

8.2.2

EE24BTECH11004 - Ankit Jainar

Question: Find the area bounded by the curves:

$$(x - 1)^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1 \quad (1)$$

Theoretical Solution:

The curves are two circles:

- Circle 1: $(x - 1)^2 + y^2 = 1$, centered at $(1, 0)$ with radius 1.
- Circle 2: $x^2 + y^2 = 1$, centered at $(0, 0)$ with radius 1.

Step 1: Finding the Points of Intersection Using Theoretical Approach

The points of intersection occur where the two circles overlap. To find the intersection points, we rewrite the equations as:

$$(x - 1)^2 + y^2 = 1 \quad (\text{Circle 1}) \quad (2)$$

$$x^2 + y^2 = 1 \quad (\text{Circle 2}) \quad (3)$$

Expanding both equations:

$$x^2 - 2x + 1 + y^2 = 1 \quad (\text{Circle 1}) \quad (4)$$

$$x^2 + y^2 = 1 \quad (\text{Circle 2}) \quad (5)$$

Subtracting the second equation from the first gives:

$$-2x + 1 = 0 \quad \implies \quad x = \frac{1}{2} \quad (6)$$

Substituting $x = \frac{1}{2}$ into the equation of Circle 2:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1 \quad (7)$$

$$\frac{1}{4} + y^2 = 1 \quad (8)$$

$$y^2 = \frac{3}{4} \quad \implies \quad y = \pm \frac{\sqrt{3}}{2} \quad (9)$$

Thus, the points of intersection are:

$$P_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad P_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad (10)$$

Matrix representation for solving this system:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \quad (11)$$

where:

$$\mathbf{V}_1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y^2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (12)$$

Step 2: Intersection of Two Circles Using Matrix Approach General Form of the Circle Equations

The general form for a circle is:

$$g(x, y) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (13)$$

For Circle 1: $(x - 1)^2 + y^2 = 1$

$$g_1(x, y) = \mathbf{x}^\top \mathbf{V}_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1 = 0 \quad (14)$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad f_1 = 0 \quad (15)$$

For Circle 2: $x^2 + y^2 = 1$

$$g_2(x, y) = \mathbf{x}^\top \mathbf{V}_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2 = 0 \quad (16)$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_2 = -1 \quad (17)$$

Subtract the Equations to Eliminate Quadratic Terms

Subtract $g_1(x, y)$ from $g_2(x, y)$:

$$g_2(x, y) - g_1(x, y) = (\mathbf{x}^\top \mathbf{V}_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2) - (\mathbf{x}^\top \mathbf{V}_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1) = 0 \quad (18)$$

$$0 = 2(\mathbf{u}_2 - \mathbf{u}_1)^\top \mathbf{x} + (f_2 - f_1) \quad (19)$$

Simplify:

$$2 \begin{pmatrix} 0 - (-1) \\ 0 - 0 \end{pmatrix}^\top \mathbf{x} + (-1 - 0) = 0 \quad (20)$$

$$2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \mathbf{x} - 1 = 0 \quad (21)$$

Line Equation Representing the Chord of Intersection

The line equation representing the chord of intersection is:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \mathbf{x} = \frac{1}{2} \quad (22)$$

This simplifies to:

$$x = \frac{1}{2}. \quad (23)$$

Substituting x into One Circle Equation

Substitute $x = \frac{1}{2}$ into the second circle equation $g_2(x, y)$: Substituting \mathbf{x} into the circle equation:

$$\begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} - 1 = 0. \quad (24)$$

Expanding the terms:

$$\frac{1}{4} + y^2 - 1 = 0. \quad (25)$$

Simplifying:

$$y^2 = \frac{3}{4}. \quad (26)$$

Thus:

$$y = \pm \frac{\sqrt{3}}{2}. \quad (27)$$

Points of Intersection

The points of intersection are:

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \quad (28)$$

Step 3: Area Calculation Using Trapezoidal Rule

The area between the two circles is symmetric about the x -axis. Therefore, we calculate the area of the upper region and multiply it by 2.

The area is given by:

$$A = 2 \int_{x=0.5}^{x=1} \left[\sqrt{1 - (x-1)^2} - \sqrt{1 - x^2} \right] dx \quad (29)$$

Expanding $1 - (x-1)^2$:

$$1 - (x-1)^2 = 1 - (x^2 - 2x + 1) \quad (30)$$

$$= 2x - x^2 \quad (31)$$

Thus, the integral becomes:

$$A = 2 \int_{x=0.5}^{x=1} \left[\sqrt{2x - x^2} - \sqrt{1 - x^2} \right] dx \quad (32)$$

Trapezoidal Rule:

We discretize the interval $[0.5, 1]$ into N equal subintervals of width h :

$$h = \frac{1 - 0.5}{N} = \frac{0.5}{N}. \quad (33)$$

The x_k values are:

$$x_k = 0.5 + k \cdot h, \quad k = 0, 1, 2, \dots, N. \quad (34)$$

The area is approximated as:

$$A \approx 2 \cdot h \cdot \left[\frac{1}{2}(f(x_0) - g(x_0)) + \sum_{k=1}^{N-1} (f(x_k) - g(x_k)) + \frac{1}{2}(f(x_N) - g(x_N)) \right], \quad (35)$$

where:

$$f(x_k) = \sqrt{2x_k - x_k^2}, \quad g(x_k) = \sqrt{1 - x_k^2}. \quad (36)$$

Difference Equation:

The iterative formula for the trapezoidal rule is:

$$A_{\text{new}} = A_{\text{old}} + h \cdot [(f(x_{k+1}) - g(x_{k+1})) + (f(x_k) - g(x_k))]. \quad (37)$$

$$A_{\text{new}} = A_{\text{old}} + h \cdot \left[\left(\sqrt{2x_{k+1} - x_{k+1}^2} - \sqrt{1 - x_{k+1}^2} \right) + \left(\sqrt{2x_k - x_k^2} - \sqrt{1 - x_k^2} \right) \right]. \quad (38)$$

The difference equation for the area is given as:

$$A_{n+1} = A_n + \frac{h}{2} [y(x_n) + h \cdot y'(x_n) + y(x_n)] \quad (39)$$

$$A_{n+1} = A_n + h \cdot \left[y(x_n) + \frac{h}{2} y'(x_n) \right] \quad (40)$$

$$A_{n+1} = A_n + h \cdot y(x_n) + \frac{h^2}{2} y'(x_n) \quad (41)$$

The curves given are:

$$(x - 1)^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1. \quad (42)$$

Rearranging the equations for $y(x)$, the piecewise function for the bounded region is:

$$y(x) = \begin{cases} \sqrt{1 - x^2} & 0 \leq x \leq \frac{1}{2} \\ \sqrt{1 - (x - 1)^2} & \frac{1}{2} \leq x \leq 1 \end{cases} \quad (43)$$

Substituting $y(x)$ into the Difference Equation:

By substituting the piecewise function $y(x)$ into the difference equation, we get:

$$A_{n+1} = \begin{cases} A_n + h \sqrt{1 - x_n^2} + \frac{h^2}{2} \left(-\frac{x_n}{\sqrt{1 - x_n^2}} \right) & 0 \leq x_n \leq \frac{1}{2}, \\ A_n + h \sqrt{1 - (x_n - 1)^2} + \frac{h^2}{2} \left(-\frac{(x_n - 1)}{\sqrt{1 - (x_n - 1)^2}} \right) & \frac{1}{2} \leq x_n \leq 1. \end{cases} \quad (44)$$

The value of x_{n+1} is updated as:

$$x_{n+1} = x_n + h. \quad (45)$$

Computational Area:

Using the trapezoidal rule with a small step size h , the computational area is: 1.2284

Theoretical Area:

Using the analytical approach, the theoretical area is: $\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.22837$

By substituting the values of x_k , $f(x_k)$, and $g(x_k)$ into the trapezoidal rule, we compute the area iteratively.

