The z-transform

Recall

- The response of LTI systems to complex exponentials z^n

$$y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

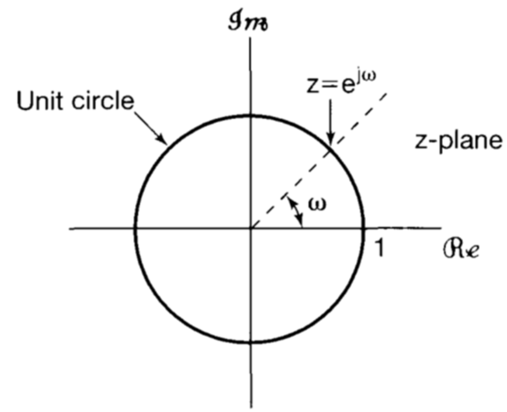
Definition

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$\boxed{X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}}$$

The z-transform

Z-transform vs Fourier transform



$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$z = e^{j\omega}$
$|z| = 1$ ⇙

⇘ $z = re^{j\omega}$

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\}}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] (re^{j\omega})^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} \{x[n] r^{-n}\} e^{-j\omega n}$$

$$\boxed{X(re^{j\omega}) = \mathcal{F}\{x[n] r^{-n}\}}$$

The z-transform

Examples

$$x[n] = a^n u[n] \qquad X(z) = ?$$

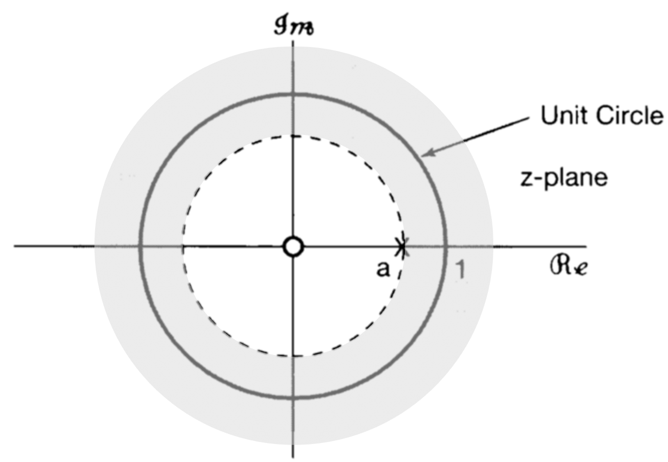
Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a} \qquad |z| > |a|$$

$$\Downarrow \; a = 1$$

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \qquad |z| > 1$$



The z-transform

Examples

$$x[n] = -a^n u[-n-1] \qquad X(z) = ?$$

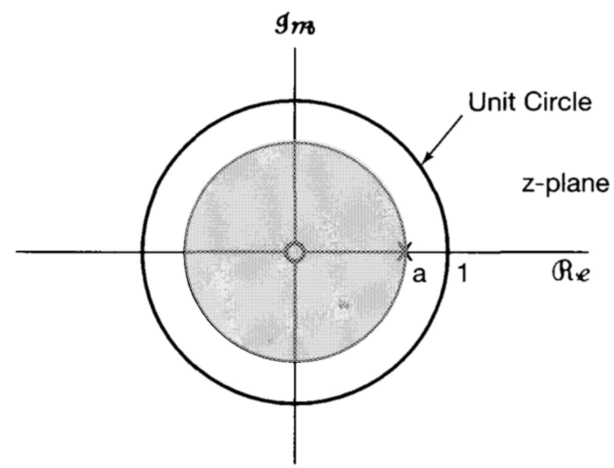
Solution

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$



$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

The z-transform

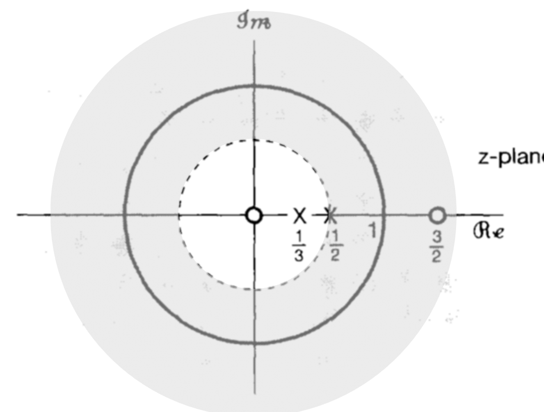
Examples

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \qquad X(z) = ?$$

Solution

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

The z-transform

Examples

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n] \qquad X(z) = ?$$

Solution

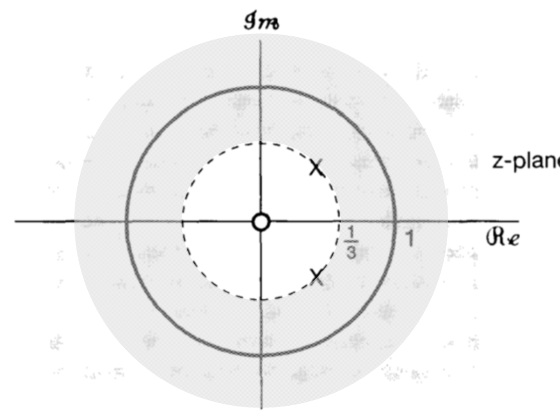
$$X(z) = \sum_{n=-\infty}^{+\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]\right\}z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{+\infty}\left(\frac{1}{3}e^{j\pi/4}\right)^n z^{-n} - \frac{1}{2j}\sum_{n=0}^{+\infty}\left(\frac{1}{3}e^{-j\pi/4}\right)^n z^{-n}$$

$$= \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}$$

For convergence,

$$\left|\frac{1}{3}e^{j\pi/4}z^{-1}\right| < 1 \;\; \& \;\; \left|\frac{1}{3}e^{-j\pi/4}z^{-1}\right| < 1 \quad \Longrightarrow \quad |z| > 1/3$$



The z-Transform
(ch.10)

- ❑ The z-transform
- ❑ The region of convergence for the z-transforms
- ❑ The inverse z-transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the z-transform
- ❑ Some common z-transform pairs
- ❑ Analysis and characterization of LTI systems using z-transforms
- ❑ System function algebra and block diagram representations
- ❑ The unilateral z-transform

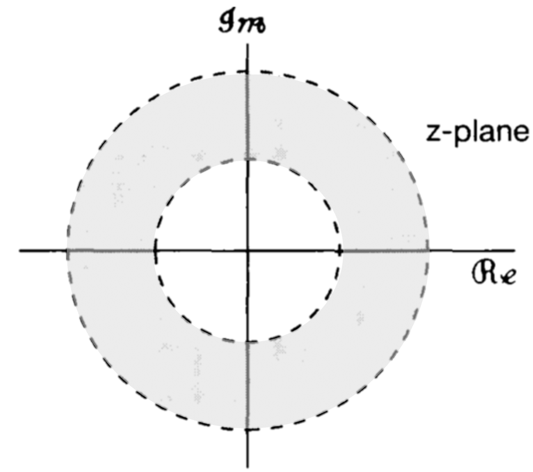
The region of convergence for z-transforms

Properties

- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

ROC of $X(z)$: $x[n]r^{-n}$ converges (absolutely summable)

$$\sum_{n=-\infty}^{+\infty} |x[n]| r^{-n} < \infty$$



- The ROC does not contain any poles.

$X(z)$ is infinite at a pole

The region of convergence for z-transforms

Properties

- If $x[n]$ is of finite duration ($x[n] \neq 0$ for $N_1 < n < N_2$), then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$

If $N_1 < 0$ and $N_2 > 0$

ROC does not include $z = 0$ or $z = \infty$

If $N_1 \geq 0$,

ROC includes $z = \infty$, not $z = 0$

If $N_2 \leq 0$,

ROC includes $z = 0$, not $z = \infty$

The region of convergence for z-transforms

Examples

$$\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1$$

ROC = the entire z-plane

$$\delta[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}$$

ROC = the entire z-plane except $z = 0$

$$\delta[n+1] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n} = z$$

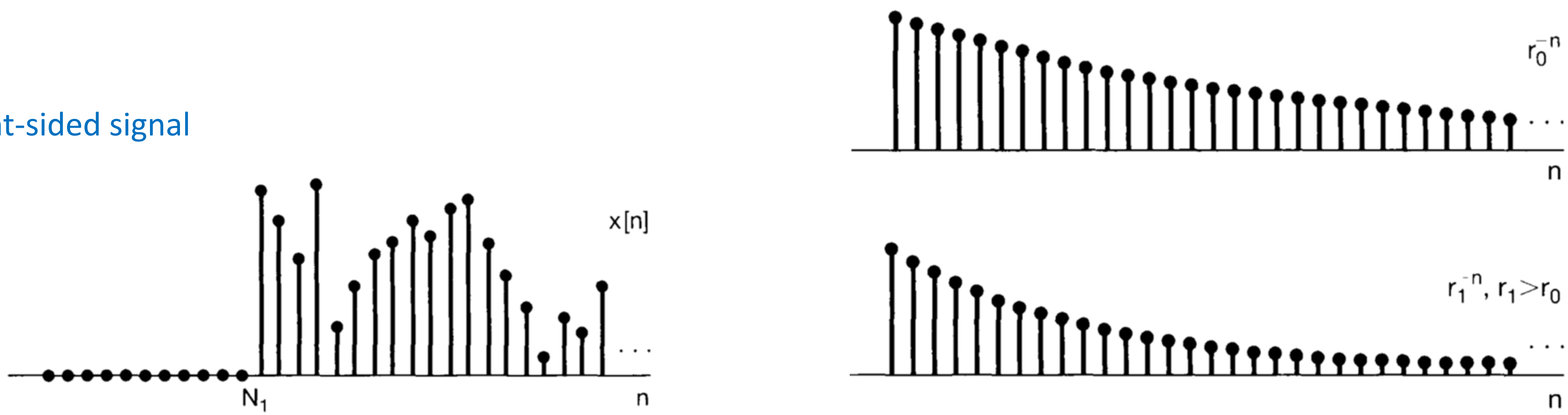
ROC = the entire finite z-plane (except $z = \infty$)

The region of convergence for z-transforms

Properties

- If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

Right-sided signal



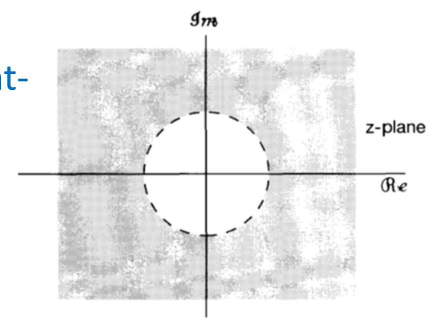
- If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $0 < |z| < r_0$ will also be in the ROC.

The region of convergence for z-transforms

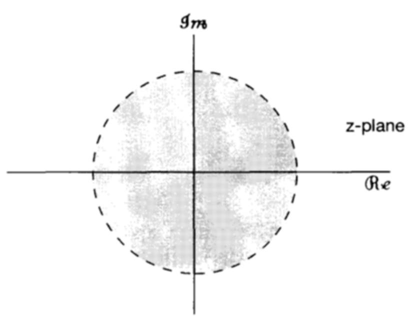
Properties

- If $x[n]$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

ROC for right-sided signal

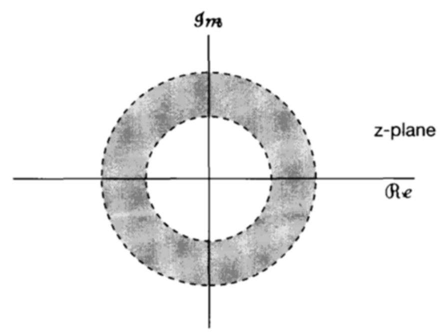


ROC for left-sided signal



$\Longrightarrow$

ROC for two-sided signal



The region of convergence for z-transforms

Examples

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1, a > 0 \\ 0 & otherwise \end{cases} \qquad X(z) = ?$$

Solution

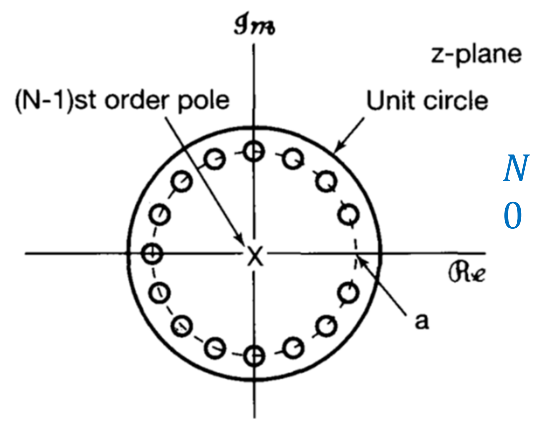
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} \quad = \sum_{n=0}^{N-1} (az^{-1})^n \quad = \frac{1-(az^{-1})^N}{1-az^{-1}} \quad = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

The N roots of the numerator polynomial:

$$z_k = ae^{j\left(\frac{2\pi k}{N}\right)}, \qquad k = 0, 1, \cdots, N-1$$

When $k = 0$, the zero cancels the pole at $z = a$

$$z_k = ae^{j\left(\frac{2\pi k}{N}\right)}, \qquad k = 1, \cdots, N-1$$

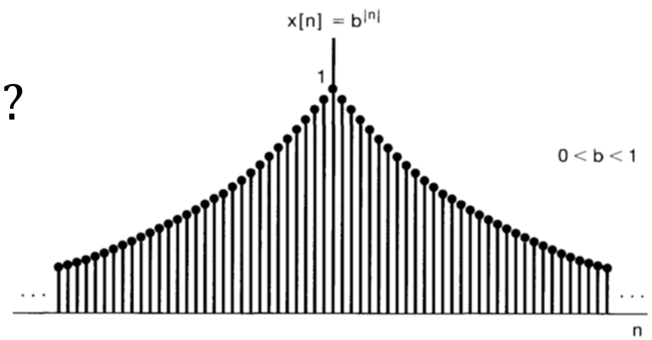


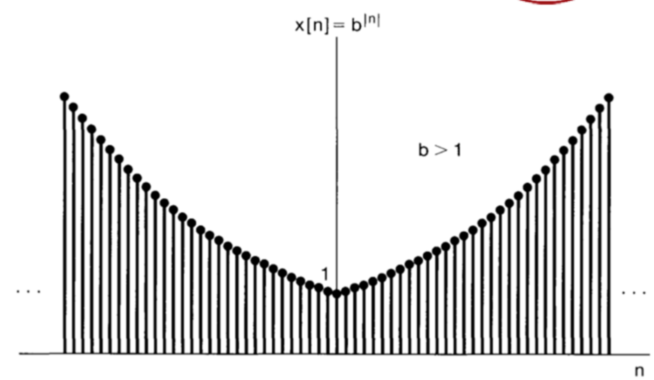
$N = 16$
$0 < a < 1$

The region of convergence for z-transforms

Examples

$x[n] = b^{|n|}, b > 0 \qquad X(z) = ?$





Solution

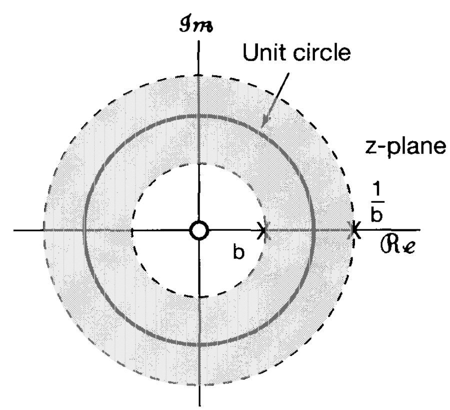
$$x[n] = b^n u[n] + b^{-n} u[-n-1]$$

$$b^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - bz^{-1}} \qquad |z| > b$$

$$b^{-n} u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{-1}{1 - b^{-1}z^{-1}} \qquad |z| < \frac{1}{b}$$

For convergence, $b < 1$

$$X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}} \qquad b < |z| < \frac{1}{b}$$



The region of convergence for z-transforms

Properties

- If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

- If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is right-sided, the ROC is the region in the z-plane outside the outer-most pole.
  If $x[n]$ is causal, the ROC also includes $z = \infty$.

- If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is left-sided, the ROC is the region in the z-plane inside the inner-most nonzero pole.
  If $x[n]$ is anti-causal, the ROC also includes $z = 0$.

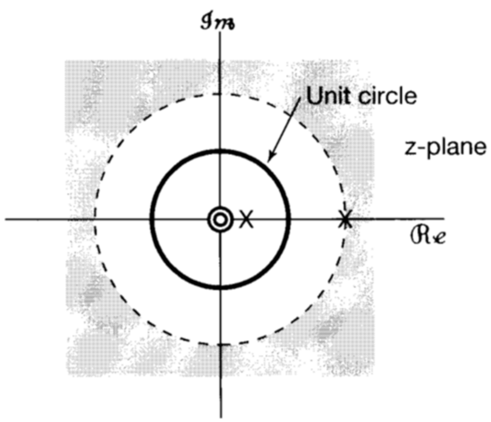
The region of convergence for z-transforms

Examples

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)(1 - 2z^{-1})}$$

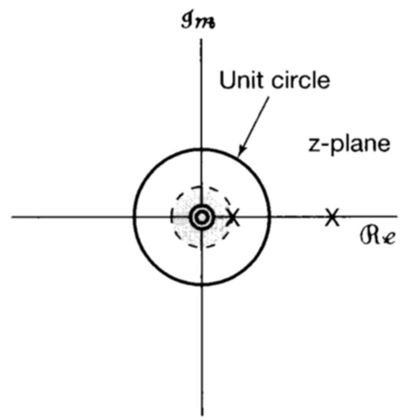
ROC ?

Solution



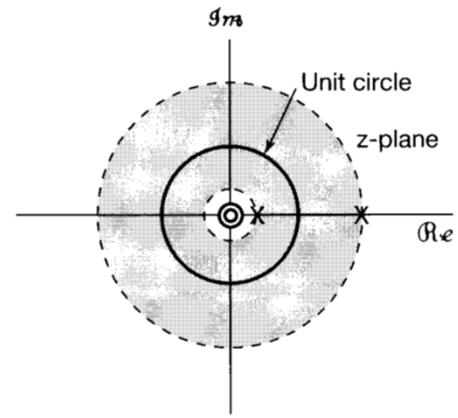
Right-sided sequence

Has no FT



Left-sided sequence

Has no FT



Two-sided sequence

FT converges

The z-Transform
(ch.10)

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The inverse z-transform

$$X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(re^{j\omega})e^{j\omega n}d\omega$$

$$x[n] = r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^n d\omega$$

$$z = re^{j\omega}$$
$$dz = jre^{j\omega}d\omega = jzd\omega$$

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz}$$

The inverse z-transform

Examples

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}, \qquad |z| > \frac{1}{3} \qquad x[n] = ?$$

Solution

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

$$\left.\begin{aligned} x_1[n] &\stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}} \quad |z| > \frac{1}{4} \\ x_2[n] &\stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}} \quad |z| > \frac{1}{3} \end{aligned}\right\} \Longrightarrow \quad x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

The inverse z-transform

Examples

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)},\qquad \frac{1}{4}<|z|<\frac{1}{3}\qquad x[n]=?$$

Solution

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}}$$

$$x[n]=x_1[n]+x_2[n]$$

$$\left.\begin{aligned} x_1[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}} \quad |z|>\frac{1}{4} \\ x_2[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{2}{1-\frac{1}{3}z^{-1}} \quad |z|<\frac{1}{3} \end{aligned}\right\} \Longrightarrow x[n]=\left(\frac{1}{4}\right)^n u[n]-2\left(\frac{1}{3}\right)^n u[-n-1]$$

Examples

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)},\qquad |z|<\frac{1}{4}\qquad\qquad x[n]=?$$

Solution

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}}$$

$$x[n]=x_1[n]+x_2[n]$$

$$\left.\begin{array}{lll} x_1[n]\overset{\mathcal{Z}}{\longleftrightarrow}\dfrac{1}{1-\frac{1}{4}z^{-1}} & & |z|<\dfrac{1}{4} \\ x_2[n]\overset{\mathcal{Z}}{\longleftrightarrow}\dfrac{2}{1-\frac{1}{3}z^{-1}} & & |z|<\dfrac{1}{3} \end{array}\right\}\Longrightarrow x[n]=-\left(\frac{1}{4}\right)^n u[-n-1]-2\left(\frac{1}{3}\right)^n u[-n-1]$$

The inverse z-transform

Examples

$$X(z) = 4z^2 + 2 + 3z^{-1}, \qquad 0 < |z| < \infty \qquad x[n] = ?$$

Solution 1

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & otherwise \end{cases}$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Solution 2

$$\delta[n+n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{n_0}$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

The inverse z-transform

Examples

$$X(z)=\frac{1}{1-az^{-1}},\qquad |z|>|a| \qquad x[n]=?$$

Solution

If $|z|>|a|$,

$$\frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\cdots$$

$$x[n]=a^nu[n]$$

$$\begin{array}{r l}
 & 1+az^{-1}+a^2z^{-2}+\cdots \\ \hline
1-az^{-1}\,) & 1 \\
 & 1-az^{-1} \\ \hline
 & \quad az^{-1} \\
 & \quad az^{-1}-a^2z^{-2} \\ \hline
 & \qquad\quad a^2z^{-2}
\end{array}$$

If $|z|<|a|$,

$$\frac{1}{1-az^{-1}}=-a^{-1}z-a^{-2}z^2+\cdots$$

$$x[n]=-a^nu[-n-1]$$

$$\begin{array}{r l}
 & -a^{-1}z-a^{-2}z^2-\cdots \\ \hline
-az^{-1}+1\,) & 1 \\
 & 1-\ a^{-1}z \\ \hline
 & \qquad a^{-1}z
\end{array}$$

The inverse z-transform

Examples

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a| \qquad x[n] = ?$$

Solution

$$\log(1 + v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v^n}{n}, \qquad |v| < 1$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \begin{cases} (-1)^{n+1} a^n / n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

$$= -\frac{(-a)^n}{n} u[n-1]$$

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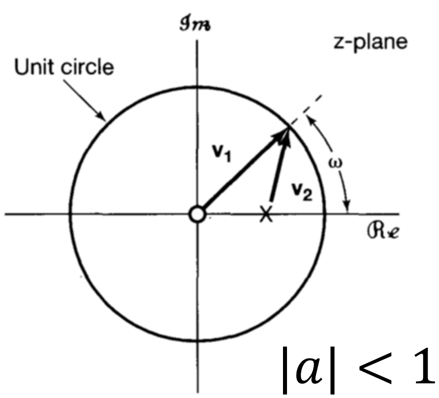
First-order systems

Consider $h[n] = a^n u[n]$

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



$|a| < 1$