Lecture 10 Wavelet Transform

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Course piazza link： piazza.com/shanghaitech.edu.cn/spring2021/cs270spring2021

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Outline

– Discrete Wavelet Transform (DWT)（小波变换）

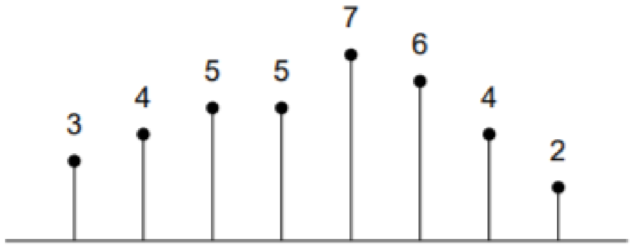
- An example for 1D-DWT
- Generalization of 1D-DWT
- 2D-DWT

Discrete Wavelet Transform (DWT)

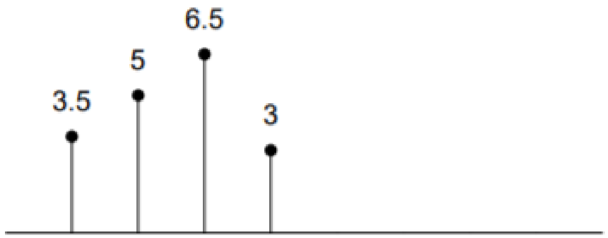
- Based on small waves called Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- Two types of DWT one for image processing （easy invertible) and one for signal processing (invertible but computational expensive).

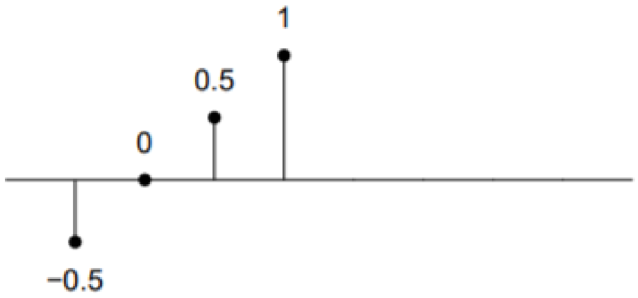
A simplest example

- We can decompose an eight-point signal x(n):



into two four-point signals:





$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1) \qquad d(n) = 0.5\,x(2n) - 0.5\,x(2n+1)$$

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A simplest example

- The above process can be represented by a block diagram:

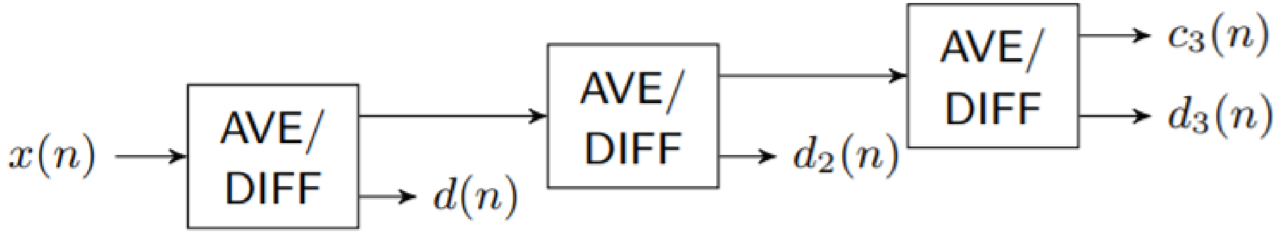
This decomposition can be easily reversed:

$$y(2n) = c(n) + d(n)$$
$$y(2n+1) = c(n) - d(n)$$

Which is also represented by a block diagram:

A simplest example

- When we repeat the simple AVE/DIFF signal decomposition:



The Haar wavelet representation of the eight-point signal x[n] is simply the set of four output signals produced by this three-level operation :

$$c_3 = [4.5]$$
$$d_3 = [-0.25]$$
$$d_2 = [-0.75, \quad 1.75]$$
$$d = [-0.5, \quad 0, \quad 0.5, \quad 1]$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Haar Transform matrix

- When N=2 we have:

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- When N=4 we have:

$$\mathbf{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

- When N=8 we have

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Haar Transform matrix

- The family of N Haar functions $\boldsymbol{h_k(t)}, (\boldsymbol{k = 0, \ldots, N-1})$ are defined on the interval $\mathbf{0 \le t \le 1}$. The shape of the specific function $\boldsymbol{h_k(t)}$ of a given index $\boldsymbol{k}$ depends on two parameters $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{k = 2^p + q - 1}$$

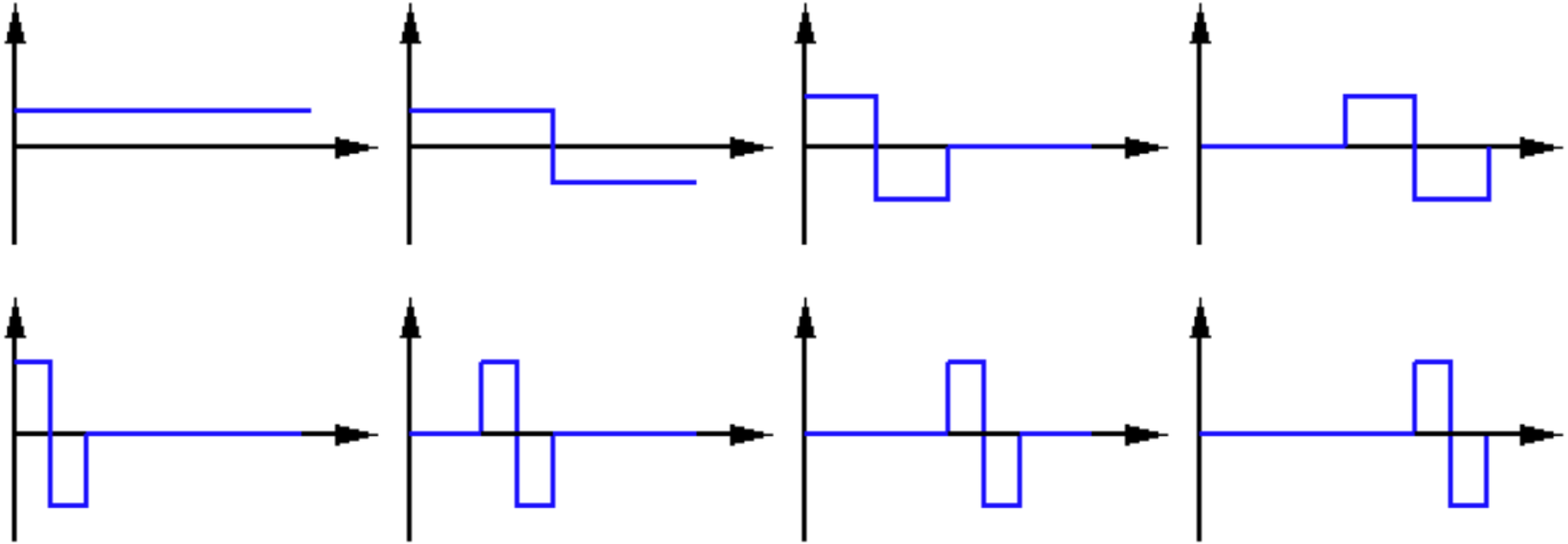
k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	0	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
q	0	1	1	2	1	2	3	4	1	2	3	4	5	6	7	8

- When $\boldsymbol{k > 0}$, the Haar function is defined by:

$$h_k(t) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \le t < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le t < q/2^p \\ 0 & \text{otherwise} \end{cases}$$

Haar Transform matrix

$$
\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2
\end{bmatrix}
$$



Generalization of 1D-DWT

- Discrete Wavelet Transform (DWT):

$$W_\varphi(j_0,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\varphi_{j_0,k}(n)$$

$$W_\psi(j,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\psi_{j,k}(n)\qquad j\geq j_0$$

- Inverse Discrete Wavelet Transform (IDWT):

$$f(n)=\frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(n)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(n)$$

Where

$\varphi_{j_0,k}(n)$: scaling function (尺度函数) $\psi_{j,k}(n)$: Wavelet (小波)

$W_\varphi(j_0,k)$: Approximation coefficients (近似系数) $W_\psi(j,k)$: detail coefficients (细节系数)

2D-DWT

- Define 2D wavelet function: Directionally sensitive wavelet

$$\psi^H(x,y)=\psi(x)\varphi(y) \qquad \psi^V(x,y)=\varphi(x)\psi(y) \qquad \psi^D(x,y)=\psi(x)\psi(y)$$

- 2D-DWT

$$W_\varphi(j_0,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$

$$W_\psi(j,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi^i_{j,m,n}(x,y) \qquad i=\{H,V,D\}$$

- 2D-IDWT

$$f(x,y)=\frac{1}{\sqrt{MN}}\sum_{m}\sum_{n} W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$

$$+\frac{1}{\sqrt{MN}}\sum_{i=\{H,V,D\}}\sum_{j=j_0}^{\infty}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} W_\psi(j,m,n)\,\psi^i_{j,m,n}(x,y)$$

Haar Transform matrix

Input: image size 8X8 $\boldsymbol{I_{in}}$

Generate a Haar matrix of 8X8 as shown right

Then clip it into 4 part:

$$\boldsymbol{H_{L1}(dim = 4 * 8); H_{L2}(dim = 2 * 8);}$$
$$\boldsymbol{H_{L3}(dim = 1 * 8); L_{L3}(dim = 1 * 8);}.$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{array}{cl} \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \hline \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ \hline 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{array}\right] & \begin{array}{l} \boldsymbol{L_{L3}} \\ \boldsymbol{H_{L3}} \\ \boldsymbol{H_{L2}} \\ \\ \\ \boldsymbol{H_{L1}} \\ \\ \\ \end{array} \end{array}$$

For computing *level 1* components:

LL_1 is down-sample of $\boldsymbol{I_{in}}$ on both X and Y direction	$\boldsymbol{HL_1 = H_{L1} * I_{in}}$ + downsample on Y direction
$\boldsymbol{LH_1 = I_{in} * H_{L1}}$ + downsample on X direction	$\boldsymbol{HH_1 = H_{L1} * I_{in} * H_{L1}}$

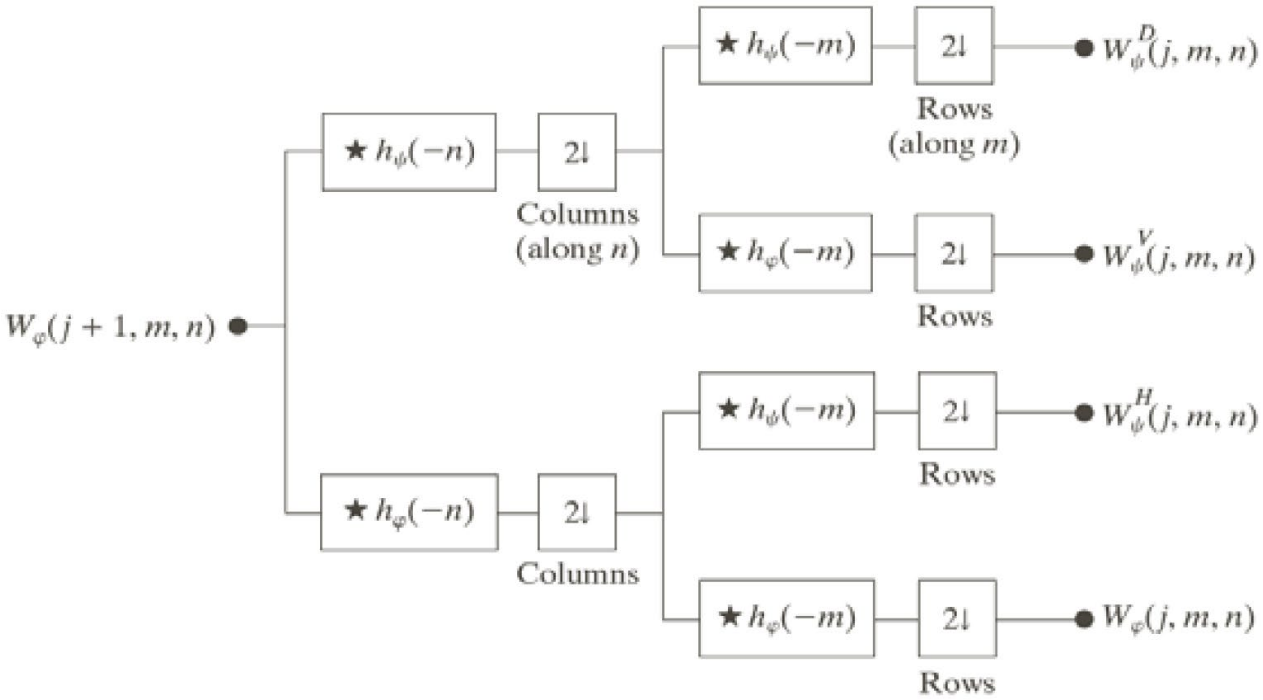
For computing *level 2* components:

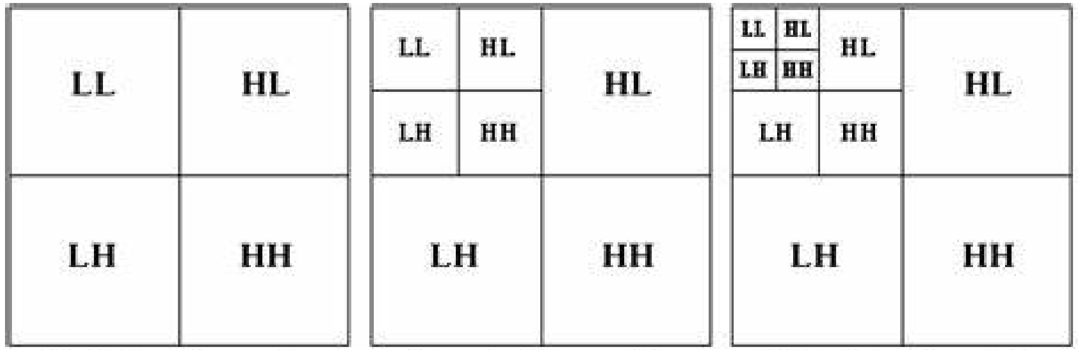
LL_2 is down-sample of LL_1 on both X and Y direction	$\boldsymbol{HL_2 = H_{L2} * I_{in}}$ + downsample **twice** on Y direction
$\boldsymbol{LH_2 = I_{in} * H_{L2}}$ + downsample **twice** on X direction	$\boldsymbol{HH_2 = H_{L2} * I_{in} * H_{L2}}$

For computing *level 3* components:

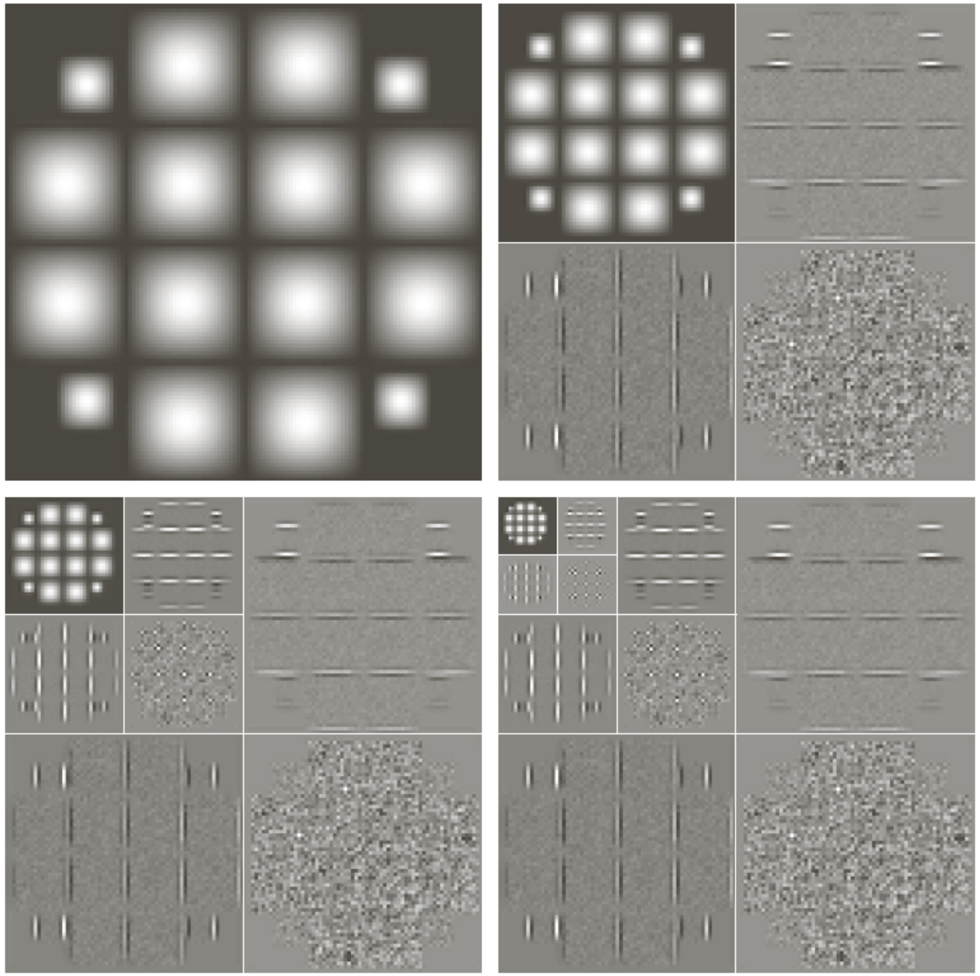
LL_3 is down-sample of LL_2 on both X and Y direction	$\boldsymbol{HL_3 = H_{L3} * I_{in}}$ + downsample **3 times** on Y direction
$\boldsymbol{LH_3 = I_{in} * H_{L3}}$ + downsample **3 times** on X direction	$\boldsymbol{HH_3 = H_{L3} * I_{in} * H_{L3}}$

2D-DWT



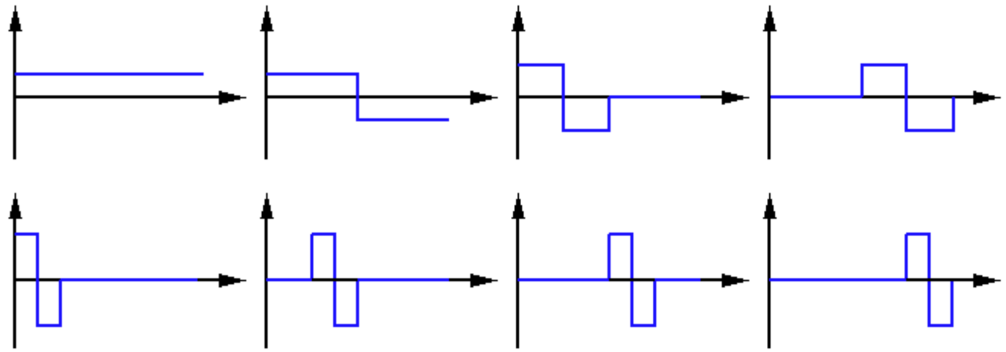


(a) Single Level Decomposition (b) Two Level Decomposition (c) Three Level Decomposition

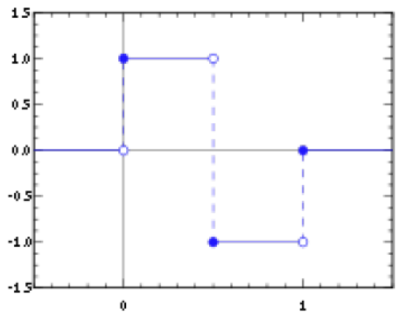


Mother Wavelet (母小波)

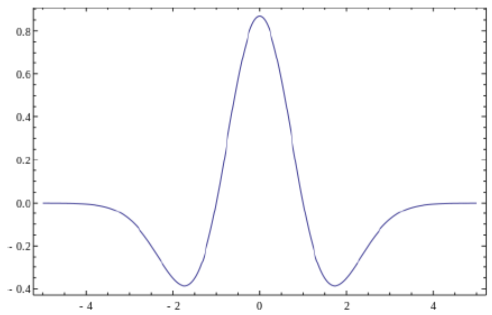
- Mother Wavelet should satisfy:
  - $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$
  - $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$
  - $\int_{-\infty}^{\infty} \psi(t) dt = 0$



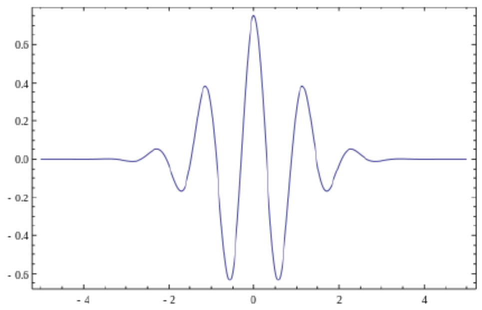
Haar



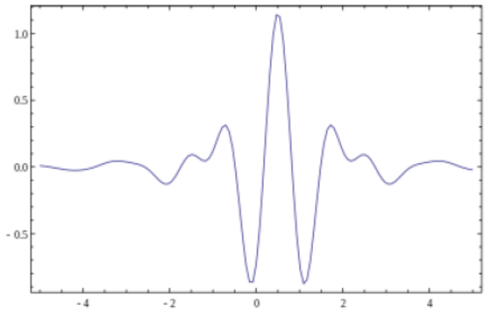
Mexican Hat



Morlet



Meyer



Take home message

- Based on small waves called Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- JPEG2000, FBI finger printing databased.