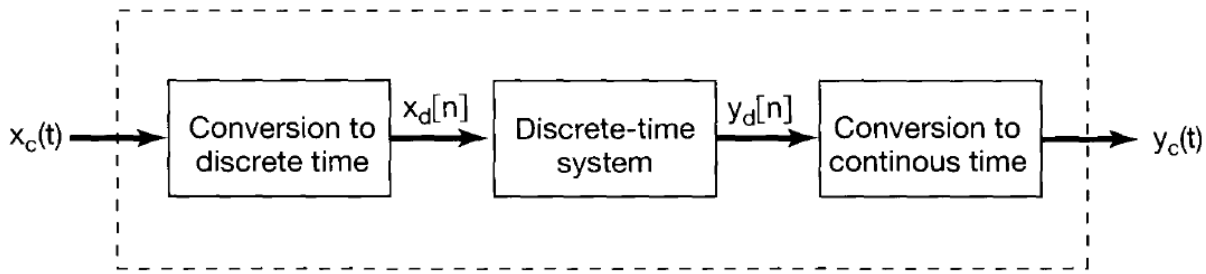
Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time signals

Discrete-Time Processing of Continuous-Time Signals

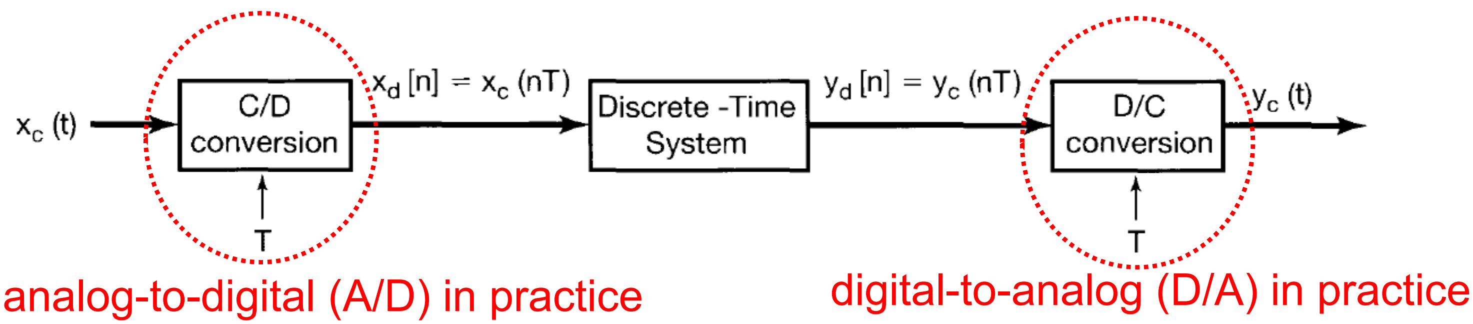
General scheme



$$x_d[n] = x_c(nT)$$

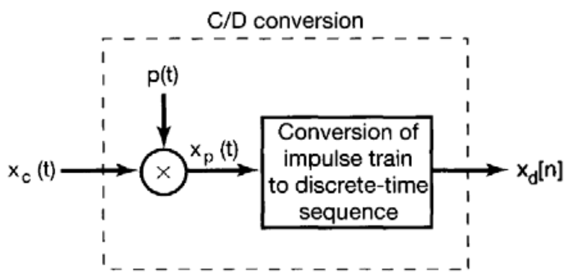
$$y_d[n] = y_c(nT)$$

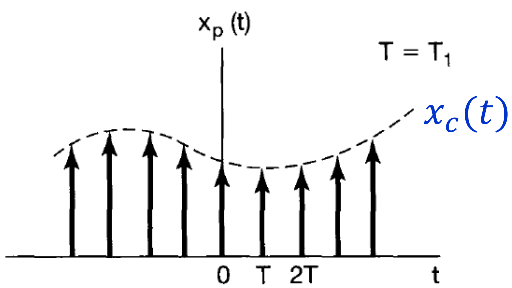
- C/D: continuous-to-discrete-time conversion
- D/C: discrete-to-continuous-time conversion



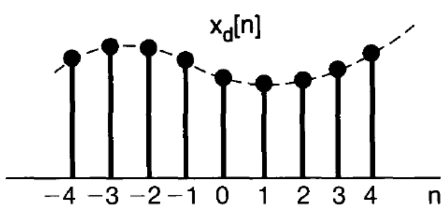
Discrete-Time Processing of Continuous-Time Signals

C/D conversion Time domain





$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

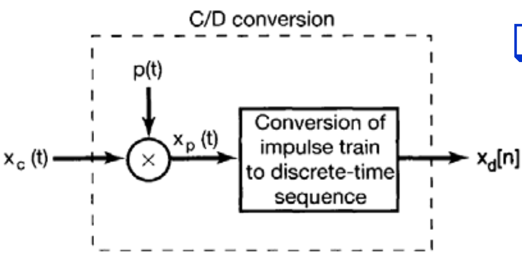


$$x_d[n] = x_c(nT)$$

Discrete-Time Processing of Continuous-Time Signals

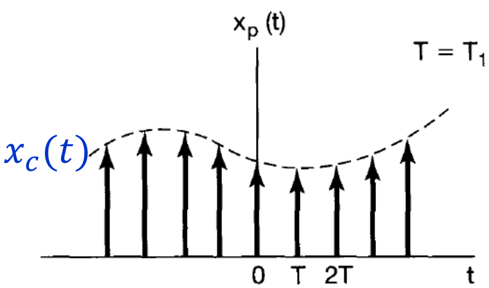
C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time



- Spectrum of $x_d[n]$

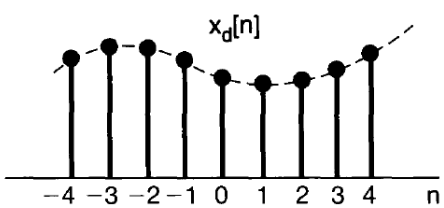
$$X_d(e^{j\Omega}) = \sum_{-\infty}^{\infty} x_d[n] e^{-jn\Omega} = \sum_{-\infty}^{\infty} x_c(nT) e^{-jn\Omega}$$



- Spectrum of $x_p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT) \implies X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot e^{-j\omega nT}$$

- If $\omega = \Omega/T$, $\boxed{X_d(e^{j\Omega}) = X_p(j\Omega/T)}$

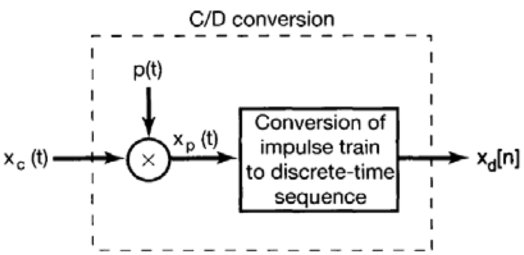


- The spectrum of $x_d[n]$ can be obtained from $X_p(j\omega)$ by replacing ω with Ω/T.

Discrete-Time Processing of Continuous-Time Signals

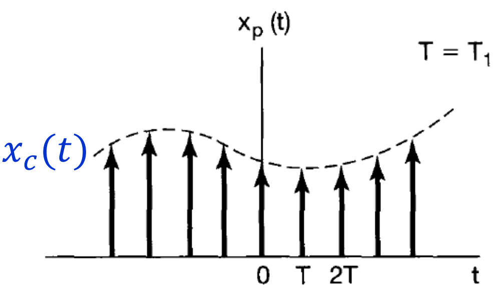
C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time

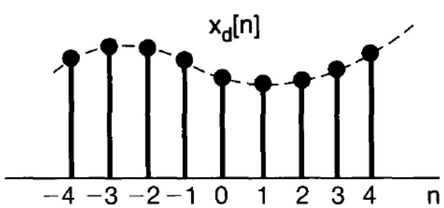


- Recall: $X_p(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\big(j(\omega - k \cdot \omega_s)\big)$

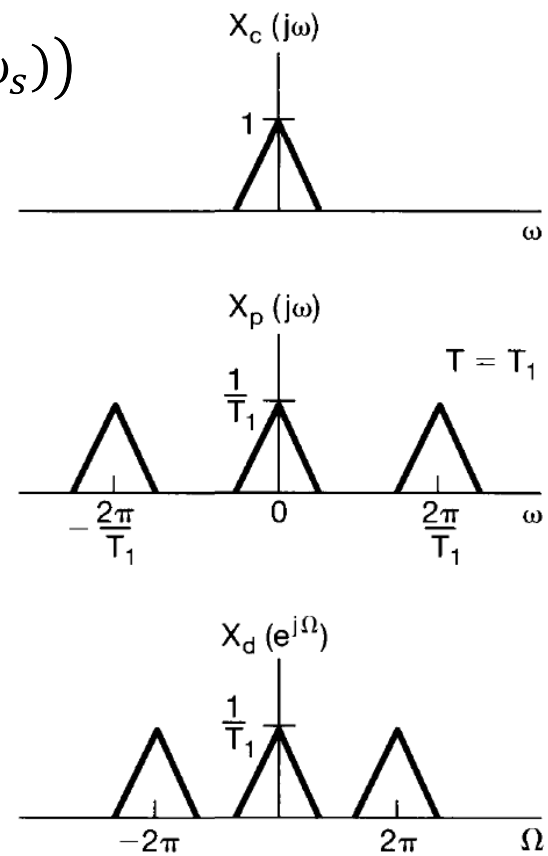
$$\boxed{\therefore X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j(\Omega - 2k\pi)/T)}$$



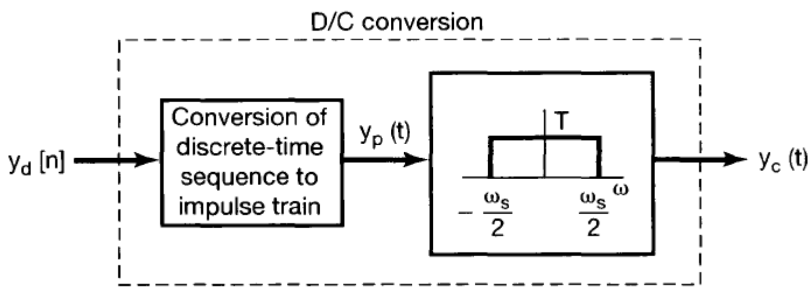
- $X_d\left(e^{j\Omega}\right)$ is a frequency-scaled version of $X_p(j\omega)$, and is periodic with period of 2π



- Informally
  - t to n: time scaling by 1/ T
  - ω to Ω: frequency scaling by T

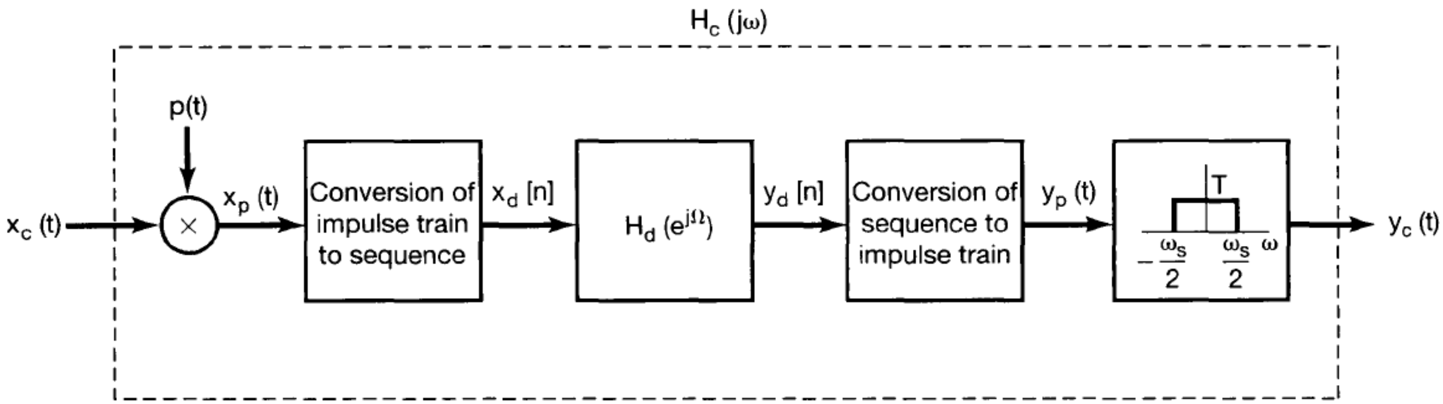


D/C conversion



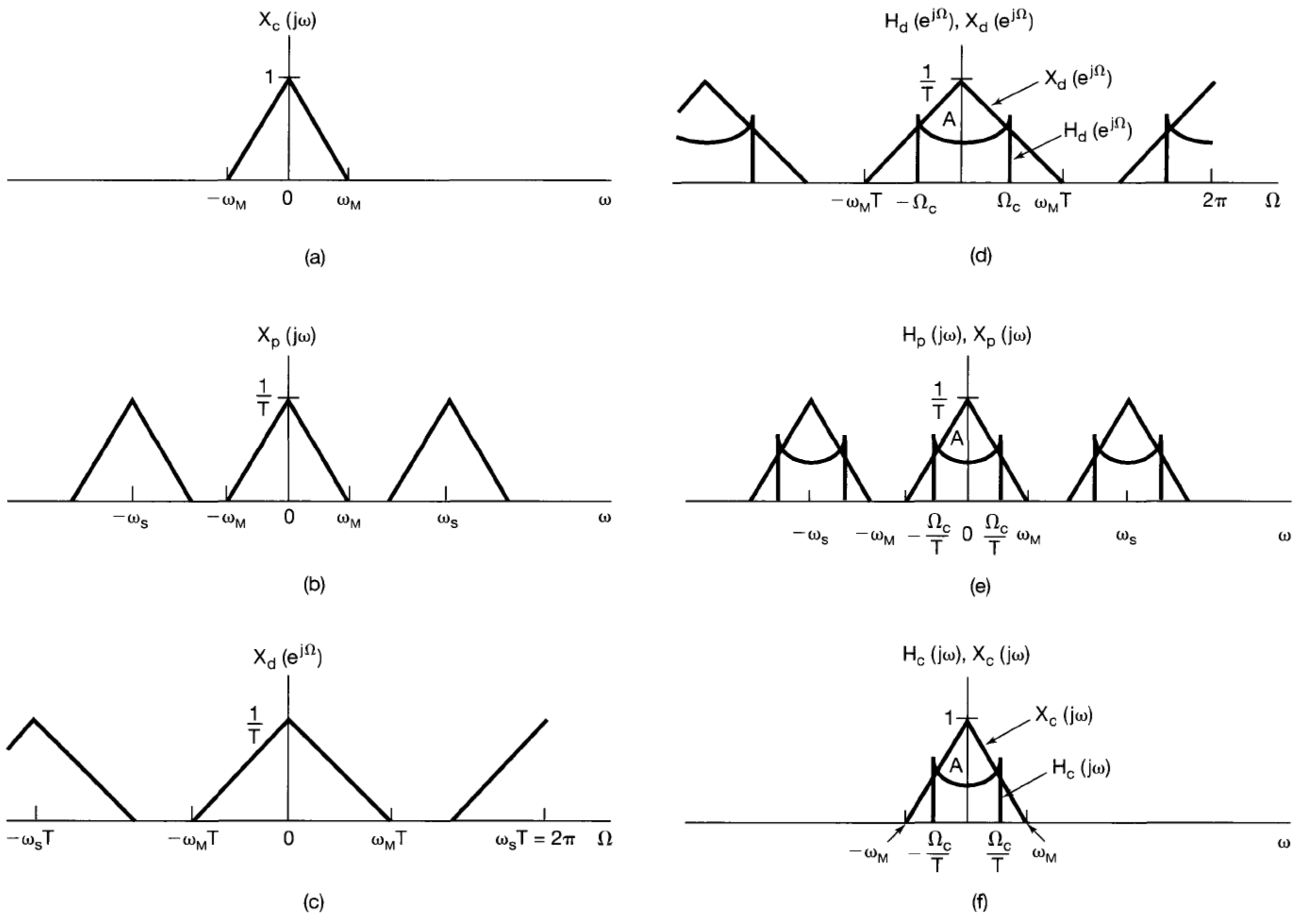
- $Y_d(e^{j\Omega})$: Spectrum of $y_d[n]$
- $Y_p(j\omega)$: Spectrum of $y_p(t)$
- $Y_p(j\omega)$ can be obtained from $Y_d(e^{j\Omega})$ by replacing Ω with ωT.

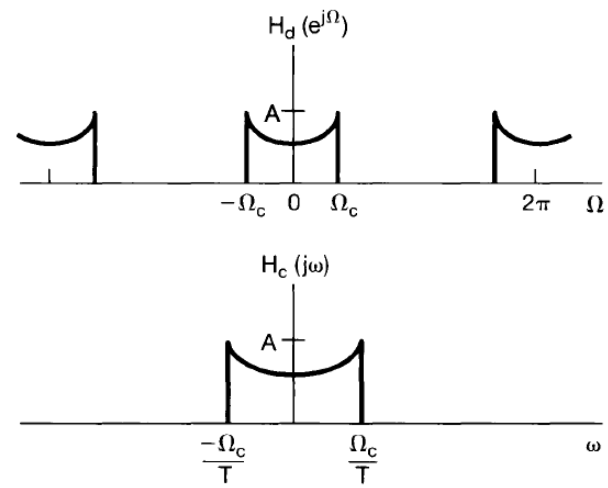
Overall system



- ❑ $x_c(t)$: input
- ❑ $y_c(t)$: output
- ❑ The overall system is equivalent to a continuous-time system with frequency response $H_c(j\omega)$
- ❑ $H_c(j\omega)$ =?

Overall system

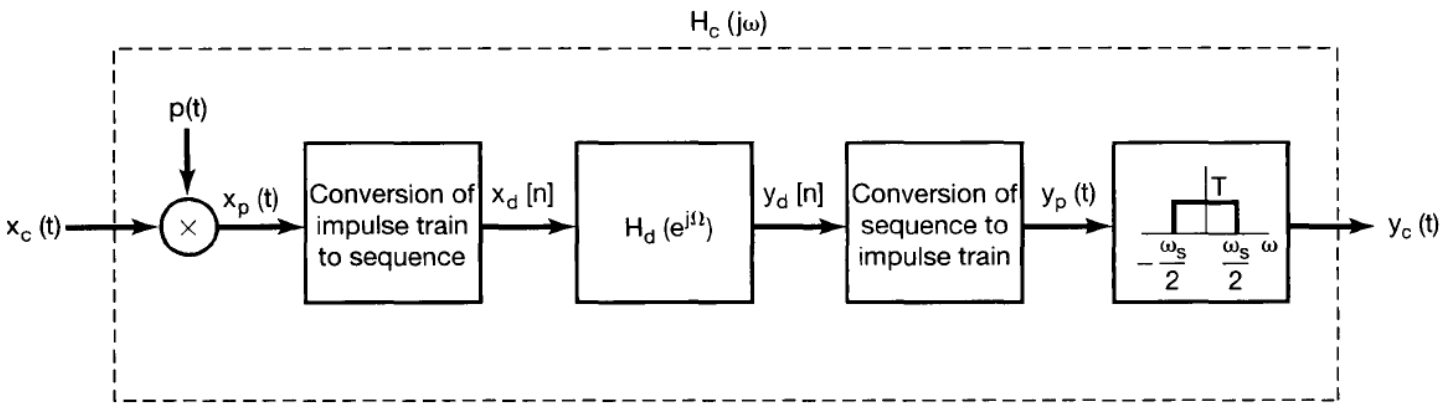




$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

Discrete-Time Processing of Continuous-Time Signals

Overall system



$$\Updownarrow \qquad \omega = \Omega/T$$

H_c (jω)

x_c (t)

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

y_c (t)

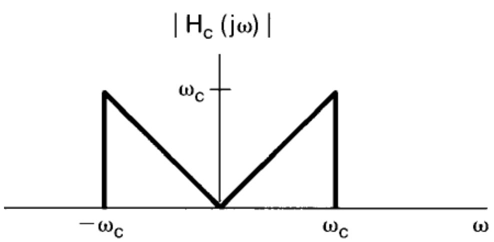
Digital differentiator: frequency response

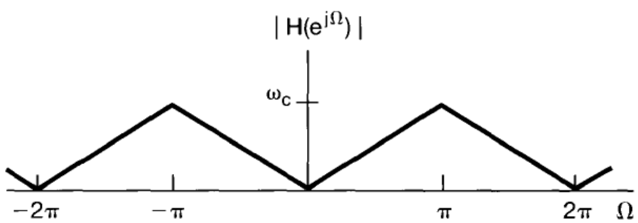
- ❑ Band-limited CT differentiator $\Longleftrightarrow$ ❑ Corresponding DT differentiator

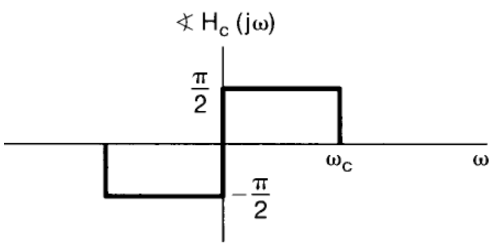
$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

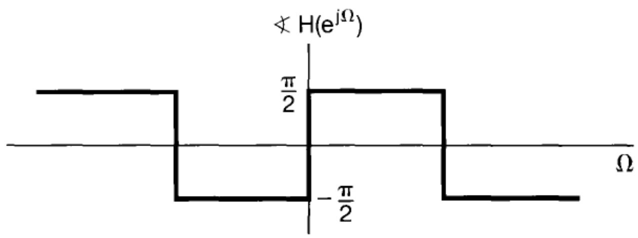
$$\boxed{H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi}$$

$\omega_c = \omega_s/2$





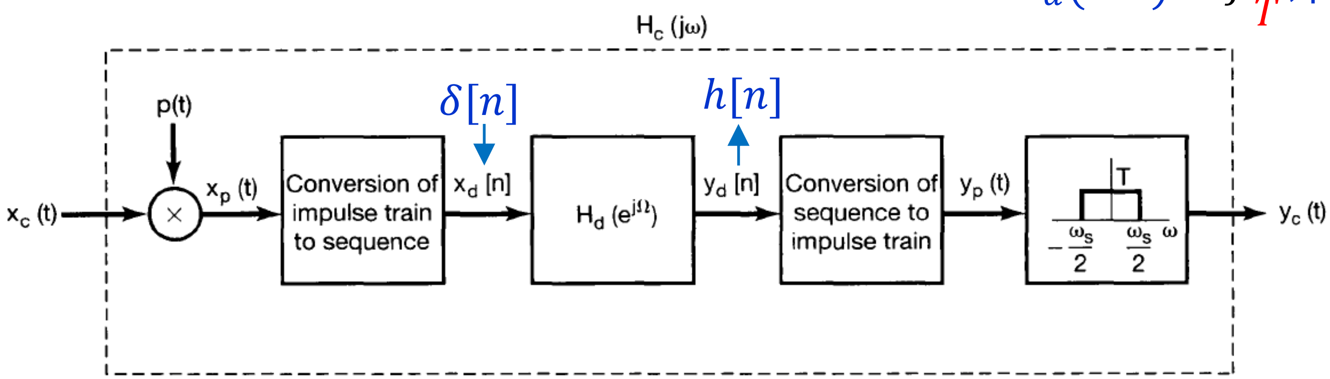




Discrete-Time Processing of Continuous-Time Signals

Digital differentiator: impulse response

$$H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi$$



- $x_c(t) = \frac{\sin(\pi t/T)}{\pi t} \Longrightarrow x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$
- $y_d[n] = y_c(nT)$ $\qquad y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}$
- $y_d[n] = \begin{cases} \frac{(-1)^n}{nT^2}, n \neq 0 \\ 0, \ n = 0 \end{cases} \Longrightarrow \boxed{\therefore h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, n \neq 0 \\ 0, \qquad n = 0 \end{cases}}$

Sampling (ch.7)

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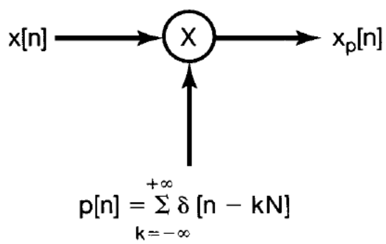
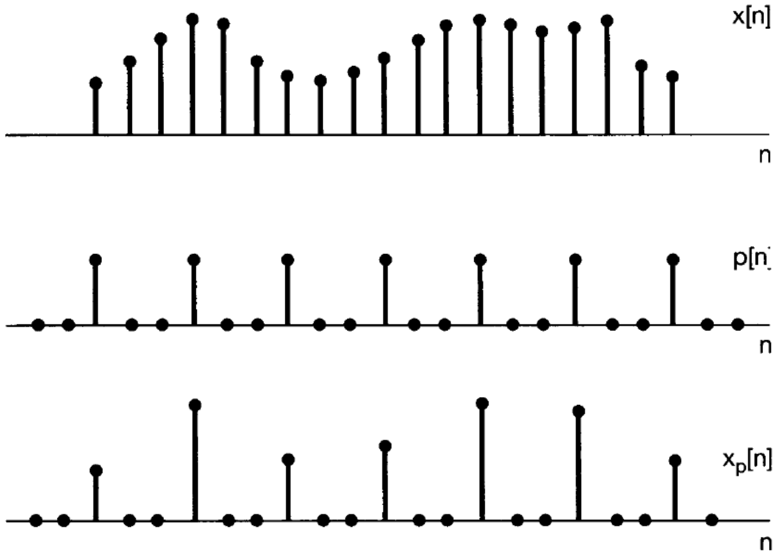
Sampling of Discrete-Time Signals

Impulse train sampling Time domain



N: sampling period



$$x_p[n] = x[n]p[n] = \sum_{-\infty}^{\infty} x[kN]\delta[n-kN]$$

$$= \begin{cases} x[n], \text{if } n \text{ is an integer multiple of N} \\ 0, \qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

Sampling of Discrete-Time Signals

Impulse train sampling Frequency domain

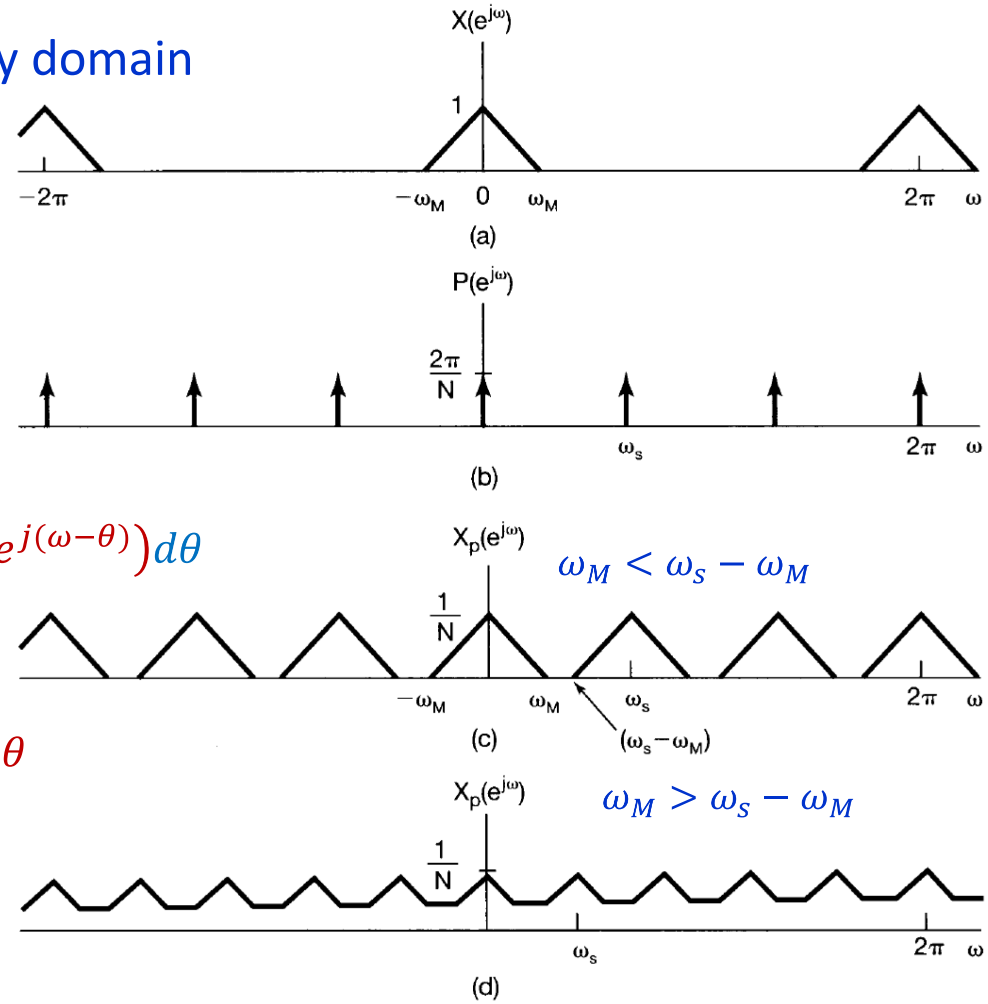
$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{K=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad \omega_s = \frac{2\pi}{N}$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X\left(e^{j(\omega-\theta)}\right)d\theta$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{N}\int_{2\pi}\left[\sum_{K=-\infty}^{\infty}\delta(\theta - k\omega_s)\right]X\left(e^{j(\omega-\theta)}\right)d\theta$$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\int_{2\pi}\delta(\theta - k\omega_s)X\left(e^{j(\omega-\theta)}\right)d\theta$$

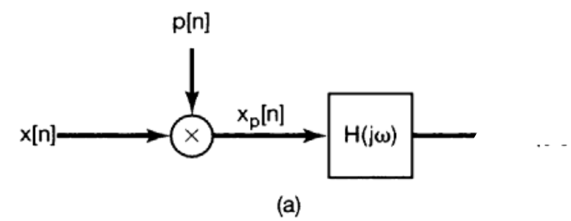
$$\boxed{X_p(e^{j\omega}) = \frac{1}{N}\sum_{K=0}^{N-1}X\left(e^{j(\omega-k\cdot\omega_s)}\right)}$$



Sampling of Discrete-Time Signals

Impulse train sampling

Reconstruction of x[n]



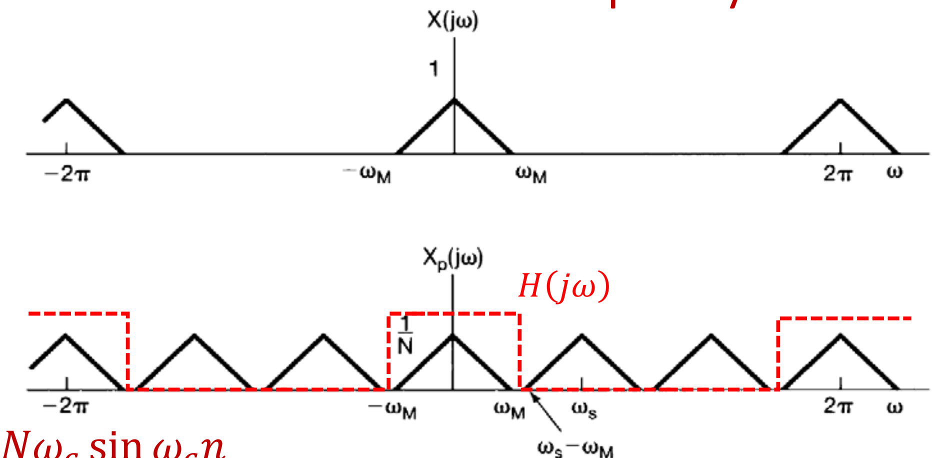
Frequency domain

$$x_r[n] = x_p[n] * h[n]$$ Time domain

$$= \left[\sum_{k=-\infty}^{\infty} x[kN]\cdot\delta[n-kN]\right] * h[n]$$

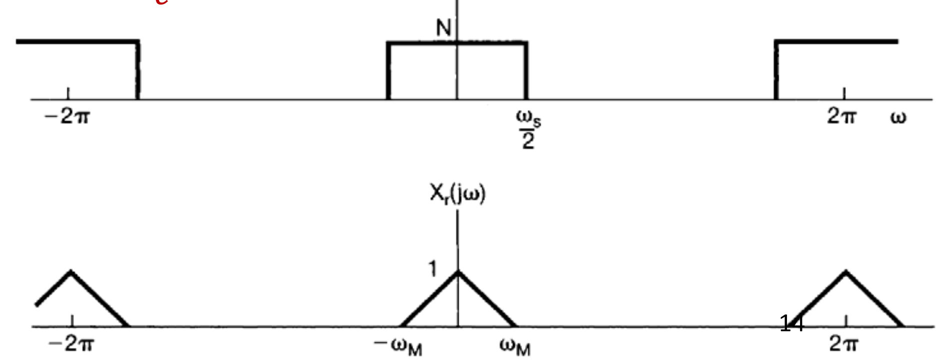
$$= \sum_{k=-\infty}^{\infty} x[kN][\delta[n-kN] * h[n]]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]h[n-kN]$$



$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}$$

$H(j\omega)$

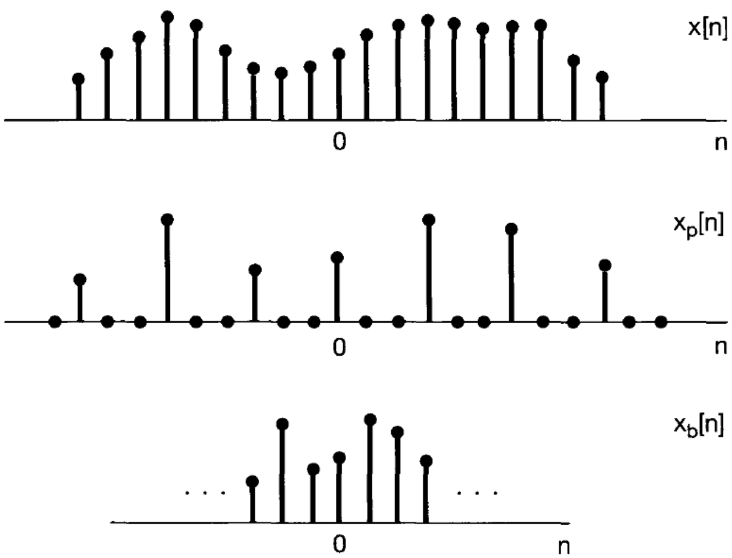


$$\boxed{x_r[n] = \sum_{k=-\infty}^{\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin\omega_c(n-kN)}{\omega_c(n-kN)}}$$

Sampling of Discrete-Time Signals

Decimation (sample rate decrease, SRD)

Time domain



$$x_b[n] = x_p[nN]$$

$$x_b[n] = x[nN]$$

Frequency domain

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_b[n]e^{-j\omega k}$$

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_p[kN]e^{-j\omega k}$$

$n = kN$

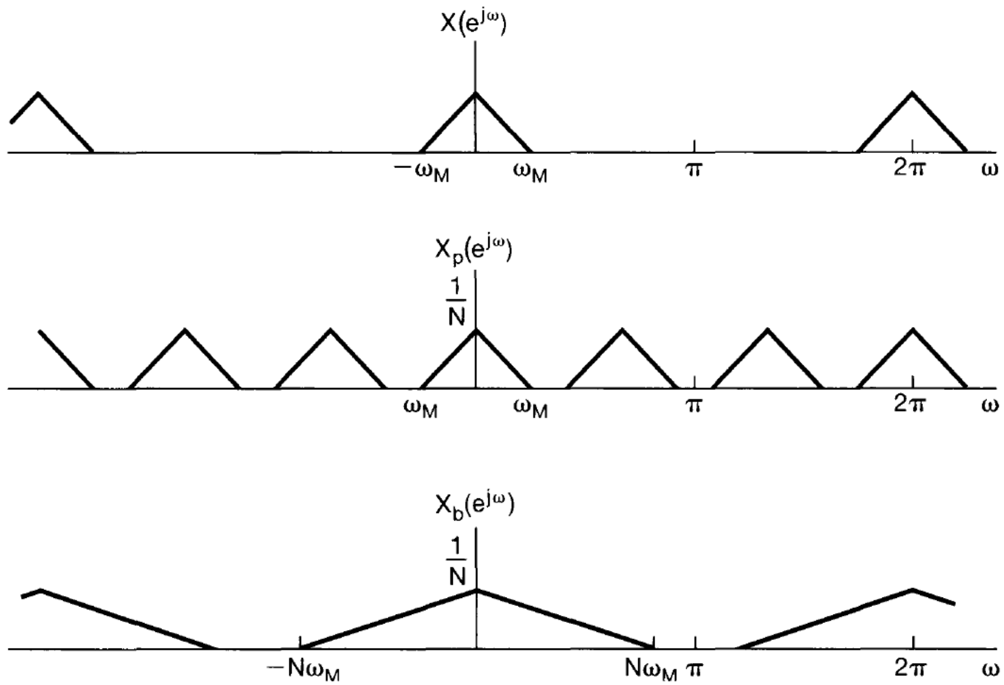
$$X_b(e^{j\omega}) = \sum_{\substack{n=\text{integer} \\ \text{number of N}}} x_p[n]e^{-j\omega n/N}$$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N}$$

$$\boxed{X_b(e^{j\omega}) = X_p(e^{j\omega/N})}$$

Sampling of Discrete-Time Signals

Decimation



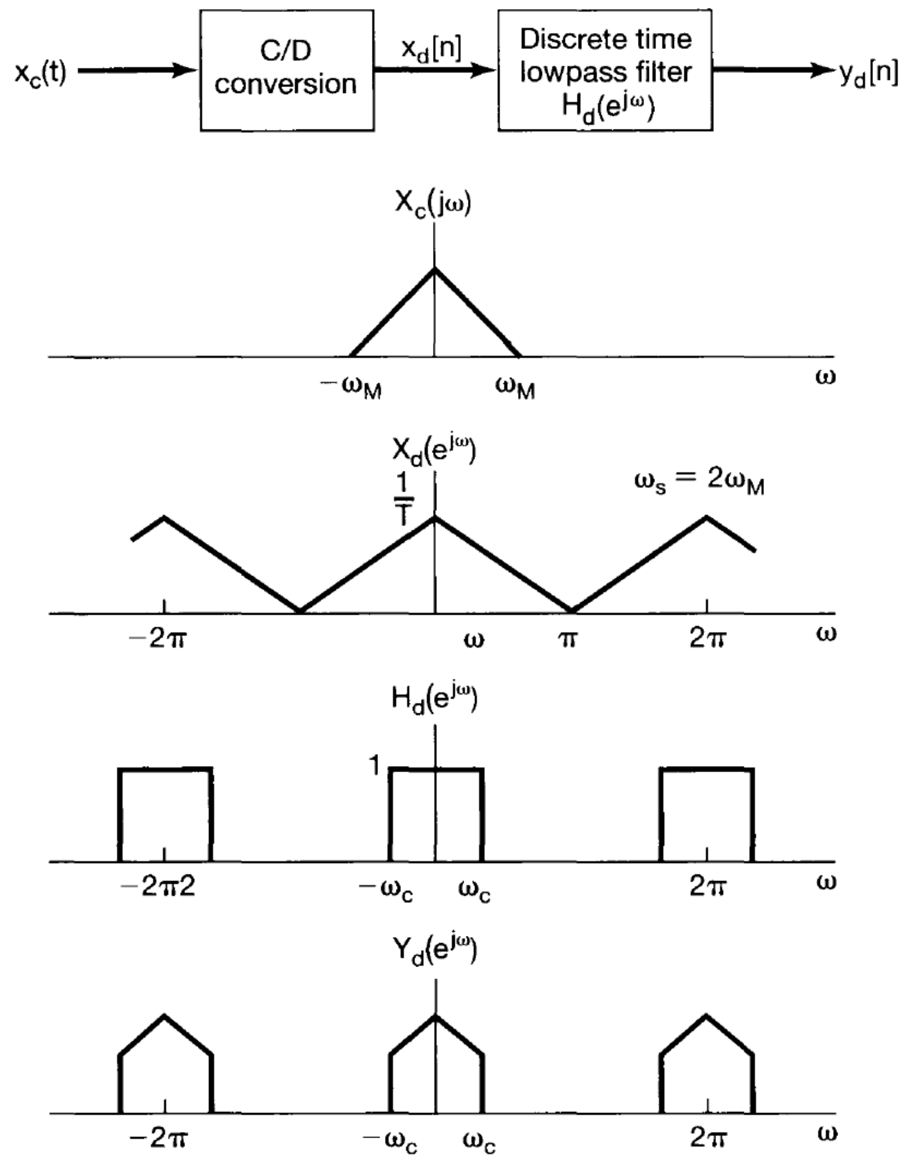
$$x_b[n] = x_p[nN] \qquad X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

Down-sampling if $\omega_s = \frac{2\pi}{N} > 2\omega_m$

Sampling of Discrete-Time

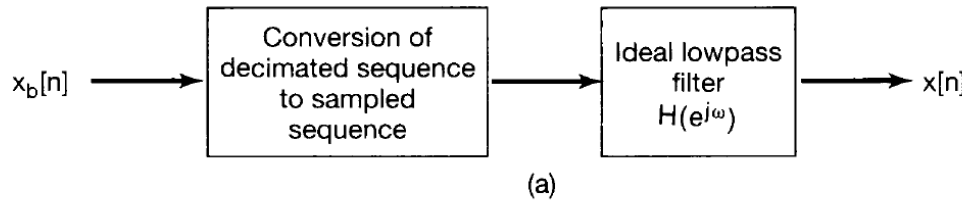
Decimation

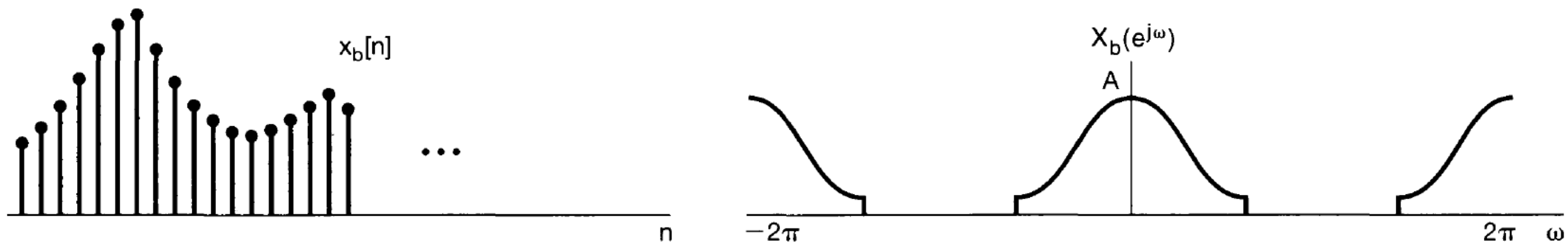
- Prevent aliasing by LPF in front of SRD $\Longrightarrow$ Decimator



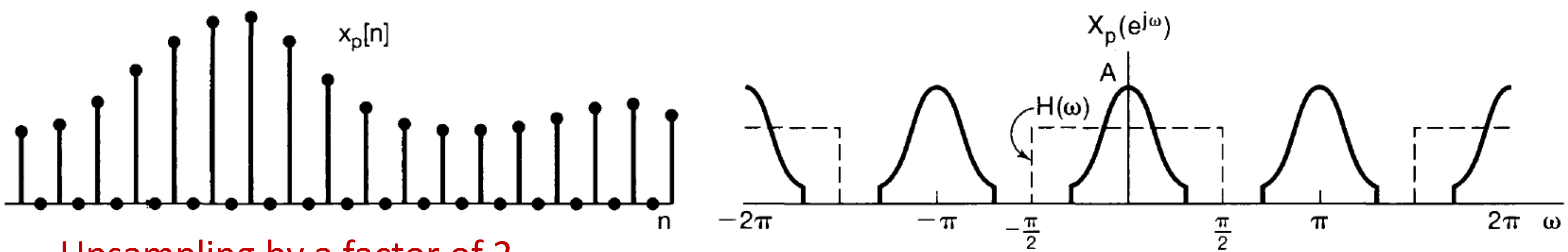
Sampling of

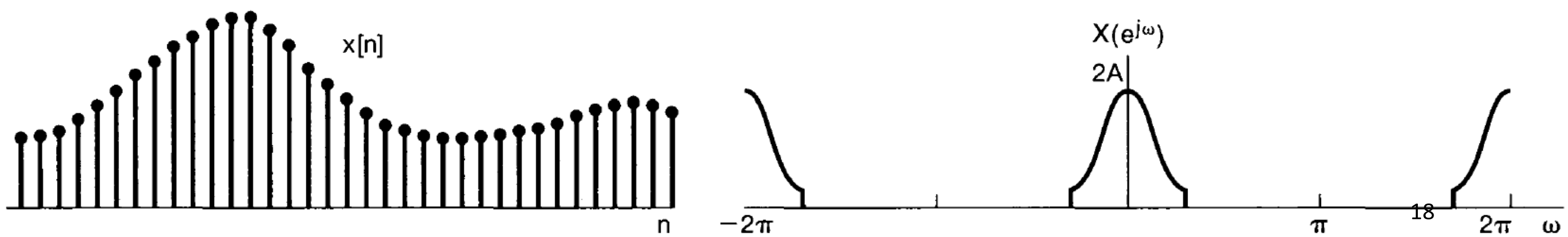
Interpolation (SRI)





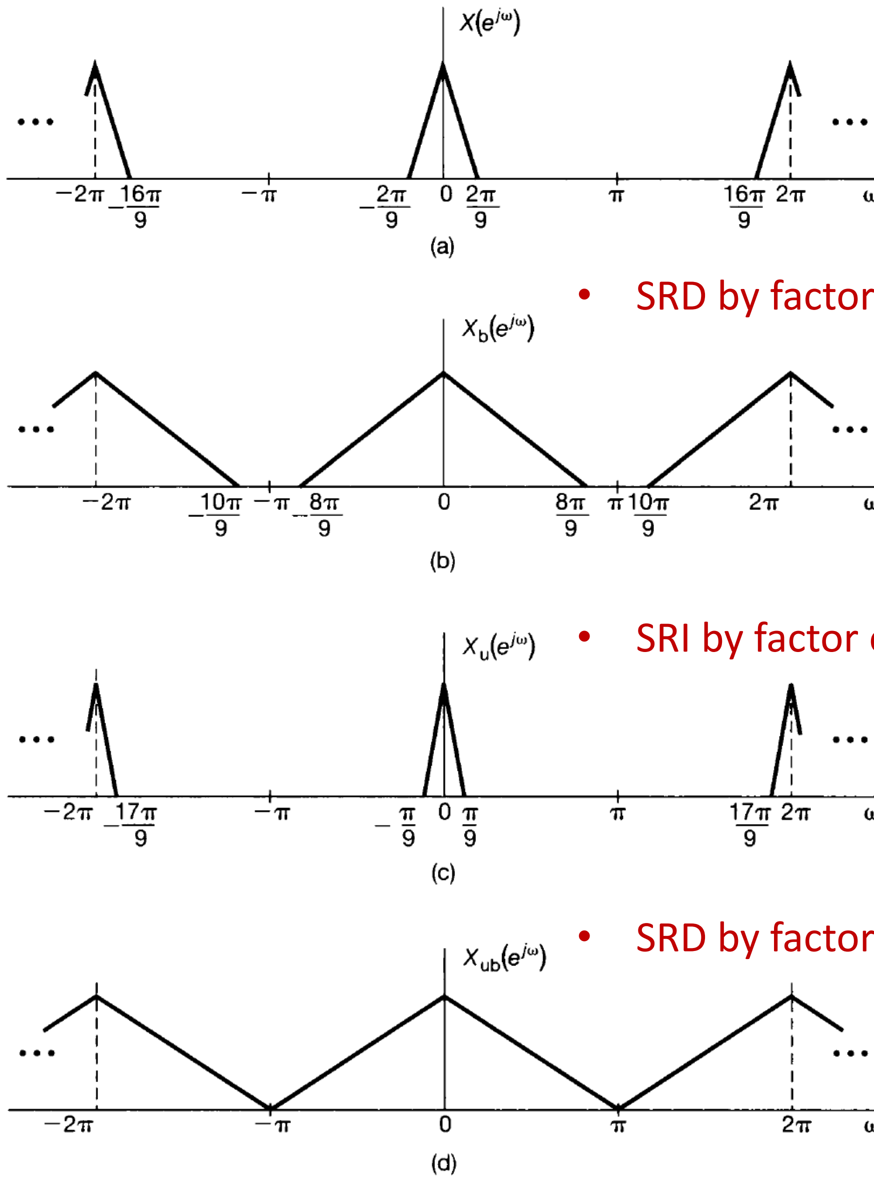
- Prevent mirrors by LPF after SRI $\Longrightarrow$ Interpolator





Sampling of Discrete-

Interpolation (SRI)



- SRD by factor of 4
- SRI by factor of 8
- SRD by factor of 9

- Overall, SRI by factor of 4.5