```
            0,1,0,0,
            1/2,0,0,1/2),nrow=4,ncol=4,byrow=TRUE)
```

to specify the transition matrix Q.

Next, we choose the number of states and the number of time periods to simulate, we allocate space for the results of the simulation, and we choose initial conditions for the chain. In this example, `x[1] <- sample(1:M,1)` says the initial distribution of the chain is uniform over all states.

```
M <- nrow(Q)
nsim <- 10^4
x <- rep(0,nsim)
x[1] <- sample(1:M,1)
```

For the simulation itself, we again use `sample` to choose a number from 1 to `M`. At time `i`, the chain was previously at state `x[i-1]`, so we must use row `x[i-1]` of the transition matrix to determine the probabilities of sampling $1, 2, \dots,$ `M`. The notation `Q[x[i-1],]` denotes row `x[i-1]` of the matrix `Q`.

```
for (i in 2:nsim){
    x[i] <- sample(M, 1, prob=Q[x[i-1],])
}
```

Since we set `nsim` to a large number, it may be reasonable to believe that the chain is close to stationarity during the latter portion of the simulation. To check this, we eliminate the first half of the simulations to give the chain time to reach stationarity:

```
x <- x[-(1:(nsim/2))]
```

We then use the `table` command to calculate the number of times the chain visited each state; dividing by `length(x)` converts the counts into proportions. The result is an approximation to the stationary distribution.

```
table(x)/length(x)
```

For comparison, the true stationary distribution of the chain is $(3/14, 2/7, 3/14, 2/7) \approx (0.214, 0.286, 0.214, 0.286)$. Is this close to what you obtained empirically?

11.7 Exercises

Markov property

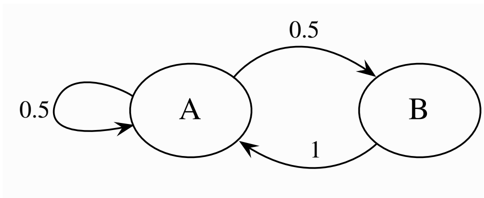
1. Ⓢ Let $X_0, X_1, X_2, \dots$ be a Markov chain. Show that $X_0, X_2, X_4, X_6, \dots$ is also a Markov chain, and explain why this makes sense intuitively.
2. Ⓢ Let $X_0, X_1, X_2, \dots$ be an irreducible Markov chain with state space $\{1, 2, \dots, M\}$,

$M \geq 3$, transition matrix $Q = (q_{ij})$, and stationary distribution $\mathbf{s} = (s_1, \dots, s_M)$. Let the initial state X_0 follow the stationary distribution, i.e., $P(X_0 = i) = s_i$.

(a) On average, how many of $X_0, X_1, \dots, X_9$ equal 3? (In terms of $\mathbf{s}$; simplify.)

(b) Let $Y_n = (X_n - 1)(X_n - 2)$. For $M = 3$, find an example of Q (the transition matrix for the *original* chain $X_0, X_1, \dots$) where $Y_0, Y_1, \dots$ is Markov, and another example of Q where $Y_0, Y_1, \dots$ is not Markov. In your examples, make $q_{ii} > 0$ for at least one i and make sure it is possible to get from any state to any other state eventually.

3. A Markov chain has two states, A and B, with transitions as follows:



Suppose we do not get to observe this Markov chain, which we'll call $X_0, X_1, X_2, \dots$. Instead, whenever the chain transitions from A back to A, we observe a 0, and whenever it changes states, we observe a 1. Let the sequence of 0's and 1's be called $Y_0, Y_1, Y_2, \dots$. For example, if the X chain starts out as

$$A, A, B, A, B, A, A, \dots$$

then the Y chain starts out as

$$0, 1, 1, 1, 1, 0, \dots$$

(a) Show that $Y_0, Y_1, Y_2, \dots$ is not a Markov chain.

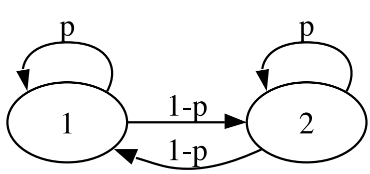
(b) In Example 11.1.3, we dealt with a violation of the Markov property by enlarging the state space to incorporate second-order dependence. Show that such a trick will not work for $Y_0, Y_1, Y_2 \dots$. That is, no matter how large m is,

$$Z_n = \{\text{the } (n - m + 1)\text{st to } n\text{th terms of the } Y \text{ chain}\}$$

is still not a Markov chain.

Stationary distribution

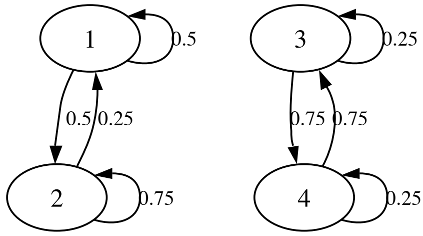
4. Ⓢ Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities.



(a) Write down the transition matrix Q for this chain.

(b) Find the stationary distribution of the chain.

(c) What is the limit of Q^n as $n \to \infty$?



5. Ⓢ Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicate transition probabilities.

   (a) Write down the transition matrix Q for this chain.

   (b) Which states (if any) are recurrent? Which states (if any) are transient?

   (c) Find two different stationary distributions for the chain.

6. Ⓢ Daenerys has three dragons: Drogon, Rhaegal, and Viserion. Each dragon independently explores the world in search of tasty morsels. Let X_n, Y_n, Z_n be the locations at time n of Drogon, Rhaegal, Viserion respectively, where time is assumed to be discrete and the number of possible locations is a finite number M. Their paths $X_0, X_1, X_2, \dots$; $Y_0, Y_1, Y_2, \dots$; and $Z_0, Z_1, Z_2, \dots$ are independent Markov chains with the same stationary distribution $\mathbf{s}$. Each dragon starts out at a random location generated according to the stationary distribution.

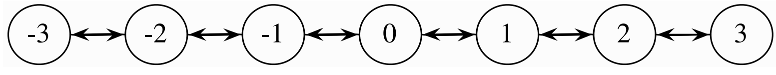
   (a) Let state 0 be home (so s_0 is the stationary probability of the home state). Find the expected number of times that Drogon is at home, up to time 24, i.e., the expected number of how many of $X_0, X_1, \dots, X_{24}$ are state 0 (in terms of s_0).

   (b) If we want to track all 3 dragons simultaneously, we need to consider the vector of positions, (X_n, Y_n, Z_n). There are M^3 possible values for this vector; assume that each is assigned a number from 1 to M^3, e.g., if $M = 2$ we could encode the states $(0, 0, 0), (0, 0, 1), (0, 1, 0), \dots, (1, 1, 1)$ as $1, 2, 3, \dots, 8$ respectively. Let W_n be the number between 1 and M^3 representing (X_n, Y_n, Z_n). Determine whether $W_0, W_1, \dots$ is a Markov chain.

   (c) Given that all 3 dragons start at home at time 0, find the expected time it will take for all 3 to be at home again at the same time.

Reversibility

7. Ⓢ A Markov chain $X_0, X_1, \dots$ with state space $\{-3, -2, -1, 0, 1, 2, 3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is $X_n - 1$ or $X_n + 1$, each with probability 1/2. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.



   (a) Is $|X_0|, |X_1|, |X_2|, \dots$ also a Markov chain? Explain.

   Hint: For both (a) and (b), think about whether the past and future are conditionally independent given the present; don't do calculations with a 7 by 7 transition matrix!

(b) Let sgn be the sign function: $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$. Is $\text{sgn}(X_0), \text{sgn}(X_1), \text{sgn}(X_2), \dots$ a Markov chain? Explain.

(c) Find the stationary distribution of the chain $X_0, X_1, X_2, \dots$.

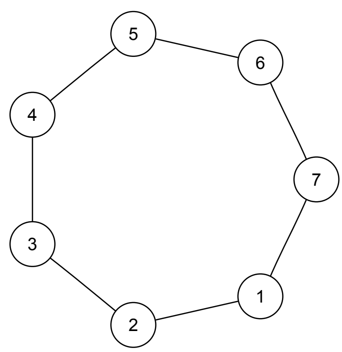
(d) Find a simple way to modify some of the transition probabilities q_{ij} for $i \in \{-3, 3\}$ to make the stationary distribution of the modified chain uniform over the states.

8. ⓢ Let G be an undirected network with nodes labeled $1, 2, \dots, M$ (edges from a node to itself are not allowed), where $M \geq 2$ and random walk on this network is irreducible. Let d_j be the degree of node j for each j. Create a Markov chain on the state space $1, 2, \dots, M$, with transitions as follows. From state i, generate a proposal j by choosing a uniformly random j such that there is an edge between i and j in G; then go to j with probability $\min(d_i/d_j, 1)$, and stay at i otherwise.

(a) Find the transition probability q_{ij} from i to j for this chain, for all states i, j.

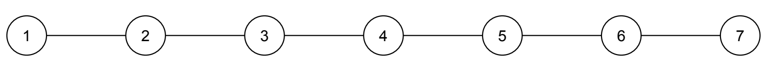
(b) Find the stationary distribution of this chain.

9. ⓢ (a) Consider a Markov chain on the state space $\{1, 2, \dots, 7\}$ with the states arranged in a "circle" as shown below, and transitions given by moving one step clockwise or counterclockwise with equal probabilities. For example, from state 6, the chain moves to state 7 or state 5 with probability 1/2 each; from state 7, the chain moves to state 1 or state 6 with probability 1/2 each. The chain starts at state 1.



Find the stationary distribution of this chain.

(b) Consider a new chain obtained by "unfolding the circle." Now the states are arranged as shown below. From state 1 the chain always goes to state 2, and from state 7 the chain always goes to state 6. Find the new stationary distribution.



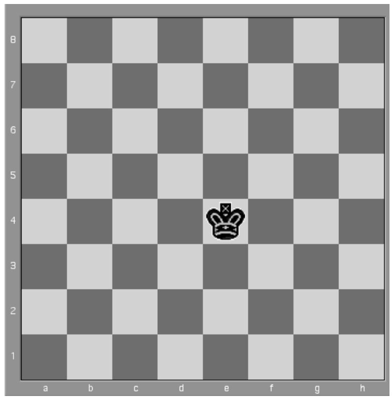
10. ⓢ Let X_n be the price of a certain stock at the start of the nth day, and assume that $X_0, X_1, X_2, \dots$ follows a Markov chain with transition matrix Q. (Assume for simplicity that the stock price can never go below 0 or above a certain upper bound, and that it is always rounded to the nearest dollar.)

ⓢ (a) A lazy investor only looks at the stock once a year, observing the values on days $0, 365, 2 \cdot 365, 3 \cdot 365, \dots$. So the investor observes $Y_0, Y_1, \dots$, where Y_n is the price after n years (which is $365n$ days; you can ignore leap years). Is $Y_0, Y_1, \dots$ also a Markov chain? Explain why or why not; if so, what is its transition matrix?

(b) The stock price is always an integer between \$0 and \$28. From each day to the next, the stock goes up or down by \$1 or \$2, all with equal probabilities (except for days when the stock is at or near a boundary, i.e., at \$0, \$1, \$27, or \$28).

If the stock is at \$0, it goes up to \$1 or \$2 on the next day (after receiving government bailout money). If the stock is at \$28, it goes down to \$27 or \$26 the next day. If the stock is at \$1, it either goes up to \$2 or \$3, or down to \$0 (with equal probabilities); similarly, if the stock is at \$27 it either goes up to \$28, or down to \$26 or \$25. Find the stationary distribution of the chain.

11. Ⓢ In chess, the king can move one square at a time in any direction (horizontally, vertically, or diagonally).



For example, in the diagram, from the current position the king can move to any of 8 possible squares. A king is wandering around on an otherwise empty 8×8 chessboard, where for each move all possibilities are equally likely. Find the stationary distribution of this chain (of course, don't list out a vector of length 64 explicitly! Classify the 64 squares into types and say what the stationary probability is for a square of each type).

12. A chess piece is wandering around on an otherwise vacant 8×8 chessboard. At each move, the piece (a king, queen, rook, bishop, or knight) chooses uniformly at random where to go, among the legal choices (according to the rules of chess, which you should look up if you are unfamiliar with them).

(a) For each of these cases, determine whether the Markov chain is irreducible, and whether it is aperiodic.

Hint for the knight: Note that a knight's move always goes from a light square to a dark square or vice versa. A *knight's tour* is a sequence of knight moves on a chessboard such that the knight visits each square exactly once. Many knight's tours exist.

(b) Suppose for this part that the piece is a rook, with initial position chosen uniformly at random. Find the distribution of where the rook is after n moves.

(c) Now suppose that the piece is a king, with initial position chosen deterministically to be the upper left corner square. Determine the expected number of moves it takes him to return to that square, fully simplified, preferably in at most 140 characters.

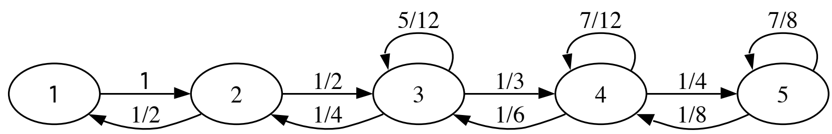
(d) The stationary distribution for the random walk of the king from the previous part is not uniform over the 64 squares of the chessboard. A recipe for modifying the chain to obtain a uniform stationary distribution is as follows. Label the squares as $1, 2, \dots, 64$, and let d_i be the number of legal moves from square i. Suppose the king is currently at square i. The next move of the chain is determined as follows:

Step 1: Generate a *proposal square* j by picking uniformly at random among the legal moves from i.

Step 2: Flip a coin with probability $\min(d_i/d_j, 1)$ of Heads. If the coin lands Heads, go to j. Otherwise, stay at i.

Show that this modified chain has a stationary distribution that is uniform over the 64 squares.

13. Ⓢ Find the stationary distribution of the Markov chain shown below, *without using matrices*. The number above each arrow is the corresponding transition probability.



14. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \dots, N\}$.

    (a) Give the transition probabilities of the chain.

    (b) Show that $(s_0, s_1, \dots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

    is the stationary distribution, by verifying the reversibility condition.

15. Nausicaa Distribution sells distribution plushies on Etsy. They have two different photos of the Evil Cauchy plushie but do not know which is more effective in getting a customer to purchase an Evil Cauchy plushie. Each visitor to their website is shown one of the two photos (call them Photo A and Photo B), and then the visitor either does buy an Evil Cauchy ("success") or does not buy one ("failure").

    Let a and b be the probabilities of success when Photo A is shown and when Photo B is shown, respectively. Even though the Evil Cauchy is irresistible, suppose that $0 < a < 1$ and $0 < b < 1$. Suppose that the following strategy is followed (note that the strategy can be followed without knowing a and b). Show the first visitor Photo A. If that visitor buys an Evil Cauchy, continue with Photo A for the next visitor; otherwise, switch to Photo B. Similarly, if the nth visitor is a "success" then show the $(n+1)$st visitor the same photo, and otherwise switch to the other photo.

    (a) Show how to represent the resulting process as a Markov chain, drawing a diagram and giving the transition matrix. The states are A1, B1, A0, B0 (use this order for the transition matrix and stationary distribution), where, for example, being at state A1 means that the current visitor was shown Photo A and was a success.

    (b) Determine whether this chain is reversible.

    Hint: First think about which transition probabilities are zero and which are nonzero.

    (c) Show that the stationary distribution is proportional to $\left(\frac{a}{1-a}, \frac{b}{1-b}, 1, 1\right)$, and find the stationary distribution.

    (d) Show that for $a \neq b$, the stationary probability of success for each visitor is strictly better than the success probability that would be obtained by independently, randomly choosing (with equal probabilities) which photo to show to each visitor.

16. This exercises considers random walk on a *weighted* undirected network. Suppose that an undirected network is given, where each edge (i, j) has a nonnegative weight w_{ij} assigned to it (we allow $i = j$ as a possibility). We assume that $w_{ij} = w_{ji}$ since the edge from i to j is considered the same as the edge from j to i. To simplify notation, define $w_{ij} = 0$ whenever (i, j) is not an edge.

    When at node i, the next step is determined by choosing an edge attached to i with probabilities proportional to the weights. For example, if the walk is at node 1 and there are 3 possible edges coming out from node 1, with weights $7, 1, 4$, then the first of these 3 edges is traversed with probability $7/12$, the second is traversed with probability $1/12$, and the third is traversed with probability $4/12$. If all the weights equal 1, then the process reduces to the kind of random walk on a network that we studied earlier.

    (a) Let $v_i = \sum_j w_{ij}$ for all nodes i. Show that the stationary distribution of node i is proportional to v_i.

    (b) Show that *every* reversible Markov chain can be represented as a random walk on a weighted undirected network.

    Hint: Let $w_{ij} = s_i q_{ij}$, where $\mathbf{s}$ is the stationary distribution and q_{ij} is the transition probability from i to j.

Mixed practice

17. Ⓢ A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to Room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

    (a) Find the stationary distributions of the cat chain and of the mouse chain.

    (b) Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases $1, 2, 3, 4$, respectively, and let Z_n be the number representing the (cat, mouse) state at time n. Is $Z_0, Z_1, Z_2, \dots$ a Markov chain?

    (c) Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

18. Let $\{X_n\}$ be a Markov chain on states $\{0, 1, 2\}$ with transition matrix

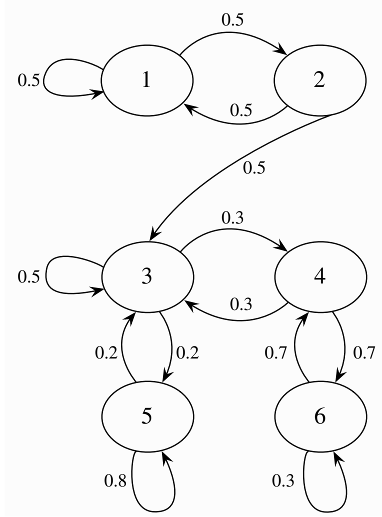
$$\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

    The chain starts at $X_0 = 0$. Let T be the time it takes to reach state 2:

$$T = \min\{n : X_n = 2\}.$$

    By drawing the Markov chain and telling a story, find $E(T)$ and $\mathrm{Var}(T)$.

19. Consider the following Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$.



(a) Suppose the chain starts at state 1. Find the distribution of the number of times that the chain returns to state 1.

(b) In the long run, what fraction of the time does the chain spend in state 3? Explain briefly.

20. Let Q be the transition matrix of a Markov chain on the state space $\{1, 2, \ldots, M\}$, such that state M is an *absorbing state*, i.e., from state M the chain can never leave. Suppose that from any other state, it is possible to reach M (in some number of steps).

(a) Which states are recurrent, and which are transient? Explain.

(b) What is the limit of Q^n as $n \to \infty$?

(c) For $i, j \in \{1, 2, \ldots, M-1\}$, find the probability that the chain is at state j at time n, given that the chain is at state i at time 0 (your answer should be in terms of Q).

(d) For $i, j \in \{1, 2, \ldots, M-1\}$, find the expected number of times that the chain is at state j up to (and including) time n, given that the chain is at state i at time 0 (in terms of Q).

(e) Let R be the $(M-1) \times (M-1)$ matrix obtained from Q by deleting the last row and the last column of Q. Show that the (i, j) entry of $(I - R)^{-1}$ is the expected number of times that the chain is at state j before absorption, given that it starts out at state i.

Hint: We have $I + R + R^2 + \cdots = (I - R)^{-1}$, analogously to a geometric series. Also, if we partition Q as

$$Q = \left(\begin{array}{c|c} R & B \\ \hline 0 & 1 \end{array}\right)$$

where B is a $(M-1) \times 1$ matrix and 0 is the $1 \times (M-1)$ zero matrix, then

$$Q^k = \left(\begin{array}{c|c} R^k & B_k \\ \hline 0 & 1 \end{array}\right)$$

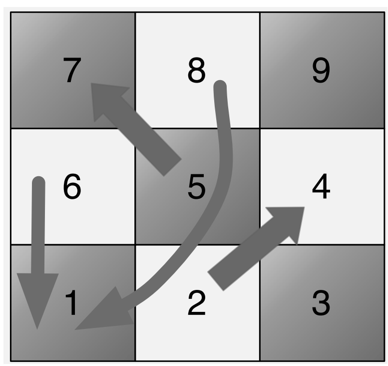
for some $(M-1) \times 1$ matrix B_k.

21. In the game called *Chutes and Ladders*, players try to be first to reach a certain destination on a board. The board is a grid of squares, numbered from 1 to the number of squares. The board has some "chutes" and some "ladders", each of which connects a pair of squares. Here we will consider the one player version of the game (this can be extended to the multi-player version without too much trouble, since with more than one player, the players simply take turns independently until one reaches the destination).

On each turn, the player rolls a fair die, which determines how many squares forward to move on the grid, e.g., if the player is at square 5 and rolls a 3, then he or she advances to square 8. If the resulting square is the base of a ladder, the player gets to climb the ladder, instantly arriving at a more advanced square. If the resulting square is the top of a chute, the player instantly slides down to the bottom of the chute.

This game can be viewed naturally as a Markov chain: given where the player currently is on the board, the past history does not matter for computing, for example, the probability of winning within the next 3 moves.

Consider a simplified version of Chutes and Ladders, played on the 3×3 board shown below. The player starts out at square 1, and wants to get to square 9. On each move, a fair coin is flipped, and the player gets to advance 1 square if the coin lands Heads and 2 squares if the coin lands Tails. However, there are 2 ladders (shown as upward-pointing arrows) and 2 chutes (shown as downward-pointing arrows) on the board.



(a) Explain why, despite the fact that there are 9 squares, we can represent the game using the 5×5 transition matrix

$$Q = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find the mean and variance for the number of times the player will visit square 7, *without* using matrices or any messy calculations.

The remaining parts of this problem require matrix calculations that are best done on a computer. You can use whatever computing environment you want, but here is some information for how to do it in R. In any case, you should state what environment you used and include your code. To create the transition matrix in R, you can use the following commands:

```
a <- 0.5
Q <- matrix(c(0,0,a,a,0,a,0,0,0,0,a,a,0,0,0,0,a,a,0,0,0,0,0,a,1),nrow=5)
```

Some useful R commands for matrices are in Appendix B.2. In particular, `diag(n)` gives the $n \times n$ identity matrix, `solve(A)` gives the inverse A^{-1}, and `A %*% B` gives the product AB (note that `A*B` does *not* do ordinary matrix multiplication). Matrix powers are not built into R, but you can compute A^k using `A %^% k` after installing and loading the `expm` package.

(c) Find the median duration of the game (defining duration as the number of coin flips).

Hint: Relate the CDF of the duration to powers of Q.

(d) Find the mean duration of the game (with duration defined as above).

Hint: Relate the duration to the total amount of time spent in transient states, and apply Part (e) of the previous problem.