1. Matrix product rule.

$$[\quad]_{A\ m\times n} \times [\quad]_{B\ n\times p} = [\quad]_{C:\ m\times p}$$

① a. regular: row·col (row in A, col in B)

$C_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{col } 4 \text{ of } B)$

$= a_{31} \times b_{14} + a_{32} b_{24} + \dots = \sum_{k=1}^{n} a_{3k} b_{k4}$.

② col:

$$\underset{A}{[\quad]} \times \underset{B}{[\ \overset{\text{col } 1}{|}\ ]} = \underset{C}{[\ \overset{\text{col } 1}{|}\ ]}$$

Col 1 of C is $\underset{\text{matrix}}{\underset{\downarrow}{\overset{\text{(all the cols)}}{A}}} \cdot \underset{\text{vector}}{\underset{\downarrow}{B_{\text{col } 1}}}$.

↳ means: Columns of C is linear combination of columns of A

③ row:

$$\underset{A}{[\ \text{—}\ ]} \cdot \underset{B}{[\quad]} = \underset{C}{[\ \text{—}\ ]}$$

row 1 of $A \cdot \overset{\text{(all the rows)}}{B}$ = row 1 in C.

↳ Rows of C is a linear combination of the rows in B

④ $A \cdot B$ = Sum of (cols of A) × (rows of B)

eg: $\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$

⑤ block:

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \cdot \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \underset{C}{\left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]}$$

Permutation matrix / 置换.

row: left product.

新矩阵的第一行有0个行1，1个行2

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P \text{ (permutation matrix)}} \cdot \begin{array}{cc} \text{行1} & \text{行2} \\ \end{array} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

column: right product.

$$\begin{array}{cc} \text{列1} & \text{列2} \\ \end{array} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{P} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

新矩阵的第一列有0个列1，1个列2.

性质: $P^{-1} = P^{T}$

$P^{T} \cdot P = I$

Inverse matrix 逆.

keep row 1.
3row1 + row 2
keep row 3

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{inverse of } E : E^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \rightarrow \text{identity}$$

For rectangle matrix:

Left inverse $\leftarrow A^{-1}A = I$

$A I = A A^{-1} \rightarrow$ right inverse

invertible / non-singular: $A^{-1}A = I = AA^{-1}$ (square matrix)

e.g: No inverse: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \rightarrow$ Non-invertible, singular. has $Ax = 0$

e.g: Inverse of $A \cdot B = B^{-1}A^{-1}$

$ABB^{-1}A^{-1} = I$

Gauss-Jordan:

Augmented matrix: $A \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ find inverse

$$\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$\quad A \quad\quad I \quad\quad\quad\quad\quad\quad\quad\quad I \quad\quad A^{-1}$

Elimination:

$EA = U$ $\qquad\qquad$ $A = LU$

↳ means If no row exchanges, multipliers go directly into L

e.g.:

$$\overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \underset{\text{diagonal}}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$A: 3\times 3$ $\quad \underbrace{E_{32} \cdot E_{31} \cdot E_{21}}_{E} A = U \Rightarrow A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{L} U$

e.g.:

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

Inverse: $\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L.$

~~Theorem~~: For transposes:

Theorem: $AA^{-1} = I \Rightarrow (A^{-1})^T \cdot A^T = I = (A^T)^{-1}$

Inverse of A transpose = transpose of A^{-1}

Transpose: $A^T_{ij} = A_{ji}$ $\qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Symmetric matrix: $A^T = A$.

property (for all matrix): $A \cdot A^T = \cancel{A^T A}\ (A \cdot A^T)^T$

↳ is always a symmtric matrix.

property for Symmetric matrix:

① the eigenvalues are real numbers $\mathbb{R}$.

②. The eigenvectors are orthogonal / perpendicular.

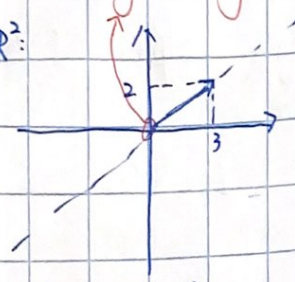
↳ can be chosen. ↳ can be orthonormal eigenvectors

* usual: $A = S\Lambda S^{-1}$ $\qquad$ Symmetric: $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$

Vector Space:

subspace : must go through origin. 必须经过原点.

e.g: R^2: subspace of $(3,2)$



subspaces of R^2: ① all of R^2

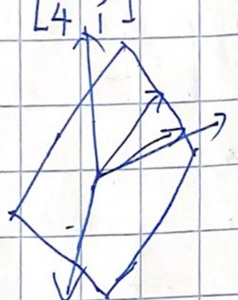
② any line through zero vector $\begin{bmatrix}0\\0\end{bmatrix}$

③ $\begin{bmatrix}0\\0\end{bmatrix}$

Column space:

e.g: columns in R^3:

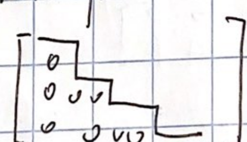
$\begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$: all their linear combination form a space



Rank: of a matrix: the number of pivots (also, pivots column)

pivot columns for A and A^T is the same

echolor form: 阶梯



reduced echolor form: 每首项系数为1, 且是所在列的唯一非零元素

$$R = \begin{bmatrix} I & F \\ 0 \dots & 0 \dots \end{bmatrix}$$

时 $RN = 0 \Rightarrow N = \begin{bmatrix} -F \\ I \end{bmatrix}$ → how to get null space

$$\begin{bmatrix} I & F \end{bmatrix} \cdot \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

Nullspace:

$Ax = 0$. $x = ?$

e.g: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ R: $\begin{array}{ccc} I & & F \end{array}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Augmented:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow X = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \begin{matrix} \rightarrow -F \\ \\ \rightarrow I \end{matrix} \quad \begin{bmatrix} -F \\ I \end{bmatrix} \rightarrow \text{Null space}$$

Solve $Ax = b$:

* when B is in the column space of A:
  And if a combination of rows in A gives zero row:
  Then same combination of entries of B must give 0

e.g $\begin{array}{c|c} A & B \\ \begin{bmatrix} 0 & 0 & 0 & 0 \end{array} & \begin{array}{c} 0 \end{bmatrix} \end{array}$

① $X_{particular}$: set all free variables to be 0.
solve $Ax=b$ for pivot variables

② $X_{nullspace}$: solve $Ax=0$, get special solutions:
把其中一个variable设成1,其它的都为0,得到一个vector

③ $X_{complete} = X_{particular} + X_{nullspace}$

e.g: $X_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$ (x_2, x_4 are free variables)

$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ 有2个C 所以free variable 数为2.

proof: $AX_p = b$, $AX_n = 0 \Rightarrow A(X_p + X_n) = 0$.

Solution state: For matrix $m \times n$.

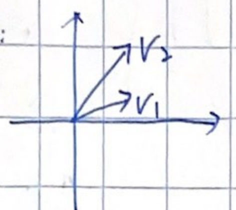
① $r = n < m$: rank = col number.
$Ax = b$ only has a particular solution / No solution

② $r = m < n$: Full rank.
$Ax = b$ for every b can be solved.

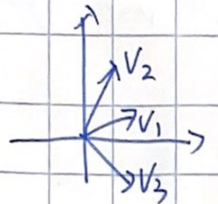
③ $r = m = n \rightarrow$ invertible.
$R = I$ $\quad Ax = b$ has a unique solution

Independence: No linear combination gives zero vector. (except 0)

e.g:



independent



exist
$c_1v_1 + c_2v_2 + c_3v_3$
~~so~~ which could be 0.
$\Rightarrow$ dependent.

In matrix: when $v_1, v_2, \dots v_n$ are columns of A:

They are independent if Nullspace of A is {zero vector}

They are dependent if $A\vec{c} = 0$ for some nonzero vector $\vec{c}$

Basis: for a space in a sequence of vectors $v_1, v_2 \dots v_n$.

1. they are independent.
2. they span the space.

e.g: space in R^3:

one basis: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\Rightarrow R^n$ n vectors give basis if the $n \times n$ matrix is invertible

dimension

* Rank(A) = pivot columns 数量 = pivots 数量 = dimension of column space of A.

4 important space for matrix $A: m \times n$.

1. Column space $C(A)$ in R^m
2. Null space $N(A)$ in R^n
3. Row space = all combinations of rows = all combinations of columns of A^T $= C(A^T)$ in R^n
4. Null space of A^T: $N(A^T)$ in R^m (Left nullspace)

$A: m \times n.$

	$C(A)$	$N(A)$	$C(A^T)$ reduced echolor form	$N(A^T)$
Basis:	~~A~~ pivot cols.	special solutions of A	first r rows of R	☆
Dimension dim	rank r	n-rank r	r	$m-r$

☆: $N(A^T)$: $A^T \cdot y = 0$

$$y^T A = 0^T$$

$$[\, y^T \,][\, A \,] = [0\,0\,0]\,..$$

~~Calcu~~ calculation of $N(A^T)$:

e.g.: $A \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$

$$A \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & 0 & 1 \end{array}\right]$$

(left block I; right side: R | E)

So $\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

but $N(A^T)$ should be 1 demension, which is

because this line is zero vector.

☆ We are finding a set of rows which contains the rows of Left nullspace.

More examples of matrix space:

① M = all 3×3 matrix $\dim M = 9$.

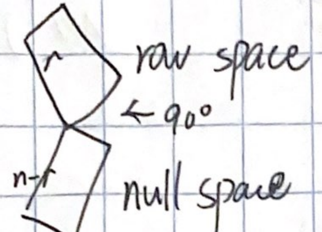
② S = all 3×3 symmetrix matrices: $\dim S = 6$

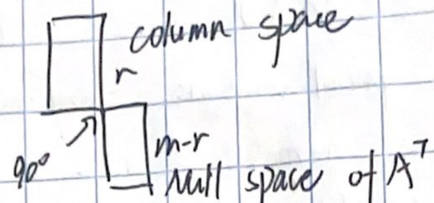
③ U = upper triangular 3×3 matrices $\dim U = 6$

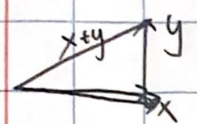
④ $S \cap U$ = symmetric upper triangular = diagnal 3×3 matrix

$\dim (S \cap U) = 3$

Orthogonal:







$x^T \cdot y = 0 \Rightarrow$ orthogonal / perpendicular

$$\underbrace{\|x\|^2}_{x^T \cdot x} + \|y\|^2 = \|x+y\|^2$$

eg: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $x^T = [1\ 2\ 3]$ $\|x\|^2 = 14$

Subspace S is orthogonal to subspace T means:
every vector in S is orthogonal to every vector in T.

Theorem 1: row space is orthogonal to null space

why: $Ax = 0$ A: $\begin{bmatrix} \text{row 1 of A} \\ \text{row 2 of A} \\ \vdots \\ \text{row m of A} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

x is orthogonal to every row.

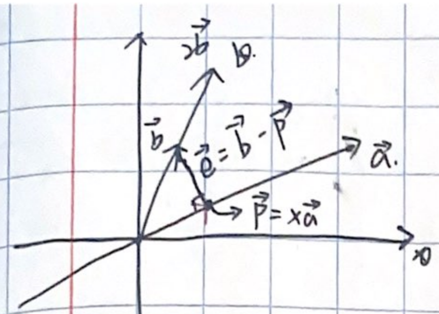
row space contains all the linear combinations of rows in A.

$$\begin{matrix} (row1)^T x = 0 \\ (row2)^T x = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 (row1)^T x = 0 \\ c_2 (row2)^T x = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 \cdot (row1)^T x + \\ c_2 (row2)^T x \end{matrix} = 0$$

Theorem 2: $A^T A$ is invertible exactly if A has independent columns.

why: if $A^T A x = 0$

$x^T \cdot A^T A x = 0 \Rightarrow (xA)^T A x = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$



$e \perp a$.

if $\vec{b} = 2\vec{b} \Rightarrow \vec{p} = 2\vec{p}$

if $\vec{a} = 2\vec{a} \Rightarrow \vec{p} = \vec{p}$

$$a^T \cdot (b - xa) = 0 \quad \rightarrow \vec{a} \perp \vec{b} - x\vec{a}$$

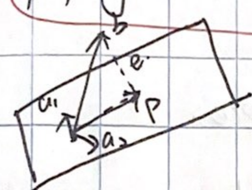
$$x a^T \cdot a = a^T b$$

$$x = \frac{a^T b}{a^T a}$$

$$\vec{p} = x\vec{a} = \frac{a^T b}{a^T a} \cdot a \qquad a^T$$

projection matrix: $P = \frac{a \cdot a^T}{a^T \cdot a}$ $\quad C(P)$ = line through a. $\quad$ rank. $R(P) = 1$

~~Theorem~~ property: $P^T = P \qquad P^2 = P$.



plane of a_1, a_2 = col space of $A\,[a_1\ a_2]$.

$$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$$

proof: $e = b - A\hat{x}$ is perpendicular to the plane

so $\begin{cases} a_1^T (b - A\hat{x}) = 0 \\ a_2^T (b - A\hat{x}) = 0 \end{cases} \Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow A^T (b - A\hat{x}) = 0$$

$$A^T \cdot e = 0$$

e in $N(A^T)$. $\quad e \perp C(A)$.

conclusion:

$\hat{x} = (A^T A)^{-1} A^T b$ $\qquad$ 如果A是向量a.

$\vec{p} = A\hat{x} = A(A^T A)^{-1} A^T b$

Matrix $P = A(A^T A)^{-1} \cdot A^T \rightarrow$ 通解. $\quad P = \frac{a \cdot a^T}{a^T \cdot a}$

If $b \rightarrow Ax$ in column space of A: $Pb = b$: $Pb = A\boxed{(A^T A)^{-1} A^T A}x = Ax$

If $b \perp$ column space of A: $Pb = 0$

Matrix $P \cdot b = \vec{P_b}$

Orthonormal: orthogonal and length = 1

Orthonormal vectors:

$$q_i^T \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$$Q = [\, q_1 \; \cdots \; q_n \,].$$

$$Q^T \cdot Q = \underbrace{\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}}_{Q^T} \cdot \underbrace{[\, q_1 \cdots q_n \,]}_{Q} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & \ddots \end{bmatrix}$$

orthogonal matrix: Q. (square)

↳ 列与列之间 orthonormal.

property: $Q^T Q = I$ $Q^T = Q^{-1}$ if Q is square.

e.g. $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Q has orthonormal columns project onto its column space:

$P_Q = Q(Q^TQ)^{-1}Q^T = QQ^T$ ($= I$ if Q is square)
↳ $Q^T = Q^{-1}$

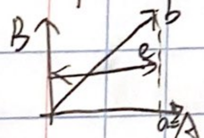
extension: $A^T A \hat{x} = A^T b$.

Now A is Q: $Q^T Q \hat{x} = Q^T b \Rightarrow \hat{x} = Q^T b$

$\hat{x}_i = q_i^T \cdot b$

Gram-Schmidt:

Independent vectors a, b → orthogonal $A \perp B$ → orthonormal.



$B = b - \frac{A^Tb}{A^TA} \cdot A$
↳ 减去A方向的分量

$q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$

$Q = [\, q_1 \;\; q_2 \,]$

a, b, c → A, B, C →

$$B = b - \frac{A^Tb}{A^TA} A$$

$$C = c - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B$$

e.g. $\underset{A}{\overset{\parallel}{a}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$Q = [\, q_1 \; q_2 \,] = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ → 和 a, b 在同一平面. 但不正交.

☆ $q_1 \cdot q_2$ always $\perp$ A.

Determinant:

property:

① $\det I = 1$. ① $\begin{bmatrix}1&0\\0&1\end{bmatrix} = 1$ ② $\begin{bmatrix}0&1\\1&0\end{bmatrix} = -1$

② Exchange rows in matrix: reverse the sign of det.

e.g: permutation P: $\det P = 1$ or -1.

e.g. $\begin{vmatrix}a&b\\c&d\end{vmatrix} = ad - cb$

③ 3a. $\begin{vmatrix}ta&tb\\c&d\end{vmatrix} = t\cdot\begin{vmatrix}a&b\\c&d\end{vmatrix}$

3b. $\begin{vmatrix}a+a'&b+b'\\c&d\end{vmatrix} = \begin{vmatrix}a&b\\c&d\end{vmatrix} + \begin{vmatrix}a'&b'\\c&d\end{vmatrix}$

linear for each row.

extension:

①. if 2 equal rows exist in matrix → $\det = 0$

②. substract $l\times$ row i from row k, det does not change

↳ 消元过程

③. Rows of zero → $\det = 0$ from 3a: $\begin{vmatrix}0\cdot a&0\cdot b\\c&d\end{vmatrix} = 0\cdot\begin{vmatrix}a&b\\c&d\end{vmatrix}$

e.g: $\begin{vmatrix}a&b\\c-la&c-lb\end{vmatrix} \overset{3b}{=} \begin{vmatrix}a&b\\c&d\end{vmatrix} + \begin{vmatrix}a&b\\-la&-lb\end{vmatrix}$

$\overset{3a}{=} \begin{vmatrix}a&b\\c&d\end{vmatrix} - l\begin{vmatrix}a&b\\ \underbrace{a\ \ b}_{0}\end{vmatrix}$

$\overset{E①}{=} \begin{vmatrix}a&b\\c&d\end{vmatrix}$ → E②

④ $U = \begin{bmatrix}d_1&&*&*&*\\0&d_2&&&\\0&0&d_3&&\\0&0&0&\ddots&\\&&&&d_n\end{bmatrix}$ upper triangle matrix:

$\det U$ = product of d's $= d_1\times d_2\times d_3\times\ldots\times d_n$

= product of pivots

⑤. $\det A = 0$ when A is singular   Singular → row is all 0 exists

$\det \neq 0$: invertible → U → Diagonal

⑥ $\det(A\cdot B) = \det A \cdot \det B$

eg: $\det A^{-1} = \frac{1}{\det A}$

$A^{-1}A = I$

$\det A^{-1} \det A = 1 = \det(A^{-1}A)$

extension: $\det A^2 = (\det A)^2$

$\det 2A = 2^n \det A$

⑦ $\det A^T = \det A$.

Calculation:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{\to 0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{\to 0} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} \cancel{0 \quad a_{12} \quad 0} \\ \cancel{a_{21} \quad 0 \quad 0} \\ \cancel{0 \quad 0 \quad a_{33}} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \overset{\text{交换}}{-} a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + \ldots$$

Regular formular:

$$\det A = \sum_{n!\ \text{terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}.$$

$(\alpha, \beta, \gamma, \ldots \omega) =$ 在 $(1, 2, 3 \ldots n)$ 列中

不重复选择.

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & \cdots \end{bmatrix}$$

eg: $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} \Rightarrow (3, 2, 1, 4).\ (4, 3, 2, 1)$

Confactors: 3×3

$$\det = \overset{\text{选}1.1.}{a_{11}} \underbrace{\overset{\text{所有 posibility}}{(a_{22}a_{33} - a_{23}a_{32})}}_{C_{11}:\ \text{cofactor}} \mp a_{12}(\ldots\ldots) + a_{13}(\quad) \cdots$$

$$C_{11} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} \qquad C_{12} = \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

along row 1: $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots\dots + a_{1n}C_{1n}$

判断正负号: a_{ij} $i+j = \text{odd} \to -$

$i+j = \text{even} \to +$

Inverse: by determinant

e.g. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \cdot \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

转置A. 求每个位置的cofactor.

$A^{-1} = \dfrac{1}{\det A} \cdot C^T$

$\begin{bmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{bmatrix} = \begin{bmatrix} ei-hf & & \\ & & \\ & & \end{bmatrix}$

Cramer's rules:

$Ax = b.$ $x = A^{-1}b = \dfrac{1}{\det A} \cdot C^T \cdot b$

$= C_{11}b_1 + C_{21}b_2 + \dots$

$x_1 = \dfrac{\det B_1}{\det A}$ $x_2 = \dfrac{\det B_2}{\det A}$ $\dots$ $x_j = \dfrac{\det B_j}{\det A}$

$B_1: \begin{bmatrix} b & a_2 & a_3 & a_4 & \dots & a_n \end{bmatrix}.$

$B_j = A$ with column j replaced by b.

Determinant application: $\det A = \text{volumn}$

Eigenvectors: find Ax parallel to x $\Rightarrow Ax = \lambda x.$

λ: eigenvalue x: eigenvector.

If A is singular $\to \lambda = 0$ is an eigenvalues

e.g.

For projection matrix:

Any x in plane: $Px = x$ $\lambda = 1$

Any $x \perp$ plane: $Px = 0(x)$ $\lambda = 0$

e.g: permutation matrix:

$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow Ax = x$ $\lambda = 1$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow Ax = -x$ $\lambda = -1$

* The sum of eigenvalues = the sum down the diagonal (trace)

Solve $Ax = \lambda x$: $\underbrace{(A - \lambda I)}_{\text{singular}} x = 0$

$\det(A - \lambda I) = 0 \rightarrow$ Find λ $\rightarrow$ Find x

e.g. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

$= \lambda^2 - \underbrace{6}_{\text{trace}}\lambda + \underbrace{8}_{\text{determinant}} = (\lambda - 4)(\lambda - 2)$

$\lambda_1 = 4$ $\quad A - 4I = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{singular}}$ $\quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$ $\quad A - 2I = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{singular}}$ $\quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

property: ① $A + nI$ → eigenvalues $+n$ → eigenvectors 不变 (e.g.)

② trace = the sum of eigenvalues.

③ eigenvectors of a symmetric matrix is orthogonal.

e.g. $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ → orthogonal

$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}$

$Q^T = -Q$ → anti-symmetric

$\det Q = 1 = \lambda_1 \lambda_2$

$\text{trace} = 0 = \lambda_1 + \lambda_2$

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ $\quad \lambda_1 = i$, $\lambda_2 = -i$

★ uppertriangle matrix in eigenvalues 就在对角线上

e.g. $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$

$\lambda_1 = \lambda_2 = 3$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → degenerate matrix diagonal: A has n independent eigenvectors ~~of~~

put them in columns of S → spectrum

$$AS = A[x_1 \ x_2 \ x_3 \ \dots \ x_n] = [\lambda_1 x_1 \ \lambda_2 x_2 \ \dots \ \lambda_n x_n]$$

$$= [x_1 \ x_2 \ \dots \ x_n] \cdot \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & & & \\ 0 & & \lambda_3 & & \\ 0 & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix} = S\Lambda \text{ lambda}$$

$$AS = S\Lambda$$

$$S^{-1}AS = \Lambda$$

$$A = S\Lambda S^{-1}$$

$$Ax = \lambda x \qquad A^2 x = \lambda A x = \lambda^2 x$$

$$A^2 = S\Lambda S^{-1} \cdot S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

$$A^k = S\Lambda^k S^{-1}$$

• property: $A^k \to 0$ as $k \to \infty$ if all $|\lambda| < 1$.

Diagonalizable: A is sure to have n independent eigenvectors.

(be diagonalizable) if all the λs are different.

algebraic multiplicity: 重复的eigenvalue数 = 2

e.g: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$

$$\lambda_1 = \lambda_2 = 2$$

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ A is not diagonalizable.

☆如果一个value对应2个vector：仍然diagonalizable

几何重数：eigenspace 的 dimension