

CS243: Introduction to Algorithmic Game Theory

Social Choice (Dengji ZHAO)

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Recap



Answer Yes/No for the following questions:

- Q1 In a simultaneous move game, given a strategy vector $s \in S$, if for each player i , and each alternate strategy vector $s' \in S$, we have $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$, then s is a **Nash equilibrium**?
- Q2 A mechanism is **truthful** means that reporting valuation function truthfully is a **dominant strategy** for all players?
- Q3 The **second price auction** for selling one item is truthful, efficient and individually rational?
- Q4 Following mechanism (f, p_1, \dots, p_n) is truthful and efficient?
- given players' valuation function report profile (v_1, \dots, v_n)

$$f(v_1, \dots, v_n) \in \operatorname{argmax}_{a \in A} \sum_i v_i(a)$$

$$p_i(v_1, \dots, v_n) = \sum_{j \neq i} v_j(f(v_{-i})) - \sum_{j \neq i} v_j(f(v_1, \dots, v_n))$$

Social Choice


- In a mechanism design setting, each player i has a valuation function $v_i : A \rightarrow \mathbb{R}$
- Now assume that a player does not have a value for each outcome $a \in A$, instead she has a preference \succsim_i over all outcomes in A . \succsim_i is a linear order on A , e.g.
 $a_1 \succsim_i a_2 \succsim_i a_3 \dots$
- We want to study the following:
 - 1 How to choose one outcome from A ?
 - 2 What is the overall ranking on A by aggregating all players' preferences?

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Social Choice (Voting)

General Social Choice Setting

- A set of n players/voters.
- A set of alternatives A (the candidates).
- Let L be the set of all linear orders on A . 
- Each voter i has a preference $\succ_i \in L$, a *total order* on A (antisymmetric, transitive). $a \succ_i b$ means i prefers a to b .

Definition

- A function $f : L^n \rightarrow A$ is called a **social choice function**.
- A function $F : L^n \rightarrow L$ is called a **social welfare function**.

Voting Methods: Social Choice Functions

- **Majority vote**
 - among **two candidates**, selects the candidate which has a majority vote, that is, more than half of the votes
- **Plurality**
 - the candidate that was placed first by the largest number of voters wins
- **Borda count**
 - each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i , and the candidate with most points wins

Majority Vote

- Consider three candidates $\{a, b, c\}$ and three voters $\{1, 2, 3\}$.
- Their preferences are the following:
 - $a \succ_1 b \succ_1 c$
 - $b \succ_2 c \succ_2 a$
 - $c \succ_3 a \succ_3 b$

Question

Selects alternatives which have a majority vote, that is, more than half of the votes: which candidate wins?

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- $a \succ b \succ c \succ a$ (Condorcet's Paradox)

Plurality

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The candidate that was placed first by the largest number of voters wins: which candidate wins?

Borda Count

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Question

Each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i , and the candidate with most points wins: which candidate wins?

Strategic Manipulations

- $a \succ_1 b \succ_1 c$
- $b \succ_2 c \succ_2 a$
- $c \succ_3 a \succ_3 b$

Assume that under tie-breaking, a is preferred to b and b is preferred to c , can a voter manipulate (**misreport preference**) to change the outcome in his/her favour?

- *Plurality*: The candidate that was placed first by the largest number of voters wins (**a wins**).
- *Borda count*: Each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i , and the candidate with most points wins (**a wins**).

Truthful (Incentive Compatible) Social Choice Function

Definition 9.4 A social choice function f can be *strategically manipulated* by voter i if for some $\prec_1, \dots, \prec_n \in L$ and some $\prec'_i \in L$ we have that $a \prec_i a'$ where $a = f(\prec_1, \dots, \prec_i, \dots, \prec_n)$ and $a' = f(\prec_1, \dots, \prec'_i, \dots, \prec_n)$. That is, voter i that prefers a' to a can ensure that a' gets socially chosen rather than a by strategically misrepresenting his preferences to be \prec'_i rather than \prec_i . f is called *incentive compatible* if it cannot be manipulated.

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Definition 9.7 Voter i is a *dictator* in social choice function f if for all $\prec_1, \dots, \prec_n \in L$, $\forall b \neq a$, $a \succ_i b \Rightarrow f(\prec_1, \dots, \prec_n) = a$. f is called a *dictatorship* if some i is a dictator in it.

Theorem 9.8 (Gibbard–Satterthwaite) Let f be an incentive compatible social choice function onto A , where $|A| \geq 3$, then f is a dictatorship.

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Gibbard-Satterthwaite Theorem is an implication of Arrow's Theorem

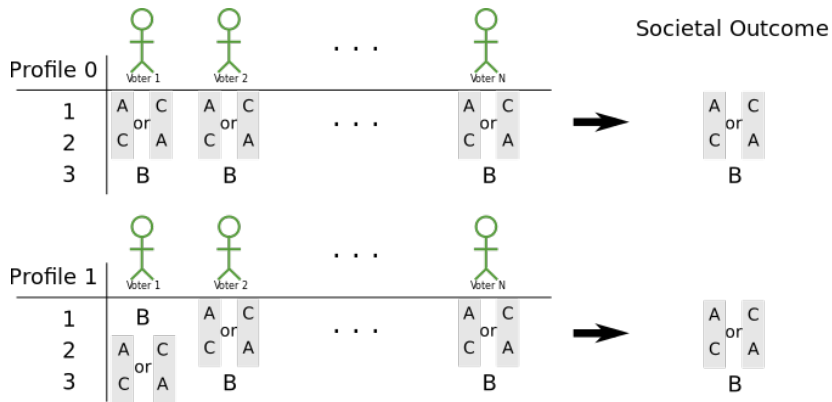
Arrow's Theorem

Theorem (Arrow's Theorem)

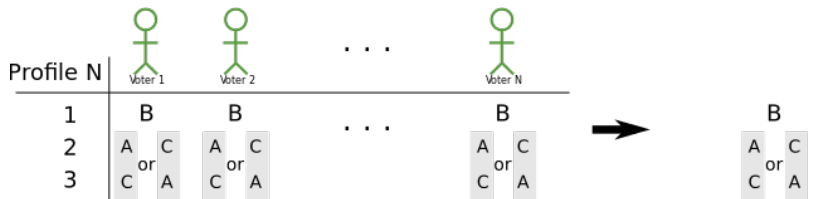
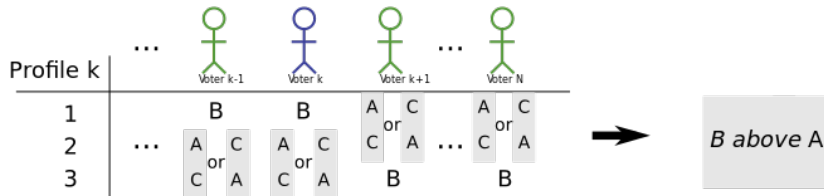
Every **social welfare function** over a set of more than 2 candidates ($|A| \geq 3$) that satisfies **unanimity** and **independence of irrelevant alternatives** is a **dictatorship**.

- F satisfies **unanimity** if for every $\succ \in L$, $F(\succ, \dots, \succ) = \succ$.
- F satisfies **independence of irrelevant alternatives** if for every $a, b \in A$, every $\succ_1, \dots, \succ_n, \succ'_1, \dots, \succ'_n \in L$, if $\succ = F(\succ_1, \dots, \succ_n)$ and $\succ' = F(\succ'_1, \dots, \succ'_n)$, then $a \succ_i b \Leftrightarrow a \succ'_i b$ for all i implies $a \succ b \Leftrightarrow a \succ' b$.
- Voter i is a **dictator** in F if for all $\succ_1, \dots, \succ_n \in L$, $F(\succ_1, \dots, \succ_n) = \succ_i$. F is not a **dictatorship** if no i is a dictator in F .

To Prove Arrow's Theorem



To Prove Arrow's Theorem

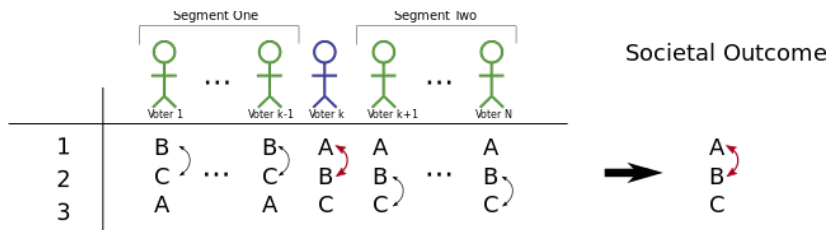


To Prove Arrow's Theorem

- There is a **pivotal voter** k for B over A.
- The **pivotal voter** for B over A is a dictator for B over C.

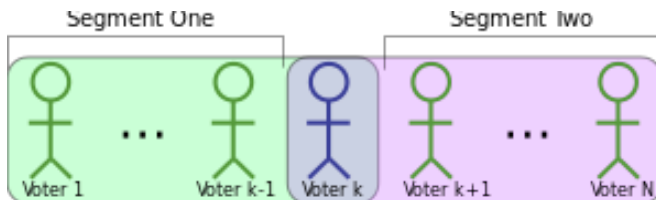
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To Prove Arrow's Theorem

- There is a **pivotal voter** k for B over A .
- The pivotal voter for B over A is a dictator for B over C .
- **There can be at most one dictator.**



Since voter k is the dictator for B over C , the pivotal voter for B over C must appear among the first k voters. That is, outside of segment two.

Likewise, the pivotal voter for C over B must appear among voters k through N . That is, outside of Segment One.

Advanced Reading

- Social Choice [AGT Chapter 9.2]