

$$Q_1: \lim_{x \rightarrow 1} [f(g(2-x)) - g(f(x^2))] = 0$$

So use L'Hôpital's rule

$$\begin{aligned} \text{Derivative of numerator: } [f(g(2-x)) - g(f(x^2))] &' \\ &= (2-x)' g'(2-x) \cdot f'(g(2-x)) - (x^2)' f'(x^2) g'(f(x^2)) \\ &= -g'(2-x) \cdot f'(g(2-x)) - 2x \cdot f'(x^2) g'(f(x^2)) \end{aligned}$$

$$x=1:$$

$$\begin{aligned} &= -g'(1) f'(g(1)) - 2f'(1) g'(f(1)) \\ &= -g'(1) f'(-2) - 2f'(1) g'(1) \\ &= -1 \cdot 3 - 2(-3) \cdot 1 = -3 + 6 = 3 \end{aligned}$$

$$\text{Derivative of denominator: } (x-1)' = 1$$

$$\lim_{x \rightarrow 1} \frac{(\dots)}{x-1} = \lim_{x \rightarrow 1} \frac{(\dots)'}{(x-1)'} = \frac{3}{1} = 3$$

b. Find $\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{x^2 - x}}$

$$= \lim_{x \rightarrow 0^-} \frac{x}{|x| \sqrt{1 - \frac{1}{x}}} = \lim_{x \rightarrow 0^-} \frac{x}{-x \sqrt{1 - \frac{1}{x}}}$$

$$= \lim_{x \rightarrow 0^-} \frac{1}{-\sqrt{1 - \frac{1}{x}}} = \frac{1}{-\infty} = 0$$

c. Calculate $\int_0^1 x \ln(x) dx$

Integration by parts: $\int u v' = uv - \int v u'$

Let $u = \ln(x)$ and $v' = x$

Then $u' = \frac{1}{x}$ and $v = \frac{x^2}{2}$

$$= \left[\ln(x) \cdot \frac{x^2}{2} \right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \left(\ln(1) \cdot \frac{1}{2} - \ln(0) \cdot 0 \right) - \left(\frac{1}{4} - 0 \right)$$

Note: $\ln(1) = 0$

$$= 0 - \frac{1}{4} = -\frac{1}{4}$$

$$f(x) = \frac{1}{\sqrt{x^2+3x} + x} = \frac{\sqrt{x^2+3x} - x}{(\sqrt{x^2+3x} + x)(\sqrt{x^2+3x} - x)} = \frac{\sqrt{x^2+3x} - x}{x^2+3x-x^2} = \frac{\sqrt{x^2+3x} - x}{3x} = \frac{|x| \cdot \sqrt{1+\frac{3}{x}} - x}{3x}$$

Break points: $\sqrt{x^2+3x} + x = 0 \quad x = 0$

Vertical Asymptotes:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2+3x} - x}{3x} = \frac{|x| \sqrt{x^2+3x} - x}{3x}$$

$$\sqrt{x^2+3x} = |x| \cdot \sqrt{1+\frac{3}{x}}$$

$$\lim_{x \rightarrow 0^+} \frac{|x| \sqrt{x^2+3x} - x}{3x} = \lim_{x \rightarrow 0^+} \frac{x \sqrt{1+\frac{3}{x}} - x}{3x} = \lim_{x \rightarrow 0^+} \frac{\sqrt{1+\frac{3}{x}} - 1}{3}$$

$$\lim_{x \rightarrow 0^-} \frac{-x \sqrt{1+\frac{3}{x}} - x}{3x} = \frac{-\sqrt{1+\frac{3}{x}} - 1}{3}$$

Only for right side:
a vertical asymptotes $x = 0$ doesn't exist

Horizontal Asymptotes:

$$\lim_{x \rightarrow \infty} \frac{|x| \sqrt{1+\frac{3}{x}} - x}{3x} \quad x > 0: \quad \lim_{x \rightarrow \infty} \frac{x \sqrt{1+\frac{3}{x}} - x}{3x} = \frac{\sqrt{1+\frac{3}{x}} - 1}{3} = 0$$

$y = 0$: one-sided

$$\lim_{x \rightarrow -\infty} \frac{-x \sqrt{1+\frac{3}{x}} - x}{3x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{1+\frac{3}{x}} - 1}{3} = \frac{-2}{3}$$

$y = -\frac{2}{3}$ one-sided

Oblique Asymptotes:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+3x} - x}{3x \cdot x} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+\frac{3}{x}} - 1}{3x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x \cdot (\sqrt{x^2+3x} + x)} \rightarrow 0$$

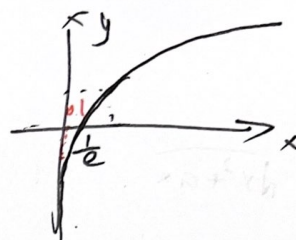
oblique Asymptotes
doesn't exist

Q3: $f(x) = x \ln(x)$ find global min/max

$f'(x) = 0$

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \frac{\ln(x) + 1}{\ln(x) = -1} = 0$$

$$x = e^{-1} = \frac{1}{e}$$



→ has global minimum. because $f(x) \rightarrow -\infty$ when $x \rightarrow 0^+$
 $\rightarrow f(x)$ decrease

$f''(x) = 0$

→ global min/max

Q4: $\int_0^{\pi/2} (\sin(x))^3 dx = \int_0^{\pi/2} (1 - \cos^2(x)) \cdot \sin(x) dx$

$\sin^2(x) + \cos^2(x) = 1 \rightarrow 1 - \cos^2(x)$

$u = \cos(x) \quad \frac{du}{dx} = -\sin(x) \quad du = -\sin(x) dx \quad dx = \frac{du}{-\sin(x)}$

$$= \int_0^{\pi/2} (1 - u^2) \cdot \frac{-du}{-\sin(x)} = \int_{\pi/2}^0 (1 - u^2) \cdot (-1) du$$

$$= \int_0^{\pi/2} (1 - u^2) du = \left[\frac{u^3}{3} - u \right]_0^{\pi/2}$$

$$= \left[\frac{\cos^3(x)}{3} - \cos(x) \right]_0^{\pi/2}$$

$$= \left(\frac{\cos^3(\pi/2)}{3} - \cos(\pi/2) \right) - \left(\frac{\cos^3(0)}{3} - \cos(0) \right)$$

$$= (0 - 0) - \left(\frac{1}{3} - 1 \right) = \frac{2}{3}$$

Q5:

5a. False.

Because Comparison Test:

When $a_n \geq b_n$, a_n ~~con~~ diverges ^{if} ~~when~~ b_n diverges.

5b: False.

① check if b_n converges \rightarrow Yes

② check if $a_n \leq b_n \rightarrow$ No. $a_n = \frac{e^n + n}{e^{n^2} - n^2} = \frac{e^n + n}{(e^n + n)(e^n - n)} = \frac{1}{e^n - n}$

$$e^n - n > e^n$$

$$\frac{1}{e^n - n} > \frac{1}{e^n}$$

$$a_n > b_n$$

5c. True.

check if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$

5d. False. not sufficient

6. $\sum_{n=1}^{\infty} n \left(\frac{2}{x}\right)^n : x \neq 0$

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{(n+1) \left(\frac{2}{x}\right)^{n+1}}{n \left(\frac{2}{x}\right)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \cdot \frac{2}{x} \right|$
 $= \lim_{n \rightarrow \infty} \left| \frac{1 + \frac{1}{n}}{1} \cdot \frac{2}{x} \right|$
 $= \left| \frac{2}{x} \right|$

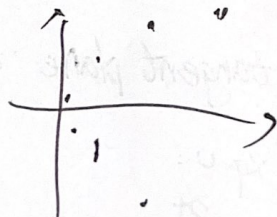
It converges if $\left| \frac{2}{x} \right| < 1$: when: $x > 2$ or $x < -2$.

diverges if $\left| \frac{2}{x} \right| > 1$: when: $0 < x < 2$, or $-2 < x < 0$

~~Not~~ Can't conclude when $x = 2$ and -2
 $\hookrightarrow \left| \frac{2}{x} \right| = 1$

when $x = 2$: $\sum_{n=1}^{\infty} n \cdot 1^n = \sum_{n=1}^{\infty} n \rightarrow$ diverges

$x = -2$: $\sum_{n=1}^{\infty} n(-1)^n$: Alternating \rightarrow diverges
 Series test.



7. Find solution $y = y(x)$.

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases}$$

$$\begin{aligned} y' + p(x)y &= f(x) \\ p(x) &= 1 \quad f(x) = x \end{aligned}$$

① $u(x) = e^{\int p(x) dx}$

$$u(x) = e^{\int 1 dx} = e^x$$

② $(u(x)y)' = u(x) \cdot f(x)$

$$(e^x y)' = e^x \cdot x$$

③ $\int (e^x y)' dx = \int e^x \cdot x dx$

$$\begin{aligned} e^x y &= x \cdot e^x - \int e^x dx \\ &= x e^x - e^x + C \end{aligned}$$

$$y = x - 1 + \frac{C}{e^x}$$

$$\Rightarrow y = x - 1 + \frac{4}{e^x}$$

$$u(x)y' + u(x) \cdot p(x)y = u(x) \cdot f(x)$$

$$p(x) \cdot y' + q(x)y = f(x)$$

$$y' + \left(\frac{q(x)}{p(x)} \right) y = \frac{f(x)}{p(x)}$$

$$\begin{aligned} u &= x & v &= e^x \\ u' &= 1 & v' &= e^x \end{aligned}$$

$$\begin{aligned} y(0) &= 3 \\ &= 0 - 1 + \frac{C}{e^0} \end{aligned}$$

$$= -1 + C = 3$$

$$C = 4$$

8. Find $f(x, y) = \sin(2x - y)$, find tangent plane at $(0, 0, f(0, 0))$

$$\frac{\partial f}{\partial x} = 2 \cos(2x - y) \quad \text{at } (0, 0): \quad \frac{\partial f}{\partial x} = 2 \cos(0 - 0) = 2$$

$$\frac{\partial f}{\partial y} = -\cos(2x - y) \quad \text{at } (0, 0): \quad \frac{\partial f}{\partial y} = -\cos(0 - 0) = -1$$

$$f(0, 0) = \sin(0 - 0) = 0$$

Tangent plane: $z = \frac{\partial f}{\partial x}(x - a) + \frac{\partial f}{\partial y}(y - b) + f(a, b)$

$$\begin{aligned} z &= 2 \cdot (x - 0) - 1 \cdot (y - 0) + 0 \\ &= 2x - y \end{aligned}$$

9. Find double integral over a triangle T with vertices $(0,0), (1,0), (1,1)$

$$0 \leq x \leq 1$$

$$0 \leq y \leq 1$$

$$\iint_T e^{2x+y} dT$$

↓

$$\int_0^1 \left[\int_0^1 e^{2x+y} dx \right] dy$$

↓ should rely on x

$$\int_0^1 \int_0^1 e^{2x+y} dx dy$$

$$= \int_0^1 \left(\frac{1}{2} e^{2x+y} \Big|_0^1 \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} e^{2+y} - \frac{1}{2} e^{0+y} \right) dy$$

$$= \frac{1}{2} \left[\int_0^1 e^{2+y} dy - \int_0^1 e^y dy \right]$$

$$= \frac{1}{2} \left(e^{2+y} \Big|_0^1 - e^y \Big|_0^1 \right)$$

$$= \frac{1}{2} (e^3 - e^2 - e^1 + e^0)$$

$$= \frac{e^3 - e^2 - e + 1}{2}$$

