



Lecture 17: Deep Generative Models II: AE & VAE

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Outline

- Representation learning
 - AutoEncoder
- Variational Autoencoders (VAEs)
 - VAE objective
 - Reparametrization trick
 - Connection to Auto-Encoders

Acknowledgement: Feifei Li et al's cs231n notes

Recall EM GMM

- MLE: maximizing the log-likelihood

$$\ell(\theta) = \sum_{i=1}^n \log p(x^{(i)}; \theta)$$

- ELBO: Evidence Lower Bound

$$\log p(\mathbf{x}) = \log \int_{\mathbf{z}} p(\mathbf{x}, \mathbf{z})$$

$$= \log \int_{\mathbf{z}} p(\mathbf{x}, \mathbf{z}) \frac{q(\mathbf{z})}{q(\mathbf{z})}$$

$$= \log \left(E_q \left[\frac{p(\mathbf{x}, \mathbf{z})}{q(\mathbf{z})} \right] \right)$$

$$\geq E_q [\log p(\mathbf{x}, \mathbf{z})] - E_q [\log q(\mathbf{z})]$$

$$= ELBO$$

$$KL(q(\mathbf{z}) || p(\mathbf{z} | \mathbf{x})) = E_q [\log q(\mathbf{z})] - E_q [\log p(\mathbf{z} | \mathbf{x})]$$

$$= E_q [\log q(\mathbf{z})] - E_q [\log p(\mathbf{z}, \mathbf{x})] + \log p(\mathbf{x})$$

$$= \log p(\mathbf{x}) - (E_q [\log p(\mathbf{z}, \mathbf{x})] - E_q [\log q(\mathbf{z})])$$

$$= \log p(\mathbf{x}) - ELBO$$



$$EBLO = E_q [\log p(\mathbf{x}, \mathbf{z})] - E_q [\log q(\mathbf{z})]$$

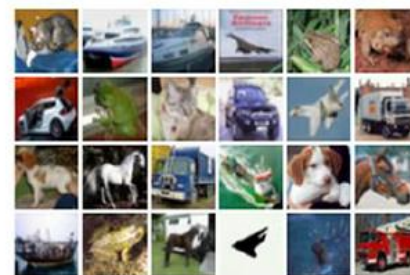
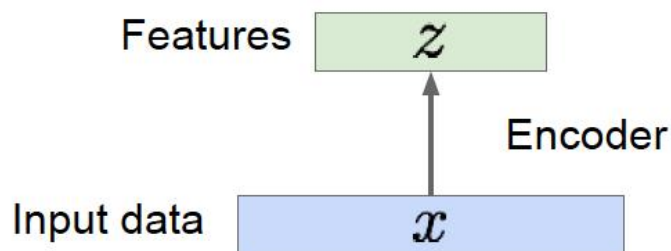
$$= E_q \left[\log \frac{p(\mathbf{x}, \mathbf{z})}{p(\mathbf{z})} \right] - E_q \left[\log \frac{q(\mathbf{z})}{p(\mathbf{z})} \right]$$

$$= E_q [\log p(\mathbf{x} | \mathbf{z})] - KL(q(\mathbf{z}) || p(\mathbf{z}))$$

Autoencoder

- Feature representation learning

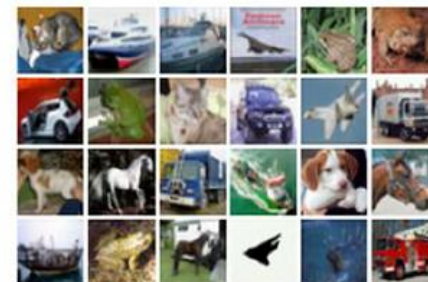
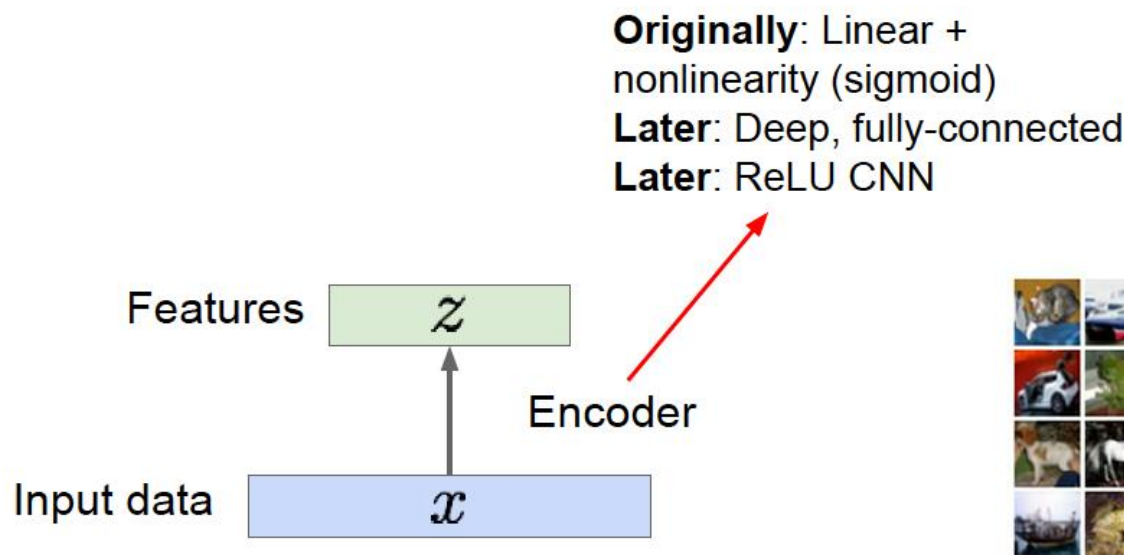
Unsupervised approach for learning a lower-dimensional feature representation from unlabeled training data



Autoencoder

■ Feature representation learning

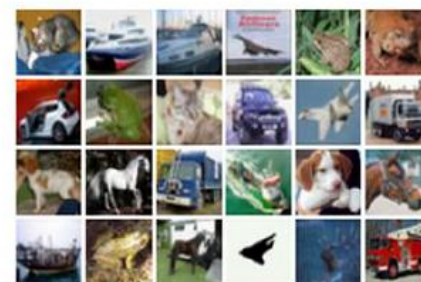
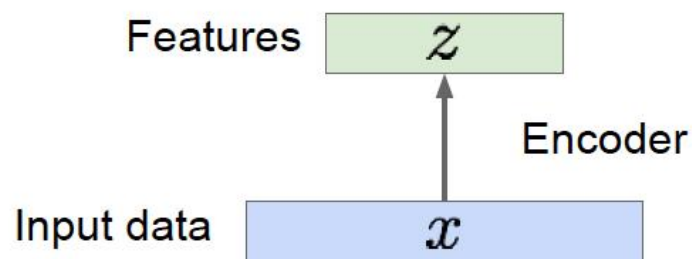
Unsupervised approach for learning a lower-dimensional feature representation from unlabeled training data



Autoencoder

- Feature representation learning

How to learn this feature representation?

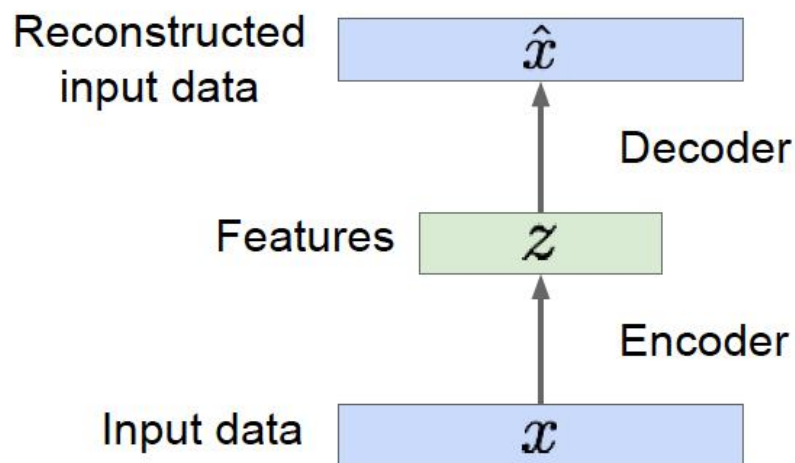


Autoencoder

■ Feature representation learning

How to learn this feature representation?

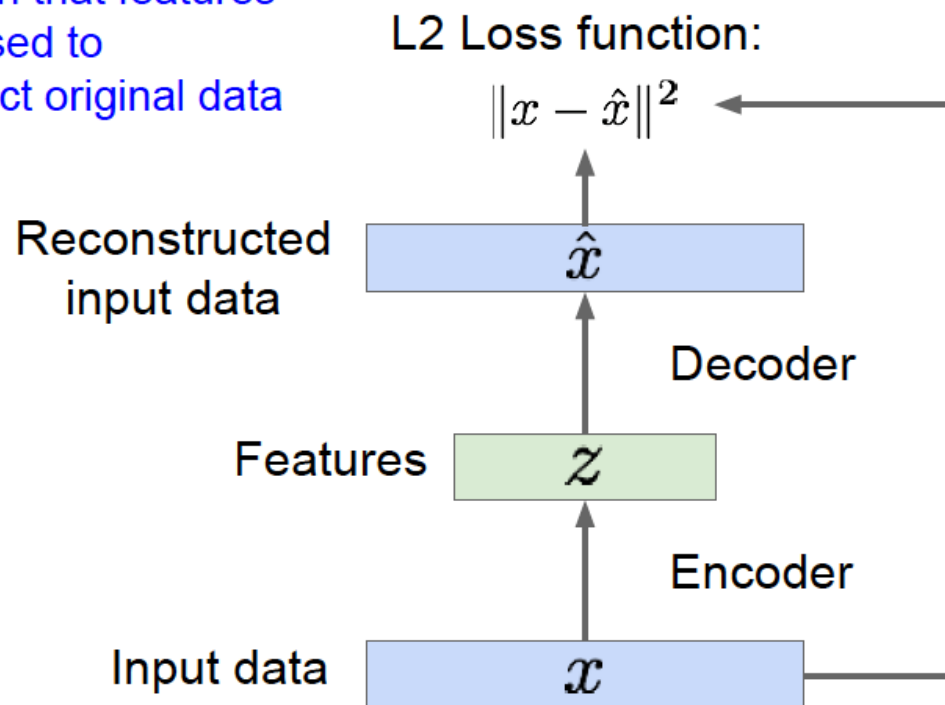
Train such that features can be used to reconstruct original data
“Autoencoding” - encoding itself



Autoencoder

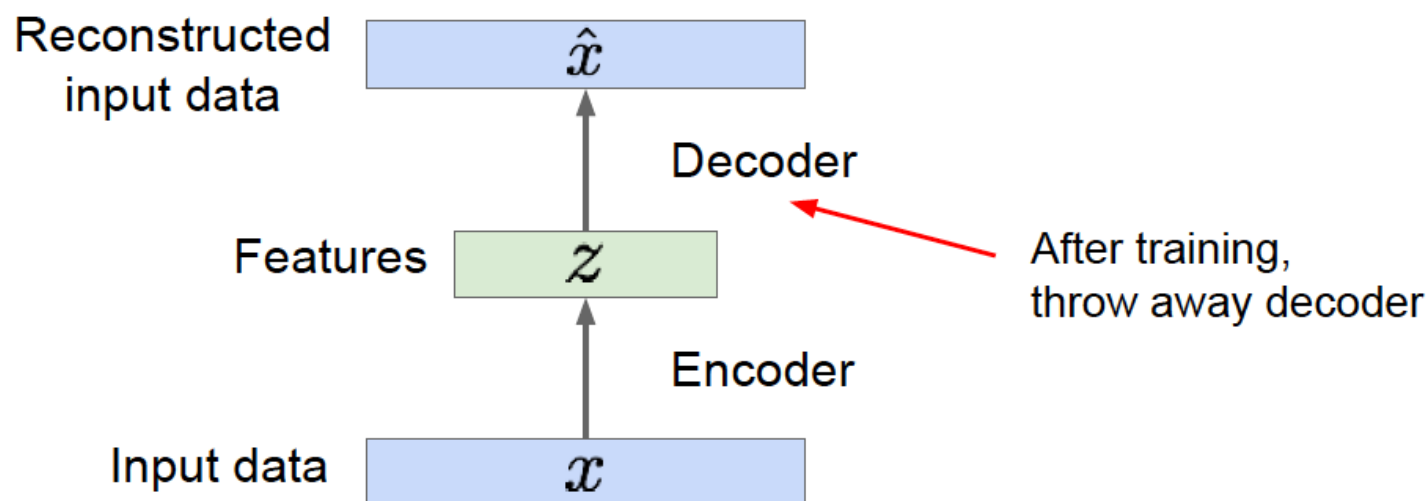
- Feature representation learning

Train such that features
can be used to
reconstruct original data



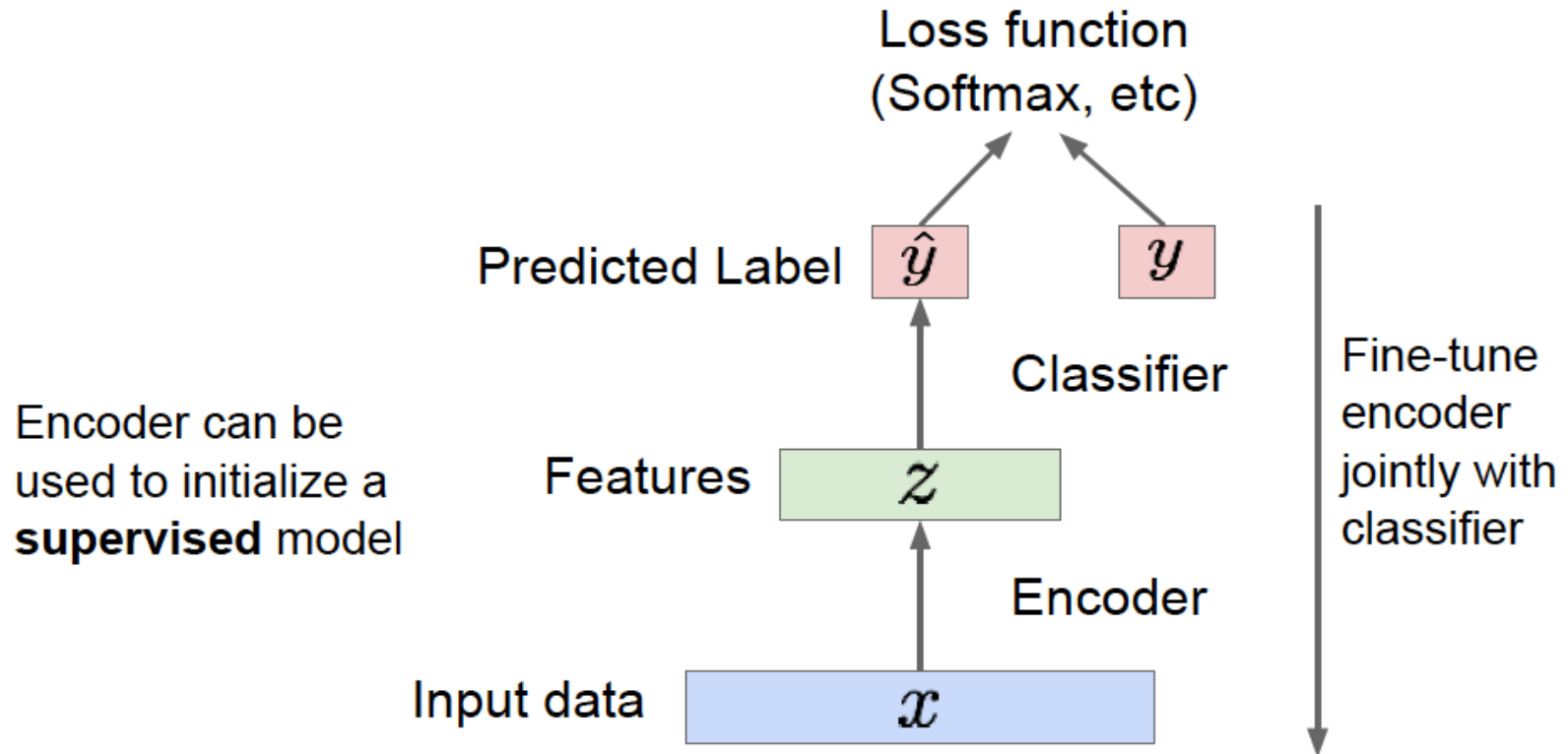
Autoencoder

- Feature representation learning



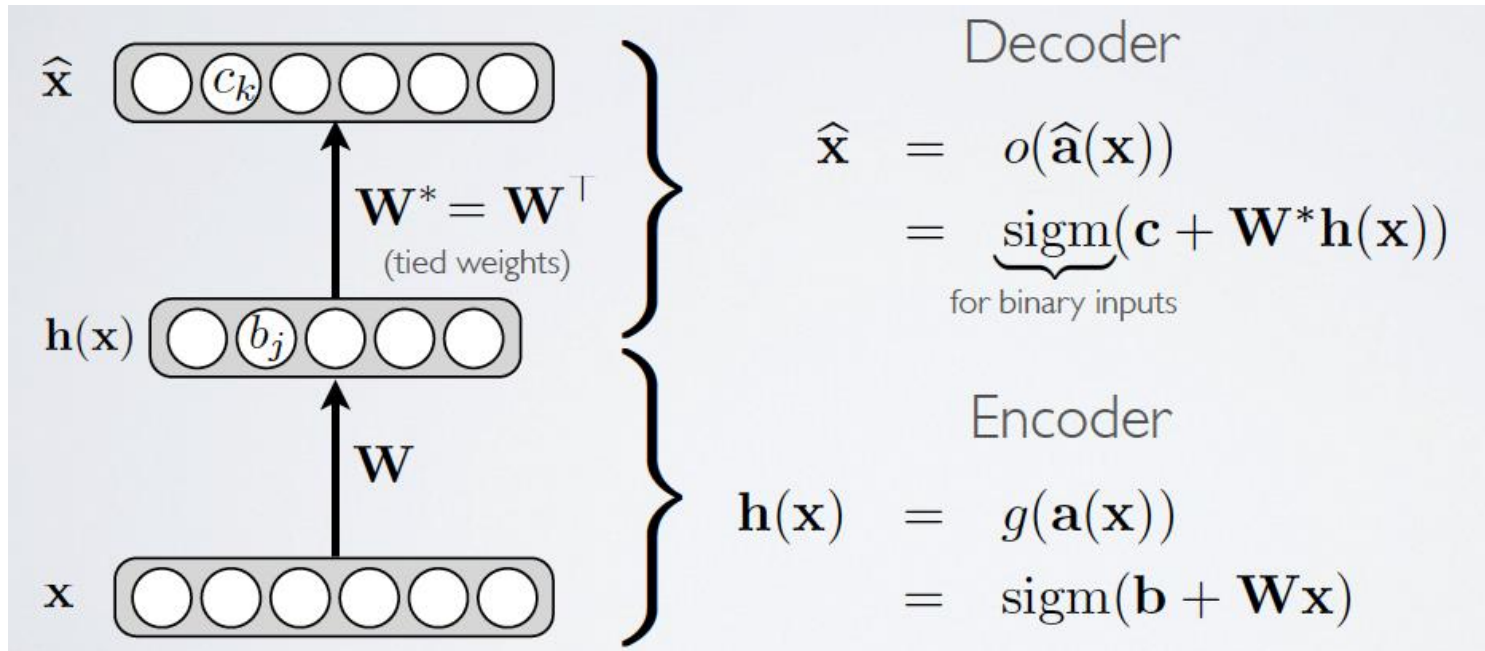
Autoencoder

- Feature representation learning



Autoencoder

■ Binary input example



$$l(f(\mathbf{x})) = - \sum_k (x_k \log(\hat{x}_k) + (1 - x_k) \log(1 - \hat{x}_k))$$

- cross-entropy (more precisely: sum of Bernoulli cross-entropies)

Regularization

- Regularized autoencoders: add regularization term that encourages the model to have other properties
 - Sparsity of the representation (sparse autoencoder)
 - Robustness to noise or to the missing inputs (denoising autoencoder)
 - Smallness of the derivative of the representation (contractive autoencoder)

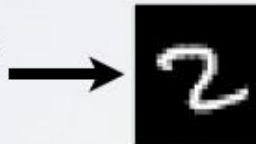
Regularization

■ Undercomplete representation

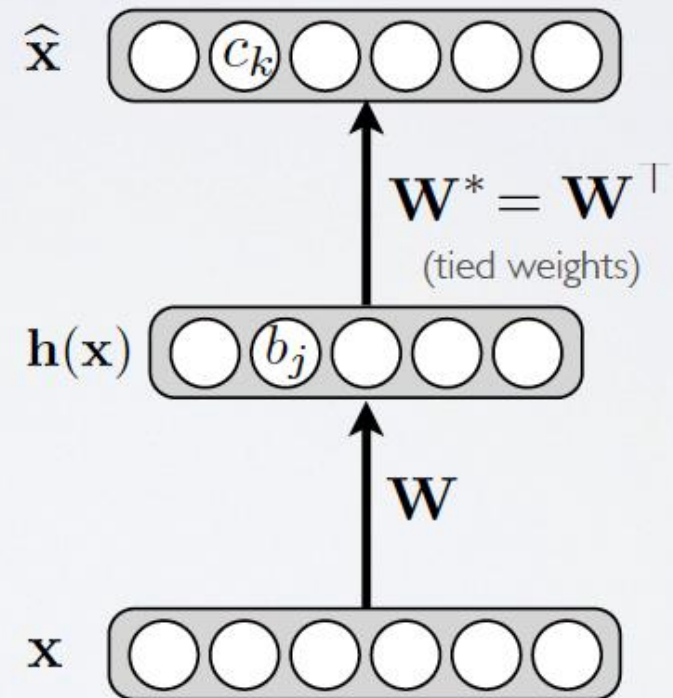
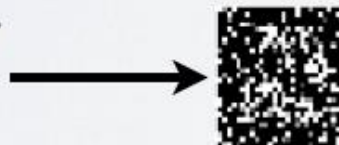
- Hidden layer is undercomplete if smaller than the input layer
 - ▶ hidden layer “compresses” the input
 - ▶ will compress well only for the training distribution

- Hidden units will be

- ▶ good features for the training distribution

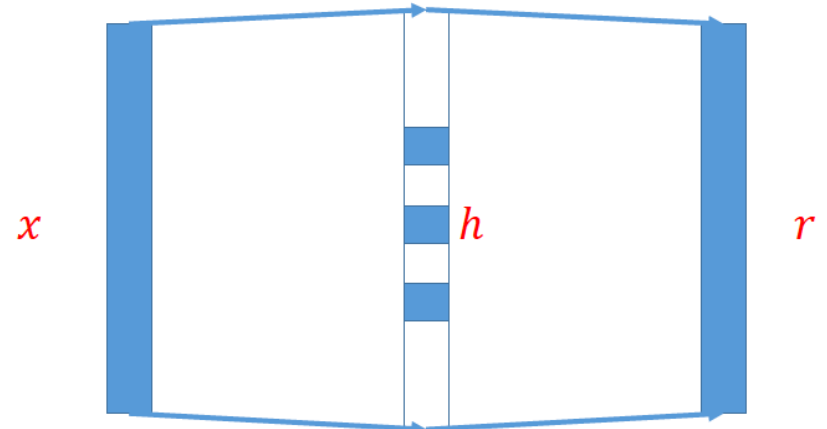


- ▶ but bad for other types of input



Regularization

$$L_R = L(x, g(f(x))) + R(h)$$



■ Sparse autoencoder

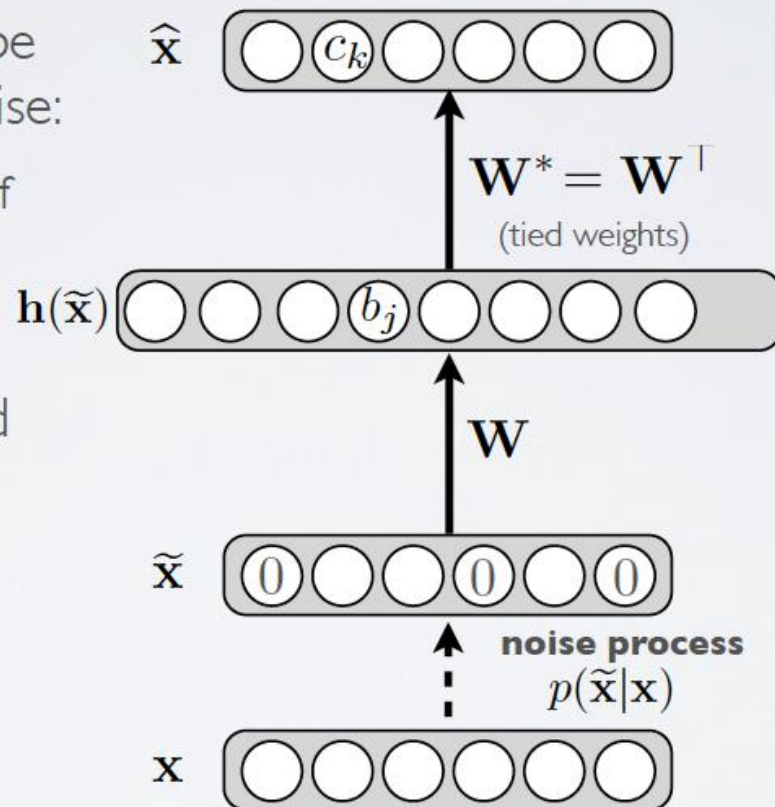
- Constrain the code to have sparsity
- Training: minimize a loss function

$$L_R = L(x, g(f(x))) + \lambda |h|_1$$

Regularization

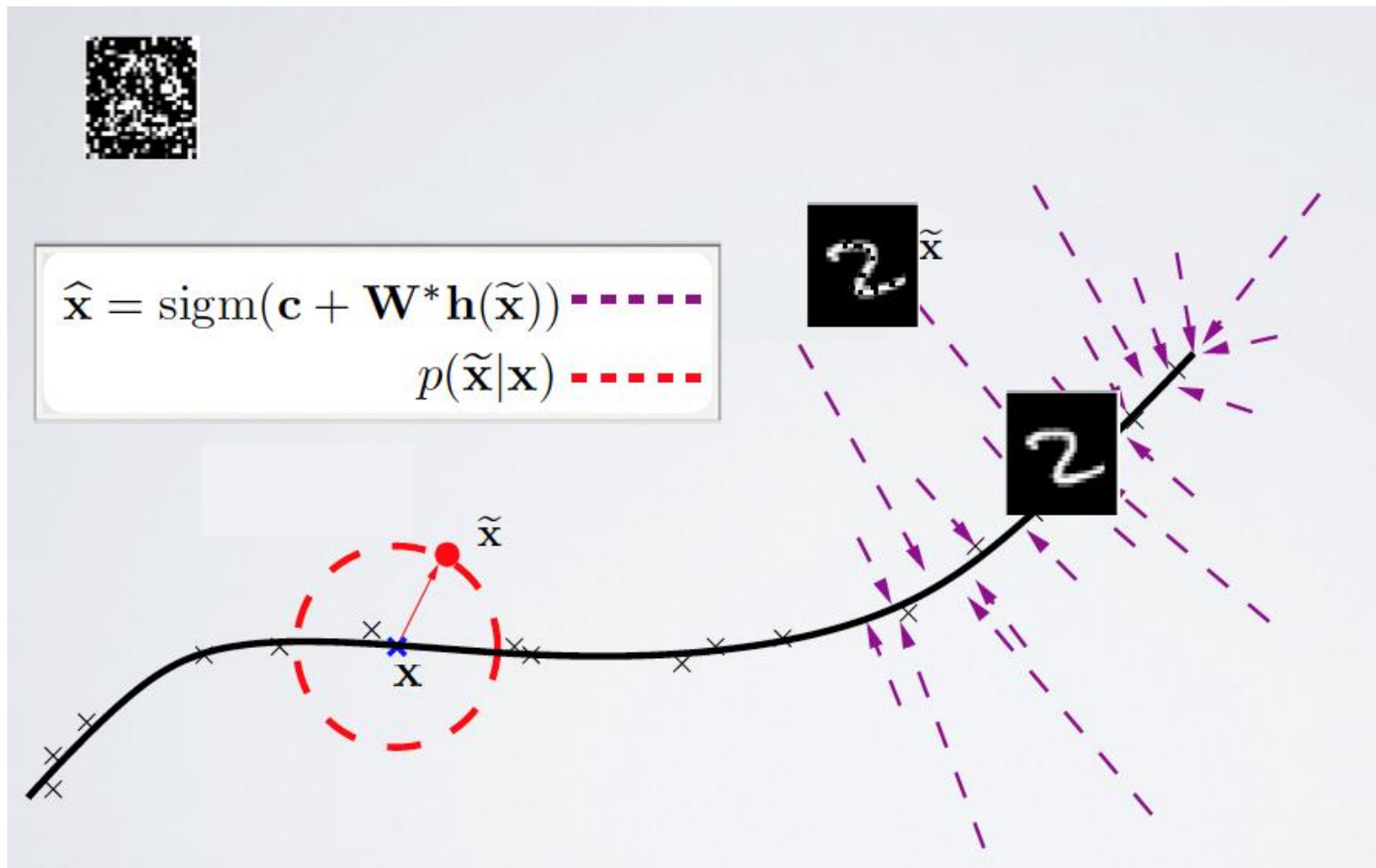
■ Denoising autoencoder

- Idea: representation should be robust to introduction of noise:
 - random assignment of subset of inputs to 0, with probability ν
 - Gaussian additive noise
- Reconstruction $\hat{\mathbf{x}}$ computed from the corrupted input $\tilde{\mathbf{x}}$
- Loss function compares $\hat{\mathbf{x}}$ reconstruction with the **noiseless input \mathbf{x}**



Regularization

- Denoising autoencoder



Outline

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Latent variable model

■ Data generation process

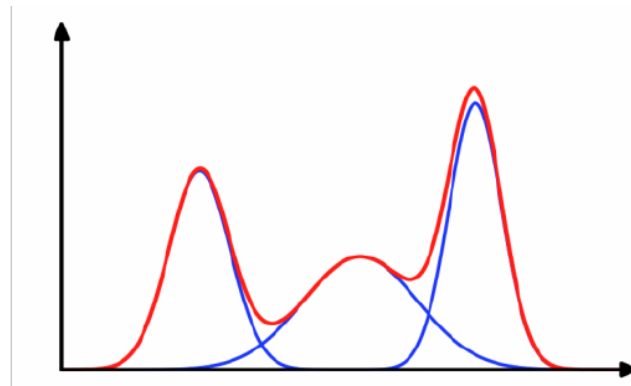
□ Latent variable z $p(z)$ = something simple

□ A mapping from the latent space to observation x

$$p(x) = \int p(x, z) dz \quad \text{where} \quad p(x, z) = p(x | z)p(z)$$

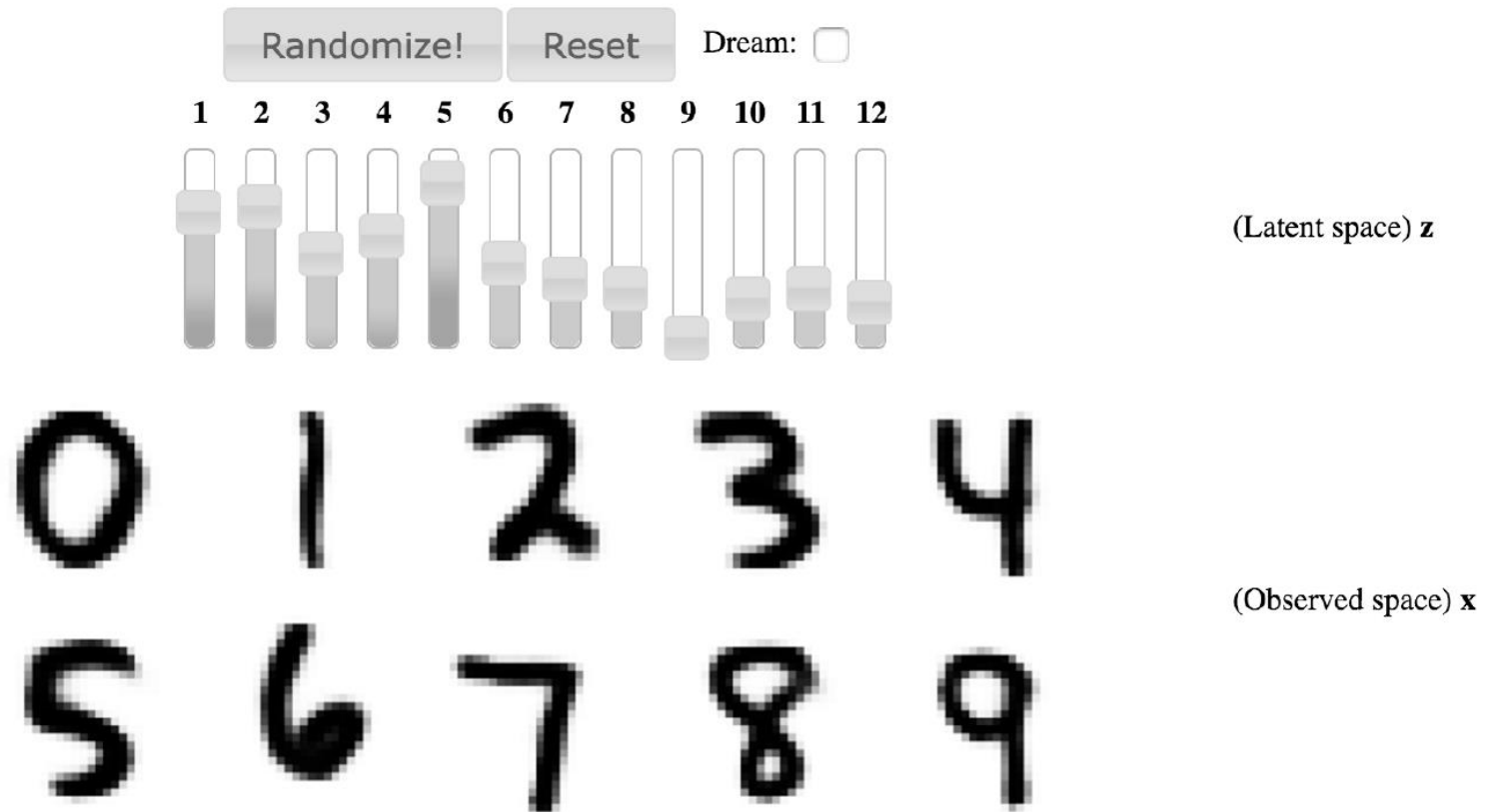
For example, a Gaussian mixture model

$$p_{\theta}(x) = \sum_{k=1}^K p_{\theta}(z = k) p_{\theta}(x|z = k)$$



An example

- Generating hand-written digits

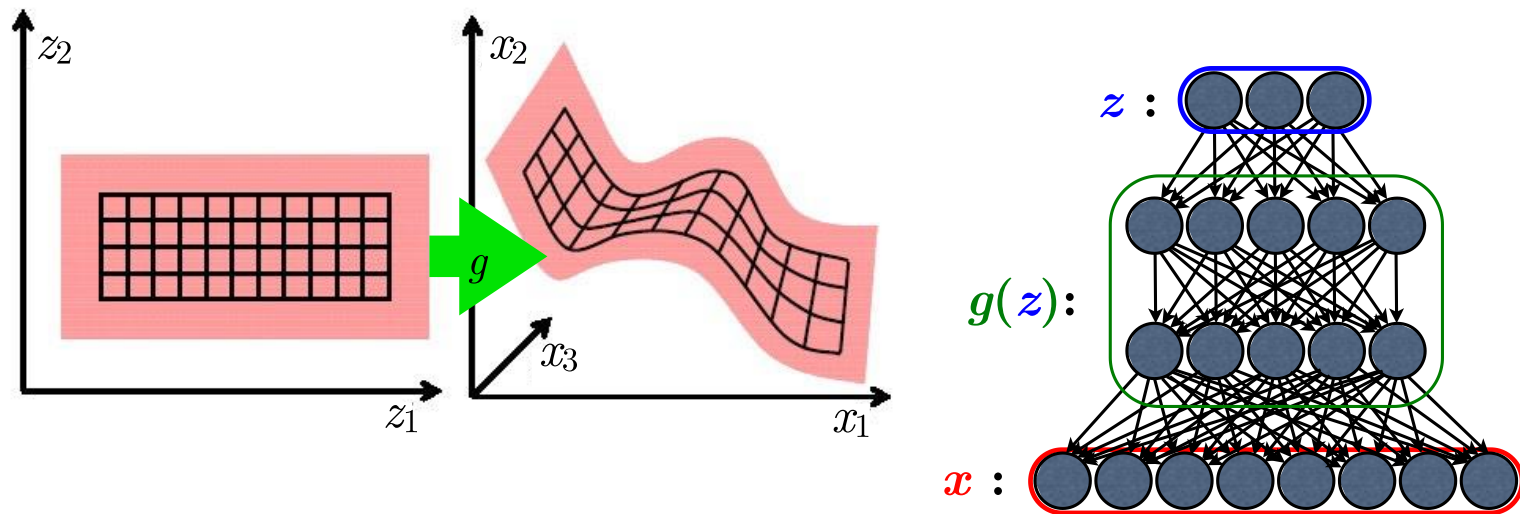


☐ http://www.dpkgingma.com/sgvb_mnist_demo/demo.html

Deep Latent Variable Models

- Leverage neural networks in a latent variable model

$$p(\mathbf{x}, \mathbf{z}) = p(\mathbf{x} \mid \mathbf{z})p(\mathbf{z}) \quad p(\mathbf{x} \mid \mathbf{z}) = g(\mathbf{z})$$



- Can represent complicated data distribution and conditional dependencies

An example

- $p(z) = N(0, I)$

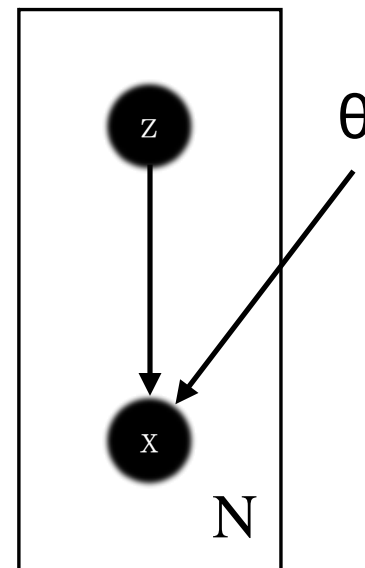
- $P_{\theta}(x|z) = N(\mu, \sigma^2)$

$\mu = f_{\theta}(z) = \text{multilayer neural net}$

- With flexible neural net,

$$p_{\theta}(x) = \int_z p_{\theta}(x|z)p(z)dz$$

- Can be arbitrarily complicated
- The multilayer network can capture complex dependencies in the data generation process



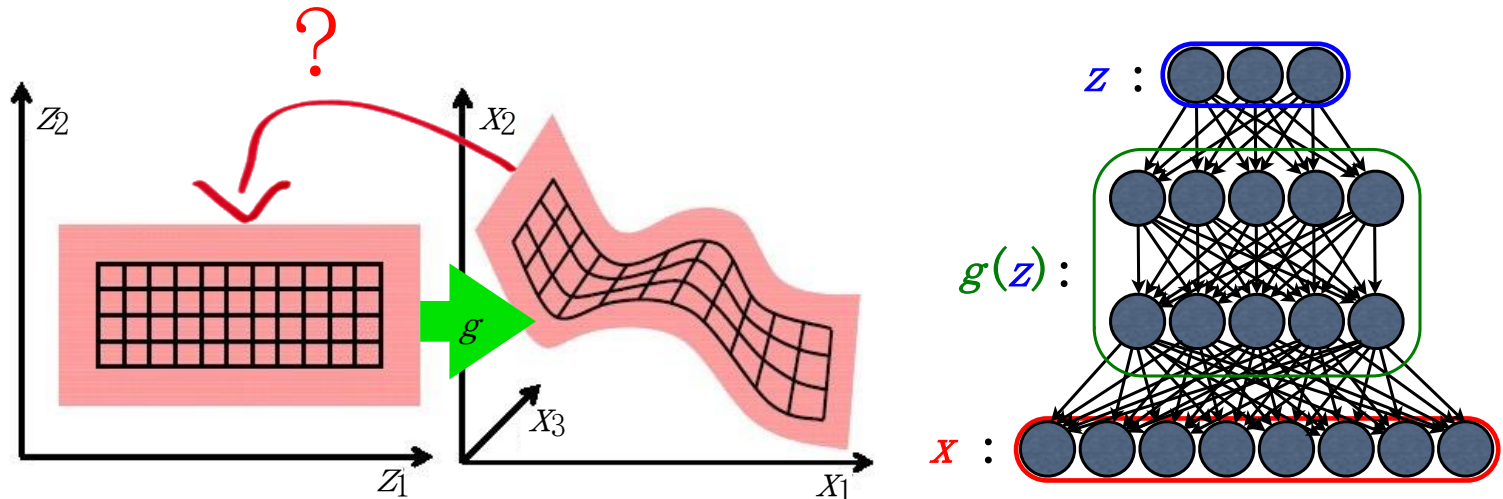
Challenges in Deep LVM

■ Inference:

- Given an observation \mathbf{x} , what is the probable \mathbf{z} ?
- Computing the posterior $p(\mathbf{z}|\mathbf{x})$ (intractable)

■ Learning:

- Given a large dataset of observations $\mathbf{X} = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}\}$
- Estimating the parameters in Deep LVM (inefficient/intractable)



The Variational Autencoder: overview

■ Inference:

- Introduce a parametric model $q_\phi(z | x)$ to approximate the true posterior $p_\theta(z | x)$
- Variational inference or Amortized inference

$$\begin{aligned} \forall x_i \quad & \arg \min_{q_i} D_{KL}(q_i(z) || p_\theta(z|x_i)) \\ \Rightarrow \forall x_i \quad & \arg \min_{\phi} D_{KL}(q_\phi(z|x_i) || p_\theta(z|x_i)) \end{aligned}$$

- Replacing datum-wise posterior distribution by a parametric family of conditional densities.

The Variational Autencoder: overview

■ Inference:

- Introduce a parametric model $q_{\phi}(z \mid x)$ to approximate the true posterior $p_{\theta}(z \mid x)$
- Variational inference or Amortized inference

■ Learning:

- Based on Maximum Likelihood

$$\max \sum_{i=1}^N \log p(x^{(i)})$$

- Direct optimization is challenging: use EM learning strategy
- Jointly learning inference model with the deep latent variable model

VAE objective

- Recall lower bound of the data log likelihood

$$\begin{aligned}\log p_{\theta}(x) &= \log \int_z p_{\theta}(x, z) dz \\ &= \log \int_z q_{\phi}(z|x) \frac{p_{\theta}(x, z)}{q_{\phi}(z|x)} dz \\ &\geq \int_z q_{\phi}(z|x) \log \frac{p_{\theta}(x, z)}{q_{\phi}(z|x)} dz \quad (\text{Jensen's Inequality}) \\ &= \mathbf{E}_{q_{\phi}(z|x)} [\log p_{\theta}(x, z) - \log q_{\phi}(z|x)] = \mathcal{L}(x; \theta, \phi)\end{aligned}$$

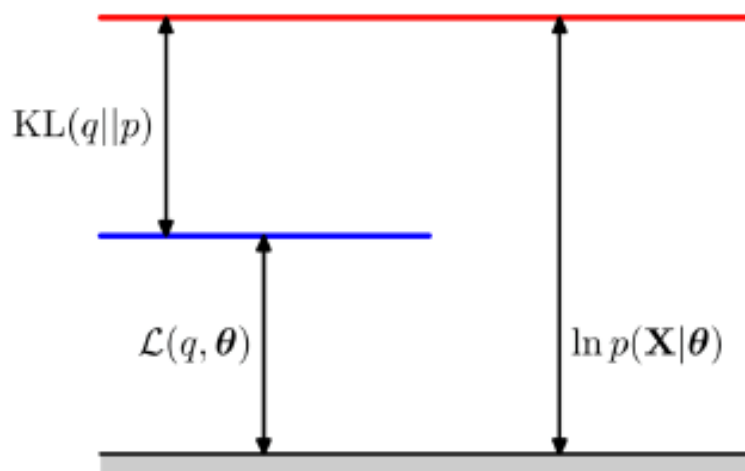
$$\log p_{\theta}(x) = \boxed{\mathcal{L}(x; \theta, \phi)} + D_{KL}(q_{\phi}(z|x) || p_{\theta}(z|x))$$

- Learning: maximize the lower bound of data likelihood
- The evidence lower bound (ELBO)

VAE objective

■ Visualizing ELBO

$$\log p_{\theta}(x) = \boxed{\mathcal{L}(x; \theta, \phi)} + D_{KL}(q_{\phi}(z|x) || p_{\theta}(z|x))$$



Bishop – Pattern Recognition and Machine Learning

- Note: all we have done so far is decompose the log probability of the data, we still have exact equality
- This holds for any distribution q

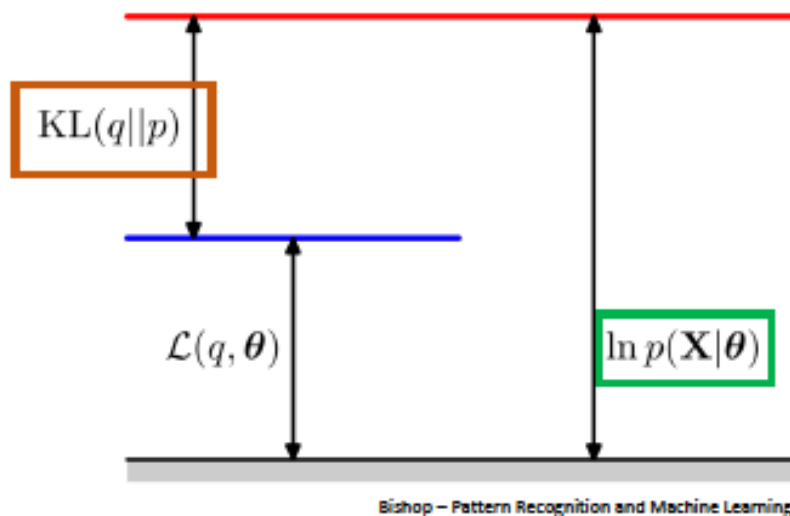
VAE learning

■ EM perspective

- Expectation Maximization alternately optimizes the ELBO, $\mathcal{L}(q, \theta)$, with respect to q (the E step) and θ (the M step)
- Initialize $\theta^{(0)}$
- At each iteration $t = 1, \dots$
 - **E step:** Hold $\theta^{(t-1)}$ fixed, find $q^{(t)}$ which maximizes $\mathcal{L}(q, \theta^{(t-1)})$
 - **M step:** Hold $q^{(t)}$ fixed, find $\theta^{(t)}$ which maximizes $\mathcal{L}(q^{(t)}, \theta)$

EM perspective

■ The E step

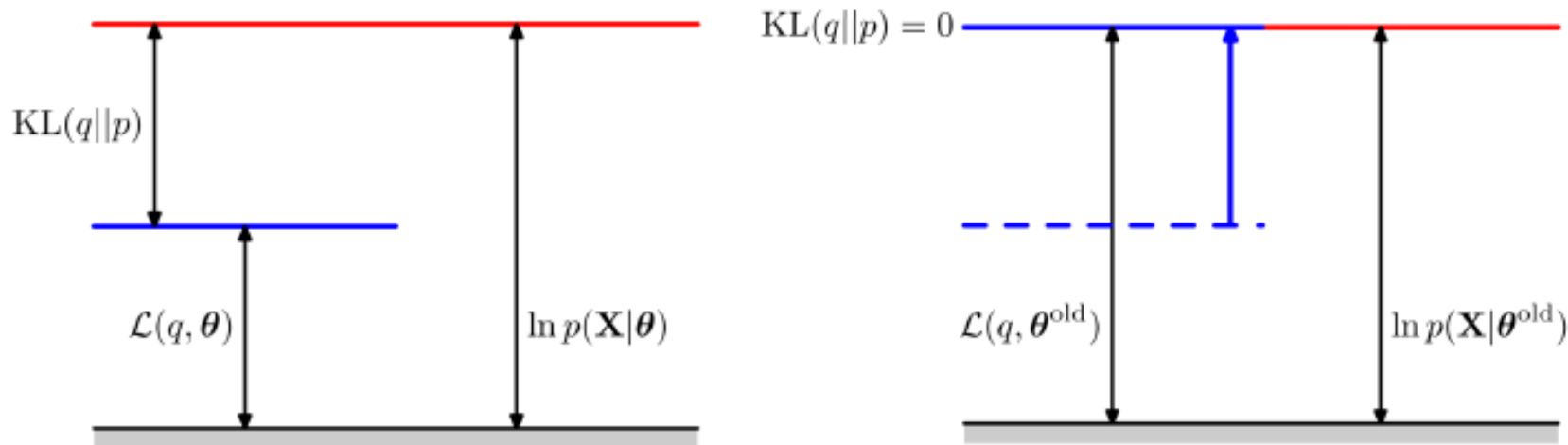


- The **first term** does not involve q , and we know the KL divergence must be non-negative
- The best we can do is to **make the KL divergence 0**
- Thus the solution is to set $q^{(t)}(z) \leftarrow p(z|x, \theta^{(t-1)})$

- Suppose we are at iteration t of our algorithm. How do we maximize $\mathcal{L}(q, \theta^{(t-1)})$ with respect to q ? We know that:
$$\operatorname{argmax}_q \mathcal{L}(q, \theta^{(t-1)}) = \operatorname{argmax}_q \log p(x|\theta^{(t-1)}) - \text{KL}(q(z) || p(z|x, \theta^{(t-1)}))$$

EM perspective

■ The E step

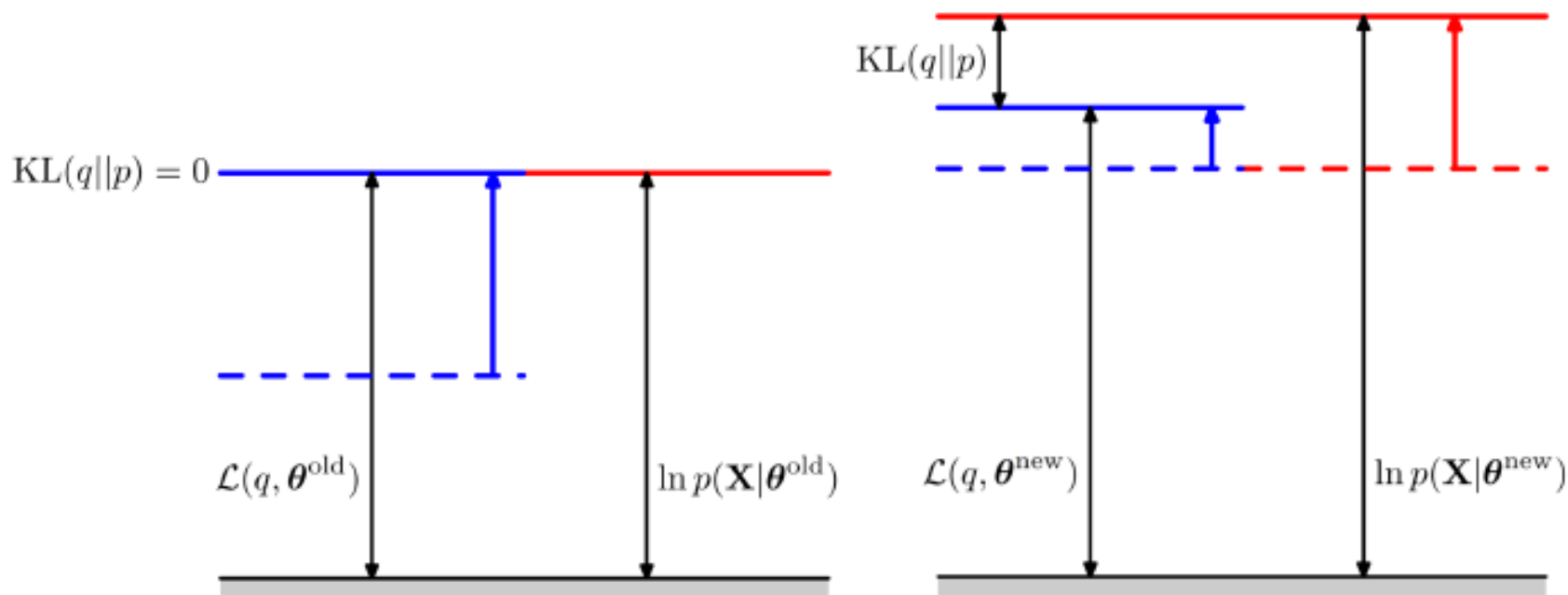


Bishop – Pattern Recognition and Machine Learning

- Suppose we are at iteration t of our algorithm. How do we maximize $\mathcal{L}(q, \theta^{(t-1)})$ with respect to q ? $q^{(t)}(z) \leftarrow p(z|x, \theta^{(t-1)})$

EM perspective

■ The M step

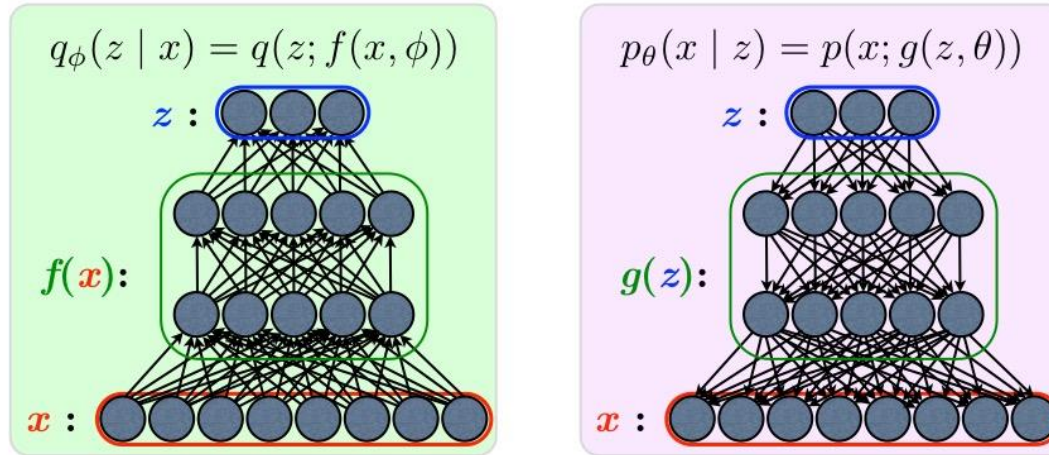


Bishop – Pattern Recognition and Machine Learning

- After applying the E step, we increase the likelihood of the data by finding better parameters according to: $\theta^{(t)} \leftarrow \mathbf{argmax}_{\theta} \mathbb{E}_{q^{(t)}(z)} [\log p(x, z | \theta)]$

VAE learning

- What is $q_\phi(z|x)$?
 - Parametrize $q_\phi(z|x)$ with another neural network



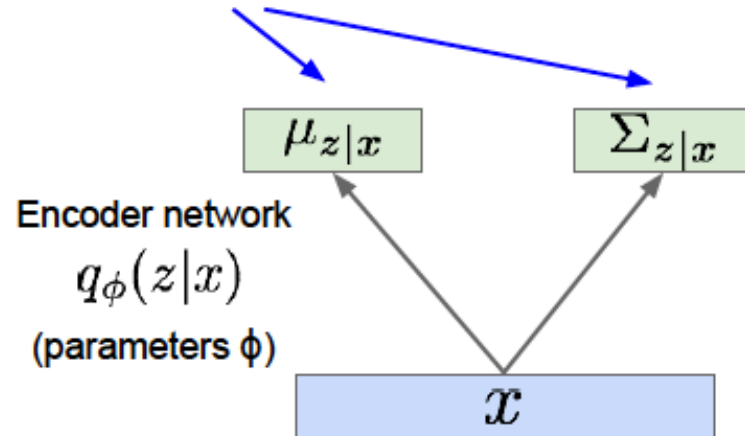
- Interpreting VAE objective

$$\begin{aligned}\mathcal{L}(\theta, \phi, x) &= \mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x, z) - \log q_\phi(z|x)] \\ &= \mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x|z) + \log p_\theta(z) - \log q_\phi(z|x)] \\ &= \underbrace{-D_{\text{KL}}(q_\phi(z|x) \| p_\theta(z))}_{\text{regularization term}} + \underbrace{\mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x|z)]}_{\text{reconstruction term}}\end{aligned}$$

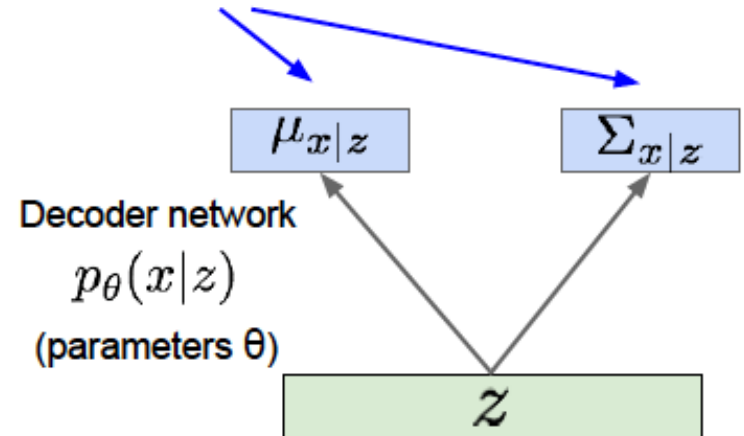
VAE Example

- Conditionals Gaussians

Mean and (diagonal) covariance of $z | x$



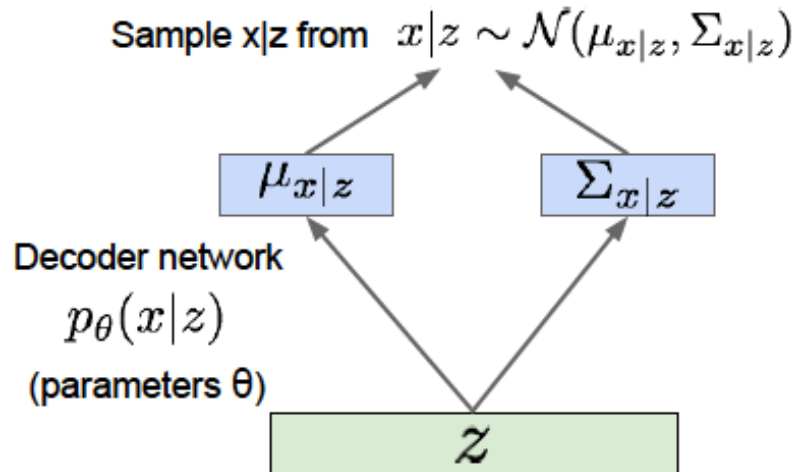
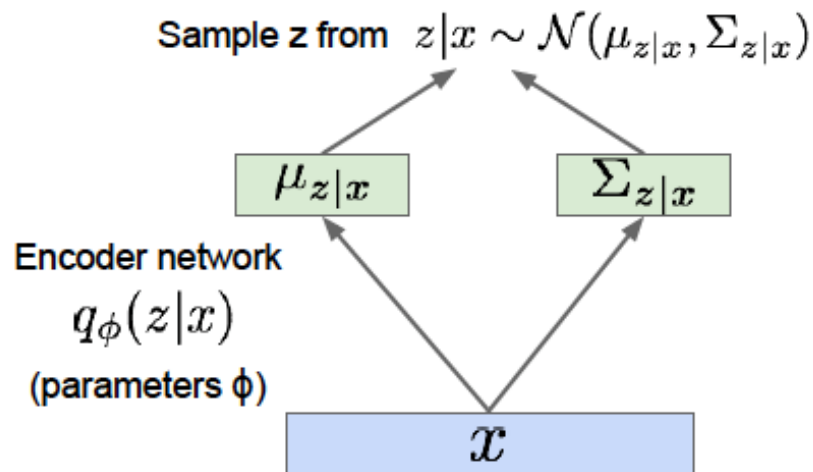
Mean and (diagonal) covariance of $x | z$



VAE Example

■ Conditionals Gaussians

Since we're modeling probabilistic generation of data, encoder and decoder networks are probabilistic



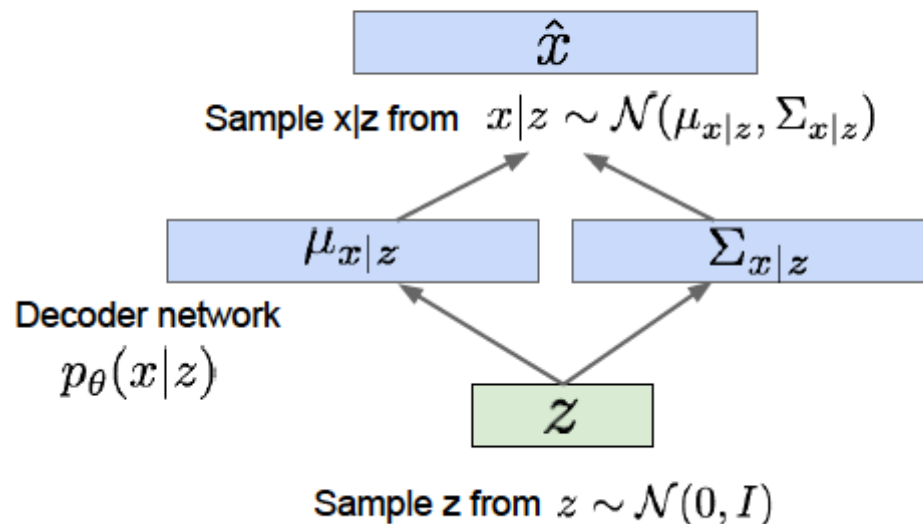
Encoder and decoder networks also called
"recognition"/"inference" and "generation" networks

Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

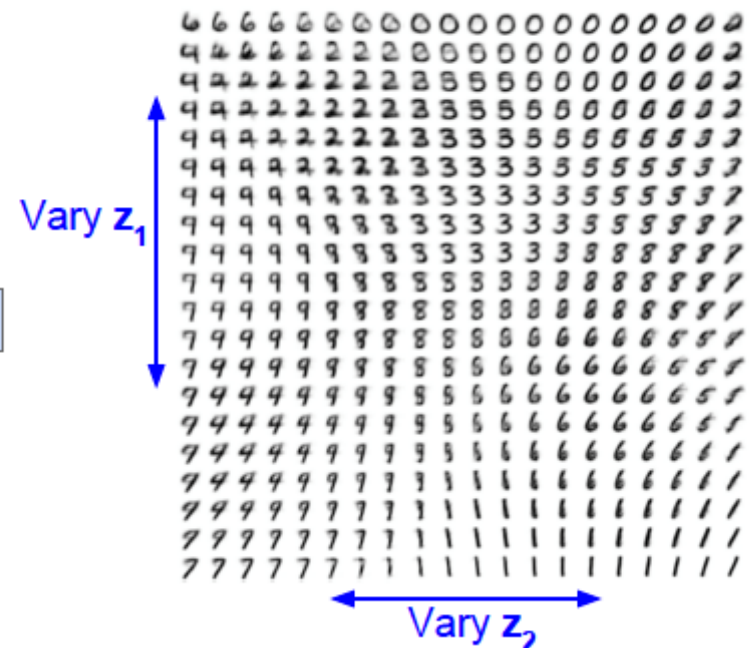
VAE Example

■ Generating data

Use decoder network. Now sample z from prior!



Data manifold for 2-d z

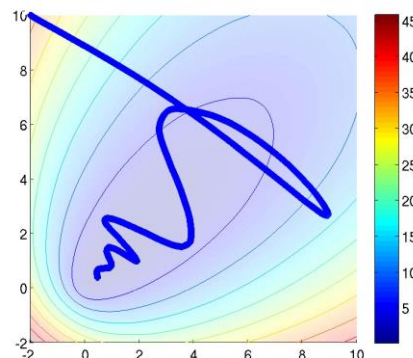


Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

Two views of Learning VAE

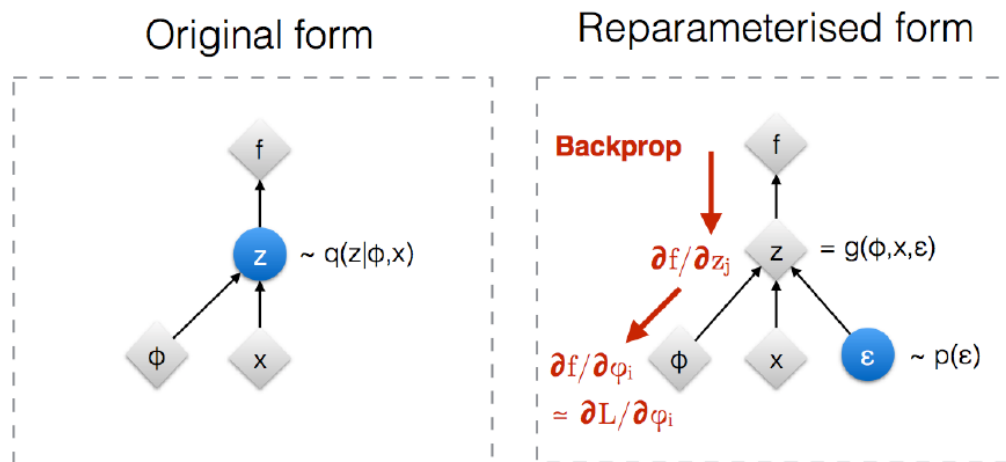
■ Optimization interpretation

- Stochastic gradient-based



■ Network interpretation

- Backpropagation



Optimization interpretation

- Recall VAE objective

$$\mathcal{L}(x, \phi, \theta) = E_{q_{\phi}(z|x)}[\log p_{\theta}(x, z) - \log q_{\phi}(z|x)]$$

- Or rewrite as $\mathcal{L}(x, \phi, \theta) = E_{q_{\phi}(z|x)}[f_{\phi, \theta}(x, z)]$

- Often no analytic solution to exact gradient

$$\nabla_{\phi, \theta} \mathcal{L}(x, \phi, \theta)$$

- Solution: stochastic gradient ascent
 - Requires unbiased estimates of gradient
 - Can use small minibatches or single point of data

$$\nabla_{\phi} \mathcal{L}(x, \phi, \theta) \approx \nabla_{\phi} f_{\phi, \theta}(x, z^{(i)}), \quad z^{(i)} \sim q_{\phi}(z|x)$$

High variance for gradient estimation

Reparameterization trick

- Reparameterize $\mathbf{z}^{(i)} \sim q_\phi(\mathbf{z}|\mathbf{x})$ using a differentiable transformation of an auxiliary noise variable ϵ

$$\mathbf{z} = g_\phi(\epsilon, \mathbf{x}) \quad \text{with} \quad \epsilon \sim q(\epsilon)$$

- Then we can write the ELBO as

$$\mathcal{L}(x, \phi, \theta) = E_{q_\phi(z|x)}[f_{\phi, \theta}(x, z)] = E_{q(\epsilon)}[f_{\phi, \theta}(x, g_\phi(\epsilon, \mathbf{x}))]$$

- And its gradient estimation with L samples

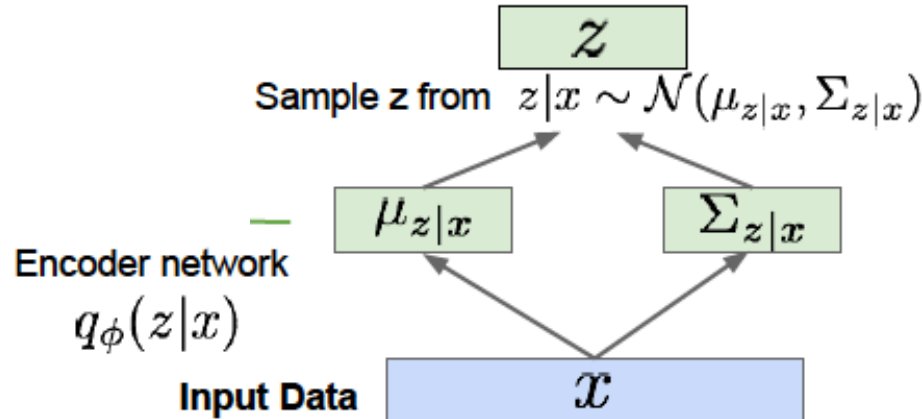
$$\nabla_\phi \mathcal{L}(x, \phi, \theta) = E_{q(\epsilon)}[\nabla_\phi f_{\phi, \theta}(x, z)] \approx \frac{1}{L} \sum_{i=1}^L \nabla_\phi f_{\phi, \theta}(x, g_\phi(\epsilon^{(i)}, x)), \quad \epsilon^{(i)} \sim q(\epsilon)$$

VAE Example

- Univariate Gaussian $z \sim p(z|x) = \mathcal{N}(\mu, \sigma^2)$

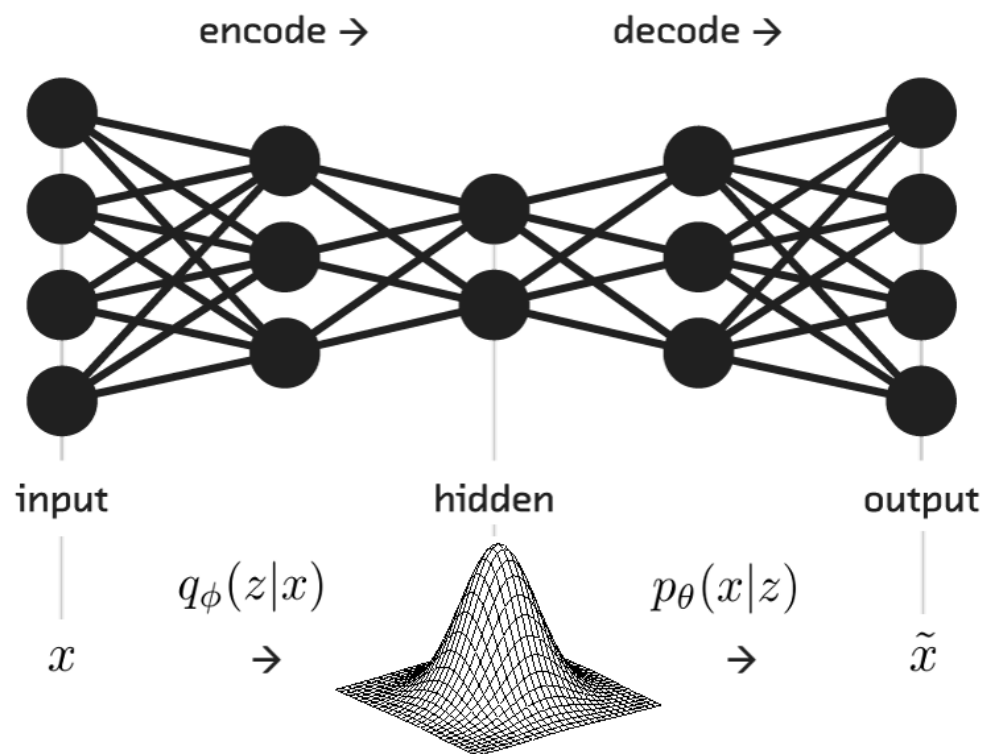
$$z = \mu + \sigma\epsilon \quad \epsilon \sim \mathcal{N}(0, 1)$$

$$\mathbb{E}_{\mathcal{N}(z;\mu,\sigma^2)} [f(z)] = \mathbb{E}_{\mathcal{N}(\epsilon;0,1)} [f(\mu + \sigma\epsilon)] \simeq \frac{1}{L} \sum_{l=1}^L f(\mu + \sigma\epsilon^{(l)})$$



Autoencoder Interpretation

- Objective $\mathcal{L}(x, \phi, \theta) = -D_{KL}(q_{\phi}(z|x)||p_{\theta}(z)) + E_{q_{\phi}(z|x)}[\log p_{\theta}(x|z)]$
Regularization term Reconstruction term



VAE Example

■ Learning objective

Putting it all together: maximizing the likelihood lower bound

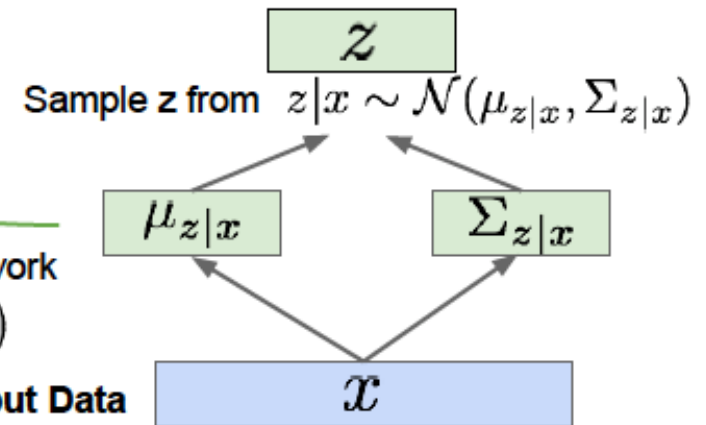
$$\underbrace{\mathbf{E}_z \left[\log p_\theta(x^{(i)} | z) \right] - D_{KL}(q_\phi(z | x^{(i)}) || p_\theta(z))}_{\mathcal{L}(x^{(i)}, \theta, \phi)}$$

Make approximate posterior distribution close to prior

Encoder network

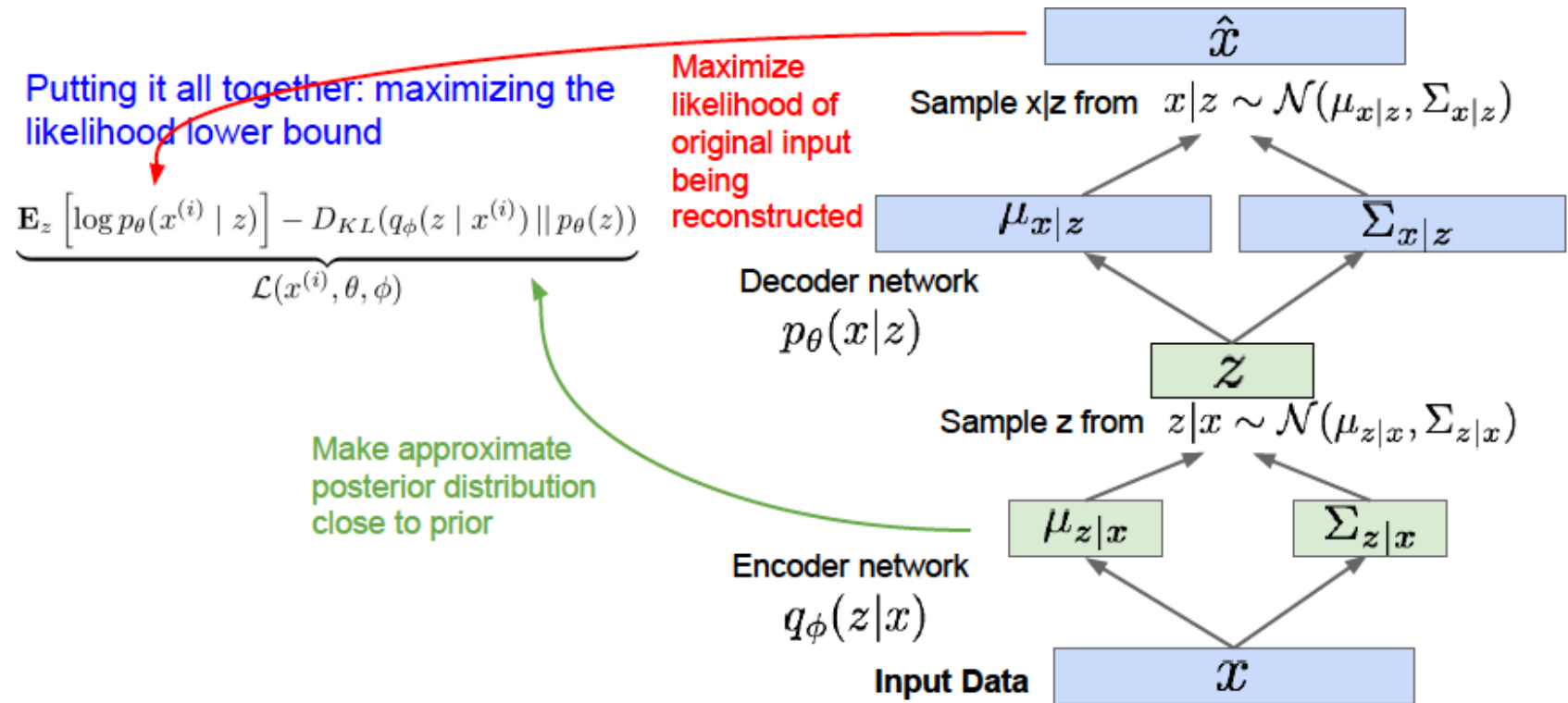
$$q_\phi(z|x)$$

Input Data



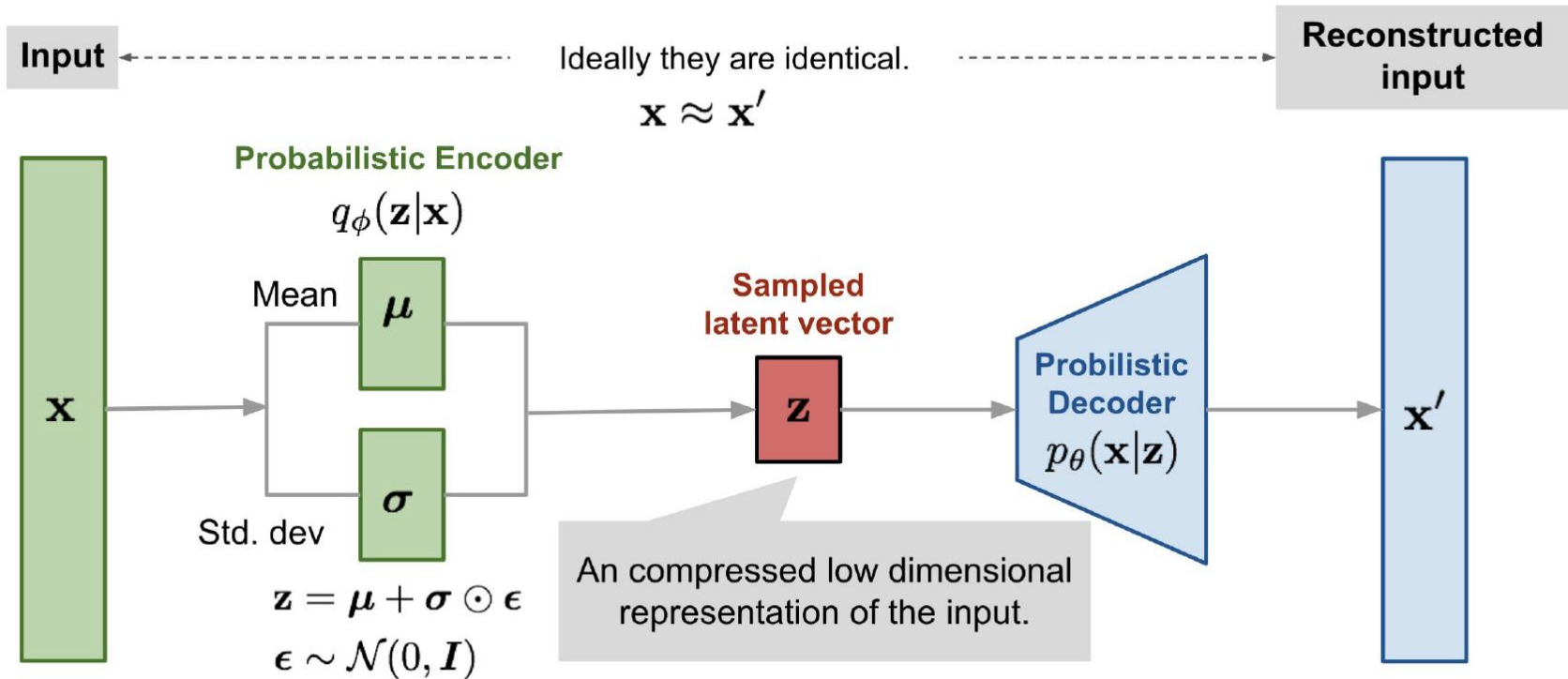
VAE Example

■ Learning objective



Autoencoder Interpretation

- Objective $\mathcal{L}(x, \phi, \theta) = \underbrace{-D_{KL}(q_\phi(z|x)||p_\theta(z))}_{\text{Regularization term}} + \underbrace{E_{q_\phi(z|x)}[\log p_\theta(x|z)]}_{\text{Reconstruction term}}$



- The objective function can be represented as an Autoencoder-like **computation graph**.

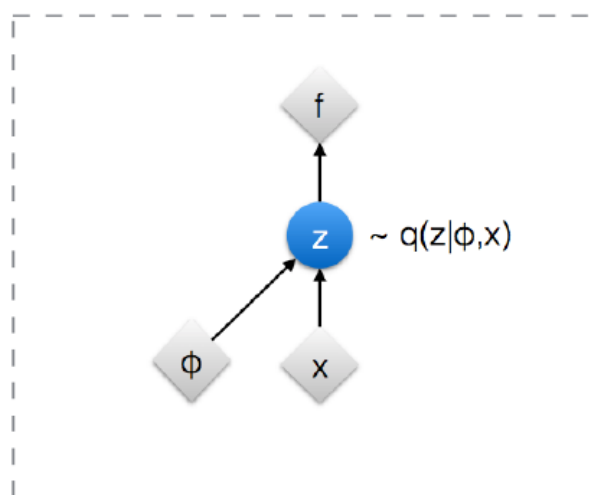
Network interpretation

$$\mathcal{L}(x, \phi, \theta) = E_{q_{\phi}(z|x)}[f_{\phi, \theta}(x, z)]$$

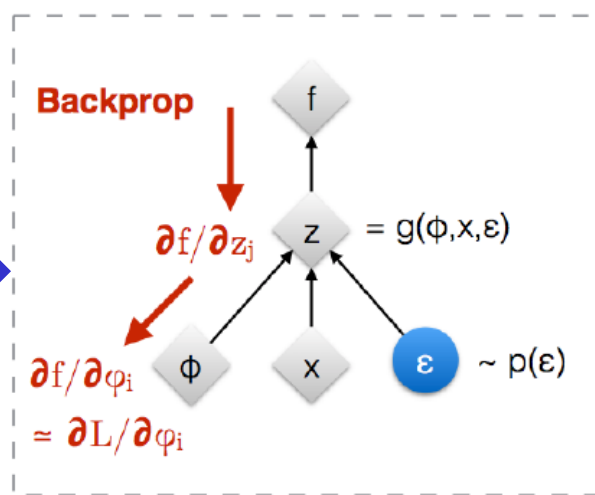


$$\mathcal{L}(x, \phi, \theta) = E_{q(\epsilon)}[f_{\phi, \theta}(x, z)] \approx \frac{1}{L} \sum_{i=1}^L f_{\phi, \theta}(x, g_{\phi}(\epsilon^{(i)}, x)), \quad \epsilon^{(i)} \sim q(\epsilon)$$

Original form



Reparameterised form



◊ : Deterministic node

● : Random node

[Kingma, 2013]

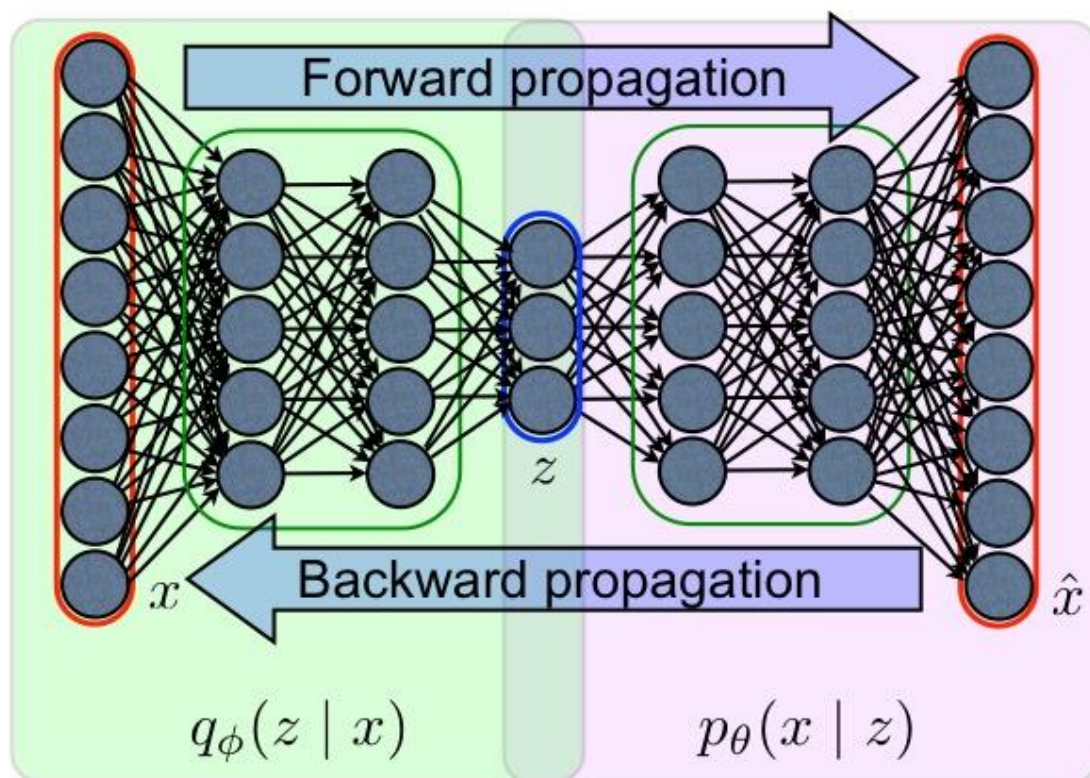
[Bengio, 2013]

[Kingma and Welling 2014]

[Rezende et al 2014]

Training with Backpropagation

- Due to **reparametrization** trick, we can simultaneously train both the **generative model** and the **inference model** by optimizing the variational bound using the gradient **backpropagation**.



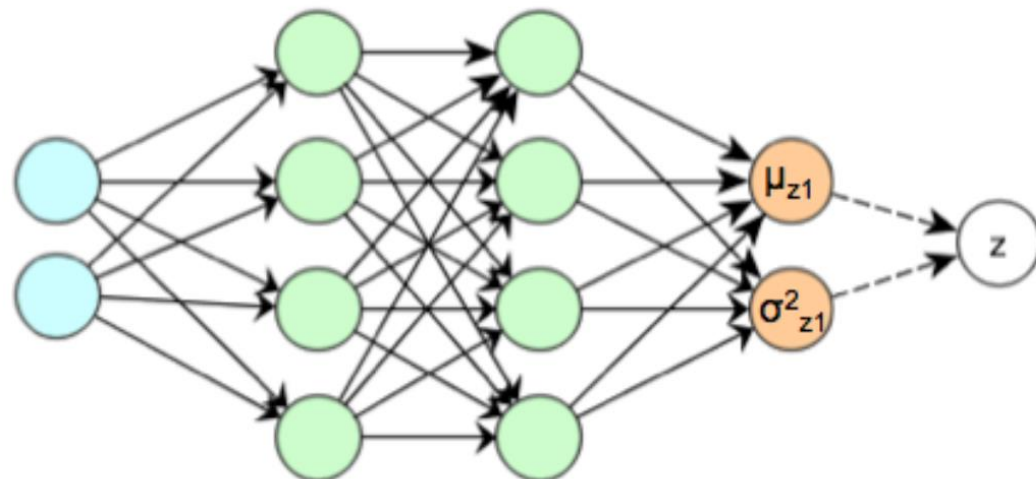
1D Gaussian Case

- We can compute the KL regularization in close form

Use $N(0,1)$ as prior for $p(z)$

$q(z|x^{(i)})$ is Gaussian with parameters $(\mu^{(i)}, \sigma^{(i)})$ determined by NN

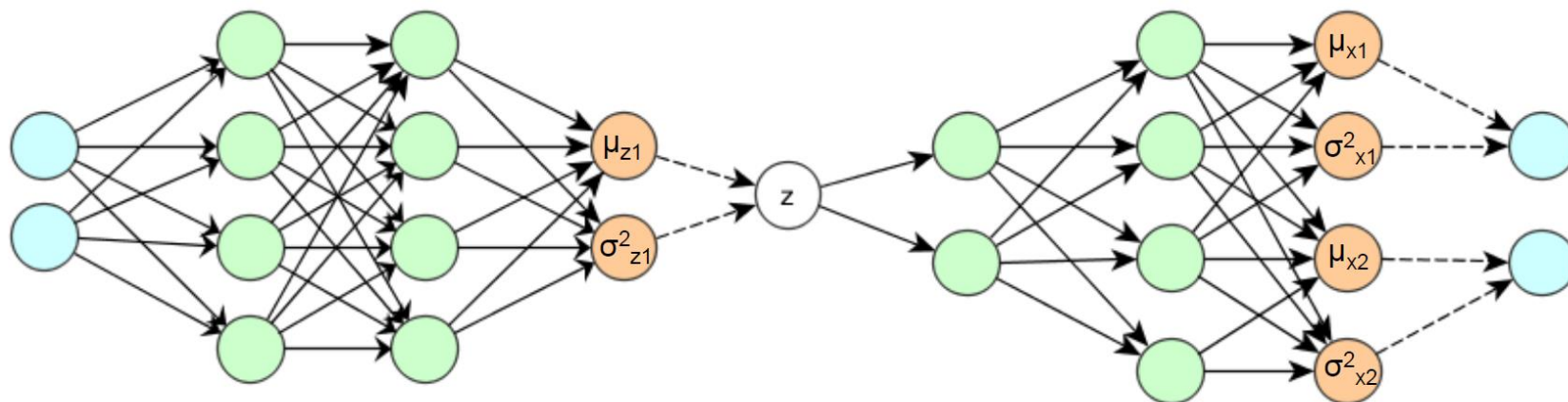
$$-D_{\text{KL}}(q(z|x^{(i)})||p(z)) = \frac{1}{2} \sum_{j=1}^J \left(1 + \log(\sigma_{z_j}^{(i)^2}) - \mu_{z_j}^{(i)^2} - \sigma_{z_j}^{(i)^2} \right)$$



1D Gaussian Case

Overall loss function for BP

Prior $p(z) \sim N(0,1)$ and p, q Gaussian, extension to $\dim(z) > 1$ trivial



Cost: Regularisation

$$-D_{\text{KL}}(q(z|x^{(i)})||p(z)) = \frac{1}{2} \sum_{j=1}^J \left(1 + \log(\sigma_{z_j}^{(i)^2}) - \mu_{z_j}^{(i)^2} - \sigma_{z_j}^{(i)^2} \right)$$

Cost: Reproduction

$$-\log(p(x^{(i)}|z^{(i)})) = \sum_{j=1}^D \frac{1}{2} \log(\sigma_{x_j}^2) + \frac{(x_j^{(i)} - \mu_{x_j})^2}{2\sigma_{x_j}^2}$$

We use mini batch gradient decent to optimize the cost function over all $x^{(i)}$ in the mini batch

Least Square for constant variance

Interpreting the latent space



<https://arxiv.org/pdf/1610.00291.pdf>

Problems of VAE

■ Model capacity

- Note that the VAE requires 2 tractable distributions to be used:
 - The prior distribution $p(z)$ must be easy to sample from
 - The conditional likelihood $p(x|z, \theta)$ must be computable
- In practice this means that the 2 distributions of interest are often simple, for example uniform, Gaussian, or even isotropic Gaussian

Problems of VAE

■ Blurry images



<https://blog.openai.com/generative-models/>

- The samples from the VAE look blurry
- Three plausible explanations for this
 - Maximizing the likelihood
 - Restrictions on the family of distributions
 - The lower bound approximation

Problems of VAE

■ Blurry images

- Recent investigations suggest that both the simple probability distributions and the variational approximation lead to blurry images
- [Kingma & colleagues: Improving Variational Inference with Inverse Autoregressive Flow](#)
- [Zhao & colleagues: Towards a Deeper Understanding of Variational Autoencoding Models](#)
- [Nowozin & colleagues: f-gan: Training generative neural samplers using variational divergence minimization](#)

Summary

- Autoencoders (AEs)
 - Representation learning
- Variational Autoencoders (VAEs)
 - VAE objective
 - Reparametrization trick
- Next time:
 - VAE: Vision applications
 - GAN
- Reading material
 - http://www.cs.columbia.edu/~blei/talks/Blei_VI_tutorial.pdf
 - http://www.cs.toronto.edu/~rgrosse/courses/csc421_2019/slides/lec17.pdf
 - <https://dvl.in.tum.de/slides/adl4cv-ws18/6.Bayesian&VAE.pdf>
 - <https://arxiv.org/pdf/1312.6114.pdf>

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