ShanghaiTech University

Spring 2018, EE111----Electric Circuit

Midterm 2

May.7th

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- This test contains 14 numbered pages, including the cover page, printed on both sides of the sheet.

- You have 120 minutes to complete this exam. The exam is closed book; no computers, phones, or calculators with programming, imagining, storage and other electronic devices are allowed.

- Please turn off all cell phones, smartwatches, and other mobile devices. Remove all hats and headphones. Put everything (except the calculator with basic function) in your backpack. Place your backpacks, laptops and jackets out of reach.

- There may be partial credit for incomplete answers; write as much of the solution as you can. We will deduct points if your solution is far more complicated than necessary.

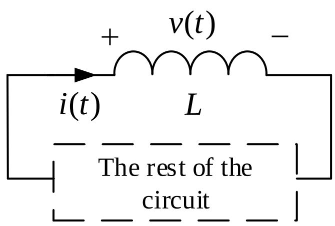
- Please prepare your submission in English only. **No Chinese submission will be accepted.**

Question	Points	Score	Viewer
1	15		
2	20		
3	15		
4	15		
5	15		
6	20		
Total	100		

1. Activation of a switch at the time $t=0$ in a certain circuit caused the voltage across a $L=20\,\text{mH}$ inductor to exhibit the voltage response:

$$v(t)=4e^{-0.2t}\text{mV} \quad t>0$$

Determine $i(t)$ for $t>0$, given the energy stored in the inductor at $t=\infty$ is 0.64mJ.



Solution:

First, according to the formula,

$$\begin{aligned} i(t) &= i(0)+\frac{1}{L}\int_0^t V(t)dt \\ &= i(0)+\frac{1}{20\times10^{-3}}\int_0^t 4e^{-0.2t}\times10^{-3}dt \\ &= i(0)+(1-e^{-0.2t}) \quad \text{(A)} \end{aligned}$$

At the time $t=\infty$, $i(\infty)=i(0)+1$ (A)

And we can get that:

$$w(\infty)=\frac{1}{2}Li^2(\infty)=0.64\times10^{-3}\,\text{J}$$

Therefore,

$$i(\infty)=\sqrt{\frac{2\times0.64\times10^{-3}}{20\times10^{-3}}}=0.25 \quad \text{(A)}$$

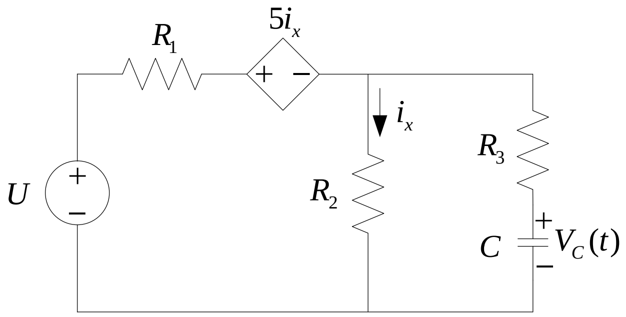
Hence,

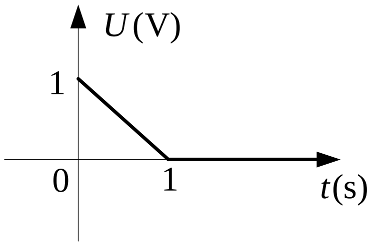
$$i(0)=i(\infty)-1=0.25-1=-0.75 \quad \text{(A)}$$

Thus,

$$\begin{aligned} i(t) &= i(0)+\frac{1}{L}\int_0^t V(t)dt \\ &= -0.75+(1-e^{-0.2t}) \\ &= 0.25-e^{-0.2t} \quad \text{(A)} \quad (t>0) \end{aligned}$$

2. In the circuit below, $R_1 = 10\ \Omega,\ R_2 = 5\ \Omega,\ R_3 = 10\ \Omega,\ C = 10\ \text{mF}$. When $t < 0$, the input voltage (U) is $1\,\text{V}$. When $t = 0$, the input voltage begins to change as shown in the plot below. When $t > 1\text{s}$, $U = 0$. Assume that the circuit reaches steady state before $t = 0$. Determine the expression for $V_C(t)$ when $t \geq 0$.





Solution:

Using KCL at the top node, we can get that: $\dfrac{V-(R_2 i_x + 5i_x)}{R_1} = i_x + i_0 \Rightarrow \dfrac{V-10i_x}{10} = i_x + i_0$.

And we can get the equation from the definition: $i_0 = C\dfrac{dV_C(t)}{dt} = 0.01\dfrac{dV_C(t)}{dt}$.

From the parallel property, we can also get that $R_2 i_x = R_3 i_0 + V_C(t) \Rightarrow 5i_x = 10i_0 + V_C(t)$.

Above all 3 equations, we can get that: $\dfrac{dV_C(t)}{dt} + 8V_C = 2V$.

Let's analyze the circuit first. When $t < 0$, the capacitor is the same as an open circuit. Thus, use KVL, $-1 + R_1 i_x + 5i_x + R_2 i_x = 0$. From this equation, we can get that $i_x = 0.05\text{A}$. Therefore, when $t < 0$, $V_C(t) = 0.05 \times 5 = 0.25\text{V} = V_C(0^+)$.

For $0 < t < 1\text{s}$, $\dfrac{dV_C(t)}{dt} + 8V_C = 2V = 2 - 2t$. Solving the equation, we can get that:

$V_C(t) = Ce^{-8t} + \dfrac{1}{4} - \dfrac{1}{32}(8t-1)$. As you know that $V_C(0^+) = 0.25\text{V}$. Therefore,

$V_C(0) = 0.25(\text{V}) = C + \dfrac{1}{4} + \dfrac{1}{32} \Rightarrow C = -\dfrac{1}{32} = -0.03125$.

Thus:

$$V_C(t) = -0.03125e^{-8t} - 0.25t + 0.28125 \quad (\text{V}) \quad (0 < t < 1s).$$

For $t > 1\text{s}$, $\dfrac{dV_C(t)}{dt} + 8V_C = 2V = 0$. Solving the equations, we can get that: $V_C(t) = Ce^{-8t}$. Since the voltage across the capacitor is continuous, thus:

$$V_C(1) = -0.03125e^{-8} - 0.25 + 0.28125 = 0.0312(\text{V}) .$$

Therefore, $V_C(1) = Ce^{-8} = 0.0312\text{V} \Rightarrow C = 93.123$.

Thus:

$$V_C(t) = 93.123e^{-8t} \quad (\text{V}) \quad (t > 1\text{s}) .$$

Above all, we can get that:

$$\boxed{V_C(t) = \begin{cases} -0.03125e^{-8t} - 0.25t + 0.28125, & (\text{V}) \quad (0 \le t \le 1s) \\ 93.123e^{-8t}, & (\text{V}) \quad (t > 1s) \end{cases}}$$

3. When the input to the circuit shown in the figure is the voltage source,

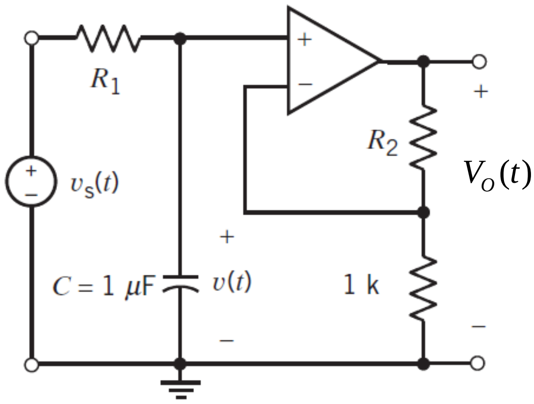
$$V_s(t) = 3 - u(t) \quad \text{V}$$

The output is the voltage

$$V_O(t) = 10 + 5e^{-50t} \quad \text{V}, \ t \ge 0.$$

Determine the values of R_1 and R_2.

(Assume that the circuit reached steady state before $t = 0$.)



Solution:

First we can get that $V_s = \begin{cases} 3\text{V}, t < 0 \\ 2\text{V}, t > 0 \end{cases}$.

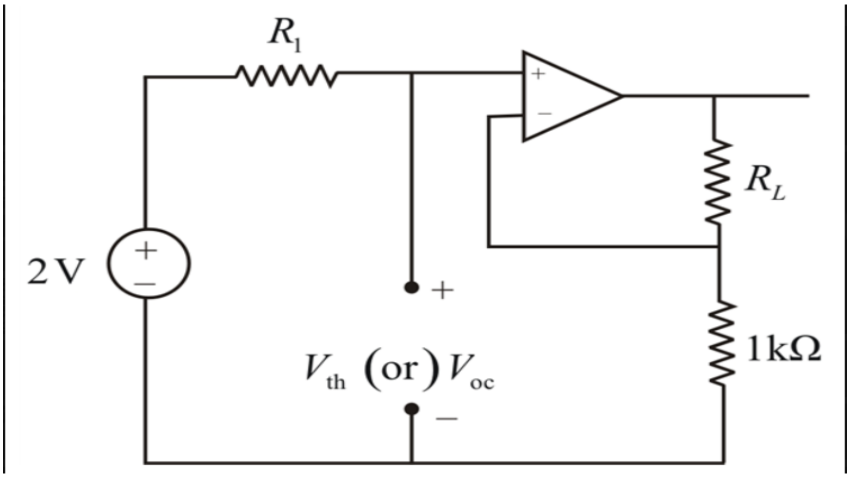
At the time $t < 0$, the circuit reached to steady state and capacitor becomes open circuited.

By applying KCL at the top node, $\dfrac{3 - V_A(0^-)}{R_1} = 0 \Rightarrow V_A(0^-) = 3\text{V}$.

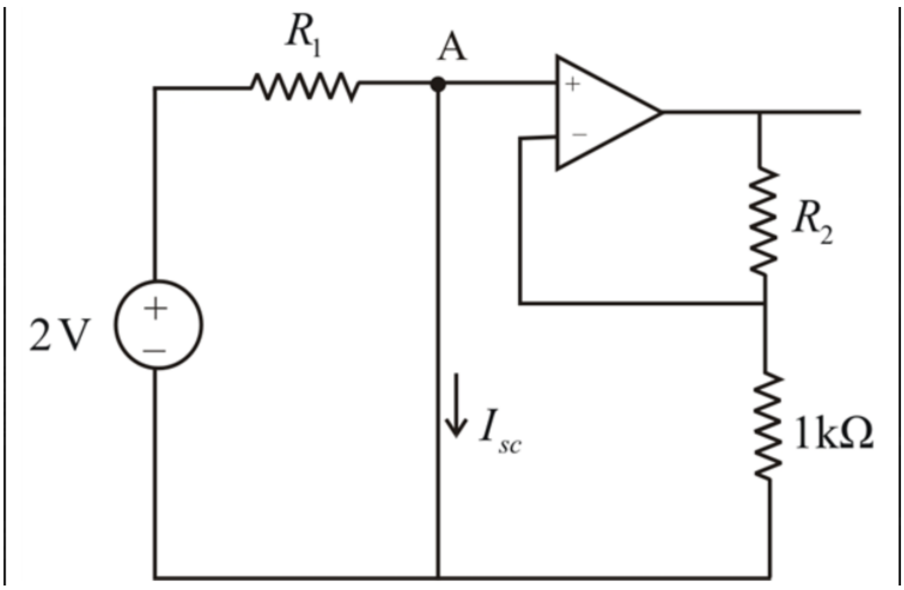
Because the capacitor voltage can not change instantaneously, $V_C(0^+) = 3\text{V}$.

To find the value of $V_C(t)$ first we need to find the value of Thevenin equivalent circuit across capacitor terminal.

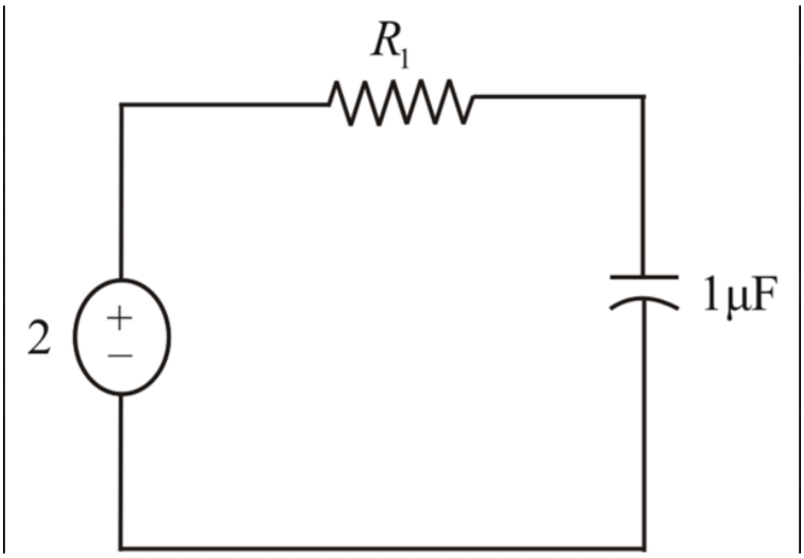
(1) Find V_{th}: Apply KCL at the top node, $\dfrac{2 - V_{th}}{R_1} = 0 \Rightarrow V_{th} = 2\text{V}$.



(2) Find I_{sc}: Apply KCL at the top node, $\dfrac{2 - V_A}{R_1} = I_{SC}, V_A = 0 \Rightarrow I_{SC} = \dfrac{2}{R_1}$ A.



(3) Find R_{th} : $R_{th} = \dfrac{2}{\dfrac{2}{R_1}} = R_1 \Omega$.



So the Thevenin circuit is below. In this RC circuit, $\tau = RC = 10^{-6} R_1$.

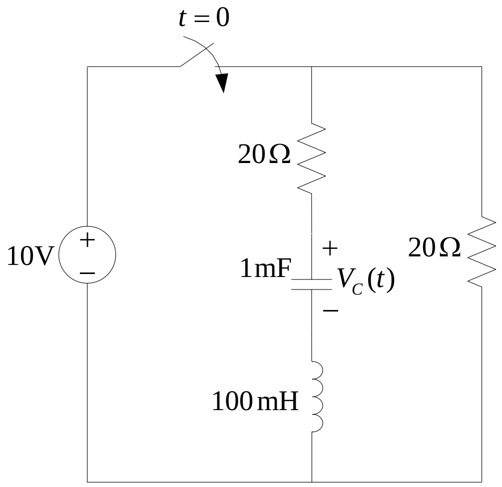
Therefore, the capacitor voltage is $V_C(t) = V_{th} + (V_C(0) - V_{th})e^{-\frac{t}{\tau}} = 2 + e^{-\frac{10^6 t}{R_1}}$ V.
From the given circuit diagram by voltage division rule and the golden rule,

$$V_C(t) = V_O(t)\frac{1000}{1000+R_2} \Rightarrow V_O(t) = \frac{1000+R_2}{1000}\left(2 + e^{-\frac{10^6 t}{R_1}}\right).$$

We are given $V_O(t) = 10 + 5e^{-50t}$ V, $t \geq 0$.

Therefore, $\boxed{R_1 = 20\text{k}\Omega, R_2 = 4\text{k}\Omega}$.

4. In the circuit below, the initial energy stored in the whole circuit is zero. At the time $t=0$, the switch is closed and a DC voltage source of $10\,\text{V}$ is applied to the circuit. Determine the expression of $V_C(t)$ for $t>0$.



Solution:

Suppose the current through the inductor is $I_C(t)$.

When $0<t<0.02\text{s}$, we can get that:

$10=20I_C(t)+V_C(t)+L\dfrac{dI_C(t)}{dt}$. And we can get that $I_C=C\dfrac{dV_C(t)}{dt}$.

Above the 2 equations, we can get that $LC\dfrac{d^2V_C(t)}{dt^2}+20C\dfrac{dV_C(t)}{dt}+V_C(t)=10$.

Thus, $s_1=s_2=-100$. Therefore, $V_C(t)=(B_1+B_2t)e^{-100t}+V(\infty)$.

From the question, we can get that $V(\infty)=10\text{V}$. And $V_C(0)=B_1+10=0$.

Therefore, $B_1=-10$.

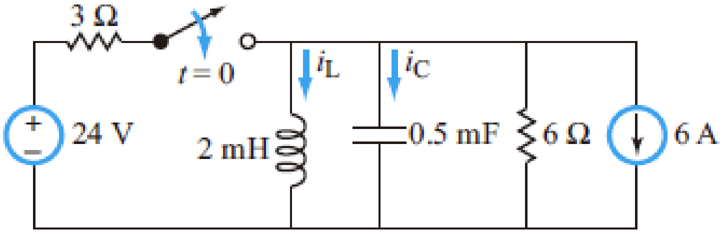
From the question,

$$I_C(0)=C\frac{dV_C(t)}{dt}=0.001(-100(B_1+B_2t)+B_2)e^{-100t}\Rightarrow B_2=-1000.$$

So, for $t>0$,

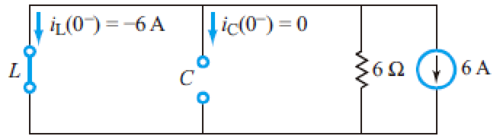
$$V_C(t)=(-10-1000t)e^{-100t}+10\quad(\text{V})$$

5. Determine $i_L(t)$ in the circuit for $t \geq 0$. Assume that the circuit reaches steady state before $t = 0$.



Solution:

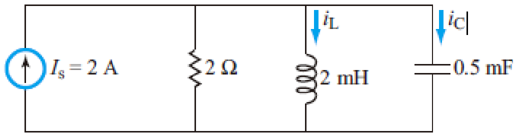
At $t = 0^-$, $i_C(0^-) = 0$, $i_L(0^-) = -6\text{A}$, $V_L(0^-) = V_C(0^-) = 0$. (Suppose i_L does not change its direction)



(a) At $t = 0^-$

At $t = 0^+$, $i_L(0^+) = i_L(0^-) = -6\text{A}$

$$\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{V_C(0^+)}{L} = 0$$



(b) At $t \geq 0$

After applying source transformation,

we can determine that: $\alpha = \dfrac{1}{2RC} = 500$, $\omega_0 = \dfrac{1}{\sqrt{LC}} = 1000\text{rad/s}$, $\alpha < \omega_0$.

Therefore, the circuit is underdamped. $\omega_d = \sqrt{\omega_0^{\ 2} - \alpha^2} = 866\text{rad/s}$.

For a parallel RLC circuit, we can get that $i(t) = I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$.

From the given information above, $i(t) = 2 + (A_1 \cos 866t + A_2 \sin 866t)e^{-500t}$.

First, $i_L(0^+) = -6 \Rightarrow 2 + (A_1) = -6 \Rightarrow A_1 = -8$.

Second, $\dfrac{di_L(0^+)}{dt} = 0 \Rightarrow -\alpha A_1 + \omega_d A_2 = 0 \Rightarrow A_2 = -4.62$

Therefore, $i(t) = 2 - (8\cos 866t + 4.62 \sin 866t)e^{-500t}$ A $(t > 0)$

6. In this basic RLC circuit, suppose the capacitor is $100\,\text{nF}$, $V = 100\text{V}$. Before $t = 0$, there is no energy stored in either the capacitor or the inductor. At $t = 0$, the switch is closed. When $t > 0$, the first time V_C exceeds (超过) $100\,\text{V}$, it reaches a peak of $176.82\,\text{V}$ at the time $t = \frac{\pi}{4}\,\text{ms}$. The second time voltage V_C exceeds 100V, it reaches a peak of $145.34\,\text{V}$ at the time $t = \frac{3\pi}{4}\,\text{ms}$.

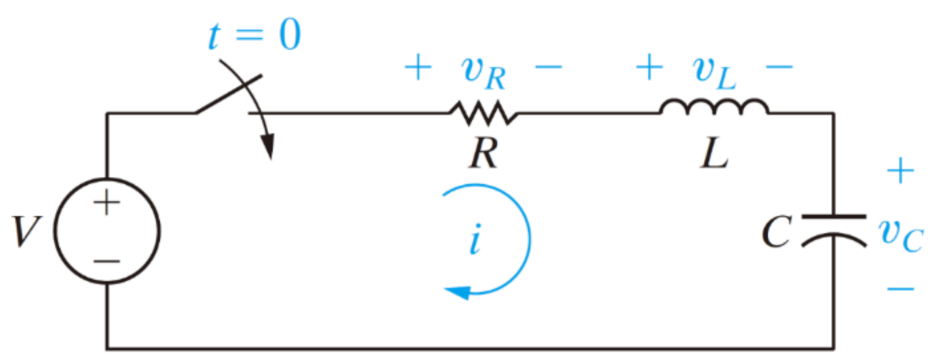
(1) Determine the circuit is overdamped, critically damped or underdamped.

(2) Choose three of the four parameters: $V,\ \alpha,\ \omega_0$ and ω_d, to express $V_C(t)$ as a function of t for $t > 0$. (Hint: The parameters $\alpha,\ \omega_0$ and ω_d are defined the same as what have been covered in class. Details are as follows.

The second-order differential equation that $V_C(t)$ should obey is: $\frac{d^2V_C(t)}{dt^2} + 2\alpha\frac{dV_C(t)}{dt} + \omega_0^2 V_C(t) = f(t)$, where $f(t)$ is a function of t and is determined by the circuit. If $\alpha < \omega_0$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.)

(3) Choose three of the four parameters: $V,\ \alpha,\ \omega_0$ and ω_d to express $\frac{dV_C(t)}{dt}$ as a function of t for $t > 0$.

(4) Find the numerical values of R and L.



Solution:

(1) The circuit is underdamped.

(2) For a basic underdamped RLC circuit,

$$V_C(t) = V + [B_1 \cos\omega_d t + B_2 \sin\omega_d t]e^{-\alpha t}$$

$$\frac{dV_C(t)}{dt} = [(\omega_d B_2 - \alpha B_1)\cos\omega_d t - (\alpha B_2 + \omega_d B_1)\sin\omega_d t]e^{-\alpha t}$$

First, we know that $V_C(0^+) = 0$.

Since the initial stored energy is zero, thus we can get that:

$$\frac{dV_C(0^+)}{dt} = i_C(0^+)C = i_L(0^+)C = 0$$

Therefore, we can solve that $B_1 = -V, B_2 = \frac{\alpha B_1}{\omega_d}$.

Thus,

$$\boxed{V_C(t) = V + \left[-V\cos\omega_d t - \frac{\alpha V}{\omega_d}\sin\omega_d t\right]e^{-\alpha t}}$$

(3) From the task (2), we can easily get that

$$\boxed{\frac{dV_C(t)}{dt} = \left(\frac{\alpha^2}{\omega_d} + \omega_d\right)Ve^{-\alpha t}\sin\omega_d t = \left(\frac{\omega_0^{\,2}}{\omega_d}\right)Ve^{-\alpha t}\sin\omega_d t}$$

(4) The peak or valley occurs if and only if $\frac{dV_C(t)}{dt} = 0$. This means that $\sin\omega_d t = 0$.

Therefore, $t = \frac{n\pi}{\omega_d}, n = 0, 1, 2, \cdots$. At the same time, we can get that the peak occurs at the time $t = \frac{n\pi}{\omega_d}, n = 1, 3, 5, \cdots$.

Using $\frac{n\pi}{\omega_d}$ to represent t_n, then we can get that $V_c(t_n) = V\left[1 - (-1)^n e^{-\frac{\alpha n\pi}{\omega_d}}\right]$.

For the first time V_C reaches peak, then we can know that

$$t_1 = \frac{\pi}{\omega_d} = \frac{\pi}{4}\times 10^{-3}\text{s},\ \ V_C(t_1) = V\left[1 + e^{-\frac{\alpha\pi}{\omega_d}}\right] = 176.82\text{V}$$

Similarly, for the second time V_C reaches peak, then we can know that

$$t_3 = \frac{3\pi}{\omega_d} = \frac{3\pi}{4} \times 10^{-3}\text{s},\quad V_C(t_3) = V\left[1 + e^{-\frac{3\alpha\pi}{\omega_d}}\right] = 145.34\text{V}$$

From the above, we can get that

$$\omega_d = 4000\text{rad/s}$$

And you can choose one of three methods to get the answer:

First methods:

$$\alpha = -\ln\left(\frac{V_C(t_1)}{V} - 1\right)\frac{\omega_d}{\pi} = 335.760$$

$$\omega_0^{\,2} = \omega_d^2 + \alpha^2 = (4000)^2 + (335.760)^2 = 16112734.67\text{rad}^2/\text{s}^2$$

$$\boxed{L = \frac{1}{\omega_0^2 C} = \frac{1}{(16112734.67)(0.1\times 10^{-6})} = 0.621\text{H},\quad R = 2\alpha L = 416.76\Omega}$$

Second methods:

$$\alpha = -\ln\left(\frac{V_C(t_3)}{V} - 1\right)\frac{\omega_d}{3\pi} = 335.703$$

$$\omega_0^{\,2} = \omega_d^2 + \alpha^2 = (4000)^2 + (335.703)^2 = 16112696.21\text{rad}^2/\text{s}^2$$

$$\boxed{L = \frac{1}{\omega_0^2 C} = \frac{1}{(16112696.21)(0.1\times 10^{-6})} = 0.621\text{H},\quad R = 2\alpha L = 416.69\Omega}$$

Third methods:

$$\alpha = \frac{1}{t_3 - t_1}\ln\frac{V_C(t_1) - V}{V_C(t_3) - V} = \frac{4000}{2\pi}\ln\frac{76.82}{45.34} = 335.674$$

$$\omega_0^{\,2} = \omega_d^2 + \alpha^2 = (4000)^2 + (335.674)^2 = 16112676.99\text{rad}^2/\text{s}^2$$

$$\boxed{L = \frac{1}{\omega_0^2 C} = \frac{1}{(16112676.99)(0.1\times 10^{-6})} = 0.621\text{H},\quad R = 2\alpha L = 416.66\Omega}$$

Note:

In this problem, students should **really understand** the plot of underdamped and the property in an underdamped circuit. In an underdamped circuit, the output voltage is an oscillation function, which has peaks and valleys. But after a long time, in basic RLC, the output voltage on the circuit is almost equal to the input voltage.

The peak is shown as in the figure pointed by green circles. (The figure may not be correct, but the shape is similar.)