EE150 Signals and Systems
– Part 3: Fourier Analysis for Continuous-Time Signal & System

$\downarrow$ Week 3, Thu, 20180315

Objective

- **Recall:**
  Previously, we use the weighted sum (integral) of shifted impulses to represent an input and then derive the convolution sum (integral).

★ In this chapter:
we use different basic signal, the complex exponential, to represent the input.

★ Why use complex exponential?

Eigen-function of LTI System

A signal for which the system output is just a constant (possibly complex) times the input is referred to as an eigenfunction of the system.

$$f(t) \longrightarrow \boxed{H} \longrightarrow C \cdot f(t)$$

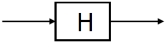
C:constant $\rightarrow$ the eigenvalue

Objective

The output to an input $x(t)$ can be found easily if $x(t)$ can be expressed as weighted sum of the eigenfunctions.

Eigen-function of LTI System

Consider e^{st}

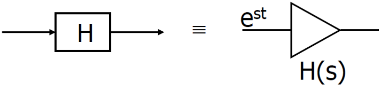


$$y(t) = \int h(\tau) e^{s(t-\tau)} d\tau$$
$$= e^{st} \int h(\tau) e^{-s\tau} d\tau$$

Let $\quad H(s) \equiv \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

$$\therefore y(t) = \underbrace{H(s)}_{\text{constant}} e^{st}$$

Thus $\quad e^{st}$



Hence, complex exponentials are eigenfunctions of LTI systems.

Remarks

1. $$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + \cdots$$
   $$\Rightarrow y(t) = a_1 e^{s_1 t} H(s_1) + a_2 e^{s_2 t} H(s_2) + \cdots$$
2. $H(s)$ is the eigenvalue of e^{st}
3. Purely imaginary exponential $e^{j\omega t} \equiv cos(\omega t) + jsin(\omega t)$

$$e^{j\omega t} \longrightarrow \boxed{} \longrightarrow H(j\omega)e^{j\omega t}$$

sinusoid

Sinusoid with the same frequency

Remarks cont.

The transfer function, $H(j\omega)$, is then
$H(j\omega) \equiv |H(j\omega)|\, e^{j\phi} \qquad \textit{with} \quad \phi \equiv \angle H(j\omega)$

The output is $\rightarrow \quad H(j\omega)e^{j\omega t} = |H(j\omega)|\, e^{j(\omega t+\phi)}$

Remarks cont.

1. Sinusoids are eigenfunctions for any LTI system.
2. $H(j\omega)$ characterizes the "frequency response" of a linear system.

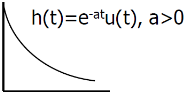
$$H(j\omega) \equiv \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$H(j\omega)$ is said to be the Fourier Transform of the time domain function $h(t)$.

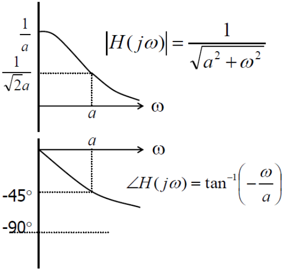
Both $h(t)$ and $H(j\omega)$ can be used to find the output for a particular input.

Remarks cont.

3. (a)



$$H(j\omega) \equiv \int_0^\infty e^{-(a+j\omega)\tau}\, d\tau = \frac{1}{a+j\omega}$$

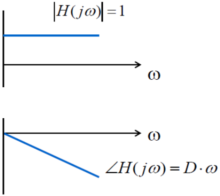


Remarks cont.

4. (b). Pure Delay

$h(t) = \delta(t-D) \quad D>0$

$H(j\omega) = e^{-j\omega D} \quad \Rightarrow$



Periodic signals & Fourier Series Expansion

Jean Baptiste Joseph Fourier
March 21 1768 - May 16 1830
Born Auxerre, France. Died Paris, France.

- Using “trigonometric sum” to describe periodic signal can be tracked back to Babylonians who predicted astronomical events similarly.
- L. Euler (in 1748) and Bernoulli (in 1753) used the “normal mode concept to describe the motion of a vibrating string; though JL Lagrange strongly criticized this concept.
- Fourier (in 1807) had found series of harmonically related sinusoids to be useful to describe the temperature distribution through body, and he claimed any periodic signal can be represented by such series.
- Dirichlet (in 1829) provide a precise condition under which a periodic signal can be represented by a Fourier series

Aside: an orthonormal set

- Consider the set, S, of $x(t)$ satisfying $x(t) = x(t + T_0)$
- Dot-product (inner-product) defined as

$$< x_1(t), x_2(t) >= \frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x_1(t) x_2^*(t) dt$$

- Consider the set, B, of functions in S

$$\mu_k(t) = e^{jk\omega_0 t};\ \omega_0 = \frac{2\pi}{T_0}, k \in \mathbb{N}$$

- Observe that they are orthonormal

$$\frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} e^{jk_1\omega_0 t} e^{-jk_2\omega_0 t} dt = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

Fourier's Idea

- The span of the orthonormal functions, B, covers most of S. i.e. $span(B) \approx S$
- More precisely, under mild assumptions: $x(t)$ is sum of sinusoids, i.e.

$$x(t) = \sum_k a_k e^{jk\omega_0 t}, \textit{where } a_k = \frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(\tau) e^{-jk\omega_0 \tau} d\tau$$

When is it valid

- Conditions for convergence of Fourier series is a deep subject
  - Forms the basis of harmonic analysis
  - Sufficient conditions for convergence at a point
    - If it is differentiable, or
    - If it does not fluctuate too much, i.e. finite number of extrema in a given interval

Check: `https://en.wikipedia.org/wiki/Dini_test`

$\uparrow$ Week 3, Thu, 20180315

$\downarrow$ Week 4, Tue, 20180320

Periodic signals & Fourier Series Expansion cont.

- Theorem (for reasonable functions):
  $x(t)$ may be expressed as a Fourier series:

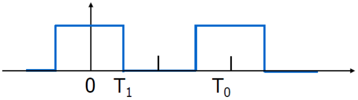
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

(sum of sinusoids whose frequencies are multiple of ω_0, the "fundamental frequency".)
Where a_k can be obtained by

$$a_k = \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau \quad \text{- } \textit{Fourier series coefficient.}$$

Note: $e^{jk\omega_0 t}$, for $k = -\infty$ to ∞, are orthonormal function. (Normal basic signal)

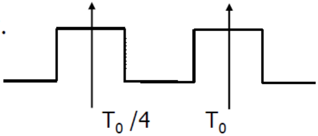
Example 1. Square Wave



When $T_1 = (1/4)T_0 \rightarrow$ 50% duty-cycle square wave

$$a_k = \frac{sin(k\omega_0 T_1)}{k\pi}, k \neq 0. \quad If \;\; T_1 = \frac{T_0}{4} \Rightarrow a_k = \frac{sin(\frac{k\pi}{2})}{k\pi}$$

Example 1. Square Wave cont.

$\therefore$ 

$$\equiv \frac{1}{2} + \frac{2}{\pi}\cos\omega_0 t - \frac{2}{3\pi}\cos 3\omega_0 t + \cdots$$

k	...	-5	-4	-3	-2	-1	0	1	2	3	4	5
a_k	...	$1/5\pi$	0	$-1/3\pi$	0	$1/\pi$	$1/2$	$1/\pi$	0	$-1/3\pi$	0	$1/5\pi$

Fourier Series

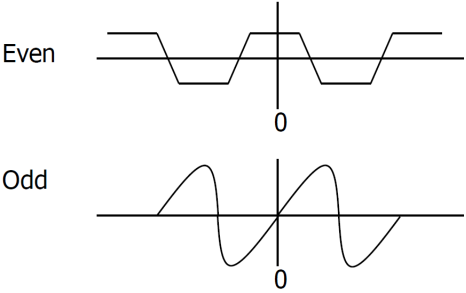
$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} (a_k cos(k\omega_0 t) + j a_k sin(k\omega_0 t)) \\
&= a_0 + \sum_{k>0} ((a_k + a_{-k}) cos(k\omega_0 t) + j(a_k - a_{-k}) sin(k\omega_0 t))
\end{aligned}
$$

In general, a_k is complex. Therefore, this is not the real and imaginary decomposition.

However, this is the even and odd decomposition

Fourier Series cont.

1. Odd/Even periodic functions



$\Rightarrow$ $a_k = a_{-k}$

$\Rightarrow$ all sine terms vanish

$\Rightarrow$ $a_k = -a_{-k}$

Fourier Series cont.

2. Approximation by Truncating Higher Harmonics

$$If \quad a_k (for\ |k| > N)\ are\ small, x(t) \approx \hat{x}(t) \equiv \sum_{-N}^{N} a_k e^{jk\omega_0 t}$$

The approximation error is

$$e(t) = x(t) - \hat{x}(t)$$

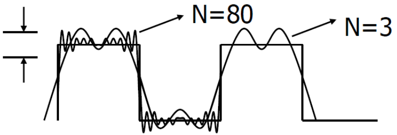
How good is the approximation?

- Relative mean square error metric

$$err = \frac{< e(t), e(t) >}{< x(t), x(t) >} = \frac{\int_{\frac{-\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} e(t)e^*(t)dt}{\int_{\frac{-\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(t)x^*(t)dt} = \frac{\sum_{|k|>N} |a_k|^2}{\sum_{|k|=-\infty}^{\infty} |a_k|^2}$$

Fourier Series cont.

e.g. square wave



Overshoot $\approx 9\%$ as N goes to ∞

Another metric for approximation: $max_t|e(t)/x(t)|$

Properties of Continuous-Time Fourier Series

Assume $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$.

$x(t)$ and its Fourier-series coefficients a_k are denoted by

$$x(t) \xleftrightarrow{FS} a_k$$

Assume $\quad x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k \quad$ (using same T_0)

1. Linearity: $z(t) = \alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha a_k + \beta b_k$
2. Time-shift: $\quad x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$

$\uparrow$ Week 4, Tue, 20180320

$\downarrow$ Week 4, Thu, 20180322

Properties of C-T Fourier Series cont.

3. Time-reverse:: $x(-t) \xleftrightarrow{FS} a_{-k}$
4. Time-scaling: $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$
5. Multiplication:
   $x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} \quad - Convolution!$

Properties of C-T Fourier Series cont.

6. conjugation & conjugate symmetry:

$$x^* \xleftrightarrow{FS} a^*_{-k}$$

$$\begin{aligned}
&\text{If } x(t) \text{ real} \quad && \rightarrow x(t) = x^*(t) \\
& && \rightarrow a_k^* = a_{-k} \\
&\text{If } x(t) \text{ is real and even,} && \\
& && \rightarrow a_k = a_{-k} = a_k^* \\
& && \rightarrow \text{Fourier Series are real \& even}
\end{aligned}$$

Properties of C-T Fourier Series cont.

7. $\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k \qquad \int x(t)dt \xleftrightarrow{FS} \frac{a_k}{jk\omega_0}$

8. Parseval's Identity: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Proof.

$$\begin{aligned}
\frac{1}{T}\int_T |x(t)|^2 dt &= \frac{1}{T}\int_T \sum_{k_1,k_2} a_{k_1} a_{k_2}^* e^{j(k_1-k_2)\omega_0 t} dt \\
&= \sum_{k_1,k_2} a_{k_1} a_{k_2}^* \delta(k_1 - k_2) \\
&= \sum_k |a_k|^2
\end{aligned}$$

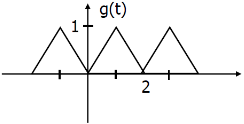
$\square$

Example:

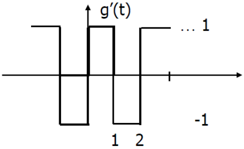
1. $x(t) = \cos \omega_0 t$

   $\rightarrow x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

   $\therefore a_0 = 0, a_1 = \frac{1}{2}, a_{-1} = \frac{1}{2}, a_k = 0 \quad otherwise$

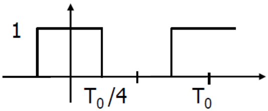
2. 

   $\rightarrow$

Example: cont.

Recall from the previous example, we have

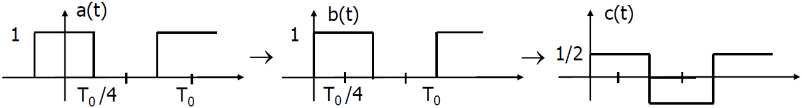


$\Rightarrow$

$$a_0 = \frac{1}{2}$$

$$a_k = \frac{\sin(k\omega_0 \cdot \frac{T_0}{4})}{k\pi} = \frac{\sin(\frac{k\pi}{2})}{k\pi}$$

By changing of variable, we can obtain



$$c(t) = a(t - \frac{T_0}{4}) - \frac{1}{2}$$

Example: cont.

Assume: $a(t) \xleftrightarrow{FS} a_k, \quad b(t) \xleftrightarrow{FS} b_k, \quad c(t) \xleftrightarrow{FS} c_k$

$$\Rightarrow c_k = \begin{cases} a_0 - \dfrac{1}{2}, & k = 0 \\ a_k e^{-jk\omega_0 \frac{T_0}{4}}, & k \neq 0 \end{cases} \quad \textit{where } \omega_0 = 2\pi \frac{1}{T_0}$$

$$\therefore c_k = \begin{cases} 0 \quad , & k = 0 \\ \dfrac{\sin(k\frac{\pi}{2})}{k\pi} \times e^{-j\frac{k\pi}{2}}, & k \neq 0 \end{cases}$$

Example: cont.

Assume: $g^{'}(t) \xleftrightarrow{FS} d_k$

$$\because g^{'}(t) = 2 \cdot c(t) \quad with \;\; T_0 = 2$$
$$\therefore d_k = 2 \cdot c_k$$

Assume: $g(t) \xleftrightarrow{FS} e_k$

$$\therefore e_k = \frac{d_k}{jk\omega_0} = \frac{2c_k}{jk\omega_0} = \frac{2c_k}{jk\pi \frac{2}{T_0}}$$

For $k \neq 0 \Rightarrow e_k = \frac{2\sin(\frac{k\pi}{2})}{j(k\pi)^2} e^{-j\frac{k\pi}{2}} \quad (T_0 = 2)$

For $k = 0 \quad e_0 = \frac{The \;\; area \;\; under \;\; g(t) \;\; in \;\; one \;\; period}{period} = \frac{1}{2}$

$\uparrow$ Week 4, Thu, 20180322

$\downarrow$ Week 5, Tue, 20180327

Fourier Series for Discrete-Time Periodic Signal

- Difference:
  1. continuous-time: infinite series
     discrete-time: finite series
  2. no convergence issue in discrete-time

★ Recall: $x[n]$ is periodic with period N if $x[n] = x[n+N]$

The fundamental period is the smallest positive N which satisfies the above eq.; and $\omega_0 = 2\pi/N$ is the fundamental frequency.

Aside-2: an orthonormal set

- Consider the set, T, of $x(n)$ satisfying $x(n) = x(n+N)$
- Dot-product (inner-product) defined as

$$< x_1(n), x_2(n) >= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) x_2^*(m)$$

- Consider the set, C, of N functions in T

$$\mu_k(n) = e^{jk\omega_0 n}; \omega_0 = \frac{2\pi}{N}, 0 \leq k \leq N-1$$

- Observe that they are orthonormal

$$\frac{1}{N} \sum_{m=0}^{N-1} e^{jk_1\omega_0 m} e^{-jk_2\omega_0 m} = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

Fourier Series for Discrete-Time Periodic Signal

Theorem

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}, \ \ where \ a_k = \frac{1}{N}\sum_{m=0}^{N-1} x[m] e^{-jk\omega_0 m}$$

Proof:

$$\begin{aligned}
& x[n] = \sum_{k=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} x(m) e^{-jk\omega_0 m} \right) e^{jk\omega_0 n}, \\
\Leftrightarrow \quad & x[n] = \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} e^{-jk\omega_0 m} e^{jk\omega_0 n}, \\
\Leftrightarrow \quad & x[n] = \frac{1}{N} \sum_{m=0}^{N-1} x(m) \delta(n - m - qN).
\end{aligned}$$

Fourier Series and LTI-System

- Recall: eigenfunction

$$x(t)=e^{st} \longrightarrow \boxed{H} \longrightarrow y(t)=H(s)\,e^{st}$$

where $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

$h(\tau)$: impulse response of the system

For discrete-time, similarly we have

$$x[n]=z^{n} \longrightarrow \boxed{H} \longrightarrow y[n]=H(z)\,z^{n}$$

where $H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k}$

Fourier Series and LTI-System cont.

$H(s)$ and $H(z)$ are referred to as "system function" (transfer functions).

$(C - T)$ continuous-time: $\quad Re\{s\} = 0 \rightarrow s = j\omega$

$(D - T)$ discrete-time:

$$|z| = 1 \rightarrow z = e^{j\omega} \Rightarrow \begin{cases} H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt,, & C - T, \\ H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}, & D - T. \end{cases}$$

$H(j\omega)$ and $H(e^{j\omega})$ are the "frequency response" of the continuous time and discrete-time system, respectively.

Fourier Series and LTI-System cont.

Note: For C.T., periodic signal, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

where $H(e^{jk\omega_0}) = H(s)|_{s=jk\omega_0}$. So, $y(t) = \sum_{k=-\infty}^{\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$

Similarly, $y[n] = \sum_{k=<N>}^{\infty} a_k H(e^{-j2\pi k/N}) e^{-jk(2\pi/N)n}$

Filter

- Filtering: A process that changes the relative amplitude (or phase) of some frequency components.

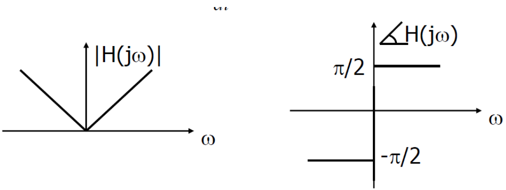
★ e.g.
  frequency-shaping filter
    (like equalizer in a Hi-Fi system)

  frequency-selective filter
    (like low-pass, band-pass, high-pass filters)

Frequency Shaping Filter

E.g. Differentiator (a HPF or LPF?)

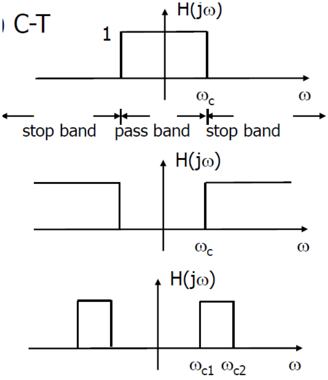
$$y(t) = \frac{dx(t)}{dt} \rightarrow H(j\omega) = j\omega$$



$\rightarrow$high frequency component is amplified & low frequency component is suppressed.$\rightarrow$HPF (high pass filter)
(a.k.a. as edge-enhancement filter in image processing)

Select some bands of frequencies and reject others.

1.



Ideal low-pass filter (LPF)

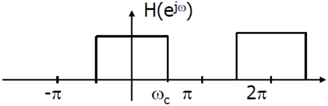
$$H(j\omega) = \begin{cases} 1 \text{ in pass band} \\ 0 \text{ in stop band} \end{cases}$$

Ideal high-pass filter (HPF)

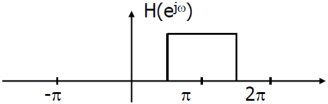
Ideal band-pass filter (BPF)

2. D-T

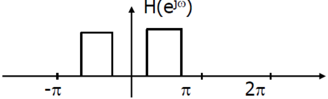
LPF



HPF



BPF



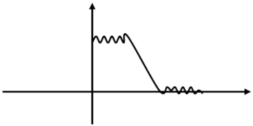
Frequency-Selective Filter cont.

For D-T: $H(e^{j\omega})$ is periodic with period 2π
low frequencies: at around $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$
high frequencies: at around $\omega \pm \pi, \pm 3\pi, \ldots$

Note: ideal filters are not realizable;

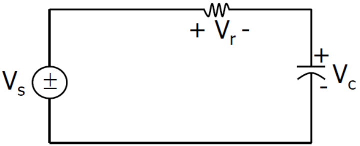
Practical filters have transition band, and may have ripple in stopband and passband

e.g. LPF



★Example

1. simple RC LPF



$$RC\frac{dV_c(t)}{dt} + V_c(t) = V_s(t)$$

$\therefore$ *input* : $V_s(t)$

$\therefore$ *output* : $V_c(t)$

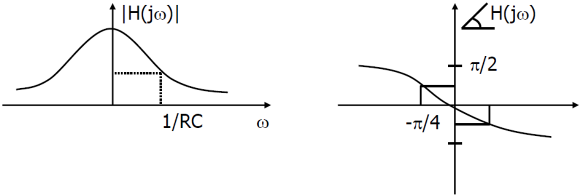
How to find $H(j\omega)$, the frequency response?
Note that, by definition, with $V_s(t) = e^{j\omega t}$, $V_c = H(j\omega)e^{j\omega t}$

Frequency-Selective Filter cont.

$$\therefore RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\therefore RC \cdot j\omega \cdot H(j\omega)e^{j\omega t} + H(jw)e^{j\omega t} = e^{j\omega t}$$

$$\rightarrow H(j\omega) = \frac{1}{1 + RCj\omega}$$



Frequency-Selective Filter cont.

$$\text{at } \omega = 0 \quad \Rightarrow |H(j\omega) = 1| \quad (max.)$$
$$\text{at } \omega \to \infty \quad \Rightarrow |H(j\omega) = 0| \quad (min.)$$
$$\therefore \text{ It is a low-pass filter.}$$

The impulse response: $h(t) = \frac{1}{RC} e^{-t/RC} \mu(t)$

step response: $\mu(t) = \left[1 - e^{-t/RC}\right] \mu(t)$

Frequency-Selective Filter cont.

★Example

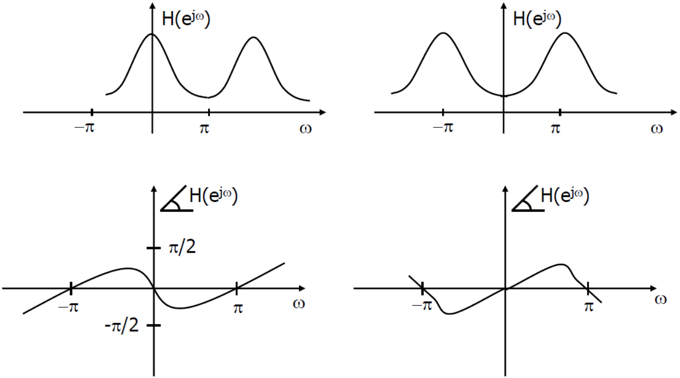
2. Discrete-time Filter described by Difference Equations.

   IIR filter: recursive, with infinite length impulse response
   FIR filter: non-recursive, with finite length impulse response

   1. IIR (Recursive)
      $y[n] - ay[n-1] = x[n]$
      we know if $x[n] = e^{j\omega n}$ then $y[n] = H(e^{j\omega})e^{j\omega n}$
      $\rightarrow H(e^{jw})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$
      $\rightarrow H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$ $\qquad (e.g.\ 3.7 - 3.9)$

Frequency-Selective Filter cont.

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$



(a) a=0.6 $\Rightarrow$ LPF

(b) a=-0.6 $\Rightarrow$ HPF

FIR

- FIR (Non-recursive)

General Form $\quad y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k] \qquad N, M \geq 0$

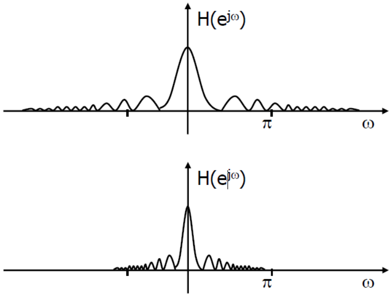
-average over N+M+1 neighboring points

-LPF

$$h[n] = \begin{cases} \dfrac{1}{N+M+1}, & for \; -N \leq n \leq M \\ \quad 0 \quad , & otherwise \end{cases}$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-j\omega k}$$

FIR cont.



a) N=M=16

b) N=M=32

So by adjusting the parameters $N\&M$, the passband can be adjusted.

Summary

- Developed Fourier series representation for both C-T and D-T systems.
- Properties of Fourier Series
- Eigenfunction and Eigenvalue of LTI systems
- System function (transfer function) and frequency response
- Filtering