# Convex Sets

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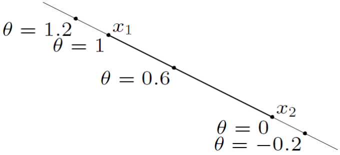
# Outline

# Definition of Affine Set

- **Line:** through $\boldsymbol{x}_1, \boldsymbol{x}_2$: all points

$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \quad (\theta \in \mathbb{R})$$




- **Affine set:** contains the line through any two distinct points in the set
- **Example:** solution set of linear equations $\{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$
  (conversely, every affine set can be expressed as solution set of system of linear equations)

# Definition of Convex Set

❧ **Line segment:** between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: all points

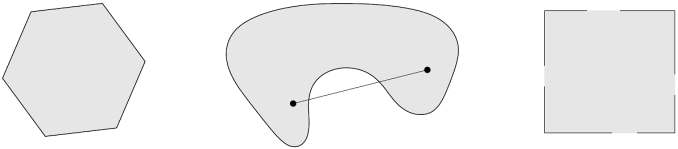
$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2$$

with $0 \leq \theta \leq 1$

❧ **Convex set:** contains line segment between any two points in the set $C$

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2 \in C$$

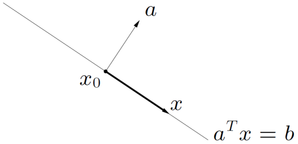
❧ **Examples** (one convex, two nonconvex sets)



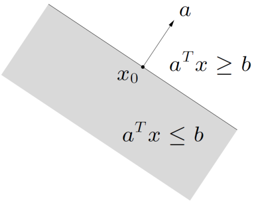
# Outline

# Examples: Hyperplanes and Halfspaces

- **Hyperplane:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\} (\boldsymbol{a} \neq \boldsymbol{0})$




- **Halfspace:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} \leq b\} (\boldsymbol{a} \neq \boldsymbol{0})$



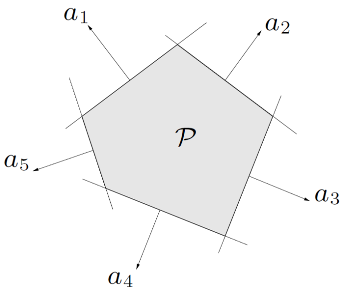

  - $\boldsymbol{a}$ is the normal vector
  - hyperplanes are affine and convex; halfspaces are convex

# Example: Polyhedra

Solution set of finitely many linear inequalities and equalities

$$\boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \qquad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$$

($\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{p\times n}, \preceq$ is componentwise inequality)




polyhedron is intersection of finite number of halfspaces and hyperplanes

# Examples: Euclidean Balls and Ellipsoids

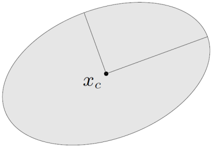
❧ **(Euclidean) Ball** with center $\boldsymbol{x}_c$ and radius $r$:

$$B(\boldsymbol{x}_c, r) = \{\boldsymbol{x} | \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \leq r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} | \|\boldsymbol{u}\|_2 \leq 1\}$$

❧ **Ellipsoid:** set of the form

$$\begin{aligned} E(\boldsymbol{x}_c, \boldsymbol{P}) &= \{\boldsymbol{x} | (\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1} (\boldsymbol{x} - \boldsymbol{x}_c) \leq 1\} \\ &= \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} | \|\boldsymbol{u}\|_2 \leq 1\} \end{aligned}$$

with $\boldsymbol{P} \in \mathbb{S}^n_{++}$ (i.e., $\boldsymbol{P}$ symmetric positive definite), $\boldsymbol{A}$ square and nonsingular



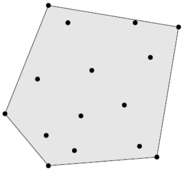
# Convex Combination and Convex Hull

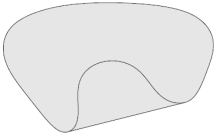
- **Convex combination** of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$: any point $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2 + \cdots + \theta_k \boldsymbol{x}_k$$

  with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

- **Convex hull** $\mathsf{conv}\, S$: set of all convex combinations of points in $S$



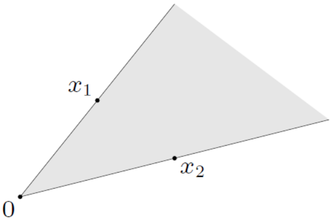


# Conic Combination and Convex Cone

- **Conic (nonnegative) combination** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: any point of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2$$

  with $\theta_1 \geq 0, \theta_2 \geq 0$




- **Convex cone:** set that contains all conic combinations of points in the set

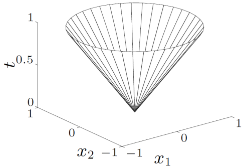
# Convex Cones: Norm Balls and Norm Cones

- **Norm:** a function $\|\cdot\|$ that satisfies
  - $\|\boldsymbol{x}\| \geq 0$; $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$
  - $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$ for $t \in \mathbb{R}$
  - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

  notation: $\|\cdot\|$ general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ a particular norm

- **Norm ball** with center $\boldsymbol{x}_c$ and radius $r$: $\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\| \leq r\}$

- **Norm cone:** $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| \leq t\}$




Euclidean norm cone or second-order cone (aka ice-cream cone)

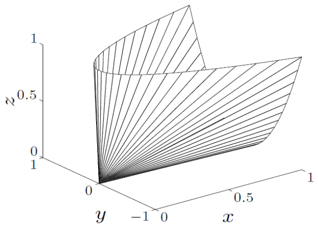
# Positive Semidefinite Cone

- **Notation**
  - $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
  - $\mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$\boldsymbol{X} \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad \boldsymbol{z}^\top \boldsymbol{X} \boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z}$$

    $\mathbb{S}^n_+$ is a convex cone
  - $\mathbb{S}^n_{++} = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succ 0\}$: positive definite $n \times n$ matrices

- **Example:** $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$



# Outline

# Operations that Preserve Convexity

How to establish the convexity of a given set $C$

- Apply the definition (can be cumbersome)

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

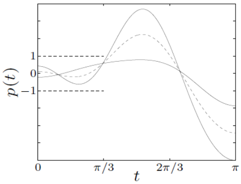
- Show that $C$ is obtained from simple convex sets(hyperplanes, halfspaces, norm balls,$\cdots$) by operations that preserve convexity
    - intersection
    - affine functions
    - perspective function
    - linear-fractional functions

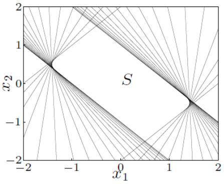
# Intersection

- **Intersection:** if $S_1, S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex ($k$ can be any positive integer)

- Example 1: a polyhedron is the intersection of halfspaces and hyperplanes

- Example 2:
$$S = \{\boldsymbol{x} \in \mathbb{R}^m \,|\, |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$
where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$




for $m = 2$

# Affine Function

suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine ($f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

- the image of a convex set under $f$ is convex
$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in S\} \text{ convex}$$
- the inverse image $f^{-1}(C)$ a convex set under $f$ is convex
$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in C\} \text{ convex}$$

### Examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{\boldsymbol{x} | x_1\boldsymbol{A}_1 + \cdots + x_m\boldsymbol{A}_m \preceq \boldsymbol{B}\}$ (with $\boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}^p$)
- $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \leq t\}$ is convex, so is
$$\{\boldsymbol{x} \in \mathbb{R}^n | \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\| \leq \boldsymbol{c}^T\boldsymbol{x} + d\}$$

# Perspective and Linear-fractional Function I

- **Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(\boldsymbol{x}, t) = \boldsymbol{x}/t, \quad \mathrm{dom} P = \{(\boldsymbol{x}, t) | t > 0\}$$

  images and inverse images of convex sets under perspective are convex

- **Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$

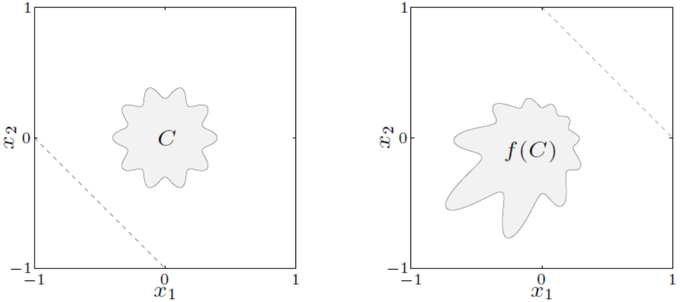
$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T \boldsymbol{x} + d}, \quad \mathrm{dom} f = \{\boldsymbol{x} | \boldsymbol{c}^T \boldsymbol{x} + d > 0\}$$

  images and inverse images of convex sets under linear-fractional functions are convex

# Perspective and Linear-fractional Function II

❧ **Examples** of a linear-fractional function

$$f(\boldsymbol{x}) = \frac{1}{x_1 + x_2 + 1}\boldsymbol{x}$$



# Outline

# Generalized Inequalities I

- A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if
  - $K$ is closed (contains its boundary)
  - $K$ is solid (has nonempty interior)
  - $K$ is pointed (contains no line)

- **Examples**
  - nonnegative orthant

$$K = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$$

  - positive semidefinite cone

$$K = \mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} | \boldsymbol{X} = \boldsymbol{X}^T \succeq \mathbf{0}\}$$

  - nonnegative polynomials on $[0, 1]$:

$$K = \{\boldsymbol{x} \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

# Generalized Inequalities II

**Generalized inequality** defined by a proper cone $K$:

$$\boldsymbol{y} \succeq_K \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{y} - \boldsymbol{x} \succeq_K \mathbf{0} \text{ or } \boldsymbol{y} - \boldsymbol{x} \in K$$

**Examples**

- Componentwise inequality ($K = \mathbb{R}^n_+$)

$$\boldsymbol{y} \succeq_{\mathbb{R}^n_+} \boldsymbol{x} \quad \Longleftrightarrow \quad y_i \geq x_i, \quad i = 1, \cdots, n$$

- Matrix inequality ($K = \mathbb{S}^n_+$)

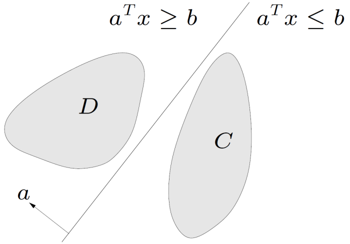
$$\boldsymbol{Y} \succeq_{\mathbb{S}^n_+} \boldsymbol{X} \quad \Longleftrightarrow \quad \boldsymbol{Y} - \boldsymbol{X} \text{ positive semidefinite}$$

# Outline

# Separating Hyperplane Theorem

If $C$ and $D$ are nonempty disjoint convex sets, there exist $\boldsymbol{a} \neq \mathbf{0}$ and $b$, such that

$$\boldsymbol{a}^T \boldsymbol{x} \leq b \text{ for } \boldsymbol{x} \in C, \quad \boldsymbol{a}^T \boldsymbol{x} \geq b \text{ for } \boldsymbol{x} \in D$$



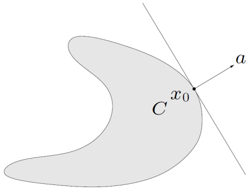

the hyperplane $\{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} = b\}$ separates $C$ and $D$

# Supporting Hyperplane Theorem

**Supporting hyperplane** to set $C$ at boundary point $\boldsymbol{x}_0$:

$$\{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}^T \boldsymbol{x}_0\}$$

where $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{a}^T \boldsymbol{x} \leq \boldsymbol{a}^T \boldsymbol{x}_0$ for all $\boldsymbol{x} \in C$




**Supporting hyperplane theorem:** if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$

# Dual Cones and Generalized Inequalities

- **Dual cone** of a cone $K$:

$$K^* = \{\boldsymbol{y} | \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in K\}$$

- Examples
  - $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
  - $K = \mathbb{S}^n_+$: $K^* = \mathbb{S}^n_+$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_1 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_\infty \leq t\}$

  First three examples are **self-dual** cones

- Dual cones of proper cones are proper, hence define generalized inequalities:

$$\boldsymbol{y} \succeq_{K^*} \mathbf{0} \quad \Longleftrightarrow \quad \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \succeq_K \mathbf{0}$$

# Reference

**Chapter 2 of:**


- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.