Sampling (ch.7)

- Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of a Signal from Its Samples Using Interpolation
- The Effect of Undersampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

The Sampling Theorem

- What is sampling?

  Converting continuous-time signals to discrete-time signals
- Why sampling?

  To use the well-developed digital technology
- But, a signal could not always be uniquely specified by equally-spaced samples

$$x_1(kT) = x_2(kT) = x_3(kT)$$

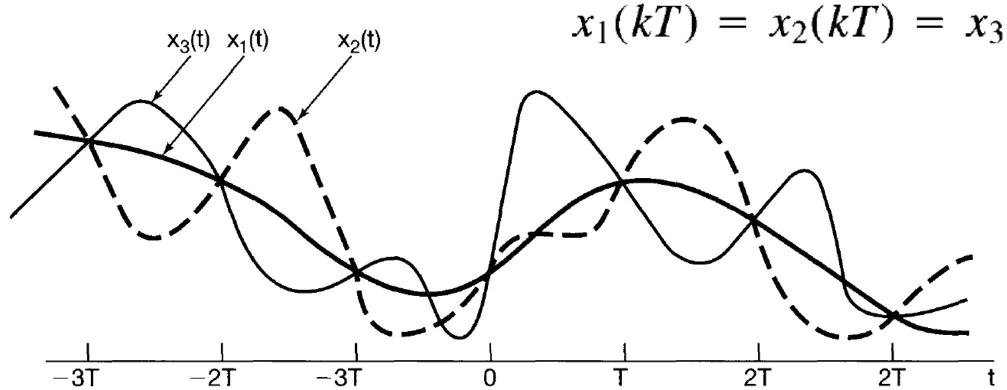
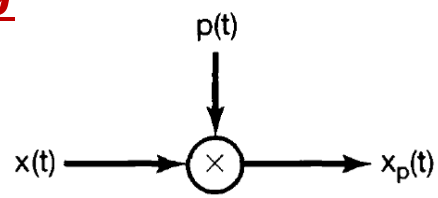


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

- The sampling theorem should be satisfied

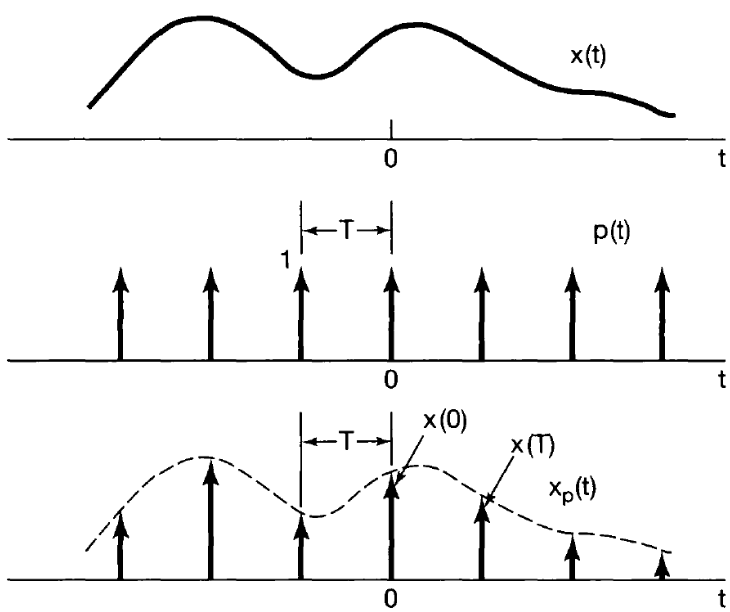
The Sampling Theorem

Impulse-Train Sampling



$$x_p(t) = x(t) \cdot p(t)$$

- Time domain

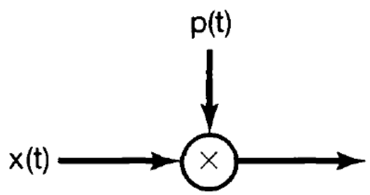


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

The Sampling Theorem

Impulse-Train Sampling



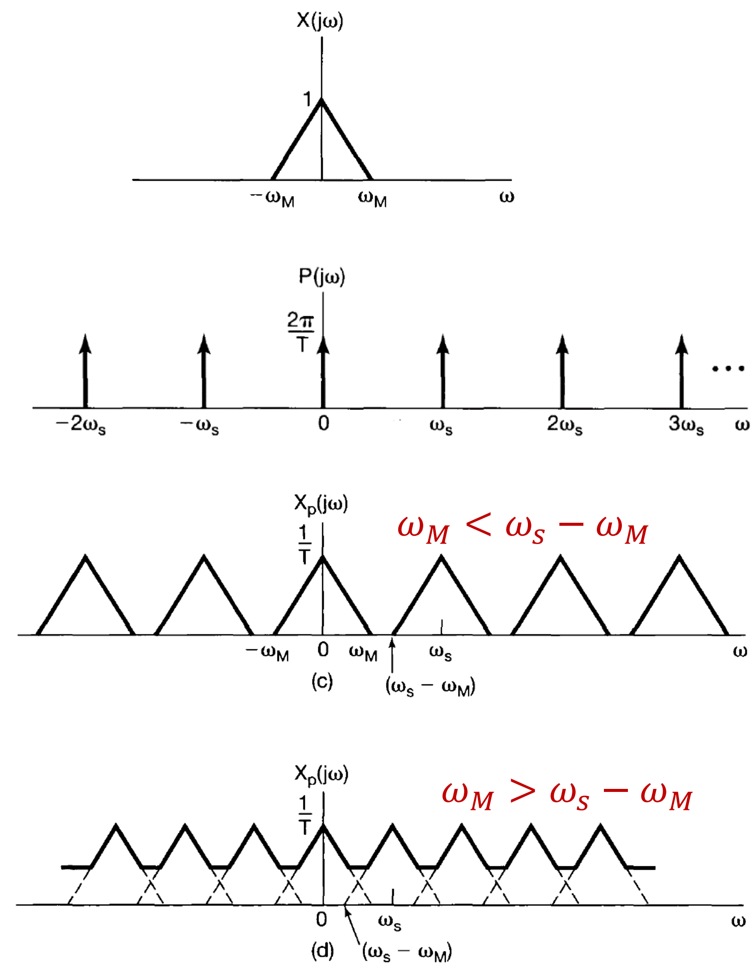
$$x_p(t) = x(t) \cdot p(t)$$

❑ Frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\big(j(\omega - k \cdot \omega_s)\big)$$



The Sampling Theorem

Sampling Theorem

Sampling Theorem:
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M,$$

where

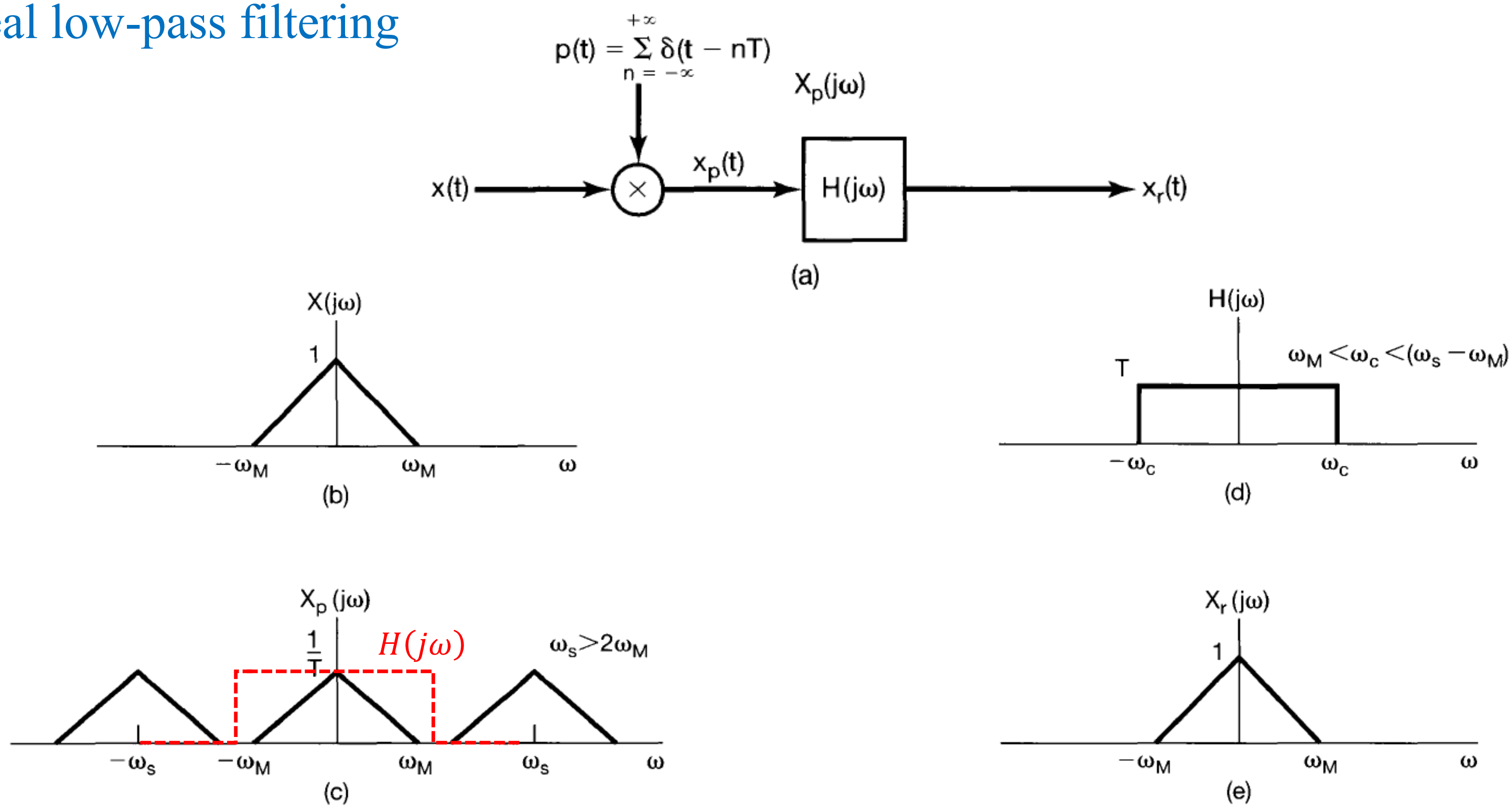
$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

The Sampling Theorem

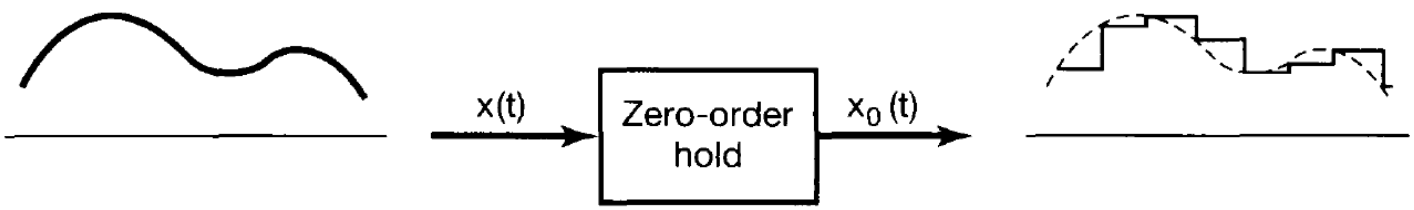
Recovery of the CT signal

- Ideal low-pass filtering



Sampling with a Zero-order Hold

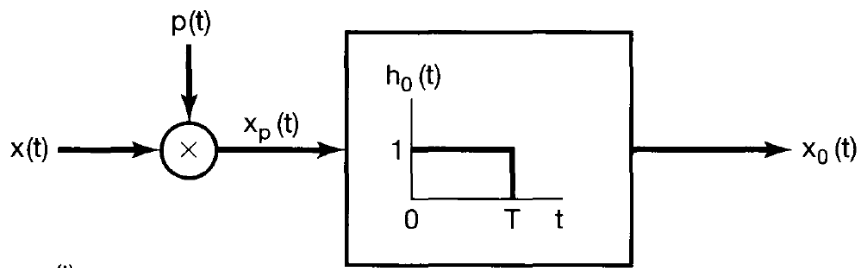
- Why: Impulse-train is difficult to generate
- Principle: Samples $x(t)$ at a given instant and holds that value until the next instant

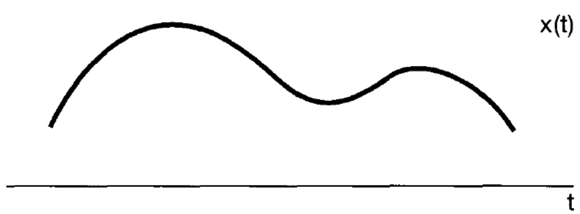


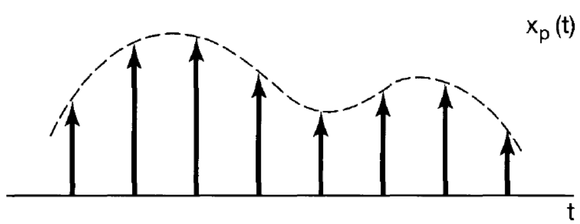
The Sampling Theorem

Sampling with a Zero-order Hold

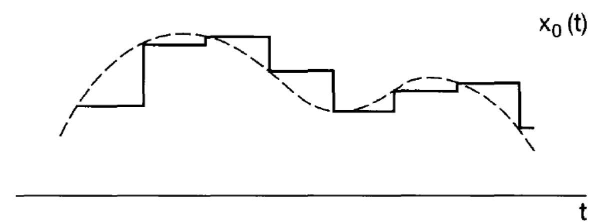
- Equivalent: Impulse-train sampling + an LTI system with a rectangular impulse response







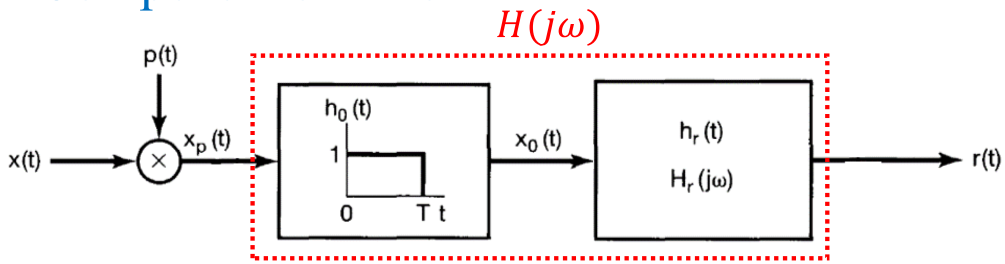
$$x_0(t) = x_p(t) * h(t) \Longrightarrow$$



The Sampling Theorem

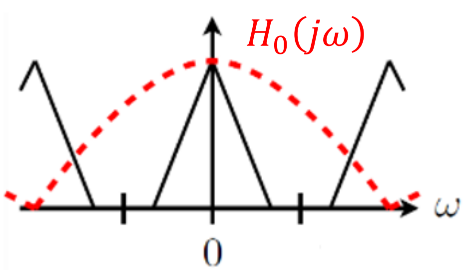
Sampling with a Zero-order Hold

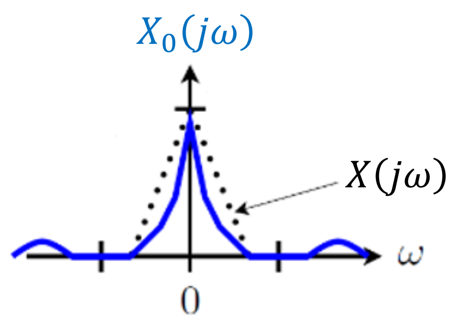
- ❑ Compensation filter



$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin \omega T}{\omega}\right]$$

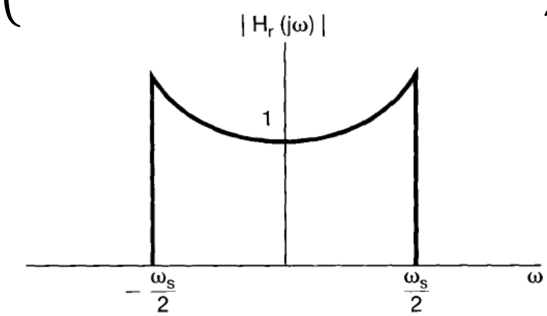
$X_p(j\omega)$

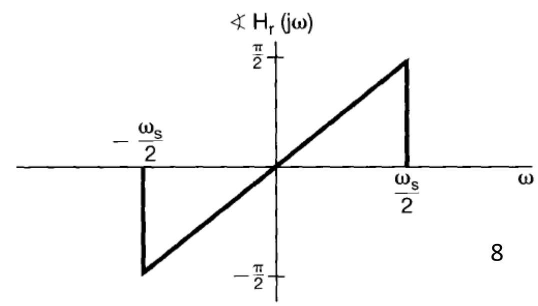




Let $H_0(j\omega)H_r(j\omega) = H(j\omega)$

$$H_r(j\omega) = \begin{cases} e^{j\omega T/2}\Big/\left[\dfrac{2\sin \omega T}{\omega}\right], & |\omega| \le \dfrac{\omega_s}{2} \\ 0, & |\omega| > \dfrac{\omega_s}{2} \end{cases}$$





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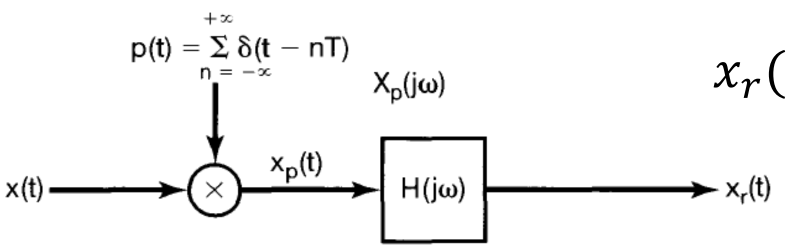
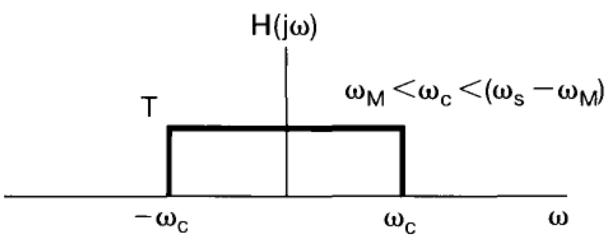
Reconstruction of a Signal Using Interpolation

Band-limited interpolation: (ideal low-pass filter)





$$h(t) = \frac{T\omega_c}{\pi} \frac{\sin \omega_c t}{\omega_c t}$$

$$x_r(t) = x_p(t) * h(t) = \left[\underbrace{\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)}_{x_p(t)} \right] * h(t)$$

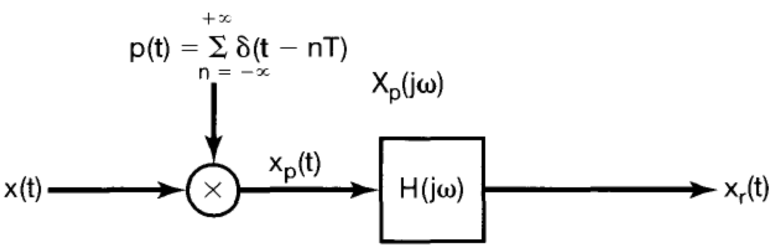
$$= \sum_{n=-\infty}^{\infty} x(nT)[\delta(t - nT) * h(t)]$$

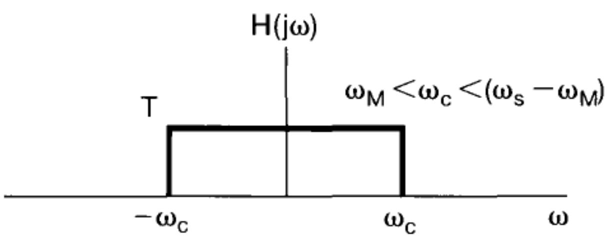
$$= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \frac{T\omega_c}{\pi} \frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$

Reconstruction of a Signal Using Interpolation

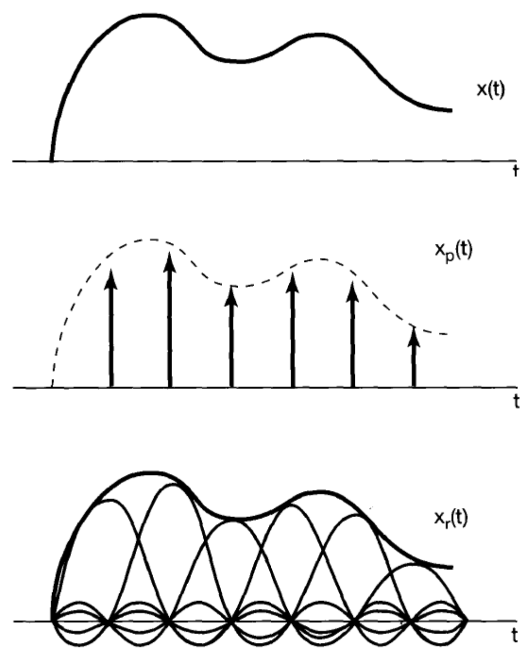
Band-limited interpolation: (ideal low-pass filter)





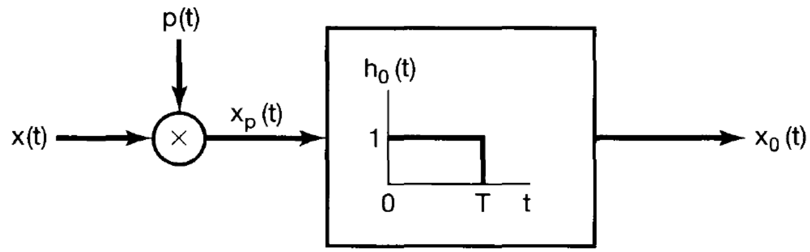
$$h(t) = \frac{T\omega_c}{\pi} \frac{\sin \omega_c t}{\omega_c t}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T\omega_c}{\pi} \frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$

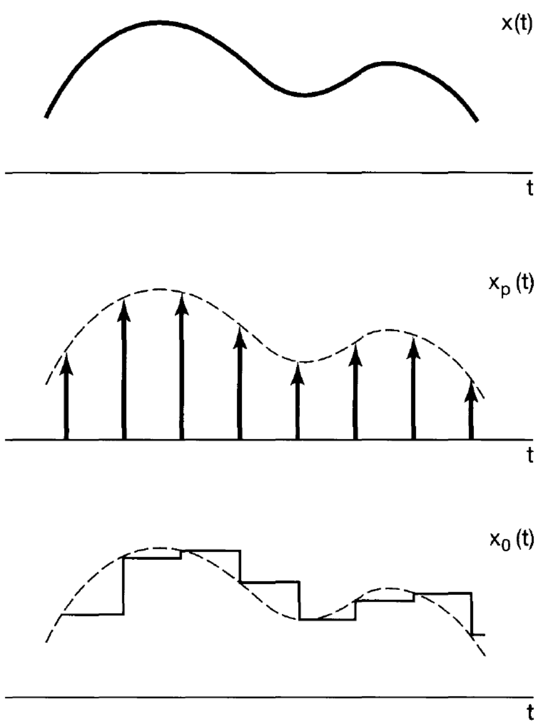


Reconstruction of a Signal Using Interpolation

Zero-order hold

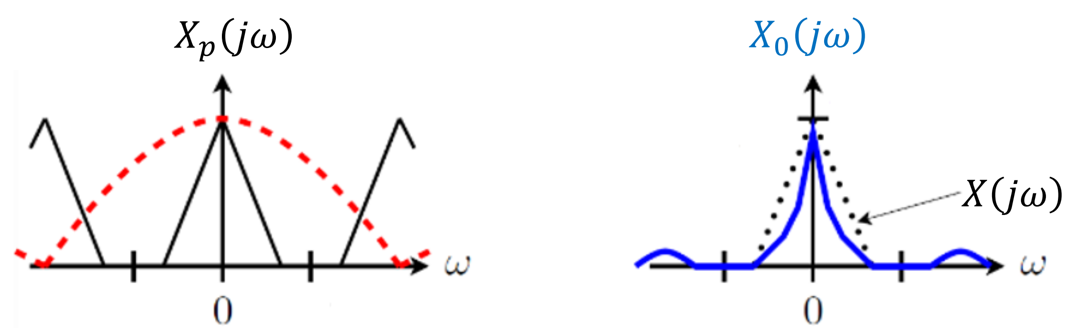


- Time domain



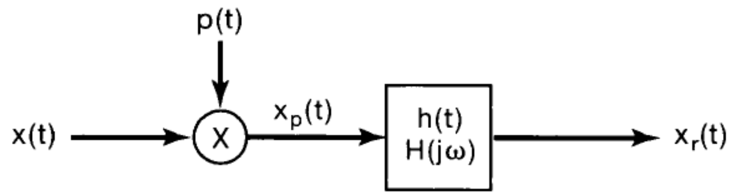
- Frequency domain

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin \omega T}{\omega}\right]$$

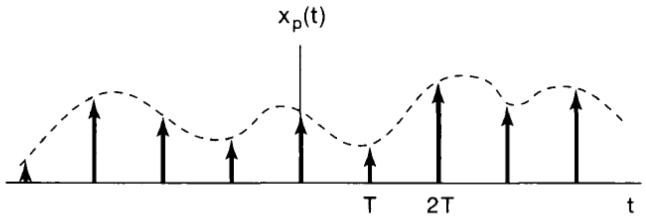


Reconstruction of a Signal Using Interpolation

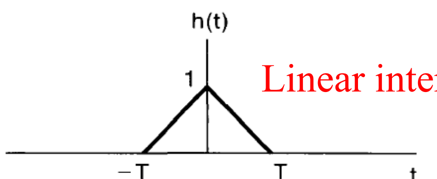
First-order hold: Impulse-train sampling + an LTI system with a tri angular impulse response



- Time domain

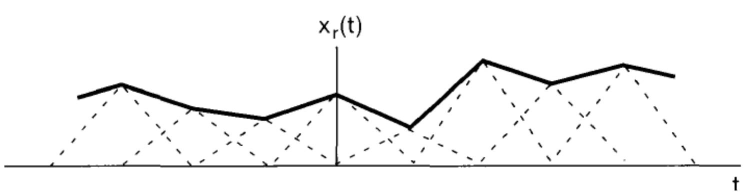


(b)



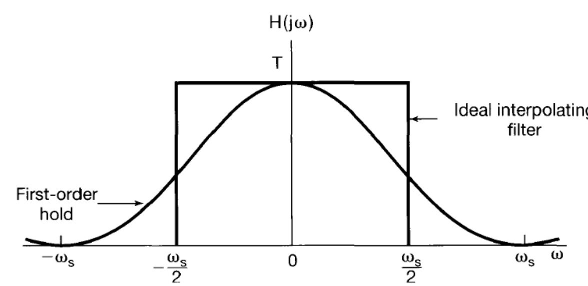
(c)

Linear interpolation



- Frequency domain

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$



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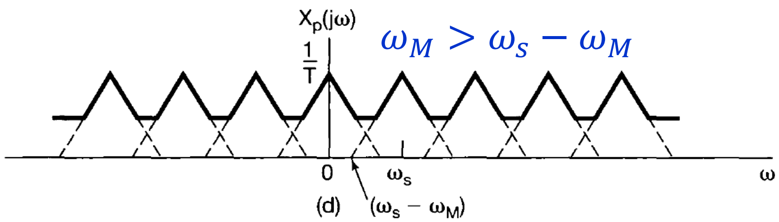
Aliasing

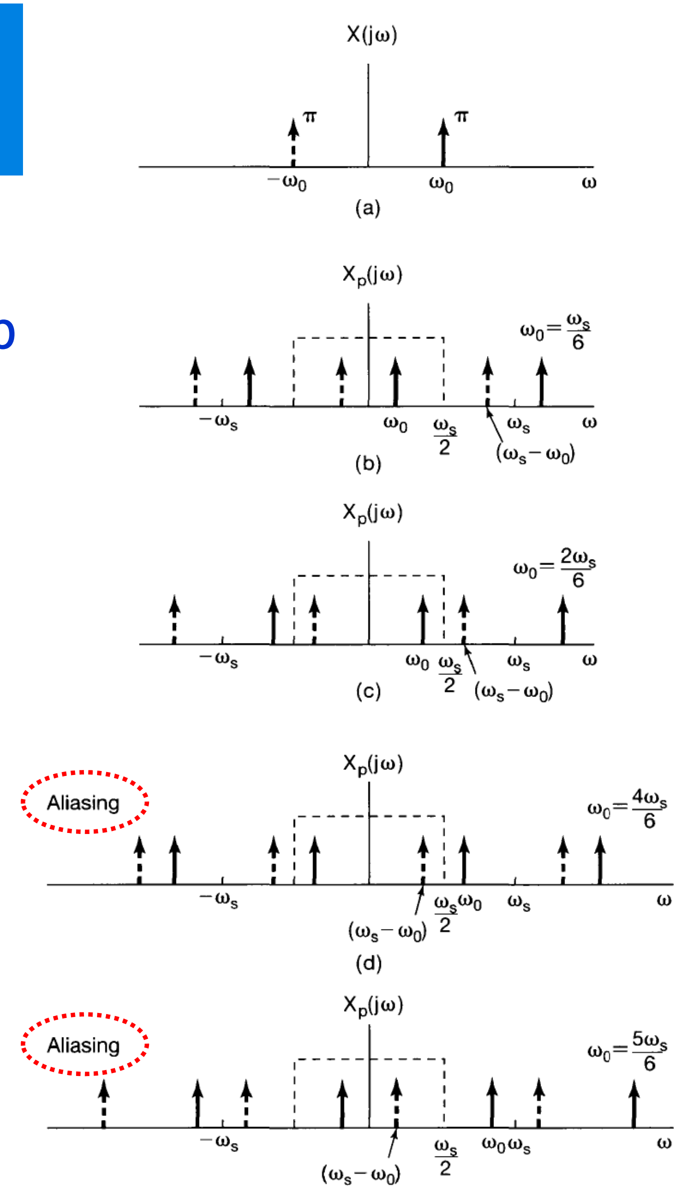
Aliasing

- When $\omega_s < 2\omega_M$, the individual spectrums overlap



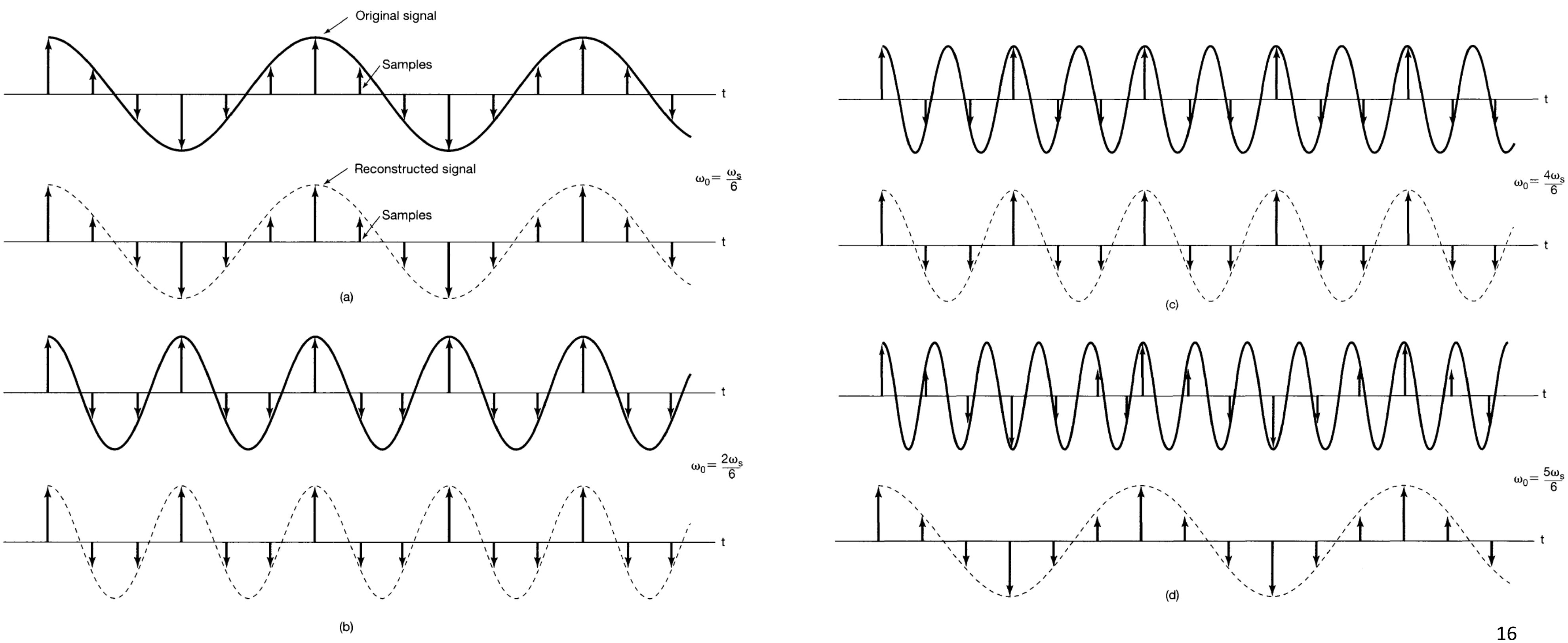
- Consider original signal is $x(t) = \cos\omega_0 t$, with different ω_0 but sampled at same ω_s

  - When aliasing occurs, the original frequency ω_0 takes on the identity of lower frequency $(\omega_s - \omega_0)$.



Aliasing

$x(t) = \cos \omega_0 t$ Time domain



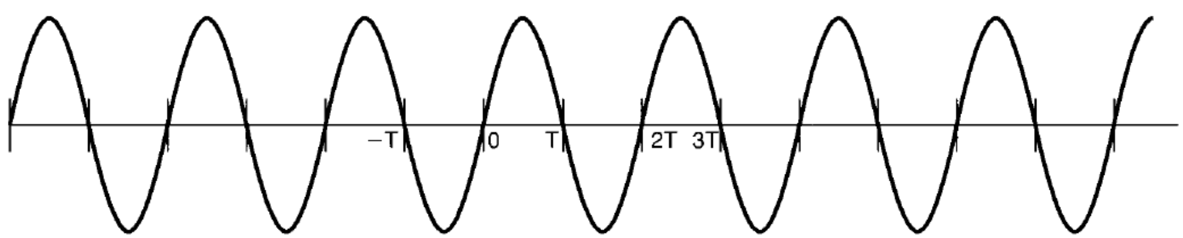
Aliasing

Aliasing

- $\omega_s = 2\omega_M$ is not sufficient to avoid aliasing
  - Consider a signal $x(t) = \cos(\omega_0 t + \emptyset)$ is sampled using impulse sampling with $\omega_s = 2\omega_0$
  - The reconstructed signal using ideal low-pass filter is

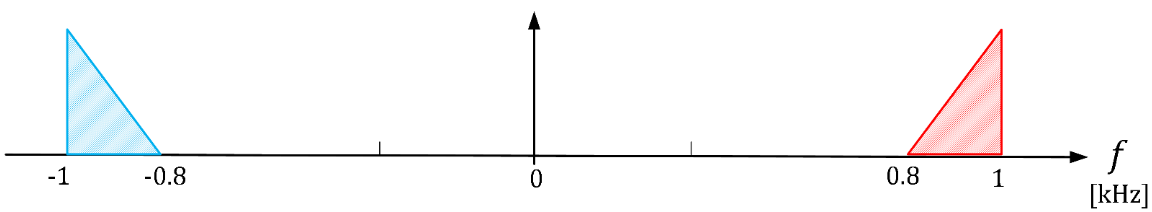
$$x_r(t) = \cos(\emptyset)\cos(\omega_0 t) \ \ {\color{red} = x(t) \text{ only if } \emptyset = 2k\pi}$$

  - Particularly, if $\emptyset = -\pi/2$, then $x(t) = \sin\omega_0 t$ and $x_r(t) = 0$



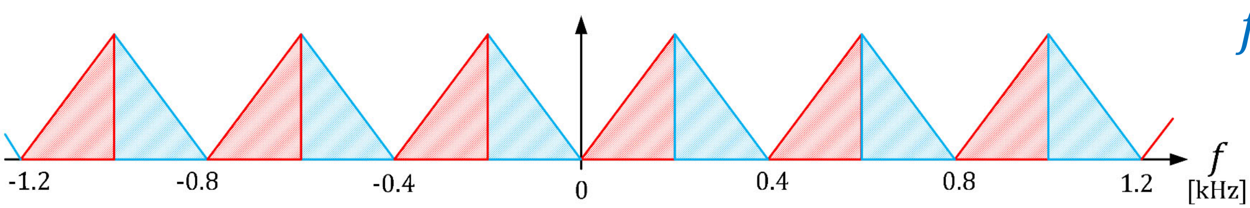
Aliasing

- For signal with $f_c > B/2$, where $f_c = (f_h + f_l)/2$ and $B = f_h - f_l$



$f_l = 800$ Hz, $f_h = 1000$ Hz

Determine the lowest f_s with no aliasing

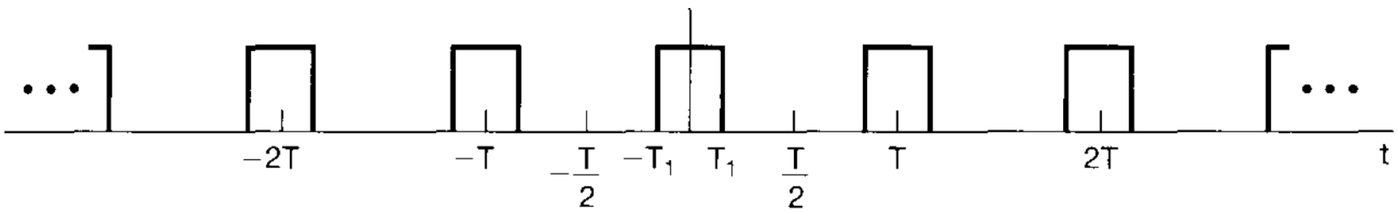


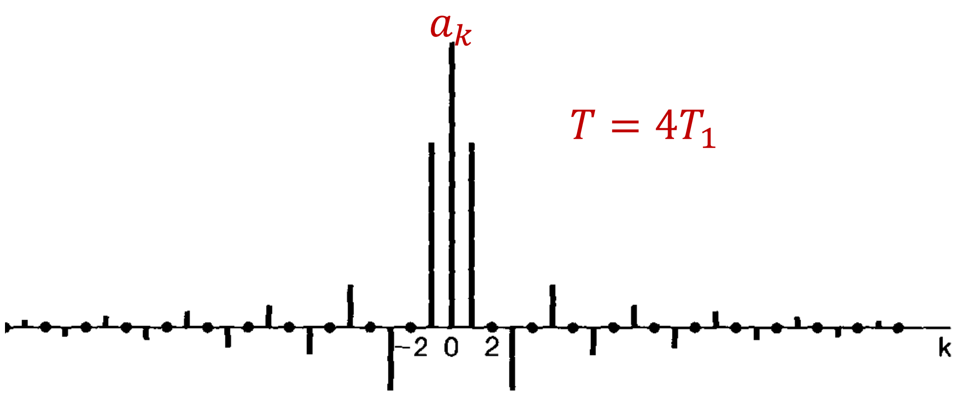
$f_s = 400$ Hz

Q: What about $f_l = 850$ Hz?

Aliasing

- For harmonic related signal, e.g., a square wave





- $\omega_s > 2K\omega_0$, with K the kth harmonics you want to include
- Low-pass filtering before sampling

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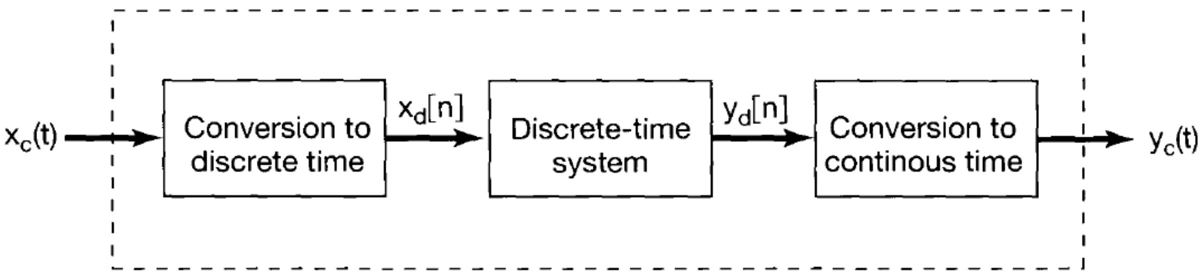
Discrete-Time Processing of Continuous-Time Signals

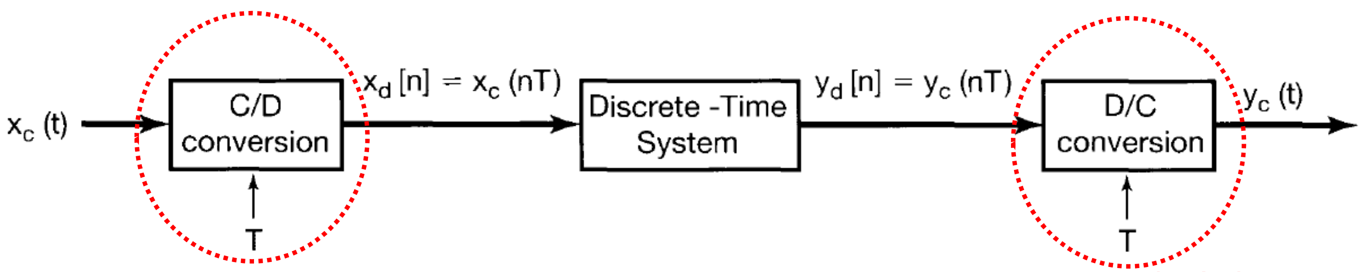
General scheme



$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

- C/D: continuous-to-discrete-time conversion
- D/C: discrete-to-continuous-time conversion

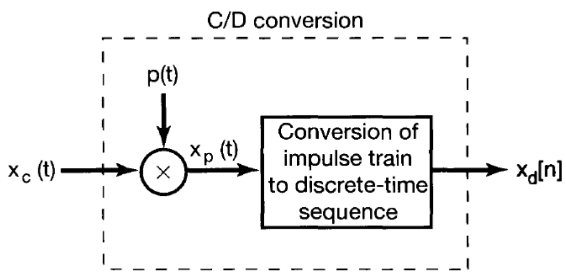


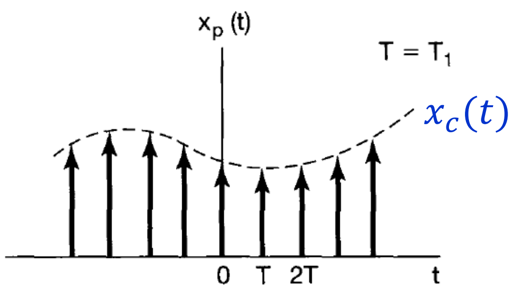
analog-to-digital (A/D) in practice

digital-to-analog (D/A) in practice

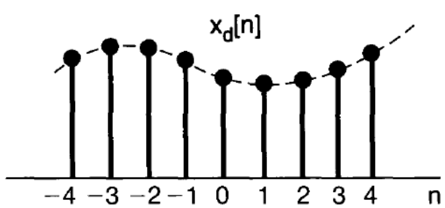
Discrete-Time Processing of Continuous-Time Signals

C/D conversion Time domain





$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

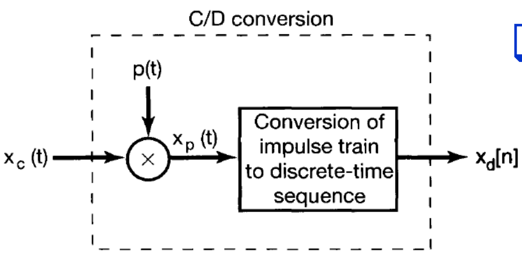


$$x_d[n] = x_c(nT)$$

Discrete-Time Processing of Continuous-Time Signals

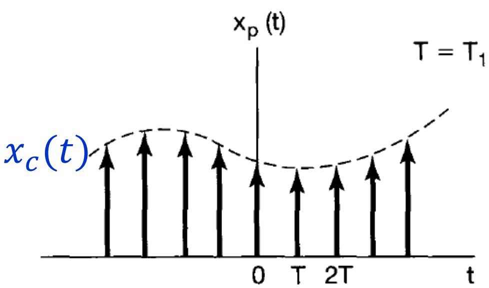
C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time



- Spectrum of $x_d[n]$

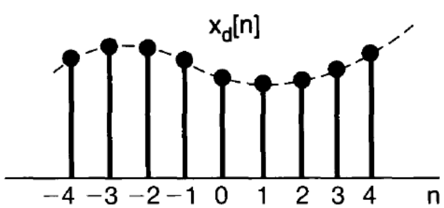
$$X_d(e^{j\Omega}) = \sum_{-\infty}^{\infty} x_d[n]e^{-jn\Omega} = \sum_{-\infty}^{\infty} x_c(nT)e^{-jn\Omega}$$



- Spectrum of $x_p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\cdot\delta(t-nT) \implies X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\cdot e^{-j\omega nT}$$

- If $\omega = \Omega/T$, $\boxed{X_d(e^{j\Omega}) = X_p(j\Omega/T)}$

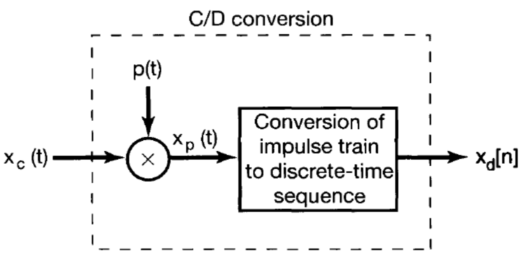


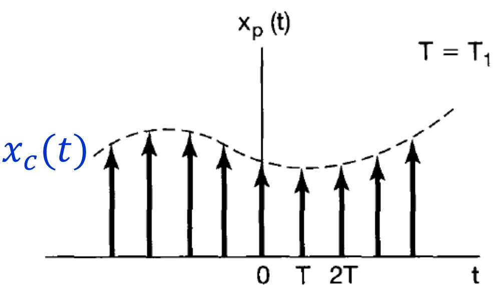
- The spectrum of $x_d[n]$ can be obtained from $X_p(j\omega)$ by replacing ω with Ω/T.

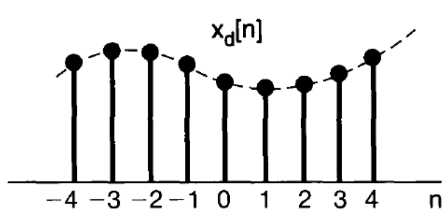
Discrete-Time Processing of Continuous-Time Signals

C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time





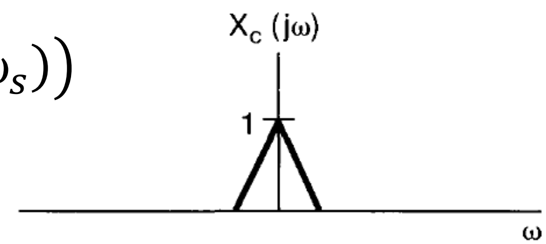


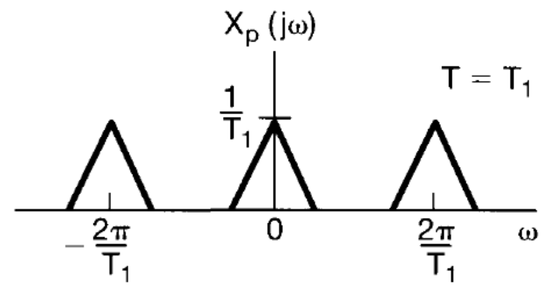
- Recall: $X_p(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\big(j(\omega - k\cdot\omega_s)\big)$

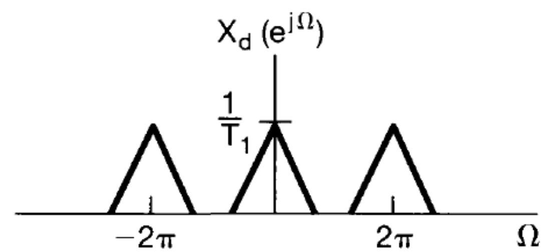
$$\therefore X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j(\Omega - 2k\pi)/T)$$

- $X_d\left(e^{j\Omega}\right)$ is a frequency-scaled version of $X_p(j\omega)$, and is periodic with period of 2π

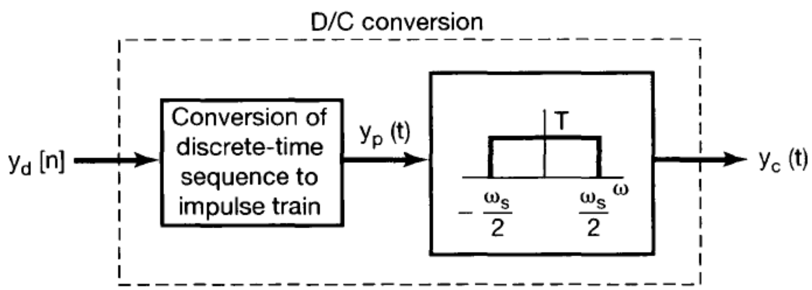
- Informally
  - t to n: time scaling by $1/\,T$
  - ω to Ω: frequency scaling by T







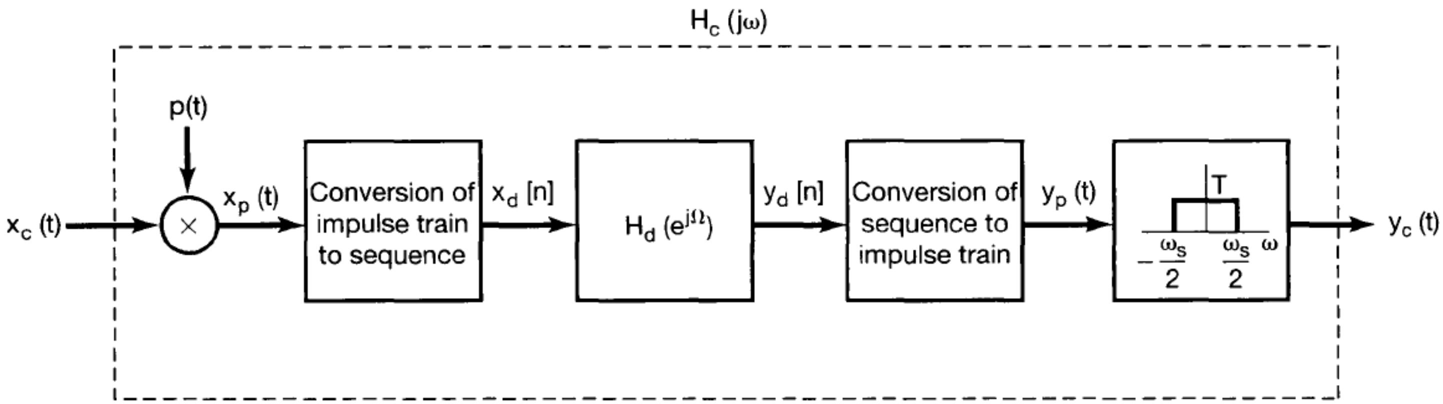
D/C conversion



- $Y_d(e^{j\Omega})$: Spectrum of $y_d[n]$
- $Y_p(j\omega)$: Spectrum of $y_p(t)$
- $Y_p(j\omega)$ can be obtained from $Y_d(e^{j\Omega})$ by replacing Ω with ωT.

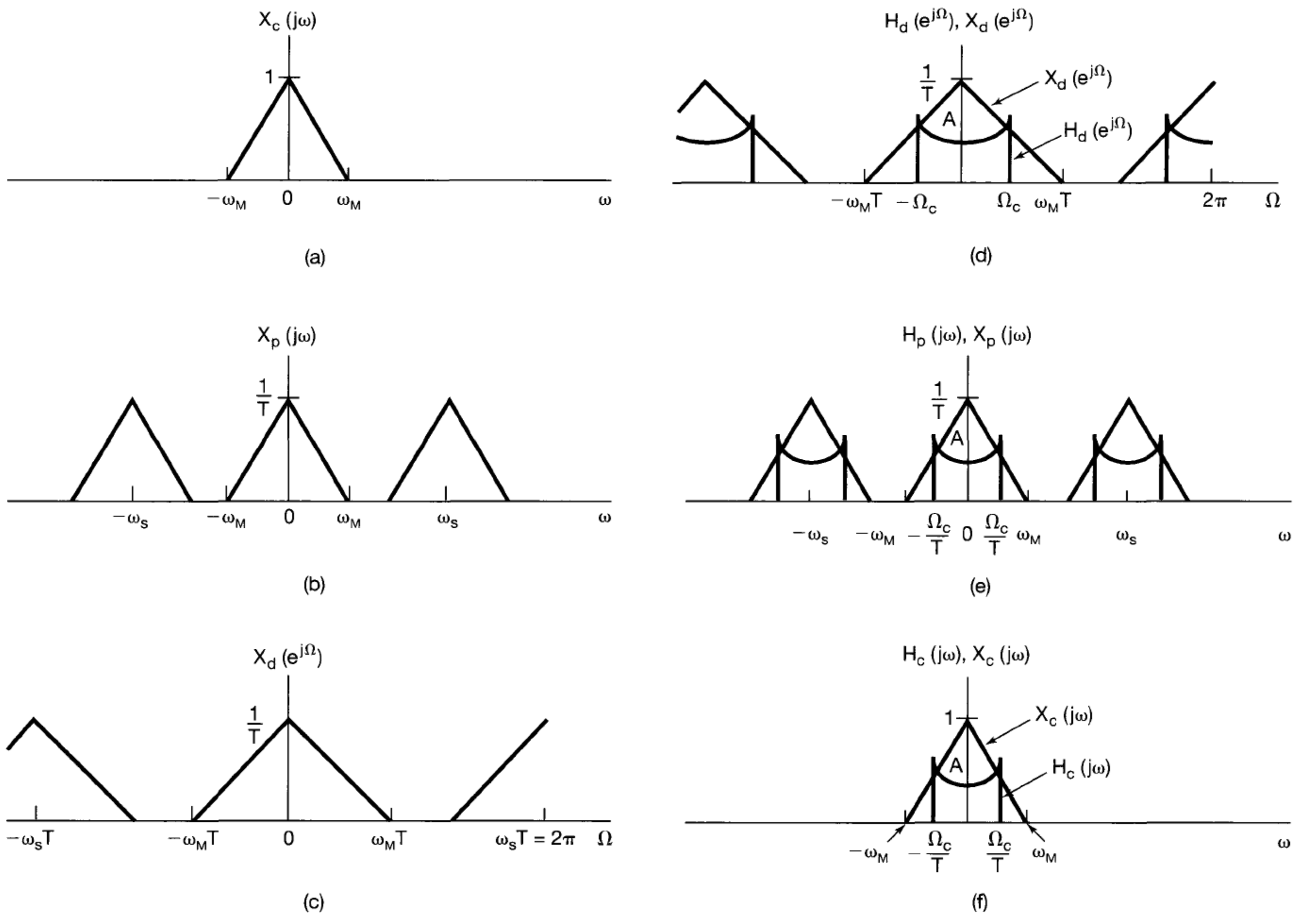
Discrete-Time Processing of Continuous-Time Signals

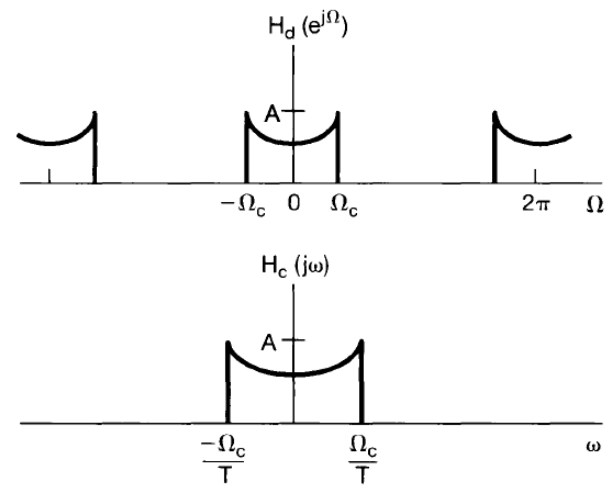
Overall system



- $x_c(t)$: input
- $y_c(t)$: output
- The overall system is equivalent to a continuous-time system with frequency response $H_c(j\omega)$
- $H_c(j\omega) = ?$

Overall system

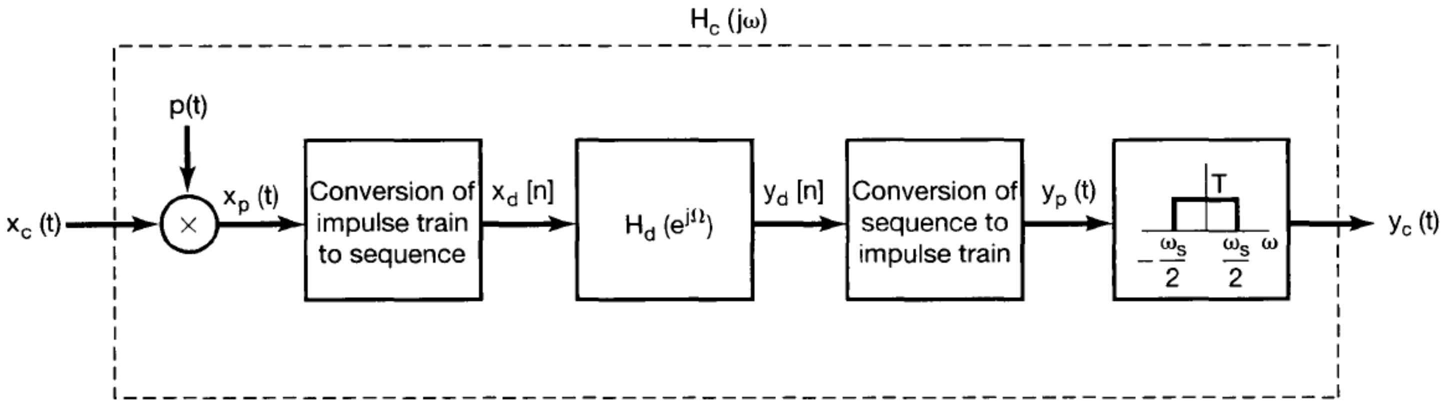




$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

Discrete-Time Processing of Continuous-Time Signals

Overall system



$$\Updownarrow \qquad \omega = \Omega/T$$

$H_c(j\omega)$

$x_c(t)$

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

$y_c(t)$

Discrete-Time Processing of Continuous-Time Signals

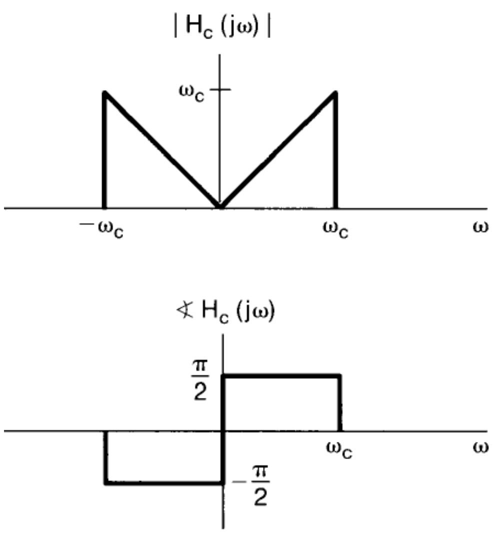
Digital differentiator: frequency response

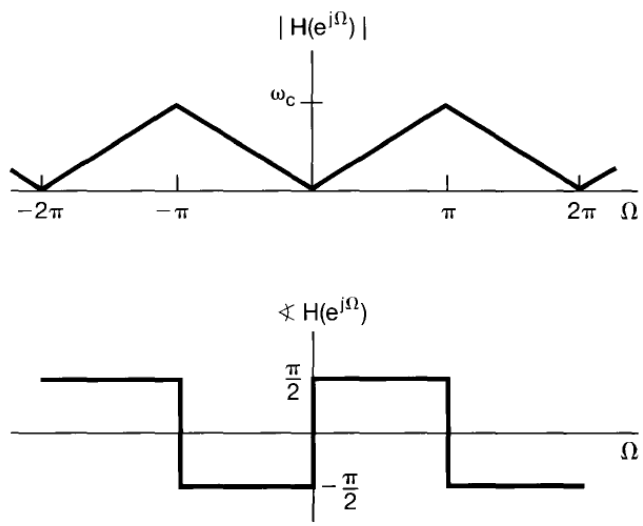
- ❑ Band-limited CT differentiator $\Longleftrightarrow$ ❑ Corresponding DT differentiator

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$\boxed{H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi}$$

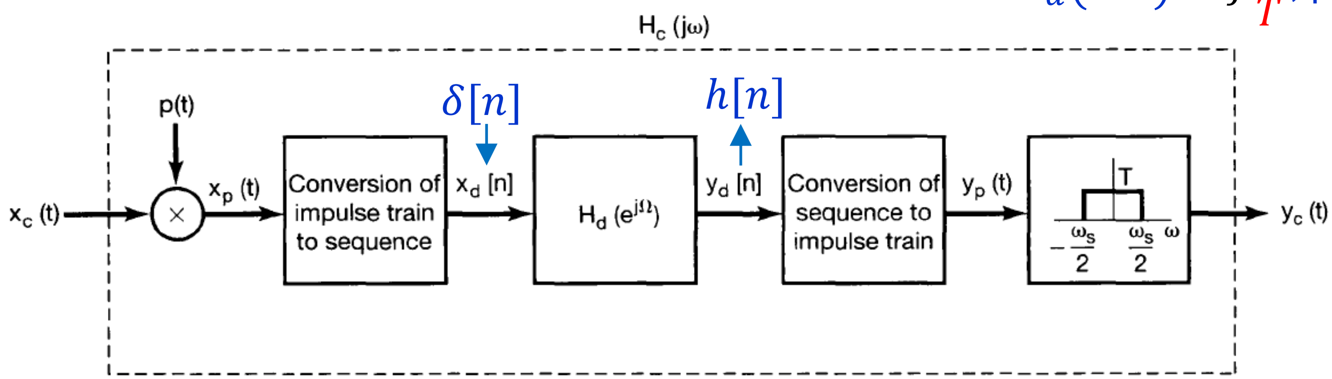
$\omega_c = \omega_s/2$





Digital differentiator: impulse response

$H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi$



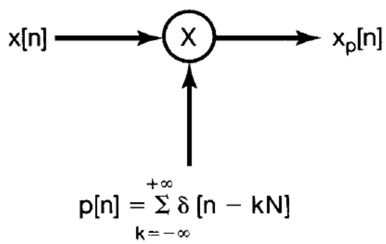
- $x_c(t) = \frac{\sin(\pi t/T)}{\pi t} \Longrightarrow x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$
- $y_d[n] = y_c(nT)$ $\qquad y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}$
- $y_d[n] = \begin{cases} \frac{(-1)^n}{nT^2}, n \neq 0 \\ 0, \; n = 0 \end{cases} \Longrightarrow \boxed{\therefore h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, n \neq 0 \\ 0, \qquad n = 0 \end{cases}}$

Sampling (ch.7)

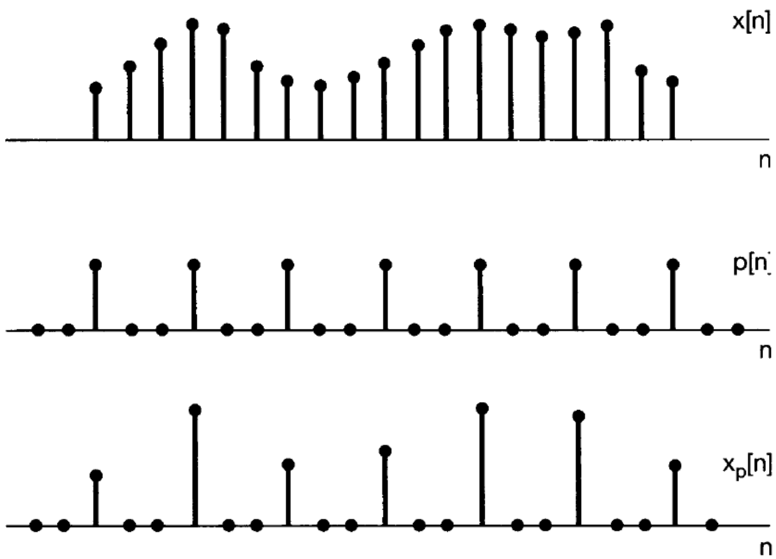
- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time signals

Sampling of Discrete-Time Signals

Impulse train sampling Time domain



N: sampling period



$$x_p[n] = x[n]p[n] = \sum_{-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$= \begin{cases} x[n], \text{if } n \text{ is an integer multiple of N} \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

Sampling of Discrete-Time Signals

Impulse train sampling

Frequency domain

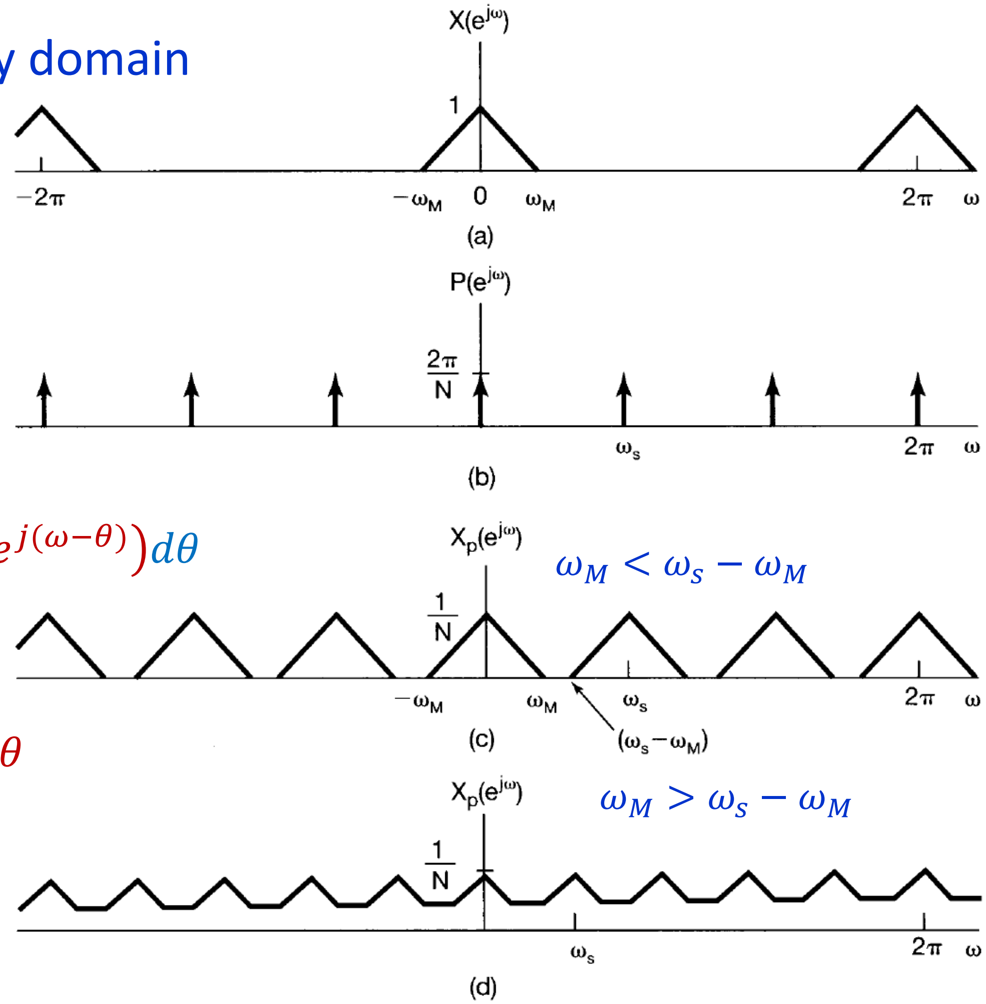
$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad \omega_s = \frac{2\pi}{N}$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{2\pi} \cdot \frac{2\pi}{N} \int_{2\pi} \left[\sum_{K=-\infty}^{\infty} \delta(\theta - k\omega_s) \right] X(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} \int_{2\pi} \delta(\theta - k\omega_s) X(e^{j(\omega-\theta)}) d\theta$$

$$\boxed{X_p(e^{j\omega}) = \frac{1}{N} \sum_{K=0}^{N-1} X(e^{j(\omega - k\cdot\omega_s)})}$$

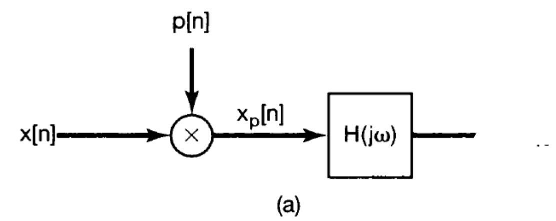


$\omega_M < \omega_s - \omega_M$

$\omega_M > \omega_s - \omega_M$

Sampling of Discrete-Time Signals

Impulse train sampling Reconstruction of x[n]



Frequency domain

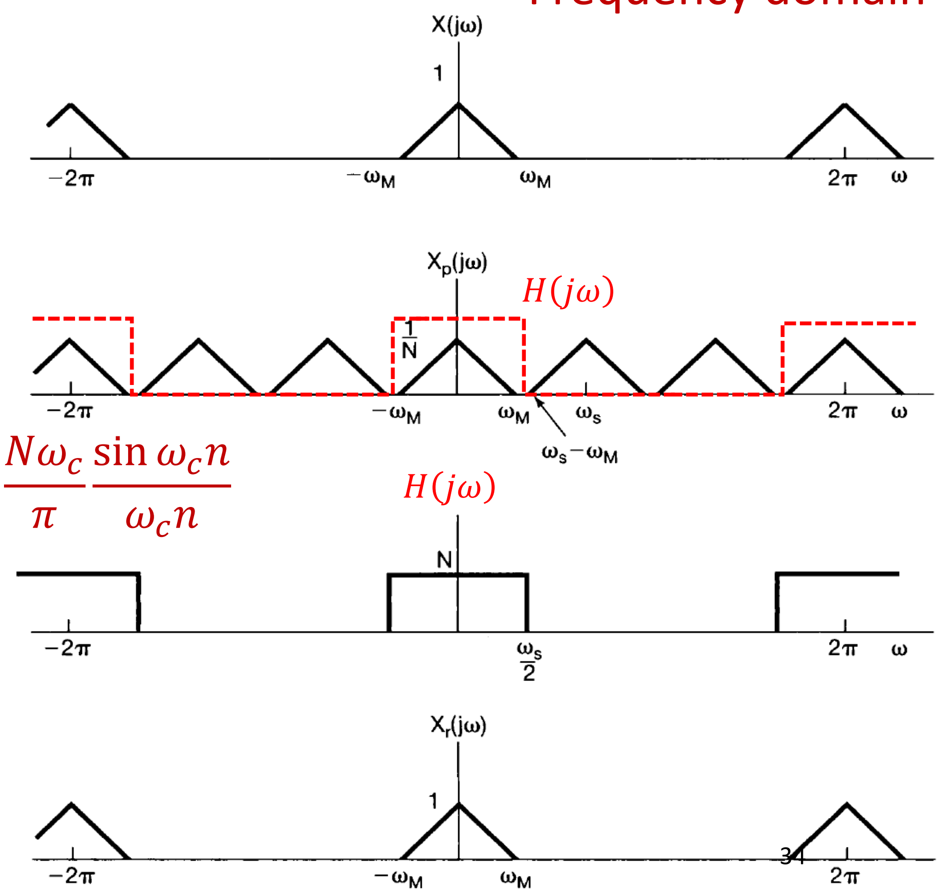
$$x_r[n] = x_p[n] * h[n]$$ Time domain

$$= \left[\sum_{k=-\infty}^{\infty} x[kN]\cdot\delta[n-kN]\right] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[kN][\delta[n-kN] * h[n]]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]h[n-kN]$$

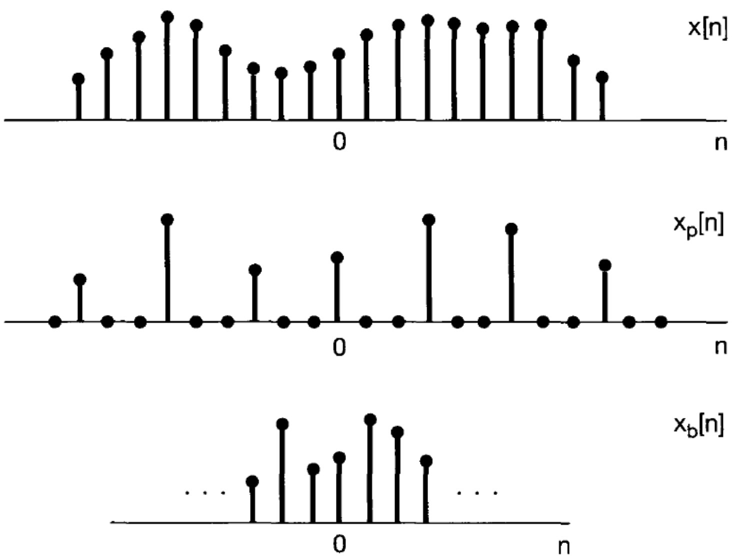
$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}$$



$$\boxed{x_r[n] = \sum_{k=-\infty}^{\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin\omega_c(n-kN)}{\omega_c(n-kN)}}$$

Decimation (sample rate decrease, SRD)

Time domain



$$x_b[n] = x_p[nN]$$

$$x_b[n] = x[nN]$$

Frequency domain

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_b[n]e^{-j\omega k}$$

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_p[kN]e^{-j\omega k}$$

$n = kN$

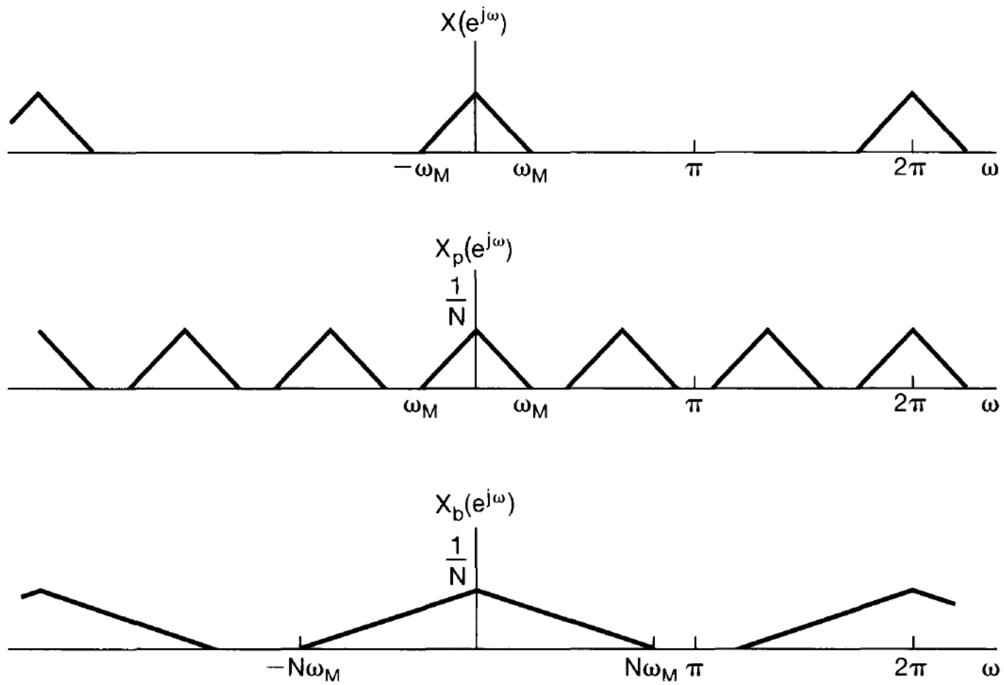
$$X_b(e^{j\omega}) = \sum_{\substack{n=\text{integer} \\ \text{number of N}}} x_p[n]e^{-j\omega n/N}$$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N}$$

$$\boxed{X_b(e^{j\omega}) = X_p(e^{j\omega/N})}$$

Sampling of Discrete-Time Signals

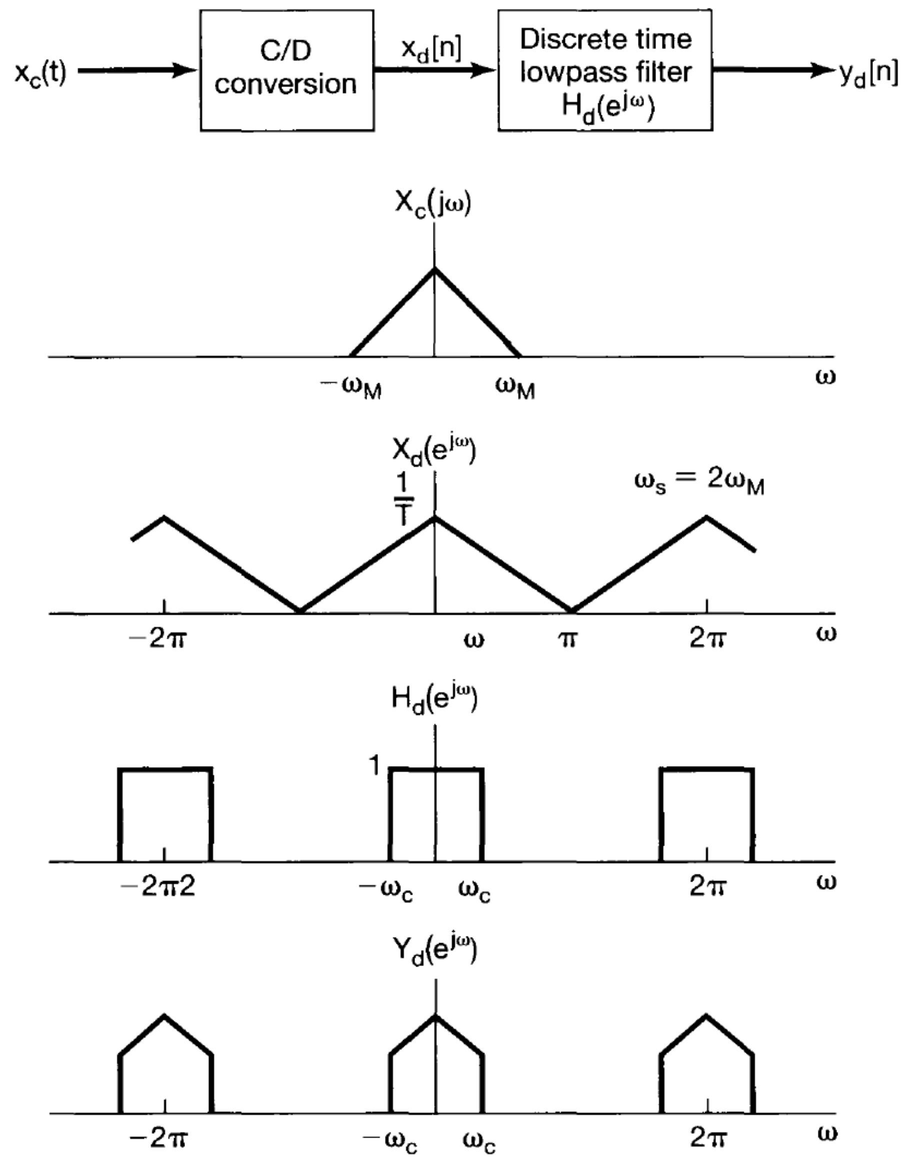
Decimation



$$x_b[n] = x_p[nN] \qquad X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

Down-sampling if $\omega_s = \frac{2\pi}{N} > 2\omega_m$

Sampling of Discrete-Time

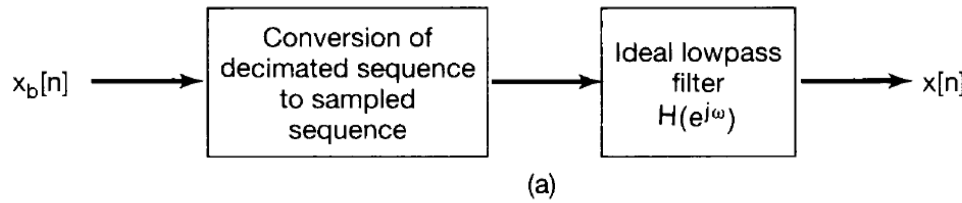


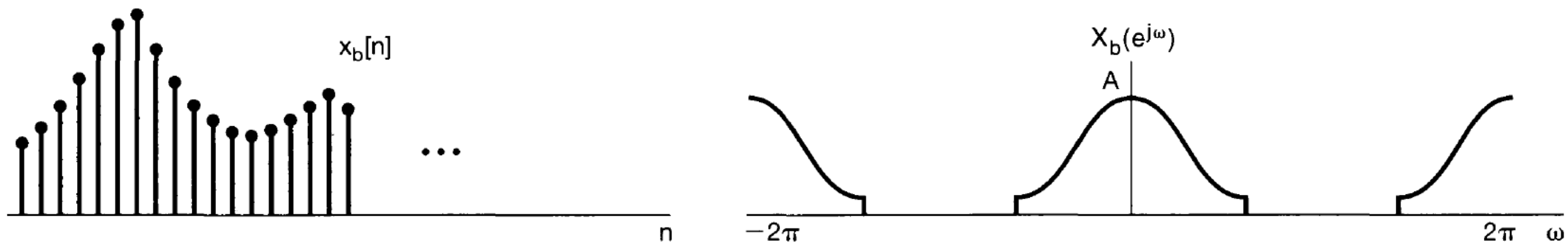
Decimation

- Prevent aliasing by LPF in front of SRD $\Longrightarrow$ Decimator

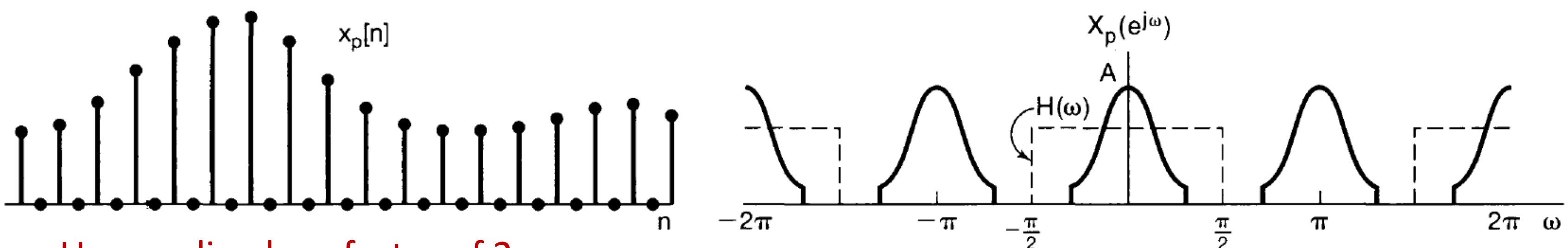
Sampling of

Interpolation (SRI)

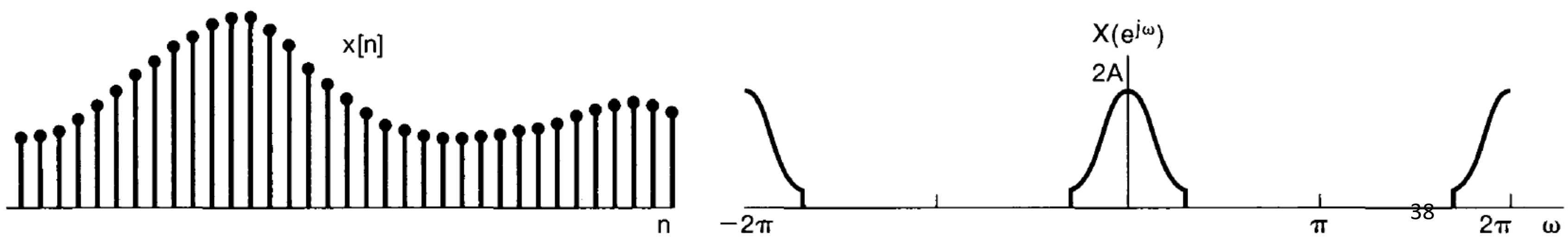




- Prevent mirrors by LPF after SRI $\Longrightarrow$ Interpolator

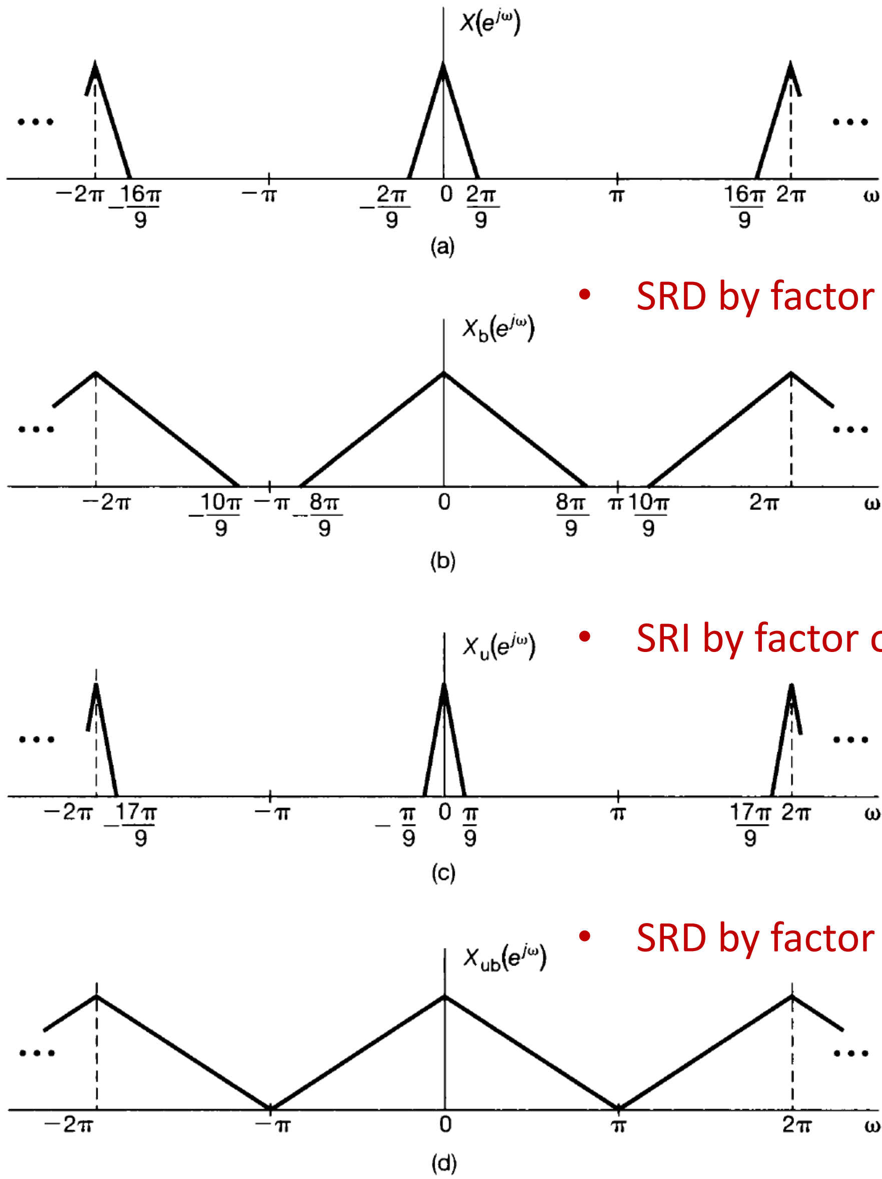


Upsampling by a factor of 2



Sampling of Discrete-

Interpolation (SRI)



- SRD by factor of 4
- SRI by factor of 8
- SRD by factor of 9

- Overall, SRI by factor of 4.5