Signals and Systems Homework 4 Solution

1. Suppose that we are given the following information about a signal $x[n]$

   1. $x[n]$ is a real and even signal.
   2. $x[n]$ has a period $N = 10$ and Fourier coefficients a_k.
   3. $a_{11} = 5$.
   4. $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$.

   Show that $x[n] = Acos(Bn + C)$, and specify numerical values for the constants A, B and C.

   Solution:

   Since the Fourier series coefficients of $x[n]$ has the same period with $x[n]$, we can get

   $$a_1 = a_{11} = 5$$

   Furthermore, since $x[n]$ is a real and even signal, then a_k is also real and even, so

   $$a_{-1} = a_1 = 5$$

   And it also implies

   $$a_9 = a_{-1} = 5$$

   We are also given

   $$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$$

   With $Parseval'stheorem$,

   $$\sum_{k=0}^{9}|a_k|^2 = 50$$

   For

   $$a_1^2 + a_9^2 = 50$$

   Then $a_k = 0$, for $k = 0, 2, 3..., 8$ (in a period 0,...,9), so

   $$x[n] = \sum_{k\in N} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{9} a_k e^{j\frac{2\pi}{10}kn} = 5e^{j\frac{2\pi}{10}n} + 5e^{j\frac{18\pi}{10}n} = 10\cos(\frac{\pi}{5}n)$$

   $$A = 10, B = \frac{\pi}{5}, C = 0.$$

2. Each of the two sequences $x_1[n]$ and $x_2[n]$ has a period $N = 4$, and the corresponding Fourier series coefficients are specified as

   $$x_1[n] \longleftrightarrow a_k, \quad x_2[n] \longleftrightarrow b_k$$

   where

   $$a_0 = a_3 = \frac{1}{2}a_1 = \frac{1}{2}a_2 = 1, \quad b_0 = b_1 = b_2 = b_3 = 1.$$

   Using the multiplication property of Fourier series, determine the Fourier series coefficients c_k for the signal $g[n] = x_1[n]x_2[n]$.

   Solution:

   Using the multiplication property, we have

   $$x_1[n]x_2[n] \stackrel{FS}{\longleftrightarrow} \sum_{l\in N} a_l b_{k-l} = \sum_{k=0}^{3} a_l b_{k-l} = b_k + 2b_{k-1} + 2b_{k-2} + 2b_{k-2}$$

   Since b_k is 1 for all values of k, it is clear that $b_K + 2b_{k-1} + 3b_{k-3}$ will be equal for all values of k, therefore

   $$g[n] = x_1[n]x_2[n] \stackrel{FS}{\longleftrightarrow} c_k = 6, \quad for \quad all \quad k$$

3. Let

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$$

be a period signal with fundamental period $T = 2$ and the Fourier coefficients a_k.

(a) Determine the value of a_0.

(b) Determine the Fourier series representation of $dx(t)/dt$.

(c) Use the result of part(b) and the differentiation property of the countinuous-time Fouries series to help determine the Fourier series coefficients of $x(t)$.

Solution:

(a) We have

$$a_0 = \frac{1}{2}\int_0^1 tdt + \frac{1}{2}\int_1^2 (2-t)dt = \frac{1}{2}$$

(b) The signal $g(t) = dx(t)/dt$ is

$$g(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 \le t \le 2 \end{cases}$$

And it is also with period $T = 2$, so the FS coefficients b_k is

$$b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

and

$$b_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt - \frac{1}{2}\int_1^2 e^{-j\pi kt}dt = \frac{1}{j\pi k}(1 - e^{-j\pi k})$$

(c) Note that

$$\frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T}a_k = jk\pi a_k$$

Then

$$a_k = \frac{1}{jk\pi}b_k = -\frac{1}{\pi^2 k^2}(1 - e^{-j\pi k})$$

4. Let

$$x[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & 8 \le n \le 9 \end{cases}$$

be a periodic signal with fundamental period $N = 10$ and Fourier sieres coefficients a_k. Also, let

$$g[n] = x[n] - x[n-1]$$

(a) Show that $g[n]$ has a fundamental period of 10.

(b) Determine the Fourier series coefficients of $g[n]$.

(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property (**page 222, Chapter 3.7.2 of Oppenheim's book), determine a_k for $k \neq 0$.**

Solution:

(a) For $0 \le n \le 9$, we have

$$g[n] = x[n] - x[n-1] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 7 \\ -1, & n = 8 \\ 0, & n = 9 \end{cases}$$

This period begin to show again in the following 10 points, its clearly to draw the conclusion that $g[n]$ is periodic with period of 10.

(b)) It is known that $T = 10$. So that the FS coefficients of $g[n]$ is b_k, which is

$$
\begin{aligned}
b_k &= \frac{1}{N}\sum_{N} g[n]e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{10}\sum_{n=0}^{9} g[n]e^{-jk\frac{\pi}{5}n} \\
&= \frac{1}{10}(1 - e^{-j\frac{8\pi}{5}k})
\end{aligned}
$$

(c) Since $g[n] = x[n] - x[n-1]$, , the FS coefficients a_k and b_k must be related as

$$b_k = a_k(1 - e^{-j\frac{\pi}{5}k})$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-jk\pi/5}} = \frac{1}{10}\frac{1 - e^{-jk8\pi/5}}{1 - e^{-jk\pi/5}}$$

5. Consider the following three continuous-time signals with a fundamental period of $T = \frac{1}{2}$:

$$x(t) = \cos(4\pi t)$$

$$y(t) = \sin(4\pi t)$$

$$z(t) = x(t)y(t)$$

(a) Determine the Fourier series coefficients of $x(t)$.

(b) Determine the Fourier series coefficients of $y(t)$.

(c) Use the result of part(a) and (b), along with the multiplication property of the countinuous-time Fourier series , to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.

(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part(c).

Solution:

(a)

$$x(t) = \cos(4\pi t) = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t}$$

So that the nonzero FS coefficients of $x(t)$ are $a_{-1} = a_1 = \frac{1}{2}$.

(b)

$$y(t) = \sin(4\pi t) = \frac{1}{2j}e^{j4\pi t} - \frac{1}{2j}e^{-j4\pi t}$$

So that the nonzero FS coefficients of $y(t)$ are $b_1 = \frac{1}{2j}, b_{-1} = -\frac{1}{2j}$

(c) Using the $multiplication property$, we know that

$$z(t) = x(t)y(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Therefore,

$$c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = \frac{1}{4j}\sigma[k-2] - \frac{1}{4j}\sigma[k+2]$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}, c_{-2} = -\frac{1}{4j}$

(d) We have

$$z(t) = \sin(4\pi t)\cos(4\pi t) = \frac{1}{2}\sin(8\pi t) = \frac{1}{4j}e^{j8\pi t} - \frac{1}{4j}e^{-j8\pi t}$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}, c_{-2} = -\frac{1}{4j}$, it is the same as part(c).

6. Consider the following three dicrete-time signals with a fundamental period of 6:

$$x[n] = 1 + cos(\frac{2\pi}{6}n), \quad y[n] = sin(\frac{2\pi}{6}n + \frac{\pi}{4}), \quad z[n] = x[n]y[n]$$

   (a) Determine the Fourier series coefficients of $x[n]$.
   (b) Determine the Fourier series coefficients of $y[n]$.
   (c) Use the result of part(a) and (b), along with the multiplication property of the discrete-time Fourier series , to determine the Fourier series coefficients of $z[n] = x[n]y[n]$.
   (d) Determine the Fourier series coefficients of $z[n]$ through direct evaluation, and compare your result with that of part(c).

Solution:

(a)

$$x[n] = 1 + \cos(\frac{2\pi}{6}n) = 1 + \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$$

So that the nonzero FS coefficients of $x[n]$ are $a_0 = 1, a_1 = a_{-1} = \frac{1}{2}$.

(b)

$$y[n] = \sin(\frac{2\pi}{6}n + \frac{\pi}{4}) = \frac{e^{j\pi/4}}{2j}e^{j2\pi n/6} - \frac{e^{-j\pi/4}}{2j}e^{-j2\pi n/6}$$

So that the nonzero FS coefficients of $y[n]$ are $b_1 = \frac{e^{j\pi/4}}{2j}, b_{-1} = -\frac{e^{-j\pi/4}}{2j}$

(c) Using the *multiplicationproperty*, we know that

$$z[n] = x[n]y[n] \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Therefore,

$$c_k = a_k * b_k = \sum_{l \in N} a_l b_{k-l} = a_{-1}b_{k+1} + a_0 b_k + a_{-1}b_{k-1}$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are

$$c_{-2} = -\frac{e^{-j\pi/4}}{4j}, c_{-1} = -\frac{e^{-j\pi/4}}{2j}, c_0 = \frac{\sqrt{2}}{4}, c_1 = \frac{e^{j\pi/4}}{2j}, c_2 = \frac{e^{j\pi/4}}{4j}$$

(d) For

$$z[n] = x[n]y[n]$$

Since $N = 6$, then take the value of $z[n]$ for $n = -3, -2, ..., 2$, then we get

$$z[-3] = 0, z[-2] = -\frac{1}{2}\sin\frac{5\pi}{12}, z[-1] = -\frac{3}{2}\sin\frac{\pi}{12}, z[0] = \sqrt{2}, z[1] = \frac{3}{2}\cos\frac{\pi}{12}, z[2] = \frac{1}{2}\cos\frac{5\pi}{12}$$

So substituting these $z[n]$ to

$$c_k = \frac{1}{N}\sum_{N} z[n]e^{-jk\frac{2\pi n}{N}}$$

The nonzero Fourier series coefficients of $z[n]$ are

$$c_{-2} = -\frac{e^{-j\pi/4}}{4j}, c_{-1} = -\frac{e^{-j\pi/4}}{2j}, c_0 = \frac{\sqrt{2}}{4}, c_1 = \frac{e^{j\pi/4}}{2j}, c_2 = \frac{e^{j\pi/4}}{4j}$$