EE150 - Signals and Systems, Spring 2020-21

Homework Set #8

Instructor: Prof. Lin Xu

Acknowledgements:

1) Total Score: **100**.
2) Deadline: **23:59, 12 June, 2021**.
3) Tutorial Time: **19:50, 13 June, 2021, TC101**.
4) Please notice that **no late submission is accepted** for this homework.

Problem 1. (3 × 5 points)

Determine the z-transform for each of the following sequences. Sketch the pole zero plot and indicate the ROC.

1) $2^n u[-n] + (\frac{1}{2})^n u[n-1]$
2) $4^n cos[\frac{\pi}{3}n + \frac{\pi}{4}]u[-n-1]$
3) $n(\frac{1}{2})^{|n|}$

Solution.

1)

$$x_1[n] = 2^n u[-n],$$

$$X_1(Z) = \sum_{n=-\infty}^{0} (2)^n z^{-n} = \sum_{n=0}^{+\infty} (2)^{-n} z^n = \frac{-2z^{-1}}{1-2z^{-1}}, |z| < 2.$$

$$x_2[n] = \frac{1}{2}^n u[n-1],$$

$$X_2(Z) = \sum_{n=1}^{+\infty} \frac{1}{2}^n z^{-n} = \sum_{n=0}^{+\infty} \frac{1}{2}^{n+1} z^{-n-1} = \frac{z^{-1}/2}{1-(1/2)z^{-1}}, |z| > \frac{1}{2}.$$

$$x[n] = x_1[n] + x_2[n],$$

$$X(Z) = \frac{-2z^{-1}}{1-2z^{-1}} + \frac{\frac{z^{-1}}{2}}{1-\frac{1}{2}z^{-1}} = \frac{3z}{(2-z)(2z-1)}$$

ROC: $\frac{1}{2} < |z| < 2$.

Pole: $2, \frac{1}{2}$, Zero:0.

2)

$$x[n] = 4^n \frac{e^{j\frac{\pi n}{3}+\frac{\pi}{4}} + e^{-j\frac{\pi n}{3}+\frac{\pi}{4}}}{2} u[-n-1],$$

$$X(Z) = \frac{e^{j\pi/4}}{2}\frac{1}{4e^{\frac{j\pi}{3}}z^{-1}-1} + \frac{e^{-j\pi/4}}{2}\frac{1}{4e^{\frac{-j\pi}{3}}z^{-1}-1}, |z| < 4.$$

ROC: $Z < |4|$.

Pole: $4e^{\frac{j\pi}{3}}, 4e^{\frac{-j\pi}{3}}$; Zero: $0,\ 2+2\sqrt{3}$.

3)

$$x[n] = n(\frac{1}{2})^{|n|} = n(\frac{1}{2})^n u[n] + n2^n u[-n-1]$$

$$X(Z) = \frac{\frac{z^{-1}}{2}}{(1-\frac{1}{2}z^{-1})^2} - \frac{2z^{-1}}{(1-2z^{-1})^2}.$$

ROC: $\frac{1}{2} < |z| < 2$.

Zero: 0, 1, -1. Pole: $\frac{1}{2}$, 2.

Problem 2. (2 × 5 points)

Suppose we are given the following facts about a particular LTI system S with impulse response $h[n]$ and z-transform $H(z)$.

- $h[n]$ is real.
- $h[n]$ is right-sided.
- $\lim_{z \to +\infty} H(z) = 1$.
- $H(z)$ has two zeros.
- $H(z)$ has one of its poles at a non-real location on the circle defined by $|z| = \frac{3}{4}$.

Answer the following two questions with your analysis:

1) Is S causal?
2) Is S stable?

Solution.

1) Since $\lim_{z \to +\infty} H(z) = 1$, $H(z)$ has no poles at infinity. Furthermore, since $h[n]$ is right sided, $h[n]$ has to be casual.
2) Since $h[n]$ is causal, the numerator and denominator polynomials of $H(z)$ have the same order. Since H$H(z)$ is given to have two zeros, we may conclude that it also has two poles.

   Since $h[n]$ is real, the poles must occur in conjugate pairs. Also, it is given that one of the poles lies on the circle defined by $|z| = \frac{3}{4}$. Therefore, the other pole also lies on this circle.

   From above analysis, we can conclude that ROC of $H(z)$ will be of form $|z| > \frac{3}{4}$, which include the unit circle. As a result, the system is stable.

Problem 3. (3 × 5 points)

A causal LTI discrete-time system is described by the difference equation

$$y[n] = 0.4y[n-1] + 0.05y[n-2] + 3x[n]$$

where x[n] and y[n] are, respectively, the input and output sequences of the system.

1) Determine the transfer function $H(z)$ of the system.
2) Determine the impulse response $h[n]$ of the system.
3) Determine the step response $s[n]$ of the system.

Solution.

1) After z-transform,

$$Y(z) = 0.4Y(z)z^{-1} + 0.05Y(z)z^{-2} + 3X(z) \tag{1}$$

Therefore, we could obtain:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1 - 0.4z^{-1} - 0.05z^{-2}} \tag{2}$$

Since the system is casual, ROC: |z| > 0.5.

2) From 1) we could get

$$H(z) = \frac{3}{1 - 0.4z^{-1} - 0.05z^{-2}} = \frac{0.5}{1 + 0.1z^{-1}} + \frac{2.5}{1 - 0.5z^{-1}}$$

Because this is a causal LTI discrete-time system, we can get the impulse response as

$$h[n] = (0.5 \times (-0.1)^n + 2.5 \times 0.5^n)\mu[n] \tag{3}$$

3)

$$X(z) = \frac{1}{1 - z^{-1}}$$

$$S(Z) = X(Z)H(Z) = \frac{3}{(1 + 0.1z^{-1})(1 - 0.5z^{-1})(1 - z^{-1})} = \frac{1/22}{1 + 0.1z^{-1}} - \frac{5/2}{1 - 0.5z^{-1}} + \frac{60/11}{1 - z^{-1}}.$$

ROC: |z| > 1.

$$s[n] = (\frac{1}{22}(-0.1)^n - \frac{5}{2}(0.5)^n + \frac{60}{11})u[n]$$

Problem 4. (3×10 points)

Consider the system function corresponding to casual LTI systems:

$$H(Z) = \frac{1}{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2})}.$$

1) Draw a direct-form block diagram.
2) Draw a block diagram that corresponds to the cascade connection of two second-order block diagrams.
3) Determine whether there exists a block diagram which is the cascade of four first-order block diagrams with the constraint that all the coefficient multipliers must be real. If false, state the reason. If true, draw the diagram.

Solution.

See Figure 1.

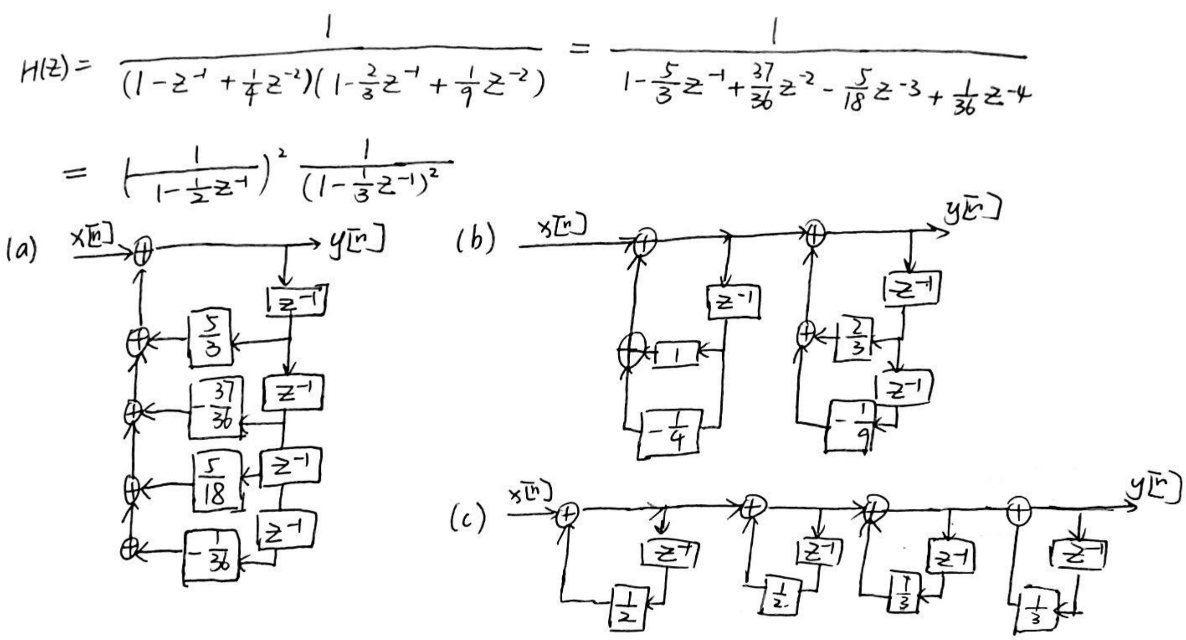

Figure 1: P4: Block Diagram

Problem 5. (3×10 points)

Consider a system whose input x[n] and output y[n] are related by

$$y[n-1] + 2y[n] = x[n].$$

1) Determine the zero input response of this system if $y[-1] = 2$.
2) Determine the zero state response of this system to the input $x[n] = (\frac{1}{4})^n u[n]$.
3) Determine the output of the system for $n \geq 0$ when $x[n] = (\frac{1}{4})^n u[n]$ and $y[-1] = 2$.

Solution. Applying the unilateral z-transform to the given differential equation, we have

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = \mathcal{X}(z).$$

1) $x[n] = 0$ for the zero input response. Since $y[-1] = 2$,

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = 0,$$
$$\mathcal{Y}(z) = \frac{-1}{1 + \frac{1}{2}z^{-1}}.$$

Taking the inverse unilateral z-transform,

$$y[n] = -(-\frac{1}{2})^n u[n].$$

2) $y[-1] = 0$ for the zero state response. Since $x[n] = (\frac{1}{4})^n u[n]$ and $y[-1] = 0$,

$$\mathcal{X}(z) = \mathcal{UZ}((\frac{1}{4})^n u[n]) = \frac{1}{1 - \frac{1}{4}z^{-1}}, |z| > \frac{1}{4}.$$

Therefore,

$$\mathcal{Y}(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} \frac{1}{2 + z^{-1}}.$$

Using partial fraction expansion followed by inverse z transform,

$$y[n] = \frac{1}{3}(-\frac{1}{2})^n u[n] + \frac{1}{6}(\frac{1}{4})^n u[n].$$

3) The total response is the sum of the zero state and zero input responses, so

$$y[n] = -\frac{2}{3}(-\frac{1}{2})^n u[n] + \frac{1}{6}(\frac{1}{4})^n u[n].$$