Parallel Search Algorithms

CS121 Parallel Computing

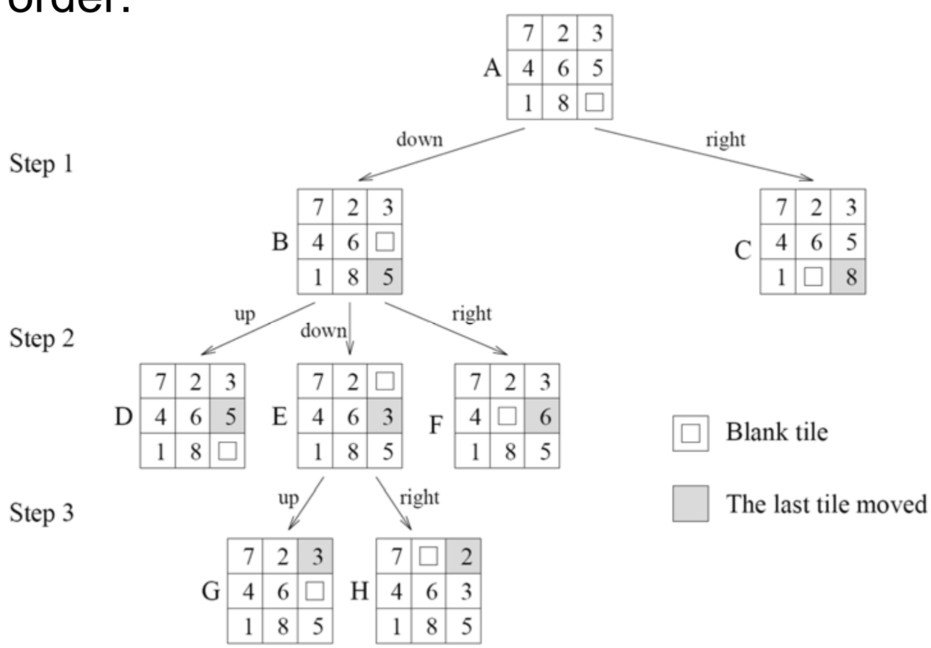
Spring 2019

Discrete optimization problems

- A discrete optimization problem (DOP) consists of a tuple (S, f), where S is a (finite or infinite) set feasible solutions satisfying certain constraints, and f is a cost function for each solution, $f\colon S \to \mathbb{R}$.
- Ex Planning and scheduling, optimal layout in VLSI, logistics and control, satisfiability problems.
- We consider the minimization problem, i.e. find $x^* \in S$ s.t. $\forall x \in S\colon f(x^*) \leq f(x)$.

Searching for solutions

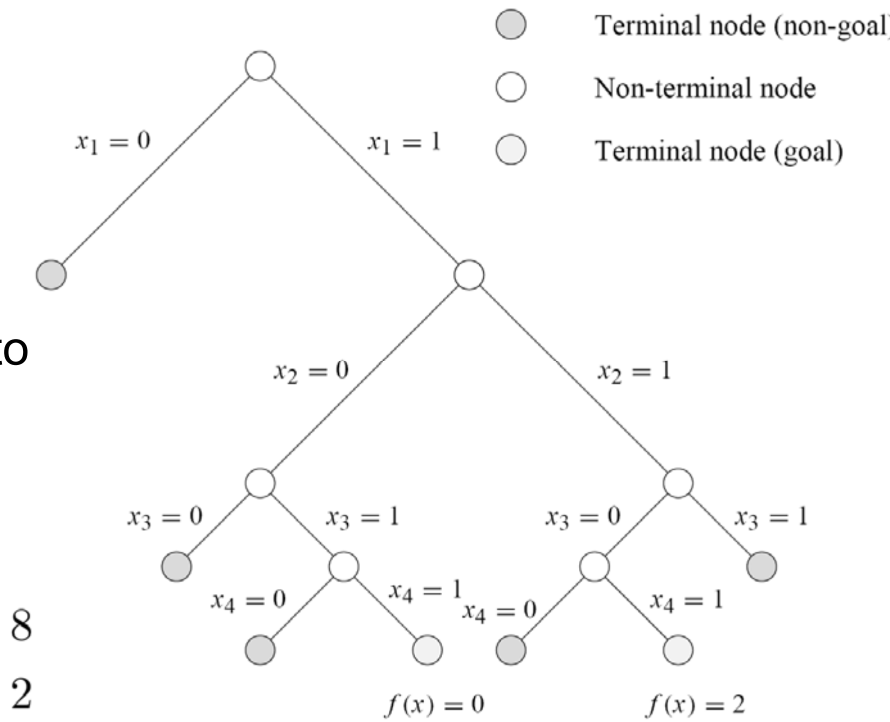
- Many interesting DOPs are NP-hard, so we either rely on heuristics or search algorithms.
- Searching a DOP can be modeled by a graph, with nodes (aka states) being (possibly infeasible) solutions and edges between states that are "close enough".
  - Cost of states can be modeled using weights on nodes and edges.
  - Search process traverses states in some order to find min cost feasible state.
    - State with no successor in traversal called terminal state. Otherwise it's a nonterminal state.
- Ex 8-puzzle. From starting configuration, move blank tile to lower right so all tiles are in order.



Searching for solutions

- Ex A mixed integer program (MIP) consists of a linear objective function and a system of linear constraints, where some of the variables are required to be integers.
  - States are assignments of values to the variables.
  - MIPs can model many interesting DOPs, including NP-hard ones.
  - Ex $\min \; 2x_1 + x_2 - x_3 - 2x_4 \quad \text{s.t.}$

$$\begin{aligned} 5x_1 + 2x_2 + x_3 + 2x_4 &\geq 8 \\ x_1 - x_2 - x_3 + 2x_4 &\geq 2 \\ 3x_1 + x_2 + x_3 + 3x_4 &\geq 5 \end{aligned}$$

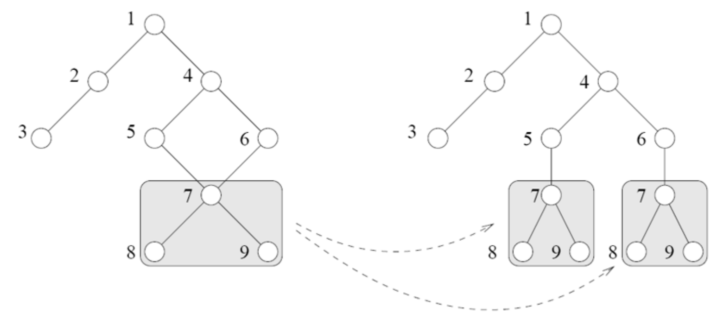


  - Feasible solutions must satisfy

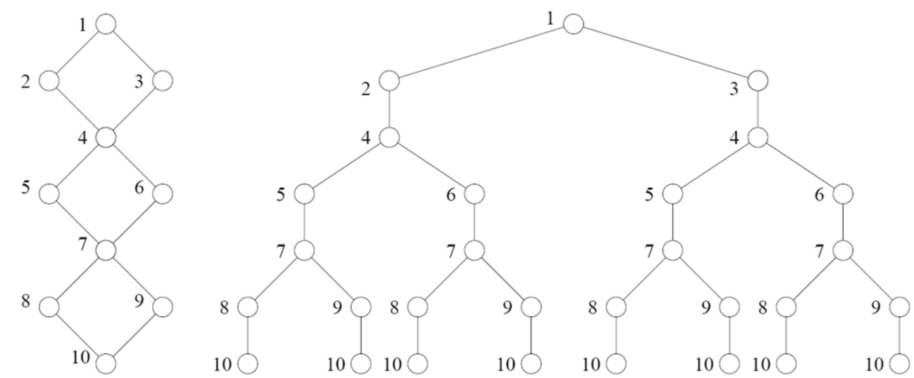
$$\sum_{x_j \text{ is free}} \max\{A[i,j], 0\} + \sum_{x_j \text{ is fixed}} A[i,j]x_j \geq b_i, i = 1, \ldots, m$$

Structure of search graph

- When the search graph is a tree, there's only one way to arrive at each state, and the state has a unique cost determined by the root-state path.
- When the search graph is not a tree, there can be multiple paths to each state.
- Can unfold graph into a tree to use tree based search methods.
  - Sometimes unfolded tree much larger than original graph.
- Same state may be discovered multiple times.
  - Use duplicate detection to detect if node has already been explored.
    - Ex Store explored nodes in hash table.
  - Cost to the state is the minimum among all the paths.
    - Update the cost as different paths discovered.
    - Ex Hash table store both a state and its current min cost.



(a)



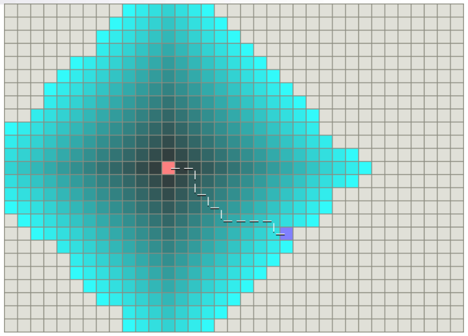
(b)

Branch and bound

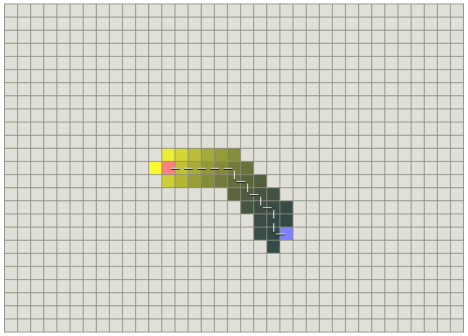
- Traverse the search graph in some order. At any time, we have a frontier of deepest nodes that have been explored.
- For each node p on the frontier, maintain a pair $(m(p), M(p))$, where $m(p) \leq$ min possible value of any feasible descendant of p, and $M(p) \geq$ max possible value of any feasible descendant of p.
  - m and M can be computed using a fast inexact heuristic.
  - Ex In knapsack, m can be the value of current knapsack, M can be the value if all remaining items are placed in knapsack.
- If $m(p) \geq M(q)$ for nodes p, q on the frontier, then don't need to explore any descendants of p.
  - The best (smallest) solution starting from p is worse (larger) than the worst possible solution starting from q, so we shouldn't explore p.

A* search

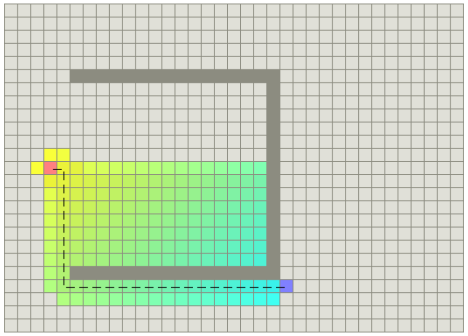
- An improvement of Dijkstra's shortest path (i.e. min cost) algorithm that searches "most promising" paths first.
- For each node n in search graph, let
  - g(n) = cost from root to n.
  - h(n) = a lower bound on the cost from n to a solution (aka admissible heuristic).
  - f(n) = g(n) + h(n), i.e. a lower bound on a solution from n.
- A* search expands nodes in order of nondecreasing f(n).
  - Dijkstra's expands nodes in nondecreasing order of g(n).
- Can be implemented in a similar way to Dijkstra.
  - Keep a closed list of explored nodes. Their values are already minimal.
  - Keep an open list of nodes whose values are tentative and can still decrease.
  - Repeatedly explore min f(n) node. Then update g, h and f functions.
- Guaranteed to find min cost solution.
- If h(n) close to real cost from n to a solution, then A* searches few nodes and is fast.
  - Ex h(n) for grid search can be Manhattan distance from n to goal.



Dijkstra's algorithm



A* search with Manhattan distance heuristic

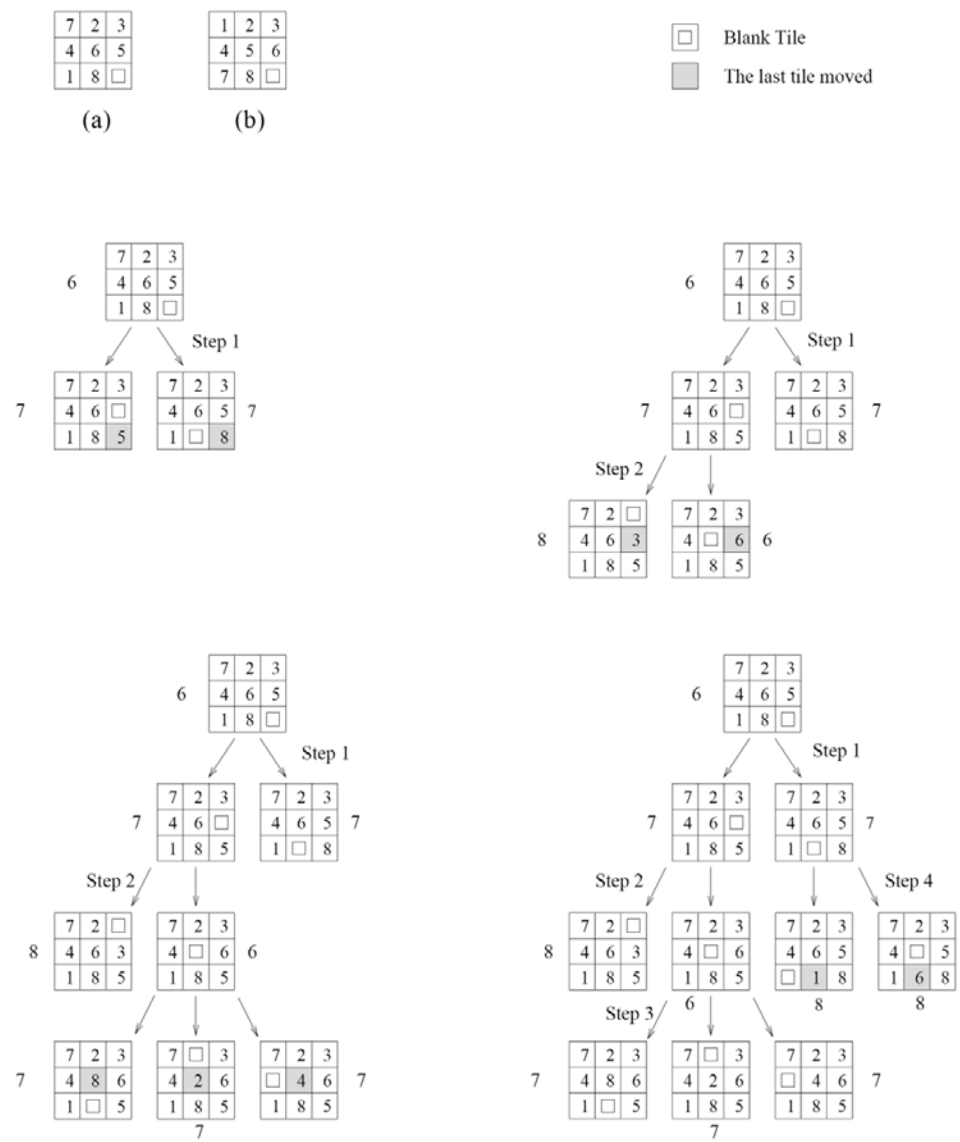


A* search with an obstacle

Source: http://theory.stanford.edu/~amitp/GameProgramming/AStarComparison.html

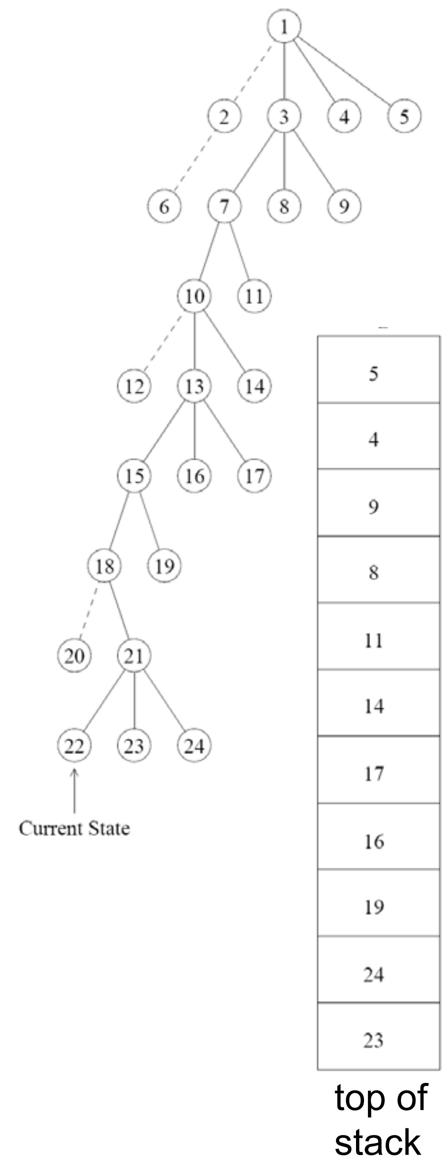
A* search

- Ex For 8 puzzle, one possible $h_1(n)$ is number of tiles in incorrect positions.
- Ex Another $h_2(n)$ is the sum of the Manhattan distances of each tile from its final position.
  - Each move of the blank tile moves one numbered tile one position.
- Decreases number of states searched by many orders of magnitude.
- Can also use h(n) that's slightly larger than cost from n to goal.
  - Further reduces number of nodes searched.
  - Can produce somewhat suboptimal solution.



DFS and variants

- Depth first search
  - Do DFS on the search graph. Backtrack from infeasible terminal nodes.
  - Visit node at top of stack. Add new unvisited nodes to top of stack.
- Depth first branch and bound
  - Search in DFS order, but cut off branches using BB.
- Iterative deepening A*
  - DFS can search very deep in part of the tree, whereas a better solution exists higher up.
  - Instead, do multiple rounds of DFS, each round searching deeper.
  - For each search, set an upper bound B for cost. If current node n has f(n) > B, then backtrack.
    - On the next round, increase B to smallest non-explored f(n) value from last round.
    - For first round, set B = f(root).
- Other more advanced algorithms used in practice include cutting plane methods and branch and cut.

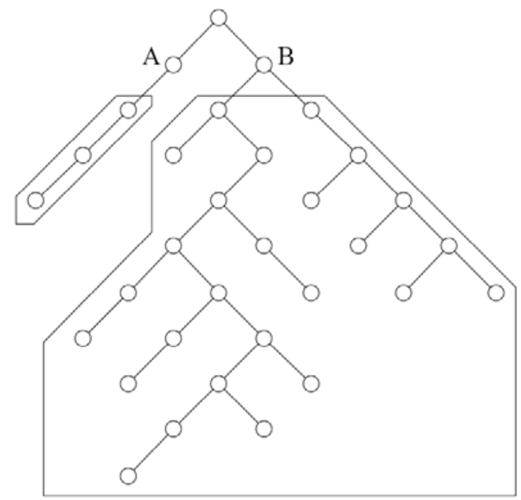


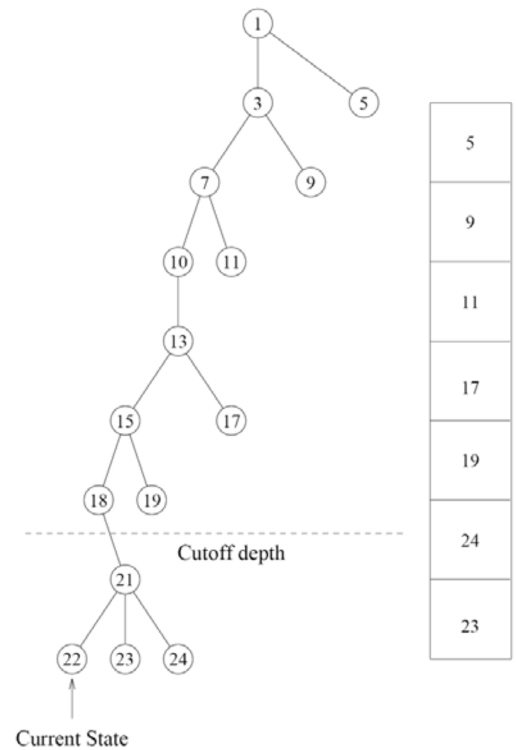
Parallel search

- Tree and graph searches can be very slow, so we can parallelize these algorithms.
- Parallel search algorithms have several sources of overhead, including communication, load imbalance and contention for shared data structures.
- Parallel algorithms can explore different part of search tree than the sequential algorithm, since the exploration order is different.
- Let W_s, W_p be the number of nodes explored by sequential and parallel algorithms, resp.
  - Search overhead factor is $\frac{W_p}{W_s}$.
  - Overhead may be >, = or < than 1.
    - As discussed later, when overhead < 1, we have a speedup anomaly.

Parallel DFS

- To parallelize DFS, can assign different processors parts of the DFS stack to explore.
- Static assignments cause poor load balancing, since different brances have different sizes.
- Hard to estimate the sizes of branches. So instead, do dynamic load balancing.
  - Can use e.g. the work-stealing algorithm from previous lecture.
- Control how much to steal.
  - Stealing too little causes overhead from many steals.
  - Stealing too much can cause poor load balancing.
- Also choose who to steal from.
  - In asynchronous round robin, processors choose victims independently, and round robin through them.
  - In global round robin, processors maintain victim in a global variable, and round robin through them.
  - In randomized work stealing, nodes pick victim randomly.
- To avoid sending very small amounts of work, set a cutoff depth and don't send work past cutoff.



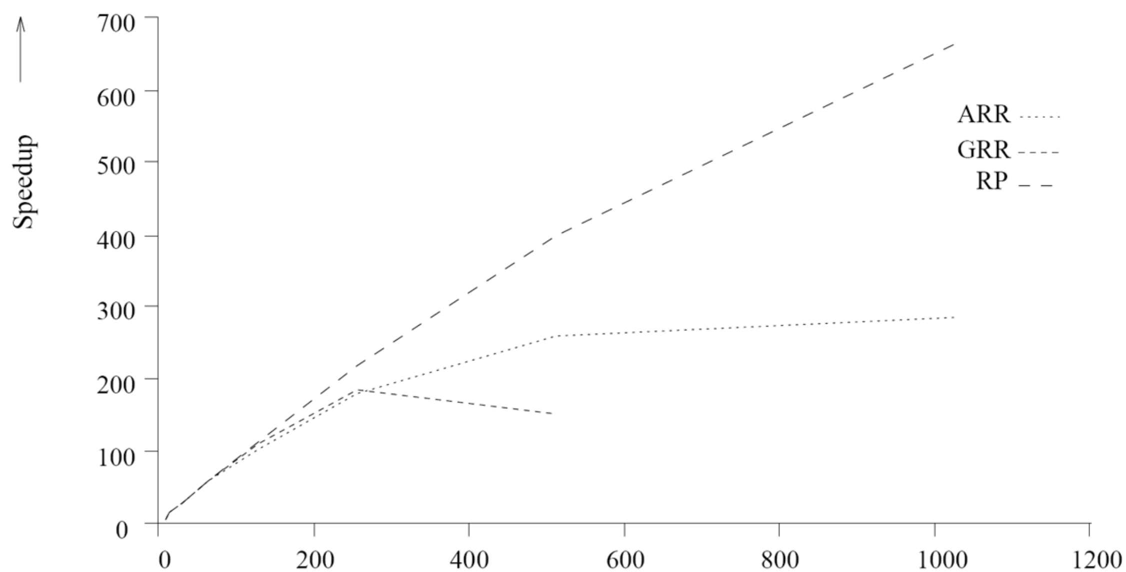


Load balancing overhead

- Since load is generated dynamically, it's hard to estimate the precise overhead. So we make the following simplifying assumptions.
  - If victim processor has W amount of work, after work stealing the victim and stealing processors each have $\geq \alpha W$ amount of work, for $\alpha > 0$.
    - I.e. the work stealing roughly balances the work between the thief and victim.
    - After load balancing, both processors have $\leq (1-\alpha)W$ work.
  - If a processor has $\leq \epsilon$ work, for a small $\epsilon > 0$, it doesn't do load balancing.
- Suppose there are p processors, and let $V(p)$ be total number of steals before every processor receives one steal attempt.
  - $V(p)$ depends on the particular work stealing algorithm.
- Suppose the max amount of work at any processor is currently W. Then after V(p) steals, the max amount is $\leq (1-\alpha)W$, by assumption above.
  - After $2V(p)$ steals, max work at any processor is $\leq (1-\alpha)^2 W$. In general, after $kV(p)$ steals, it's $\leq (1-\alpha)^k W$.
  - So after $\log_{\frac{1}{1-\alpha}}\left(\frac{W}{\epsilon}\right) V(p)$ steals, each processor has $\leq \epsilon$ amount of work.
  - So total overhead from work stealing is $O(V(p) \log W)$.
  - Efficiency is $1/(1 + \frac{t_{comm} V(p) \log W)}{W})$.

Load balancing overhead

- For asynchronous round robin, $p \leq V(p) \leq p^2$, because each processor round robins through victim processors.
  - For isoefficiency need $W = \Omega(V(p) \log W) = \Omega(p^2 \log p)$.
- For global round robin, $V(p) = p$.
  - Each process accesses the victim variable $O(V(p) \log W)$ times.
  - Since the victim variable is shared, the accesses are serialized. So each process takes $O(p \log W)$ time for its accesses.
  - For isoefficiency, need work per process $\frac{W}{p} = \Omega(p \log W)$, so $W = \Omega(p^2 \log p)$.
- For randomized, can prove expected $V(p) = O(p \log p)$.
  - For isoefficiency need $W = \Omega(p \log^2 p)$.



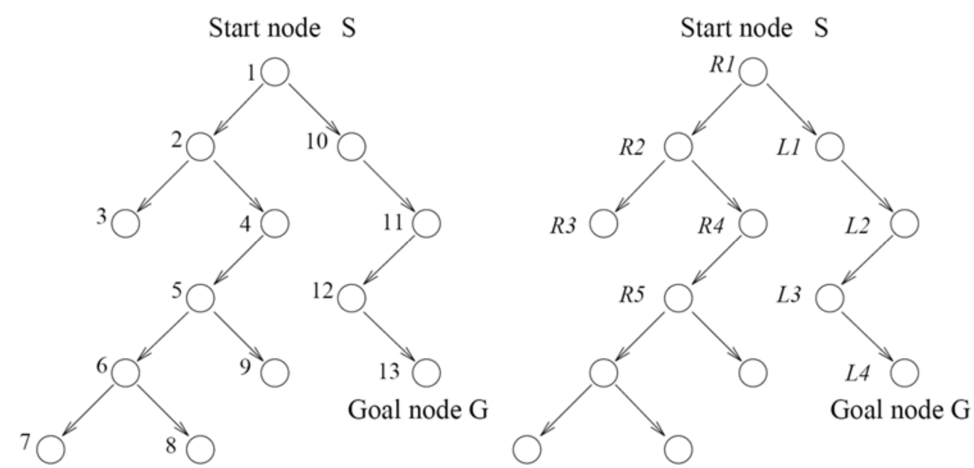
Parallel DFBB and IDA*

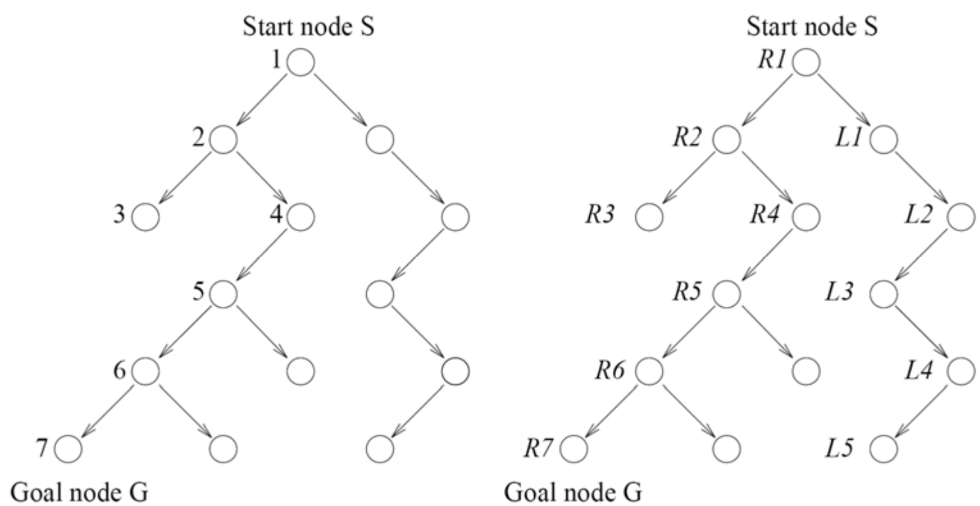
- To parallelize depth first branch and bound, processors store the globally best solution.
  - If a processor finds a better solution, it broadcasts it.
  - Since the best solution changes infrequently, there's not much overhead.
- For parallel IDA*, broadcast the bound for each round of DFS, then parallelize the DFS.

Parallel A*

- In A*, the cost of nodes are stored in a priority queue.
  - Parallelizing A* requires accessing the queue in parallel.
- With p processors, expand smallest p nodes in queue.
  - This expands some nodes that aren't expanded in the sequential algorithm, leading to redundant work.
- Another problem is contention when accessing the queue.
  - To decrease contention, can give each processor its own copy of the queue.
  - If a processor finds a good node (with small f(n)), it should update other nodes so they don't expand non-minimal nodes.
  - More updates reduces redundancy but increases communication, so need to find a tradeoff.
- If the search graph is not a tree, then can visit the same node multiple times, so need duplicate detection.
  - Can store visited nodes in a hash table and distribute hash table among processors.
  - Causes communication for each node visited.
    - Can be amortized if amount of computation per node is large.

Speedup anomalies





- Since parallel search explores nodes in a different order than sequential search, it can explore fewer or more nodes.
- Ex In the upper figures, the sequential search on the left explores 13 nodes before finding the goal. On the right, parallel search using two processors R and L explores 9 nodes.
  - There is a superlinear speedup of 13/5 > 2.
  - This is called an acceleration anomaly.
- Ex In the lower figures, the left sequential search explores 7 nodes, but the right parallel search explores 12 nodes.
  - This is called a deceleration anomaly.

Average speedup in DFS

- In certain simple settings we can analyze the expected speedup using parallel DFS.
- Assume the following
  - The search graph is a tree with M leaves. All the solutions are at the leaves, and there is at least one solution.
  - The number of nodes explored is proportional to the number of leaves explored.
  - The parallel DFS partitions the tree into m (= number processor) equal parts. All processors run at the same speed.
  - The sequential DFS orders the partitions randomly and searches each completely before searching the next partition.
  - In the i’th partition, each leaf has an independent probability ρ_i of being a solution.
  - Both sequential and parallel DFS stop after finding one solution.

Average speedup in DFS

- The expected number of leaves (and hence nodes) searched in the i'th partition is $1/\rho_i$.
- The expected running time W_s of sequential DFS is proportional to $\frac{1}{m}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}+\cdots+\frac{1}{\rho_m}\right)$, since it searches a random partition.
- In parallel DFS, in each parallel step one leaf from each partition is searched. The probability that at least one of them is a solution (and hence DFS stops) is $1-\Pi_{i=1}^{m}(1-\rho_i) \approx \rho_1+\rho_2+\cdots+\rho_m$ for small ρ's.
  - The expected running time W_m of parallel DFS is proportional to $\frac{m}{\rho_1+\rho_2+\cdots+\rho_m}$.
- $W_m = O(1/$ (arithmetic mean of $\rho_1, \ldots, \rho_m$)), and $W_s = O(1/$(harmonic mean of $\rho_1, \ldots, \rho_m$)).
  - Since arithmetic mean ≥ harmonic mean, then $W_m \leq W_s$.
  - $W_m = W_s$ only if $\rho_1 = \rho_2 = \cdots = \rho_m$.
  - If the ρ's aren't equal, then $W_m < W_s$. So we can get superlinear speedup if the solutions aren't equally distributed in the search partitions.