# CS290: Introduction to Algorithmic Game Theory

Week 5.2, Matching II (Dengji ZHAO)

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# Recap: Matching

Matching (Mechanism Design without Money)

- Agents in two sides.
- A matching: each agent is assigned to at most one agent on the other side.

*Two-sided matching*

- Agent in one set has strict preferences over agents in other set, e.g. students to schools

*One-sided matching*

- Only one side has strict preference on the other side, e.g. house allocation

# Recap: One-sided Matching: House Allocation

- Without initial allocation
  - Serial dictatorship mechanism: pareto optimal
- With initial allocation
  - Top-trading-cycle (TTC) mechanism: pareto optimal, truthful

# Two-sided Matching

**Definition**

A stable matching is a matching with no blocking pair, a blocking pair is two agents who prefer to match with each other.

Stable Matchings:

- Boy-Proposing Deferred Acceptance: stable
- Girl-Proposing Deferred Acceptance: stable

**Question**

Is Deferred Acceptance truthful?

# Truthful Stable Matching

**Theorem**

*The direct mechanism associated with the male propose algorithm is truthful for the males.*

**Question**

Is there a mechanism that is both stable and truthful for both the males and females?

| $\succ_{m_1}$ | $\succ_{m_2}$ | $\succ_{m_3}$ | $\succ_{w_1}$ | $\succ_{w_2}$ | $\succ_{w_3}$ |
|---|---|---|---|---|---|
| $w_2$ | $w_1$ | $w_1$ | $m_1$ | $m_3$ | $m_1$ |
| $w_1$ | $w_3$ | $w_2$ | $m_3$ | $m_1$ | $m_3$ |
| $w_3$ | $w_2$ | $w_3$ | $m_2$ | $m_2$ | $m_2$ |

# No Truthful and Stable Matching Mechanism

**Theorem**

*There exists no mechanism that is both stable and truthful (in two-sided matching).*

**Proof.**

- Consider two boys and two girls with the following preference profile:
  - $b_1 : g_1 \succ_{b_1} g_2 \succ_{b_1} b_1; b_2 : g_2 \succ_{b_2} g_1 \succ_{b_2} b_2$
  - $g_1 : b_2 \succ_{g_1} b_1 \succ_{g_1} g_1; g_2 : b_1 \succ_{g_2} b_2 \succ_{g_2} g_2$
- Only two stable matchings: $(b_1, g_1), (b_2, g_2)$ and $(b_1, g_2), (b_2, g_1)$, if the mechanism chooses the first matching, then $g_1$ will misreport $b_2 \succ_{g_1} g_1 \succ_{g_1} b_1$ to force the mechanism to choose the other matching.

$\square$

# Kidney Disease

- Kidney failure: a serious medical problem
- Preferred treatment: kidney transplant
  - Cadaver kidneys
  - Donation from live healthy people/relatives
  - Must be blood- and tissue-type compatible

# Kidney Disease

http://optn.transplant.hrsa.gov

**118,241**

people need a lifesaving organ transplant (total waiting list candidates). Of those, **75,814** people are active waiting list candidates. Totals as of today 9:58am

**5,367**

organ transplants performed so far in 2017

Total Transplants January - February 2017

as of 03/19/2017

**2,553**

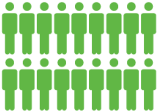
donors

Total Donors January - February 2017

as of 03/19/2017

## Organ donation and transplantation can save lives

Every ten minutes, someone is added to the national transplant waiting list.



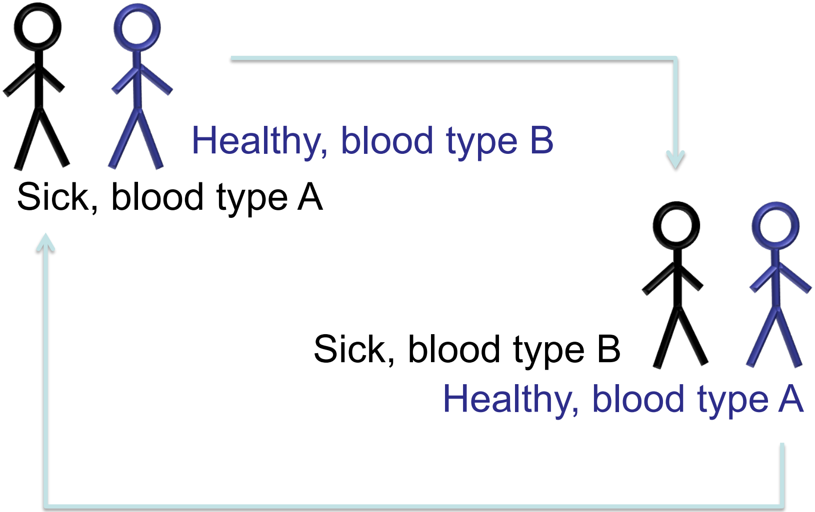
On average, 22 people die each day while waiting for a transplant.


One organ donor can save eight lives. Sign up to be a donor in your state.

# Kidney Donation and Kidney Exchange: One-sided Matching

- Incompatible pairs
  - a patient and a donor (they are incompatible)
- Kidney exchanges
  - incompatible pairs participate in swaps

# 2-cycle Swap



Healthy, blood type B
Sick, blood type A
Sick, blood type B
Healthy, blood type A

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- What will happen if there is one extra donor without patient?

# Many-to-One Matching: College Admissions

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- A set of colleges $C$, each $c \in C$ has a capacity $q_c$.
- A set of students $I$, each $i \in I$ has a preference $\succeq_i$ over $C$.
- Each college $c \in C$ has a preference $\succeq_c$ over $2^I$.

# Matching in College Admissions

## Definition

A matching for college admissions is $\mu : C \cup I \Rightarrow 2^{C \cup I}$ such that:

- $\mu(c) \subseteq I$ such that $|\mu(c)| \leq q_c$ for all $c \in C$,
- $\mu(i) \subseteq C$ such that $|\mu(i)| \leq 1$ for all $i \in I$, and
- $i \in \mu(c)$ if and only if $\mu(i) = \{c\}$ for all $c \in C$ and $i \in I$.

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A matching $\mu$ is

- **blocked by a college** $c \in C$ if there exists $i \in \mu(c)$ such that $\emptyset \succ_c \{i\}$.
- **blocked by a student** $i \in I$ if $\emptyset \succ_i \mu(i)$.
- **individually rational** if it is not blocked by any college or student.

## Stable Matching

A matching $\mu$ is **blocked by a pair** $(c, i) \in C \times I$ if

1. $c \succ_i \mu(i)$, and
2. either there exists $j \in \mu(c)$ such that $\{i\} \succ_c \{j\}$, or $|\mu(c)| < q_c$ and $\{i\} \succ_c \emptyset$.

**Definition**

A matching is stable if it is not blocked by any student, college or pair.

# College-Proposing Deferred Acceptance Algorithm

1. Each college $c$ proposes to its top $q_c$ acceptable students.
2. Each student rejects any unacceptable proposals and, if more than one acceptable proposal is received, she "holds" the most-preferred and rejects the rest.
3. Repeat until no more rejections. Each student is matched with the college she has been holding in the last step.

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**Quiz**

Does the college-proposing deferred acceptance algorithm give a stable matching? Yes

## Stable and Truthful

**Theorem**

*There exists no mechanism that is stable and truthful.*

**Theorem**

*Truth-telling is a weakly dominant strategy for all students under the Student-Proposing Deferred Acceptance mechanism.*

**Theorem**

*There exists no stable mechanism where truth-telling is a weakly dominant strategy for all colleges.*

# Not Truthful for Colleges

There are 2 colleges $c_1, c_2$ with $q_{c_1} = 2, q_{c_2} = 1$, and 2 students $i_1, i_2$. The preferences are as follows:

- $\succeq_{i_1}$: $\{c_1\} \succeq_{i_1} \{c_2\} \succeq_{i_1} \emptyset$;
- $\succeq_{i_2}$: $\{c_2\} \succeq_{i_2} \{c_1\} \succeq_{i_2} \emptyset$;
- $\succeq_{c_1}$: $\{i_1, i_2\} \succeq_{c_1} \{i_2\} \succeq_{c_1} \{i_1\} \succeq_{c_1} \emptyset$;
- $\succeq_{c_2}$: $\{i_1\} \succeq_{c_2} \{i_2\} \succeq_{c_2} \emptyset$

The only stable matching is $(c_1, i_1), (c_2, i_2)$.

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- $\succeq_{c_1}$: $\{i_1, i_2\} \succeq_{c_1} \{i_2\} \succeq_{c_1} \{i_1\} \succeq_{c_1} \emptyset$;
- $\succeq_{c_2}$: $\{i_1\} \succeq_{c_2} \{i_2\} \succeq_{c_2} \emptyset$

The only stable matching is $(c_1, i_1), (c_2, i_2)$.

**Question**

Is there any way for college $c_1$ to manipulate to receive a better matching?

- Yes, e.g. $\succeq'_{c_1}$: $\{i_2\} \succeq'_{c_1} \emptyset \succeq'_{c_1} \{i_1, i_2\} \succeq'_{c_1} \{i_1\}$

## Advanced Reading

- *Matching Markets: Theory and Practice* by Atila Abdulkadirog and Tayfun Sonmez