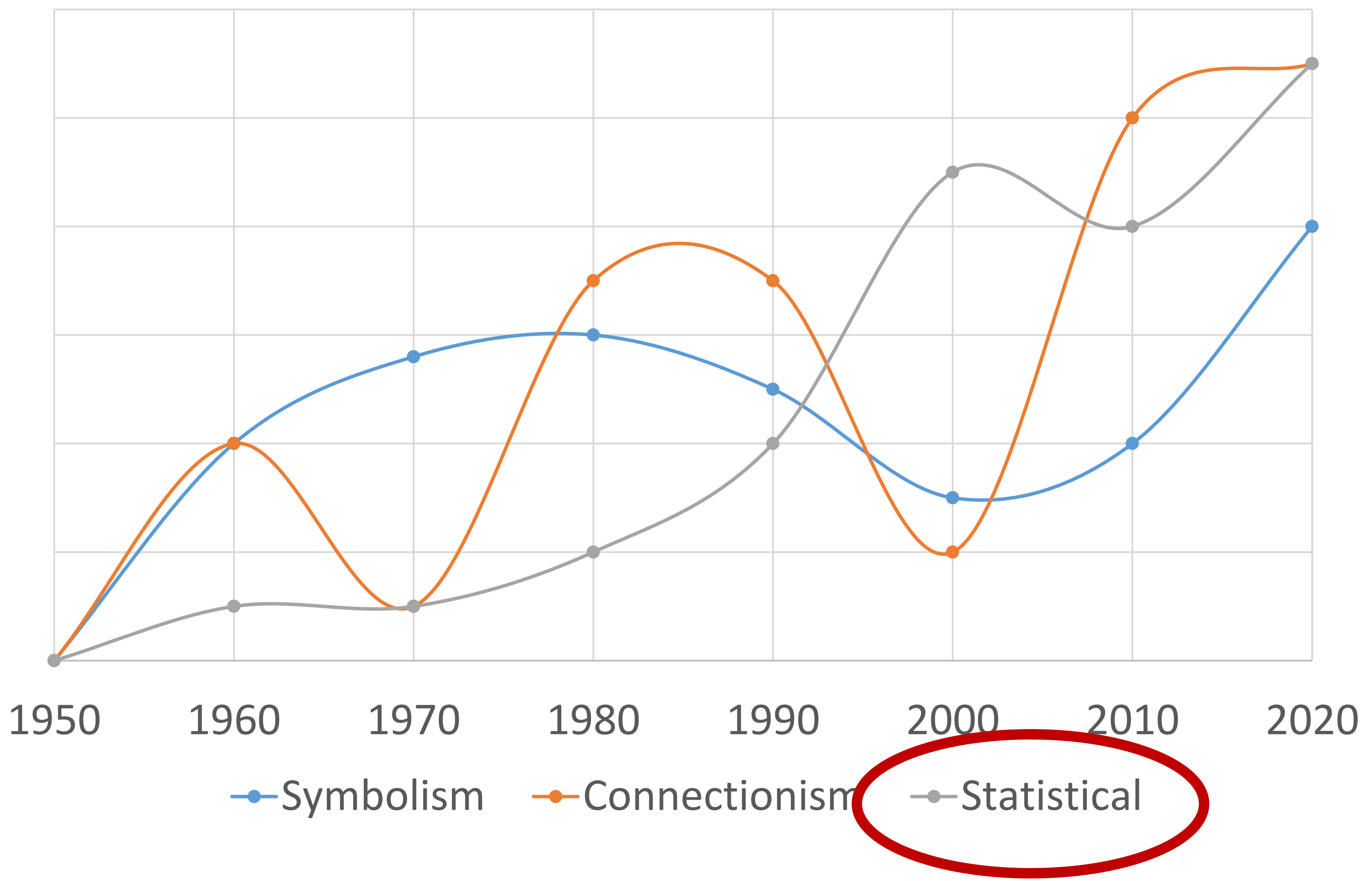
# Announcement

- Midterm @Nov. 12 (in class)
  - Location: TBA
  - Format
    - Closed-book. You can bring an A4-size cheat sheet and nothing else.
    - Around 5 problems
  - Grade
    - 25% of the total grade

# Three types of (strong) AI approaches



# Probability

AIMA Chapter 13

[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley (ai.berkeley.edu).]

# Uncertainty

- My flight to New York is scheduled to leave at 11:25
  - Let action $A_t$ = leave home $t$ minutes before flight and drive to the airport
  - Will $A_t$ ensure that I catch the plane?
- Problems:
  - noisy sensors (radio traffic reports, Google maps)
  - uncertain action outcome (car breaking down, accident, etc.)
  - partial observability (other drivers' plans, etc.)
  - immense complexity of modelling and predicting traffic, security line, etc.

# Probability

- Probability
  - Given the available evidence and the choice $A_{120}$, I will catch the plane with probability 0.92
- ***Subjective*** or ***Bayesian*** probability:
  - Probabilities relate propositions to one's own state of knowledge
    - ignorance: lack of relevant facts, initial conditions, etc.
    - laziness: failure to list all exceptions, compute detailed predictions, etc.
  - Not claiming a "probabilistic tendency" in the actual situation (traffic is not like quantum mechanics)

# Decisions

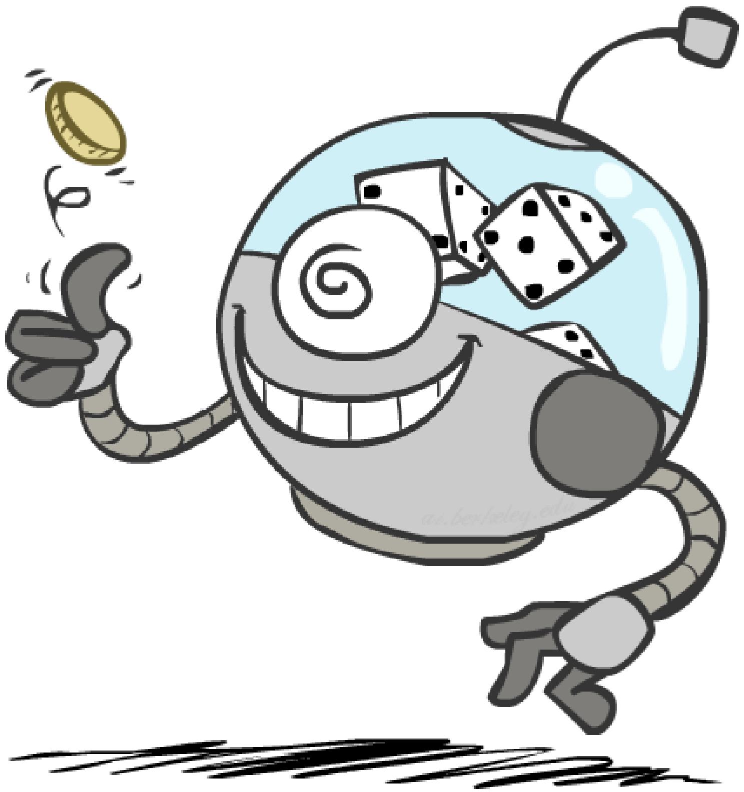
- Suppose I believe
  - $P(CatchPlane \mid A_{60}, \text{all my evidence...}) = 0.51$
  - $P(CatchPlane \mid A_{120}, \text{all my evidence...}) = 0.97$
  - $P(CatchPlane \mid A_{1440}, \text{all my evidence...}) = 0.9999$
- Which action should I choose?
- Depends on my ***preferences*** for, e.g., missing flight, airport food, etc.
- ***Utility theory*** is used to represent and infer preferences
- ***Decision theory*** = utility theory + probability theory
- ***Maximize expected utility*** : $a^* = \text{argmax}_a \sum_s P(s \mid a)\, U(s)$

# Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the pacman?

- We denote random variables with capital letters

- Like variables in a CSP, random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}



# Probability Distributions

- Associate a probability with each value of a random variable
  - Temperature:

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

  - Weather:

$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.1 |
| fog | 0.3 |
| meteor | 0.0 |

- A probability is a single number

$$P(W = rain) = 0.1$$

Shorthand notation: $P(rain) = P(W = rain)$,

- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

# Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey: $P(x_1, x_2, \ldots x_n) \geq 0$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

- Size of distribution for n variables with domain size d? $d^n$
  - For all but the smallest distributions, cannot write out by hand!

# Probabilistic Models

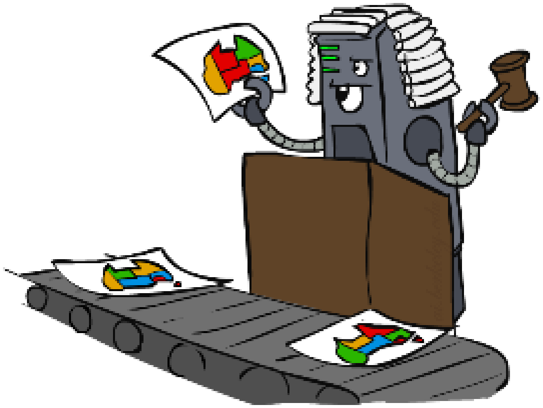
- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Joint distributions: say whether assignments (outcomes) are likely
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

Distribution over T,W

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |


Constraint over T,W

| T | W | P |
|---|---|---|
| hot | sun | T |
| hot | rain | F |
| cold | sun | F |
| cold | rain | T |



# Probabilities of events

- An *event* is a set E of outcomes

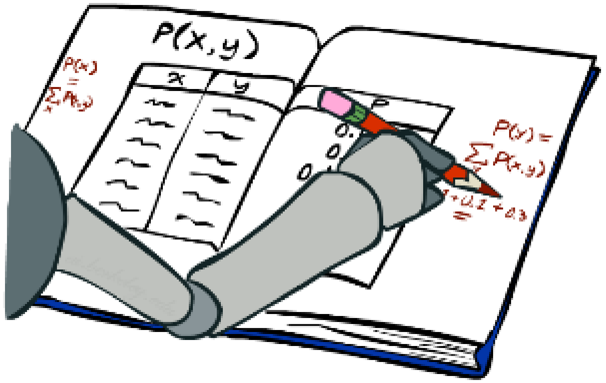
$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- Given a joint distribution over all variables, we can compute any event probability!
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

$P(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

# Marginal Distributions




- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$P(t) = \sum_w P(t,w)$$

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$$P(w) = \sum_t P(t,w)$$

$P(W)$

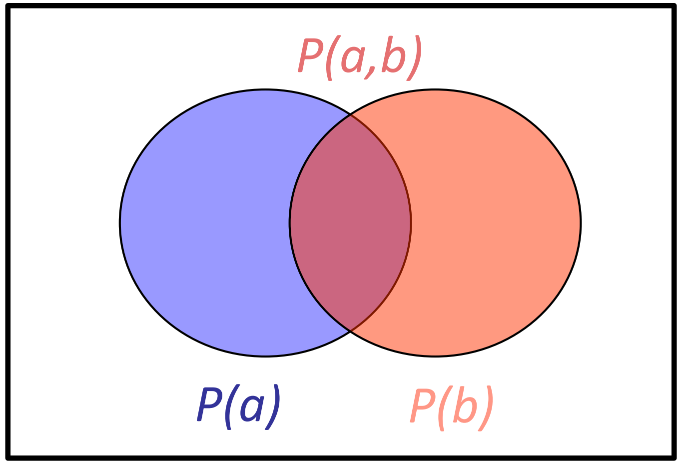
| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.4 |

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

# Conditional Probabilities

- The probability of an event given that another event has occurred

$$P(a|b) = \frac{P(a,b)}{P(b)}$$




$$P(T,W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 \;\; = 0.5$$

# Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Joint Distribution

$$P(T,W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

Conditional Distributions

$$P(W|T)$$

$$P(W|T=hot)$$

| W | P |
|---|---|
| sun | 0.8 |
| rain | 0.2 |

$$P(W|T=cold)$$

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

# Probabilistic Inference

- Probabilistic inference
  - compute a desired probability from other known probabilities (e.g. conditional from joint)
- We generally compute conditional probabilities
  - These represent the agent’s beliefs given the evidence
  - P(on time | no reported accidents) = 0.90

# Inference by Enumeration

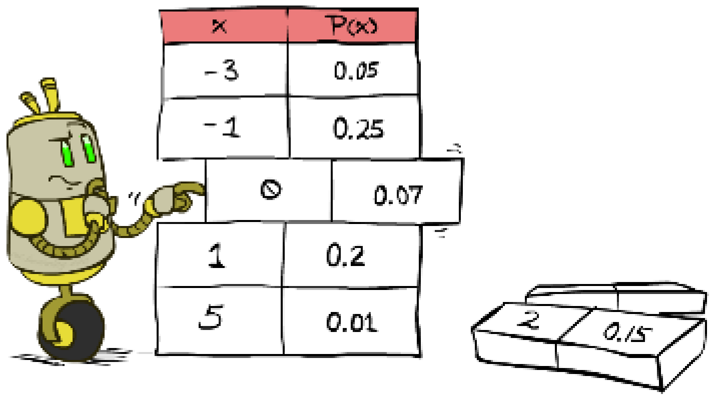
- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query variable: $Q$
  - Hidden variables: $H_1 \dots H_r$

$$\left.\begin{array}{l} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{array}\right\} \begin{array}{c} X_1, X_2, \dots X_n \\ \textit{All variables} \end{array}$$

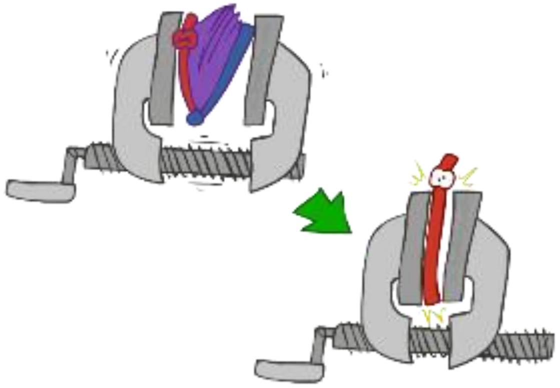
- We want:

$$P(Q|e_1 \dots e_k)$$

- Step 1: Select the entries consistent with the evidence




- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

# Inference by Enumeration

1. Select the entries consistent with the evidence
2. Sum out H to get joint of Query and evidence
3. Normalize

- P(W | winter)? sun: 0.5, rain: 0.5
- P(W | winter, hot)? sun: 0.67, rain: 0.33

| S | T | W | P |
|---|---|---|---|
| summer | hot | sun | 0.30 |
| summer | hot | rain | 0.05 |
| summer | cold | sun | 0.10 |
| summer | cold | rain | 0.05 |
| winter | hot | sun | 0.10 |
| winter | hot | rain | 0.05 |
| winter | cold | sun | 0.15 |
| winter | cold | rain | 0.20 |

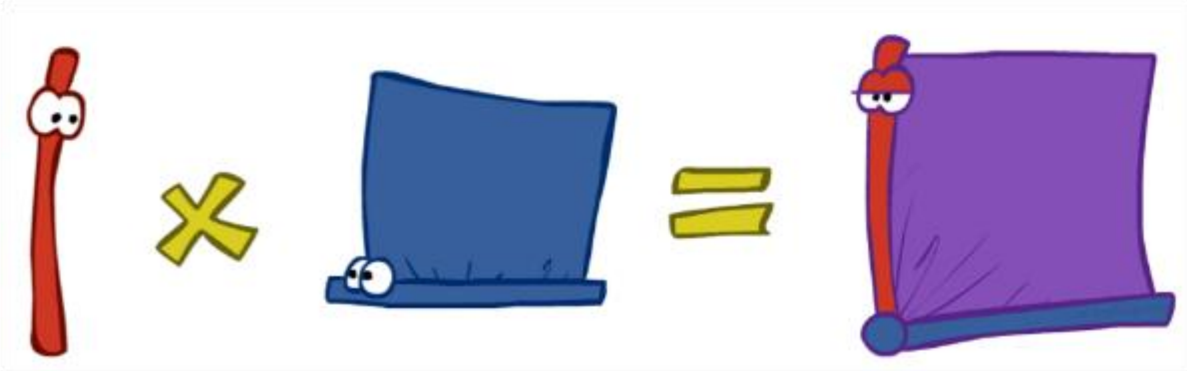
# Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

# The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x,y)}{P(y)}$$



# The Product Rule

$$P(y)P(x|y) = P(x, y)$$

- Example:

$P(W)$

| W | P |
|---|---|
| sun | 0.8 |
| rain | 0.2 |

$P(D|W)$

| D | W | P |
|---|---|---|
| wet | sun | 0.1 |
| dry | sun | 0.9 |
| wet | rain | 0.7 |
| dry | rain | 0.3 |

$P(D, W)$

| D | W | P |
|---|---|---|
| wet | sun | |
| dry | sun | |
| wet | rain | |
| dry | rain | |

# The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

# Bayes’ Rule

- Two ways to factor a joint distribution over two variables:

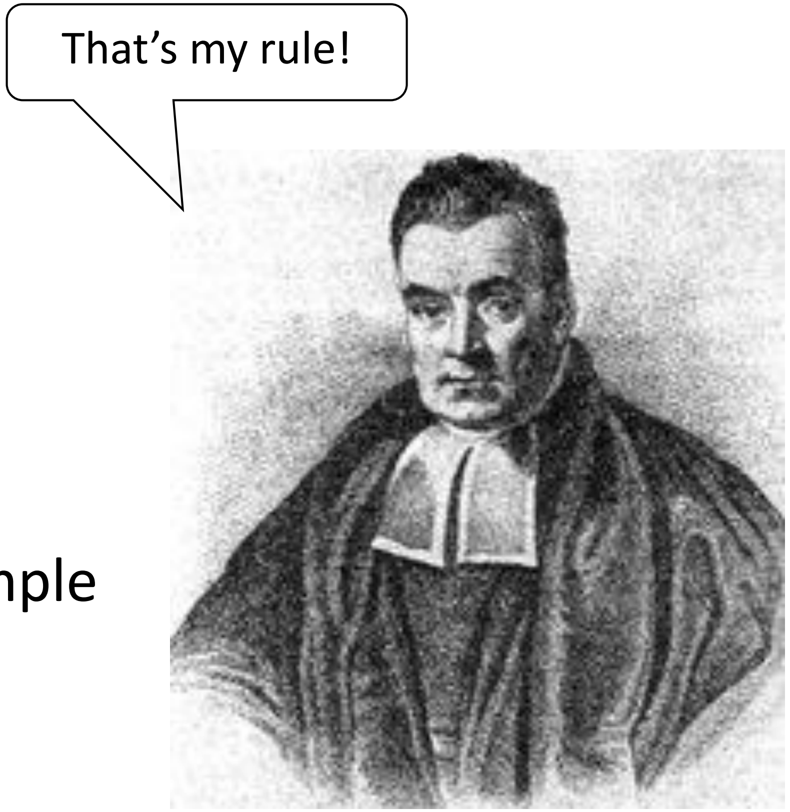
$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we’ll see later

- In the running for most important AI equation!



# Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

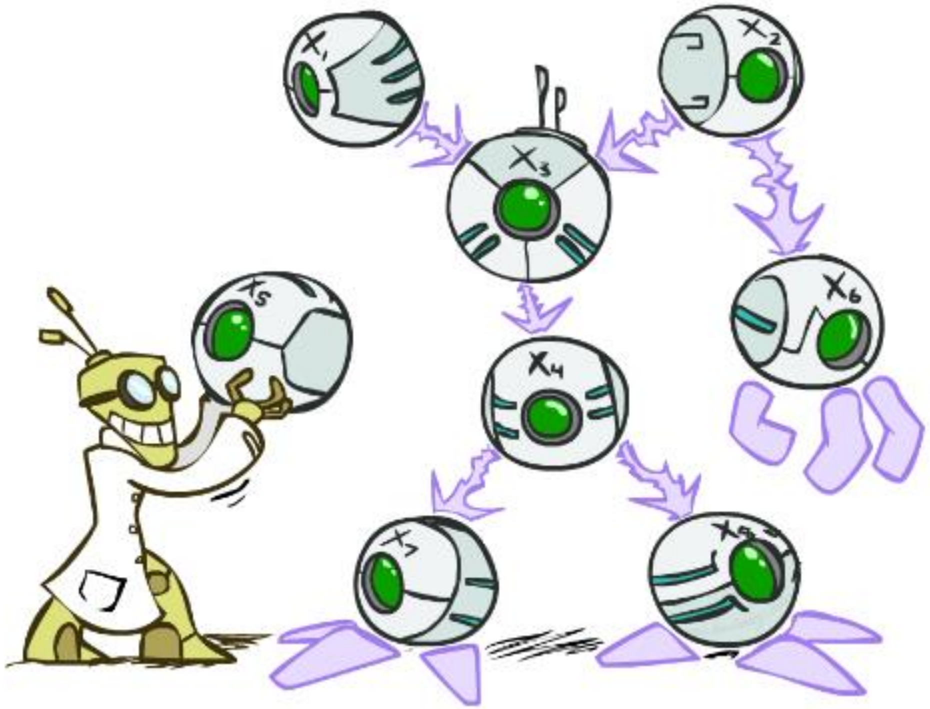
$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.9999}$$

# Bayesian Networks



AIMA Chapter 14.1, 14.2

[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley (ai.berkeley.edu)
and from slides by Markus Svensén & Christopher M Bishop]

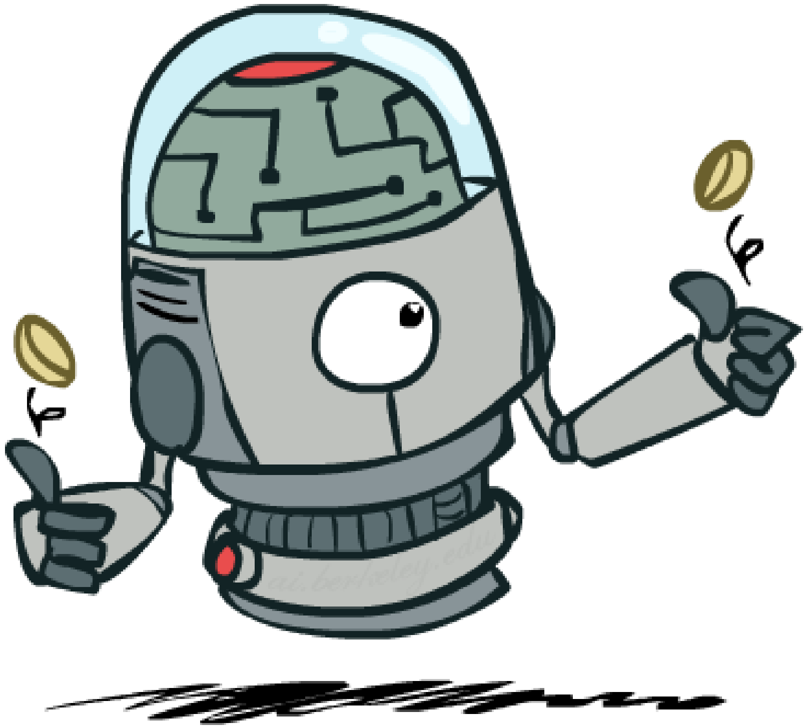
# Additional Reference

- [PRML] Pattern Recognition and Machine Learning, Christopher Bishop, Springer 2006.
  - Chapter 8.1 - 8.3

# Probabilistic Models

- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: making decisions based on expected utility
- How do we build models, avoiding the $d^n$ blowup?

# Independence



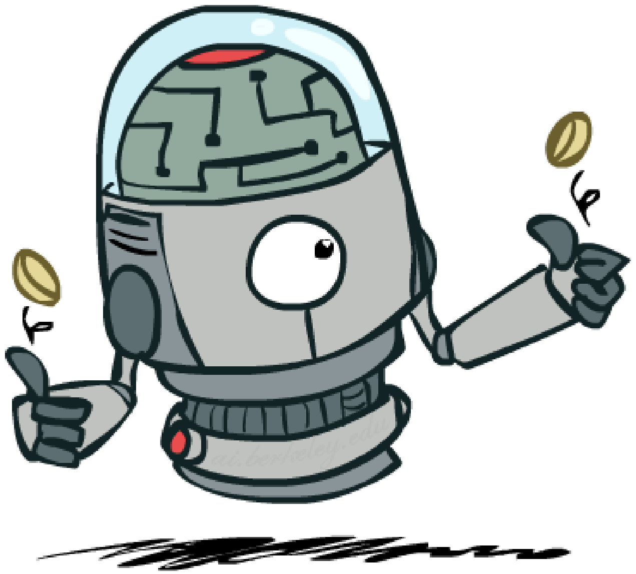
# Independence

- Two variables X and Y are (absolutely) ***independent*** if

$$\forall x,y \qquad P(x,y) = P(x)P(y)$$

  - This says that their joint distribution ***factors*** into a product of two simpler distributions
  - Combine with product rule $P(x,y) = P(x|y)P(y)$ we obtain another form:

$$\forall x,y\ P(x|y) = P(x) \quad \text{or} \quad \forall x,y\ P(y|x) = P(y)$$



- Example: two dice rolls $Roll_1$ and $Roll_2$
  - $P(Roll_1=5, Roll_2=5) = P(Roll_1=5)P(Roll_2=5) = 1/6 \times 1/6 = 1/36$
  - $P(Roll_2=5 \mid Roll_1=5) = P(Roll_2=5)$

# Conditional Independence

- Unconditional (absolute) independence is rare

- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.

- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y | Z$

  if and only if:

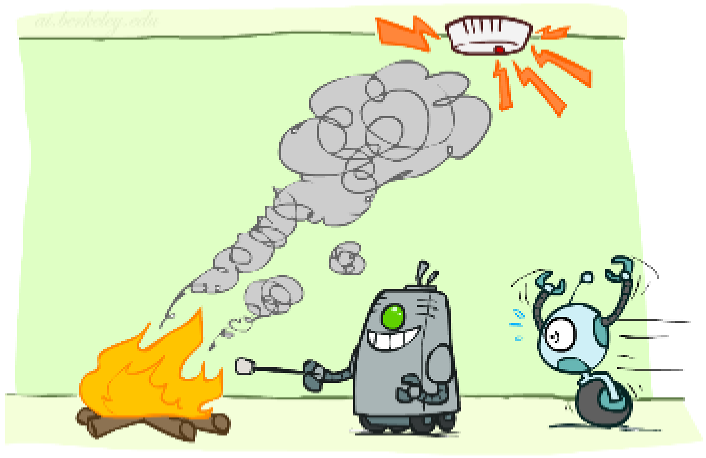
  $\forall x,y,z \quad P(x \mid y,z) = P(x|z)$

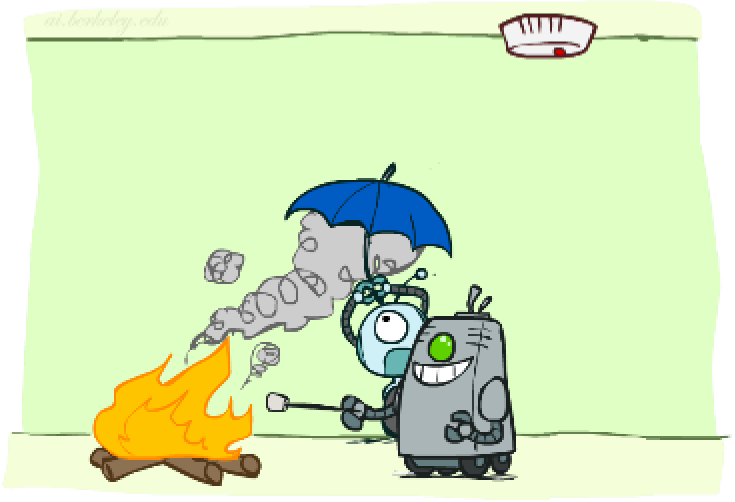
  or, equivalently, if and only if

  $\forall x,y,z \quad P(x,y \mid z) = P(x|z)P(y|z)$

# Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm (smoke detector)





# Conditional Independence

- What about this domain:
  - Traffic
  - Umbrella
  - Raining

# Conditional Independence and the Chain Rule

- Chain rule: $$P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$$

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

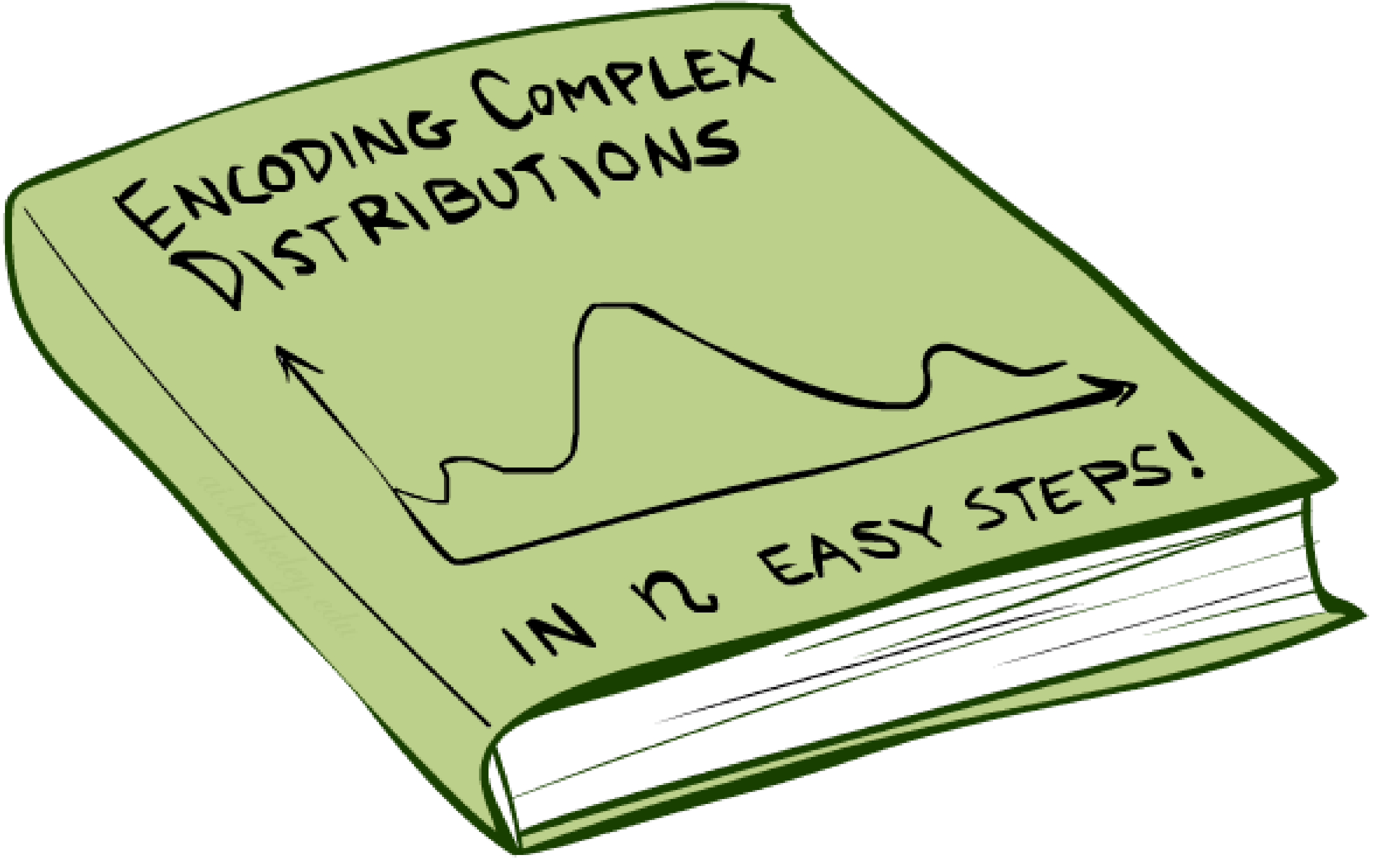
- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})\boxed{P(\text{Umbrella}|\text{Rain})}$$

*Requires less space to encode!*

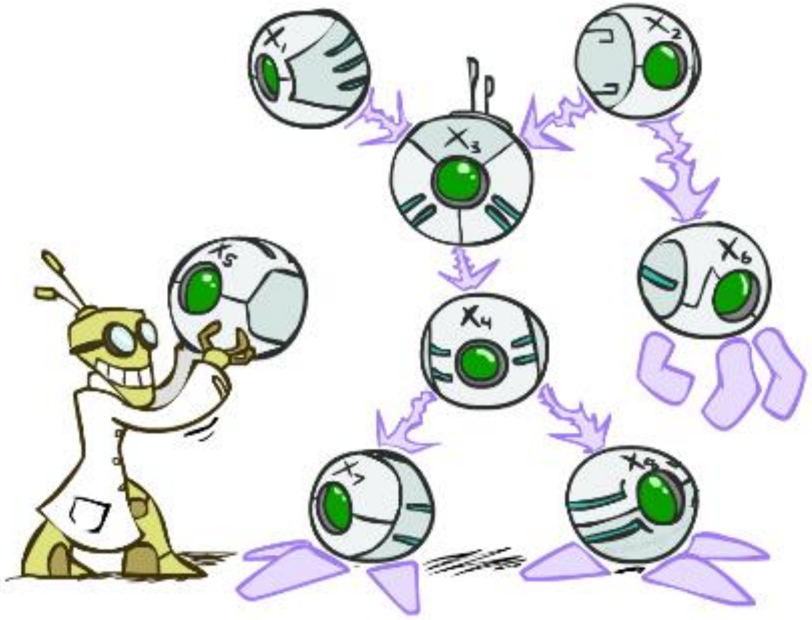
- BayesNets / graphical models help us express conditional independence assumptions

# Bayesian Networks: Big Picture



# Bayesian Networks: Big Picture

- Full joint distribution tables answer every question, but:
  - Size is exponential in the number of variables
  - Need gazillions of examples to learn the probabilities
  - Inference by enumeration (summing out hiddens) is too slow
- Bayesian networks:
  - Express all the conditional independence relationships in a domain
  - Factor the joint distribution into a product of small conditionals
  - Often reduce size from exponential to linear
  - Faster learning from fewer examples
  - Faster inference (linear time in some important cases)



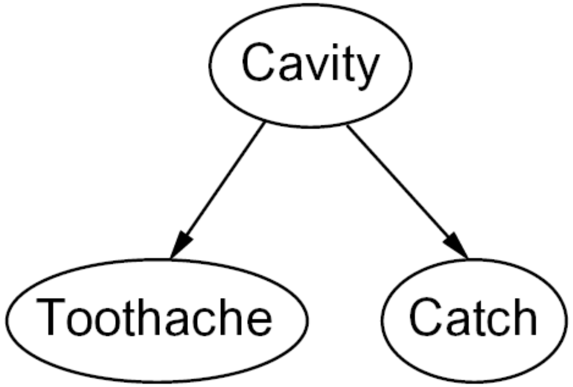
# Bayesian Networks Syntax

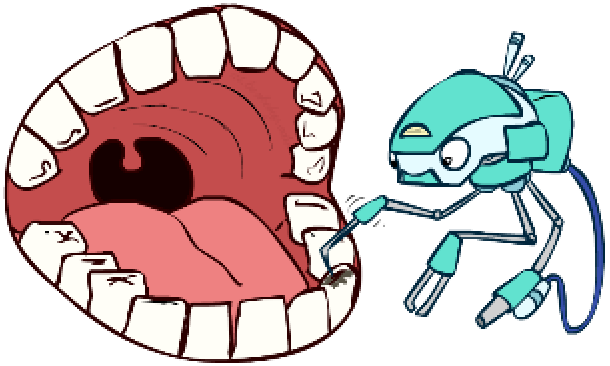
# Bayesian Networks Syntax

- Nodes: variables (with domains)

- Arcs: interactions
  - Indicate “direct influence” between variables
  - For now: imagine that arrows mean direct causation (in general, they may not!)
  - Formally: encode conditional independence (more later)

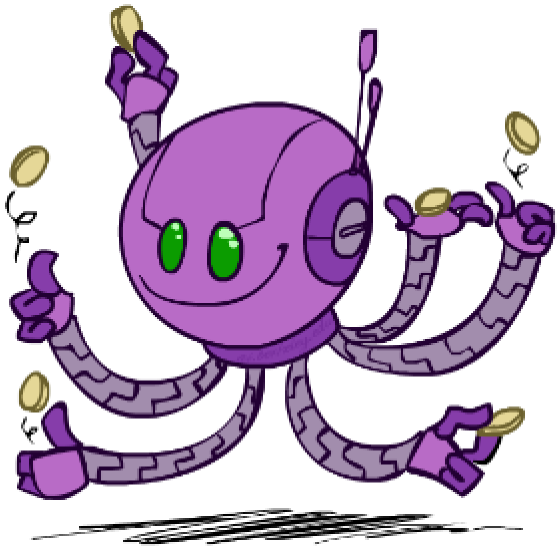
- No cycle is allowed!

# Example: Coin Flips

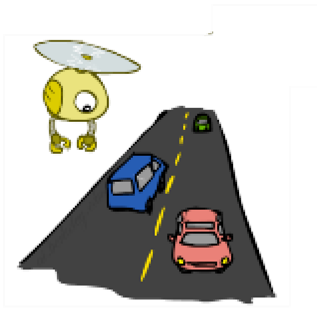
- N independent coin flips

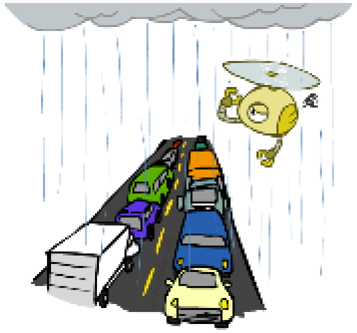



- No interactions between variables: absolute independence

# Example: Traffic

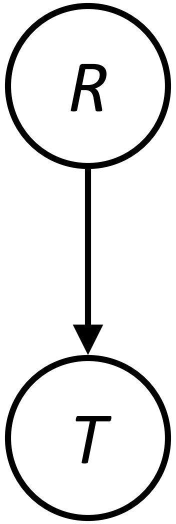
- Variables:
  - R: It rains
  - T: There is traffic





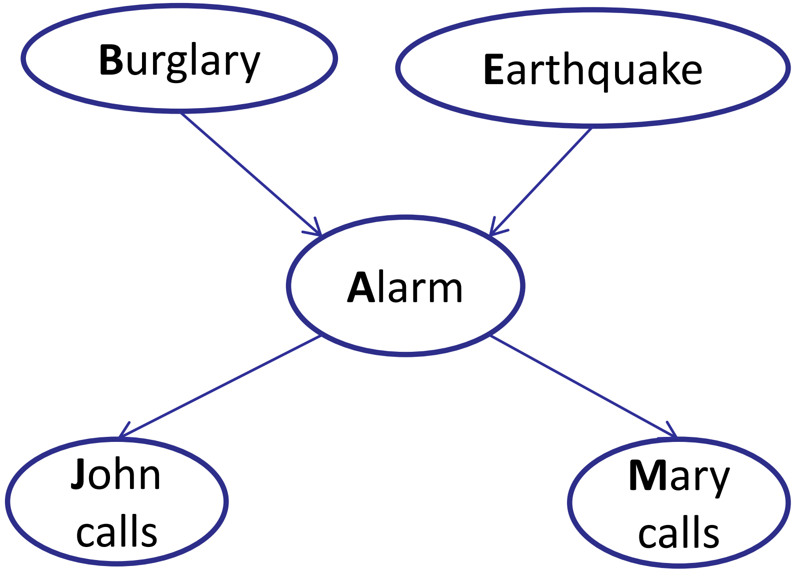
- Model 1: independence

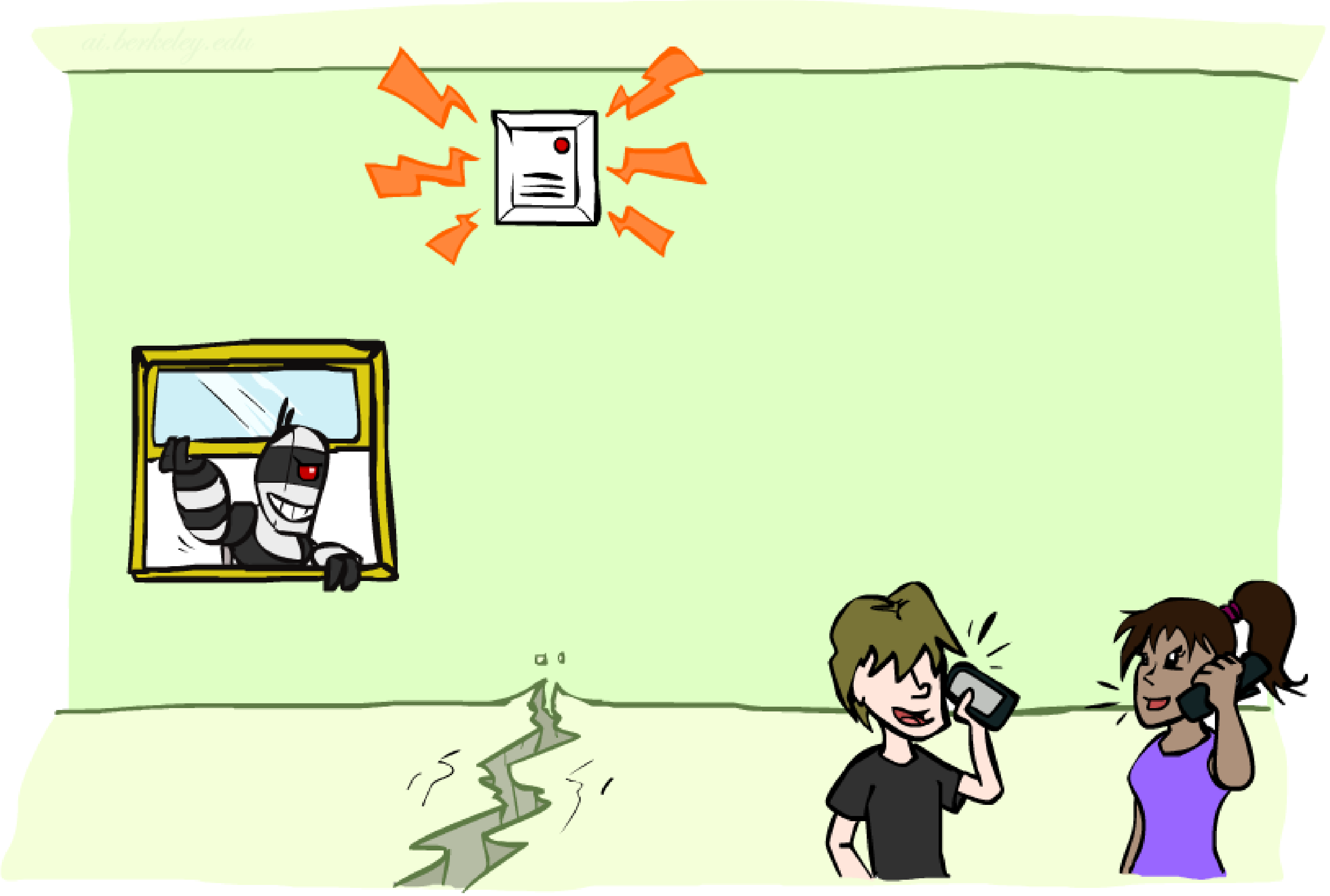

- Model 2: rain causes traffic



# Example: Alarm Network

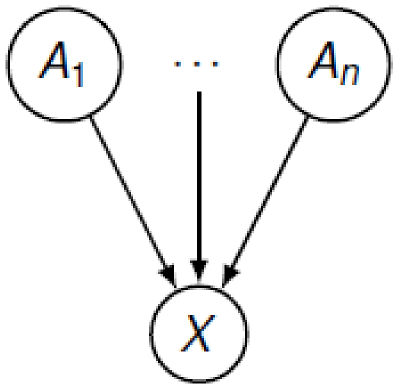
- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!



# Bayesian Networks Syntax

- A directed, acyclic graph
- Conditional distributions for each node given its ***parent variables*** in the graph
  - ***CPT***: conditional probability table: each row is a distribution for child given a configuration of its parents
  - Description of a noisy "causal" process




$$P(X|A_1, \cdots, A_n)$$

*A Bayes net = Topology (graph) + Local Conditional Probabilities*

# Example: Coin Flips

$X_1$ $\quad$ $X_2$ $\quad \cdots \quad$ $X_n$

$P(X_1)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

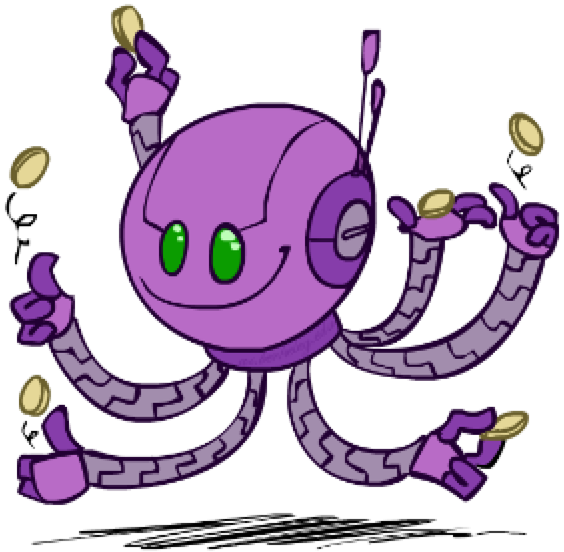
$P(X_2)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

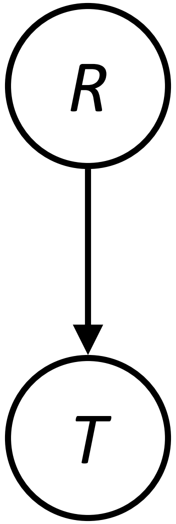
$\cdots$

$P(X_n)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |



# Example: Traffic




$P(R)$

| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

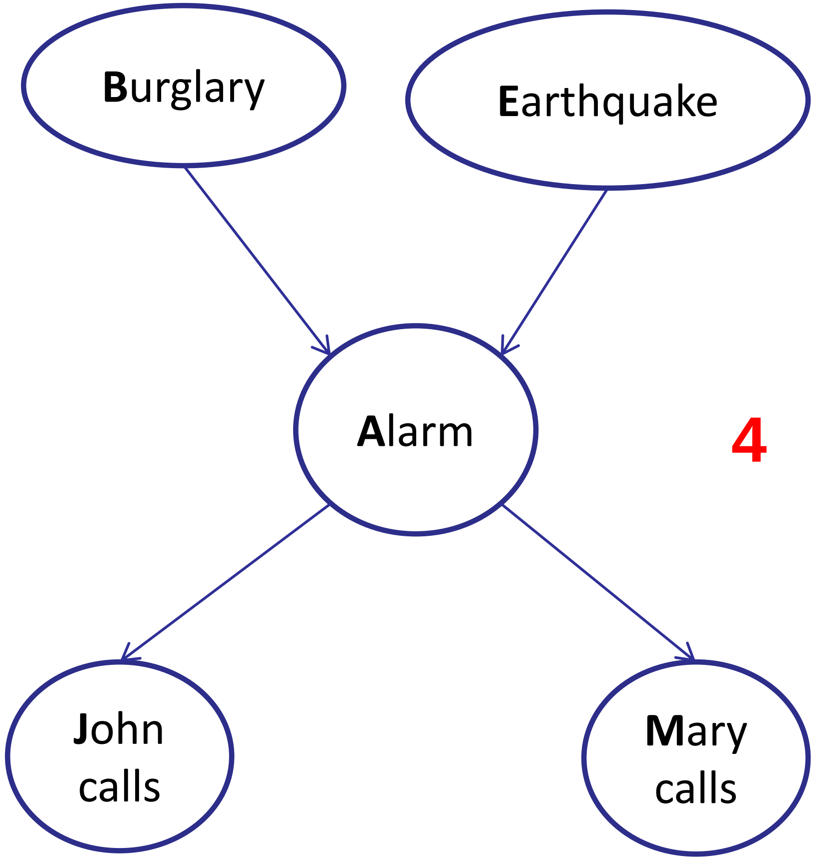
$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

# Example: Alarm Network

| P(B) | |
|---|---|
| true | false |
| 0.001 | 0.999 |

1




1

| P(E) | |
|---|---|
| true | false |
| 0.002 | 0.998 |

4

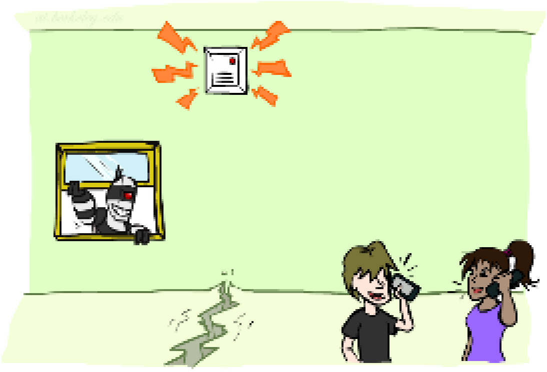
| B | E | P(A\|B,E) | |
|---|---|---|---|
| | | true | false |
| true | true | 0.95 | 0.05 |
| true | false | 0.94 | 0.06 |
| false | true | 0.29 | 0.71 |
| false | false | 0.001 | 0.999 |

| A | P(J\|A) | |
|---|---|---|
| | true | false |
| true | 0.9 | 0.1 |
| false | 0.05 | 0.95 |

2

2

| A | P(M\|A) | |
|---|---|---|
| | true | false |
| true | 0.7 | 0.3 |
| false | 0.01 | 0.99 |



Number of free parameters in each CPT:

- Parent domain sizes $d_1, \ldots, d_k$
- Child domain size $d$
- Each table row must sum to 1

$(d-1)\ \Pi_i\ d_i$

# General formula for sparse BNs

- Suppose
  - *n* variables
  - Maximum domain size is *d*
  - Maximum number of parents is *k*
- Full joint distribution has size $O(d^n)$
- Bayes net has size $O(n \cdot d^{k+1})$
  - Linear scaling with *n* as long as causal structure is local

# Bayesian Networks Semantics

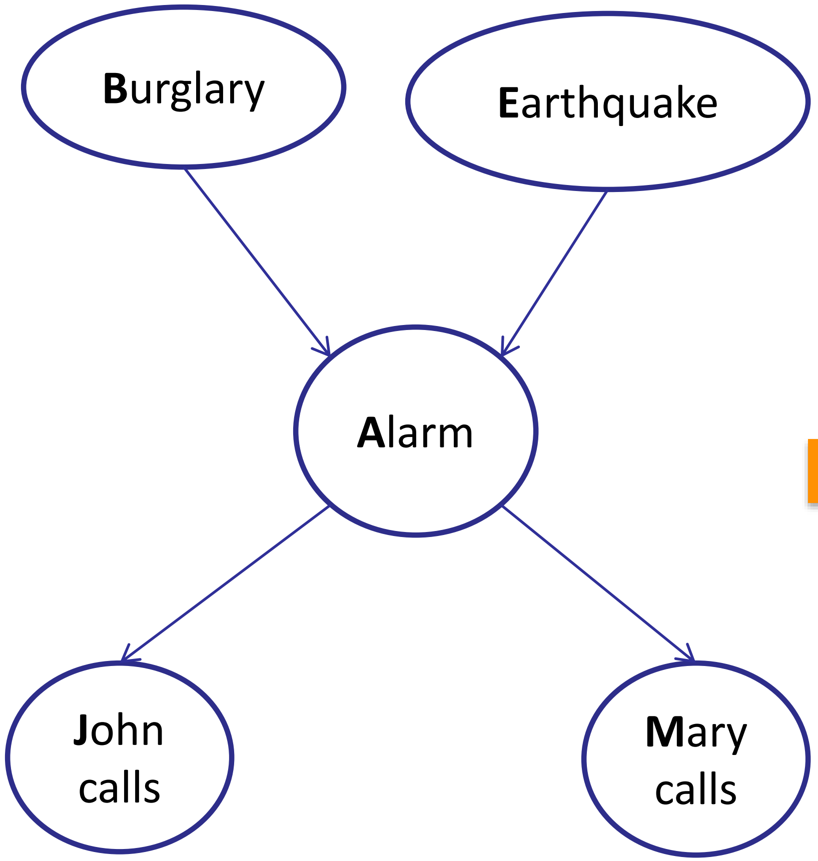
# Bayesian networks global semantics

- Bayes nets encode joint distributions as product of conditional distributions on each variable:

$$P(X_1,..,X_n) \;=\; \prod_i P(X_i \mid Parents(X_i))$$

# Example

| P(B) | |
|---|---|
| true | false |
| 0.001 | 0.999 |




| P(E) | |
|---|---|
| true | false |
| 0.002 | 0.998 |

$P(b, \neg e, a, \neg j, \neg m) =$

$P(b)\ P(\neg e)\ P(a|b, \neg e)\ P(\neg j|a)\ P(\neg m|a)$

$=.001 x .998 x .94 x .1 x .3 = .000028$

| B | E | P(A\|B,E) | |
|---|---|---|---|
| | | true | false |
| true | true | 0.95 | 0.05 |
| true | false | 0.94 | 0.06 |
| false | true | 0.29 | 0.71 |
| false | false | 0.001 | 0.999 |

| A | P(J\|A) | |
|---|---|---|
| | true | false |
| true | 0.9 | 0.1 |
| false | 0.05 | 0.95 |

| A | P(M\|A) | |
|---|---|---|
| | true | false |
| true | 0.7 | 0.3 |
| false | 0.01 | 0.99 |

# Probabilities in BNs

- Global semantics: $P(X_1,..,X_n) = \prod_i P(X_i \mid Parents(X_i))$

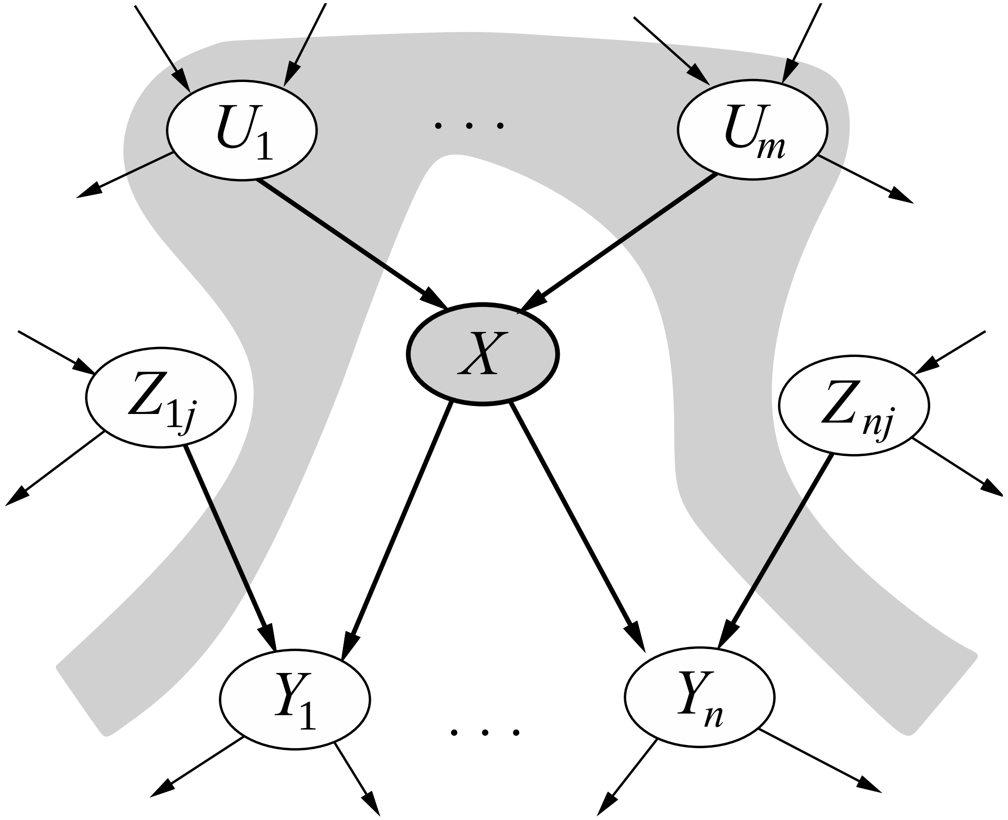
- Chain rule (valid for all distributions): $P(X_1,..,X_n) = \prod_i P(X_i \mid X_1,\ldots,X_{i-1})$

- So for any *i*, we have: $P(X_i \mid X_1,\ldots,X_{i-1}) = P(X_i \mid Parents(X_i))$
  - Conditional independence: parents “shield” node $X_i$ from the other predecessors

- So the network topology implies that certain conditional independencies hold

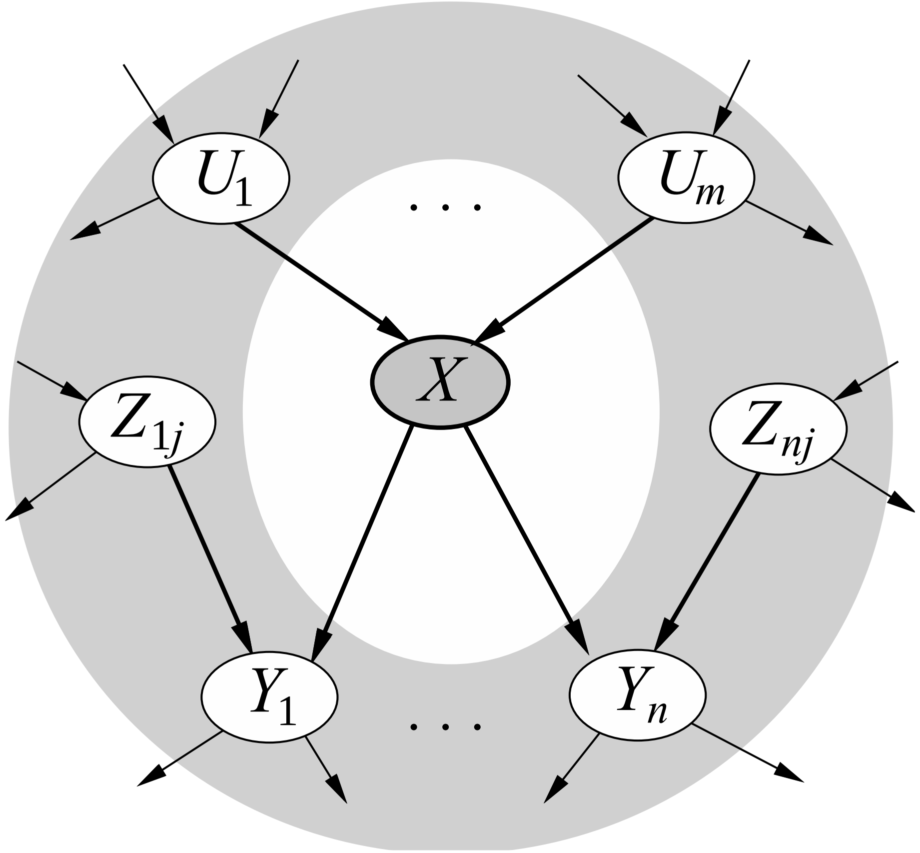
# Conditional independence semantics

- ***Every variable is conditionally independent of its non-descendants given its parents***
- Conditional independence semantics <=> global semantics



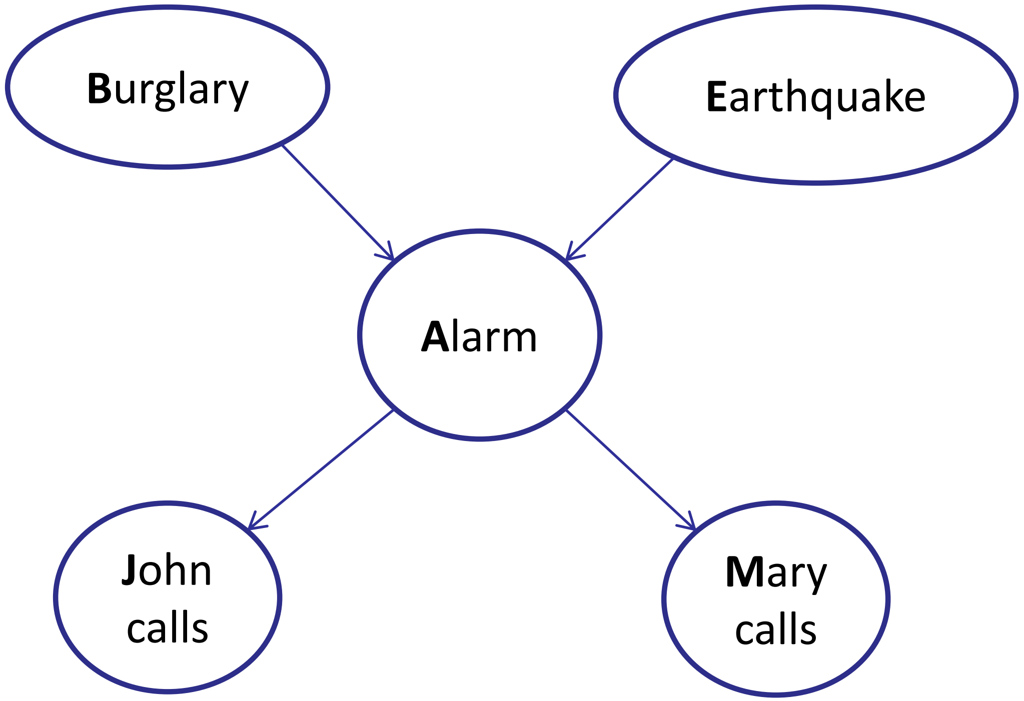
# Markov blanket

- A variable's Markov blanket consists of parents, children, children's other parents
- ***Every variable is conditionally independent of all other variables given its Markov blanket***



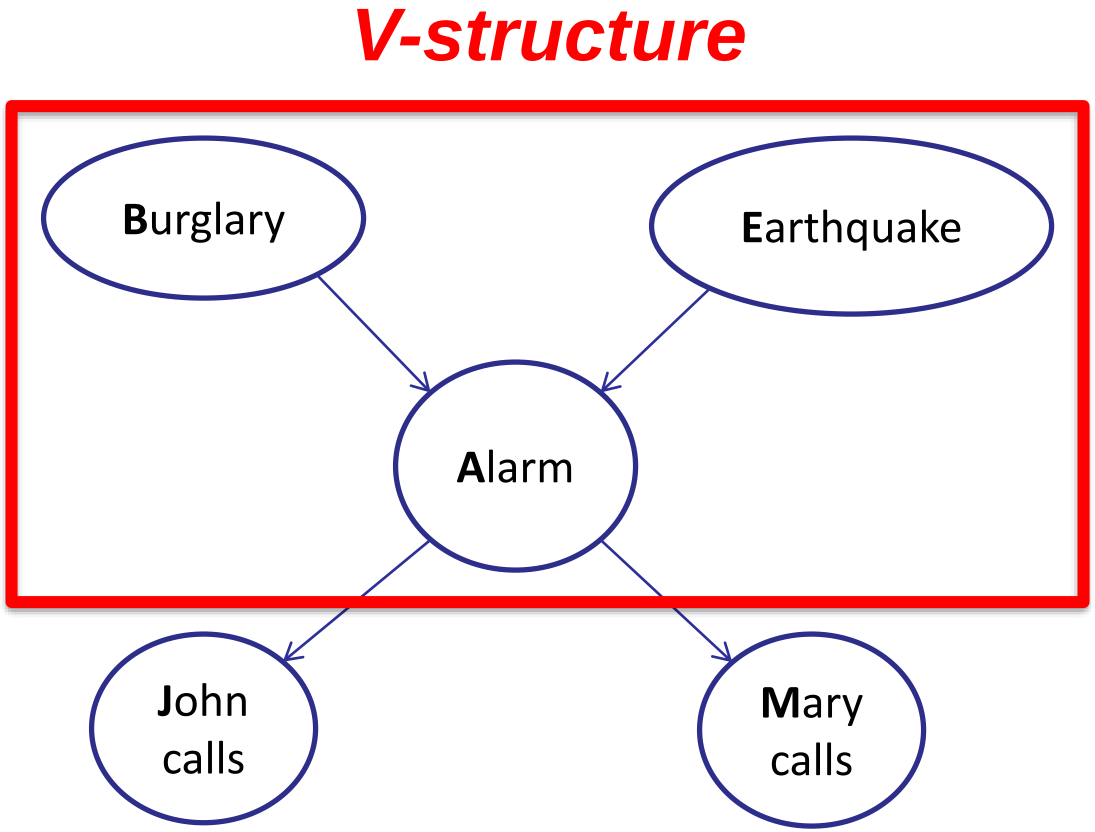
# Example

- JohnCalls independent of Burglary given Alarm?
  - Yes
- JohnCalls independent of MaryCalls given Alarm?
  - Yes
- Burglary independent of Earthquake?
  - Yes

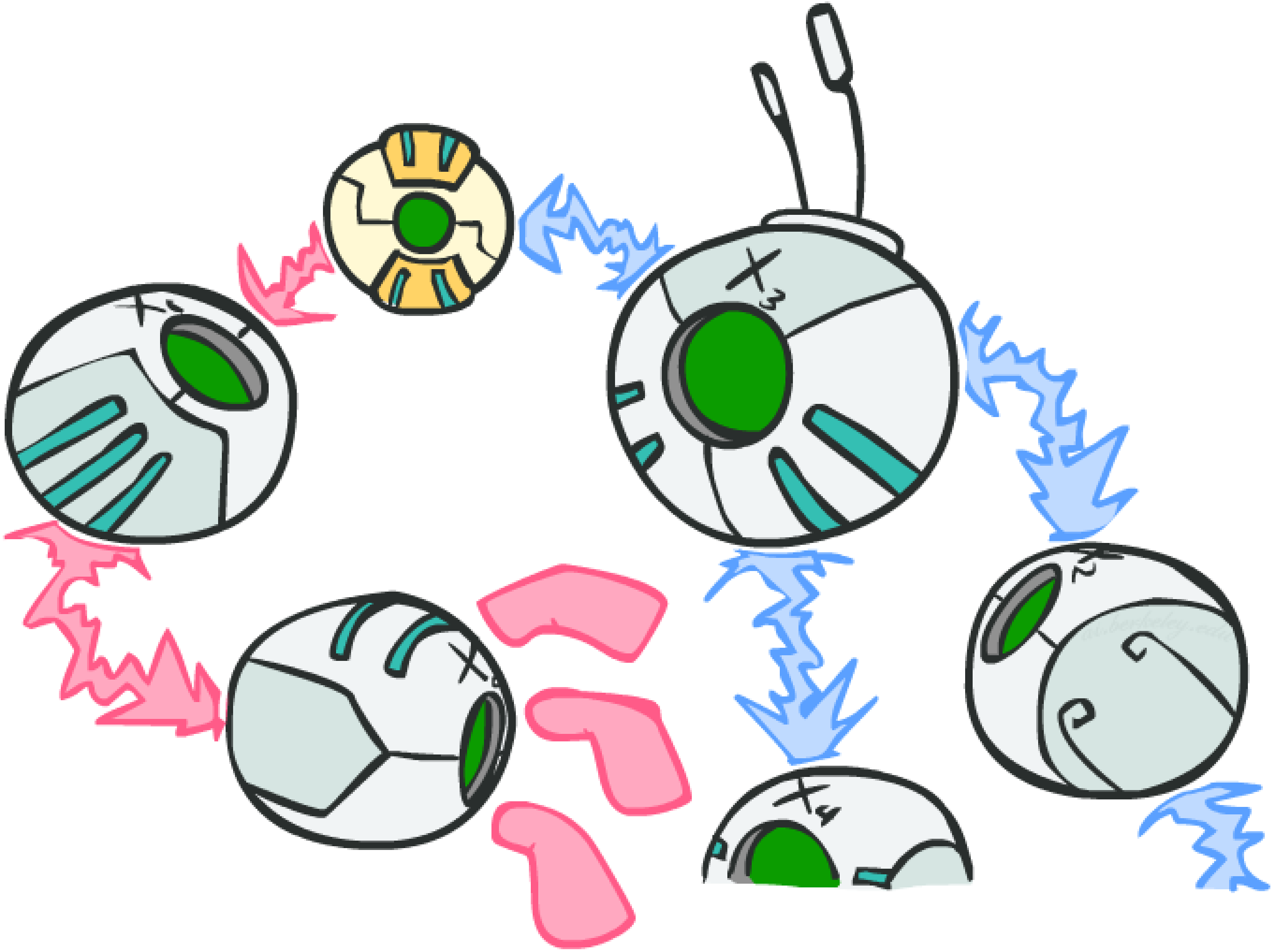


# Example

- Burglary independent of Earthquake given Alarm?
  - NO!
  - Given that the alarm has sounded, both burglary and earthquake become more likely
  - But if we then learn that a burglary has happened, the alarm is ***explained away*** and the probability of earthquake drops back
- Burglary independent of Earthquake given JohnCalls?
- Any simple algorithm to determine conditional independence?



# D-separation



# Causal Chains

- This configuration is a "causal chain"


X: Low pressure Y: Rain Z: Traffic

Global semantics:

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? *No!*
- Guaranteed X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

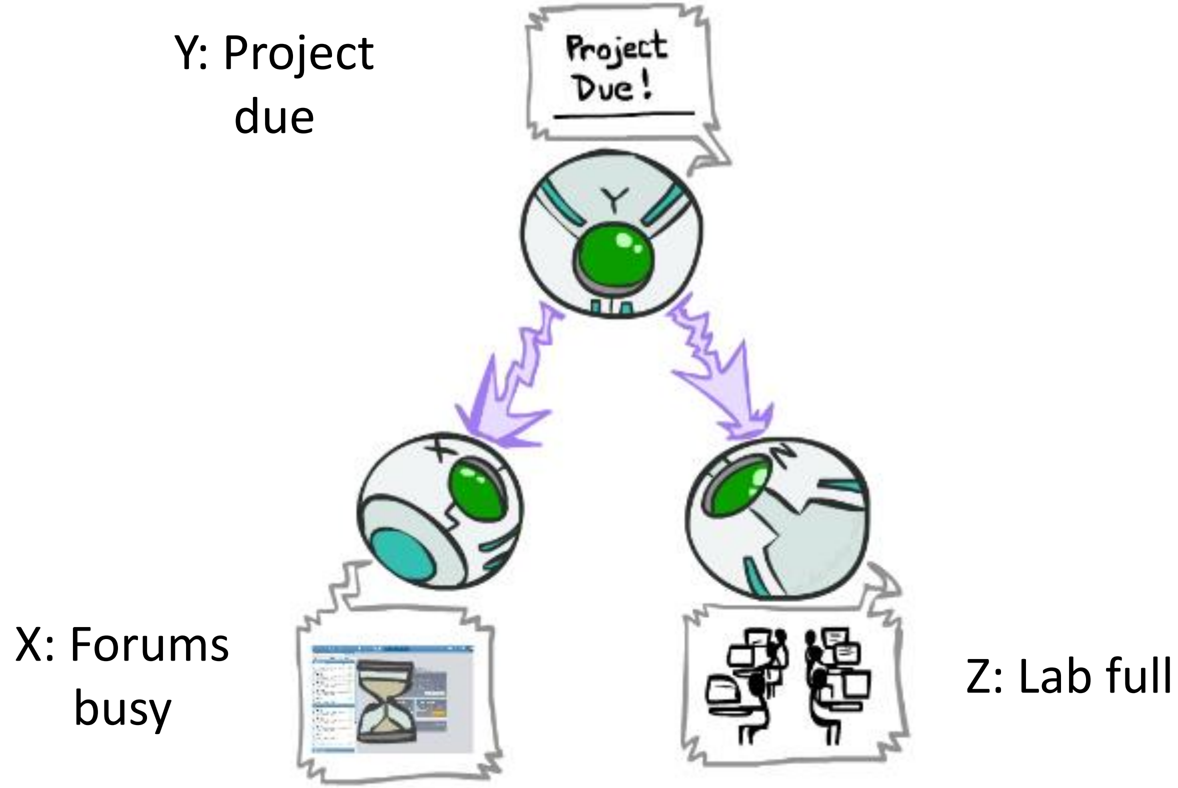
$$= P(z|y)$$

*Yes!*

- Evidence along the chain "blocks" the influence

# Common Cause

- This configuration is a "common cause"




Global semantics:

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? *No!*
- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

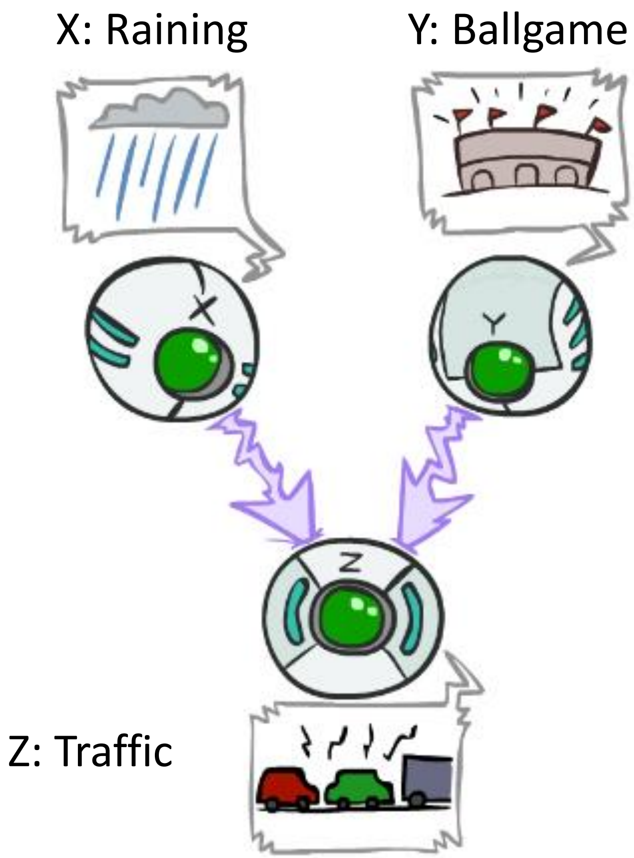
$$= P(z|y)$$

*Yes!*

- Observing the cause blocks influence between effects.

# Common Effect

- Last configuration: two causes of one effect (v-structures)

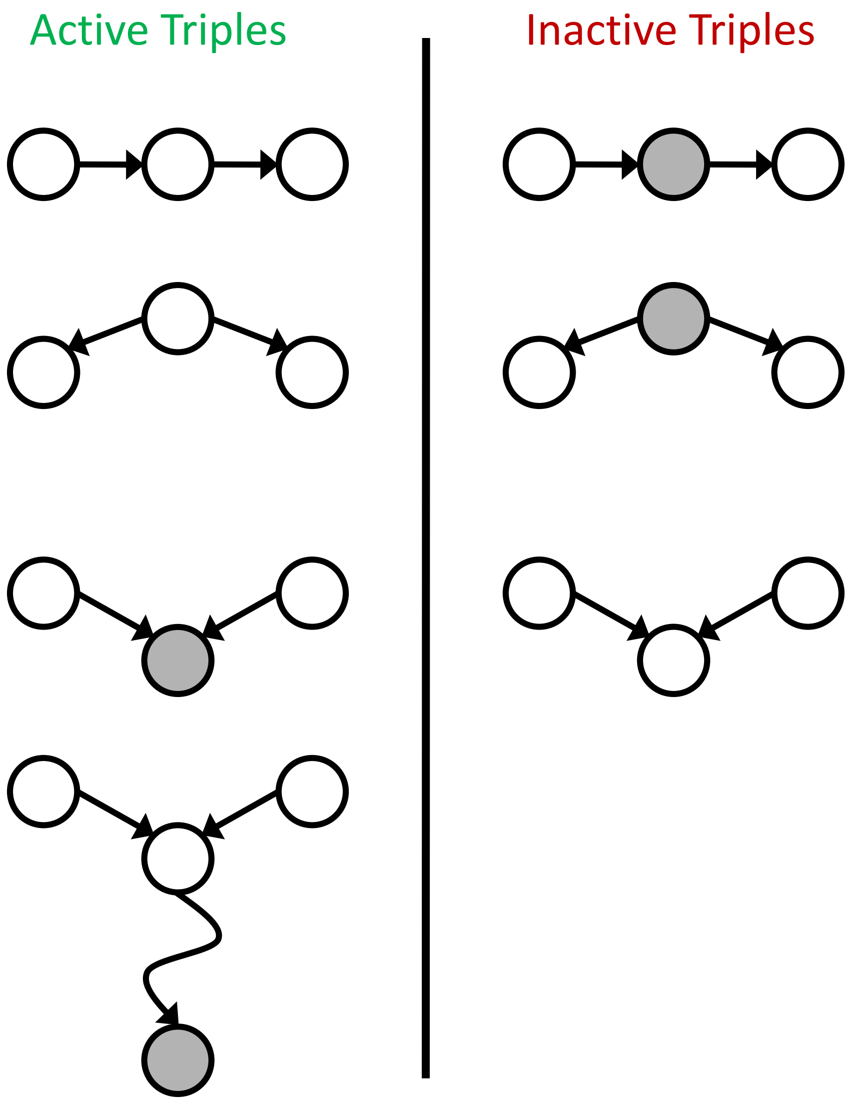



- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

# D-separation - the General Case

# Active / Inactive Paths

- Question: X, Y, Z are non-intersecting subsets of nodes. Are X and Y conditionally independent given Z?
- A triple is active in the following three cases
  - Causal chain A $\rightarrow$ B $\rightarrow$ C where B is unobserved (either direction)
  - Common cause A $\leftarrow$ B $\rightarrow$ C where B is unobserved
  - Common effect (aka v-structure)
    A $\rightarrow$ B $\leftarrow$ C where B *or one of its descendents* is observed
- A path is active if each triple along the path is active
- A path is blocked if it contains a single inactive triple
- If all paths from X to Y are blocked, then X is said to be "**d-separated**" from Y by Z
- If d-separated, then X and Y are conditionally independent given Z

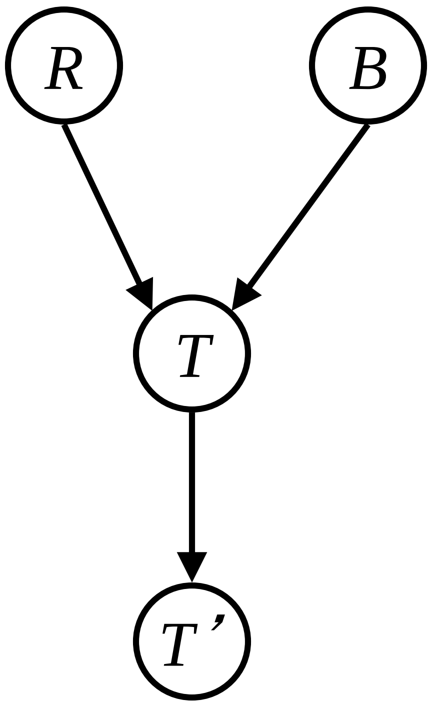


# Example

$R \perp\!\!\!\perp B$ *Yes*

$R \perp\!\!\!\perp B | T$

$R \perp\!\!\!\perp B | T'$



# Example

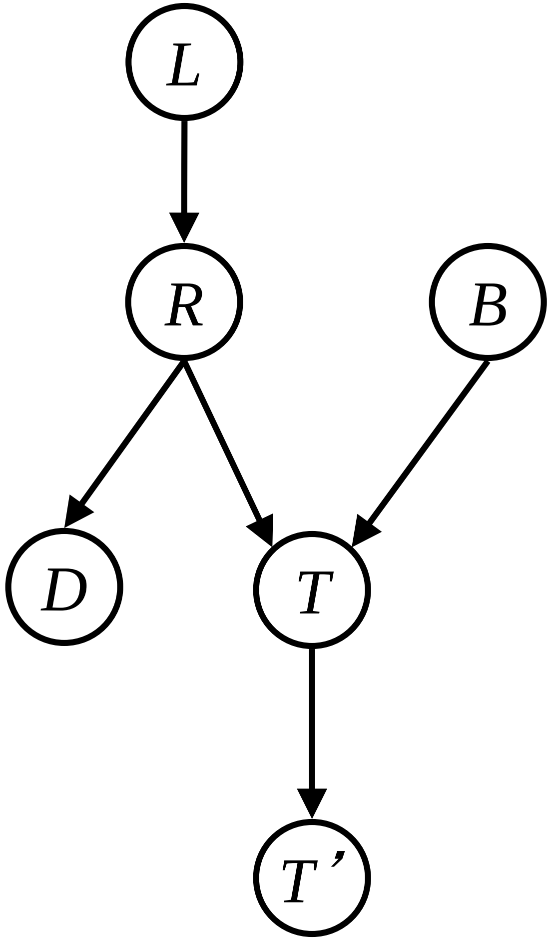
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



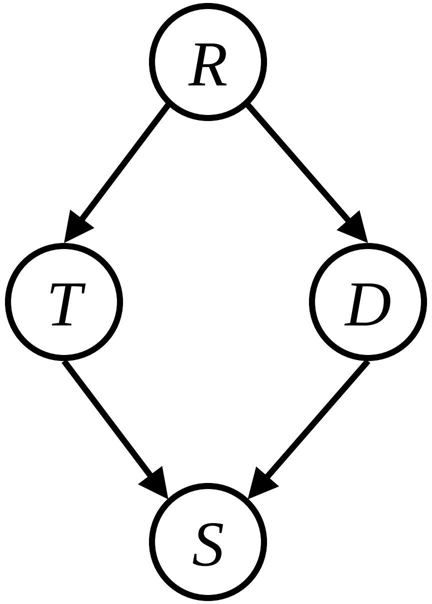
# Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:

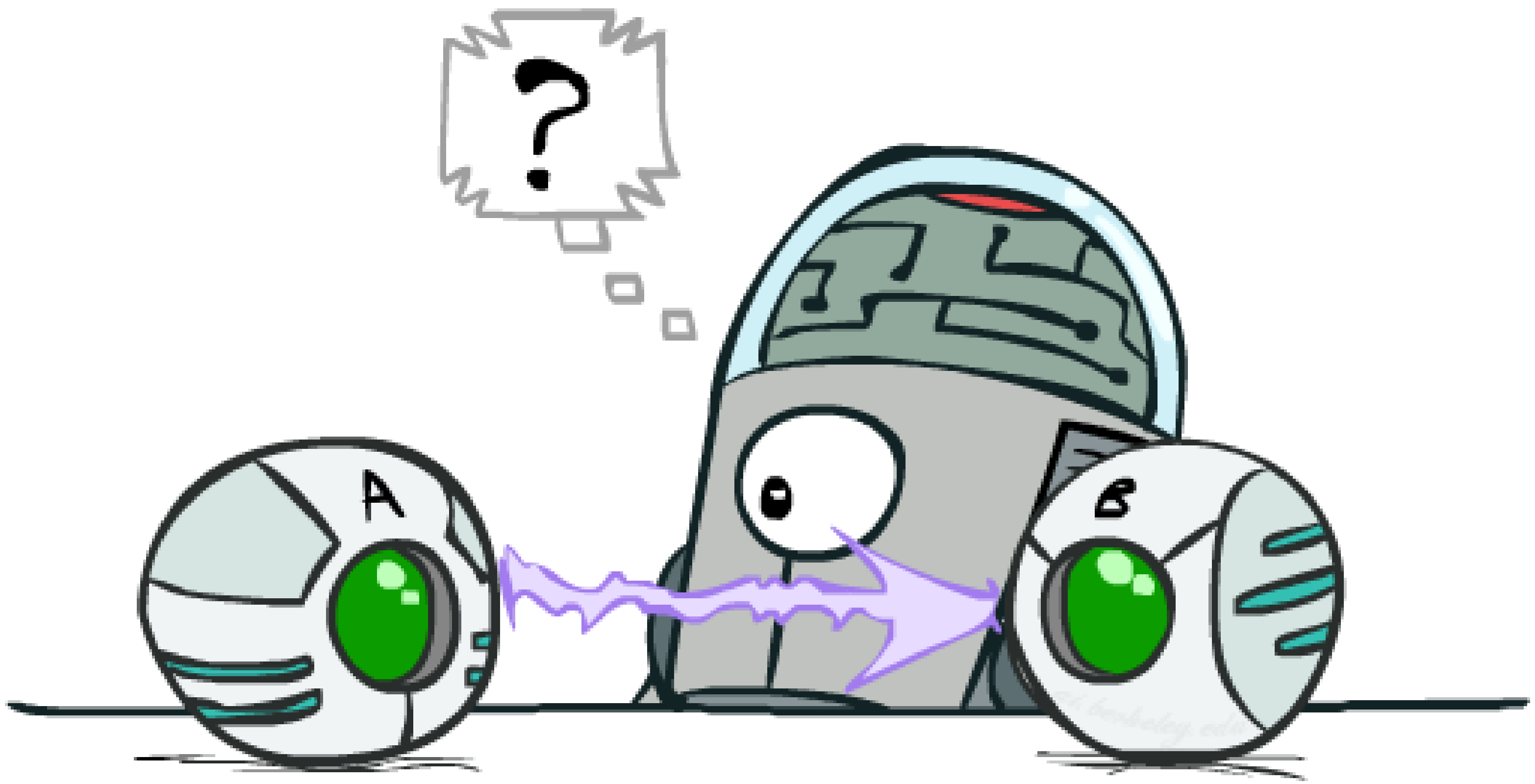
$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R$$ *Yes*

$$T \perp\!\!\!\perp D | R, S$$

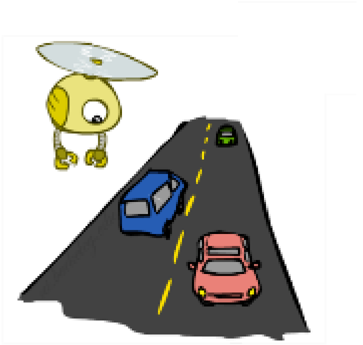


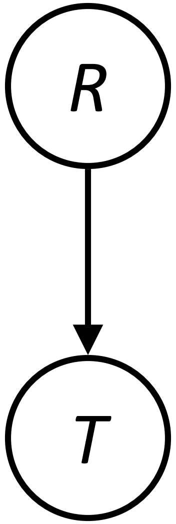
# Node Ordering



# Example: Traffic

- Causal direction






$P(R)$

| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

$P(T|R)$

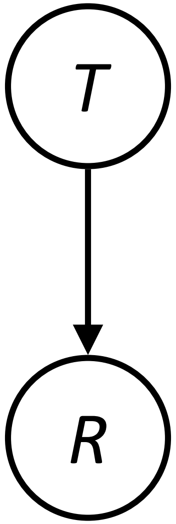
| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

$P(T, R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

# Example: Reverse Traffic

- Reverse causality?




$P(T)$

| | |
|---|---|
| +t | 9/16 |
| -t | 7/16 |

$P(R|T)$

| | | |
|---|---|---|
| +t | +r | 1/3 |
| | -r | 2/3 |

| | | |
|---|---|---|
| -t | +r | 1/7 |
| | -r | 6/7 |

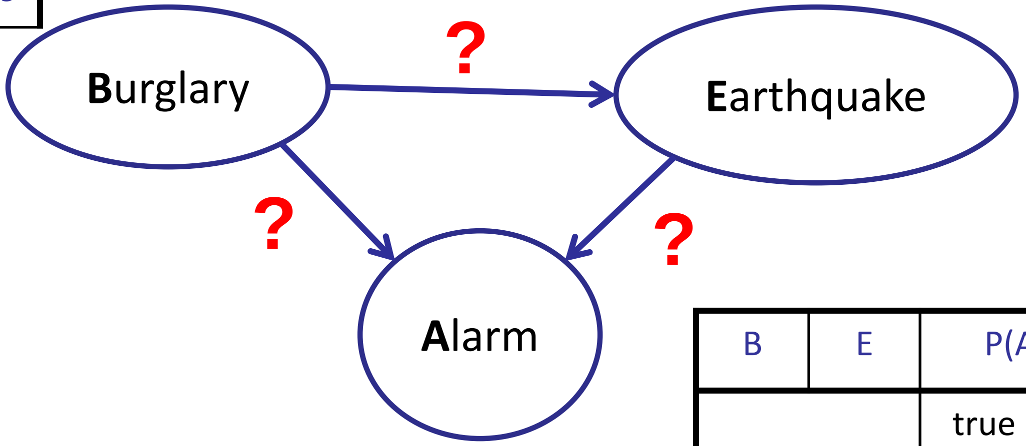
$P(T, R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

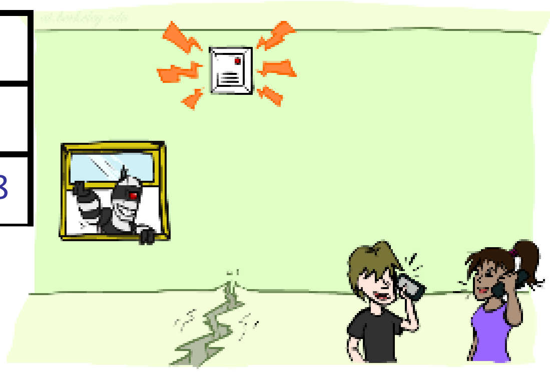
# Example: Burglary

- Burglary
- Earthquake
- Alarm

| P(B) | |
|---|---|
| true | false |
| 0.001 | 0.999 |




| P(E) | |
|---|---|
| true | false |
| 0.002 | 0.998 |



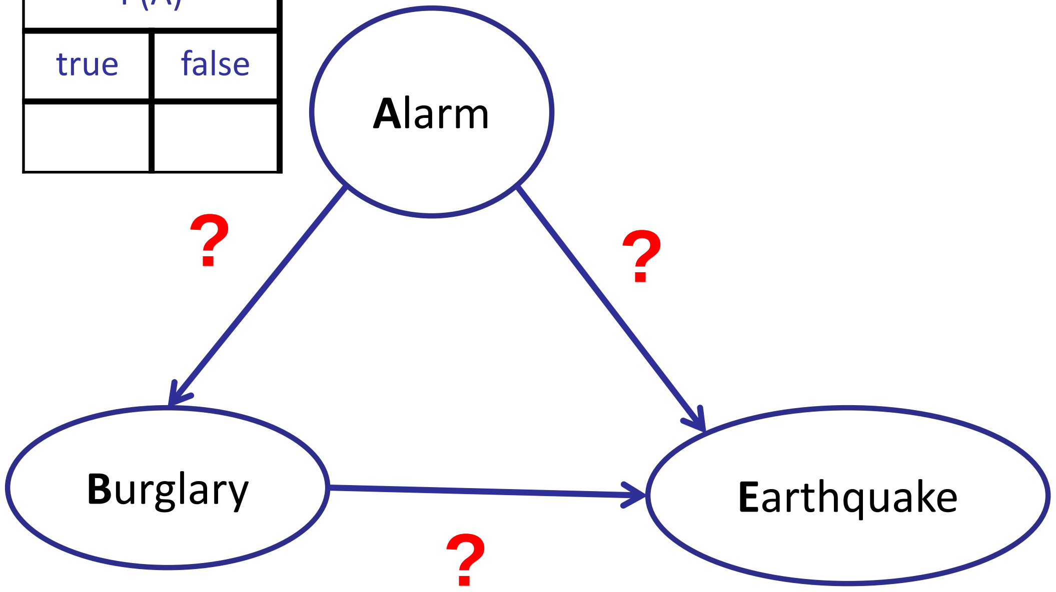
| B | E | P(A\|B,E) | |
|---|---|---|---|
| | | true | false |
| true | true | 0.95 | 0.05 |
| true | false | 0.94 | 0.06 |
| false | true | 0.29 | 0.71 |
| false | false | 0.001 | 0.999 |

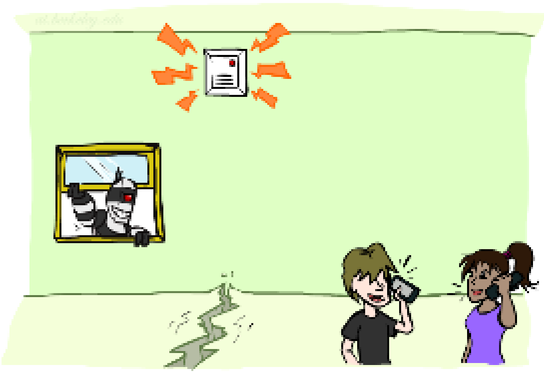
2 edges, 6 free parameters

# Example: Burglary

- Alarm
- Burglary
- Earthquake

| P(A) | |
|---|---|
| true | false |
| | |



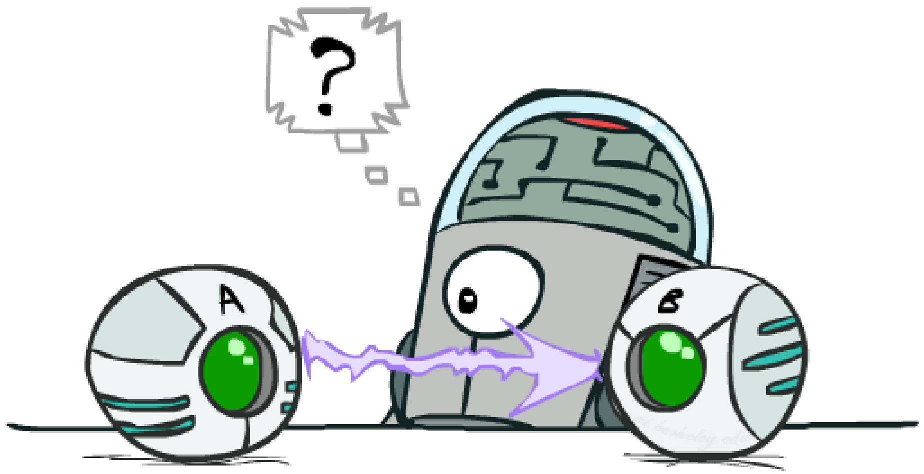
| A | P(B\|A) | |
|---|---|---|
| | true | false |
| true | | |
| false | | |

| A | B | P(E\|A,B) | |
|---|---|---|---|
| | | true | false |
| true | true | | |
| true | false | | |
| false | true | | |
| false | false | | |

3 edges, 7 free parameters

# Causality?

- When Bayes nets reflect the true causal patterns:
  (e.g., Burglary, Earthquake, Alarm)
  - Often simpler (fewer parents, fewer parameters)
  - Often easier to assess probabilities
  - Often more robust: e.g., changes in frequency of burglaries should not affect the rest of the model!




- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Umbrella*
  - End up with arrows that reflect correlation, not causation

- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology really encodes conditional independence:
    $P(X_i \mid X_1,\ldots,X_{i-1}) = P(X_i \mid Parents(X_i))$

# Example Application: Topic Modeling

[Containing some materials from slides by David Blei at Princeton and by Hongning Wang at UVa]

# Introduction

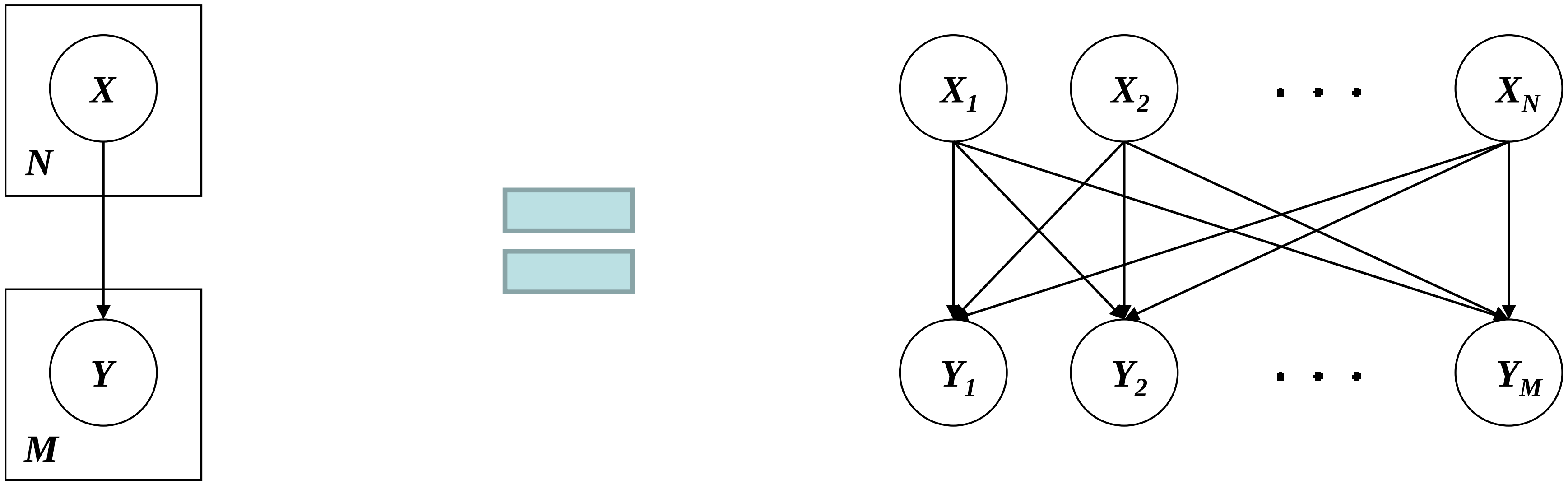
- A large body of text available online
  - It is difficult to find and discover what we need.
- Topic models
  - Approaches to discovering the main themes of a large unstructured collection of documents
  - Can be used to automatically organize, understand, search, and summarize large electronic archives
  - Latent Dirichlet Allocation (LDA) is the most popular

# Plate Notation

- Representation of repeated subgraphs in a Bayesian network

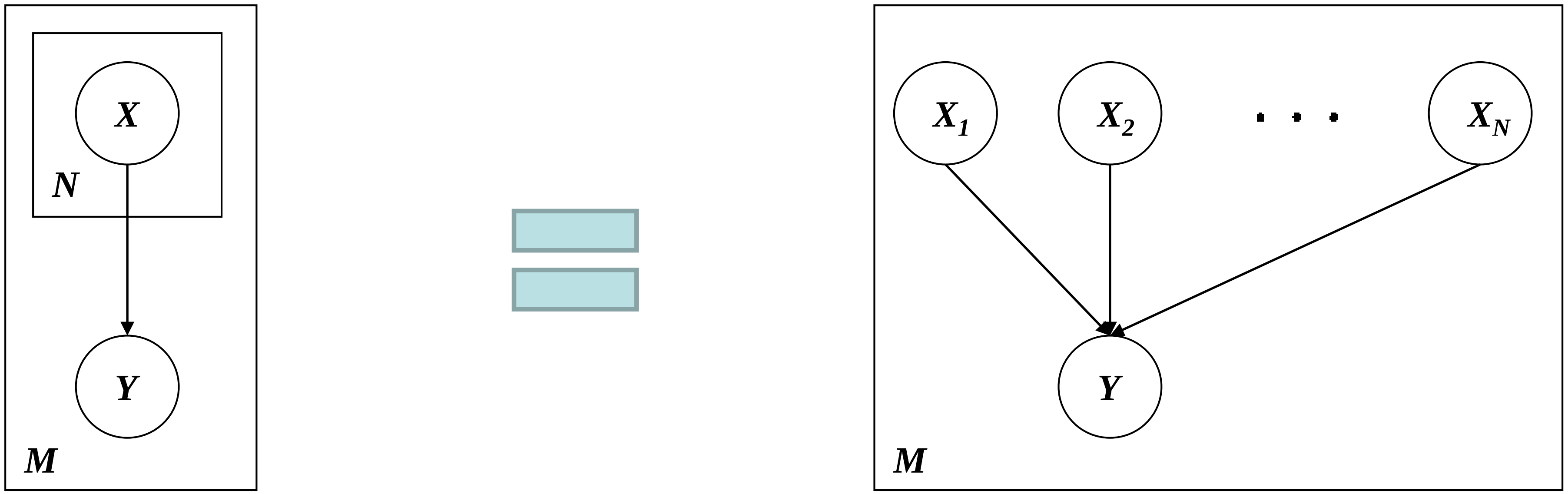
# Plate Notation

- Representation of repeated subgraphs in a Bayesian network

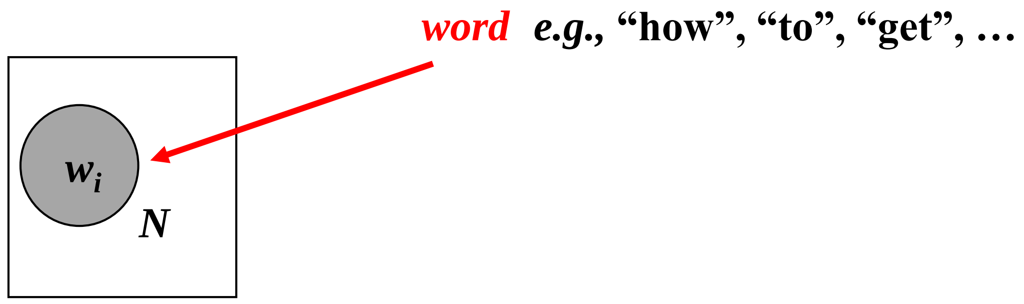


# Plate Notation

- Representation of repeated subgraphs in a Bayesian network



# How to generate a document

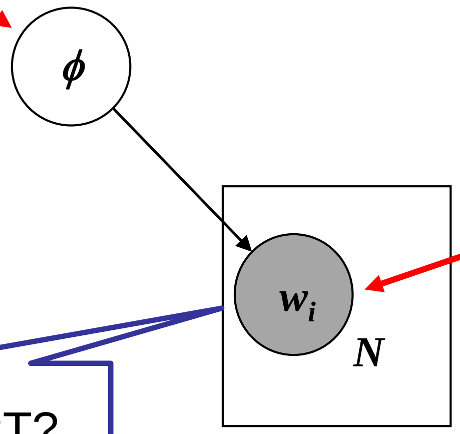


# How to generate a document

*distribution over words*

*e.g.*, [0.1, 0.3, 0.2, …]

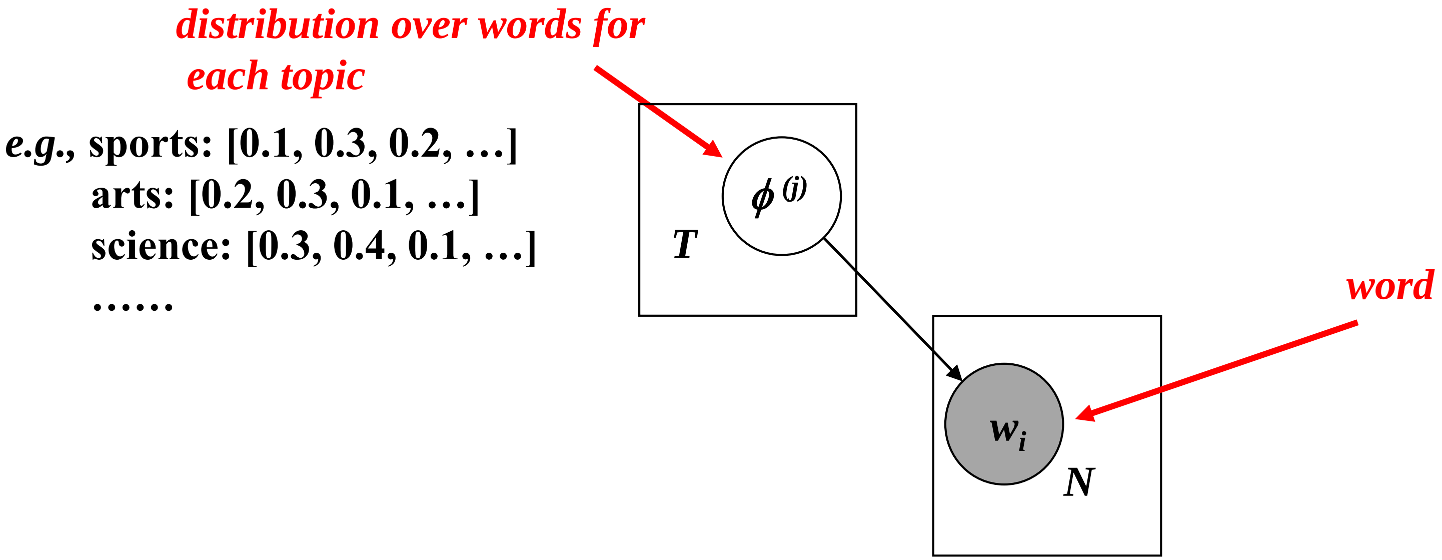
| | |
|---|---|
| how | 0.1 |
| to | 0.3 |
| get | 0.2 |
| … | … |



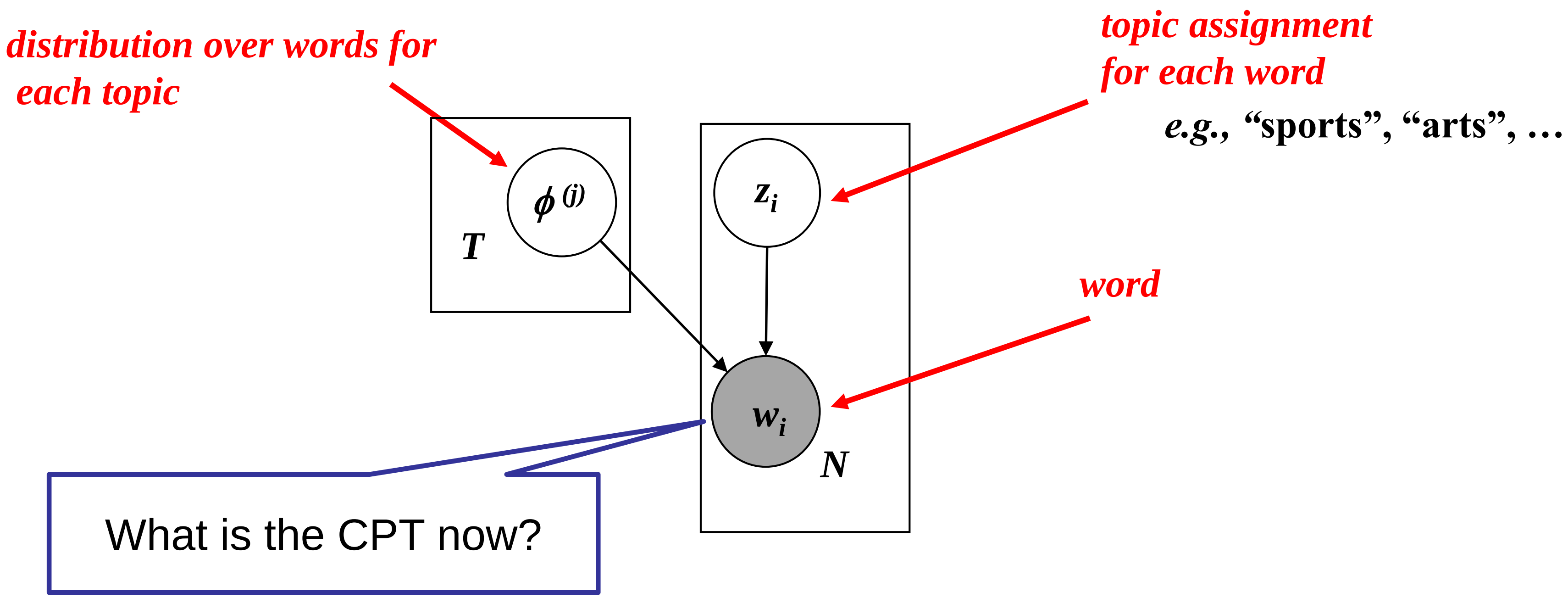

*word*

What is the CPT?

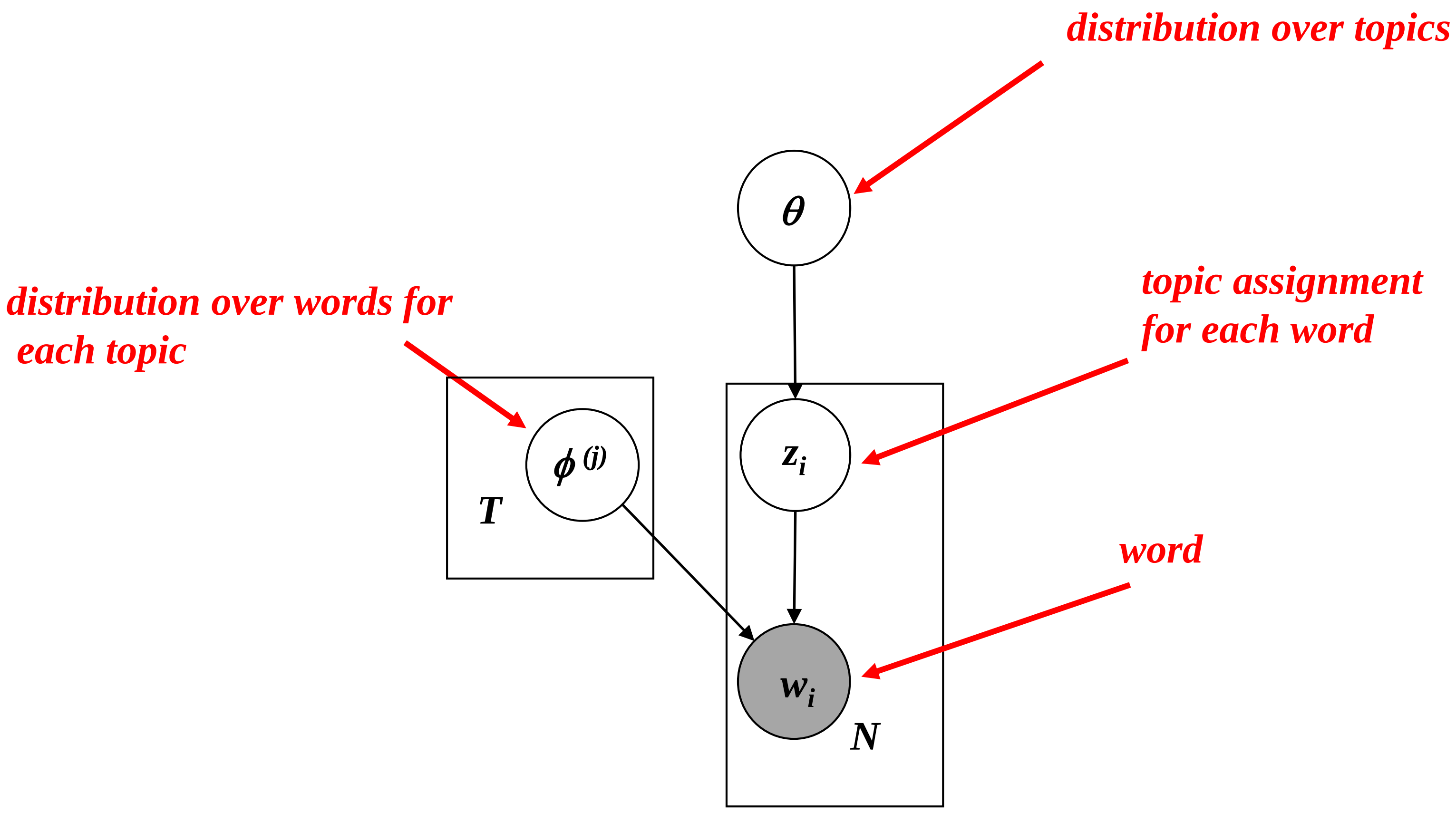
# How to generate a document



# How to generate a document



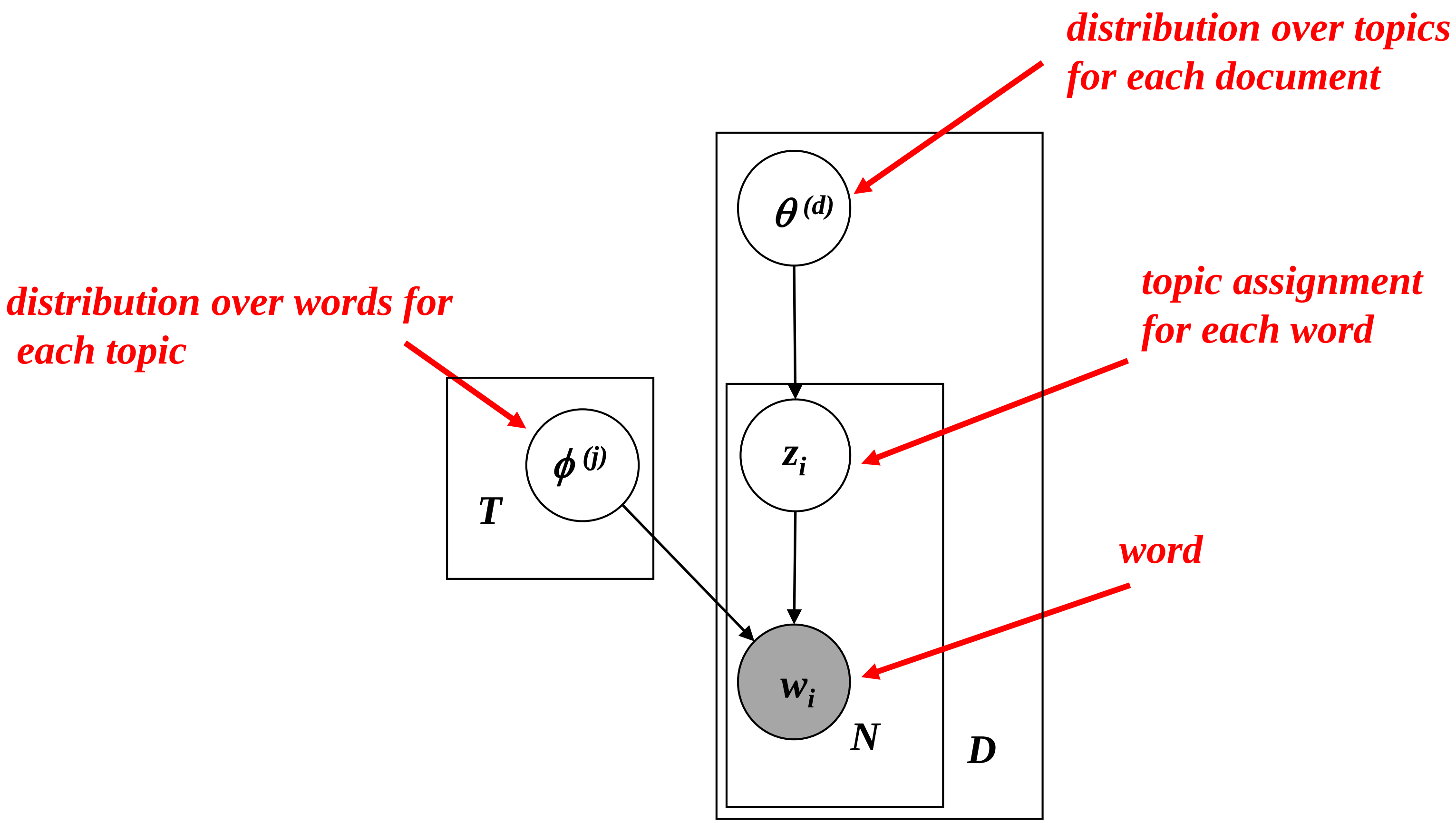
# How to generate a document



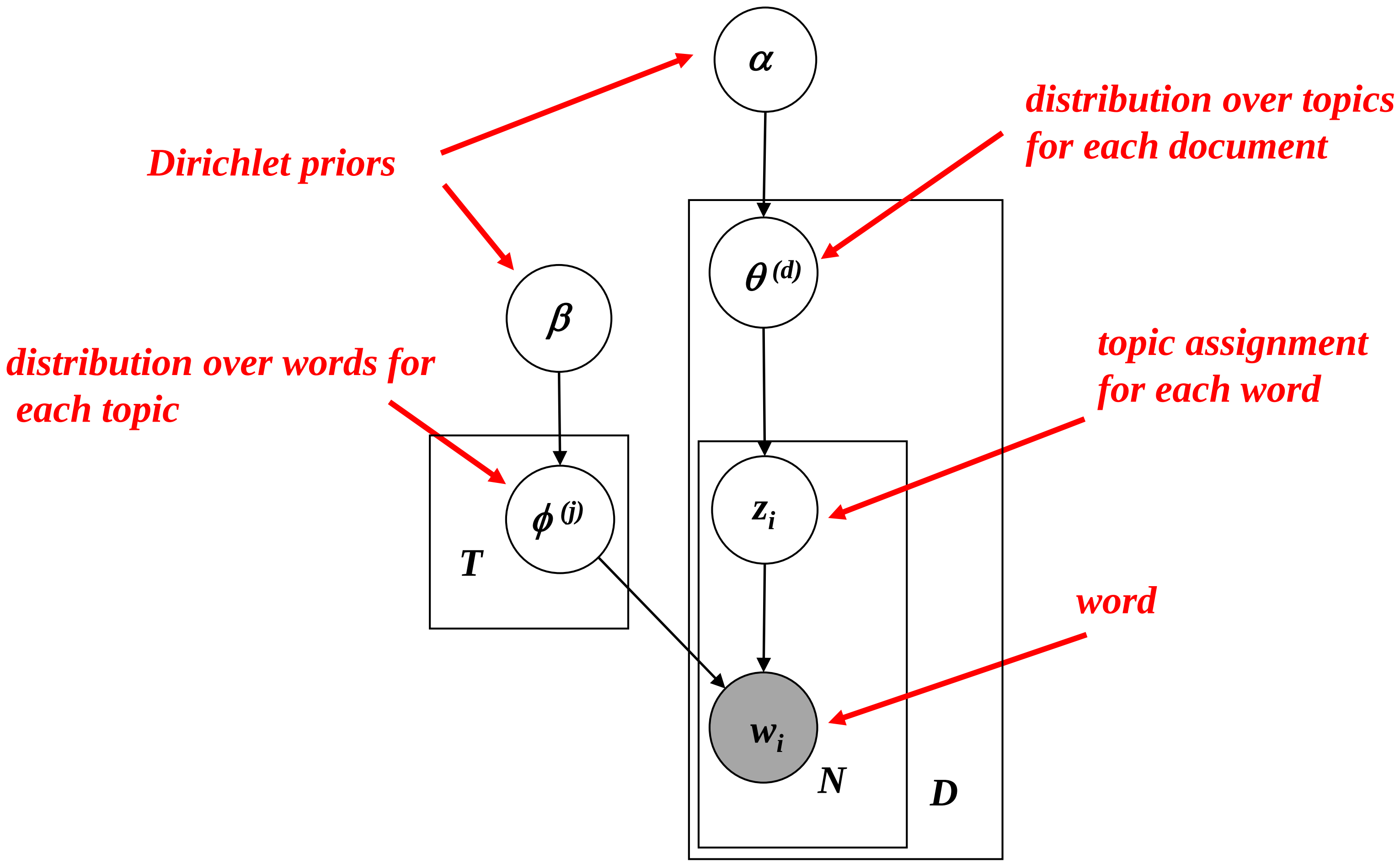

*e.g.*, [0.2, 0.3, 0.2, …]

| | |
|---|---|
| sports | 0.1 |
| arts | 0.3 |
| science | 0.2 |
| … | … |

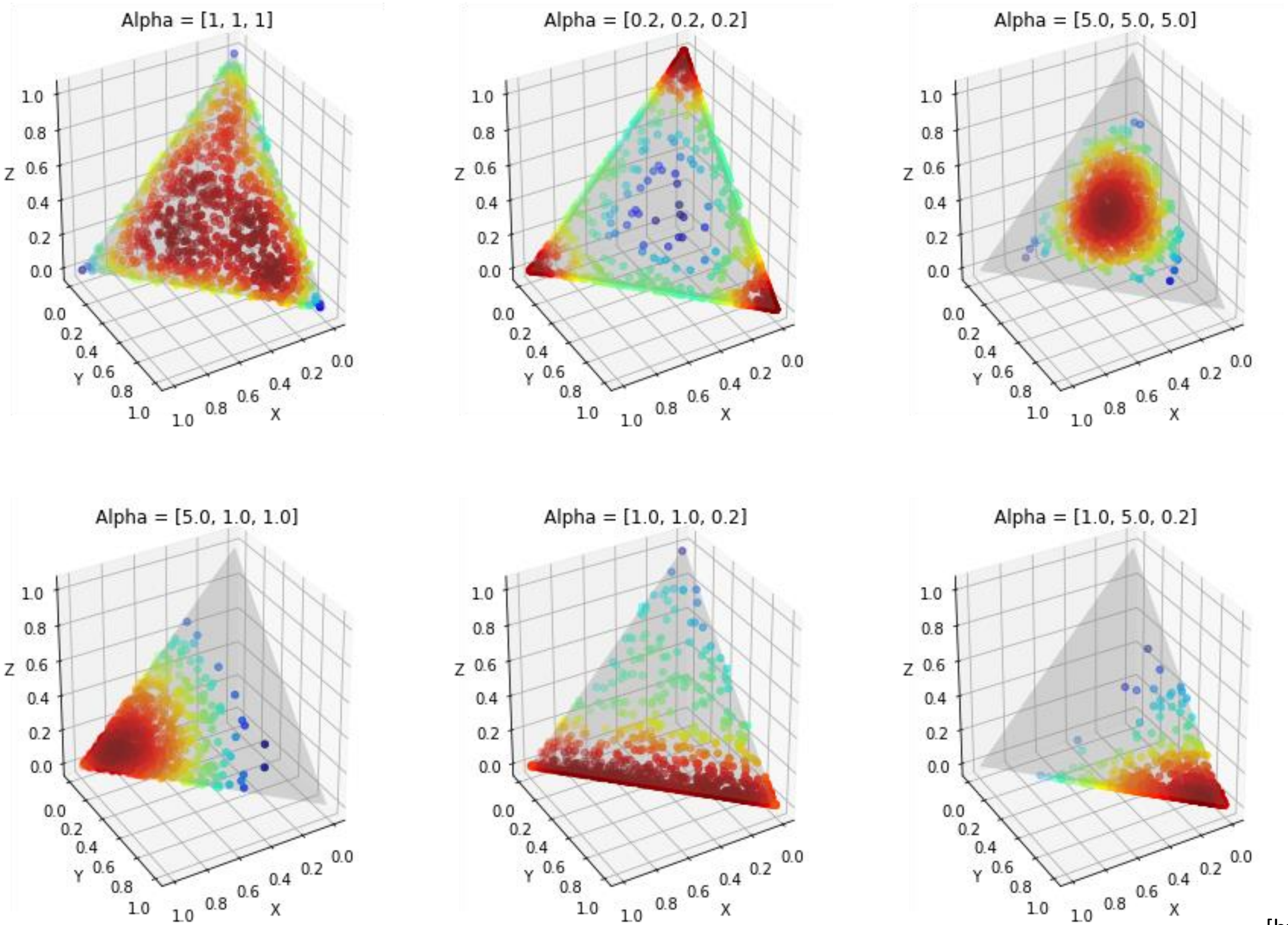
# How to generate documents



# How to generate documents

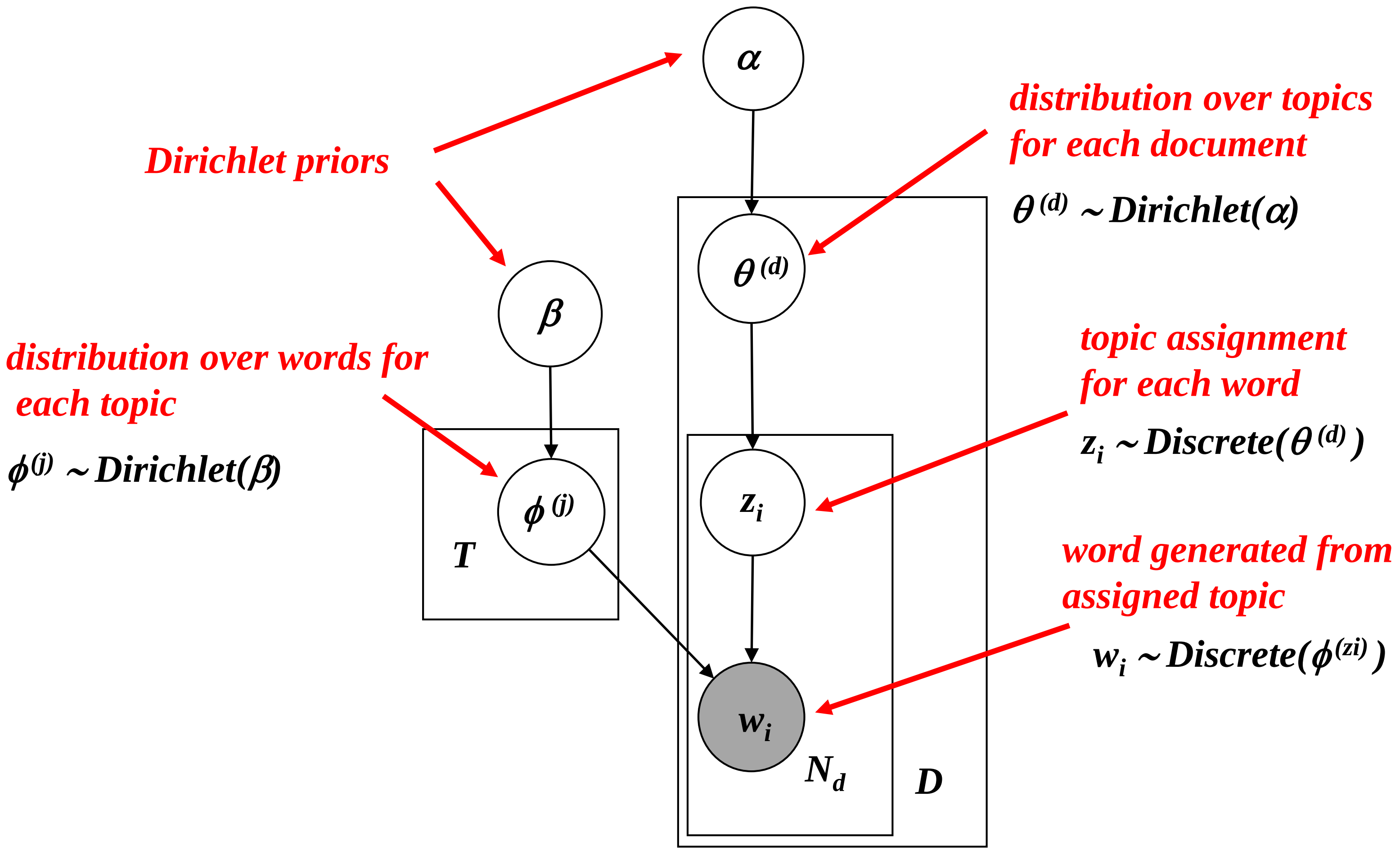


# Dirichlet Distribution

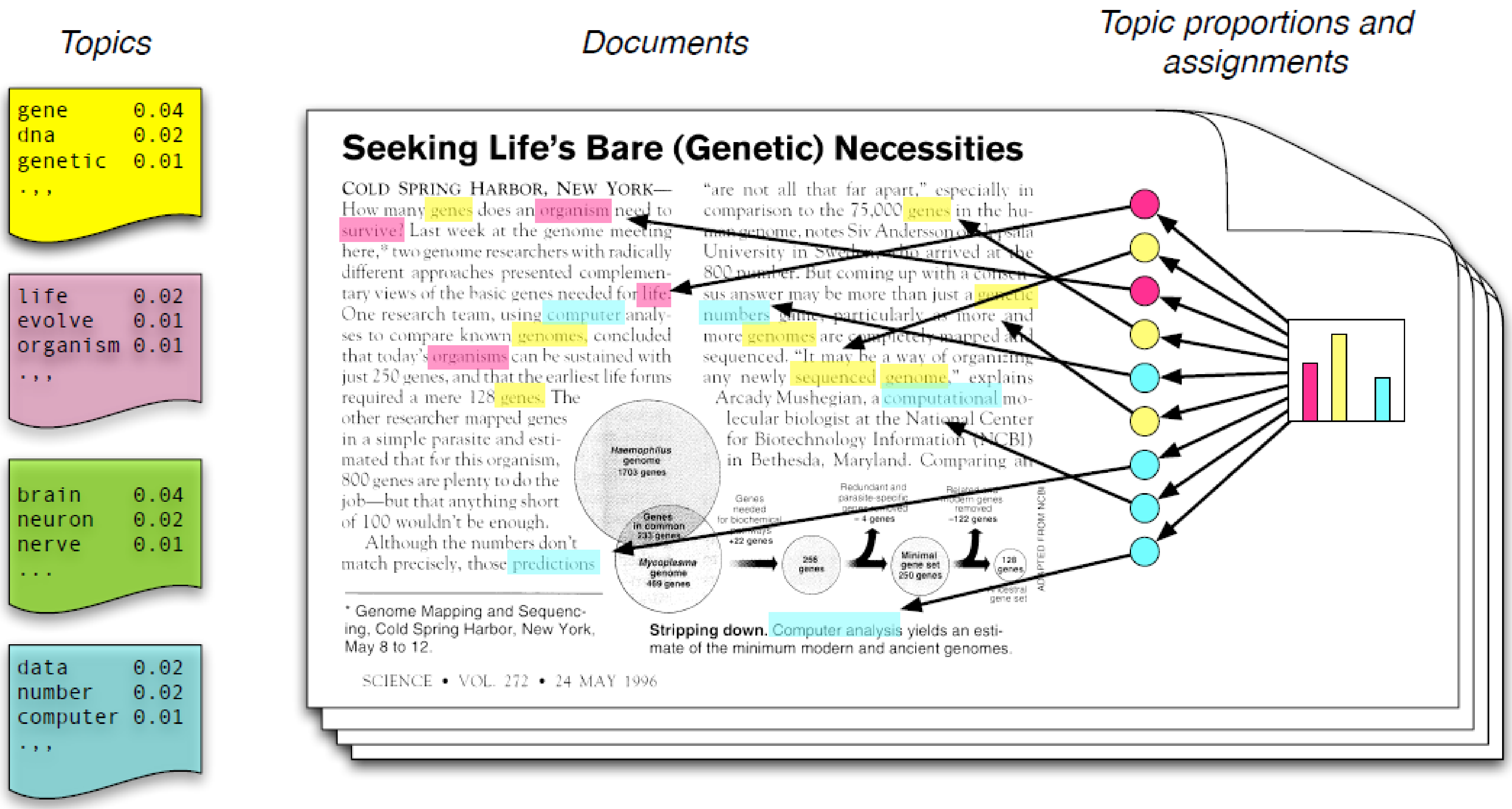



[by Thushan Ganegedara at USydney]

# Latent Dirichlet Allocation (LDA)

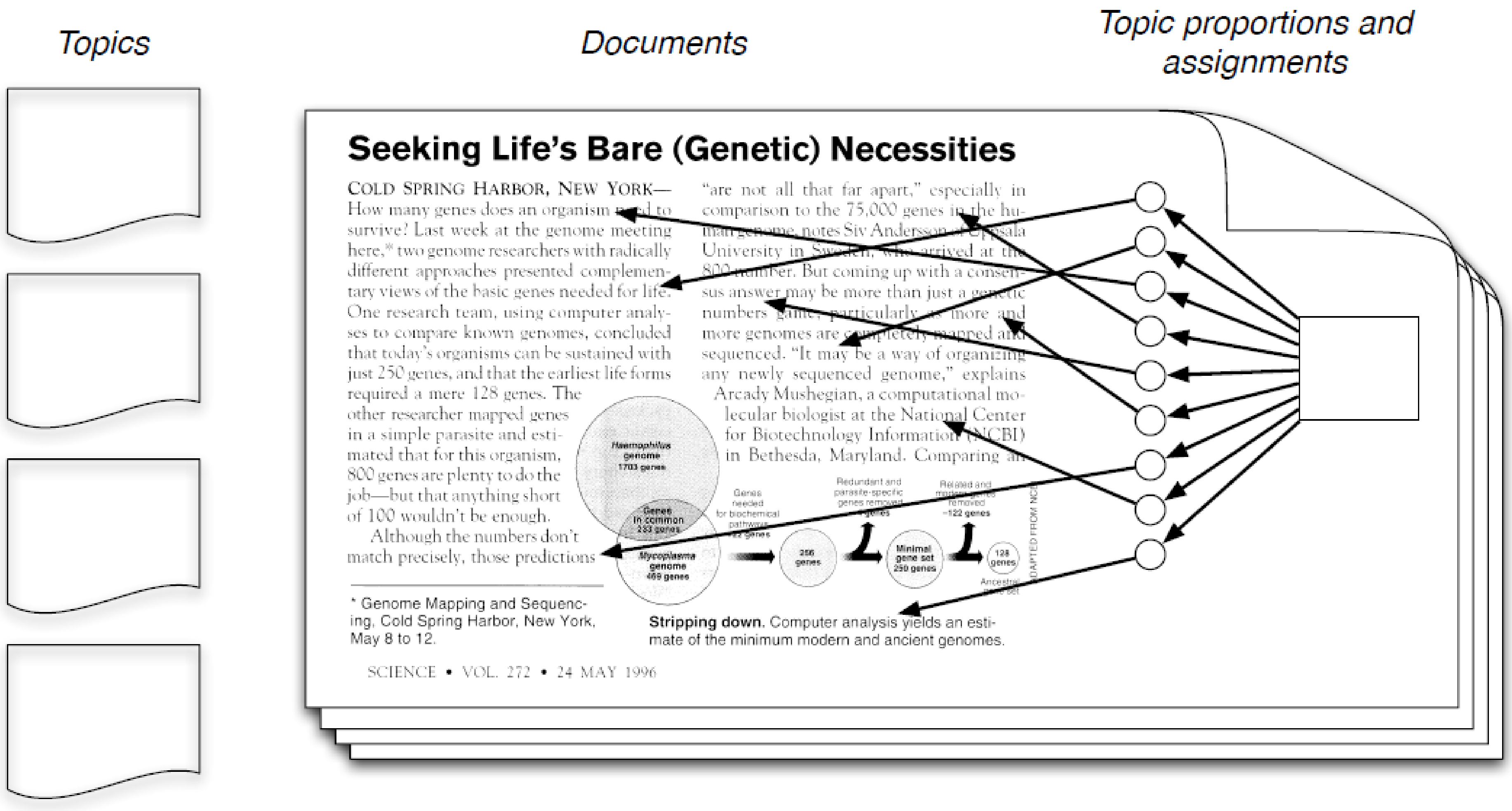


# Illustration




- Each **topic** is a distribution of words; each **document** is a mixture of corpus-wide topics; and each **word** is drawn from one of those topics.

# Illustration

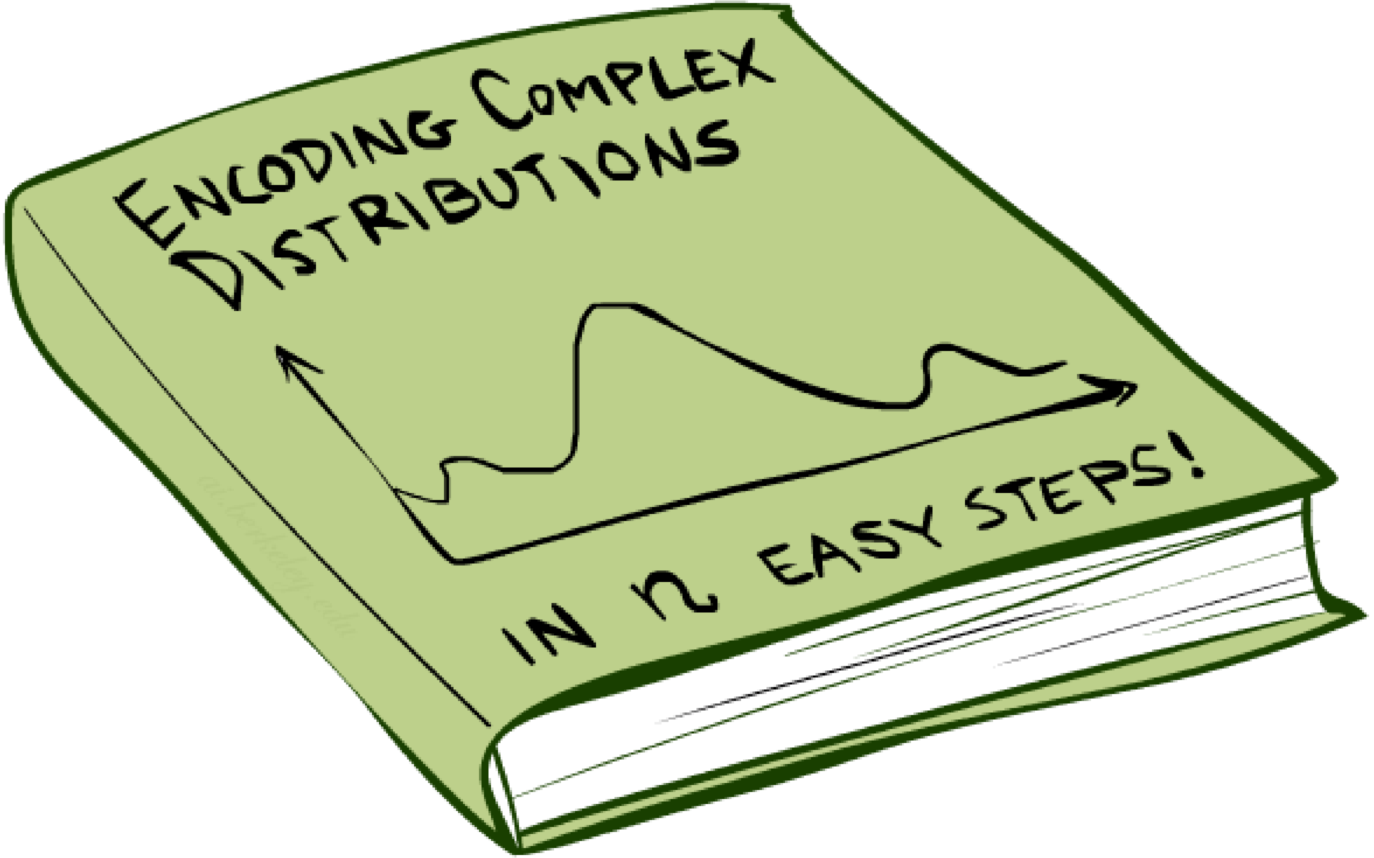



- In reality, we only observe documents. The other structures are hidden variables that must be inferred. (We will discuss inference later.)

# Topics inferred by LDA

| "Arts" | "Budgets" | "Children" | "Education" |
|---|---|---|---|
| NEW | MILLION | CHILDREN | SCHOOL |
| FILM | TAX | WOMEN | STUDENTS |
| SHOW | PROGRAM | PEOPLE | SCHOOLS |
| MUSIC | BUDGET | CHILD | EDUCATION |
| MOVIE | BILLION | YEARS | TEACHERS |
| PLAY | FEDERAL | FAMILIES | HIGH |
| MUSICAL | YEAR | WORK | PUBLIC |
| BEST | SPENDING | PARENTS | TEACHER |
| ACTOR | NEW | SAYS | BENNETT |
| FIRST | STATE | FAMILY | MANIGAT |
| YORK | PLAN | WELFARE | NAMPHY |
| OPERA | MONEY | MEN | STATE |
| THEATER | PROGRAMS | PERCENT | PRESIDENT |
| ACTRESS | GOVERNMENT | CARE | ELEMENTARY |
| LOVE | CONGRESS | LIFE | HAITI |

# Markov Networks



# Markov Networks

- A Bayesian network encodes a joint distribution with a directed acyclic graph
  - A CPT captures uncertainty between a node and its parents
- A Markov network (or Markov random field) encodes a joint distribution with an undirected graph
  - A potential function captures uncertainty between a clique of nodes

# Markov Networks

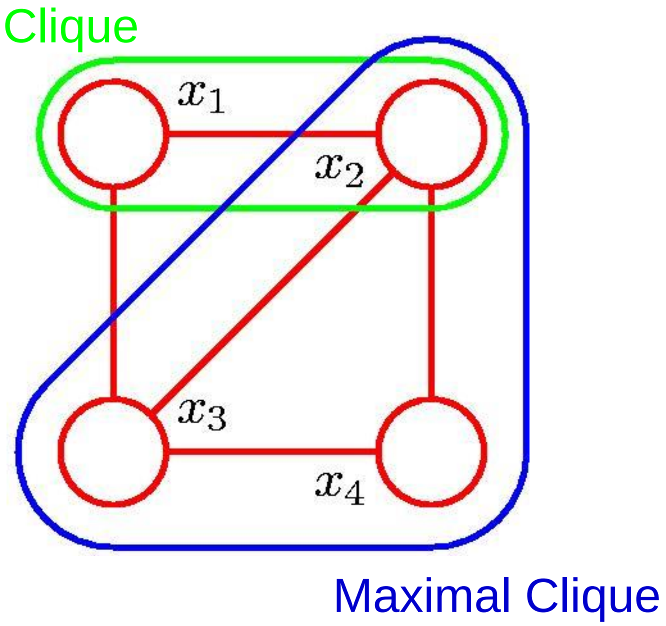
- Markov network = undirected graph + potential functions
  - For each clique (or max clique), a potential function is defined
    - A potential function is not locally normalized, i.e., it doesn't encode probabilities
  - A joint probability is proportional to the product of potentials

$$p(\mathbf{x}) = \frac{1}{Z}\prod_C \psi_C(\mathbf{x}_C)$$

where $\psi_C(\mathbf{x}_C)$ is the potential over clique C and

$$Z = \sum_{\mathbf{x}}\prod_C \psi_C(\mathbf{x}_C)$$

is the normalization coefficient (aka. partition function).



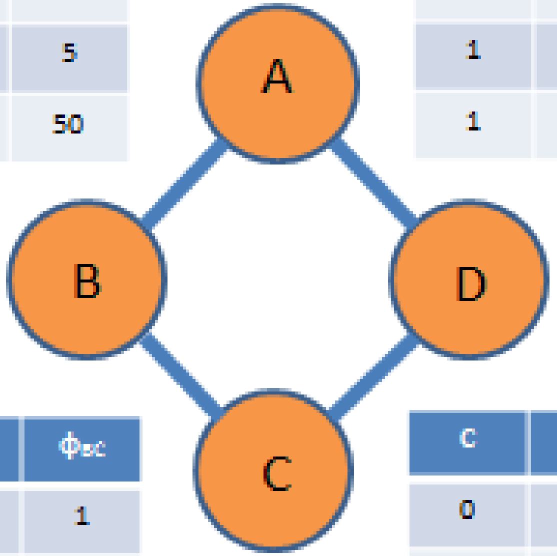
# Markov Networks

| A | B | C | D | $\phi_{AB}\phi_{BC}\phi_{CD}\phi_{AD}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 250 |
| 0 | 0 | 0 | 1 | 37500 |
| 0 | 0 | 1 | 0 | 50000 |
| 0 | 0 | 1 | 1 | 625000 |
| 0 | 1 | 0 | 0 | 1125 |
| 0 | 1 | 0 | 1 | 168750 |
| 0 | 1 | 1 | 0 | 50000 |
| 0 | 1 | 1 | 1 | 625000 |
| 1 | 0 | 0 | 0 | 250 |
| 1 | 0 | 0 | 1 | 375 |
| 1 | 0 | 1 | 0 | 50000 |
| 1 | 0 | 1 | 1 | 6250 |
| 1 | 1 | 0 | 0 | 112500 |
| 1 | 1 | 0 | 1 | 168750 |
| 1 | 1 | 1 | 0 | 5000000 |
| 1 | 1 | 1 | 1 | 625000 |

$Z = 7520750$

| A | B | $\phi_{AB}$ |
|---|---|---|
| 0 | 0 | 50 |
| 0 | 1 | 5 |
| 1 | 0 | 5 |
| 1 | 1 | 50 |

| A | D | $\phi_{AD}$ |
|---|---|---|
| 0 | 0 | 5 |
| 0 | 1 | 50 |
| 1 | 0 | 50 |
| 1 | 1 | 5 |

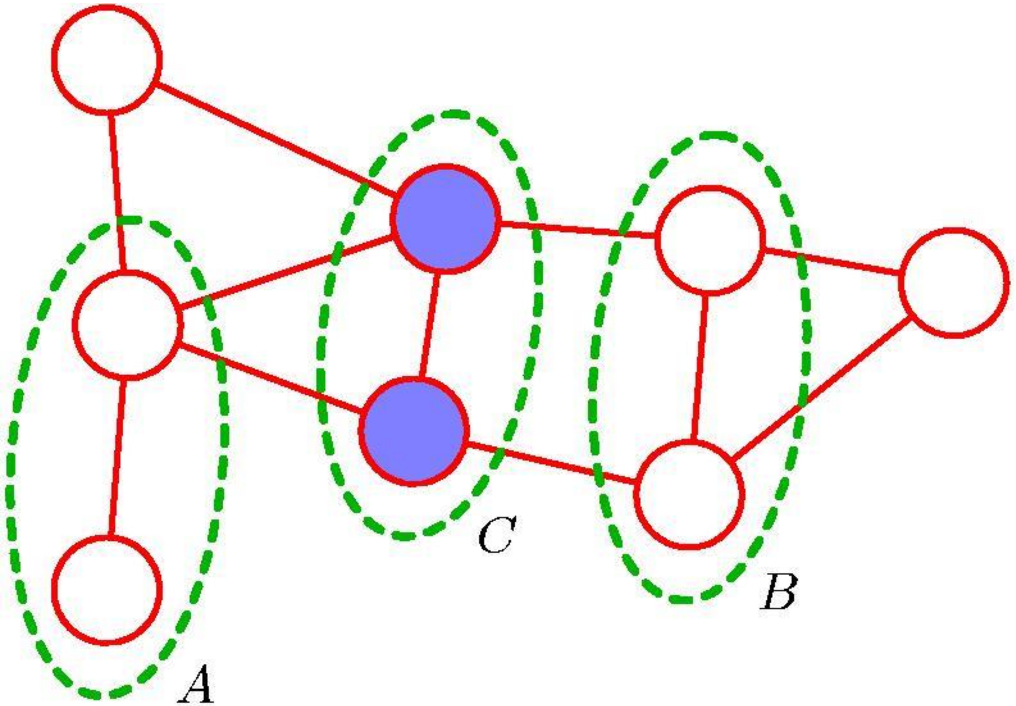



| B | C | $\phi_{BC}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 5 |
| 1 | 0 | 45 |
| 1 | 1 | 50 |

| C | D | $\phi_{CD}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 15 |
| 1 | 0 | 40 |
| 1 | 1 | 50 |

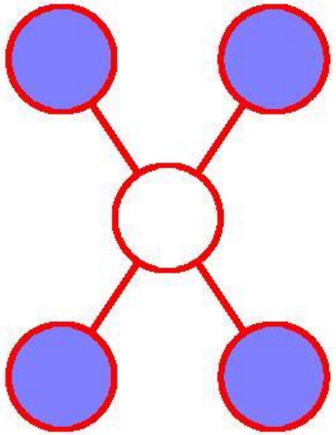
# Markov Networks

- Conditional independence and Markov blanket in MN



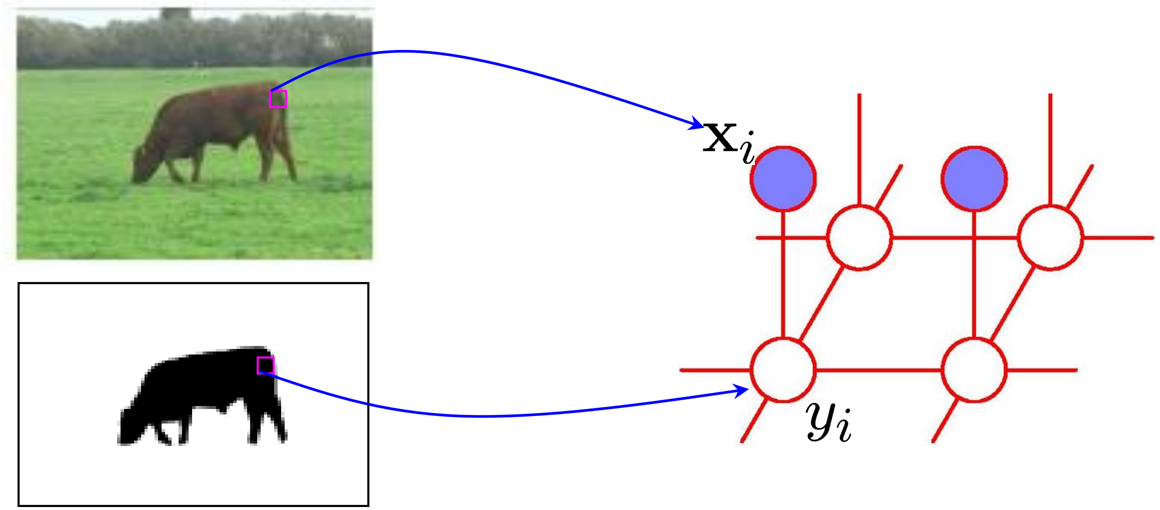

$$A \perp\!\!\!\perp B \mid C$$

Markov Blanket



# Example – Image Segmentation

- Binary segmentation



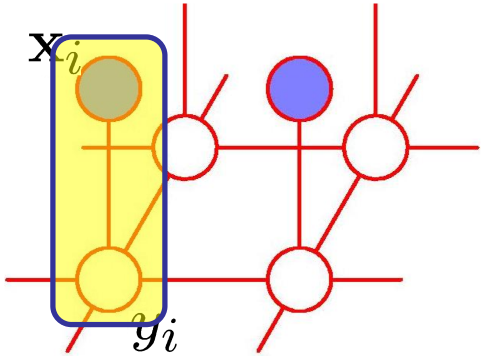

$\mathbf{x}_i$ image pixel

$y_i \in \{0, 1\}$

0: background
1: foreground

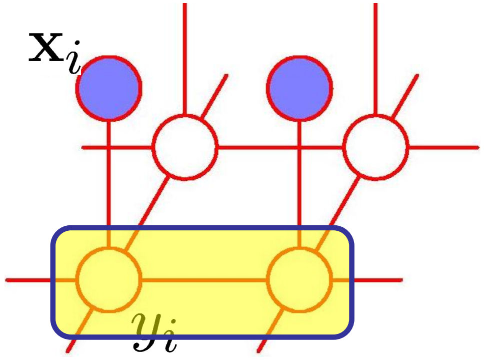
# Example – Image Segmentation

- Unary potential
  - Indicating how likely a pixel is a background vs. foreground
    - Ex: $\psi(\mathbf{x}_i, y_i) = \exp\left(w^T \phi(\mathbf{x}_i, y_i)\right)$, where $\phi(\mathbf{x}_i, y_i)$ is a feature vector
    - Ex: we may assign a large weight to the feature: $\{\mathbf{x}_i$ is dark and $y_i = 0\}$

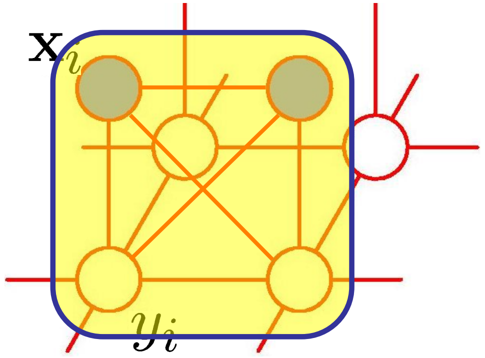


# Example – Image Segmentation

- Pairwise potential
  - Encouraging adjacent pixels to have same labels (smoothing)
    - Ex. $\psi(y_i, y_j) = \exp\left(\alpha\, I(y_i = y_j)\right)$




  - A better design is to incorporate pixel info, e.g., similar pixels are more likely to have same labels.
    - Need to change the graph structure



# Example – Image Segmentation

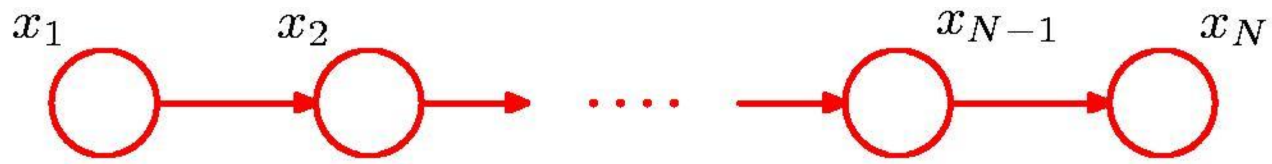
- Inferring labels from image pixels


X Y Y

# Graphical Models

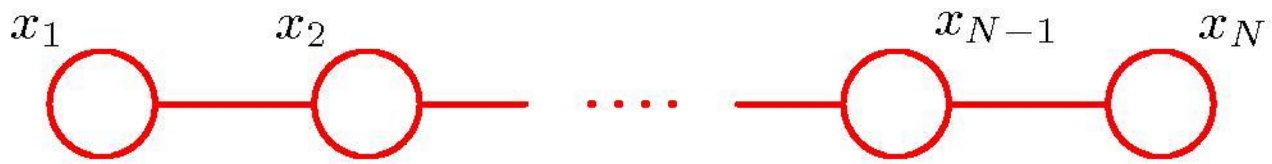
- A graphical model is a probabilistic model for which a graph expresses conditional dependence between random variables
  - Bayesian networks: directed acyclic graph
  - Markov networks: undirected graph
  - Factor graphs, conditional random fields, etc.

# Converting Directed to Undirected Graphs (1)



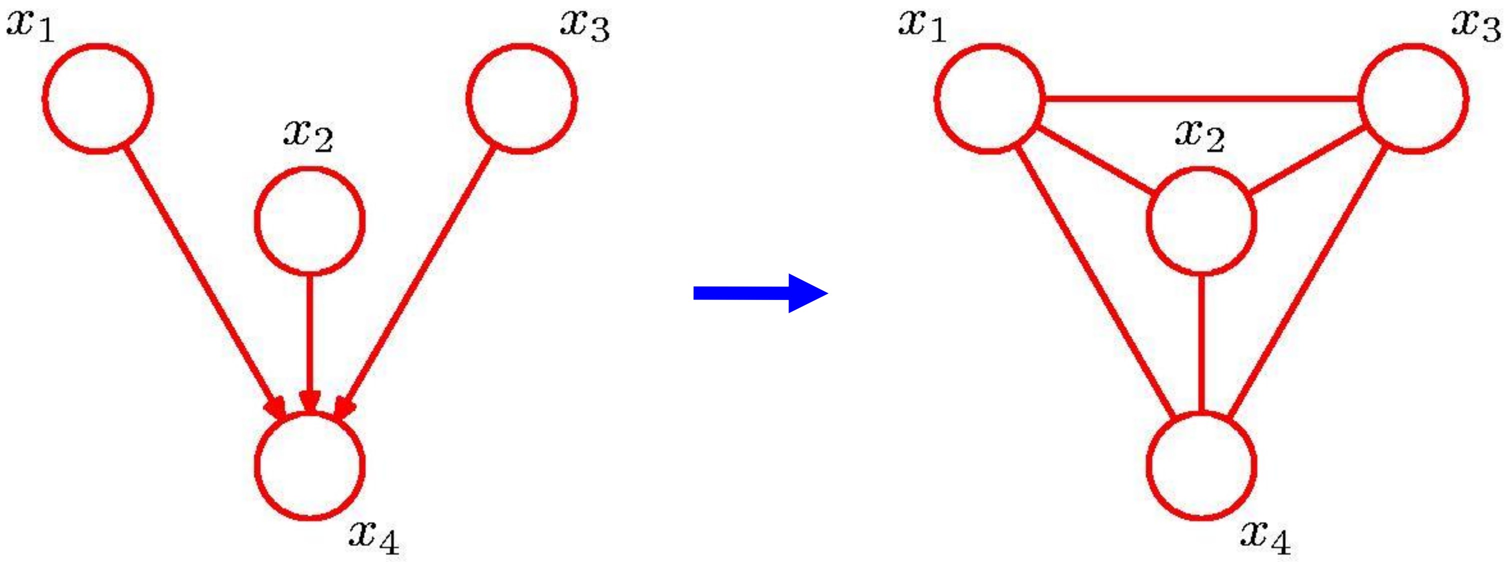

$$p(\mathbf{x}) = \underbrace{p(x_1)p(x_2|x_1)}\, p(x_3|x_2) \cdots p(x_N|x_{N-1})$$

$$p(\mathbf{x}) = \frac{1}{Z}\, \psi_{1,2}(x_1, x_2)\, \psi_{2,3}(x_2, x_3) \cdots \psi_{N-1,N}(x_{N-1}, x_N)$$



# Converting Directed to Undirected Graphs (2)

- Additional links (moralization)

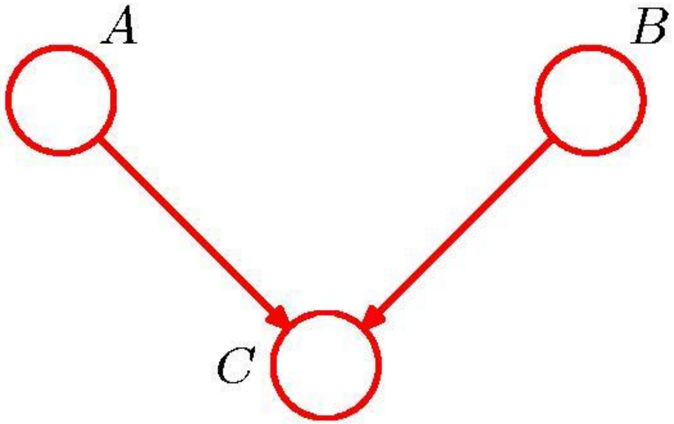


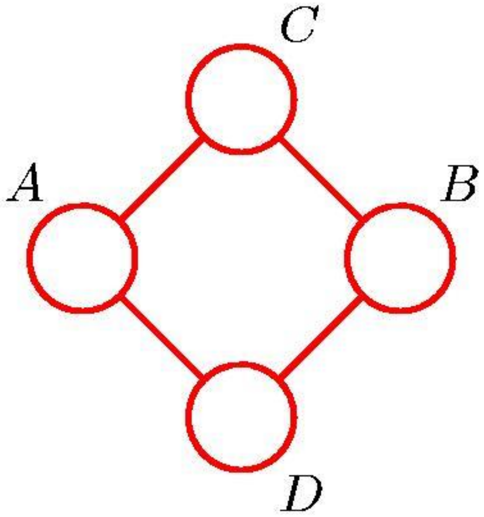

$$
\begin{aligned}
p(\mathbf{x}) &= p(x_1)p(x_2)p(x_3)p(x_4|x_1,x_2,x_3) \\
&= \frac{1}{Z}\psi(x_1,x_2,x_3,x_4)
\end{aligned}
$$

# Bayesian Network → Markov Network

- Steps
  1. Moralization
  2. Construct potential functions from CPTs
- The BN and MN encode the same distribution
- Do they encode the same set of conditional independence?

# Encoding Conditional Independence




$$A \perp\!\!\!\perp B \mid \emptyset$$

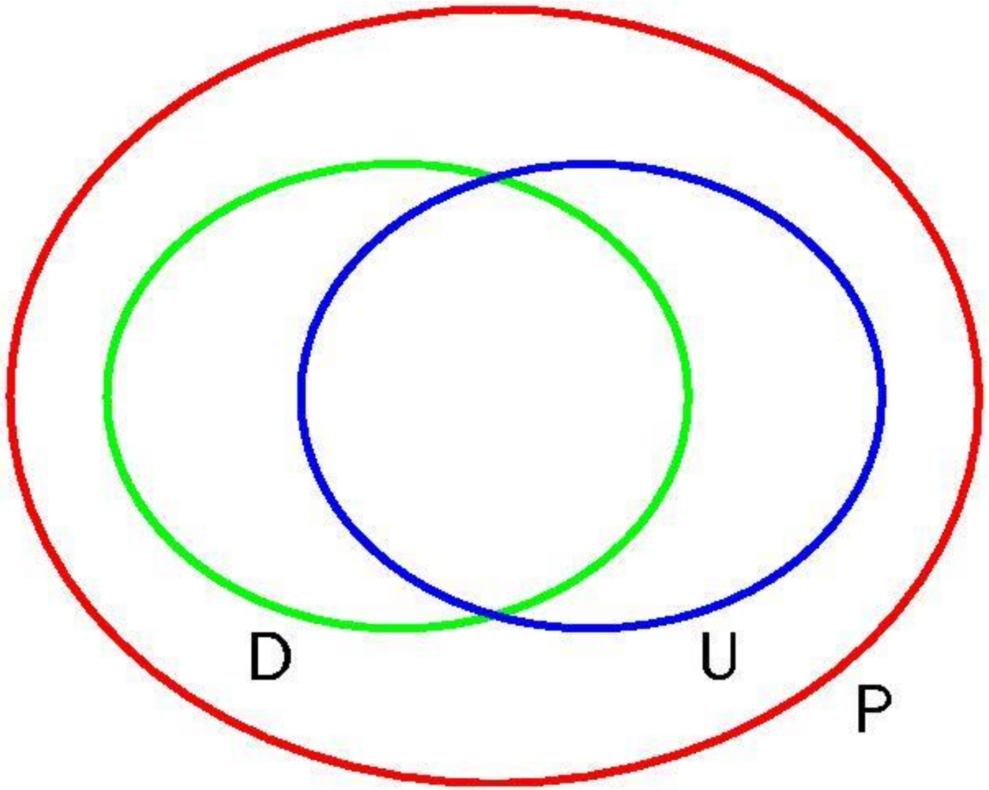
$$A \not\!\perp\!\!\!\perp B \mid C$$

$$A \not\!\perp\!\!\!\perp B \mid \emptyset$$

$$A \perp\!\!\!\perp B \mid C \cup D$$

$$C \perp\!\!\!\perp D \mid A \cup B$$

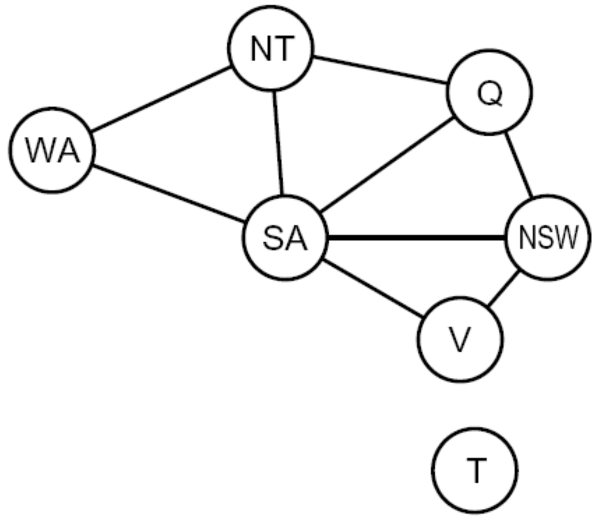
# Encoding Conditional Independence

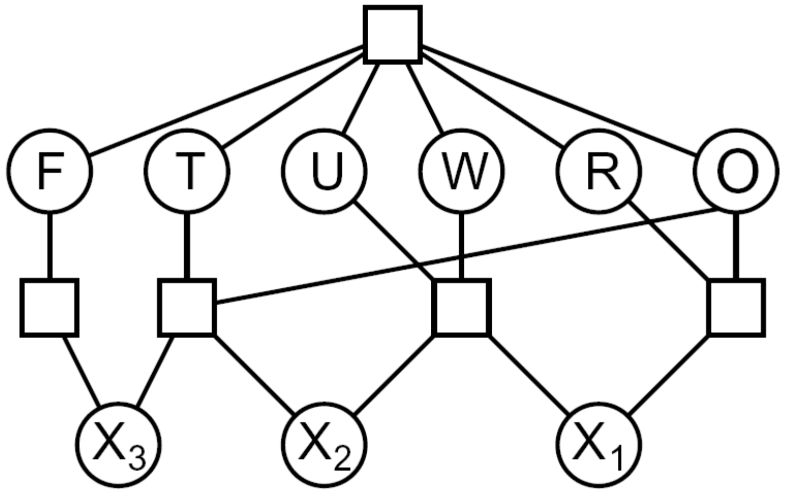



The set of distributions whose conditional independence can be exactly (i.e., no more, no less) represented by a **directed**/**undirected** graph

# Markov networks vs. Constraint graphs

- Constraint graphs can be seen as Markov networks with 0/1 potentials



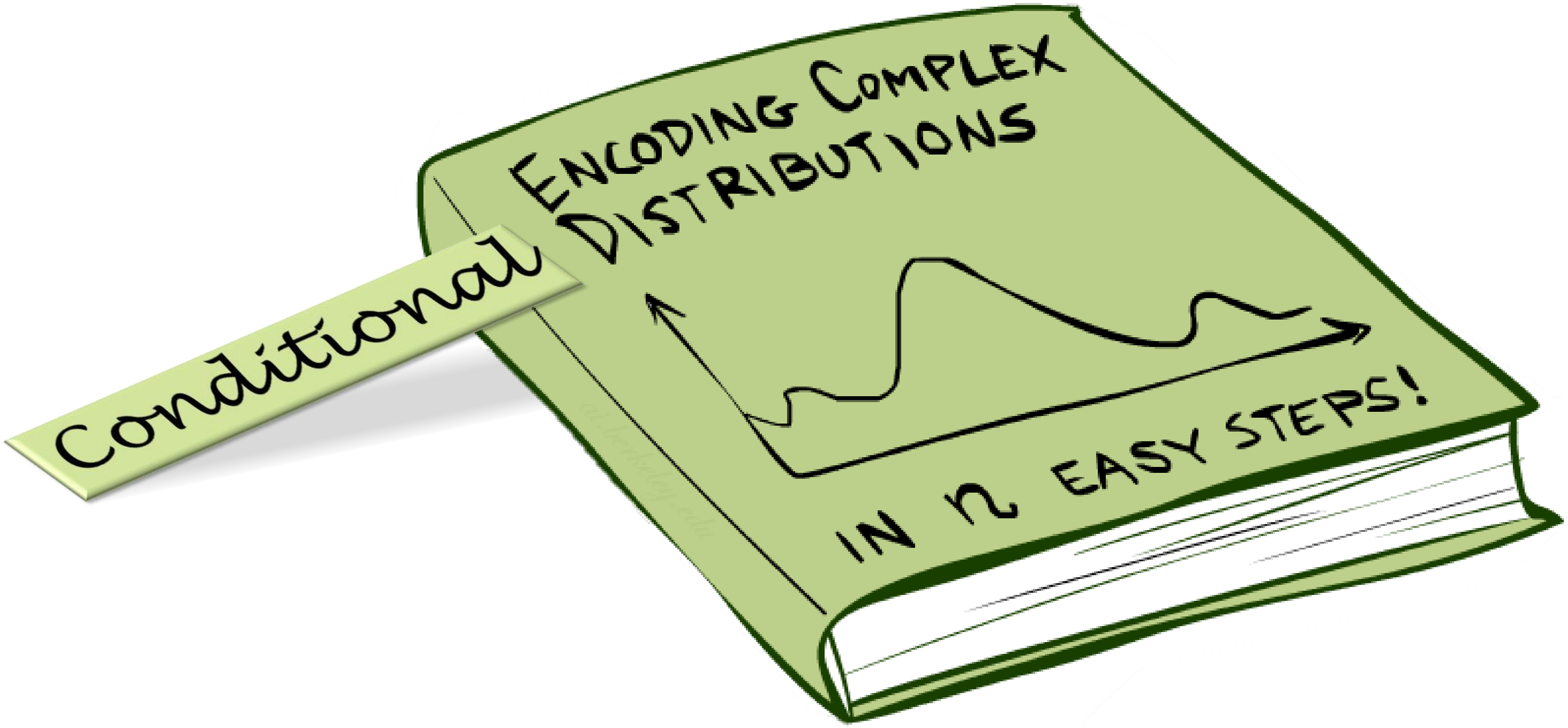



# BN/MN vs. Logic

- Which logic is BN/MN more similar to: PL? FOL?
  - Boolean nodes represent propositions
  - No explicit representation of objects, relations, quantifiers

- BN/MN can be seen as a probabilistic extension of PL
- PL can be seen as BN/MN with deterministic CPTs/potentials

# Conditional Random Fields



# Generative vs. Discriminative Models

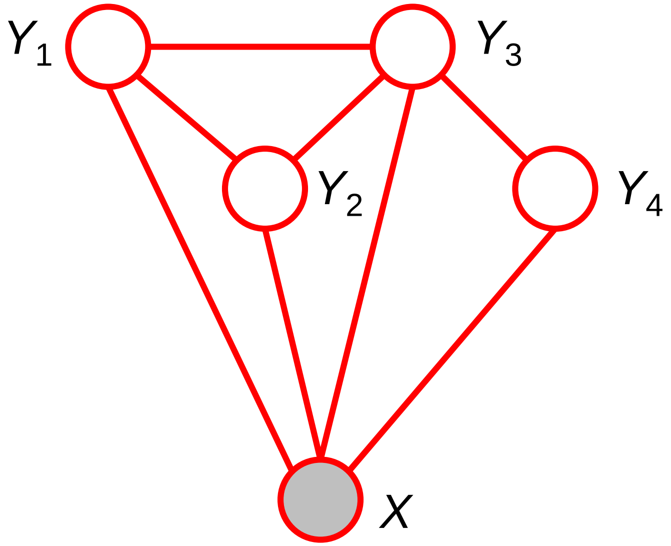
- Generative models
  - A generative model represents a joint distribution $P(X_1, X_2, \ldots, X_n)$
  - Both BN and MN are generative models

- Discriminative models
  - In some scenarios, we only care about predicting queries from evidence
    - E.g., image segmentation
  - A discriminative model represents a conditional distribution $P(Y_1, Y_2, \ldots, Y_n | X)$
  - It does not model $P(X)$

# Conditional Random Fields (CRF)

- An extension of MN (aka. Markov random field) where everything is conditioned on an input

$$P(\mathrm{y}|x) = \frac{1}{Z(x)} \prod_{C} \psi_C(\mathrm{y}_C, x)$$



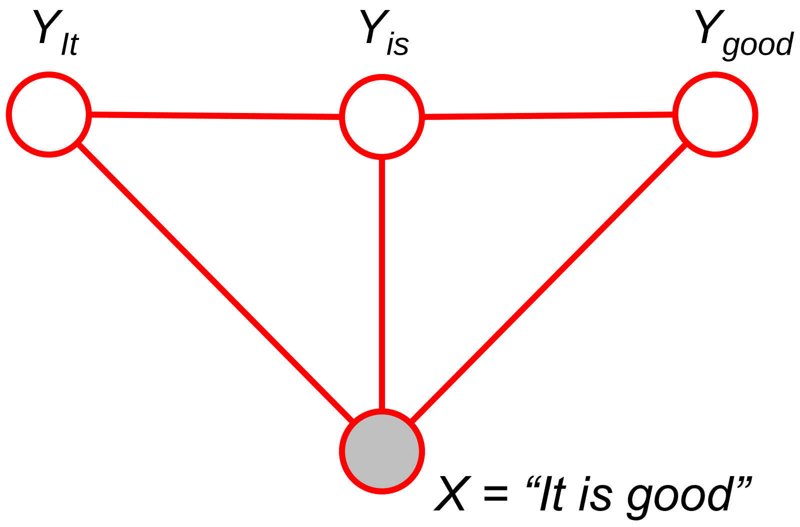

where $\psi_C(y_C, x)$ is the potential over clique C and

$$Z(x) = \sum_{\mathrm{y}} \prod_{C} \psi_C(\mathrm{y}_C, x)$$

is the normalization coefficient.

# CRF Applications

- NLP
  - POS tagging
  - Named entity recognition
  - Syntactic parsing
- CV
  - Image segmentation
  - Posture recognition



# Summary

- A Bayesian network encodes a joint distribution
  - Syntax: DAG+CPTs
  - Semantics
    - Global semantics
    - Conditional independence semantics
    - D-separation

- Markov networks
  - Syntax: undirected graph + potentials
  - Semantics
  - Extension: CRF