Controllability of Linear Systems

- Reachable Sets

- Controllability

Contents

Reachable points

Let $x_0 \in \mathbb{R}^{n_x}$ be given. Consider

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{with} \quad x(0) = x_0 \; .$$

A point $x_T \in \mathbb{R}^{n_x}$ is called reachable from the point x_0 in time T, if there exists a control control input u with $u(t) \in \mathbb{U}$ such that $x(T) = x_T$.

Reachable sets

The set of reachable points at time $t \geq 0$ can be written in the form

$$S(t) = \left\{ G(t,0)x_0 + \int_0^t G(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau \,\middle|\, \forall \tau \in [0,t],\ u(\tau) \in \mathbb{U} \right\} .$$

Here, $G(t,\tau)$ denotes the fundamental solution and $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ the control constraint set.

- $S(t)$ can be interpreted as the set of all points $x_T \in \mathbb{R}^{n_x}$ to which we can steer the dynamic system.

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Example

For the scalar LTI system

$$\dot{x}(t) = ax(t) + bu(t)$$

with control bounds $\mathbb{U} = [-1, 1]$, the reachable set $S(t)$ is an interval,

$$S(t) = \left[e^{at}x_0 + \int_0^t e^{a(t-\tau)} b \, \mathrm{d}\tau \, , \, e^{at}x_0 - \int_0^t e^{a(t-\tau)} b \, \mathrm{d}\tau \right] \, .$$

Properties of reachable sets

1. If the set $\mathbb{U}$ is bounded, then the set $S(t)$ is for every given t bounded.
2. If the set $\mathbb{U}$ is point symmetric, then the set $S(t)$ is point symmetric.
3. If the set $\mathbb{U}$ is convex, then the set $S(t)$ is convex.
4. If the set $\mathbb{U}$ is convex and compact in $\mathbb{R}^{n_u}$, then the set $S(t)$ is convex and compact in $\mathbb{R}^{n_x}$.

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Reachable set for unconstrained linear systems

If $\mathbb{U} = \mathbb{R}^{n_u}$, the reachable sets of linear systems can be characterized explicitly (assume $x_0 = 0$).

- If $s \neq 0$ is in $S(t)$, then we have $\alpha s \in S(t)$.
- If $0 \neq s_1, s_2 \in S(t)$ can be reached with controls u_1, u_2, then $s_1 + s_2 \in S(t)$ can be reached with $u_1 + u_2$.

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Reachable set for unconstrained linear systems

Putting these two properties together, we know that $S(t)$ must be a vector space: there must exist a (potentially rank-deficient) matrix $P(t) \in \mathbb{R}^{n_x \times n_x}$ such that

$$S(t) = \{P(t)v \mid v \in \mathbb{R}^{n_x}\} \ .$$

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Controllability Grammian

Idea: show that the matrix

$$P(t) = \int_0^t G(t,\tau)B(\tau)B(\tau)^\mathsf{T}G(t,\tau)^\mathsf{T}\,\mathrm{d}\tau\ ,$$

has the desired properties.

Step 1: The point $P(t)v$ with $v \in \mathbb{R}^{n_x}$ can be reached using the input $u(\tau) = B(\tau)^\mathsf{T}G(t,\tau)^\mathsf{T}v$,

$$x(t) = \int_0^t G(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau = P(t)v$$

Thus, $S(t) \supseteq \{P(t)v \mid v \in \mathbb{R}^{n_x}\}$.

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Step 2: Assume that we can reach a point $s \notin \{P(t)v \mid v \in \mathbb{R}^{n_x}\}$. In this case we can find a vector c with $c^\intercal s \neq 0$ but $P(t)c = 0$,

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\begin{aligned}
0 < \left(c^\intercal s\right)^2 &= \left(\int_0^t c^\intercal G(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau\right)^2 \\
&\leq \left(\int_0^t c^\intercal G(t,\tau)B(\tau)B(\tau)^\intercal G(t,\tau)^\intercal c \mathrm{d}\tau\right) \int_0^T \|u(\tau)\|_2^2\,\mathrm{d}\tau \\
&= \underbrace{c^\intercal P(t) c}_{=0} \int_0^T \|u(\tau)\|_2^2\,\mathrm{d}\tau \;=\; 0\;,
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This is a contradiction. Thus, $S(t) \subseteq \{P(t)v \mid v \in \mathbb{R}^{n_x}\}$.

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Lyapunov differential equation

The matrix

$$P(t) = \int_0^t G(t,\tau)B(\tau)B(\tau)^{\mathsf{T}}G(t,\tau)^{\mathsf{T}}\,\mathrm{d}\tau\ ,$$

can alternatively be computed from the (inhomogeneous) Lyapunov differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^{\mathsf{T}} + B(t)B(t)^{\mathsf{T}} \quad \text{with} \quad P(0) = 0\ .$$

Controllability: definitions and properties

Lyapunov differential equation for the "Controllability Grammian" $P(t)$:

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^\mathsf{T} + B(t)B(t)^\mathsf{T} \quad \text{with} \quad P(0) = 0 \; .$$

- If $P(t)$ is positive definite for a $t > 0$, we can steer the system to any desired target point in $\mathbb{R}^{n_x}$. In this case, the system is called controllable (without control constraints).
- If $P(t) \succ 0$, then $P(t') \succ 0$ for all $t' \geq t$.
- If $P(t)$ is only positive definite, the set $\{P(t)v \mid v \in \mathbb{R}^{n_x}\}$ is called the controllable subspace of the system.

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