Nonconvex Optimization for Smart Grid

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Outline

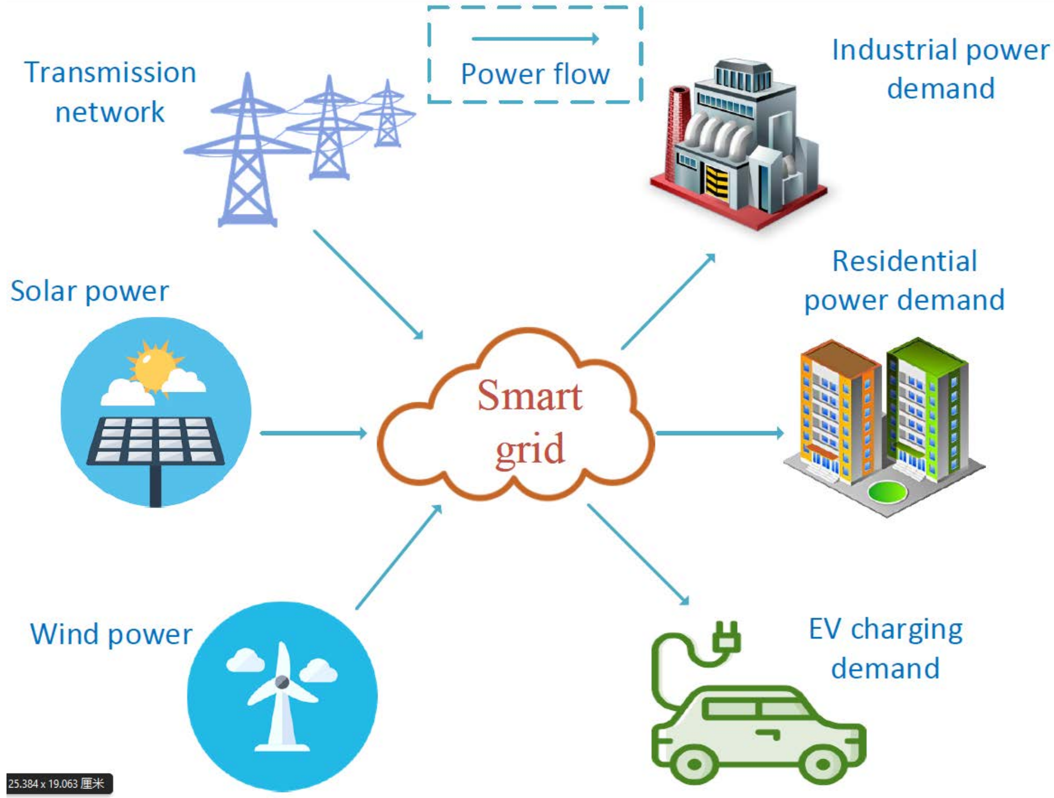
Smart Grid

Optimal Power flow (OPF), Large-scale OPF

EV charging, Demand response

PMU placement

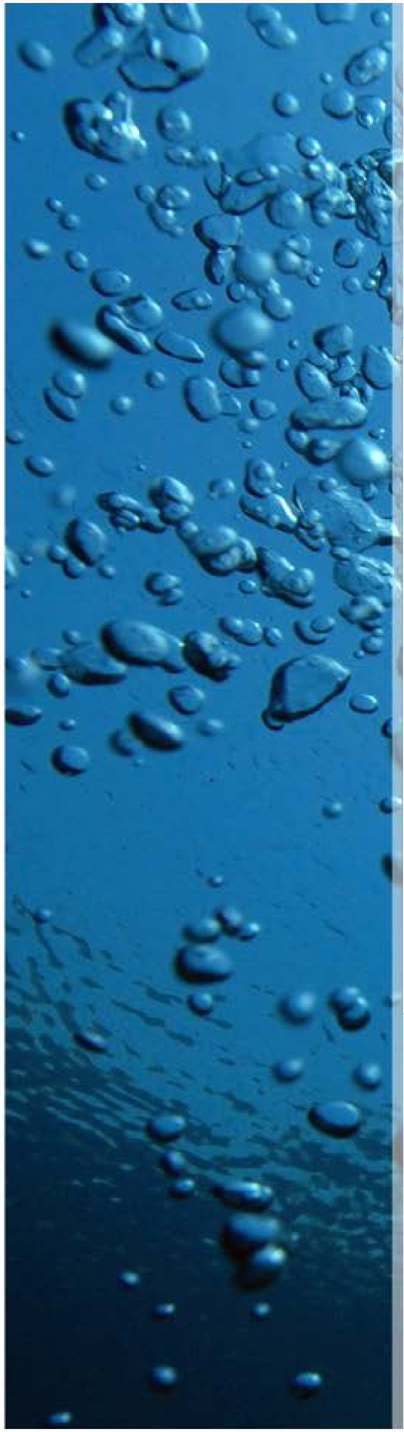
General structure of smart grid



The “smart grid” will save energy, reduce costs and increase reliability by delivering electricity from suppliers to consumers using two-way communication that can control appliances such as the charging of electric vehicles and the power flow from renewable sources at customers’ homes.

Potential Saving in Smart grid

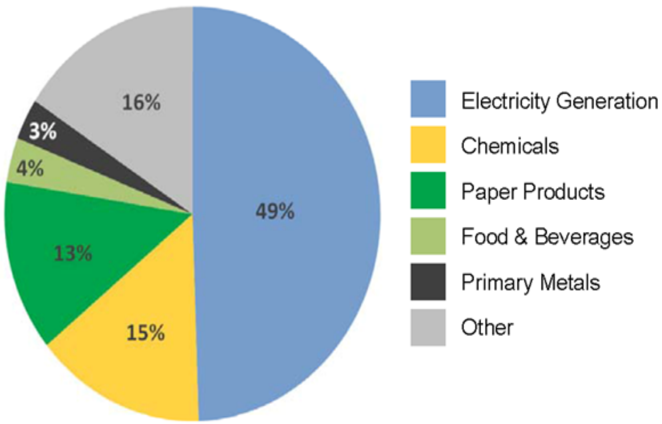
- A planned modernization of the U.S. national power grid will cost up to $476 billion over the next 20 years but will provide up to $2 trillion in customer benefits over that time, according to industry experts.



Environmental Urgency

Toxic Industrial Air Pollution in the U.S.

Toxic Air Pollution by Sector



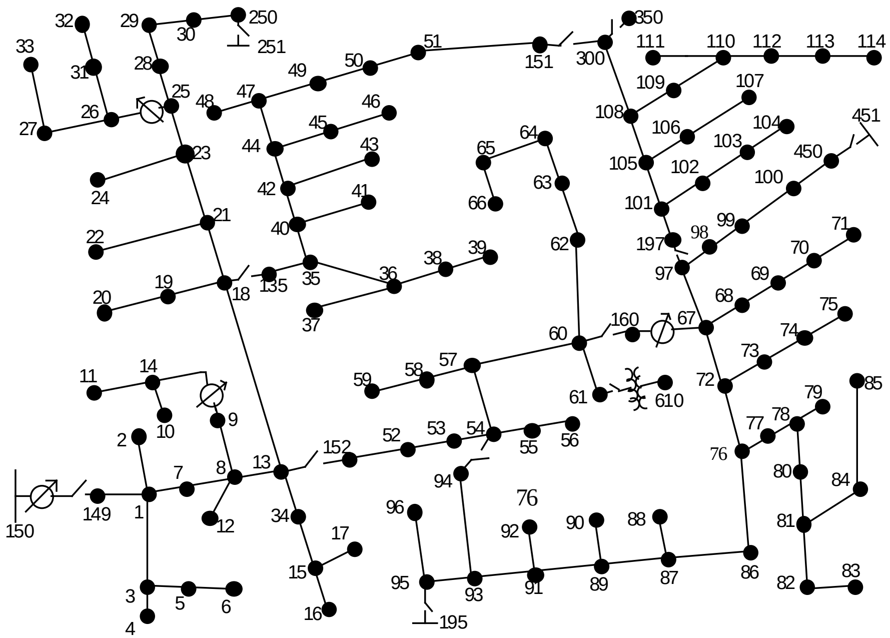
Sector	Toxic Air Pollution (lbs)	% of National Air Pollution
Electricity Generation	381,740,601	49%
Chemicals	112,870,057	15%
Paper Products	103,249,010	13%
Food & Beverages	26,908,977	3%
Primary Metals	24,923,246	3%
Other	121,888,815	16%
Total	**771,580,707**	**100%**

Half of the air pollutions come from electricity generation.

Optimal Power Flow

1. Optimal power flow problem in smart grid

To locate nodal buses' voltages for minimizing the cost of generation subject to operating constraints.



Mathematical formulation of the OPF

Objective function:

– Minimize total generating cost

$$
\begin{aligned}
f(V) \;=\; & \sum_{k\in\mathcal{G}} [c_{k2}(P_{L_k} + \Re(\sum_{m\in\mathcal{N}(k)} V_k V_m^* y_{km}^*))^2 \\
& + c_{k1}(P_{L_k} + \Re(\sum_{m\in\mathcal{N}(k)} V_k V_m^* y_{km}^*)) + c_{k0}].
\end{aligned}
$$

Equality constraints:

– Power balance at each node:

$$
-P_{L_k} - jQ_{L_k} = \sum_{m\in\mathcal{N}(k)} V_k V_m^* y_{km}^*, \forall k \in \mathcal{N} \setminus \mathcal{G}
$$

Mathematical formulation of the OPF

- Inequality constraints:
  - Limits on active and reactive power at each generator:

$$P_{G_k}^{min} \leq P_{L_k} + \Re\Big(\sum_{m \in \mathcal{N}(k)} V_k V_m^* y_{km}^* \Big) \leq P_{G_k}^{max}, \forall k \in \mathcal{G}$$

$$Q_{G_k}^{min} \leq Q_{L_k} + \Im\Big(\sum_{m \in \mathcal{N}(k)} V_k V_m^* y_{km}^* \Big) \leq Q_{G_k}^{max}, \forall k \in \mathcal{G}$$

Mathematical formulation of the OPF

- Inequality constraints:
  - Limits on voltage at each node:

$$V_k^{min} \leq |V_k| \leq V_k^{max}, \forall k \in \mathcal{N}$$

$$|\arg(V_k) - \arg(V_m)| \leq \theta_{km}^{max}, \forall (k,m) \in \mathcal{L}$$

Indefinite Quadratic Formulation

$$\min_{V \in \mathbb{C}^n} f(V) \quad \text{s.t.} \quad (5a)$$

$$-P_{L_k} - jQ_{L_k} = \sum_{m \in \mathcal{N}(k)} V_k {V_m}^* y^*_{km}, \forall k \in \mathcal{N} \setminus \mathcal{G} \; (5b)$$

$$P^{min}_{G_k} \leq P_{L_k} + \Re\Big(\sum_{m \in \mathcal{N}(k)} V_k {V_m}^* y^*_{km}\Big) \leq P^{max}_{G_k}, \forall k \in \mathcal{G} \; (5c)$$

$$Q^{min}_{G_k} \leq Q_{L_k} + \Im\Big(\sum_{m \in \mathcal{N}(k)} V_k {V_m}^* y^*_{km}\Big) \leq Q^{max}_{G_k}, \forall k \in \mathcal{G} \; (5d)$$

$$V^{min}_k \leq |V_k| \leq V^{max}_k, \forall k \in \mathcal{N} \; (5e)$$

$$|S_{km}| = |V_k {V_m}^* y^*_{km}| \leq S^{max}_{km}, \forall (k,m) \in \mathcal{L} \; (5f)$$

$$|V_k - V_m| \leq V^{max}_{km}, \forall (k,m) \in \mathcal{L} \; (5g)$$

$$|\arg(V_k) - \arg(V_m)| \leq \theta^{max}_{km}, \forall (k,m) \in \mathcal{L} \; (5h)$$

It is obvious that (5) is minimization of nonconvex objective function over quadratic equality constraints (5b) and (nonconvex) indefinite quadratic constraints (5c)-(5h).

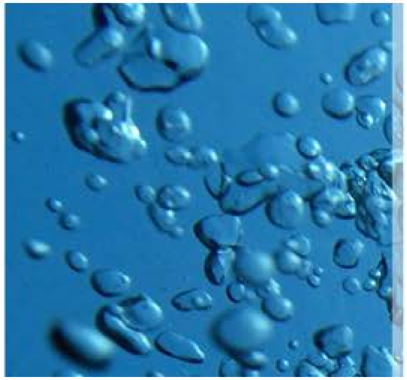
Introduce new variable

Define the Hermitian symmetric matrix of outer product:

$$W = VV^H \in \mathbb{C}^{n \times n}$$

which must satisfy

$$W \succeq 0 \text{ and } \mathsf{rank}(W) = 1$$



Recast to linear constraints + rank-one constraint

$$\min_{W \in \mathbb{C}^{n \times n}} \quad F(W) \quad \text{s.t.} \quad (6a)$$

$$-P_{L_k} - jQ_{L_k} = \sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^* \; \forall k \in \mathcal{N} \setminus \mathcal{G}, \quad (6b)$$

$$P_{G_k}^{min} \leq P_{L_k} + \Re\Big(\sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^*\Big) \leq P_{G_k}^{max}, \forall k \in \mathcal{G} \quad (6c)$$

$$Q_{G_k}^{min} \leq Q_{L_k} + \Im\Big(\sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^*\Big) \leq Q_{G_k}^{max}, \forall k \in \mathcal{G} \quad (6d)$$

$$(V_k^{min})^2 \leq W_{kk} \leq (V_k^{max})^2, \forall k \in \mathcal{N} \quad (6e)$$

$$|W_{km} y_{km}^*| \leq S_{km}^{max}, \forall (k,m) \in \mathcal{L} \quad (6f)$$

$$W_{kk} + W_{mm} - W_{km} - W_{mk} \leq (V_{km}^{max})^2, \forall (k,m) \in \mathcal{L} \quad (6g)$$

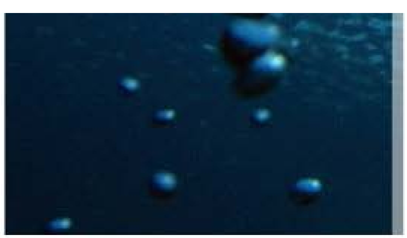
$$\Im(W_{km}) \leq \Re(W_{km}) \tan \theta_{km}^{max}, \forall (k,m) \in \mathcal{L} \quad (6h)$$

$$W \succeq 0, \quad (6i)$$

$$\text{rank}(W) = 1, \quad (6j)$$

Dropping the rank-one constraint to treat OPF as a semi-definite program.

- When the optimal solution is not of rank-one, it has no physical meaning.
- Even when there is an optimal rank-one solution, the existing software often output rank-more-than-one solution.



Our Method

We express the discrete rank-one constraint by d.c. constraint

rank(W) = 1 + $W \geq 0$ = $\text{Trace}(W) = \lambda_{max}(W)$

As Trace(W) ≥ $\lambda_{max}(W)$ is always, the rank-one constraint is just the reverse convex constraint

$$\lambda_{max}(W) - \text{Trace}(W) \geq 0$$

Then we incorporate the reverse convex constraint into the objective to consider the penalized optimization.

Exact penalized optimization

$$\min\ F(W) + \mu(\mathrm{Trace}(W)\text{-}\lambda_{max}(W))$$

s.t. linear constraints

- F(W) is convex in W, so the above objective is a d.c. function (but nonsmooth).

- As the problem is high dimensional, it is challenging to use global search methods for computation.

Exact penalized optimization

Go to local search iterations: initialized by point $W^{(0)}$ of the SDR problem, at the κ –th iteration to solve the SDP

$$\min \ F(W) + \mu(\mathrm{Trace}(W) - {w_k}^H W w_k)$$
$$\text{s.t.} \quad \text{linear constraints}$$

By doing this, the rank of W^{κ} will tend to 1, yielding an optimal solution of OPF

How to know the found solution is global

Network	Lower Bound	Found Value
WB5	946.53	946.58
Case9	5305.5	5305.68
Case14	8088.71	8088.71
Case30	574.87	574.87
Case57	41,213.99	41,313.72
Case118mod	129,682.50	129,686.03

Practical global optimal solutions found.

Large-scale
Optimal Power Flow

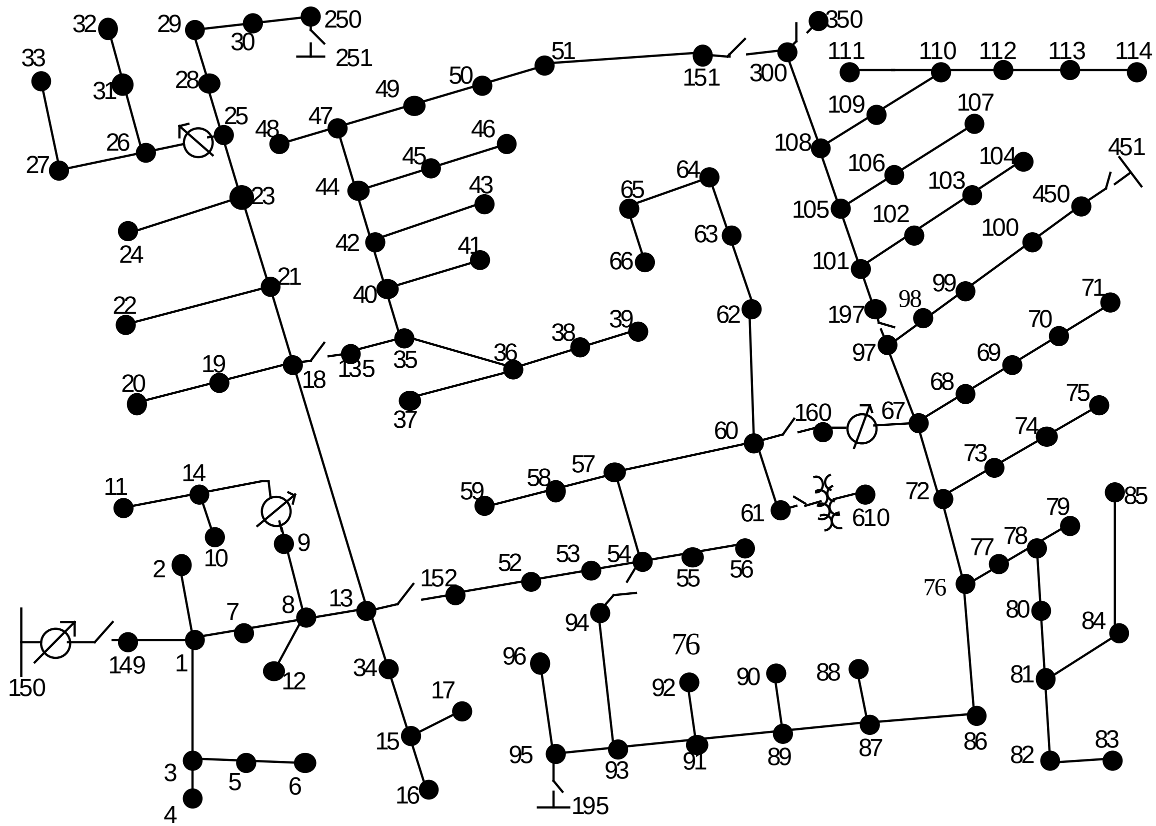
2. OPF for large scale smart grid

- For large scale algorithm, there are more than 2000 nodes, i.e. the dimension of V is more than 2000 and the dimension of W=VV^H is more that 2,000,000.

SDP solvers are unable to address.

- Fortunately, large scale grids' connection are sparse.
- It is impossible to design large scale grids with many connections.

IEEE 123 nodes test feeder



What does the sparseness mean?

$$\min_{W \in \mathbb{C}^{n \times n}} \quad F(W) \quad \text{s.t.} \quad W \succeq 0 \tag{4a}$$

$$-P_{L_k} - jQ_{L_k} = \sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^*, \; k \in \mathcal{N} \setminus \mathcal{G} \tag{4b}$$

$$P_{G_k}^{min} \le P_{L_k} + \Re\left(\sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^* \right) \le P_{G_k}^{max}, k \in \mathcal{G} \tag{4c}$$

$$Q_{G_k}^{min} \le Q_{L_k} + \Im\left(\sum_{m \in \mathcal{N}(k)} W_{km} y_{km}^* \right) \le Q_{G_k}^{max}, k \in \mathcal{G} \tag{4d}$$

$$(V_k^{min})^2 \le W_{kk} \le (V_k^{max})^2, k \in \mathcal{N} \tag{4e}$$

$$|W_{km} y_{km}^*| \le S_{km}^{max}, (k,m) \in \mathcal{L} \tag{4f}$$

$$W_{kk} + W_{mm} - W_{km} - W_{mk} \le (V_{km}^{max})^2, (k,m) \in \mathcal{L} \tag{4g}$$

$$\Im(W_{km}) \le \Re(W_{km}) \tan \theta_{km}^{max}, (k,m) \in \mathcal{L} \tag{4h}$$

$$\text{rank}(W) = 1, \tag{4i}$$

Sparseness means that there are not so many W_{km}.

Bags Decomposition

- Decompose the set of buses' voltages $\mathcal{N} := \{V_1, \dots, V_n\}$ into I possibly overlapped subsets $\mathcal{N}_i = \{V_{i_1}, \dots, V_{i_{\mathrm{N}_i}}\}$ of connected buses called by bags of voltages.

- Reset bags such that they are of similar sizes.

New rank-one formulations

- Define

$$V_{\mathrm{N}_i} = [V_{i_1}, \dots, V_{i_{\mathrm{N}_i}}]^H$$
$$W^i = V_{\mathrm{N}_i}(V_{\mathrm{N}_i})^H$$

- Entries of W^i may be overlapped

- Replacing $W^i_{km} = V_{i_k}V^H_{i_m}$, we have the following equivalent optimization reformulation

$$\min \; F(W) \;\; s.t.(3b) - (3h), rank\left(W^i\right) = 1$$

Advantages and challenges

- Advantages: Each W^i is of moderate dimension and the total dimension of all W^i is dramatically reduced compared with W.
- Challenges: many rank-one constraints (many d.c. constraints), not easy to handle.

For example, consider

$$\min F(W) + \mu \sum_{i=1}^{N} \left(Trace\left(W^i\right) - \lambda_{max}\left(W^i\right)\right)$$

and use iterations:

$$\min F(W) + \mu \sum_{i=1}^{N} \left(Trace\left(W^i\right) - (w_{max}^i)^H W^i w_{max}^i\right)$$

It is difficult to get all W^i of rank-one.

Large Scale OPF Iteration

- After reaching rank-one how to keep W^i of rank-one?

$$\text{Trace}(W^i) - (w_{\max}^{i,(\kappa)})^H W^i w_{\max}^{i,(\kappa)} \leq \epsilon_{tol}, i \in \mathcal{L}^{(\kappa)}$$

- Iteration

$$\min_{W=\text{diag}\{W^i\}} F(W) + \mu \sum_{i \notin \mathcal{L}^{(\kappa)}} [\text{Trace}(W^i)$$
$$-(w_{\max}^{i,(\kappa)})^H W^i w_{\max}^{i,(\kappa)}] \quad \text{s.t.} \quad (4b)-(4h), (6b),$$

Simulation results

Global optimality tolerance (GOT) of its found solution defined as:

$$GOT = \frac{\text{found value} - \text{lower bound}}{\text{lower bound}}$$

System	Lower bound	Found value	GOT
Polish-2383wp	1.8490 E6	1.8408 E6	4.3267 E-4
Polish-2736sp	1.3041 E6	1.3042 E6	7.6681 E-5
Polish-2737sop	7.7571 E5	7.7572 E5	1.2891 E-5
Polish-2746wop	1.2039 E6	1.2040 E6	8.3063 E-5
Polish-2746wp	1.6266 E6	1.6266 E6	6.1478 E-7
Polish-3012wp	2.5717 E6	2.5727 E6	3.8885 E-4
Polish-3120sp	2.1314 E6	2.1391 E6	3.6321 E-3

Global optimal solution found.

Joint OPF and EV charging

3. **Joint OPF and EV charging**

Motivations:

-Electrical vehicles (EVs) as a promising solution to resolve both the economic and environmental concerns in the transportation industry.

-Using a smart power grid to serve residences and charge EVs constitutes one of the most important applications of the smart grid technology.

-The massive integration of plug-in EVs (PEVs) into the grid causes many potential impacts such as voltage deviation, increased load variations and power loss of the grid.

China:

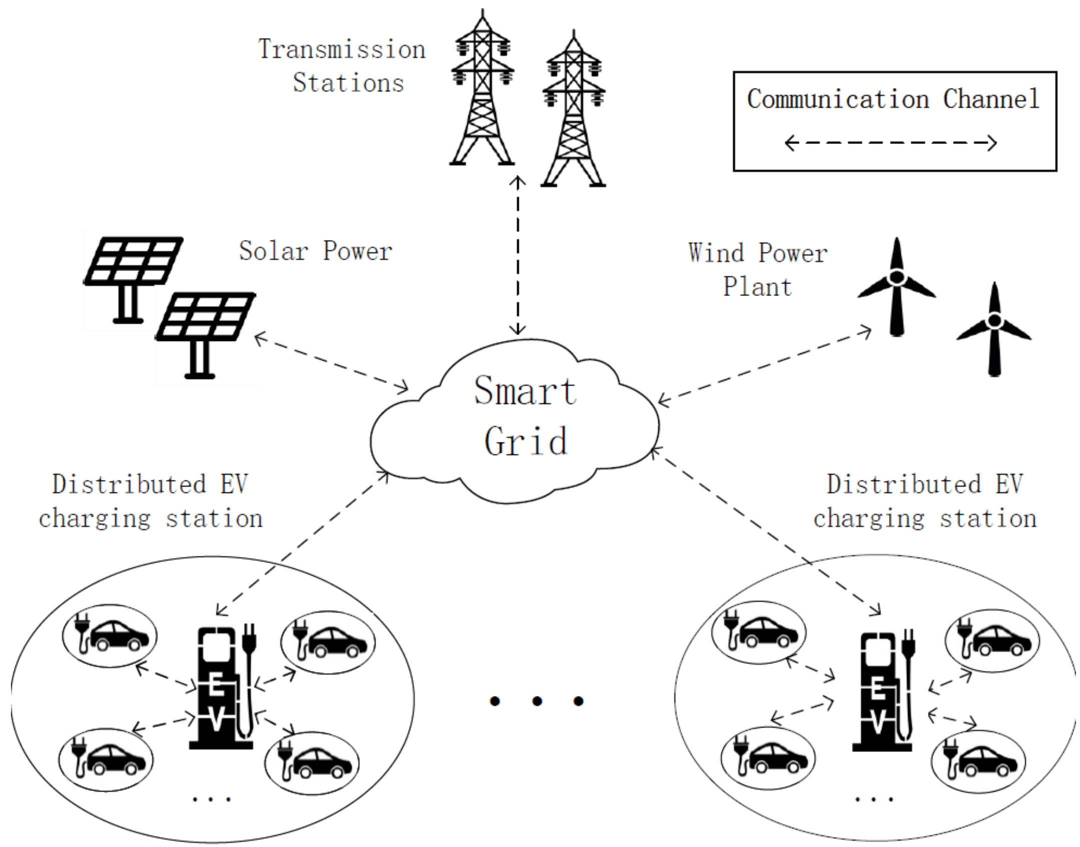
By 2025, New buses and Heavy-duty trucks will be fully electrified.

By 2030, EVs will be fully popularized.

Europe：

By 2040, No new diesel and petrol vehicles permitted.

Structure


Transmission
Stations
Communication Channel
Solar Power
Wind Power
Plant
Smart
Grid
Distributed EV
charging station
Distributed EV
charging station
EV
EV

Joint OPF-PEV Charging

Objective

- Generation costs
- PEV charging costs

Constraints

- Grid operation constraints
- PEV charging constraints

New challenges

- EVs arrive and depart randomly at different CSs.
- The arrival and departure information of EVs are not known in advance. So the conventional model predictive control (MPC) is not applicable.
- Our contribution is to develop a novel MPC-based approach to address this problem.

Objective

$$F(\mathcal{R}, \mathcal{P}^{PEV}) = \sum_{t'\in\mathcal{T}} \sum_{k\in\mathcal{G}} f(P_{g_k}(t')) + \sum_{t'\in\mathcal{T}} \sum_{k\in\mathcal{N}} \sum_{n\in\mathcal{H}_k} \beta_t P_{k_n}(t'),$$

where $f(P_{g_k}(t'))$ is the cost function of real power generation by DGs, which is linear or quadratic in $P_{g_k}(t')$, and β_t is the known PEV charging price during the time interval $(t', t'+1]$.

Dynamic PEV charging

$$\sum_{t'=t_{k_n,a}}^{t_{k_n,d}} u_h P_{k_n}(t') = C_{k_n}(1 - s_{k_n}^0), \qquad (2)$$

where u_h is the charging efficiency of the battery and $P_{k_n}(t')$ is a decision variable representing the power charging rate of PEV $k_n \in \mathcal{H}_k$ at time t'.

Power balance

$$V_k(t')\Big(\sum_{m\in\mathcal{N}(k)} y_{km}(V_k(t') - V_m(t'))\Big)^* = (P_{g_k}(t') - P_{l_k}(t') - \sum_{n\in\mathcal{H}_k} P_{k_n}(t')) + j(Q_{g_k(t')} - Q_{l_k}(t')), k \in \mathcal{G}.$$

Similarly,

$$V_k(t')\Big(\sum_{m\in\mathcal{N}(k)} y_{km}(V_k(t') - V_m(t'))\Big)^* = -P_{l_k}(t') - jQ_{l_k}(t'), k \notin \mathcal{G}.$$

Prediction horizon

For each $k_n \in C(t)$, let $\mathcal{P}_{k_n}(t)$ be its remaining demand for charging by the departure time $t_{k_n,d}$. Define

$$\Psi(t) = \max_{k_n \in C(t)} t_{k_n,d}. \tag{12}$$

At time t we solve the optimal power flow (OPF) problem over the prediction horizon [t, Ψ(t)] but then take only V (t), P(t), R(t) for online updating solution.

Joint OPF-PEV charging formulation

$$\min_{V(t'),R(t'),P_{k_n}(t'),t'\in[t,\Psi(t)],k_n\in C(t)} F_{[t,\Psi(t)]} \quad (13a)$$

$$\text{s.t.} \quad \text{network balance and bound constraints in} \quad [t,\Psi(t)], \quad (13b)$$

$$V_k(t')\Big(\sum_{m\in\mathcal{N}(k)} y_{km}(V_k(t')-V_m(t'))\Big)^* = (P_{g_k}(t') -$$

$$P_{l_k}(t') - \sum_{k_n\in C(t)} P_{k_n}(t')) + j(Q_{g_k}(t') - Q_{l_k}(t')), \quad (13c)$$

$$\sum_{t'=t}^{t_{k_n,d}} u_h P_{k_n}(t') = \mathcal{P}_{k_n}(t), \quad (13d)$$

Joint OPF-PEV charging Re-cast

$$\min_{W(t'),R(t'),P_{k_n}(t'),t'\in[t,\Psi(t)],k_n\in C(t)} F_{[t,\Psi(t)]} \quad (14a)$$

$$\text{s.t.} \quad \text{network balance and bound constraints in} \quad [t,\Psi(t)], \quad (14b)$$

$$\sum_{m\in\mathcal{N}(k)} (W_{kk}(t') - W_{km}(t'))y^*_{km} = (P_{g_k}(t') - P_{l_k}(t') - \sum_{k_n\in C(t)} P_{k_n}(t')) + j(Q_{g_k}(t') - Q_{l_k}(t')), \quad k\in\mathcal{G}, \quad (14c)$$

$$\sum_{m\in\mathcal{N}(k)} (W_{kk}(t') - W_{km}(t'))y^*_{km} = -P_{l_k}(t') - jQ_{l_k}(t'), k\notin\mathcal{G}, \quad (14d)$$

$$\underline{V}_k^2 \le W_{kk}(t') \le \overline{V}_k^2, \quad k\in\mathcal{N}, \quad (14e)$$

$$\Im(W_{km}(t')) \le \Re(W_{km}(t'))\tan(\theta_{km}^{max}), (k,m)\in\mathcal{L}, \quad (14f)$$

$$|(W_{kk}(t') - W_{km}(t'))y^*_{km}| \le S_{km}, \quad (14g)$$

$$W(t') \succeq 0, \quad (14h)$$

$$\text{rank}(W(t')) = 1. \quad (14i)$$

Our method

Relax the rank constraint (14i), we solve the SDR:

$$\min_{W(t'),R(t'),P_{k_n}(t')} F_{[t,\Psi(t)]} \quad \text{s.t. } (14b)-(14h). \qquad (15)$$

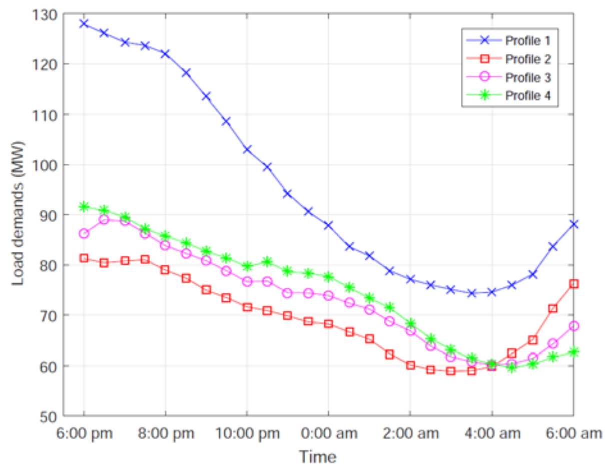
If the solution of W is not rank-one, solve the following problem by previous algorithm for OPF at time t.

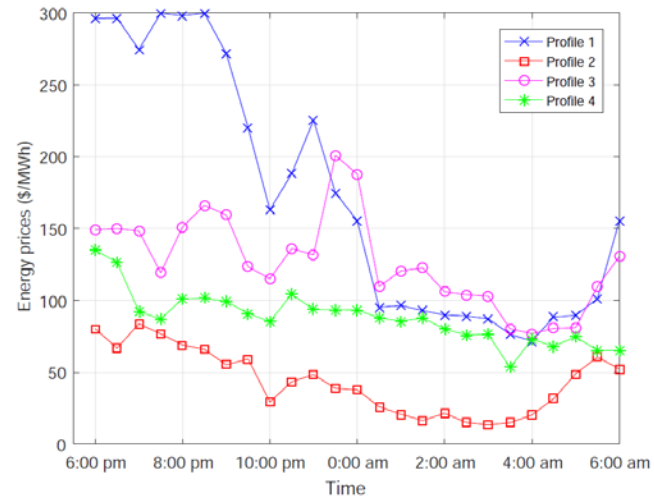
$$\begin{aligned}
&\min_{W(t),R(t)} \quad F(P_g(t))) := \sum_{k\in\mathcal{G}} f(P_{g_k}(t)) && \text{(16a)}\\
&\text{s.t.} \quad (3)-(4),(7),(14d)-(14h) \quad \text{for } t'=t, && \text{(16b)}\\
&\sum_{m\in\mathcal{N}(k)} (W_{kk}(t)-W_{km}(t))y_{km}^{*} = (P_{g_k}(t)-P_{l_k}(t) \\
&-\sum_{k_n\in C(t)} \hat{P}_{k_n}(t)) + j(Q_{g_k}(t)-Q_{l_k}(t)), \quad k\in\mathcal{G}, && \text{(16c)}\\
&\text{rank}(W(t))=1. && \text{(16d)}
\end{aligned}$$

Global solutions found

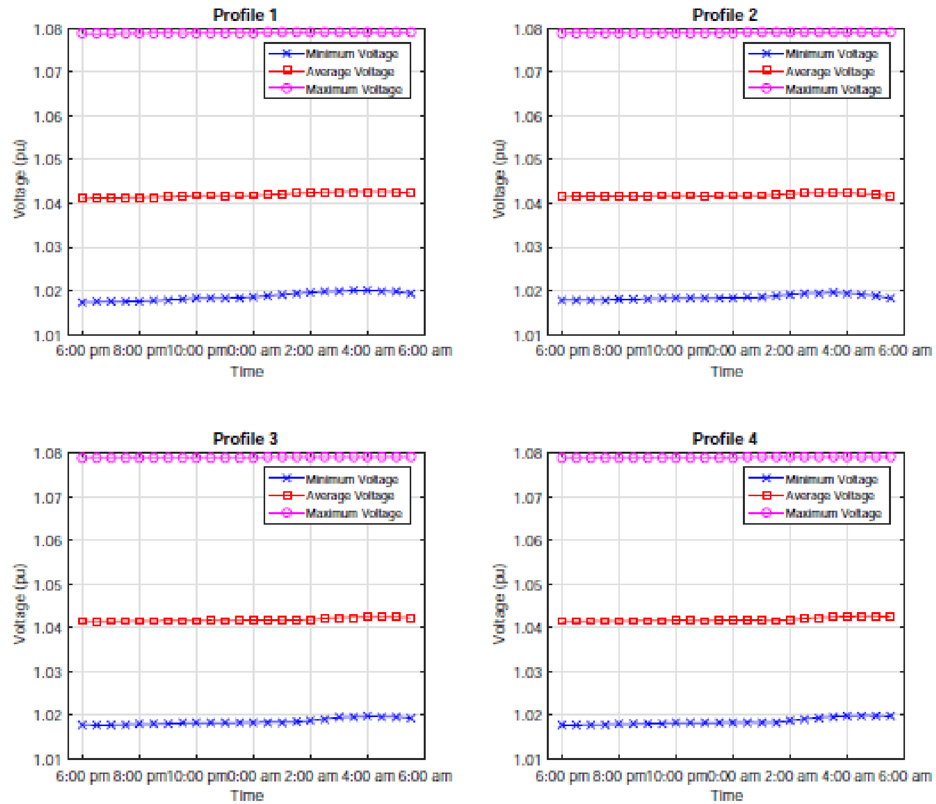
	Rank	μ	Lower bound	Computed value	Opt. degree
Case9	9	1	27978.1	27978.1	100%
Case14	1	-	40800.7	40800.7	100%
Case30	1	-	4935.6	4935.6	100%
Case118mod	2	50	644225.3	644233.9	99.999%

Load demands and Energy Price



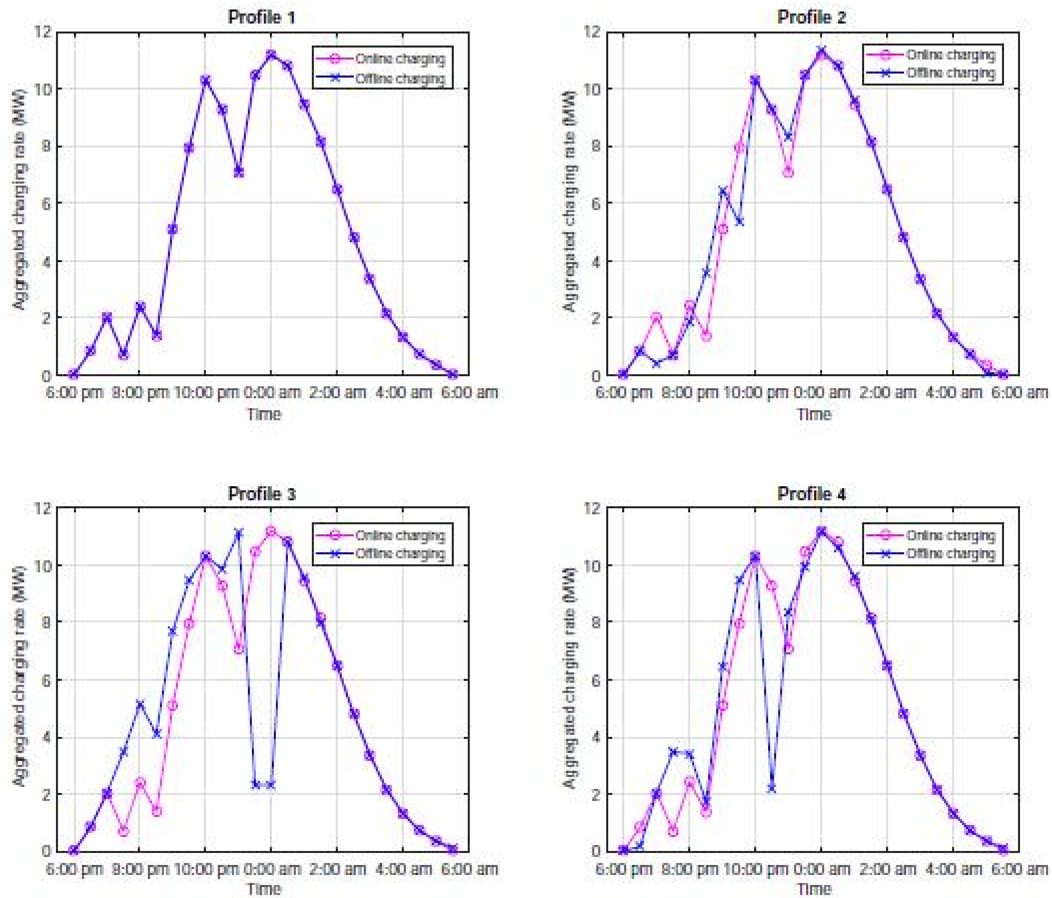


Voltage



Profile 1
Profile 2
Profile 3
Profile 4
Minimum Voltage
Average Voltage
Maximum Voltage
Voltage (pu)
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
6:00 pm 8:00 pm10:00 pm0:00 am 2:00 am 4:00 am 6:00 am
Time

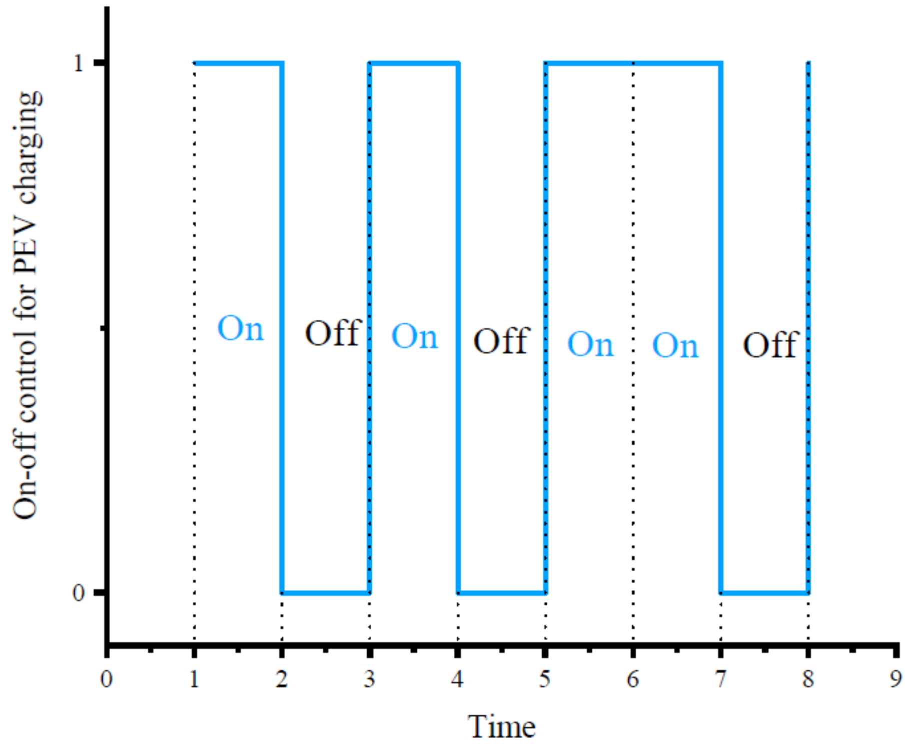
Charging rate



On-off EV charging

On-off charging

- Easily control
- Simple implementation



Challenges

- Dynamic AC optimal power flow
  - Nonconvex nodal voltage constraints
- PEVs arrive and depart randomly at different Charging Stations
  - Random nature
- On-off strategy of PEV charging
  - Binary charging variables

Challenges

- Dynamic AC optimal power flow
  - Nonconvex constraints in terms of nodal voltage

$$
\begin{aligned}
&V_k(t')[\sum_{m\in\mathcal{N}(k)} y_{km}(V_k - V_m)]^* = [P_{g_k}(t') - P_{l_k}(t') \\
&- \sum_{n\in\mathcal{H}_k} \bar{P}_{k_n}\tau_{k_n}(t')] + j[Q_{g_k}(t') - Q_{l_k}(t')], k \in \mathcal{G},
\end{aligned}
$$

Challenges

- PEVs arrive and depart randomly at different Charging Stations
  - Random nature

- Model predictive control:
  - During the computational procedure, we solve the dynamic OPF-PEV problem over the prediction horizon [t, T], while only the solution at time slot t is required for online updating.

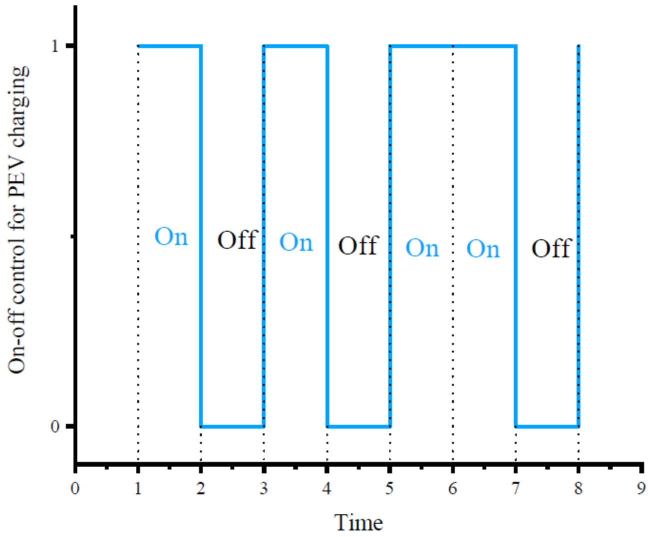
Challenges

- On-off strategy of PEV charging
  - Binary charging constraints

$$\tau_{k_n}(t') \in \{0,1\}$$

$$\sum_{t'=t_{k_n,a}}^{t_{k_n,d}} u_h \bar{P}_{k_n} \tau_{k_n}(t') \geq C_{k_n}(1 - s_{k_n}^0).$$

To make PEV k_n fully charged at its departure.



Smart grid operation constraints

The following constraints are about power balance, power generation, voltage and phase balance, and line capacity:

$$
\begin{aligned}
\sum_{m\in\mathcal{N}(k)} (W_{kk}(t') - W_{km}(t'))y^*_{km} &= [P_{g_k}(t') - P_{l_k}(t') \\
- \sum_{k_n\in C(t)} \bar{P}_{k_n}\tau_{k_n}(t')] &+ j(Q_{g_k}(t') - Q_{l_k}(t')), \quad k \in \mathcal{G}, \\
\underline{P}_{g_k} \leq P_{g_k}(t') &\leq \overline{P}_{g_k}, \ k \in \mathcal{G}, \\
\underline{Q}_{g_k} \leq Q_{g_k}(t') &\leq \overline{Q}_{g_k}, \ k \in \mathcal{G}, \\
\underline{V}_k^2 \leq W_{kk}(t') &\leq \overline{V}_k^2, \ k \in \mathcal{N}, \\
\Im(W_{km}(t')) &\leq \Re(W_{km}(t'))\tan(\theta_{km}^{max}), (k,m) \in \mathcal{L}, \\
|(W_{kk}(t') - W_{km}(t'))y^*_{km}| &\leq S_{km}, (k,m) \in \mathcal{L},
\end{aligned}
$$

MINLP

$$
\begin{aligned}
&\min_{\mathcal{W}_P(t),\mathcal{R}_P(t),\tau_P(t)} \quad F_P(\mathcal{R}_P(t),\tau_P(t)) \\
\text{s.t.} \quad & (1)-(5),(9)-(10) \quad \text{for } t' \in [t,\Psi(t)], && \text{(16a)} \\
& \sum_{t'=t}^{t_{k_n,d}} u_h \bar{P}_{k_n} \tau_{k_n}(t') \geq d_{k_n}(t), k_n \in C(t), && \text{(16b)} \\
& \tau_{k_n}(t') \in \{0,1\}, t' \in [t, t_{k_n,d}], k_n \in C(t), && \text{(16c)} \\
& W(t') \succeq 0, \quad \text{for } t' \in [t,\Psi(t)] && \text{(16d)} \\
& \text{rank}(W(t')) = 1, \quad \text{for } t' \in [t,\Psi(t)]. && \text{(16e)}
\end{aligned}
$$

- The MPC is solved over [t, Ψ(t)] at each time t, only R(t), W(t) and τ(t) are used for updating the solution.
- (16b) and (16c) are the integer constraints.
- (16e) is the nonconvex rank constraint.

Two-Stage Optimization

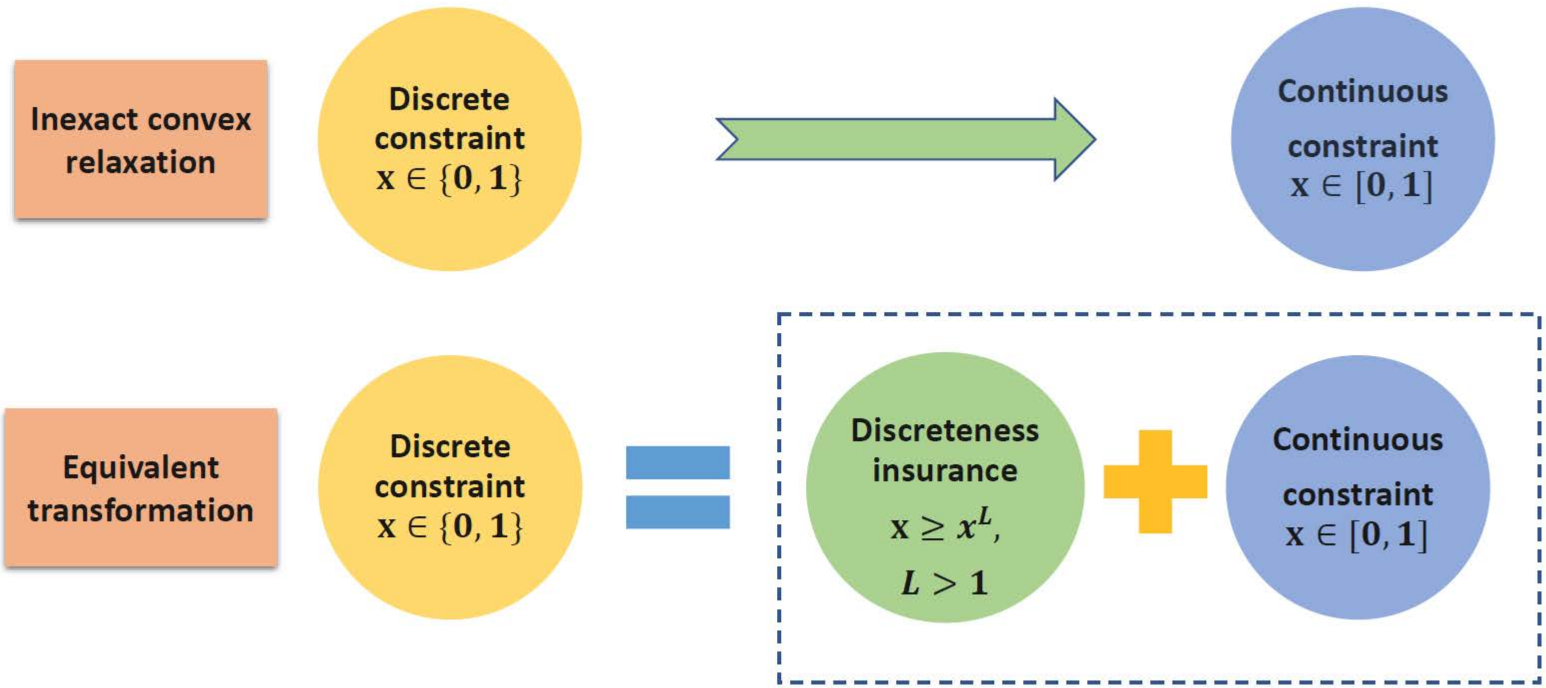
- Stage 1: Mixed-integer convex programming (MICP)
  - Relax the rank-one constraint (16e) and solve the MICP.

$$\min_{W(t),R(t)} \quad F(P_g(t)) + \nu(\mathrm{Trace}(W(t)) - \lambda_{\max}(W(t))), \quad (28a)$$

$$\text{s.t.} \quad (18b), W(t) \succeq 0. \; (28b)$$

- Stage 2: Nonconvex rank-one optimization
  - If the solution of Stage 1 is rank-one, then skip Stage 2; otherwise, use the following penalized optimization to handle the rank-one constraints, which was published in our previous work.

Equivalent transformation for MICP



Lemma and Proposition

Lemma 1: The binary constraint (14) can be fulfilled with the following continuous constraints with linear constraint (15),

$$0 \le \tau_{k_n}(t') \le 1, t' \in [t, t_{k_n,d}], k_n \in C(t), \tag{19}$$

$$\begin{aligned} g(\tau_P(t)) &\ge \bar{\tau}(t) \\ &\triangleq \sum_{k_n \in C(t)} \bar{\tau}_{k_n}(t), \end{aligned} \tag{20}$$

for $L > 1$ and $g(\tau_P(t)) \triangleq \sum_{k_n \in C(t)} \sum_{t'=t}^{t_{k_n,d}} \tau_{k_n}^L(t')$.

Proposition 1: Under the linear constraint (15), the function

$$g_1(\tau_P(t)) \triangleq \frac{1}{g(\tau_P(t))} - \frac{1}{\bar{\tau}(t)} \tag{21}$$

is a measure to evaluate the satisfaction of binary constraint (14) with $g_1(\tau_P(t)) \ge 0 \ \forall \ \tau_{k_n}(t') \in [0, 1]$ and $g_1(\tau_P(t)) = 0$ if and only if $\tau_{k_n}(t')$ are binary (i.e. satisfying (14)).

Find a tight lower bound

$$\begin{aligned} g(\tau_P(t)) &\ge g^{(\kappa)}(\tau_P(t)) \\ &\triangleq g(\tau_P^{(\kappa)}(t)) + \langle \nabla g(\tau_P^{(\kappa)}(t)), \tau_P(t) - \tau_P^{(\kappa)}(t) \rangle \\ &= -(L-1) \sum_{k_n \in C(t)} \sum_{t'=t}^{t_{k_n,d}} (\tau_{k_n}^{(\kappa)}(t'))^L \\ &\quad + L \sum_{k_n \in C(t)} \sum_{t'=t}^{t_{k_n,d}} (\tau_{k_n}^{(\kappa)}(t'))^{L-1} \tau_{k_n}(t'). \end{aligned} \tag{23}$$

Hence, an approximation of the upper bounding for $g_1(\tau_P(t))$ at the variable $\tau_P^{(\kappa)}(t)$ can be easily obtained as

$$g_1(\tau_P(t)) \le g_1^{(\kappa)}(\tau_P(t)) \triangleq \frac{1}{g^{(\kappa)}(\tau_P(t))} - \frac{1}{\bar{\tau}(t)} \tag{24}$$

over the trust region

$$g^{(\kappa)}(\tau_P(t)) > 0. \tag{25}$$

Path following algorithm

Then, the following convex problem is solved at the κ-th iteration to obtain the next iterative point $(\mathcal{W}_P^{(\kappa+1)}(t), \mathcal{R}_P^{(\kappa+1)}(t), \tau_P^{(\kappa+1)}(t))$:

$$\min_{\mathcal{W}_P(t),\mathcal{R}_P(t),\tau_P(t)} \Phi^{(\kappa)}(\mathcal{R}_P(t), \tau_P(t)) \triangleq$$
$$F_P(\mathcal{R}_P(t), \tau_P(t)) + \mu g_1^{(\kappa)}(\tau_P(t)) \quad \text{s.t.} \quad (1)-(5)$$
$$\text{for } t' \in [t, \Psi(t)], (15), (16b), (16c), (19), (25). \qquad (26)$$

Algorithm 1 MICP Solver

Set $\kappa = 0$, choose a feasible $\tau_P^{(0)}(t)$ for the following optimization problem:

$$\begin{aligned} &\min_{\mathcal{W}_P(t),\mathcal{R}_P(t),\tau_P(t)} F_P(\mathcal{R}_P(t), \tau_P(t)) \\ &\quad \text{s.t. } (15), (16b), (16c), (19). \end{aligned} \qquad (27)$$

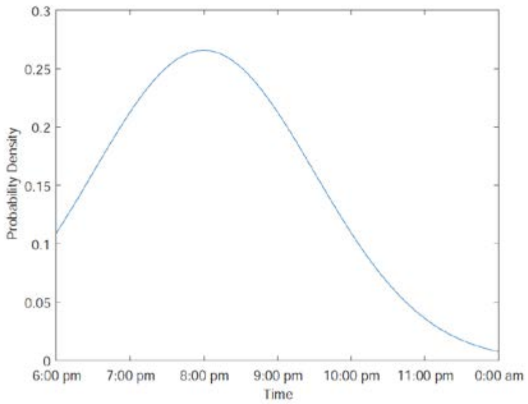
κ-th iteration. Solve the optimization problem (26),
if

$$\sum_{k_n \in C(t)} \sum_{t'=t}^{t_{k_n,d}} \left(\tau_{k_n}^{(\kappa+1)}(t') - \left(\tau_{k_n}^{(\kappa+1)}(t') \right)^L \right) \approx 0,$$

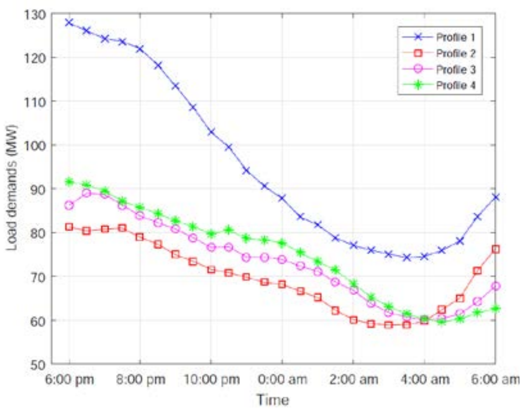
then accept $\tau_P^{(\kappa+1)}(t)$ as the found solution.
else $\kappa = \kappa + 1$, go to the next iteration.
end if

Simulation

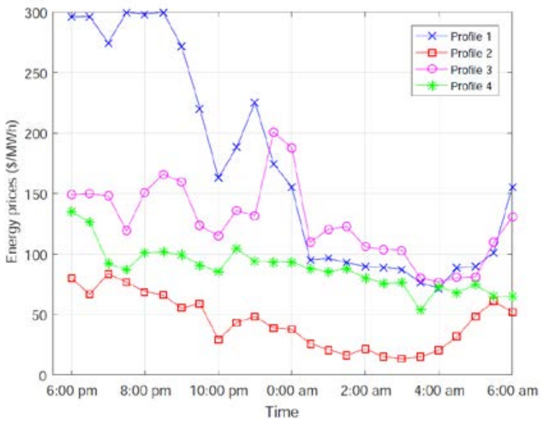
PEV distribution



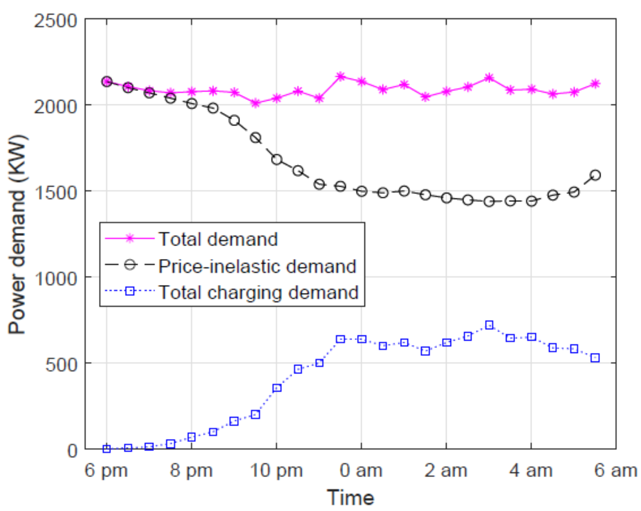
Residential load demands



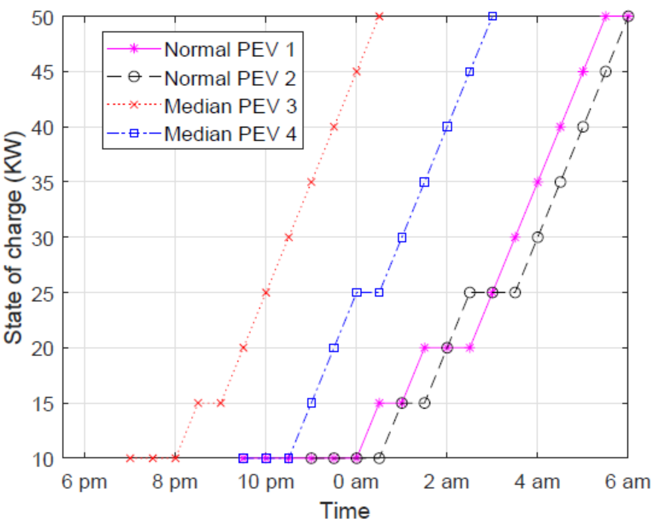
Energy Price



Simulation



Total power demand, charging power demand and real price-inelastic demand under Profile 1 during the serving time period



The Sate of charge of PEVs under Profile 1 during the serving time period

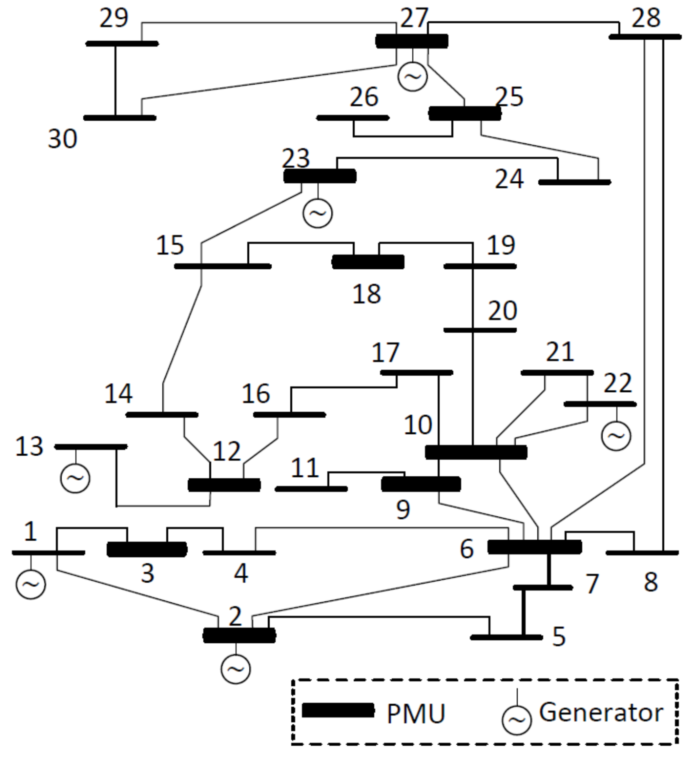
PMU placement

PMU placement

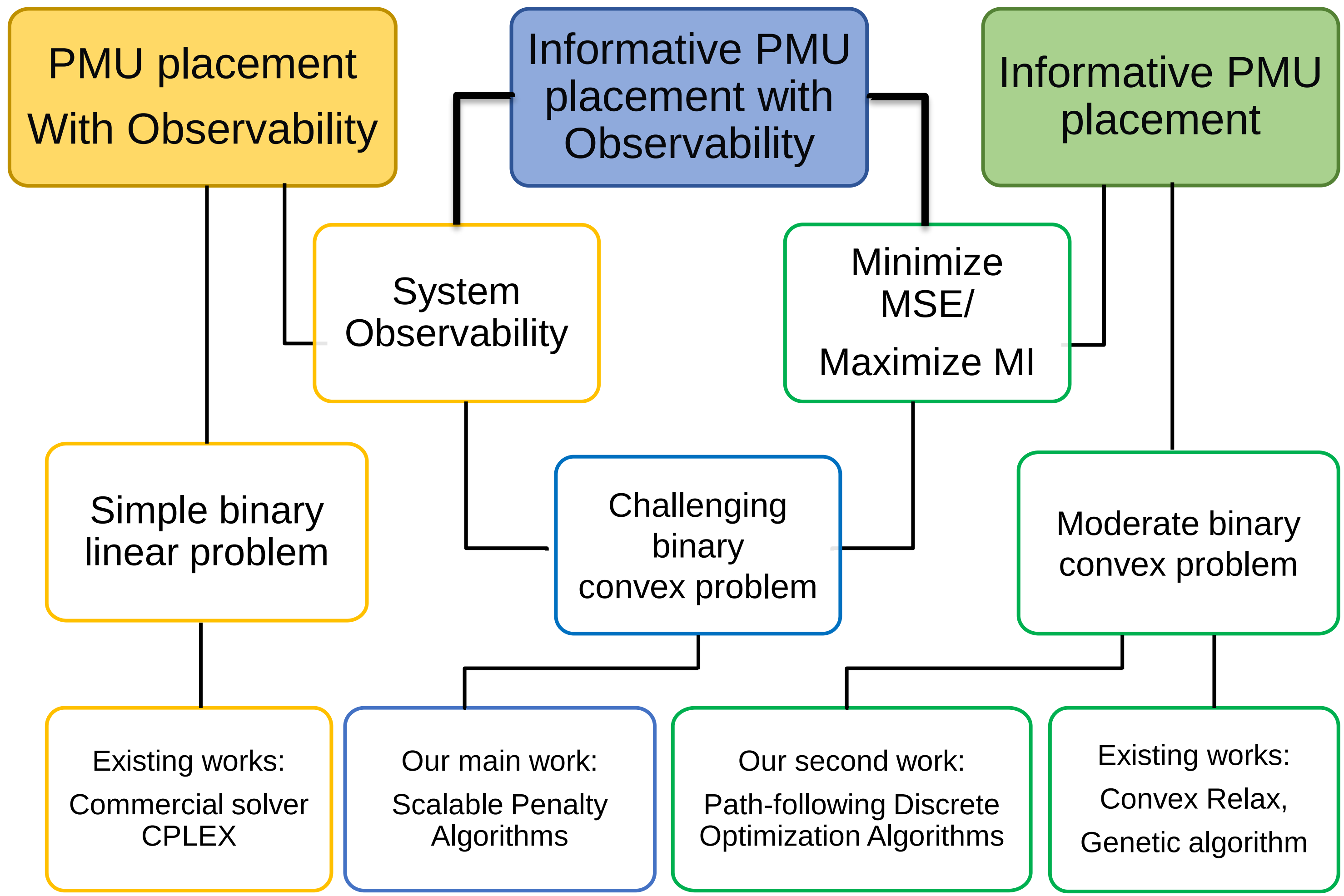
Phasor measurement unit (PMU) is an advanced digital meter, which is used in smart power grids for real-time monitoring of grid operations.

PMU placement

1) System Observability
2) Informative state estimation



PMU placement



PMU placement model

Select **S** PMUs $\mathcal{D}_S := \{\boldsymbol{x} \in \{0,1\}^N \ : \ \sum_{k=1}^{N} x_k = S\},$

to guarantee

1) system observability $\mathcal{A}\boldsymbol{x} \geq 1_N,$

2) Minimize Mear

$$f_e(\boldsymbol{x}) \quad = \quad \text{Trace}\left(\left(B^T \Sigma_P^{-1} B + \sum_{k=1}^{N} x_k H_k^T \mathcal{R}_{w_k}^{-1} H_k\right)^{-1}\right), \tag{11}$$

3) Maximize Mutual information

$$\begin{aligned} f_{MI}(\boldsymbol{x}) : \quad &= \quad -\ln|\mathcal{R}_e(\boldsymbol{x})| \\ &= \quad \ln|B^T \Sigma_P^{-1} B + \sum_{k=1}^{N} x_k H_k^T R_{w_k}^{-1} H_k|, \end{aligned} \tag{13}$$

Existing models of PMU placement

PMU placement With Observab

$$\min_{\boldsymbol{x}} \sum_{k=1}^{N} x_k \ : \ \boldsymbol{x} \in \{0,1\}^N \quad \mathcal{A}\boldsymbol{x} \geq 1_N,$$

Informative PMU placement:

1) Convex relaxation

2) Greedy algorithm

In optimizing the so called gain matrix [14], the work [15] considered the simple binary convex problem

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{x} \in \mathcal{D}_S, \tag{25}$$

by solving its convex relaxation problem, which is

$$\min_{\boldsymbol{x}\in\mathbb{R}_+^N, \mathbf{T}\in\mathbb{R}^{N\times N}} \text{Trace}(\mathbf{T}) \quad \text{s.t.} \quad \boldsymbol{x} \in \mathsf{Poly}(\mathcal{D}_S), \tag{26a}$$

$$\begin{bmatrix} B^T\Sigma_P^{-1}B + \sum_{k=1}^N x_k H_k^T R_{w_k}^{-1} H_k & I_N \\ I_N & \mathbf{T} \end{bmatrix} \succeq 0, \tag{26b}$$

when $f = f_e$, or

$$\max_{\boldsymbol{x}\in\mathbb{R}_+^N} \ \ln|B^T\Sigma_P^{-1}B + \sum_{k=1}^N x_k H_k^T R_{w_k}^{-1} H_k| \quad \text{s.t.} \quad \boldsymbol{x} \in \mathsf{Poly}(\mathcal{D}_S), \tag{27}$$

when $f = f_{MI}$, for

$$\mathsf{Poly}(\mathcal{D}_S) = \{\boldsymbol{x} \in [0,1]^N : \sum_{k=1}^N x_k = S\}, \tag{28}$$

Scale Penalty Algorithms for optimization placement

- Based on our previous work, we transform the nonconvex and discrete binary constraint to continuous linear constraint.

$$\min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}} f(\boldsymbol{x}) + \mu P^{(\kappa)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}), \quad \text{s.t. } (34b), (35), (36), (37), \ (38)$$

with

$$P^{(\kappa)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}) := \left(\frac{1}{g_1^{(\kappa)}(\boldsymbol{x})} - \frac{1}{S}\right) + \left(\frac{1}{g_2^{(\kappa)}(\boldsymbol{y})} - \frac{1}{|\mathcal{Z}|}\right) + \left(\frac{1}{g_3^{(\kappa)}(\boldsymbol{c})} - \frac{1}{N-|\mathcal{Z}|}\right).$$

- Propose a scalable algorithm to handle the f(x). Although the function f(x) is already convex, it is not easy to optimize it. For instance, when f = f_e, usually f_e is expressed by Trace(T), where T is a slack symmetric matrix variable of size N X N satisfying the semi-definite constraint (26b), which is not scalable to x.

$$f_e(\boldsymbol{x}) \le f_e^{(\kappa)}(\boldsymbol{x}) := a_0^{(\kappa)} + \sum_{k=1}^{N} \frac{a_k^{(\kappa)}}{x_k + \epsilon} \text{ for}$$

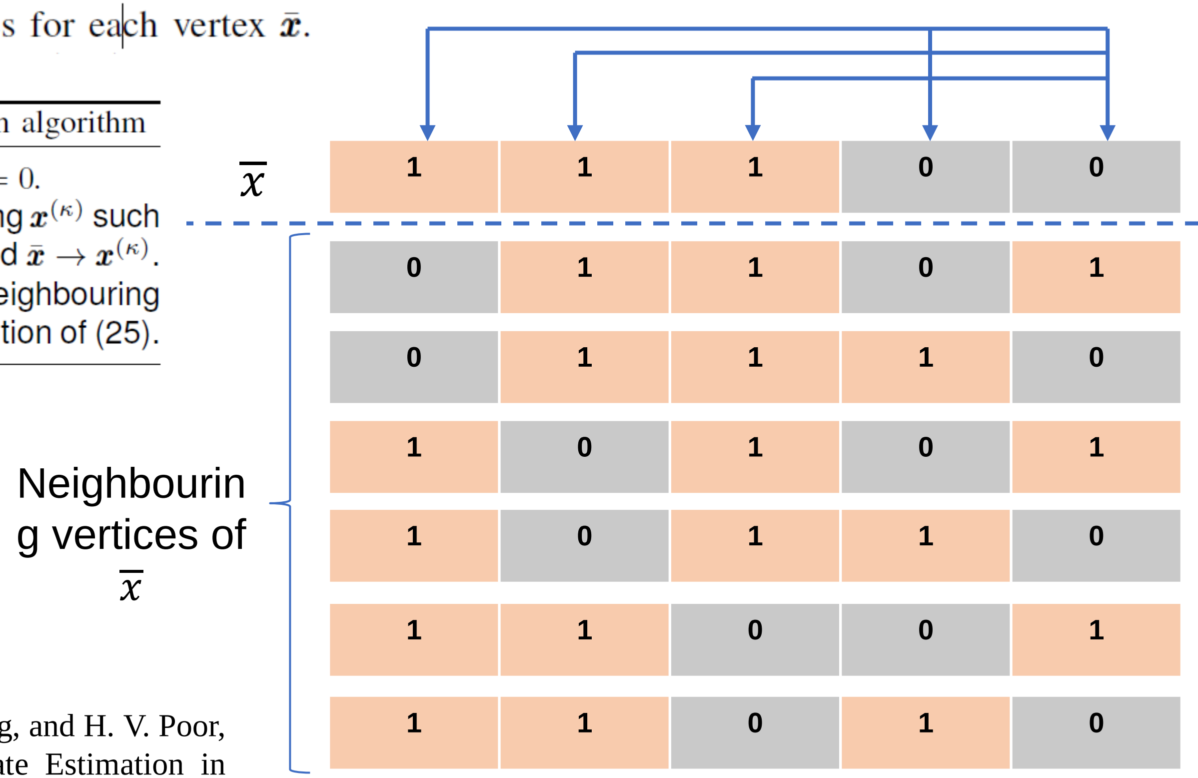
$$0 < a_0^{(\kappa)} := \mathrm{Trace}((\mathcal{R}_e(\boldsymbol{x}^{(\kappa)}))^2 \mathcal{A}_\epsilon),$$
$$0 < a_k^{(\kappa)} := (x_k^{(\kappa)} + \epsilon)^2 \mathrm{Trace}((\mathcal{R}_e(\boldsymbol{x}^{(\kappa)}))^2 \times H_k^T R_{w_k}^{-1} H_k), k = 1, \ldots, N.$$

Tailed path-following discrete optimization algorithms

Recall that point $\boldsymbol{x}$ is a vertex neighbouring the vertex $\bar{\boldsymbol{x}}$ if and only if there exists a pair (i, j) such that $x_i = 0 \neq \bar{x}_i = 1$ and $x_j = 1 \neq \bar{x}_j = 0$ and $x_\ell = \bar{x}_\ell$ whenever $\ell \neq i$ and $\ell \neq j$, i.e. $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ are exactly different in two entries and there are $S(N - S)$ neighbouring vertices for each vertex $\bar{\boldsymbol{x}}$.

Algorithm 2 Path-following discrete optimization algorithm

Initialization. Start from a $\boldsymbol{x}^{(0)} \in \mathcal{D}_S$. Set $\kappa = 0$.
κ-th iteration. If there is a $\bar{\boldsymbol{x}} \in \mathcal{D}_S$ neighbouring $\boldsymbol{x}^{(\kappa)}$ such that $f(\bar{\boldsymbol{x}}) < f(\boldsymbol{x}^{(\kappa)})$ then reset $\kappa + 1 \to \kappa$ and $\bar{\boldsymbol{x}} \to \boldsymbol{x}^{(\kappa)}$. Otherwise, if $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^{(\kappa)})$ for all $\boldsymbol{x} \in \mathcal{D}_S$ neighbouring $\boldsymbol{x}^{(\kappa)}$ then stop: $\boldsymbol{x}^{(\kappa)}$ is the global optimal solution of (25).



- Y. Shi, H. D. Tuan, A. A. Nasir, T. Q. Duong, and H. V. Poor, "PMU Placement Optimization for E_cient State Estimation in Smart Grid" accepted by IEEE Journal on Selected Areas in Communications, 2019. (Impact factor **9.302**)

Distributed MPC
for joint OPF and DR

Distributed MPC for joint coordination of OPF and DR

Motivations

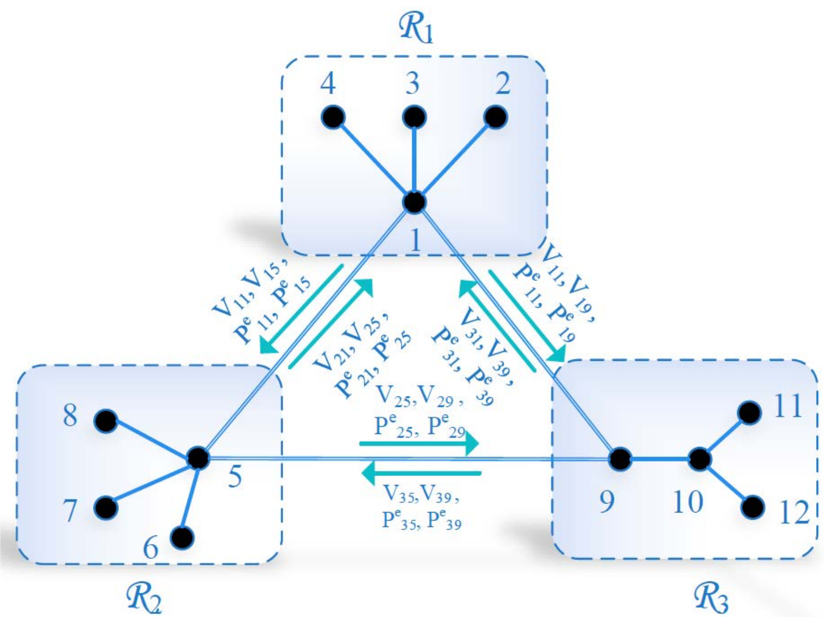
- The system is prone to disruption once the centralized controller is in outage.

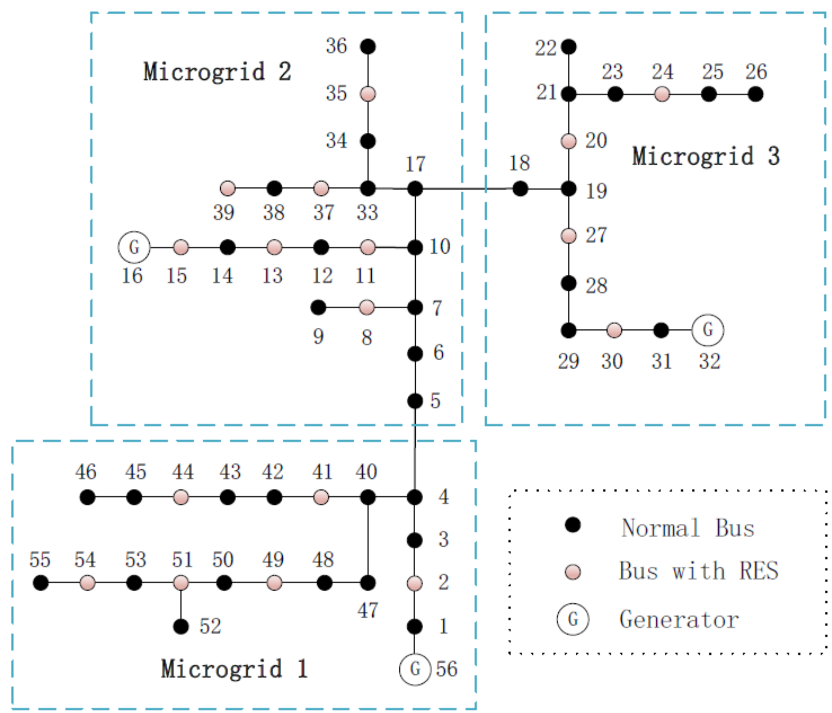
- Distributed controller is capable of improving the system robustness and reliability even when contingency occurs.

- The computational and communication cost increases dramatically with the increase of system size since the centralized controller has to exchange the information with all the participants.

- Importantly, unlike the centralized controller, which relies on gathering users' private data such as load profiles, the distributed controller can alleviate the potential concerns of privacy and security problems.

Distributed Networks





ADMM-based DMPC

Algorithm 1 ADMM-based DMPC (19)

---

1) Initialization: Set $\kappa = 0$ and initialize $\mathbf{W}_{ji}^{(\kappa)}(t)$, $(\mathbf{P}_{ji}^{e}(t))^{(\kappa)}$, $\Gamma_{ij}^{(\kappa)}(t)$ and $\gamma_{ij}^{(\kappa)}(t)$ for all $i \in \mathcal{K}$.

2) Update the primal variables: For each region i, input $\mathbf{W}_{ji}^{(\kappa)}(t)$, $(\mathbf{P}_{ji}^{e}(t))^{(\kappa)}$, $\Gamma_{ij}^{(\kappa)}(t)$ and $\gamma_{ij}^{(\kappa)}(t)$ and update $\mathbf{W}_{i}^{(\kappa+1)}(t)$, $\mathcal{R}_{i}^{(\kappa+1)}(t)$ and $(\mathbf{P}_{i}^{e}(t))^{(\kappa+1)}$ concurrently by solving the sub-problem:

$$\min F^{i}_{[t',T]}(\{\mathbf{W}_i(t)\}, \{(\mathbf{P}_i^e(t))\}, \{\mathbf{W}_{ji}^{(\kappa)}(t)\}, \{\Gamma_{ij}^{(\kappa)}(t)\}, \{(\mathbf{P}_{ji}^{e}(t))^{(\kappa)}(t)\}, \{\gamma_{ij}^{(\kappa)}(t)\}) \text{ s.t. } (21b), (21c). \quad (22)$$

3) Update the auxiliary variables: Update $\mathbf{W}_{ji}^{(\kappa+1)}(t)$ and $(\mathbf{P}_{ji}^{e}(t))^{(\kappa+1)}$ by solving the unconstrained problem:

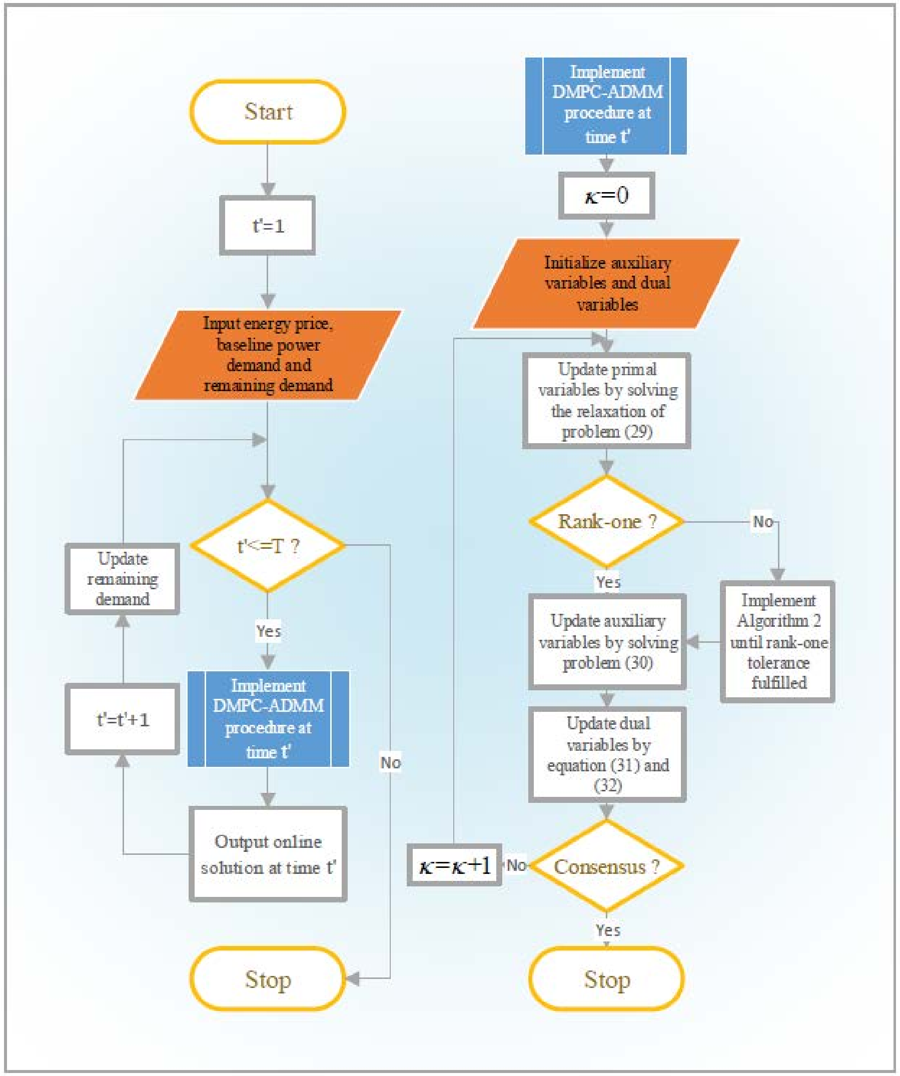
$$\min \sum_{i=1}^{K} (\mathrm{Trace}(\Gamma_{ij}^{H}(t') \otimes (\mathbf{W}_{ij}^{(\kappa+1)}(t') - \mathbf{W}_{ji}(t'))) + \gamma_{ij}^{H}(t')((\mathbf{P}_{ij}^{e}(t'))^{(\kappa+1)} - \mathbf{P}_{ji}^{e}(t')) + \frac{\delta}{2}||\mathbf{W}_{ij}^{(\kappa+1)}(t') - \mathbf{W}_{ji}(t')||^2 + \frac{\delta}{2}||(\mathbf{P}_{ij}^{e}(t'))^{(\kappa+1)} - \mathbf{P}_{ji}^{e}(t')||^2). \quad (23)$$

4) Update the dual variables: For each region i, update $\gamma_{il}^{(\kappa+1)}$ by the following procedure,

$$\Gamma_{ij}^{(\kappa+1)}(t') = \Gamma_{ij}^{(\kappa)}(t') + \delta(\mathbf{W}_{ij}^{(\kappa+1)}(t') - \mathbf{W}_{ji}^{(\kappa+1)}(t')),$$
$$\gamma_{ij}^{(\kappa+1)}(t') = \gamma_{ij}^{(\kappa)}(t') + \delta((\mathbf{P}_{ij}^{e}(t'))^{(\kappa+1)} - (\mathbf{P}_{ji}^{e}(t'))^{(\kappa+1)}).$$

5) Stopping criterion: If $||\mathbf{W}_{ij}^{(\kappa+1)}(t') - \mathbf{W}_{ji}^{(\kappa+1)}(t')|| \le \epsilon$ and $||(\mathbf{P}_{ij}^{e}(t'))^{(\kappa+1)} - (\mathbf{P}_{ji}^{e}(t'))^{(\kappa+1)}|| \le \epsilon$, stop the algorithm and output $\mathbf{W}_{i}^{(\kappa+1)}(t')$, $\mathcal{R}_{i}^{(\kappa+1)}(t')$ and $(\mathbf{P}_{i}^{e}(t'))^{(\kappa+1)}$ as the solution of (19); otherwise $\kappa = \kappa + 1$, go to step 2.

ADMM-based DMPC



4. Reduced H_∞ Control in LTI System

- Consider a linear continuous time (LTI) system

$$
\begin{aligned}
\dot{x} &= Ax + B_\infty w + Bu \\
z &= C_\infty x + D_\infty w + D_{\infty u} u \\
y &= Cx + D_{y\infty} w
\end{aligned}
$$

In control system, the design of reduced-order output feedback controller is a significant application, which in general is a nonconvex control problem and can be transformed into an optimization problem with bilinear matrix inequality (BMI) constraints.

O.Toker etc, proved that optimization problems with BMI constraints are NP-hard.

State-of-art BMI solver

- Initialize from a reduced-order stabilizing controller and then move within a convex feasibility subset containing this initialized point.

Drawbacks of BMI solver

1. Finding an initial reduced-order stabilizing controller is a NP-hard problem.
2. The feasibility set is highly nonconvex and disconnected in general. The stabilizing controller may be trapped by local minima.
3. Convergence is slow and dependent very much on the local geometry around such initial point.

BMI Formulation

Nonconvexity:
KCX

$$\min_{\gamma,\mathbf{X},\mathbf{K}} \quad \gamma \quad \text{s.t.}$$

$$\begin{pmatrix} (A_0 + \mathcal{B}\mathbf{K}\mathcal{C})\mathbf{X} + * & * & * \\ (B_{0,\infty} + \mathcal{B}\mathbf{K}\mathcal{D}_{2,\infty})^T & -\gamma I & * \\ (C_{0,\infty} + \mathcal{D}_{12}\mathbf{K}\mathcal{C})\mathbf{X} & D_\infty + \mathcal{D}_{12}\mathbf{K}\mathcal{D}_{2,\infty} & -\gamma I \end{pmatrix} \prec 0, \tag{5a}$$

$$\mathbf{X} \succ 0, \quad \gamma > 0. \tag{5b}$$

BMI Formulation

- Introduce the new variable

$$W = KCX$$

$$\min_{\gamma,\mathbf{X},\mathbf{W},\mathbf{K}} \gamma \quad \text{s.t.,}$$

$$\begin{pmatrix} (A_0\mathbf{X} + \mathcal{B}\mathbf{W}) + * & * & * \\ (B_{0,\infty} + \mathcal{B}\mathbf{K}\mathcal{D}_{2,\infty})^T & -\gamma I & * \\ C_{0,\infty}\mathbf{X} + \mathcal{D}_{12}\mathbf{W} & D_\infty + \mathcal{D}_{12}\mathbf{K}\mathcal{D}_{2,\infty} & -\gamma I \end{pmatrix} \prec 0, \tag{7a}$$

$$\mathbf{X} \succ 0, \quad \gamma > 0, \tag{7b}$$

$$\mathbf{W} = \mathbf{K}\mathcal{C}\mathbf{X}. \tag{7c}$$

Nonconvex Spectral Algorithm On H_∞ Control

- The BMI constraint $W = KCX$ is fulfilled by

$$\begin{pmatrix} \mathbf{W}_{11} & \mathbf{W} & \mathbf{K} \\ \mathbf{W}^T & \mathbf{W}_{22} & \mathbf{X}^T\mathcal{C}^T \\ \mathbf{K}^T & \mathcal{C}\mathbf{X} & I \end{pmatrix} \geq 0 \tag{9}$$

$$\mathrm{Trace}(\mathbf{W}_{11} - \mathbf{K}\mathbf{K}^T) \leq 0. \tag{10}$$

- By Schur's complement, (10) is equivalent to (12)

$$\mathrm{rank}\,(\mathcal{W}_K) = n_y + n_K \tag{12}$$

$$\mathcal{W}_K := \begin{pmatrix} \mathbf{W}_{11} & \mathbf{K} \\ \mathbf{K}^T & I_{n_y+n_K} \end{pmatrix}$$

The rank constraint

$$\operatorname{rank}(\mathcal{W}_K) = n_y + n_K \qquad \Longleftrightarrow$$

$$\mathrm{Trace}(\mathcal{W}_K) - \lambda_{[n_y+n_K]}(\mathcal{W}_K) = 0 \qquad (13)$$

where $\mathrm{Trace}(\mathcal{W}_K)$ is the trace of $\mathcal{W}_K$, $\lambda_{[n_y+n_K]}(\mathcal{W}_K)$ is the summation of $(n_y + n_K)$ largest eigenvalues of $\mathcal{W}_K$.

Function $\lambda_{[n_y+n_K]}(\mathcal{W}_K)$ is nonsmooth but is lower bounded by

$$\begin{aligned} \lambda_{[n_y+n_K]}(\mathcal{W}_K) &= \sum_{i=1}^{n_y+n_K} (w_i)^T \mathcal{W}_K w_i \\ &\geq \sum_{i=1}^{n_y+n_K} (w_i^{(\kappa)})^T \mathcal{W}_K w_i^{(\kappa)}, \end{aligned} \qquad (16)$$

where $w_1^{(\kappa)}, \ldots.., w_{n_y+n_K}^{(\kappa)}$ are the normalized eigenvectors corresponding to $n_y + n_K$ largest eigenvalues of $\mathcal{W}_K^{(\kappa)}$.

Nonconvex Spectral Algorithm On H_∞ Control

Algorithm 1 Nonconvex Spectral Optimization Algorithm for Solving BMI on $\mathcal{H}_\infty$ control

Initialize $\kappa := 0$ and solve

$$\min_{\gamma,\mathbf{X},\mathbf{W},\mathbf{K}} \gamma \quad \text{s.t.} \quad (7b),(7c),(9). \tag{18}$$

to find its optimal solution $(\gamma^{(\kappa)},\mathbf{X}^{(\kappa)},\mathbf{W}^{(\kappa)},\mathbf{K}^{(\kappa)})$, stop the algorithm if

$$\text{Trace}(\mathcal{W}_K^{(\kappa)}) - \sum_{i=1}^{n_y+n_K} (w_i^{(\kappa)})^T \mathcal{W}_K^{(\kappa)} w_i^{(\kappa)} \le \epsilon \tag{19}$$

and accept $(\gamma^{(0)},\mathbf{X}^{(0)},\mathbf{W}^{(0)},\mathbf{K}^{(0)})$ as the optimal solution of the nonconvex program (14).

repeat

Solve the convex program (17), to find the optimal solution $(\gamma^{(\kappa+1)},\mathbf{X}^{(\kappa+1)},\mathbf{W}^{(\kappa+1)},\mathbf{K}^{(\kappa+1)})$

Set $\kappa := \kappa + 1$.

until

$$\text{Trace}(\mathcal{W}_K^{(\kappa)}) - \sum_{i=1}^{n_y+n_K} (w_i^{(\kappa)})^T \mathcal{W}_K^{(\kappa)} w_i^{(\kappa)} \le \epsilon \tag{20}$$

Accept $(\gamma^{(\kappa)},\mathbf{X}^{(\kappa)},\mathbf{W}^{(\kappa)},\mathbf{K}^{(\kappa)})$ as a found solution of (14).

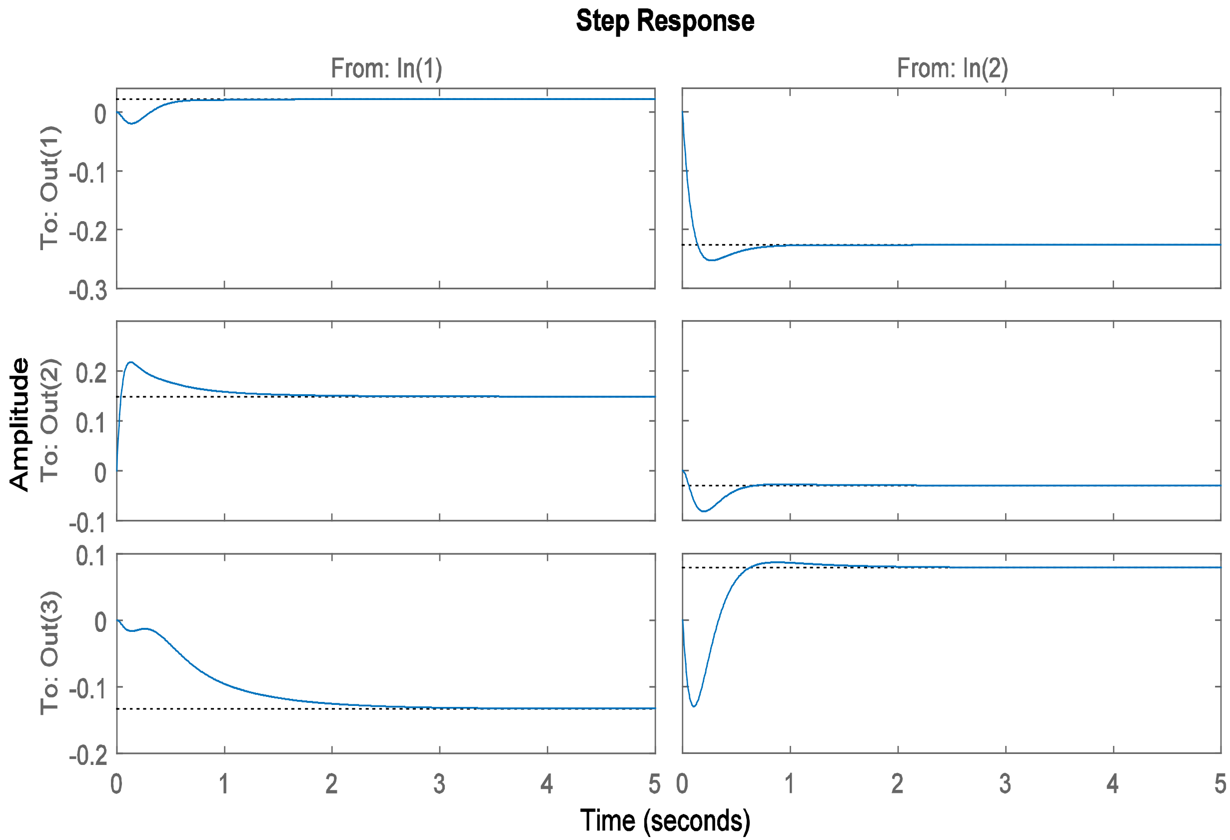
Simulation Results

TABLE I: Numerical results compared with [13]

Cases	$[n_x, n_y, n_u, n_K]$	μ	γ	# iter	γ in [13]
Helicopter	$[4, 1, 2, 2]$	1	0.104	2	0.133
Chemical reactor	$[4, 3, 2, 2]$	2	1.037	4	1.142
Transport aircraft	$[10, 3, 2, 1]$	2	1.237	5	2.860

P. Apkarian, D. Noll, and O. Prot, "A trust region spectral bundle method for nonconvex eigenvalue optimization," SIAM Journal on Optimization, vol. 19, no. 1, pp. 281–306, 2008.

Chemical reactor step response



Transport Aircraft step response