Introduction to Algorithmic Game Theory Final Exam Question Sheet [Spring 2017]

Instructor: Dengji Zhao

Time: 8:15am - 9:45am, June 2, 2017
Venue: SIST 1D-104

Candidate Information

- *Student ID*:
- *Name (Last, First)*:
- *Seat number*:
- *Signature*:

Format: open book

- You may bring any printed papers or books, but no electronic devices except for regular calculators.
- You are not allowed to discuss or share anything with others.
- You are not allowed to ask any questions regarding the exam questions, except for printing issues of this question sheet.
- **Answer the questions in the *Answer Sheet* only**.

1 Mechanism Design (6 credits)

In a mechanism design setting, there is a set of allocations A, a set of n agents denoted by N, each agent $i \in N$ has a valuation function $v_i : A \to \mathbb{R}$. For each agent $i \in N$, there is a (publicly known) weight $w_i \geq 0$. Under this setting, we define the following mechanism:

Weighted Mechanism:
Given all agents' valuation function report profile $(v'_1, ..., v'_n)$:

- the allocation function f satisfies:

$$f(v'_1, ..., v'_n) \in \arg\max_{a \in A} \sum_{i \in N} w_i v'_i(a)$$

- assume that $f(v'_1, ..., v'_n) = a^*$, the payment for each agent $i \in N$ is defined as:

$$p_i(v'_1, ..., v'_n) = -\sum_{j \in N, j \neq i} \frac{w_j}{w_i} v'_j(a^*)$$

Questions:

1. (**1 credit**) Under the Weighted Mechanism, what is the utility for agent $i \in N$, when all agents' valuation function report profile is $(v'_1, ..., v'_n)$ and the allocation is a^*? (*note that their truthful valuation function profile is* $(v_1, ..., v_n)$)
2. (**5 credits**) Is the Weighted Mechanism truthful?
   - If your answer is yes, show the formal proof.
   - Otherwise, show a counter example (where an agent can misreport to gain a higher utility), or show a formal proof.

2 School Choice (6 credits)

Consider a school choice problem (matching students to schools) where each school has capacity one (i.e. each school can accept at most one student). Consider the following preference orders for three students a_1, a_2 and a_3, and priority orders for three schools b_1, b_2 and b_3:

$$\succ_{a_1}: b_2\ b_1\ b_3$$
$$\succ_{a_2}: b_1\ b_2\ b_3$$
$$\succ_{a_3}: b_1\ b_2\ b_3$$
$$\succ_{b_1}: a_1\ a_3\ a_2$$
$$\succ_{b_2}: a_2\ a_1\ a_3$$
$$\succ_{b_3}: a_2\ a_1\ a_3$$

Questions:

1. (**2 credits**) What is the stable matching of using student-proposing deferred acceptance? Use "(a_1, b_1)" to represent a_1 is matched with b_1.

2. (**4 credits**) The Boston mechanism (used in Boston high schools until 2005) is defined as follows:

> **Boston Mechanism**:
> - In step one, each student proposes to her first choice school, and students are matched with a school in order of school priority while there remains capacity.
> - In each subsequent step $k > 1$: each un-matched student proposes to her k-th most preferred school, and students are matched with a school in order of school priority while there remains capacity. The mechanism terminates when all students are matched.

Run the Boston mechanism on the above example.

(a) (3 credits) Show that student a_2 has a useful misreport (i.e. a misreport that will give a_2 a better match).

(b) (1 credit) Give a general description of the kind of manipulation that can be useful in the Boston mechanism.

3 Cooperative Game (7 credits)

Let the pair (N, v) be a *cooperative game* in characteristic function form, where N is the set of players (let $|N| = n$) and the value/characteristic function $v : 2^N \to \mathbb{R}$ is defined on N's power set (the set of all subsets of N). Consider the following instance of the game:

- $N = \{1, 2, 3, 4\}$
- The value function is defined as:

$$v(S) = \begin{cases} 100, & \text{if } S \in \{\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\},\{2,3,4\}\} \\ 0, & \text{otherwise} \end{cases}$$

Questions:

1. (**4 credits**) What is the Shapley value of all players in the above game?
   - the definition of the Shapley value of player i is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S))$$

2. (**3 credits**) Is their Shapley value in the *core* of the game?
   - If your answer is yes, justify why?
   - Otherwise, show an example where a group of players form a smaller coalition and there exists a payoff distribution in the smaller coalition such that the payoff of each player in the group is not less than her Shapley value and at least one player's payoff is greater than her Shapley value.

4 Repeated Game (6 credits)

Consider a Prisoners' Dilemma game with the payoff matrix given in Figure 1. Two players $\{1, 2\}$ play this PD game for $k \geq 1$ times/rounds. Define the discounted payoff to player $i \in \{1, 2\}$ in stage game t to be $\pi_{i,t}\delta^t$, where $\pi_{i,t}$ is the actual payoff in round $t \in [0, k-1]$ and $0 \leq \delta < 1$ is the discount factor. We analyse strategies that consist of decision rules about which action to play in each stage game. The discounted average payoff of a strategy for player i is defined as

$$\overline{\pi}_i = (1-\delta)\sum_{t=0}^{k-1} \pi_{i,t}\delta^t$$

	C	D
C	3, 3	-1, 5
D	5, -1	1, 1

Fig. 1: The payoff matrix of the Prisoners' Dilemma

Consider the following three strategies:

S1: Always play C until the other player played D, then play D for all the rest stage games.
S2: Play C in the first round ($t = 0$), then repeat what the other player did in the last round.
S3: Play D in the first round, then alternatively play C and D, i.e. the action sequence is $DCDCDCDCDC$....

Questions:

1. (**1 credit**) When $k = 1$, is there any dominant strategy for each player? If your answer is yes, specify all dominant strategies.
2. (**2 credits**) When $k = 4$, player 1 plays strategy $S2$ and player 2 plays $S3$, what is the discounted average payoff of player 2?
3. (**3 credits**) When $k = 4$ and player 1 plays strategy $S2$, under what condition $S3$ gives a higher discounted average payoff than $S1$ for player 2 to play? Justify your answer or show how you get your answer.