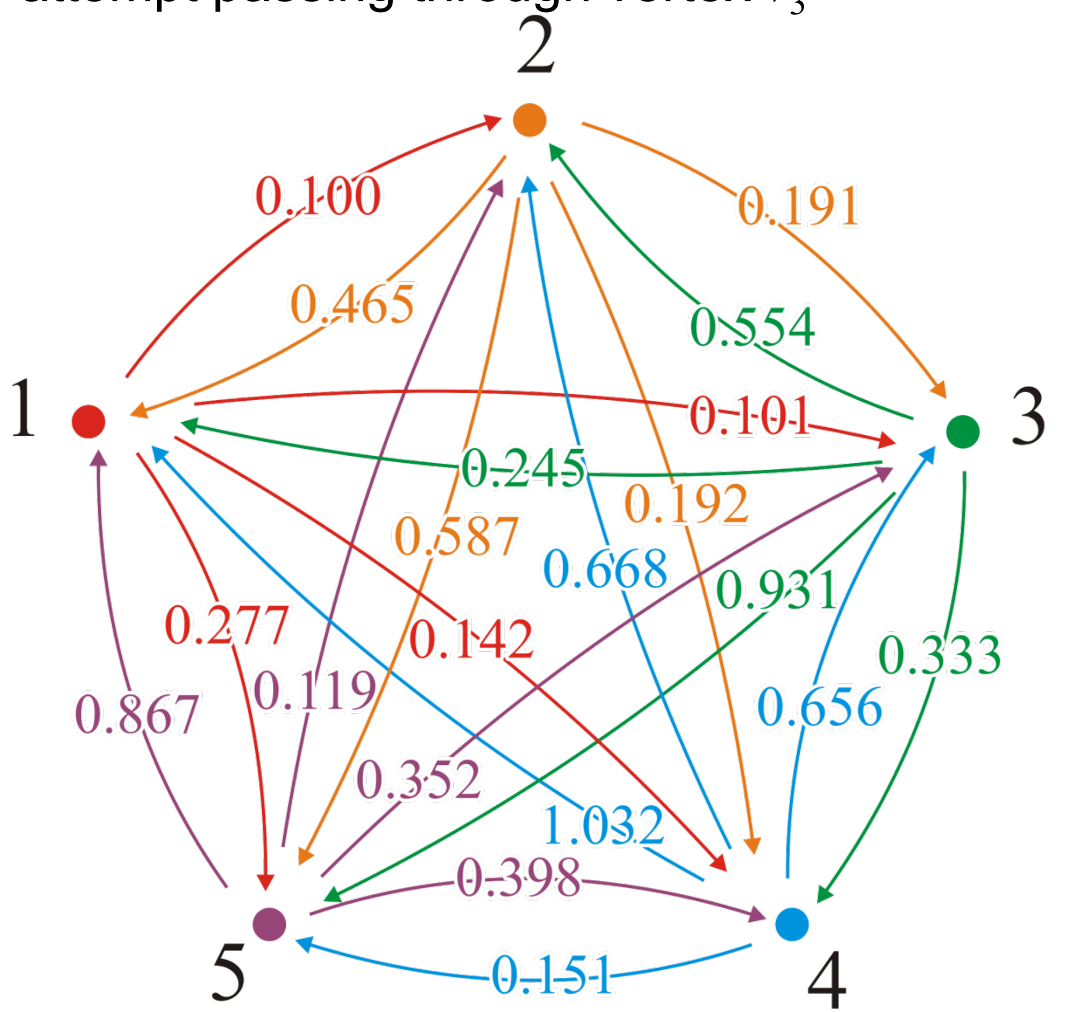


Example

With the next pass, $k = 3$, we attempt passing through vertex v_3

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 0.901 & 0.668 & 0.656 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$

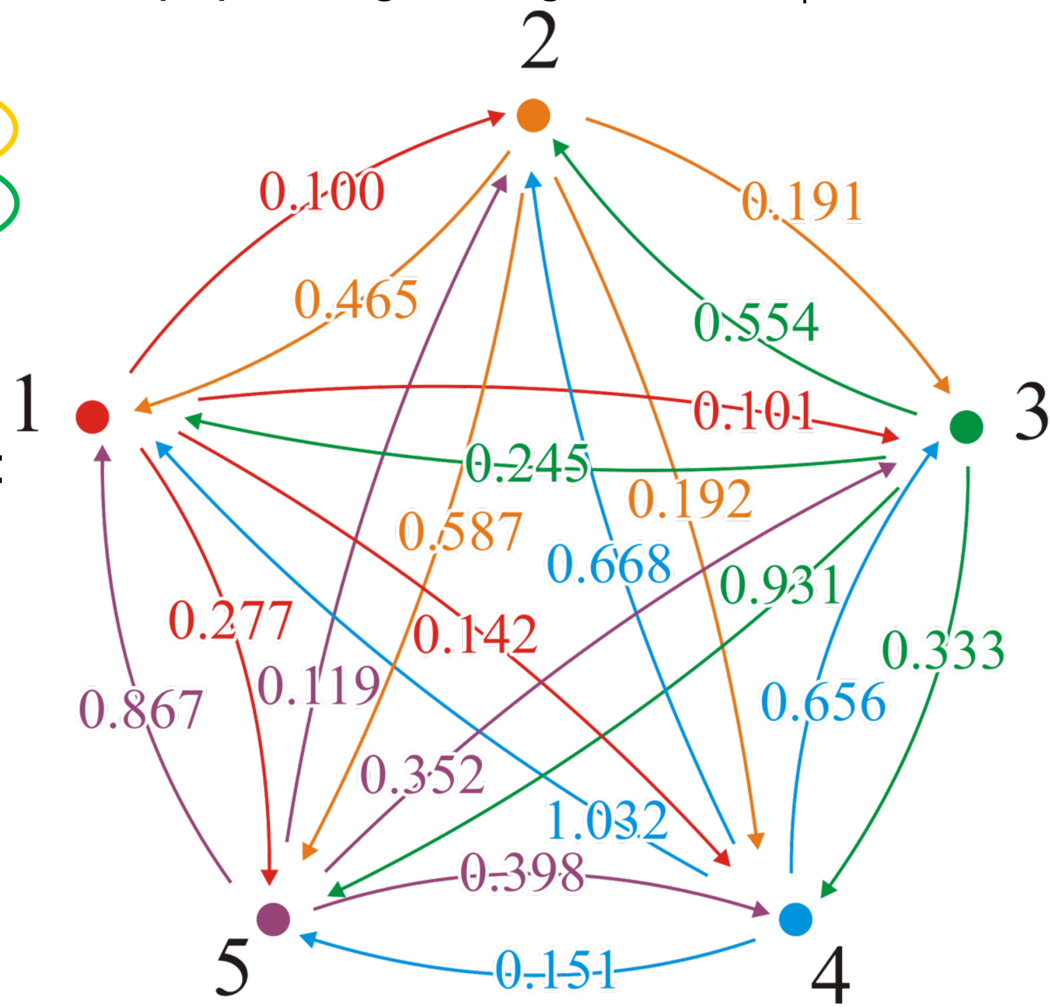
We update the table



Example

With the next pass, $k = 4$, we attempt passing through vertex v_4

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.587 \\ 0.245 & 0.345 & 0 & 0.333 & 0.522 \\ 0.901 & 0.668 & 0.656 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$



There are two shorter paths:

$(2, 5) \rightarrow (2, 4, 5)$

$$0.587 > 0.192 + 0.151$$

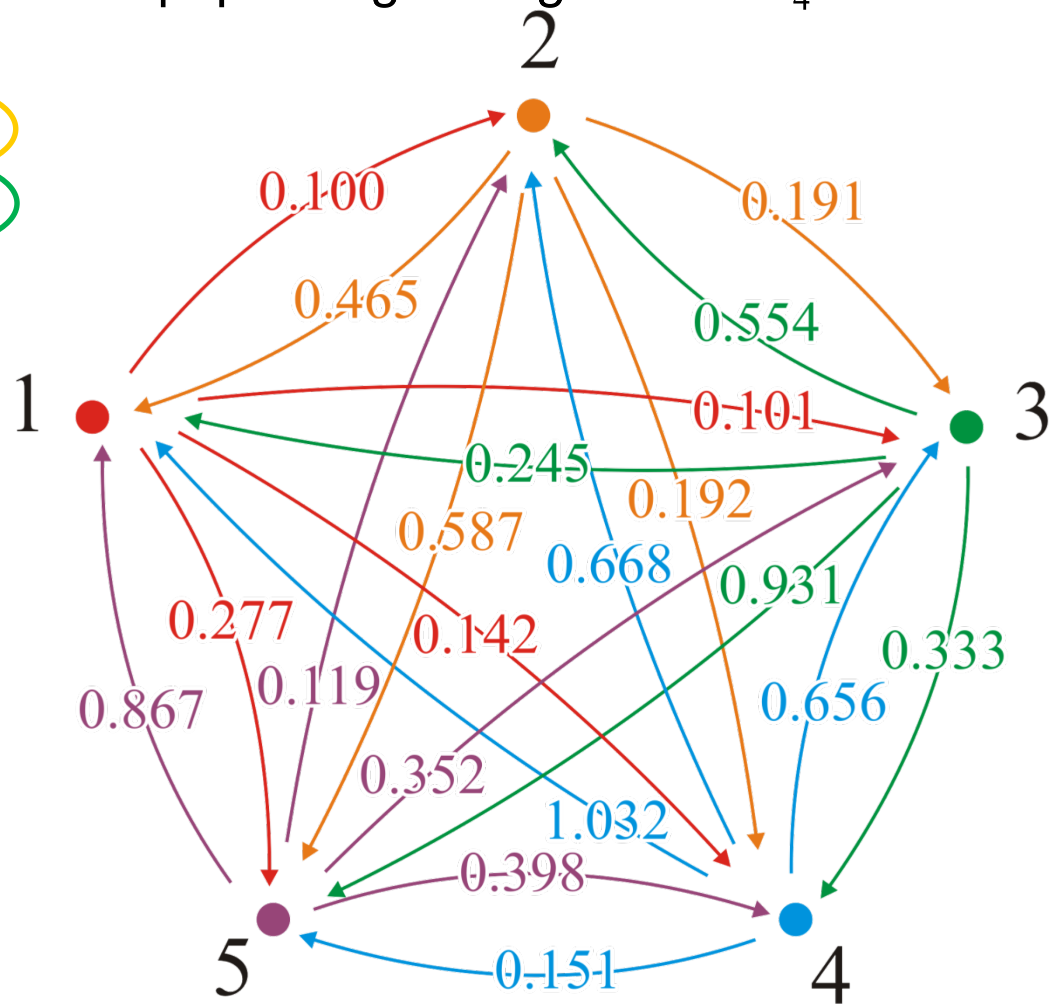
$(3, 5) \rightarrow (3, 4, 5)$

$$0.522 > 0.333 + 0.151$$

Example

With the next pass, $k = 4$, we attempt passing through vertex v_4

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.343 \\ 0.245 & 0.345 & 0 & 0.333 & 0.484 \\ 0.901 & 0.668 & 0.656 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$



We update the table

Example

With the last pass, $k = 5$, we attempt passing through vertex v_5

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.343 \\ 0.245 & 0.345 & 0 & 0.333 & 0.484 \\ 0.901 & 0.668 & 0.656 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$

There are three shorter paths:

$(4, 1) \to (4, 5, 1)$

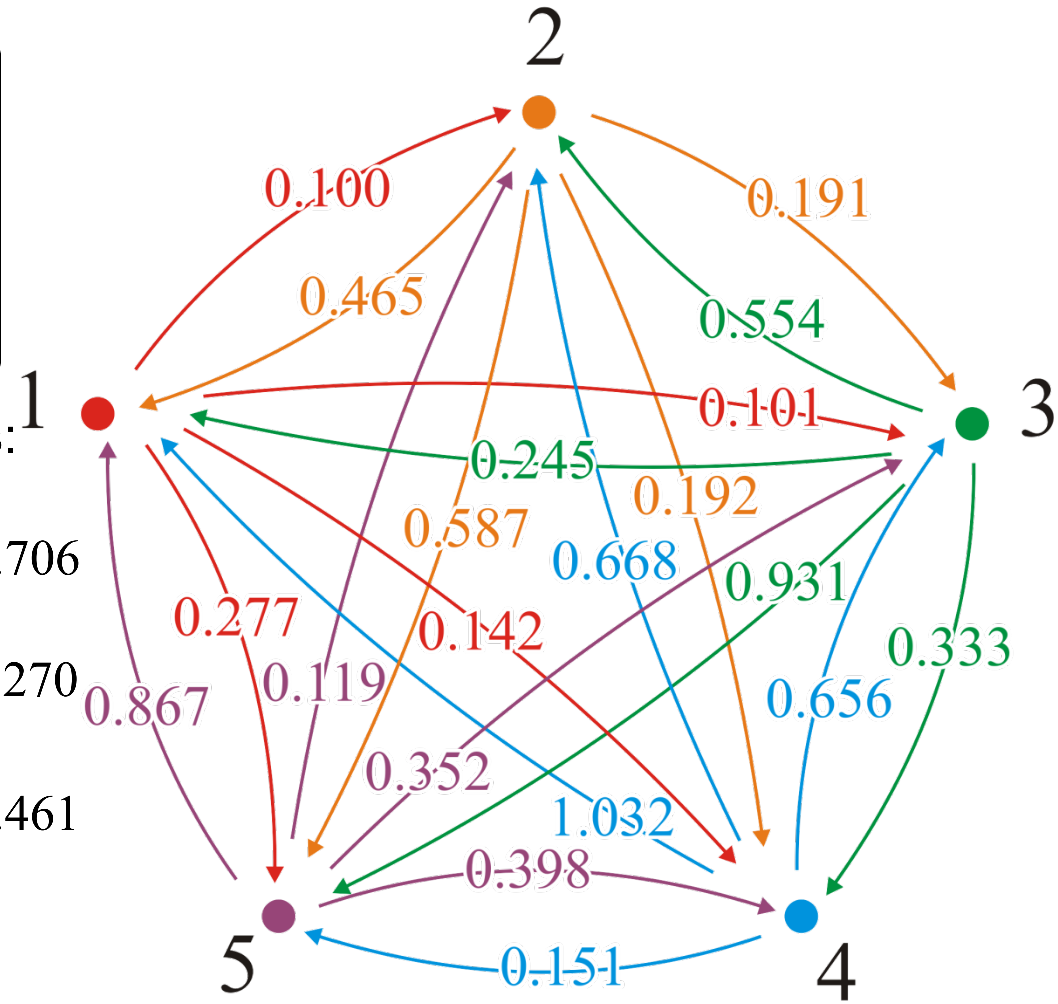
$$0.901 > 0.151 + 0.555 = 0.706$$

$(4, 2) \to (4, 5, 2)$

$$0.668 > 0.151 + 0.119 = 0.270$$

$(4, 3) \to (4, 5, 3)$

$$0.656 > 0.151 + 0.310 = 0.461$$

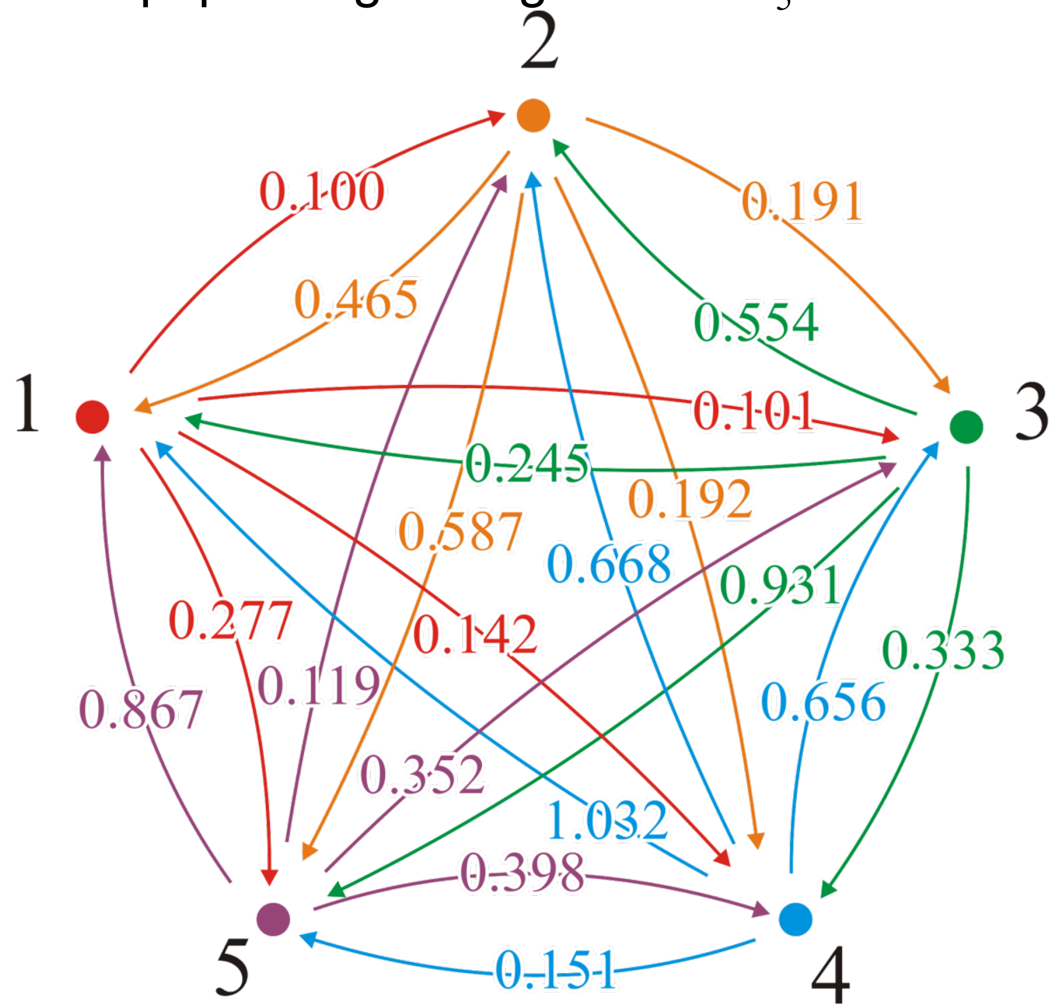


Example

With the last pass, $k = 5$, we attempt passing through vertex v_5

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.343 \\ 0.245 & 0.345 & 0 & 0.333 & 0.484 \\ 0.706 & 0.270 & 0.461 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$

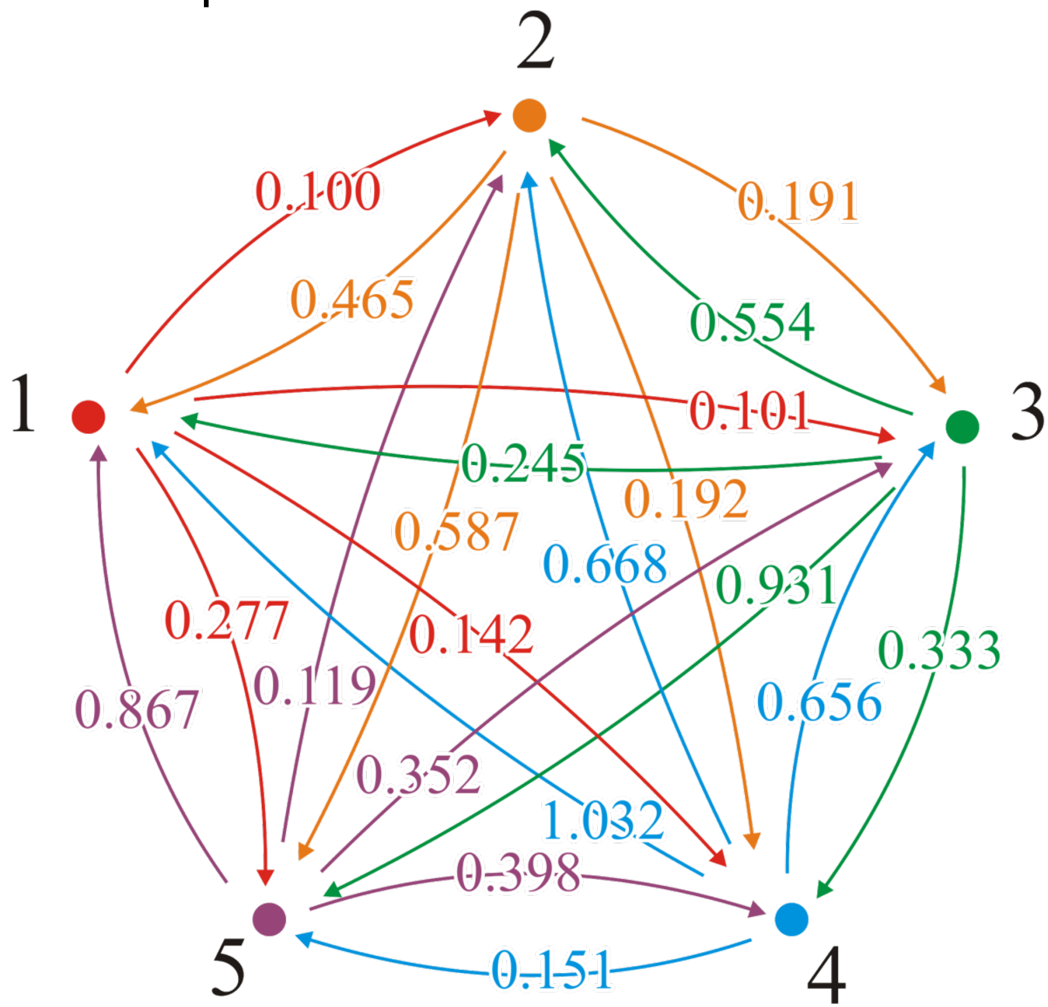
We update the table



Example

Thus, we have a table of all shortest paths:

$$\begin{pmatrix} 0 & 0.100 & 0.101 & 0.142 & 0.277 \\ 0.436 & 0 & 0.191 & 0.192 & 0.343 \\ 0.245 & 0.345 & 0 & 0.333 & 0.484 \\ 0.706 & 0.270 & 0.461 & 0 & 0.151 \\ 0.555 & 0.119 & 0.310 & 0.311 & 0 \end{pmatrix}$$



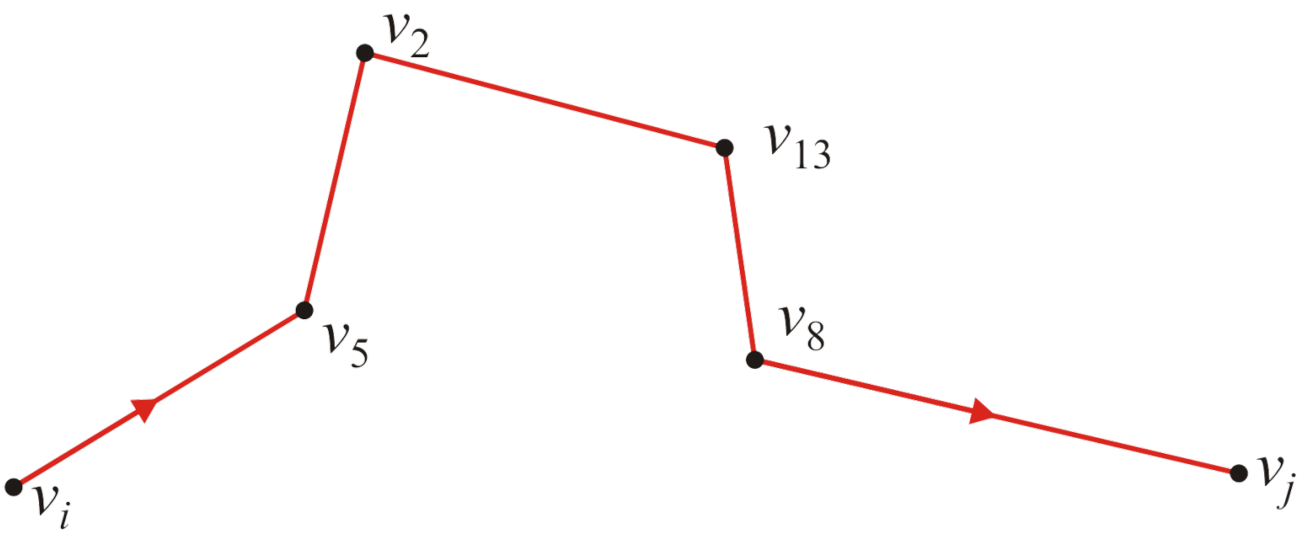
What Is the Shortest Path?

This algorithm finds the shortest distances, but what are the paths corresponding to those shortest distances?

- Recall that with Dijkstra's algorithm, we could find the shortest paths by recording the previous node
- Here we use a similar approach, but we choose to store the next node instead of the previous node

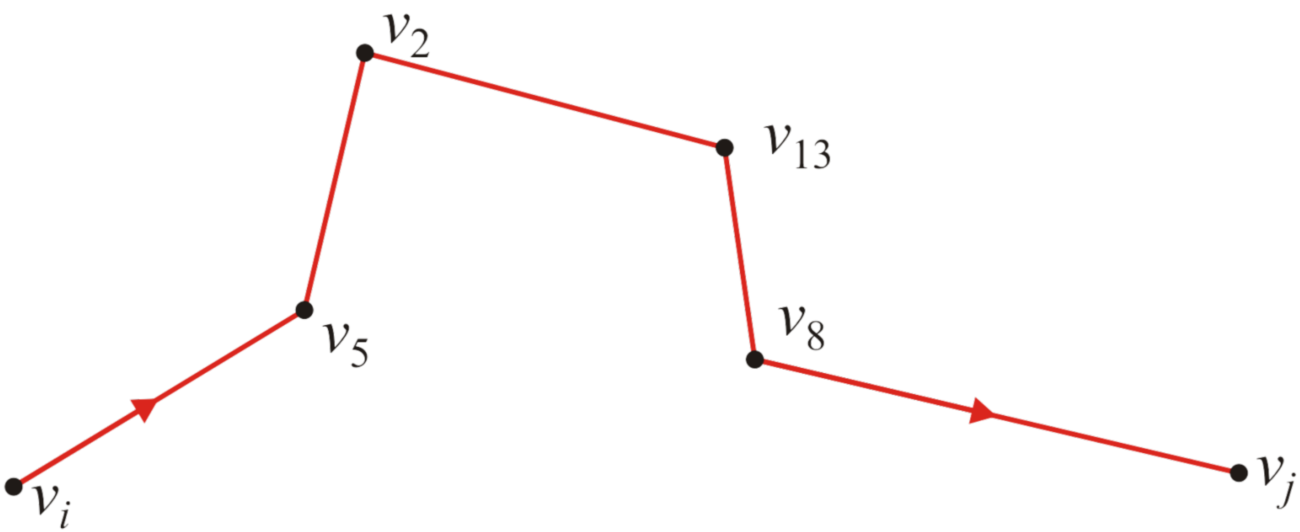
What Is the Shortest Path?

Suppose the shortest path from v_i to v_j is as follows:



What Is the Shortest Path?

Does this path consist of (v_i, v_5) and the shortest path from v_5 to v_j?

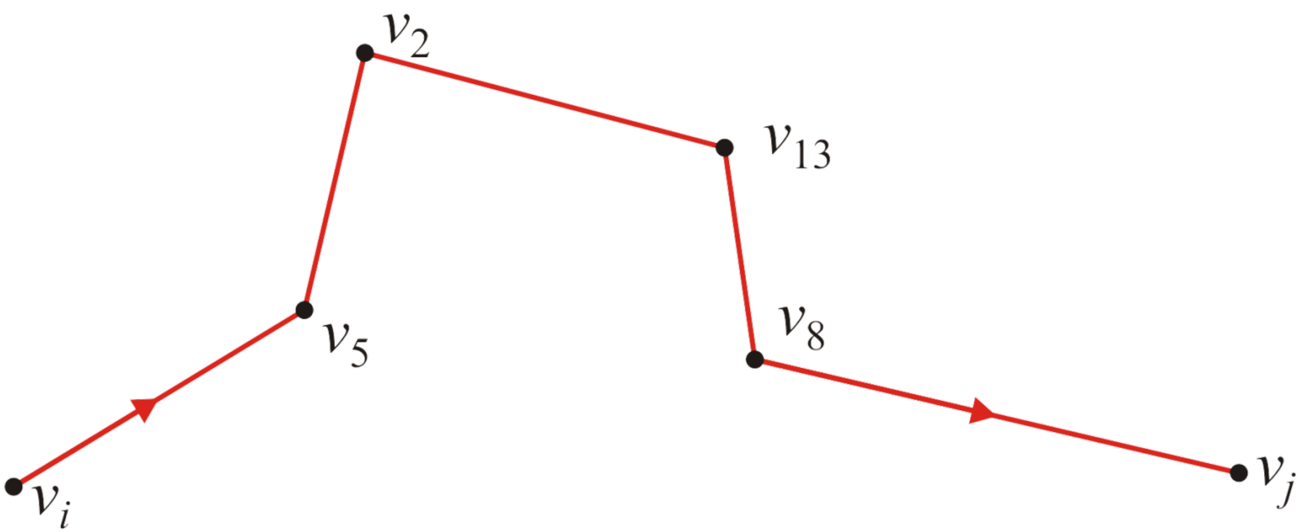


Yes

– If there was a shorter path from v_5 to v_j, then we would also find a shorter path from v_i to v_j

What Is the Shortest Path?

Does this path consist of (v_i, v_5) and the shortest path from v_5 to v_j?

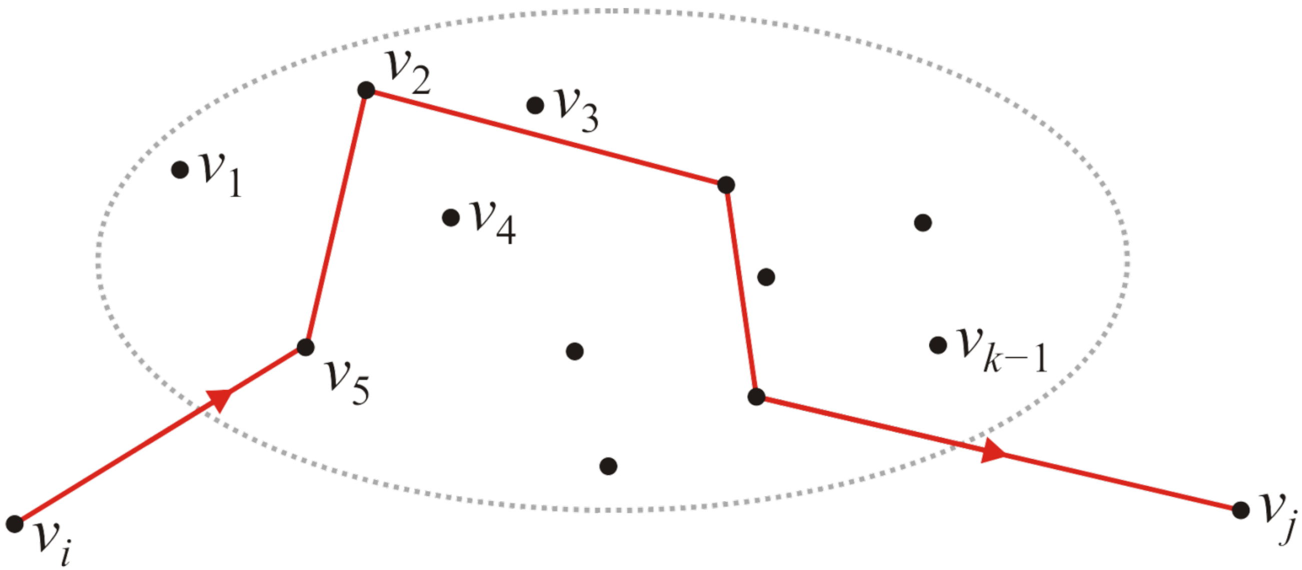


To find the shortest path from v_i to v_j, we only need to know that v_5 is the next vertex in the path — the rest of the path would be recursively recovered as the shortest path from v_5 to v_j

What Is the Shortest Path?

Now, suppose we have the shortest path from v_i to v_j which passes through the vertices $v_1, v_2, \ldots, v_{k-1}$

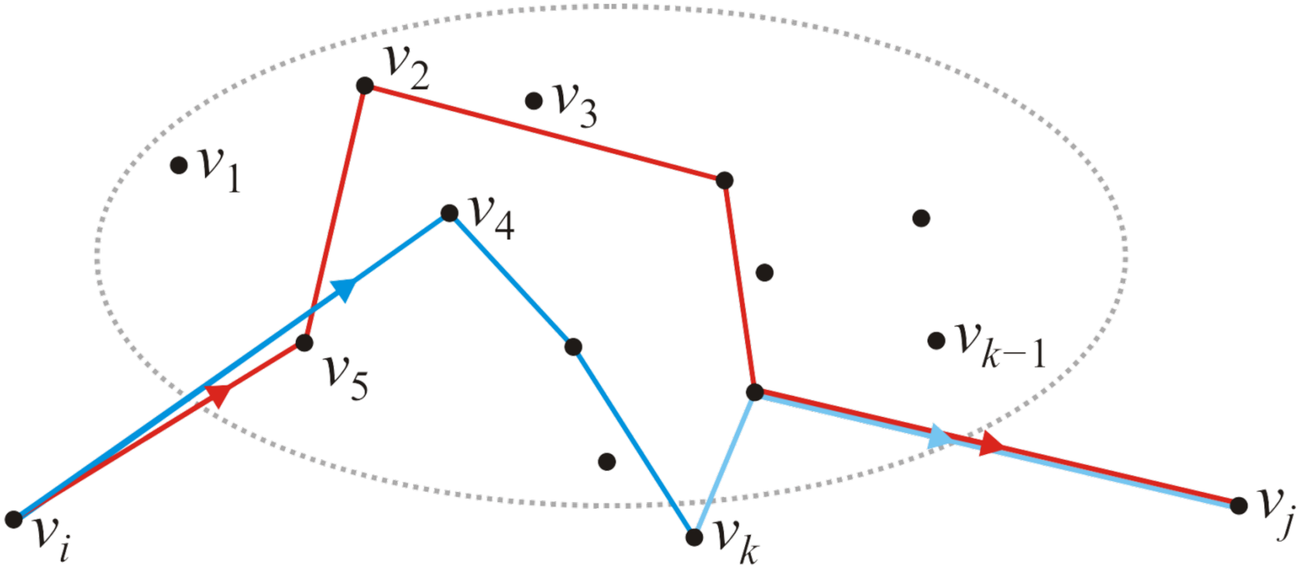
– In this example, the next vertex in the path is v_5



What Is the Shortest Path?

What if we find a shorter path passing through v_k?

– Now the next vertex in the new path should be the next vertex in the shortest path from v_i to v_k , which is v_4 in this example



What Is the Shortest Path?

Let us store the next vertex in the shortest path. Initially:

$$p_{i,j} = \begin{cases} \varnothing & \text{If } i = j \\ j & \text{If there is an edge from } i \text{ to } j \\ \varnothing & \text{Otherwise} \end{cases}$$

What Is the Shortest Path?

When we find a shorter path, update the next node:

$$p_{i,j} = p_{i,k}$$

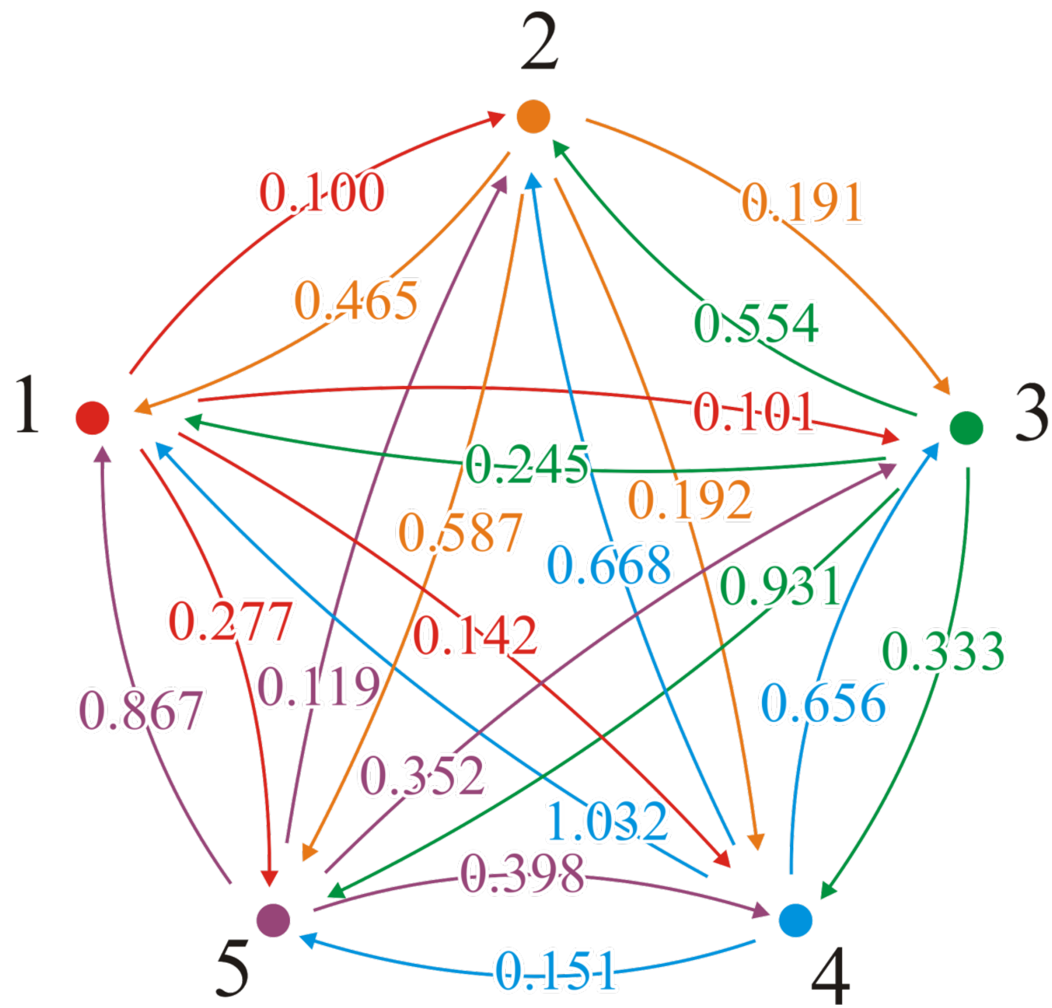
```
// Initialize the matrix p
//   ...

for ( int k = 0; k < num_vertices; ++k ) {
    for ( int i = 0; i < num_vertices; ++i ) {
        for ( int j = 0; j < num_vertices; ++j ) {
            if ( d[i][j] > d[i][k] + d[k][j] ) {
                p[i][j] = p[i][k];
                d[i][j] = d[i][k] + d[k][j];
            }
        }
    }
}
```

Example

In our original example, initially, the next node is exactly that:

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 1 & - & 3 & 4 & 5 \\ 1 & 2 & - & 4 & 5 \\ 1 & 2 & 3 & - & 5 \\ 1 & 2 & 3 & 4 & - \end{pmatrix}$$

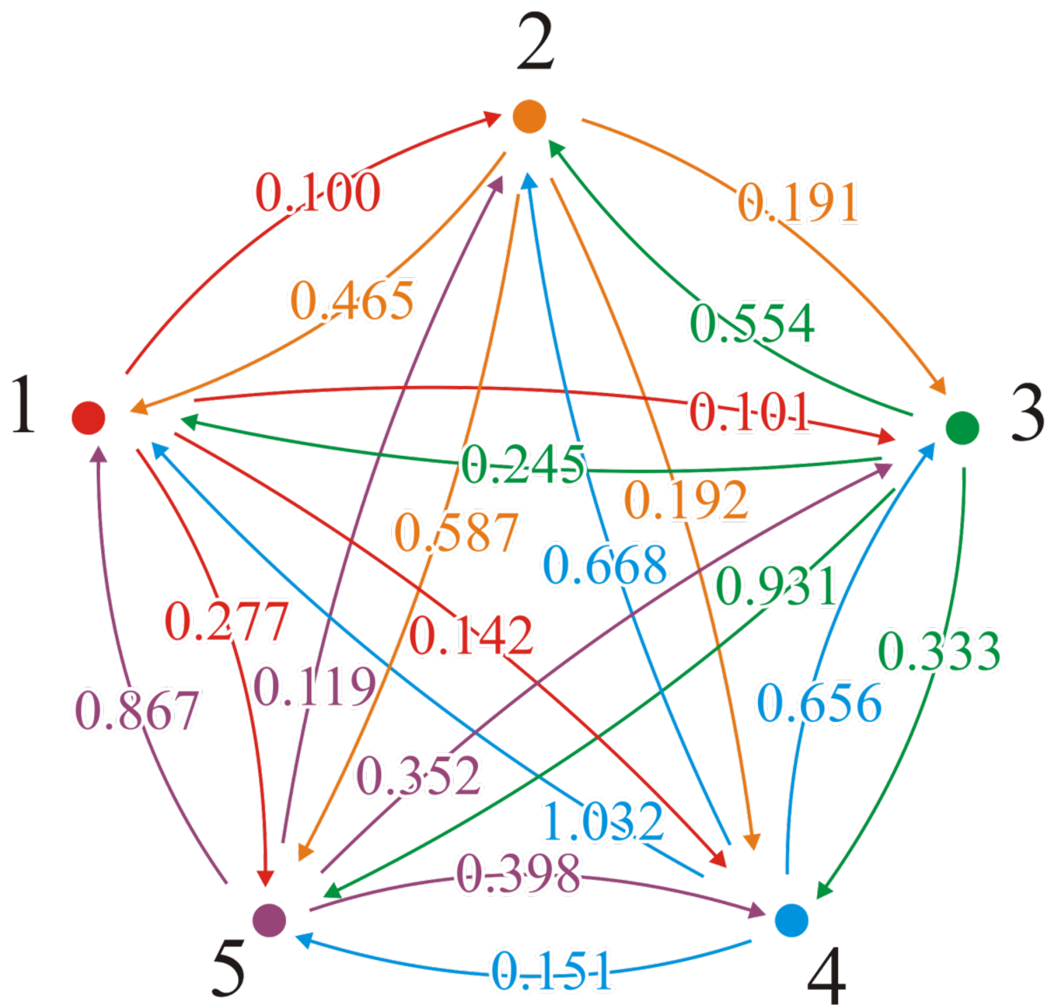
This would define our matrix $\mathbf{P} = (p_{ij})$



Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 1 & - & 3 & 4 & 5 \\ 1 & 2 & - & 4 & 5 \\ 1 & 2 & 3 & - & 5 \\ 1 & 2 & 3 & 4 & - \end{pmatrix}$$



There are two shorter paths:

$(3, 2) \rightarrow (3, 1, 2)$

$$0.554 > 0.245 + 0.100$$

$(3, 5) \rightarrow (3, 1, 5)$

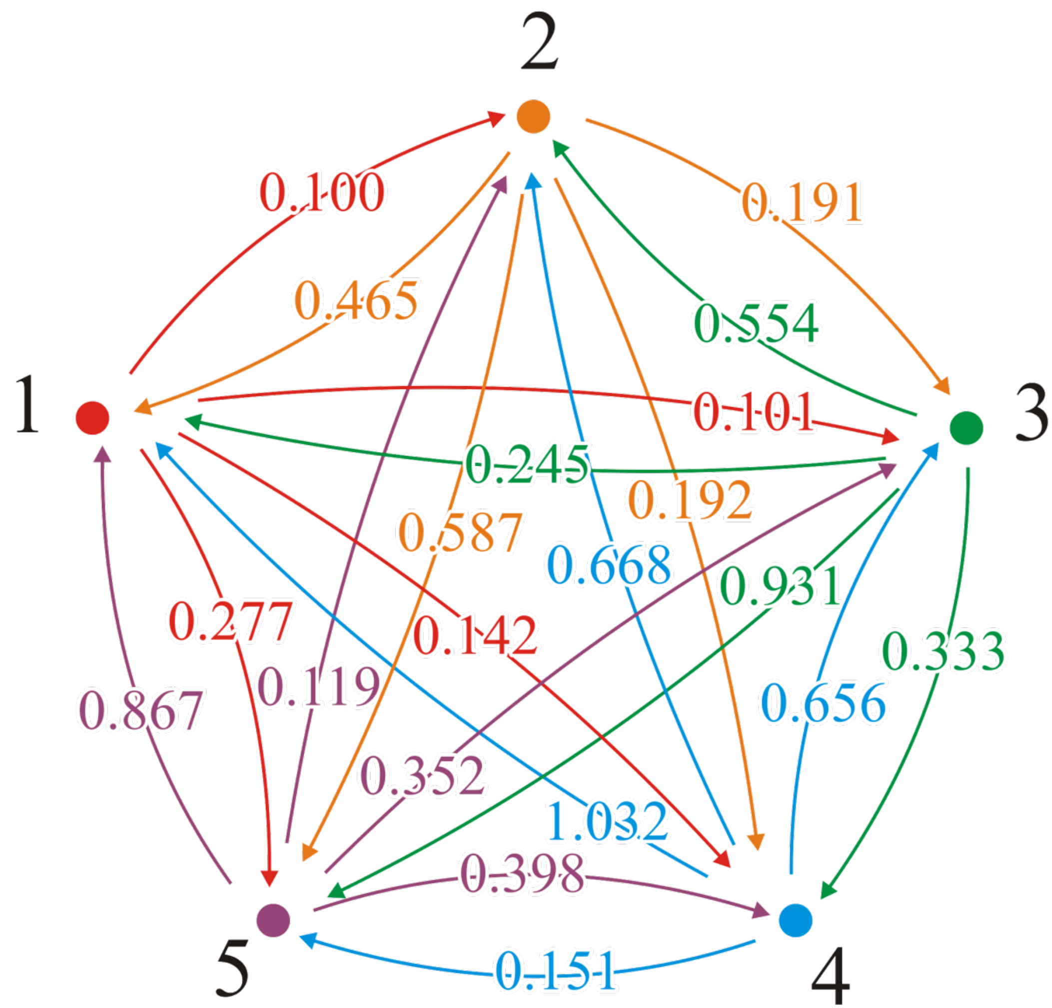
$$0.931 > 0.245 + 0.277$$

Example

With the first pass, $k = 1$, we attempt passing through vertex v_1

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 1 & - & 3 & 4 & 5 \\ 1 & 1 & - & 4 & 1 \\ 1 & 2 & 3 & - & 5 \\ 1 & 2 & 3 & 4 & - \end{pmatrix}$$

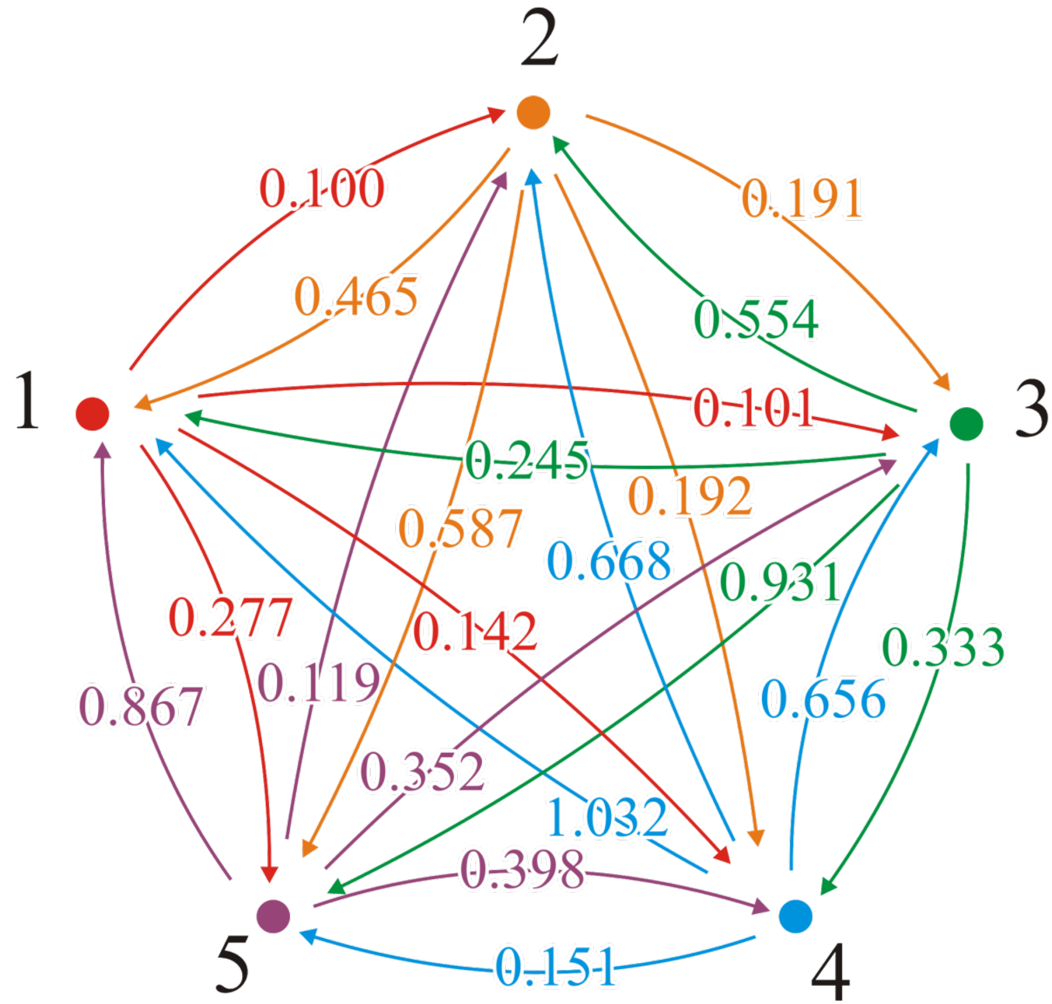
We update each of these



Example

After all the steps, we end up with the matrix $\boldsymbol{P} = (p_{i,j})$:

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$



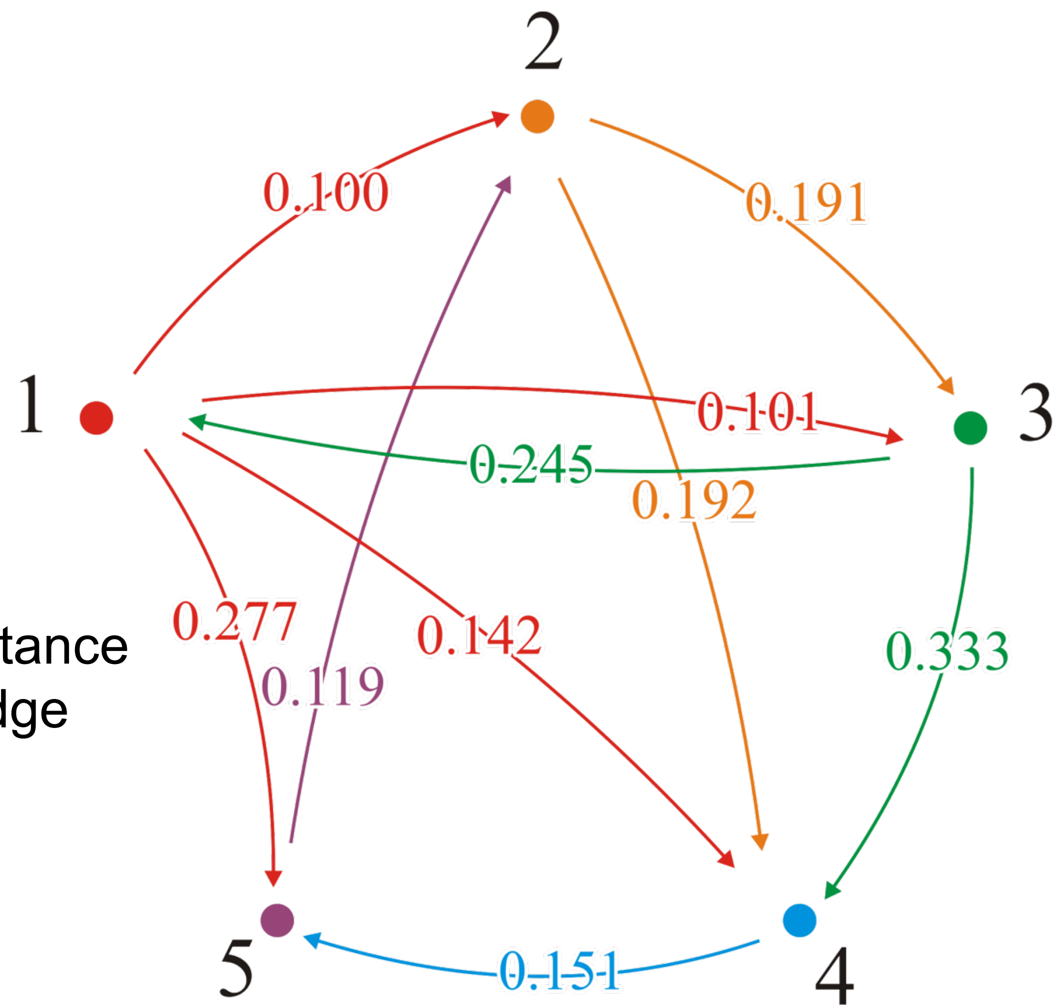
Example

These are all the adjacent edges that are still the shortest distance

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$

For each of these, $p_{i,j} = j$

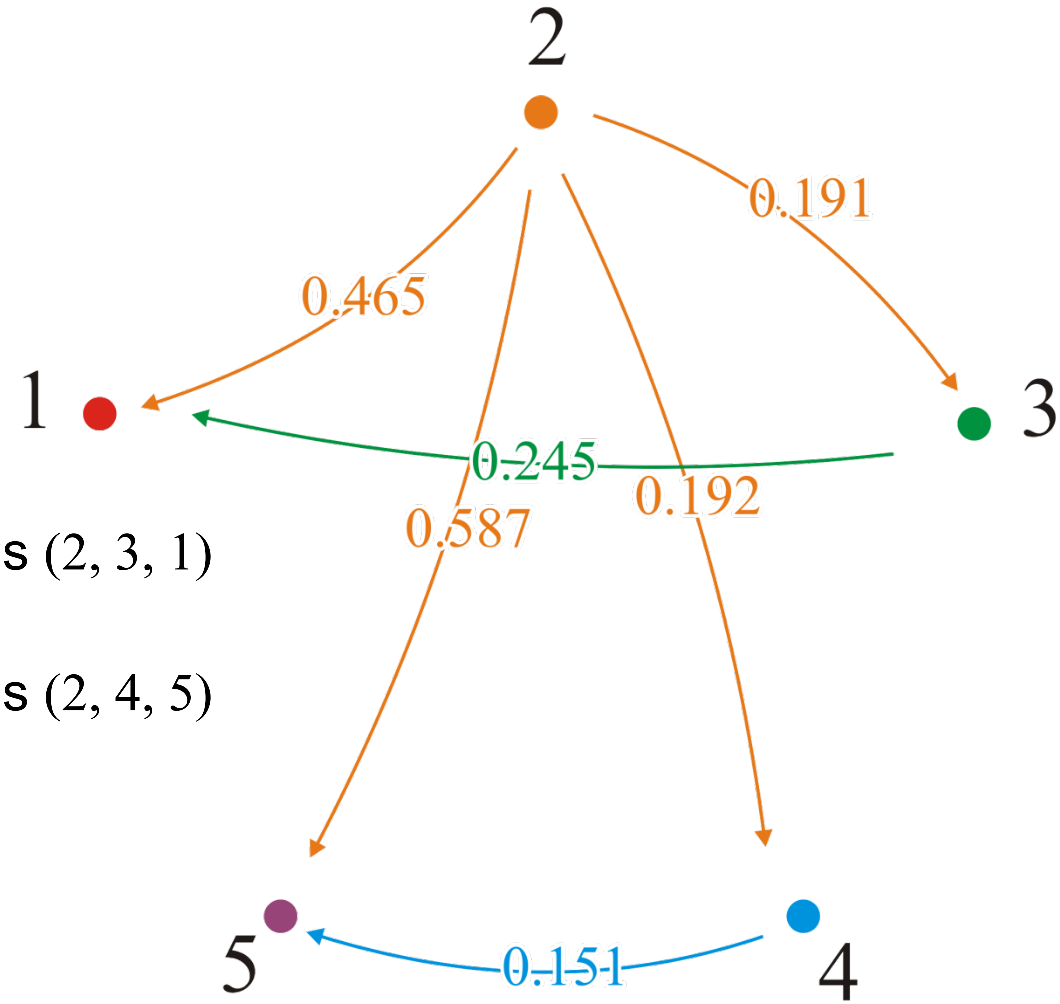
In all cases, the shortest distance from vertex 1 is the direct edge



Example

From vertex v_2, $p_{2,3} = 3$ and $p_{2,4} = 4$; we go directly to vertices v_3 and v_4

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$



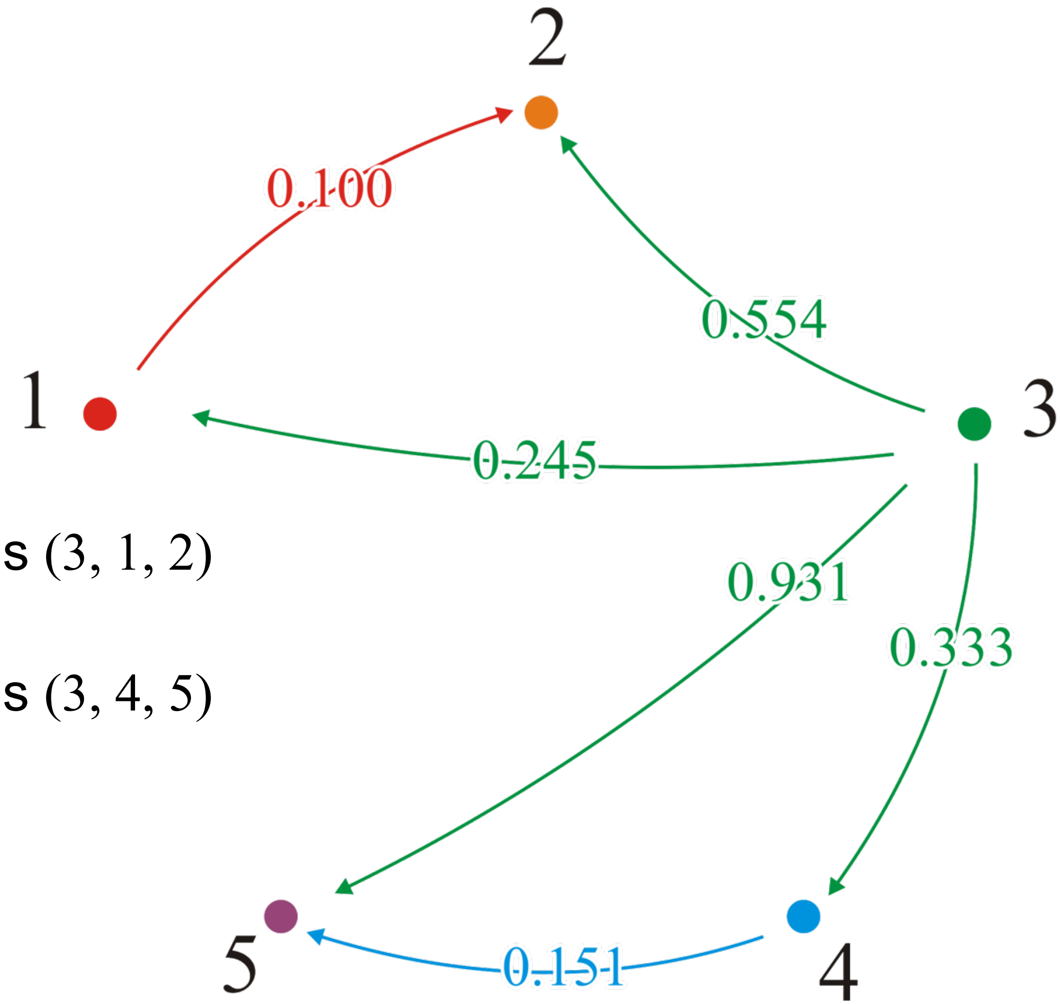
But $p_{2,1} = 3$ and $p_{3,1} = 1$;
the shortest path to v_1 is $(2, 3, 1)$

Also, $p_{2,5} = 4$ and $p_{4,5} = 5$;
the shortest path to v_5 is $(2, 4, 5)$

Example

From vertex v_3, $p_{3,1} = 1$ and $p_{3,4} = 4$; we go directly to vertices v_1 and v_4

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$



But $p_{3,2} = 1$ and $p_{1,2} = 2$;
the shortest path to v_2 is $(3, 1, 2)$

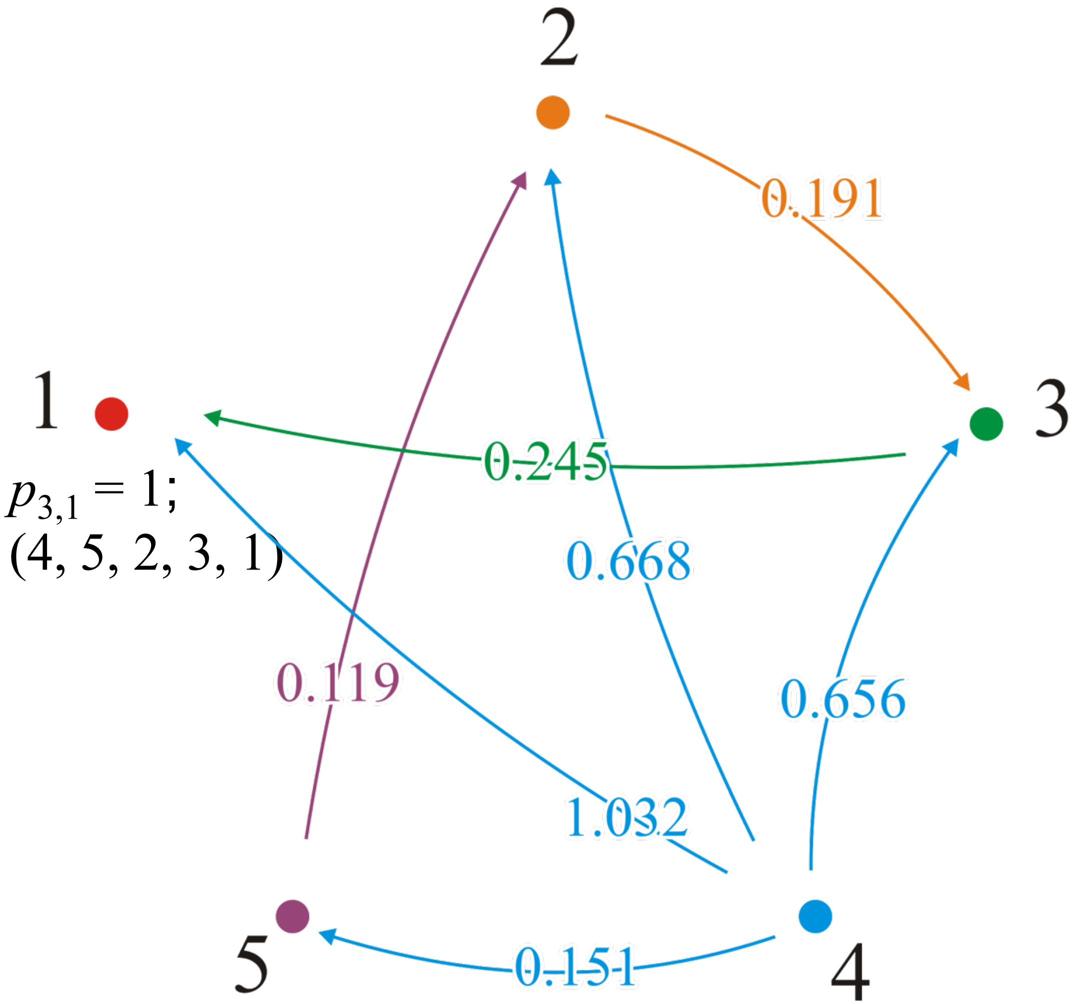
Also, $p_{3,5} = 4$ and $p_{4,5} = 5$;
the shortest path to v_5 is $(3, 4, 5)$

Example

From vertex v_4, $p_{4,5} = 5$; we go directly to vertex v_5

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$

But $p_{4,1} = 5, p_{5,1} = 2, p_{2,1} = 3, p_{3,1} = 1$;
the shortest path to v_1 is $(4, 5, 2, 3, 1)$

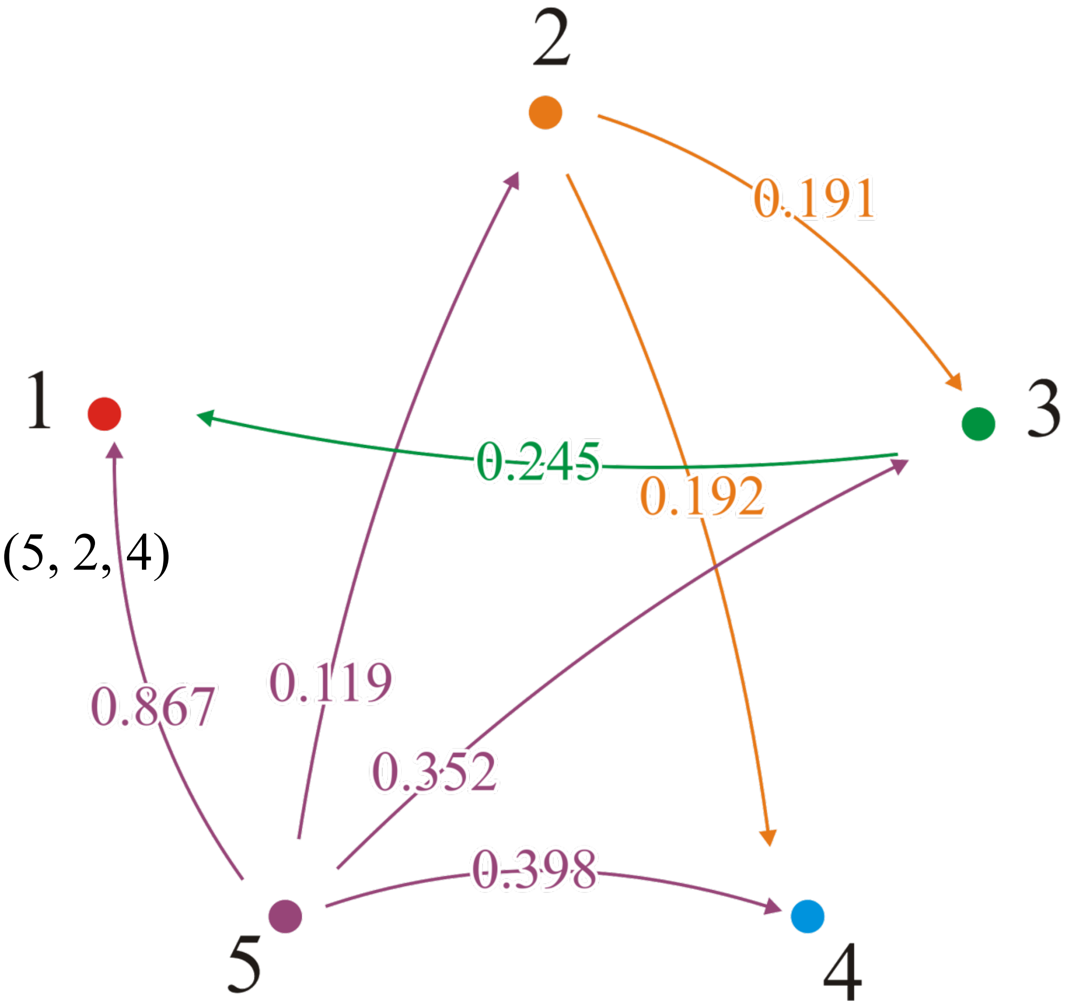


Example

From vertex v_5, $p_{5,2} = 2$; we go directly to vertex v_2

$$\begin{pmatrix} - & 2 & 3 & 4 & 5 \\ 3 & - & 3 & 4 & 4 \\ 1 & 1 & - & 4 & 4 \\ 5 & 5 & 5 & - & 5 \\ 2 & 2 & 2 & 2 & - \end{pmatrix}$$

But $p_{5,4} = 2$ and $p_{2,4} = 4$;
the shortest path to v_4 is $(5, 2, 4)$



Which Vertices are Connected?

Finally, what if we only care if a connection exists?

- Recall that with Dijkstra's algorithm, we could find the shortest paths by recording the previous node
- In this case, can make the observation that:

  A path from v_i to v_j exists if either:

  A path exists through the vertices from v_1 to v_{k-1}, or

  A path, through those same nodes, exists from v_i to v_k and a path exists from v_k to v_j

Which Vertices are Connected?

The *transitive closure* is a Boolean graph:

```
bool tc[num_vertices][num_vertices];

// Initialize the matrix tc:  Theta(|V|^2)
//   ...

// Run Floyd-Warshall
for ( int k = 0; k < num_vertices; ++k ) {
    for ( int i = 0; i < num_vertices; ++i ) {
        for ( int j = 0; j < num_vertices; ++j ) {
            tc[i][j] = tc[i][j] || (tc[i][k] && tc[k][j]);
        }
    }
}
```

Example

Consider this directed graph

– Each pair has only one directed edge

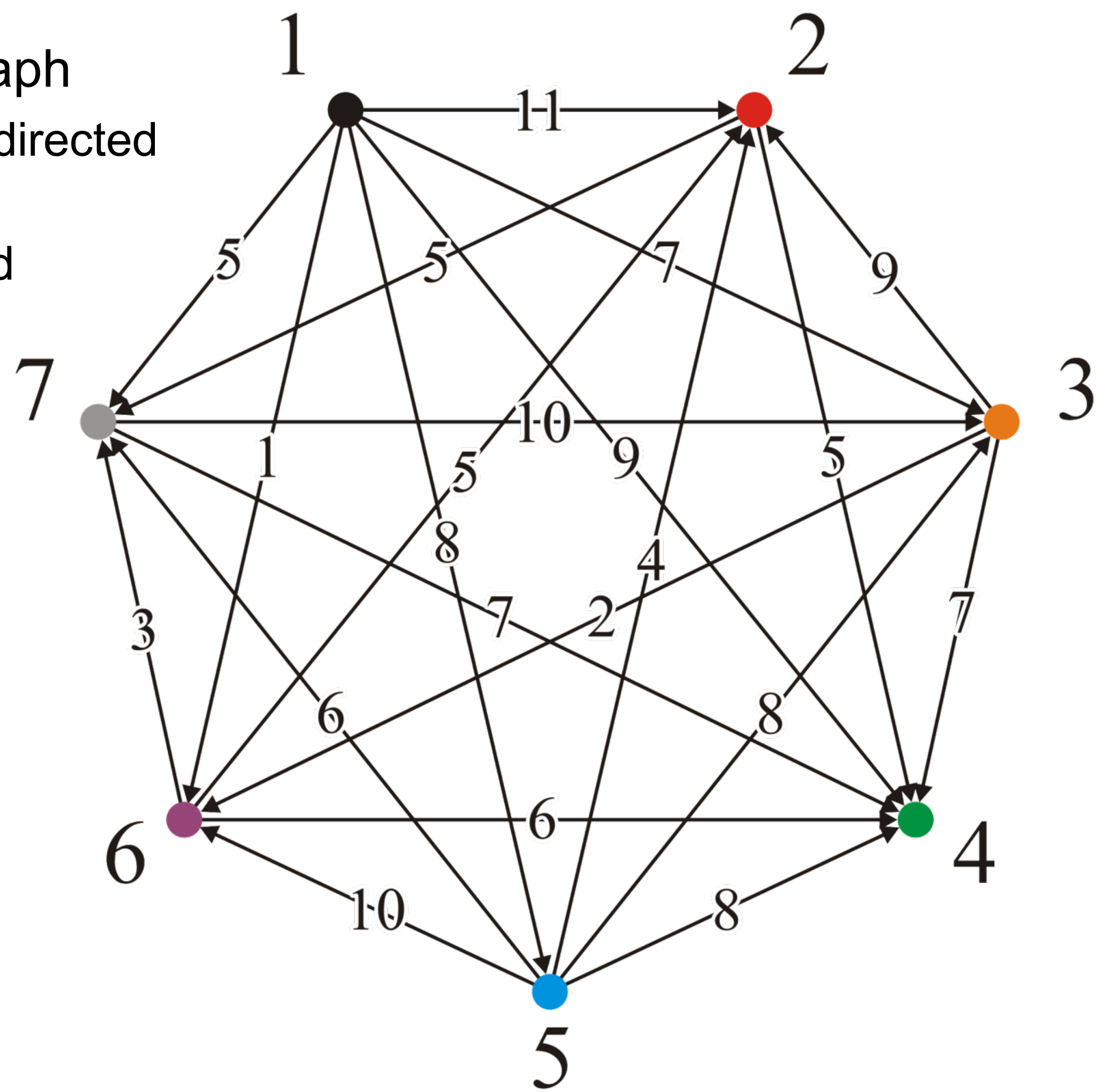
– Vertex v_1 is a source and v_4 is a sink

We will apply all three matrices

– Shortest distance

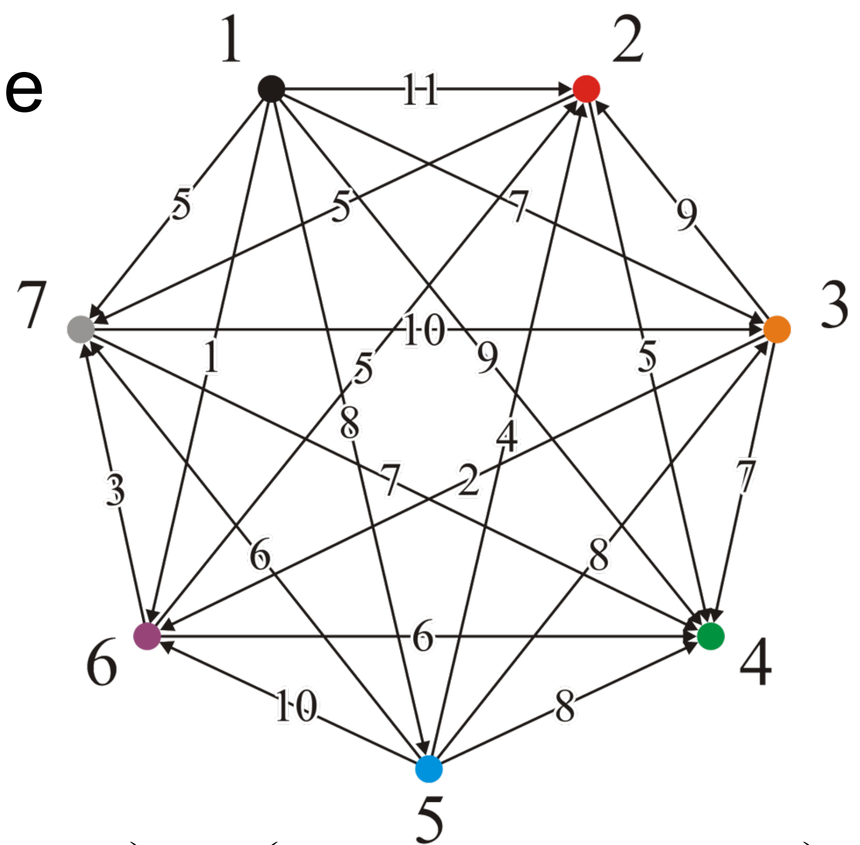
– Paths

– Transitive closure



Example

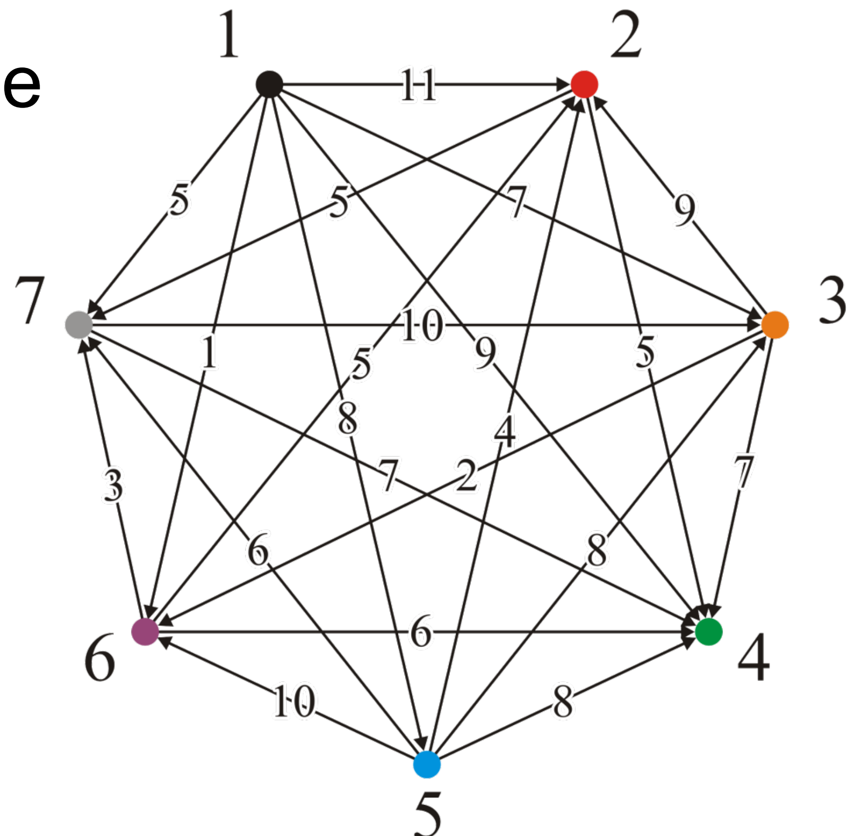
We set up the three initial matrices



$$
\begin{pmatrix}
0 & 11 & 7 & 9 & 8 & 1 & 5 \\
\infty & 0 & \infty & 5 & \infty & \infty & 5 \\
\infty & 9 & 0 & 7 & \infty & 2 & \infty \\
\infty & \infty & \infty & 0 & \infty & \infty & \infty \\
\infty & 4 & 8 & 8 & 0 & 10 & 6 \\
\infty & 5 & \infty & 6 & \infty & 0 & 3 \\
\infty & \infty & 10 & 7 & \infty & \infty & 0
\end{pmatrix}
\quad
\begin{pmatrix}
- & 2 & 3 & 4 & 5 & 6 & 7 \\
- & - & - & 4 & - & - & 7 \\
- & 2 & - & 4 & - & 6 & - \\
- & - & - & - & - & - & - \\
- & 2 & 3 & 4 & - & 6 & 7 \\
- & 2 & - & 4 & - & - & 7 \\
- & - & 3 & 4 & - & - & -
\end{pmatrix}
\quad
\begin{pmatrix}
- & T & T & T & T & T & T \\
F & - & F & T & F & F & T \\
F & T & - & T & F & T & F \\
F & F & F & - & F & F & F \\
F & T & T & T & - & T & T \\
F & T & F & T & F & - & T \\
F & F & T & T & F & F & -
\end{pmatrix}
$$

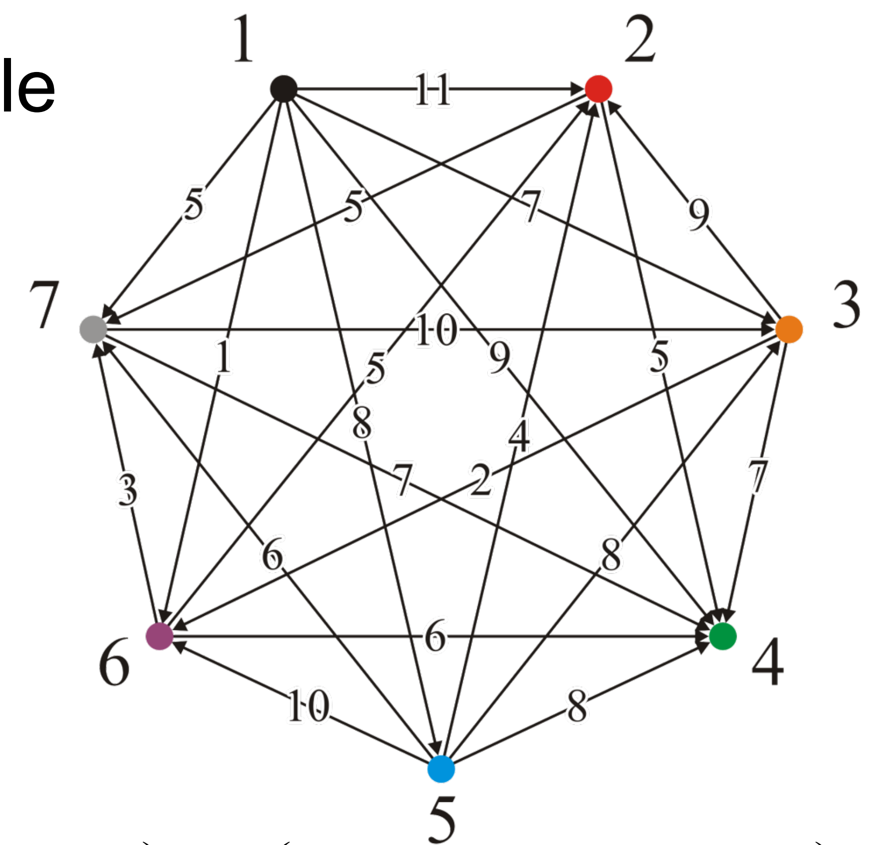
Example

At step 1, no path leads to v_1, so no shorter paths could be found passing through v_1



$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & \infty & 10 & 7 & \infty & \infty & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & - \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & - & 3 & 4 & - & - & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & F \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & F & T & T & F & F & - \end{pmatrix}$$

Example



At step 2, we find:

- A path $(3, 2, 7)$ of length 14

$$
\begin{pmatrix}
0 & 11 & 7 & 9 & 8 & 1 & 5 \\
\infty & 0 & \infty & 5 & \infty & \infty & 5 \\
\infty & 9 & 0 & 7 & \infty & 2 & \infty \\
\infty & \infty & \infty & 0 & \infty & \infty & \infty \\
\infty & 4 & 8 & 8 & 0 & 10 & 6 \\
\infty & 5 & \infty & 6 & \infty & 0 & 3 \\
\infty & \infty & 10 & 7 & \infty & \infty & 0
\end{pmatrix}
\quad
\begin{pmatrix}
- & 2 & 3 & 4 & 5 & 6 & 7 \\
- & - & - & 4 & - & - & 7 \\
- & 2 & - & 4 & - & 6 & - \\
- & - & - & - & - & - & - \\
- & 2 & 3 & 4 & - & 6 & 7 \\
- & 2 & - & 4 & - & - & 7 \\
- & - & 3 & 4 & - & - & -
\end{pmatrix}
\quad
\begin{pmatrix}
- & T & T & T & T & T & T \\
F & - & F & T & F & F & T \\
F & T & - & T & F & T & F \\
F & F & F & - & F & F & F \\
F & T & T & T & - & T & T \\
F & T & F & T & F & - & T \\
F & F & T & T & F & F & -
\end{pmatrix}
$$

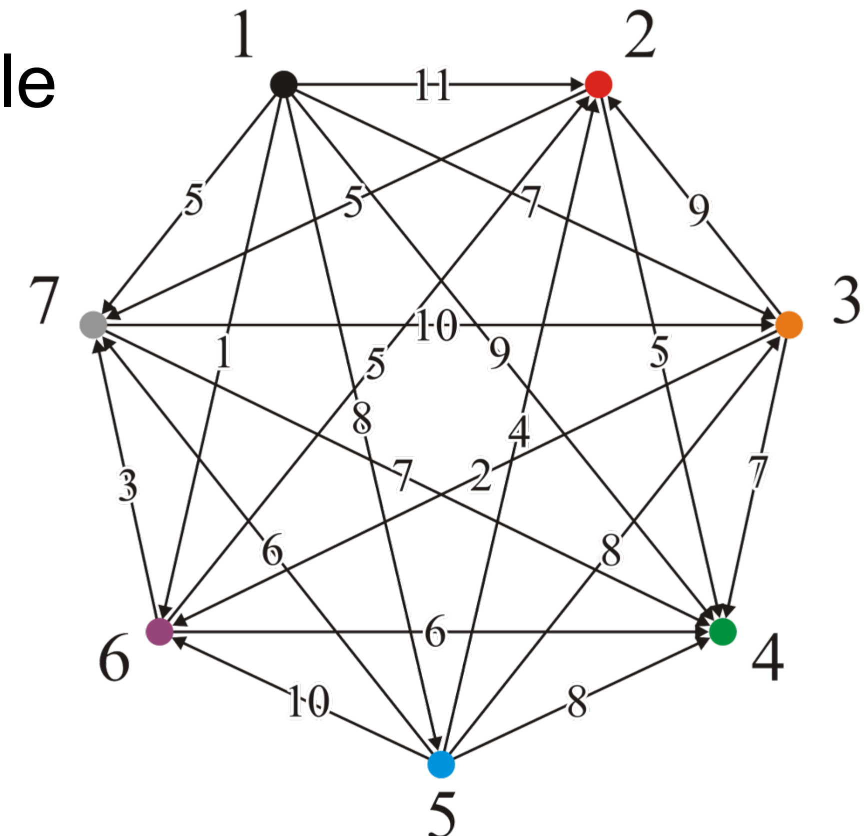
Example

At step 2, we find:

- A path $(3, 2, 7)$ of length 14

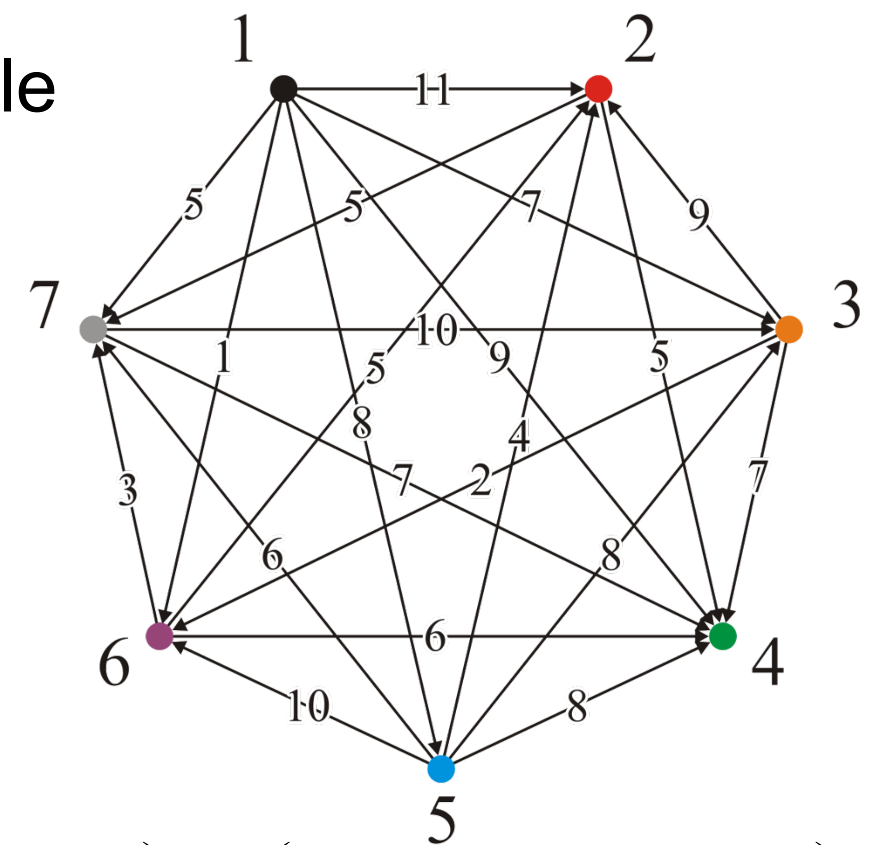
We update

$$d_{3,7} = 14,\ p_{3,7} = 2 \text{ and } c_{3,7} = T$$



$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & \infty & 10 & 7 & \infty & \infty & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & - & 3 & 4 & - & - & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & F & T & T & F & F & - \end{pmatrix}$$

Example

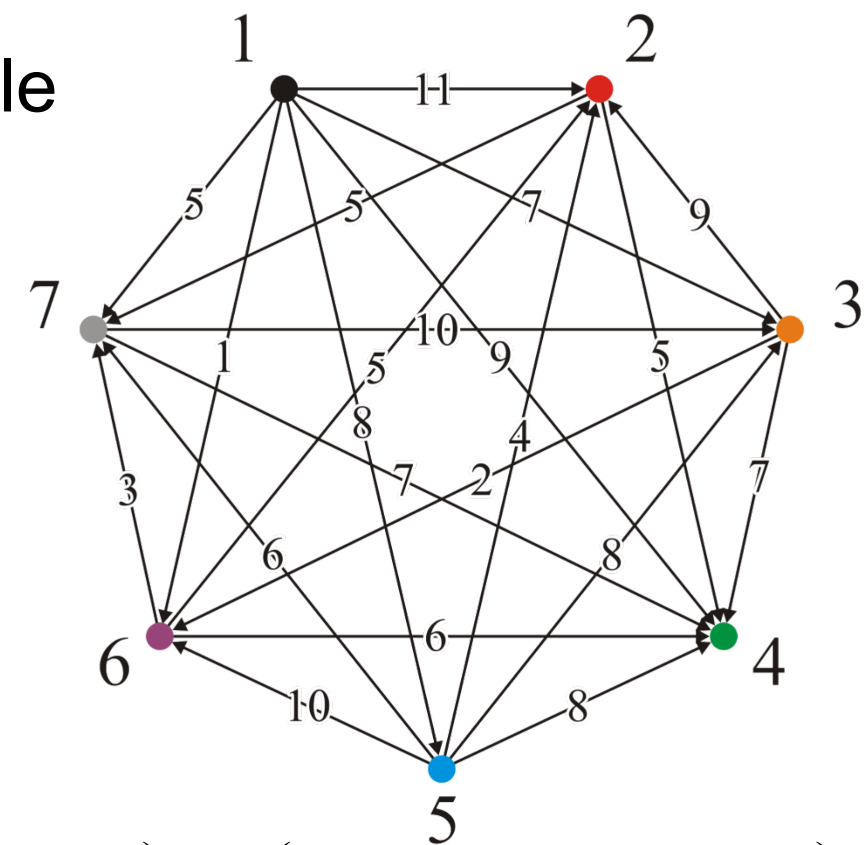


At step 3, we find:

- A path $(7, 3, 2)$ of length 19
- A path $(7, 3, 6)$ of length 12

$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & \infty & 10 & 7 & \infty & \infty & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & - & 3 & 4 & - & - & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & F & T & T & F & F & - \end{pmatrix}$$

Example



At step 3, we find:

- A path $(7, 3, 2)$ of length 19
- A path $(7, 3, 6)$ of length 12

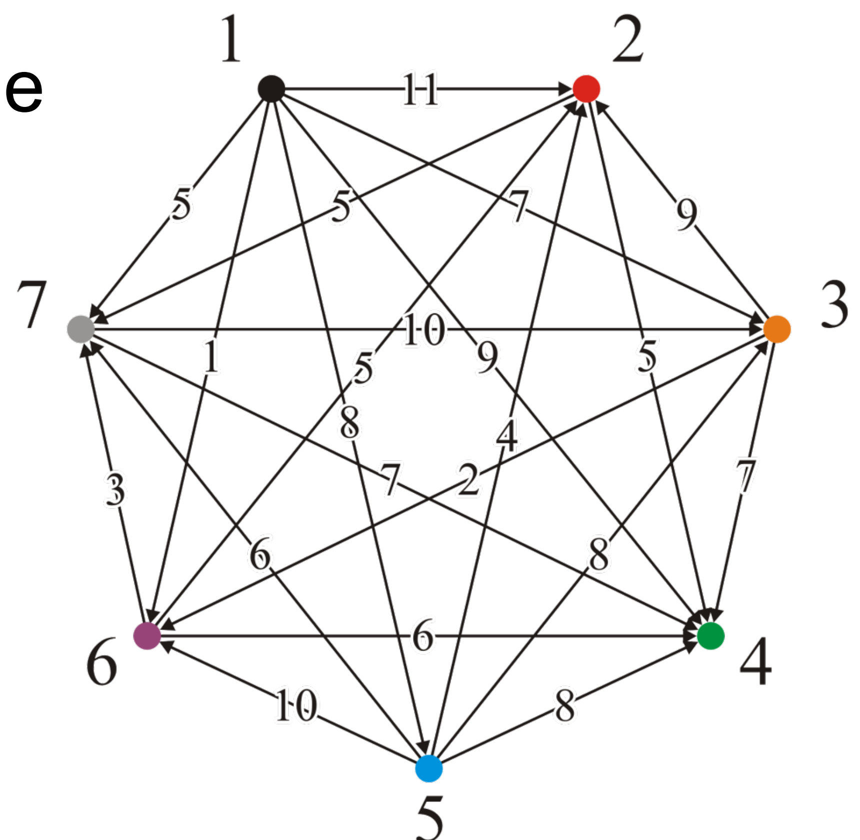
We update

$$d_{7,2} = 19,\ p_{7,2} = 3 \text{ and } c_{7,2} = T$$
$$d_{7,6} = 12,\ p_{7,6} = 3 \text{ and } c_{7,6} = T$$

$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 19 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

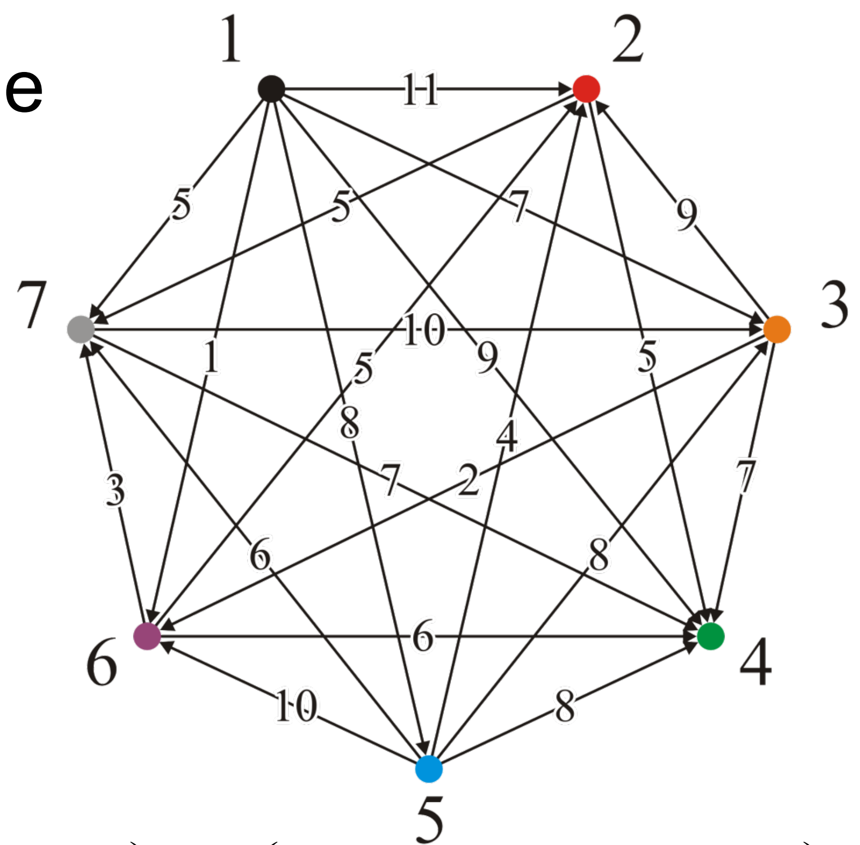
Example

At step 4, there are no paths out of vertex v_4, so we are finished



$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 19 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

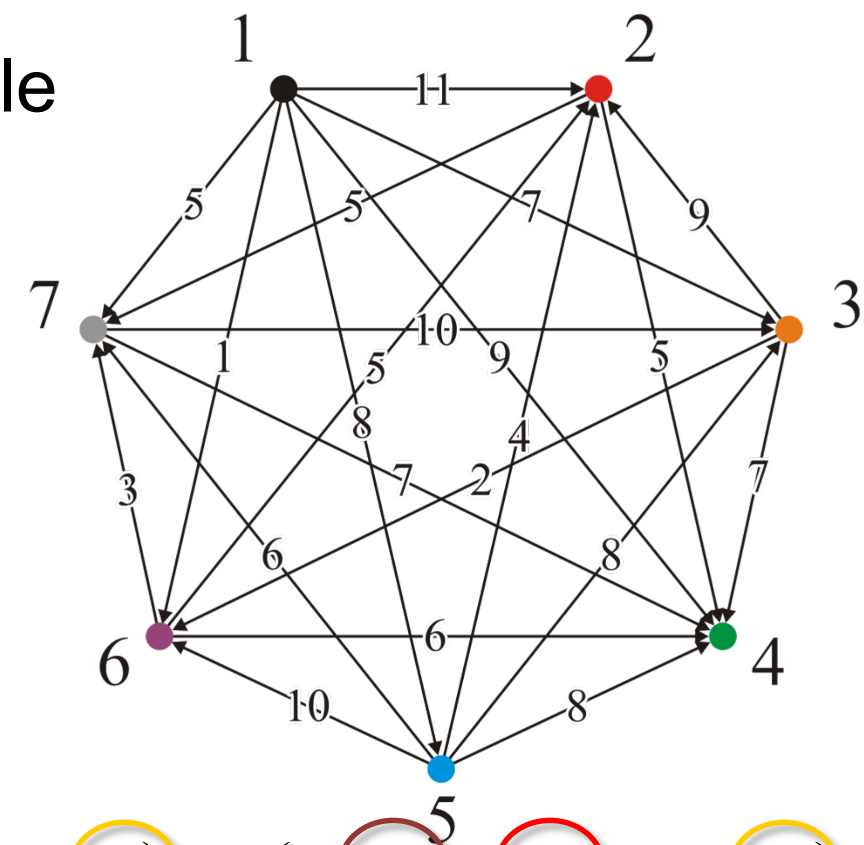
Example



At step 5, there is one incoming edge from v_1 to v_5, and it doesn't make any paths out of vertex v_1 any shorter...

$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 19 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example



At step 6, we find:

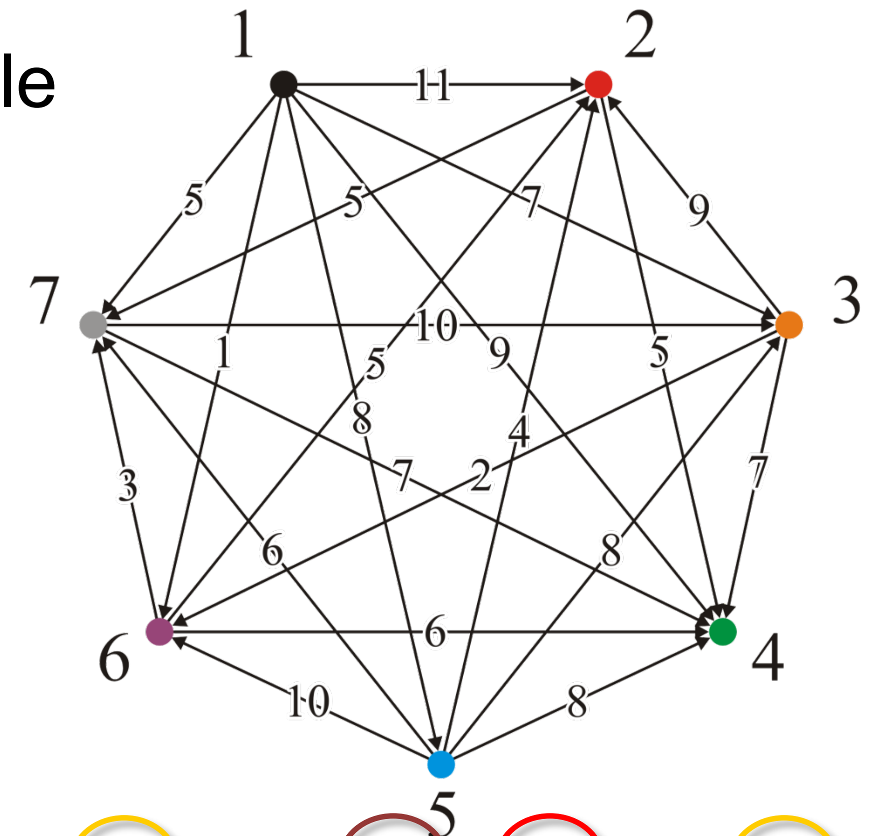
- A path $(1, 6, 2)$ of length 6
- A path $(1, 6, 4)$ of length 7
- A path $(1, 6, 7)$ of length 4
- A path $(3, 6, 2)$ of length 7
- A path $(3, 6, 7)$ of length 5
- A path $(7, 3, 6, 2)$ of length 17

$$\begin{pmatrix} 0 & 11 & 7 & 9 & 8 & 1 & 5 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 9 & 0 & 7 & \infty & 2 & 14 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 19 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 2 & 3 & 4 & 5 & 6 & 7 \\ - & - & - & 4 & - & - & 7 \\ - & 2 & - & 4 & - & 6 & 2 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example

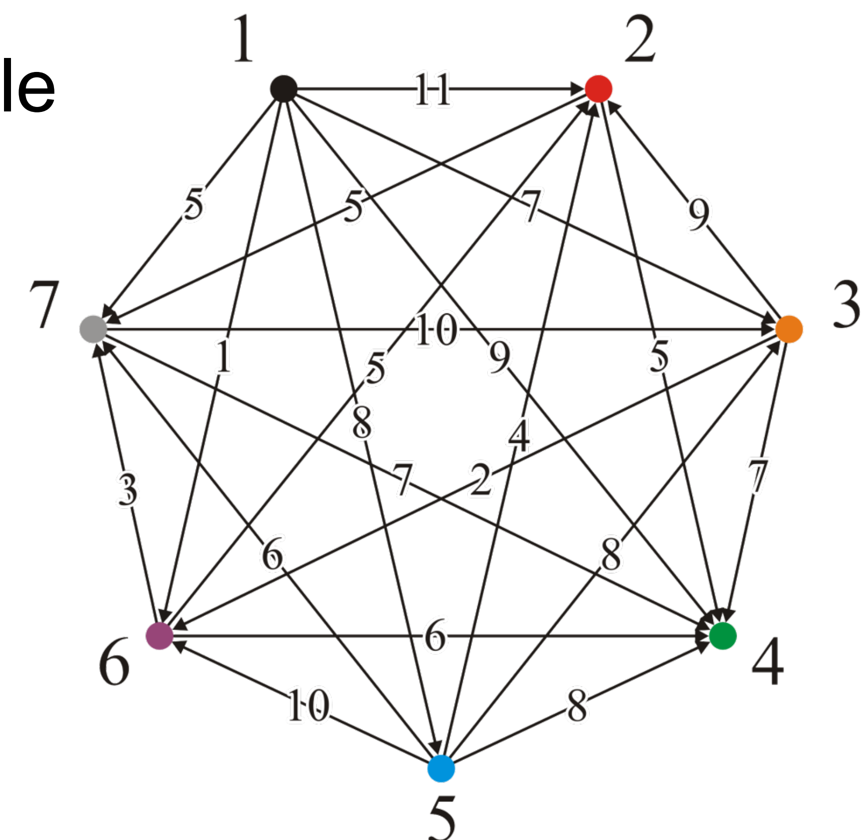
At step 6, we find:

- A path $(1, 6, 2)$ of length 6
- A path $(1, 6, 4)$ of length 7
- A path $(1, 6, 7)$ of length 4
- A path $(3, 6, 2)$ of length 7
- A path $(3, 6, 7)$ of length 5
- A path $(7, 3, 6, 2)$ of length 17



$$\begin{pmatrix} 0 & 6 & 7 & 7 & 8 & 1 & 4 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 7 & 0 & 7 & \infty & 2 & 5 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 17 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 6 & 3 & 6 & 5 & 6 & 6 \\ - & - & - & 4 & - & - & 7 \\ - & 6 & - & 4 & - & 6 & 6 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example

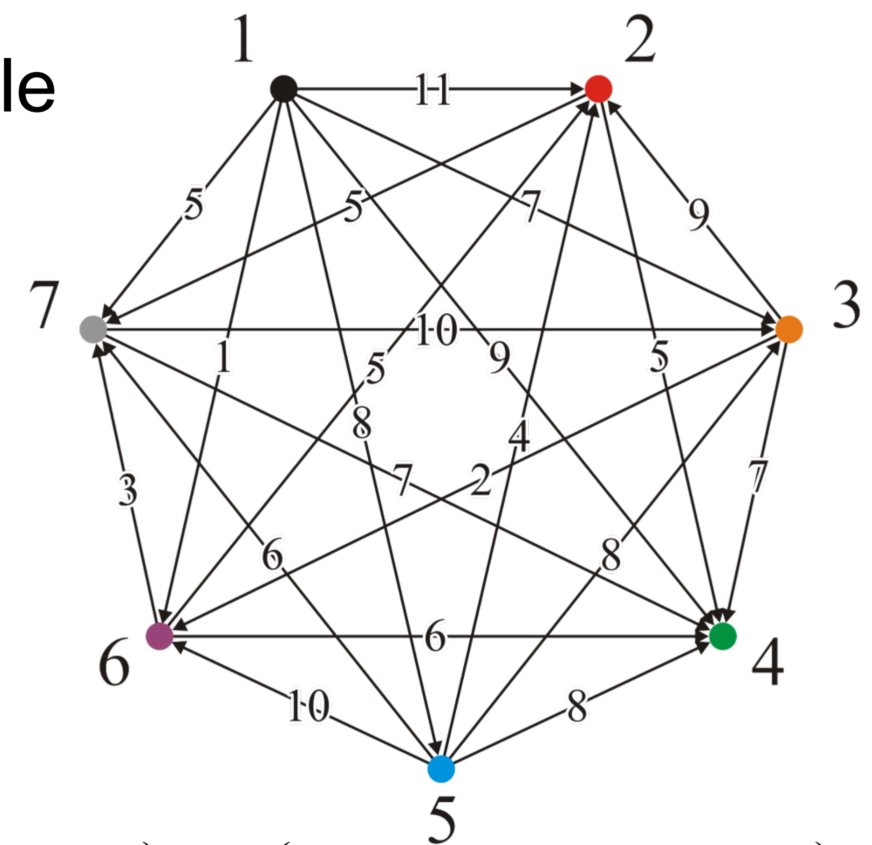


At step 7, we find:

- A path $(2, 7, 3)$ of length 15
- A path $(2, 7, 6)$ of length 17
- A path $(6, 7, 3)$ of length 13

$$\begin{pmatrix} 0 & 6 & 7 & 7 & 8 & 1 & 4 \\ \infty & 0 & \infty & 5 & \infty & \infty & 5 \\ \infty & 7 & 0 & 7 & \infty & 2 & 5 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & \infty & 6 & \infty & 0 & 3 \\ \infty & 17 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 6 & 3 & 6 & 5 & 6 & 6 \\ - & - & - & 4 & - & - & 7 \\ - & 6 & - & 4 & - & 6 & 6 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & - & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & F & T & F & F & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & F & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example

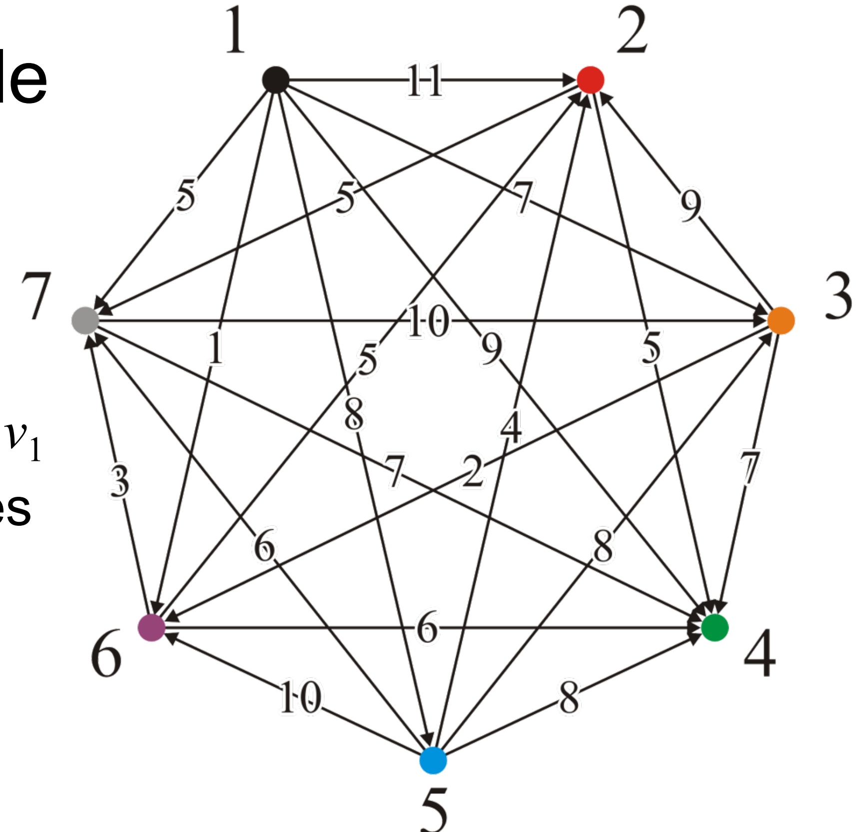


Finally, at step 7, we find:

- A path $(2, 7, 3)$ of length 15
- A path $(2, 7, 6)$ of length 17
- A path $(6, 7, 3)$ of length 13

$$\begin{pmatrix} 0 & 6 & 7 & 7 & 8 & 1 & 4 \\ \infty & 0 & 15 & 5 & \infty & 17 & 5 \\ \infty & 7 & 0 & 7 & \infty & 2 & 5 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & 13 & 6 & \infty & 0 & 3 \\ \infty & 17 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 6 & 3 & 6 & 5 & 6 & 6 \\ - & - & 7 & 4 & - & 7 & 7 \\ - & 6 & - & 4 & - & 6 & 6 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & 7 & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & T & T & F & T & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & T & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example



Note that:

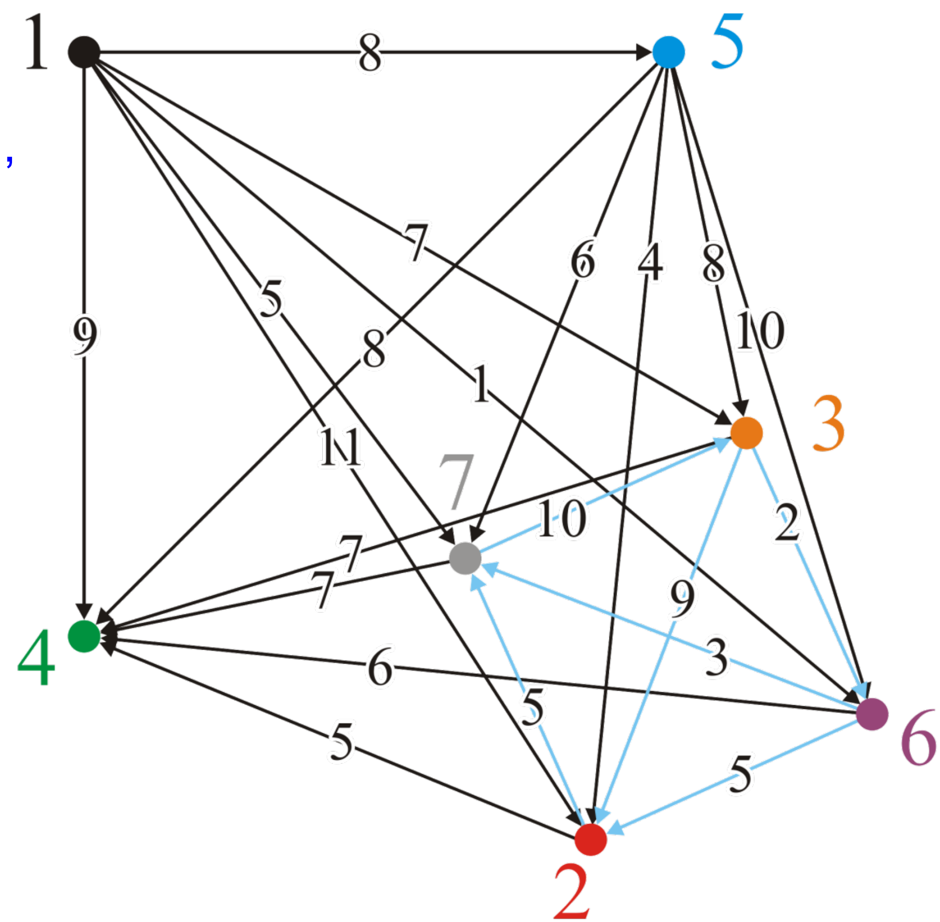
- From v_1 we can go anywhere
- From v_5 we can go anywhere but v_1
- We go between any of the vertices in the set $\{v_2, v_3, v_6, v_7\}$
- We can't go anywhere from v_4

$$\begin{pmatrix} 0 & 6 & 7 & 7 & 8 & 1 & 4 \\ \infty & 0 & 15 & 5 & \infty & 17 & 5 \\ \infty & 7 & 0 & 7 & \infty & 2 & 5 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & 13 & 6 & \infty & 0 & 3 \\ \infty & 17 & 10 & 7 & \infty & 12 & 0 \end{pmatrix} \quad \begin{pmatrix} - & 6 & 3 & 6 & 5 & 6 & 6 \\ - & - & 7 & 4 & - & 7 & 7 \\ - & 6 & - & 4 & - & 6 & 6 \\ - & - & - & - & - & - & - \\ - & 2 & 3 & 4 & - & 6 & 7 \\ - & 2 & 7 & 4 & - & - & 7 \\ - & 3 & 3 & 4 & - & 3 & - \end{pmatrix} \quad \begin{pmatrix} - & T & T & T & T & T & T \\ F & - & T & T & F & T & T \\ F & T & - & T & F & T & T \\ F & F & F & - & F & F & F \\ F & T & T & T & - & T & T \\ F & T & T & T & F & - & T \\ F & T & T & T & F & T & - \end{pmatrix}$$

Example

We could reinterpret this graph as follows:

- Vertices $\{v_2, v_3, v_6, v_7\}$ form a *strongly connected* subgraph
- You can get from any one vertex to any other
- With the transitive closure graph, it is much faster finding such strongly connected components



$$\begin{pmatrix} 0 & 6 & 7 & 7 & 8 & 1 & 4 \\ \infty & 0 & 15 & 5 & \infty & 17 & 5 \\ \infty & 7 & 0 & 7 & \infty & 2 & 5 \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty \\ \infty & 4 & 8 & 8 & 0 & 10 & 6 \\ \infty & 5 & 13 & 6 & \infty & 0 & 3 \\ \infty & 17 & 10 & 7 & \infty & 12 & 0 \end{pmatrix}$$

Summary

This topic:

- The concept of all-pairs shortest paths
- The Floyd-Warshall algorithm
- Finding the shortest paths
- Finding the transitive closure