Siganls & System: Homework #8

Due on January 6, 2022 at 23:59

HW8-solution

Problem 1. (3 × 5 points)

Determine the z-transform for each of the following sequences. Sketch the pole zero plot and indicate the ROC.

1) $2^n u[-n] + (\frac{1}{2})^n u[n-1]$
2) $4^n cos[\frac{\pi}{3}n + \frac{\pi}{4}]u[-n-1]$
3) $n(\frac{1}{2})^{|n|}$

Solution.

1)

$$x_1[n] = 2^n u[-n],$$

$$X_1(Z) = \sum_{n=-\infty}^{0} (2)^n z^{-n} = \sum_{n=0}^{+\infty} (2)^{-n} z^n = \frac{-2z^{-1}}{1-2z^{-1}}, |z| < 2.$$

$$x_2[n] = \frac{1}{2}^n u[n-1],$$

$$X_2(Z) = \sum_{n=1}^{+\infty} \frac{1}{2}^n z^{-n} = \sum_{n=0}^{+\infty} \frac{1}{2}^{n+1} z^{-n-1} = \frac{z^{-1}/2}{1-(1/2)z^{-1}}, |z| > \frac{1}{2}.$$

$$x[n] = x_1[n] + x_2[n],$$

$$X(Z) = \frac{-2z^{-1}}{1-2z^{-1}} + \frac{\frac{z^{-1}}{2}}{1-\frac{1}{2}z^{-1}} = \frac{3z}{(2-z)(2z-1)}$$

ROC: $\frac{1}{2} < |z| < 2.$

Pole: $2, \frac{1}{2}$, Zero:0.

2)

$$x[n] = 4^n \frac{e^{j\frac{\pi n}{3}+\frac{\pi}{4}} + e^{-j\frac{\pi n}{3}+\frac{\pi}{4}}}{2} u[-n-1],$$

$$X(Z) = \frac{e^{j\pi/4}}{2} \frac{1}{4e^{\frac{j\pi}{3}}z^{-1}-1} + \frac{e^{-j\pi/4}}{2} \frac{1}{4e^{\frac{-j\pi}{3}}z^{-1}-1}, |z| < 4.$$

ROC: $Z < |4|$.

Pole: $4e^{\frac{j\pi}{3}}, 4e^{\frac{-j\pi}{3}}$; Zero: $0,\ 2+2\sqrt{3}$.

3)

$$x[n] = n(\frac{1}{2})^{|n|} = n(\frac{1}{2})^n u[n] + n2^n u[-n-1]$$

$$X(Z) = \frac{\frac{z^{-1}}{2}}{(1-\frac{1}{2}z^{-1})^2} - \frac{2z^{-1}}{(1-2z^{-1})^2}.$$

ROC: $\frac{1}{2} < |z| < 2.$

Zero: 0, 1, -1. Pole: $\frac{1}{2}$, 2.

Problem 2

(2 * 5 points) Suppose we are given the following facts about a paorticular LTI system S with impluse response h[n] and z-transform H(z).

- h[n] is real.
- h[n] is right-sided.
- $\lim_{z\to+\infty} H(z) = 0$.
- H(z) has two zeros.
- H(z) has one of its poles at a non-real location on the circle defined by $|z| = \frac{3}{4}$

Determine the correctness of the following statements. Correct them if they are incorrect and give reasons:

(a) Since $\lim_{z\to+\infty} H(z) = 0$, H(z) has no poles at infinity. Furthermore, since h[n] is right sided, h[n] has to be casual.

(b) Since h[n] is causal, the numerator and denominator polynomials of H(z) have the same order. Since H(z) is given to have two zeros, we may conclude that it also has two poles. Since h[n] is real, the poles must occur in conjugate pairs. Also, it is given that one of the poles lies on the circle defined by $|z| = \frac{3}{4}$. Therefore, the other pole also lies on this circle.From above analysis, we can conclude that ROC of H(z) will be of form $|z| > \frac{3}{4}$, which include the unit circle. As a result, the system is stable.

Solution

(a)

True.

(b)

Wrong.

From the statement of (b), we can only induce that the order of the numerator cannot be greater than the order of the denominator. So there are 2 or more poles. If there is one pole located outside of the unit circle, we cannot say that the system is stable.

Problem 3. (3 × 5 points)

A causal LTI discrete-time system is described by the difference equation

$$y[n] = 0.4y[n-1] + 0.05y[n-2] + 3x[n]$$

where x[n] and y[n] are the input and output sequences of the system, respectively.

1) Determine the transfer function $H(z)$ of the system.
2) Determine the impulse response $h[n]$ of the system.
3) Determine the step response $s[n]$ of the system.

Solution.

1) After z-transform,

$$Y(z) = 0.4Y(z)z^{-1} + 0.05Y(z)z^{-2} + 3X(z) \tag{1}$$

Therefore, we could obtain:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1 - 0.4z^{-1} - 0.05z^{-2}} \tag{2}$$

Since the system is casual, ROC: |z| > 0.5.

2) From 1) we could get

$$H(z) = \frac{3}{1 - 0.4z^{-1} - 0.05z^{-2}} = \frac{0.5}{1 + 0.1z^{-1}} + \frac{2.5}{1 - 0.5z^{-1}}$$

Because this is a causal LTI discrete-time system, we can get the impulse response as

$$h[n] = (0.5 \times (-0.1)^n + 2.5 \times 0.5^n)\mu[n] \tag{3}$$

3)

$$X(z) = \frac{1}{1 - z^{-1}}$$

$$S(Z) = X(Z)H(Z) = \frac{3}{(1 + 0.1z^{-1})(1 - 0.5z^{-1})(1 - z^{-1})} = \frac{1/22}{1 + 0.1z^{-1}} - \frac{5/2}{1 - 0.5z^{-1}} + \frac{60/11}{1 - z^{-1}}.$$

ROC: |z| > 1.

$$s[n] = (\frac{1}{22}(-0.1)^n - \frac{5}{2}(0.5)^n + \frac{60}{11})u[n]$$

Problem 4. (3×10 points)

Consider the system function corresponding to casual LTI systems:

$$H(Z) = \frac{1}{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2})}.$$

1) Draw a direct-form block diagram.
2) Draw a block diagram that corresponds to the cascade connection of two second-order block diagrams.
3) Determine whether there exists a block diagram which is the cascade of four first-order block diagrams with the constraint that all the coefficient multipliers must be real. If false, state the reason. If true, draw the diagram.

Solution.

See Figure 1.

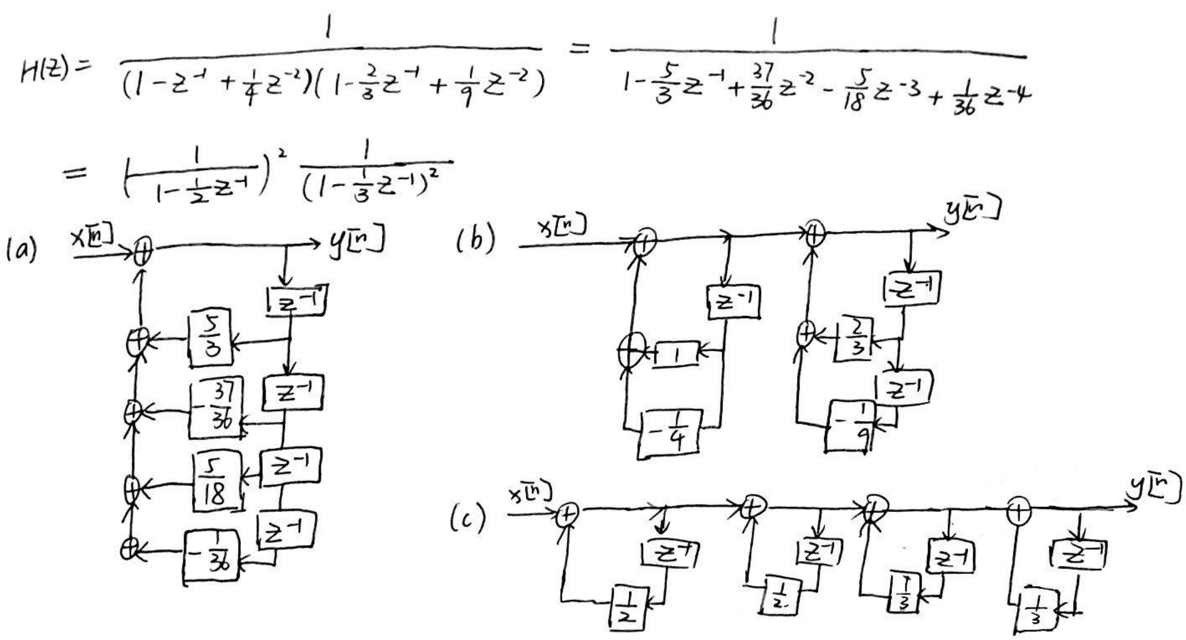

Figure 1: P4: Block Diagram

Problem 5

(3 * 10 points) Consider a system whose input x[n] and output y[n] are related by

$$y[n-2] + 5y[n-1] + 6y[n] = x[n].$$

(a) Determine the zero input response of this system if y[-2] = 6 and y[-1] = 0.

(b) Determine the zero state response of this system to the input $x[n] = -6\delta[n]$.

(c) Determine the output of this system for n≥0 when $x[n] = -6\delta[n]$, y[-2]=6 and y[-1]=0.

Solution:
(a)
Hence it's zero input response, the input X[n] = 0.
Applying the unilateral z-transform to the given differential equation, we have

$$z^{-2}(Y(z) + zy[-1] + z^2y[-2])) + 5z^{-1}(Y(z) + zy[-1]) + 6Y(z) = 0$$

$$\Rightarrow (z^{-2} + 5z^{-1} + 6)Y(z) + 6 = 0$$

$$\Rightarrow Y(z) = \frac{-6}{z^{-2}+5z^{-1}+6} = -\frac{6}{z^{-1}+2} + \frac{6}{z^{-1}+3}$$

so the zero input response is:

$$\text{y[n]} = [2(-\tfrac{1}{3})^n - 3(-\tfrac{1}{2})^n]\text{u[n]}.$$

(b)
Hence it's zero input response, the state y[-2] = y[-1] = 0.
Applying the unilateral z-transform to the given differential equation, we have

$$z^{-2}Y(z) + 5z^{-1}Y(z) + 6Y(z) = X(z) = -6$$

$$\Rightarrow Y(z) = \frac{-6}{z^{-2}+5z^{-1}+6} = \frac{-6}{z^{-2}+5z^{-1}+6} = -\frac{6}{z^{-1}+2} + \frac{6}{z^{-1}+3}$$

so the zero state response is:

$$\text{y[n]} = [2(-\tfrac{1}{3})^n - 3(-\tfrac{1}{2})^n]\text{u[n]}.$$

(c)
The total response is the sum of the zero state and zero input responses, so

$$\text{y[n]} = [4(-\tfrac{1}{3})^n - 6(-\tfrac{1}{2})^n]\text{u[n]}.$$