Lab 5 Sampling and Reconstruction

Objective

- Learn to convert an analog signal to a discrete-time sequence via sampling.
- Be able to reconstruct an analog signal from a discrete-time sequence.
- Understand the conditions when a sampled signal can uniquely represent its analog counterpart.

Content

Sampling

For continuous-time signal processing, we can also convert it to discrete-time signal processing through sampling, which is also in line with the working mode of a computer. To make sure that the sampled signal can represent the original signal, the signal should be sampled at a sufficient rate determined by the sampling theorem. In this case, we can restore the original signal from the sampled signal.

Nyquist Sampling Theorem

If a signal is band limited and its samples are taken at a sufficient rate, then the samples uniquely specify the signal and the signal can be reconstructed from those samples. This is known as the Nyquist sampling theorem.
When a real signal $\mathrm{x}(t)$ is sampled in the time domain, the sampled signal can be represented as:

$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

Since the impulse $\delta_T(t)$ is a periodic signal of period T, it can be expressed as a trigonometric Fourier series as follows (Fourier series expansion is discussed in Fourier Analysis).

$$\delta_T(t) = \frac{1}{T}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \cdots] \qquad \omega_s = \frac{2\pi}{T} = 2\pi f_s$$

Therefor

$$x_s(t) = x(t)\delta_T(t) = \frac{1}{T}[x(t) + 2x(t)\cos\omega_s t + 2\,x(t)\cos 2\omega_s t + 2\,x(t)\cos 3\omega_s t + \cdots]$$

According to the characteristics of Fourier theory, transform $x_s(t)$ in the time domain into $X_S(\omega)$ in frequency domain term by term as listed in Table 1.

Table 1 Fourier Transform of Sampled Signal

Time Domain	Frequency Domain

$\mathrm{x}(t)$	$\mathrm{X}(\omega)$
$2\mathrm{x}(t)\cos\omega_s t$	$\mathrm{X}(\omega+\omega_s)+X(\omega-\omega_s)$
$2\mathrm{x}(t)\cos2\omega_s t$	$\mathrm{X}(\omega+2\omega_s)+X(\omega-2\omega_s)$
$2\mathrm{x}(t)\cos3\omega_s t$	$\mathrm{X}(\omega+3\omega_s)+X(\omega-3\omega_s)$
…	…

From the table, it is easy to find out that the spectrum $X_S(\omega)$ consists of $\mathrm{X}(\omega)$ repeating periodically with period ω_S. Therefor $X_S(\omega)$ can be expressed as follows:

$$X_S(\omega)=\frac{1}{T}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)$$

To reconstruct $\mathrm{x}(t)$ from $x_s(t)$, it is necessary to be able to recover $\mathrm{X}(\omega)$ from $X_S(\omega)$, which requires that there is no overlap between consecutive $X_S(\omega)$. Figure 1 shows $X_S(\omega)$, from which it is easy to find that as long as the sampling frequency ω_s is greater than twice the signal bandwidth ω_b, $X_S(\omega)$ will not contain any overlap of $\mathrm{X}(\omega)$. In this case $\mathrm{x}(t)$ can be recovered from its samples $x_s(t)$. $2\omega_b$ is called the Nyquist rate and ω_s must exceed it in order to avoid aliasing.

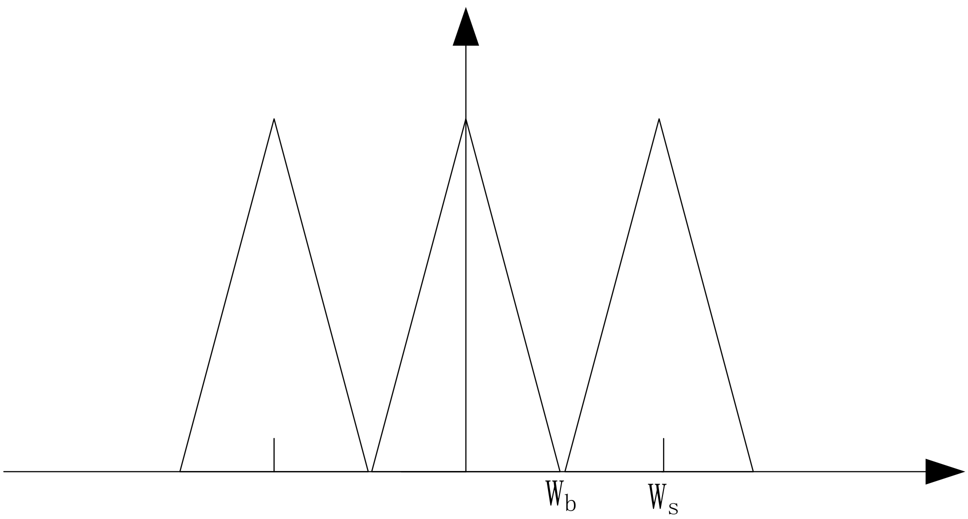


Figure 1 Nyquist frequency

Sampling of Non Band Limited Signal

In most cases, $\mathrm{x}(t)$ may not be bandlimited. At this moment an anti-aliasing filter is needed.
An anti-aliasing filter is a filter that is used before a signal sampler, to restrict the bandwidth of a signal to approximately satisfy the sampling theorem. The aim of the filter is to eliminate the frequency components beyond ${}^{f_s}/_{2}$ from $\mathrm{x}(t)$ before sampling $\mathrm{x}(t)$. Usually the relationship between the cutoff frequency of anti-aliasing filter f_c and the sample rate f_s is as follows:

$$f_c={}^{f_s}/_{2.56}$$

Function **filter(b,a,x)** can help to build an anti-aliasing filter. There are many kinds of filters. Here we take the moving-average filter and Butterworth filter as examples. Other kinds of filters will be discussed in Digital Signal Processing in detail.

Moving-average filter

A moving-average filter slides a window of length **windowSize** along the data, computing averages of the data contained in each window. The following difference equation defines a moving-average filter of a vector x:

$$\mathrm{y(n)} = \frac{1}{windowSize}(x(n) + x(n-1) + \cdots + x(n-(windowSize-1)))$$

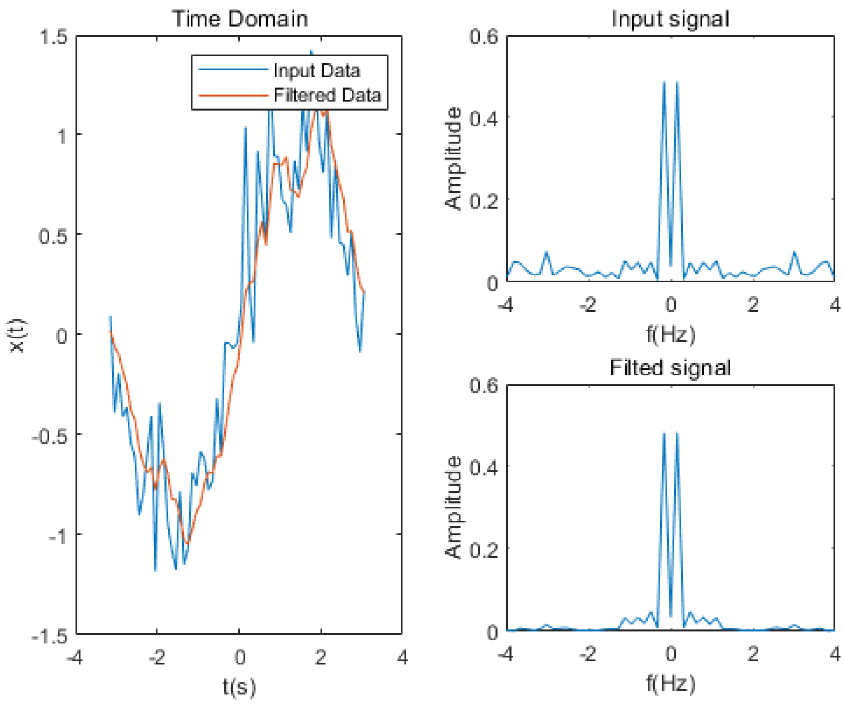
An example of a moving-average filter with windowSize=5 is given below.

```
% signal with noisy
clf;clear;
ds = 0.1;
Fs = 1/ds;
t = -pi:ds:pi;
x = sin(t)+0.25*randn(size(t));
N = length(x);

% set the filter
windowSize = 5;
b = (1/windowSize)*ones(1,windowSize);
a = 1;

% show the result in time domain
y = filter(b,a,x);
subplot(2,2,[1 3]);
plot(t,x); hold on;
plot(t,y); legend('Input Data','Filtered Data');
xlabel('t(s)');ylabel("x(t)");title('Time Domain')

% show the result in frequency domain
f = (-floor(N/2):floor(N/2))*Fs/N;
X = fftshift(fft(x))/N;
subplot(2,2,2); plot(f,abs(X)); axis([-4 4 0 0.6])
xlabel('f(Hz)');ylabel('Amplitude');title('Input signal');
Y = fftshift(fft(y))/N;
subplot(2,2,4); plot(f,abs(Y));axis([-4 4 0 0.6])
xlabel('f(Hz)');ylabel('Amplitude');title('Filted signal');
```



Butterworth filter

Function **butter** is used to design a Butterworth filter. Details are as follows:
[b, a]=butter(N,Wn) designs an Nth order lowpass digital Butterworth filter and returns the filter coefficients in vectors **b** (numerator) and **a** (denominator). The cutoff frequency Wn must be 0.0<Wn<1.0, with 1.0 corresponding to half the sample rate. That means $w_n = \frac{f_c}{f_s/2}$, f_c is the cutoff frequency of the filter and f_s is the sampling frequency.
After getting the filter coefficients with function **butter**, filter the input data with function **filter**. Here is an example:

```
% signal with noisy
% signal with noisy
clf; clear;
ds = 0.1;
fs = 1/ds;
t = -pi:ds:pi-ds;
x = sin(t)+0.25*randn(size(t));
fc = 0.5; % cutoff frequency

% set the filter
[b,a] = butter(4,fc/(fs/2));
[H w] = freqz(b,a);
plot(w/pi*fs/2,abs(H)); % the unit of the horizontal axis is *pi
rad/sample
```

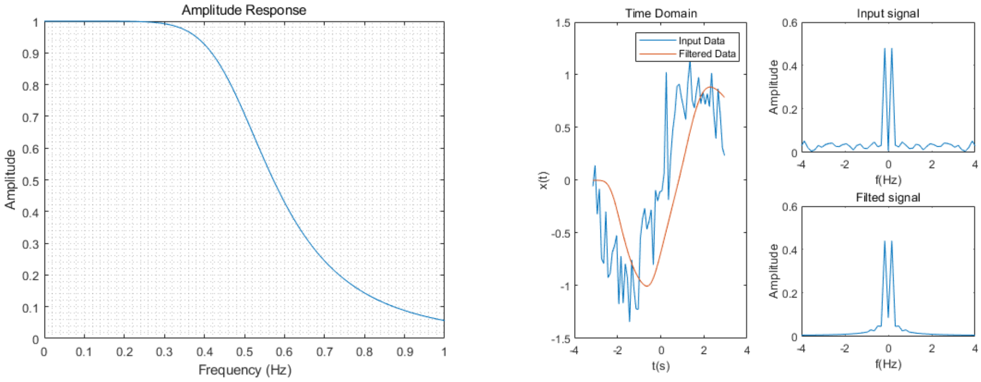
```
xlim([0 1]);grid minor;title('Amplitude Response');
xlabel('Frequency (Hz)'); ylabel('Amplitude');

% show the result in time domain
y = filter(b,a,x);
subplot(2,2,[1 3]);
plot(t,x); hold on;
plot(t,y); legend('Input Data','Filtered Data');
xlabel('t(s)');ylabel("x(t)");title('Time Domain')

% show the result in frequency domain
N = length(x);
f = (-N/2:N/2-1)*fs/N;

Fx = fftshift(fft(x))/N;
subplot(2,2,2); plot(f,abs(Fx));axis([-4 4 0 0.6]);
xlabel('f(Hz)');ylabel('Amplitude');title('Input signal');

Fy = fftshift(fft(y))/N;
subplot(2,2,4); plot(f,abs(Fy));axis([-4 4 0 0.6]);
xlabel('f(Hz)');ylabel('Amplitude');title('Filted signal');
```



Reconstruction

A band limited signal $x(t)$ can be reconstructed from its samples. To reconstruct the signal, pass the sampled signal through an ideal low pass filter with the bandwidth of ω_c, where ω_c should satisfy: $\omega_b < \omega_c < \omega_s - \omega_b$. The transfer function of the filter is expressed as follows:

$$H(\omega) = \begin{cases} T, & -\omega_c < \omega < \omega_c \\ 0, & otherwise \end{cases}$$

The relationship is displayed in Figure 2.

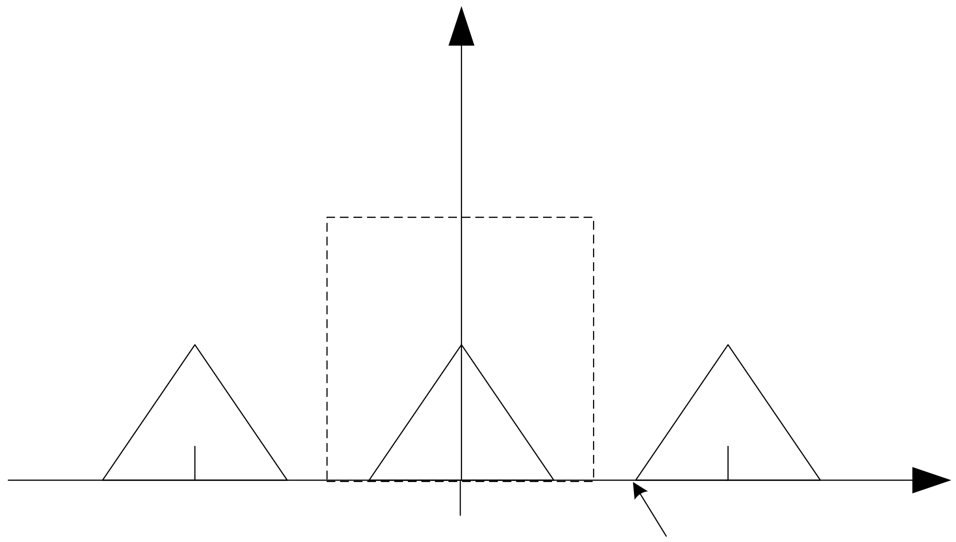


Figure 2 Relationship of $\mathrm{H}(\omega)$ and $X'(\omega)$

So in the frequency domain:

$$X_R(\omega) = X_S(\omega) \cdot H(\omega)$$

Then in time domain:

$$x_r(t) = x_s(t) * h(t)$$

For simplicity, set ω_c as the average of ω_b and $(\ \omega_s - \ \omega_b)$, that is:

$$\omega_c = \frac{\omega_s}{2} = \frac{\pi}{T_s}$$

Then:

$\mathrm{h}(t)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} T_s e^{j\omega t}\,d\omega = \frac{T_s}{2\pi}\cdot\frac{1}{jt}\cdot e^{j\omega t}\Big|_{-\pi/T_s}^{\pi/T_s} = \frac{T_s}{\pi t}\sin\frac{\pi t}{T_s} = sinc\left(\frac{t}{T_s}\right)$$

Where

$$\mathrm{sinc}(\mu) = {\sin(\pi\mu)}/{\pi\mu}.$$

So

$$x_r(t) = x_s(t) * h(t) = \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right) * h(t)$$

$$= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(nT_s)\delta(\tau - nT_s)h(t-\tau)\,d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)sinc\left(\frac{t - nT_s}{T_s}\right)$$

The interpolation formula can be verified at $\mathrm{t} = \mathrm{k}\Delta\mathrm{s}$:

$$x_r(k\Delta s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\frac{(k\Delta s - nT_s)}{T_s}$$

$$\mathrm{sinc}(k-n) = \frac{\sin\big((k-n)\pi\big)}{(k-n)\pi}$$

$$= \begin{cases} 0, & \mathrm{k} \neq \mathrm{n} \\ \lim\limits_{m\to 0}\dfrac{\sin(m\pi)}{m\pi} = \lim\limits_{m\to 0}\dfrac{\frac{d\sin(m\pi)}{dm}}{\frac{dm\pi}{dm}} = \lim\limits_{m\to 0}\dfrac{\pi\cos(m\pi)}{\pi} = 1, & \mathrm{k} = \mathrm{n} \end{cases}$$

So $x_r(k\Delta s) = x(nT_s)$, which aligns with $x_r(t) = x(t)$.

An example of sampling and reconstruction is given below.

Eg: x_samples is the sampled signal, x_time is the sample time and Ts is the sample interval.

```
dr = 0.5;
% original signal
clear;clf;
ds = 0.1;t = 0:ds:20;
F1 = 0.1;F2 = 0.2;
x = sin(2*pi*F1*t)+sin(2*pi*F2*t);
x_size = length(x);

% sampling
s = 10;    % sample rate = 1/ (ts*s)
Ts = ds*s;
x_samples = x(1:s:x_size); % gets samples of x.
x_time = (0:length(x_samples)-1)*Ts;

% Reconstruction Method 1: Follow the formula
x_recon=zeros(length(t),1);
for k=1:length(t)
   for n=1:length(x_samples)
       x_recon(k) = x_recon(k)+x_samples(n)*sinc(((k-1)*ds-(n-
1)*Ts)/Ts);
   end
end

% Reconstruction Method 2: Calculate sample by sample
x_recon = 0;
for n=1:length(x_samples)
   x_recon = x_recon+x_samples(n)*sinc((t-x_time(n))/Ts);
end
```