Homework 6

Due date:
Mar. 23rd, 2018
Turn in your homework in class

Rules:

- Work on your own. Discussion is permissible, but extremely similar submissions will be judged as plagiarism.
- Please show all intermediate steps: a correct solution without an explanation will get zero credit.
- Please submit on time. No late submission will be accepted.
- Please prepare your submission in English only. No Chinese submission will be accepted.

1. A series RLC circuit exhibits the following voltage and current responses:

$$v_c(t) = (6cos4t - 3sin4t)e^{-2t}u(t) \text{ V}$$
$$i_c(t) = -(0.24cos4t + 0.18sin4t)e^{-2t}u(t) \text{ A}$$

Determine α, ω_0, R, L and C

Solution: The coefficient of the exponential is equal to α and the argument of the sine and cosine functions is ω_d:

$$\alpha = 2 \text{ Np/s}, \quad (1)$$
$$\omega_d = 4 \text{ rad/s}. \quad (2)$$

Hence,

$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2} = \sqrt{4^2 + 2^2} = \sqrt{20}\,. \quad (3)$$

Applying $i_C(t) = C\,v_C'(t)$ to the expression for $v_C(t)$, and then comparing it with the given expression for $i_C(t)$, leads to

$$\begin{aligned} i_C(t) &= C\,v_C'(t) \\ &= C[-2e^{-2t}(6\cos 4t - 3\sin 4t) + e^{-2t}(-24\sin 4t - 12\cos 4t)] \\ &= C[-24\cos 4t - 18\sin 4t]e^{-2t}, \end{aligned}$$

which requires that

$$C = 10^{-2} \text{ F}.$$

From $\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{20}$,

$$L = 5 \text{ H}.$$

Finally,

$$\alpha = \frac{R}{2L} = 2, \qquad \text{or} \qquad R = 4L = 20\ \Omega.$$

2. Determine $v_c(t)$ in the Fig.1 for $t \geq 0$.

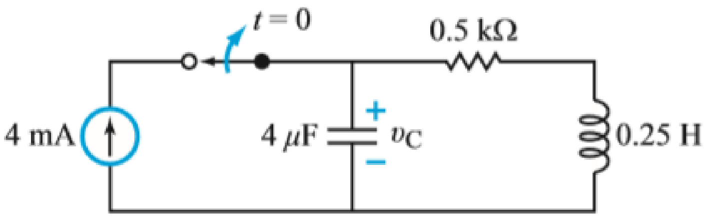
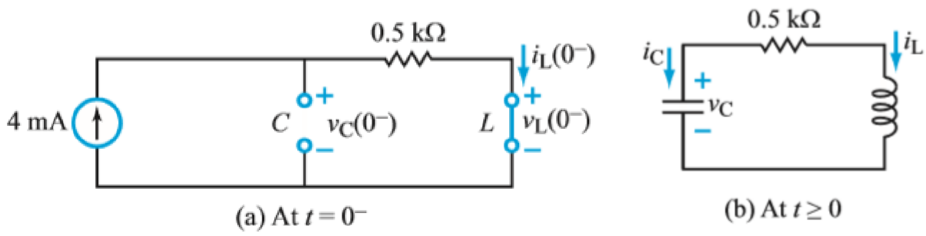


Figure 1

Solution



At $t = 0^-$,

$$v_C(0^-) = 4 \times 10^{-3} \times 0.5 \times 10^3 = 2 \text{ V},$$
$$i_L(0^-) = 4 \text{ mA}.$$

At $t = 0$,

$$v_C(0) = v_C(0^-) = 2 \text{ V},$$
$$i_C(0) = -i_L(0) = -i_L(0^-) = -4 \text{ mA}.$$

At $t \geq 0$,

$$\alpha = \frac{R}{2L} = \frac{0.5 \times 10^3}{2 \times 0.25} = 1000 \text{ Np/s},$$
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25 \times 4 \times 10^{-6}}} = 1000 \text{ rad/s}.$$

Since $\alpha = \omega_0$, the response is critically damped:

$$v_C(t) = (B_1 + B_2 t)e^{-\alpha t} = (B_1 + B_2 t)e^{-1000t}.$$

Initial condition $v_C(0) = 2$ V leads to

$$B_1 = 2 \text{ V},$$

and initial condition $i_C(0) = -4$ mA leads to

$$i_C(0) = C\, v_C'(0)$$
$$= 4 \times 10^{-6}\, [B_2 e^{-1000t} - 1000(B_1 + B_2 t)e^{-1000t}]\big|_{t=0} = -4 \times 10^{-3},$$

which reduces to

$$B_2 - 1000B_1 = -1000,$$

or

$$B_2 = 1000B_1 - 1000 = 1000 \times 2 - 1000 = 1000 \text{ V/s}.$$

Hence,

$$v_C(t) = (2 + 1000t)e^{-1000t} \quad \text{(V)}, \qquad \text{for } t \geq 0.$$

3. When t<0, no energy is stored in the capacitor in Fig.2. The switch moves from position A to position B at $t = 0$. Determine $i(t)$ for $t \geq 0$.

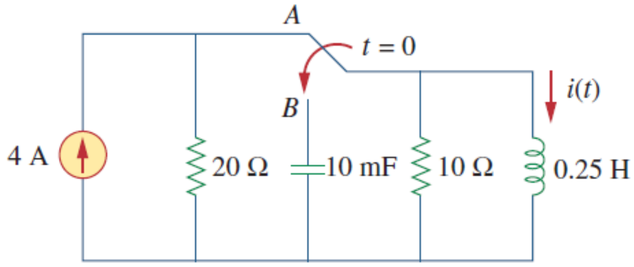


Figure 2

Solution

When the switch is in position A, the inductor acts like a short circuit so

$$i(0^-) = 4$$

When the switch is in position B, we have a source-free parallel RCL circuit

$$\alpha = \frac{1}{2RC} = \frac{1}{2\text{x}10\text{x}10\text{x}10^{-3}} = 5$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{4}\text{x}10\text{x}10^{-3}}} = 20$$

Since $\alpha < \omega_o$, we have an underdamped case.

$$s_{1,2} = -5 + \sqrt{25-400} = -5 + j19.365$$

$$i(t) = e^{-5t}\left(A_1 \cos 19.365t + A_2 \sin 19.365t\right)$$

$$i(0) = 4 = A_1$$

$$v = L\frac{di}{dt} \longrightarrow \frac{di(0)}{dt} = \frac{v(0)}{L} = 0$$

$$\frac{di}{dt} = e^{-5t}\left(-5A_1 \cos 19.365t - 5A_2 \sin 19.365t - 19.365A_1 \sin 19.365t + 19.365A_2 \cos 19.365t\right)$$

$0 = [di(0)/dt] = -5A_1 + 19.365A_2$ or $A_2 = 20/19.365 = 1.0328$

$$\mathbf{i(t) = e^{-5t}[4\cos(19.365t) + 1.0328\sin(19.365t)]\ A}$$

4. When t<0, no energy is stored in the capacitor nor the inductor in the circuit of Fig. 3. Find $i(t)$ for $t \geq 0$.

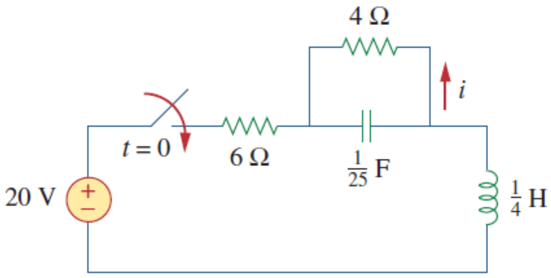
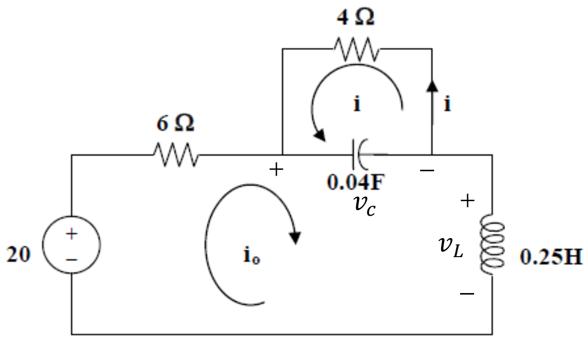


Figure 3

Solution

For t < 0, i(0) = 0 and v(0) = 0.

For t > 0, the circuit is as shown below.



initial values:

$$i(0^+) = 0$$

$$i(\infty) = -\frac{20}{6+4} = -2A$$

$$i'(0) = 0$$

From equations:

$$\begin{cases} 6i_o + v_c + v_L = 20 \\ i_o = -i + C\dfrac{dv_c}{dt} \\ v_c = -4i \\ v_L = L\dfrac{di}{dt} \end{cases}$$

We can obtain the following equation:

$$\frac{d^2i}{dt^2} + \frac{121}{4}\frac{di}{dt} + 250i = -500$$

$$\alpha = \frac{121}{8} = 15.125, \omega_0 = 5\sqrt{10}, \omega_d = \frac{3\sqrt{151}}{8} \approx 4.61$$

since $\alpha = \omega_0$, the response will be underdamped and given by

$$i(t) = e^{-15.125t}(A_1 cos4.61t + A_2 sin4.61t) - 2$$

$$A_1 = i(0) - i(\infty) = 2 \qquad -15.125A_1 + 4.61A_2 = 0 \rightarrow A_2 \approx 6.57$$

Therefore

$$i(t) = e^{-15.125t}(2cos4.61t + 6.57sin4.61t) - 2 \;\; A \;\; (t \geq 0)$$

5. Choose the value of C in the circuit of Fig. 4 so that $v_c(t)$ has a critically damped response for $t \geq 0$.

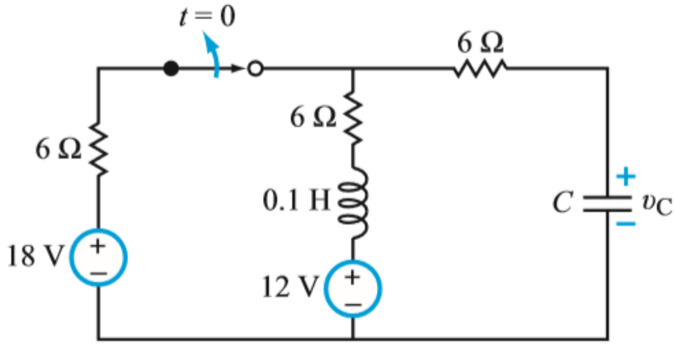
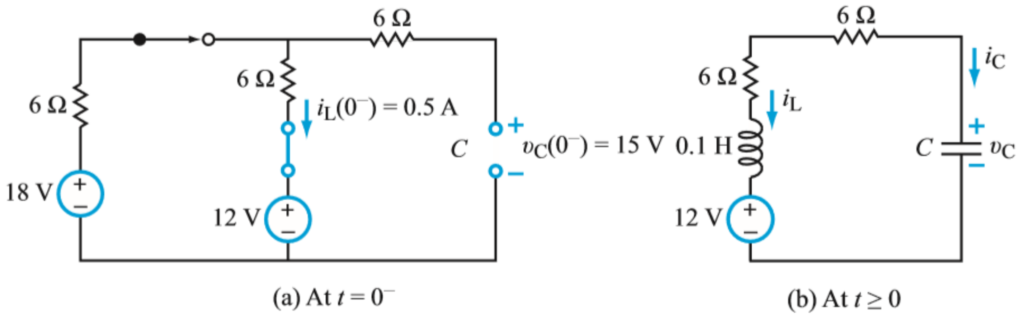


Figure 4

Solution



At $t = 0^-$:

$$i_L(0^-) = \frac{18 - 12}{6 + 6} = 0.5 \text{ A},$$

$$v_C(0^-) = 6 \times 0.5 + 12 = 15 \text{ V}.$$

At $t = 0$:

$$v_C(0) = v_C(0^-) = 15 \text{ V},$$

$$i_C(0) = -i_L(0) = -i_L(0^-) = -0.5 \text{ A},$$

$$v_C'(0) = \frac{i_C(0)}{C} = -\frac{0.5}{C}.$$

At $t \geq 0$:

$$\alpha = \frac{R}{2L} = \frac{6+6}{2 \times 0.1} = 60 \text{ Np/s}.$$

To have a critically damped response, it is necessary that $\omega_0 = \alpha$, or

$$\frac{1}{\sqrt{LC}} = 60 \text{ rad/s},$$

which requires that

$$C = \frac{1}{360} \text{ F}.$$

6. Determine $v_C(t)$ in the circuit for t ≥ 0, given that $V_0 = 12V$, $R_1 = 0.4\ \Omega$, $R_2 = 1.2\ \Omega$, L = 0.1 H and C = 0.4 F.

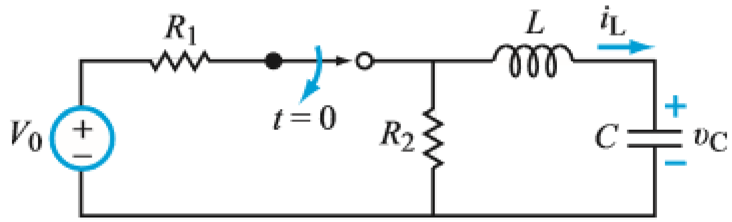
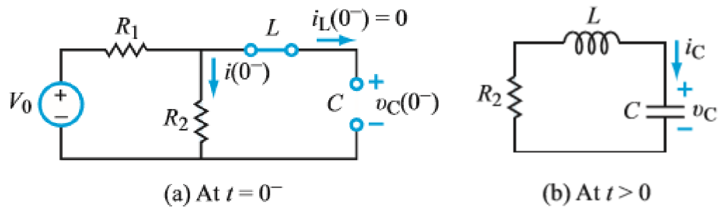


Figure 5

Solution



At $t = 0^-$,

$$i(0^-) = \frac{V_0}{R_1 + R_2} = \frac{12}{0.4 + 1.2} = 7.5\text{ A},$$

and

$$v_C(0^-) = i(0^-)R_2 = 7.5 \times 1.2 = 9\text{ V}.$$
$$i_L(0^-) = 0.$$

At $t = 0$,

$$v_C(0) = v_C(0^-) = 9\text{ V}, \quad (1)$$
$$i_C(0) = i_L(0) = i_L(0^-) = 0. \quad (2)$$

Next we determine the value of α:

$$\alpha = \frac{R_2}{2L} = \frac{1.2}{2 \times 0.1} = 6\text{ Np/s}.$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \times 0.4}} = 5\text{ rad/s}.$$

Since $\alpha > \omega_0$, the response will be overdamped, and given by

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t},$$

with

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$= -6 + \sqrt{6^2 - 5^2} = -2.68\text{ Np/s},$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -9.32\text{ Np/s}.$$

Hence,

$$\upsilon_C(t) = A_1 e^{-2.68t} + A_2 e^{-9.32t}. \tag{3}$$

The initial conditions given by Eq. (1) requires that

$$A_1 + A_2 = 9. \tag{4}$$

Similarly, the initial condition given by Eq. (2) requires that

$$i_C(0) = C\upsilon_C'(0) = C\left[-2.68A_1 e^{-2.68t} - 9.32A_2 e^{-9.32t}\right]\Big|_{t=0} = 0,$$

or

$$-2.68A_1 - 9.32A_2 = 0. \tag{5}$$

Simultaneous solution of Eqs. (4) and (5) gives

$$A_1 = 12.64 \text{ V}, \qquad A_2 = -3.64 \text{ V},$$

and

$$\upsilon_C(t) = (12.64e^{-2.68t} - 3.64e^{-9.32t}) \qquad \text{(V)}, \qquad \text{for } t \geq 0.$$

7. The initial value of the voltage v in the circuit shown in Fig. 6 is zero, and the initial value of the capacitor current, $i_C(0^+)$ is 45 mA. The expression for the capacitor current is known to be $i_C(t) = \mathrm{A}_1 e^{-200t} + \mathrm{A}_2 e^{-800t}$, $t \geq 0^+$,

   R=250Ω. Find:

   (a) The value of L, C, A_1 and A_2
   (b) The express for $v(t)$ for $t \geq 0$.

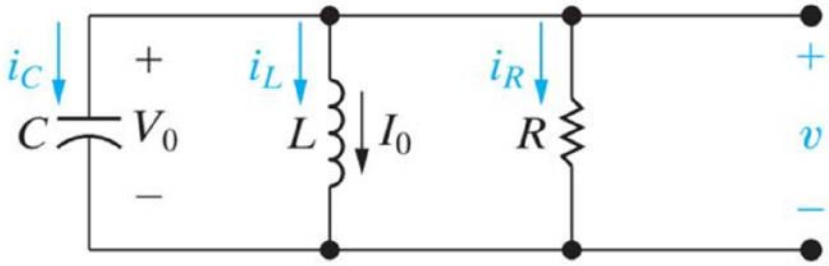


Figure 6

Solution:

a) $\begin{cases} S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -200 \\ S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -800 \end{cases}$ [2']

$\Rightarrow \begin{cases} S_1 + S_2 = -2\alpha \\ S_1 \cdot S_2 = \omega_0^2 \end{cases} \Rightarrow \begin{cases} \alpha = 500\ \mathrm{rad/s} = \frac{1}{2RC} \\ \omega_0 = 400\ \mathrm{rad/s} = \frac{1}{\sqrt{LC}} \end{cases}$.

$\Rightarrow \begin{cases} C = 4\times 10^{-6} F \text{ [2']} \\ L = 1.5625 H \text{ [2']} \end{cases}$

$i_C(0^+) = C \cdot \frac{dV_C(t)}{dt} \Rightarrow V_C(t) = \frac{1}{C}\int_0^{t_0} i_C(t)\,dt + V_C(t_0)$ [5']+[4']

$= \frac{1}{C}\left(-\frac{A_1}{200} e^{-200t} - \frac{A_2}{800} e^{-800t}\right)$

$\begin{cases} i_C(0^+) = 0.045 A \\ V_C(0^+) = 0 V \end{cases} \Rightarrow \begin{cases} A_1 + A_2 = 0.045 \text{ [1']} \\ \frac{A_1}{200} + \frac{A_2}{800} = 0 \text{ [1']} \end{cases} \Rightarrow \begin{cases} A_1 = -0.015 A \text{ [2']} \\ A_2 = 0.06 A \text{ [2']} \end{cases}$

$\Rightarrow i_C(t) = -0.015 e^{-200t} + 0.06 e^{-800t} A,\ t \geq 0^+$

b) $V_C(t) = \frac{1}{C}\int_0^{t_0} i_C(t)\,dt + V_C(0) = \frac{1}{C}\left(-\frac{A_1}{200} e^{-200t} - \frac{A_2}{800} e^{-800t}\right) + V_C(0)$ [2'+1']

$= 18.75 e^{-200t} - 18.75 e^{-800t}\ V,\ t \geq 0^+$ [2']

8. The op-amp circuit shown in Fig.7 is called a two-pole low-pass filter. If $v_{in} = Au(t)$, determine $v_{out}(t)$ for $t \geq 0$ when $A = 2\text{V}$, $R_1 = 5\text{k}\Omega$, $R_2 = 10\text{k}\Omega$, $R_3 = 12\text{k}\Omega$, $R_4 = 20\text{k}\Omega$, $C_1 = 100\mu\text{F}$, and $C_2 = 200\mu\text{F}$. (No energy is stored in the capacitors when t<0).

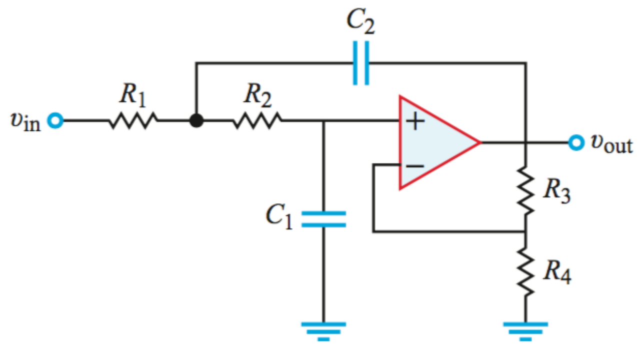
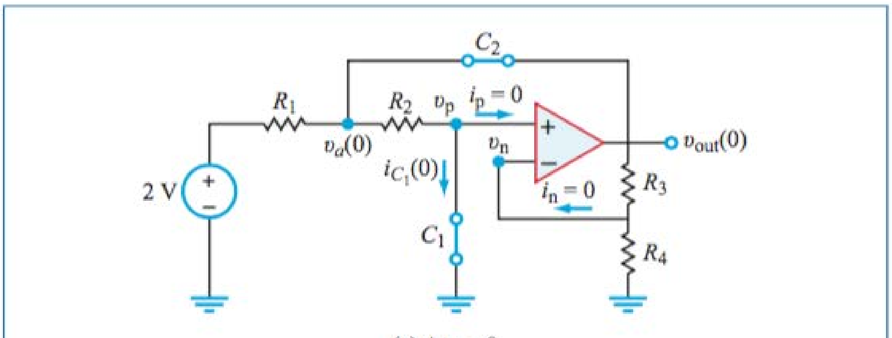


Figure 7

Solution:



(a) At $t = 0$

Prior to $t = 0$, v_{in} was equal to zero. Hence,

$$v_{C_1}(0) = v_{C_1}(0^-) = 0,$$
$$v_{C_2}(0) = v_{C_2}(0^-) = 0. \tag{1}$$

At $t = 0$ (Fig. (a)):

$$v_p(0) = 0,$$
$$v_{out}(0) = \left(\frac{R_3 + R_4}{R_4}\right) v_n(0)$$
$$= 0 \qquad \text{(because } v_n = v_p\text{)}.$$

Hence,

$$v_a(0) = v_{out}(0) = 0.$$

Consequently,

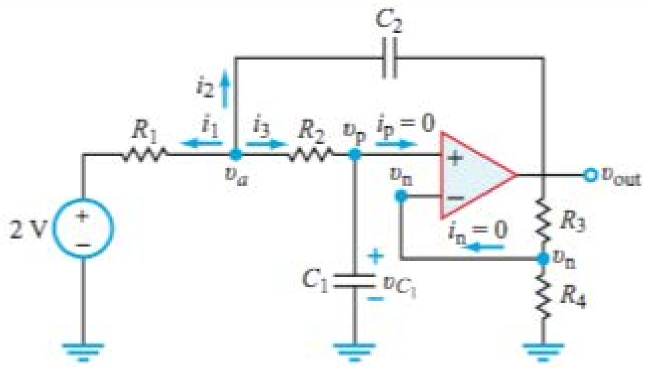
$$i_{C_1}(0) = 0,$$

and

$$v'_{C_1}(0) = \frac{i_{C_1}(0)}{C_1} = 0. \tag{2}$$

Equations (1) and (2) constitute the initial conditions we need for v_C.

At $t \geq 0$ the circuit is as shown in Fig. (b).



(b) At $t \geq 0$

At node v_a:

$$i_1 + i_2 + i_3 = 0,$$

or equivalently,

$$\frac{v_a - 2}{R_1} + C_2(v'_a - v'_{out}) + i_3 = 0. \tag{3}$$

Also,

$$i_3 = \frac{v_a - v_p}{R_2}, \tag{4}$$

$$v_p = v_{C_1} \tag{5}$$

and

$$i_3 = C_1 v'_{C_1}. \tag{6}$$

Combining Eqs. (4)–(6) gives:

$$v_a = v_{C_1} + R_2 C_1 v'_{C_1}, \tag{7}$$

and its time derivative is

$$v'_a = v'_{C_1} + R_2 C_1 v''_{C_1}. \tag{8}$$

Using Eqs. (7) and (8) in Eq. (3) to replace v_a and v'_a, respectively, and using Eq. (6) to replace i_3 in Eq. (3) leads to:

$$\frac{v_{C_1} + R_2 C_1 v'_{C_1} - 2}{R_1} + C_2(v'_{C_1} + R_2 C_1 v''_{C_1} - v'_{out}) + C_1 v'_{C_1} = 0. \tag{9}$$

Finally, we impose the op-amp voltage constraint $v_p = v_n$ (or equivalently $v_{C_1} = v_n$) and use voltage division to relate v_{out} to v_n:

$$v_{out} = \left(\frac{R_3 + R_4}{R_4}\right) v_n = \left(\frac{R_3 + R_4}{R_4}\right) v_{C_1}. \tag{10}$$

Using Eq. (10) in Eq. (9), followed with grouping of like terms, leads to

$$\upsilon''_{C_1} + a\upsilon'_{C_1} + b\upsilon_{C_1} = C, \tag{11}$$

with

$$a = \frac{R_2C_1 + R_1C_2\left[1 - \left(\frac{R_3+R_4}{R_4}\right)\right] + R_1C_1}{R_1R_2C_1C_2} = 0.9,$$

$$b = \frac{1}{R_1R_2C_1C_2} = 1,$$

$$c = \frac{2}{R_1R_2C_1C_2} = 2.$$

Hence,

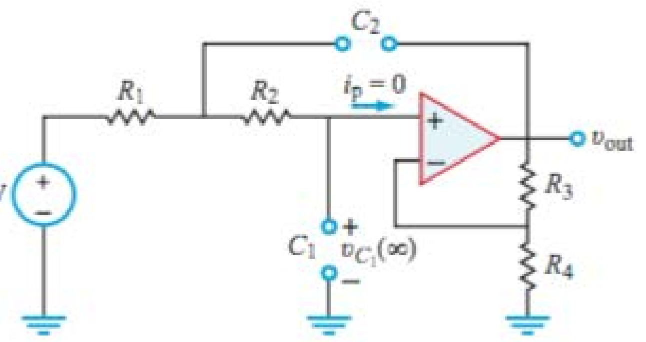
$$\alpha = \frac{a}{2} = 0.45 \text{ Np/s}, \qquad \omega_0 = \sqrt{b} = 1 \text{ rad/s}.$$

The underdamped response of υ_{C_1} is

$$\upsilon_{C_1}(t) = \upsilon_{C_1}(\infty) + [D_1 \cos\omega_d t + D_2 \sin\omega_d t]e^{-\alpha t}, \tag{12}$$

with $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{1 - 0.45^2} = 0.89$ rad/s.

At $t = \infty$ (Fig. (c)):



(c) At $t = \infty$

The capacitors act like open circuits, allowing no currents to flow through either R_1 or R_2. Hence,

$$\upsilon_{C_1}(\infty) = 2 \text{ V}. \tag{13}$$

The constants D_1 and D_2 are given by

$$D_1 = \upsilon_{C_1}(0) - \upsilon_{C_1}(\infty) = -2 \text{ V},$$

$$D_2 = \frac{\upsilon'_{C_1}(0) + \alpha[\upsilon_{C_1}(0) - \upsilon_{C_1}(\infty)]}{\omega_d} = \frac{0 + 0.45[0-2]}{0.89} = -1.01 \text{ V},$$

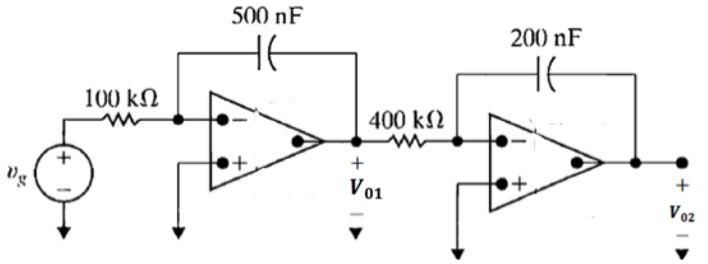
and the expression for $\upsilon_{C_1}(t)$ becomes:

$$\upsilon_{C_1}(t) = 2 - (2\cos 0.89t + 1.01 \sin 0.89t)e^{-0.45t} \qquad \text{(V)}.$$

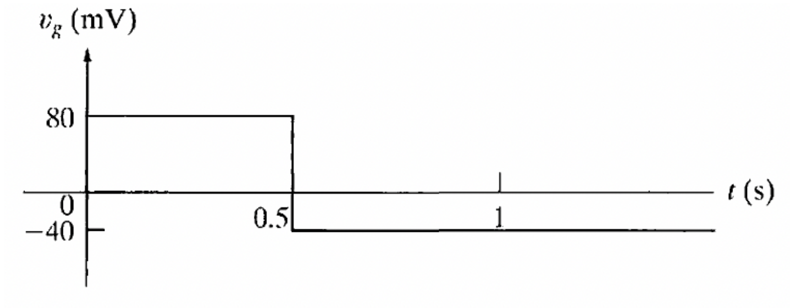
Using Eq. (10), the output voltage is given by:

$$\begin{aligned}\upsilon_{out}(t) &= \left(\frac{R_3+R_4}{R_4}\right)\upsilon_{C_1}(t)\\ &= [3.2 - (3.2\cos 0.89t - 1.61 \sin 0.89t)e^{-0.45t}] \qquad \text{(V)}, \qquad \text{for } t \geq 0.\end{aligned}$$

9. The voltage signal of Fig. 8 (b) is applied to the cascaded integrating amplifiers shown in Fig.8 (a). There is no energy stored in the capacitors at the instant the signal is applied. Derive the numerical expressions for $v_{o1}(t)$ and $v_{o2}(t)$ for the time intervals $0 < t < 1$ s. Assume the Ops are working in their linear range.



(a)



(b)

Figure 8

[a] $\dfrac{d^2v_o}{dt^2} = \dfrac{1}{R_1C_1R_2C_2}v_g$

$$\frac{1}{R_1C_1R_2C_2} = \frac{10^{-6}}{(100)(400)(0.5)(0.2) \times 10^{-6} \times 10^{-6}} = 250$$

$$\therefore \quad \frac{d^2v_o}{dt^2} = 250v_g$$

$0 \le t \le 0.5^-$:

$v_g = 80\,\text{mV}$

$$\frac{d^2v_o}{dt^2} = 20$$

$$\text{Let} \quad g(t) = \frac{dv_o}{dt}, \qquad \text{then} \quad \frac{dg}{dt} = 20 \quad \text{or} \quad dg = 20\,dt$$

$$\int_{g(0)}^{g(t)} dx = 20\int_0^t dy$$

$$g(t) - g(0) = 20t, \quad g(0) = \frac{dv_o}{dt}(0) = 0$$

$$g(t) = \frac{dv_o}{dt} = 20t$$

$$dv_o = 20t\,dt$$

$$\int_{v_o(0)}^{v_o(t)} dx = 20\int_0^t x\,dx; \qquad v_o(t) - v_o(0) = 10t^2, \quad v_o(0) = 0$$

$$v_o(t) = 10t^2\,\text{V}, \quad 0 \le t \le 0.5^-$$

$$\frac{dv_{o1}}{dt} = -\frac{1}{R_1C_1}v_g = -20v_g = -1.6$$

$$dv_{o1} = -1.6\,dt$$

$$\int_{v_{o1}(0)}^{v_{o1}(t)} dx = -1.6\int_0^t dy$$

$$v_{o1}(t) - v_{o1}(0) = -1.6t, \qquad v_{o1}(0) = 0$$

$$v_{o1}(t) = -1.6t\,\text{V}, \qquad 0 \le t \le 0.5^-$$

$0.5^+ \le t \le t_{\text{sat}}$:

$$\frac{d^2 v_o}{dt^2} = -10, \qquad \text{let} \quad g(t) = \frac{dv_o}{dt}$$

$$\frac{dg(t)}{dt} = -10; \qquad dg(t) = -10\,dt$$

$$\int_{g(0.5^+)}^{g(t)} dx = -10\int_{0.5}^{t} dy$$

$$g(t) - g(0.5^+) = -10(t-0.5) = -10t + 5$$

$$g(0.5^+) = \frac{dv_o(0.5^+)}{dt}$$

$$C\frac{dv_o}{dt}(0.5^+) = \frac{0 - v_{o1}(0.5^+)}{400 \times 10^3}$$

$$v_{o1}(0.5^+) = v_o(0.5^-) = -1.6(0.5) = -0.80\,\text{V}$$

$$\therefore \quad C\frac{dv_{o1}(0.5^+)}{dt} = \frac{0.80}{0.4 \times 10^3} = 2\,\mu\text{A}$$

$$\frac{dv_{o1}}{dt}(0.5^+) = \frac{2 \times 10^{-6}}{0.2 \times 10^{-6}} = 10\,\text{V/s}$$

$$\therefore \quad g(t) = -10t + 5 + 10 = -10t + 15 = \frac{dv_o}{dt}$$

$$\therefore \quad dv_o = -10t\,dt + 15\,dt$$

$$\int_{v_o(0.5^+)}^{v_o(t)} dx = \int_{0.5^+}^{t} -10y\,dy + \int_{0.5^+}^{t} 15\,dy$$

$$v_o(t) - v_o(0.5^+) = -5y^2\Big|_{0.5}^{t} + 15y\Big|_{0.5}^{t}$$

$$v_o(t) = v_o(0.5^+) - 5t^2 + 1.25 + 15t - 7.5$$

$$v_o(0.5^+) = v_o(0.5^-) = 2.5\,\text{V}$$

$$\therefore \quad v_o(t) = -5t^2 + 15t - 3.75\,\text{V}, \qquad 0.5^+ \le t \le t_{\text{sat}}$$

$$\frac{dv_{o1}}{dt} = -20(-0.04) = 0.8, \qquad 0.5^+ \le t \le t_{\text{sat}}$$

$$dv_{o1} = 0.8\,dt; \qquad \int_{v_{o1}(0.5^+)}^{v_{o1}(t)} dx = 0.8\int_{0.5^+}^{t} dy$$

$$v_{o1}(t) - v_{o1}(0.5^+) = 0.8t - 0.4; \qquad v_{o1}(0.5^+) = v_{o1}(0.5^-) = -0.8\,\text{V}$$

$$\therefore \quad v_{o1}(t) = 0.8t - 1.2\,\text{V}, \qquad 0.5^+ \le t \le t_{\text{sat}}$$

Summary:

$0 \le t \le 0.5^{-}\,\mathrm{s}: \qquad v_{o1} = -1.6t\,\mathrm{V}, \quad v_o = 10t^2\,\mathrm{V}$

$0.5^{+}\mathrm{s} \le t \le t_{\mathrm{sat}}: \qquad v_{o1} = 0.8t - 1.2\,\mathrm{V}, \quad v_o = -5t^2 + 15t - 3.75\,\mathrm{V}$

[b] $-12.5 = -5t_{\mathrm{sat}}^2 + 15t_{\mathrm{sat}} - 3.75$

$\therefore \quad 5t_{\mathrm{sat}}^2 - 15t_{\mathrm{sat}} - 8.75 = 0$

Solving, $\qquad t_{\mathrm{sat}} = 3.5\,\mathrm{sec}$

$v_{o1}(t_{\mathrm{sat}}) = 0.8(3.5) - 1.2 = 1.6\,\mathrm{V}$

$\omega_o > \alpha^2 \quad \therefore \qquad$ underdamped

$$s_{1,2} = -50 \pm \sqrt{50^2 - 5000} = -50 \pm j50$$

$$v_c = 60 + B_1'e^{-50t}\cos 50t + B_2'e^{-50t}\sin 50t$$

$$v_c(0) = -90 = 60 + B_1' \quad \therefore \qquad B_1' = -150$$

$$C\frac{dv_c}{dt}(0) = -5; \qquad \frac{dv_c}{dt}(0) = \frac{-5}{2\times 10^{-3}} = -2500$$

$$\frac{dv_c}{dt}(0) = -50B_1' + 50B_2 = -2500 \quad \therefore \qquad B_2' = -200$$

$$v_c = 60 - 150e^{-50t}\cos 50t - 200e^{-50t}\sin 50t\,\mathrm{V}, \quad t \ge 0$$

10. In the circuit shown in Fig.9, the switch was closed at t = 0 and re-opened at t = 0.5s. Determine the response $i_L(t)$ for $t \geq 0$, there's no energy stored in the inductor and capacitor.

Assume that V_S = 18V, R_S =1Ω, R_1 =5Ω, R_2 =2Ω, L= 2H and C_1= $\frac{1}{17}$F.

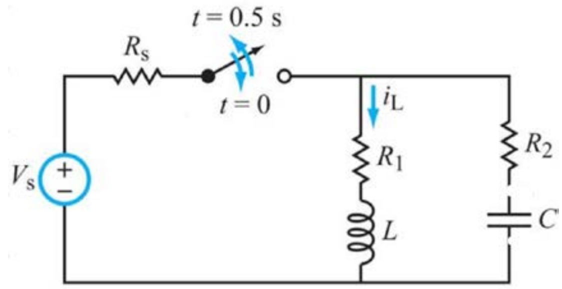


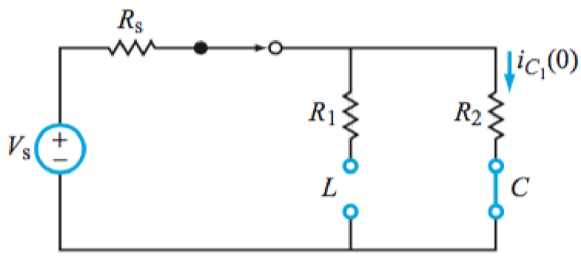
Figure 9

Solution:

Time Segment 1: $0 \leq t \leq 0.5$ **s**

Prior to $t = 0$, the circuit contained no sources. Hence,

$$i_{L_1}(0) = i_{L_1}(0^-) = 0, \qquad \text{[open-circuit equivalent]}$$
$$\upsilon_{C_1}(0) = \upsilon_{C_1}(0^-) = 0. \qquad \text{[short-circuit equivalent]}$$

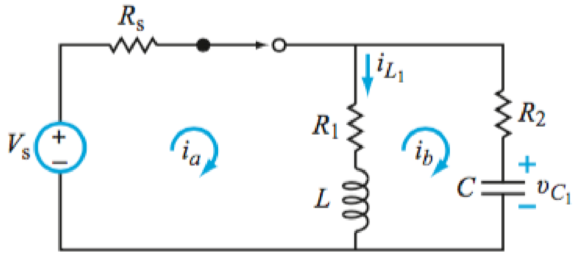


(a) At $t = 0$

At $t = 0$ (Fig. (a)):

$$i_{C_1}(0) = \frac{V_s}{R_s + R_2} = \frac{18}{1+2} = 6 \text{ A}, \tag{1}$$

$$\upsilon'_{C_1}(0) = \frac{i_{C_1}(0)}{C} = \frac{6}{\frac{1}{17}} = 102 \text{ V/s}. \tag{2}$$



(b) At $0 \le t \le 0.5$ s

At $0 \le t \le 0.5$ s (Fig. (b)):

$$-V_s + R_s i_a + i_b R_2 + v_{C_1} = 0, \qquad \text{[outer loop]} \tag{3}$$

$$i_b = C\frac{dv_{C_1}}{dt} = Cv'_{C_1}. \tag{4}$$

Using Eq. (4) in Eq. (3) and solving for i_a gives

$$i_a = \frac{V_s - v_{C_1} - R_2 C v'_{C_1}}{R_s}. \tag{5}$$

The left loop equation is:

$$-V_s + R_s i_a + R_1(i_a - i_b) + L(i'_a - i'_b) = 0. \tag{6}$$

The derivative of Eq. (5) gives

$$i'_a = \frac{-v'_{C_1} - R_2 C v''_{C_1}}{R_s}. \tag{7}$$

Using Eqs. (4), (5), and (7) in (6) gives:

$$-V_s + (V_s - v_{C_1} - R_2 C v'_{C_1}) + \frac{R_1}{R_s}(V_s - v_{C_1} - R_2 C v'_{C_1}) - R_1 C v'_{C_1} + \frac{L}{R_s}[-v'_{C_1} - R_2 C v''_{C_1}] - LCv''_{C_1} = 0. \tag{8}$$

Collecting like terms leads to:

$$v''_{C_1}\left[LC\left(1+\frac{R_2}{R_s}\right)\right] + v'_{C_1}\left[R_2 C + \frac{R_1 R_2 C}{R_s} + R_1 C + \frac{L}{R_s}\right] + v_{C_1}\left[1 + \frac{R_1}{R_s}\right] = \frac{R_1 V_s}{R_s}. \tag{9}$$

or equivalently

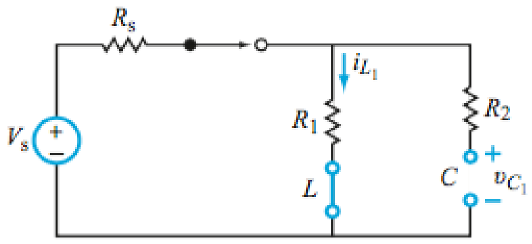
$$v''_{C_1} + av'_{C_1} + bv'_{C_1} = c, \tag{10}$$

where

$$a = \frac{R_s(R_1+R_2)C + R_1 R_2 C + L}{(R_s+R_2)LC} = \frac{1(5+2)(\frac{1}{17}) + 5\times 2\times(\frac{1}{17}) + 2}{(1+2)\times 2\times\frac{1}{17}} = 8.5,$$

$$b = \frac{R_s + R_1}{(R_s+R_2)LC} = \frac{1+5}{(1+2)\times 2\times\frac{1}{17}} = 17,$$

$$c = \frac{R_1 V_s}{(R_s + R_2)LC} = \frac{5 \times 18}{(1+2) \times 2 \times \frac{1}{17}} = 255.$$
$$\alpha = \frac{a}{2} = 4.25 \text{ Np/s},$$
$$\omega_0 = \sqrt{b} = \sqrt{17} = 4.12 \text{ rad/s},$$
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -4.25 + \sqrt{4.25^2 - 17} = -3.22 \text{ Np/s},$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -4.25 - \sqrt{4.25^2 - 17} = -5.28 \text{ Np/s}.$$



(c) At $t = \infty$ (had the switch remained closed)

Had the switch remained closed, at $t = \infty$, the circuit becomes as shown in Fig. (c), in which case
$$v_{C_1}(\infty) = i_{L_1} R_1 = \frac{V_s R_1}{R_s + R_1} = \frac{18 \times 5}{1+5} = 15 \text{ V}.$$

From Table 6-2,
$$A_1 = \frac{v'_{C_1}(0) - s_2[v_{C_1}(0) - v_{C_1}(\infty)]}{s_1 - s_2} = \frac{102 + 5.28[0 - 15]}{-3.22 + 5.28} = 11.05 \text{ V},$$
$$A_2 = -\left[\frac{v'_{C_1}(0) - s_1[v_{C_1}(0) - v_{C_1}(\infty)]}{s_1 - s_2}\right] = \frac{102 + 3.22[0 - 15]}{-3.22 + 5.28} = -26.05 \text{ V}.$$

Hence, $v_C(t)$ is given by
$$\begin{aligned} v_{C_1}(t) &= v_{C_1}(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ &= 15 + 11.05 e^{-3.22t} - 26.05 e^{-5.28t} \qquad \text{(V)}, \qquad \text{for } 0 \le t \le 0.5 \text{ s}. \end{aligned} \tag{11}$$

From Fig. (b), the current $i_{L_1}(t)$ is given by
$$i_{L_1}(t) = i_a - i_b. \tag{12}$$

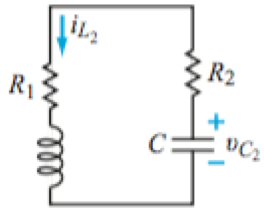
Using Eqs. (4) and (5) in Eq. (12) gives:
$$\begin{aligned} i_{L_1}(t) &= \frac{V_s}{R_s} - \frac{v_{C_1}}{R_s} - \frac{R_2 C}{R_s} v'_{C_1} - C v'_{C_1} \\ &= \frac{V_s}{R_s} - \frac{v_{C_1}}{R_s} - C\left(1 + \frac{R_2}{R_s}\right) v'_{C_1}. \end{aligned} \tag{13}$$

From Eq. (11),
$$\begin{aligned} v'_{C_1}(t) &= -3.22 \times 11.05 e^{-3.22t} + 5.28 \times 26.05 e^{-5.28t} \\ &= -35.59 e^{-3.22t} + 137.59 e^{-5.28t} \qquad \text{(V/s)}. \end{aligned} \tag{14}$$

Using Eqs. (11) and (14) in Eq. (13), and then simplifying terms, leads to

$$i_{L_1}(t) = [3 - 4.77e^{-3.22t} + 1.77e^{-5.28t}] \quad \text{(A)}, \quad \text{for } 0 \le t \le 0.5 \text{ s}. \tag{15}$$

Time Segment 2: $t > 0.5$ s



(d) After $t = 0.5$ s

After re-opening the switch, the circuit becomes a series RLC circuit as shown in Fig. (d). Since the circuit no longer contains sources,

$$i_{L_2}(\infty) = 0,$$
$$v_{C_2}(\infty) = 0.$$

From Table 6-1, the damping factors are:

$$\alpha = \frac{R}{2L} = \frac{R_1 + R_2}{2L} = \frac{5+2}{2 \times 2} = \frac{7}{4} = 1.75 \text{ Np/s},$$
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2 \times \frac{1}{17}}} = 2.92 \text{ rad/s}.$$

Since $\alpha < \omega_0$, the response will be underdamped:

$$i_{L_2}(t) = [D_1 \cos \omega_d(t - 0.5) + D_2 \sin \omega_d(t - 0.5)]e^{-\alpha(t-0.5)}, \tag{16}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 2.33 \text{ rad/s},$$

and the expression in Eq. (16) was shifted in time by 0.5 s. At $t = 0.5$ s, we require that:

$$i_{L_1}(0.5) = i_{L_2}(0.5), \tag{17a}$$
$$v_{C_1}(0.5) = v_{C_2}(0.5). \tag{17b}$$

Equating the expressions given by Eqs. (15) and (16) at $t = 0.5$ s gives:

$$3 - 4.78e^{-3.22 \times 0.5} + 1.78e^{-5.28 \times 0.5} = D_1,$$

which gives

$$D_1 = 2.17 \text{ V}. \tag{18}$$

From the circuit in Fig. (d),

$$\begin{aligned} v_{C_2}(t) &= (R_1 + R_2)i_{L_2} + Li'_{L_2} \\ &= 7[D_1 \cos \omega_d(t - 0.5) + D_2 \sin \omega_d(t - 0.5)]e^{-\alpha(t-0.5)} \end{aligned}$$

$$+2[-\omega_d D_1 \sin\omega_d(t-0.5)+\omega_d D_2 \cos\omega_d(t-0.5)$$
$$-\alpha D_1 \cos\omega_d(t-0.5)-\alpha D_2 \sin\omega_d(t-0.5)]e^{-\alpha(t-0.5)}. \quad (19)$$

At $t = 0.5$ s, Eqs. (11) and (19) give:

$$\upsilon_{C_1}(0.5) = 15 + 11.07e^{-3.22\times 0.5} - 26.07e^{-5.28\times 0.5} = 15.35 \text{ V}, \quad (20a)$$
$$\begin{aligned}\upsilon_{C_2}(0.5) &= 7D_1 + 2\omega_d D_2 - 2\alpha D_1 \\ &= 7\times 2.17 + 2\times 2.33 D_2 - 2\times 1.75\times 2.17 = 7.6 + 4.66 D_2. \quad (20b)\end{aligned}$$

Equating Eq. (20a) to Eq. (20b) leads to

$$D_2 = 1.66 \text{ V}.$$

Hence,

$$i_{L_2}(t) = [2.17\cos 2.33(t-0.5) + 1.66\sin 2.33(t-0.5)]e^{-1.75(t-0.5)} \quad \text{(A)},$$
$$\text{for } t \geq 0.5 \text{ s.} \quad (21)$$

The expressions given by Eqs. (15) and (21) constitute the complete solution.