# 8.2.2

EE24BTECH11004 - Ankit Jainar

**Question:** Find the area bounded by the curves:

$$(x-1)^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1 \tag{1}$$

**Theoretical Solution:**
The curves are two circles:

- Circle 1: $(x-1)^2 + y^2 = 1$, centered at $(1, 0)$ with radius 1.
- Circle 2: $x^2 + y^2 = 1$, centered at $(0, 0)$ with radius 1.

**Step 1: Finding the Points of Intersection Using Theoritical Approach**
The points of intersection occur where the two circles overlap. To find the intersection points, we rewrite the equations as:

$$(x-1)^2 + y^2 = 1 \quad \text{(Circle 1)} \tag{2}$$
$$x^2 + y^2 = 1 \quad \text{(Circle 2)} \tag{3}$$

Expanding both equations:

$$x^2 - 2x + 1 + y^2 = 1 \quad \text{(Circle 1)} \tag{4}$$
$$x^2 + y^2 = 1 \quad \text{(Circle 2)} \tag{5}$$

Subtracting the second equation from the first gives:

$$-2x + 1 = 0 \quad \Longrightarrow \quad x = \frac{1}{2} \tag{6}$$

Substituting $x = \frac{1}{2}$ into the equation of Circle 2:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1 \tag{7}$$
$$\frac{1}{4} + y^2 = 1 \tag{8}$$
$$y^2 = \frac{3}{4} \quad \Longrightarrow \quad y = \pm\frac{\sqrt{3}}{2} \tag{9}$$

Thus, the points of intersection are:

$$P_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad P_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{10}$$

Matrix representation for solving this system:

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}, \tag{11}$$

where:

$$V_1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y^2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{12}$$

**Step 2:Intersection of Two Circles Using Matrix Approach**

**General Form of the Circle Equations**

The general form for a circle is:

$$g(x, y) = \mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{13}$$

For Circle 1: $(x-1)^2 + y^2 = 1$

$$g_1(x, y) = \mathbf{x}^\top V_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1 = 0 \tag{14}$$

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad f_1 = 0 \tag{15}$$

For Circle 2: $x^2 + y^2 = 1$

$$g_2(x, y) = \mathbf{x}^\top V_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2 = 0 \tag{16}$$

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_2 = -1 \tag{17}$$

**Subtract the Equations to Eliminate Quadratic Terms**

Subtract $g_1(x, y)$ from $g_2(x, y)$:

$$g_2(x, y) - g_1(x, y) = (\mathbf{x}^\top V_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2) - (\mathbf{x}^\top V_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1) = 0 \tag{18}$$

$$0 = 2(\mathbf{u}_2 - \mathbf{u}_1)^\top \mathbf{x} + (f_2 - f_1) \tag{19}$$

Simplify:

$$2\begin{pmatrix} 0-(-1) \\ 0-0 \end{pmatrix}^\top \mathbf{x} + (-1-0) = 0 \tag{20}$$

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \mathbf{x} - 1 = 0 \tag{21}$$

**Line Equation Representing the Chord of Intersection**

The line equation representing the chord of intersection is:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \mathbf{x} = \frac{1}{2} \tag{22}$$

This simplifies to:

$$x = \frac{1}{2}. \tag{23}$$

**Substituting $x$ into One Circle Equation**

Substitute $x = \frac{1}{2}$ into the second circle equation $g_2(x, y)$: Substituting $\mathbf{x}$ into the circle equation:

$$\begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} - 1 = 0. \tag{24}$$

Expanding the terms:

$$\frac{1}{4} + y^2 - 1 = 0. \tag{25}$$

Simplifying:

$$y^2 = \frac{3}{4}. \tag{26}$$

Thus:

$$y = \pm\frac{\sqrt{3}}{2}. \tag{27}$$

**Points of Intersection**

The points of intersection are:

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \tag{28}$$

**Step 3: Area Calculation**

The area between the two circles is symmetric about the $x$-axis. Therefore, we calculate the area of the upper region and multiply it by 2.

The area is given by:

$$A = 2\int_{x=0.5}^{x=1} \left[ \sqrt{1-(x-1)^2} - \sqrt{1-x^2} \right] dx \tag{29}$$

Expanding $1 - (x - 1)^2$:

$$\begin{aligned} 1-(x-1)^2 &= 1-(x^2-2x+1) && (30) \\ &= 2x - x^2 && (31) \end{aligned}$$

Thus, the integral becomes:

$$A = 2\int_{x=0.5}^{x=1} \left[ \sqrt{2x-x^2} - \sqrt{1-x^2} \right] dx \tag{32}$$

**Trapezoidal Rule:**

We discretize the interval $[0.5, 1]$ into $N$ equal subintervals of width $h$:

$$h = \frac{1-0.5}{N} = \frac{0.5}{N}. \tag{33}$$

The $x_k$ values are:

$$x_k = 0.5 + k \cdot h, \quad k = 0, 1, 2, \ldots, N. \tag{34}$$

The area is approximated as:

$$A \approx 2 \cdot h \cdot \left[ \frac{1}{2}(f(x_0) - g(x_0)) + \sum_{k=1}^{N-1} (f(x_k) - g(x_k)) + \frac{1}{2}(f(x_N) - g(x_N)) \right], \tag{35}$$

where:

$$f(x_k) = \sqrt{2x_k - x_k^2}, \quad g(x_k) = \sqrt{1 - x_k^2}. \tag{36}$$

**Difference Equation:**

The iterative formula for the trapezoidal rule is:

$$A_{\text{new}} = A_{\text{old}} + h \cdot \left[ (f(x_{k+1}) - g(x_{k+1})) + (f(x_k) - g(x_k)) \right]. \tag{37}$$

$$\begin{aligned} A_{\text{new}} = A_{\text{old}} + h \cdot \Bigg[ & \left( \sqrt{2x_{k+1} - x_{k+1}^2} - \sqrt{1 - x_{k+1}^2} \right) \\ & + \left( \sqrt{2x_k - x_k^2} - \sqrt{1 - x_k^2} \right) \Bigg]. \end{aligned} \tag{38}$$

The difference equation for the area is given as:

$$A_{n+1} = A_n + \frac{h}{2} \left[ y(x_n) + h \cdot y'(x_n) + y(x_n) \right] \tag{39}$$

$$A_{n+1} = A_n + h \cdot \left[ y(x_n) + \frac{h}{2} y'(x_n) \right] \tag{40}$$

$$A_{n+1} = A_n + h \cdot y(x_n) + \frac{h^2}{2} y'(x_n) \tag{41}$$

The curves given are:

$$(x-1)^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1. \tag{42}$$

Rearranging the equations for $y(x)$, the piecewise function for the bounded region is:

$$y(x) = \begin{cases} \sqrt{1 - x^2} & 0 \le x \le \frac{1}{2} \\ \sqrt{1 - (x-1)^2} & \frac{1}{2} \le x \le 1 \end{cases} \tag{43}$$

**Substituting $y(x)$ into the Difference Equation:**

By substituting the piecewise function $y(x)$ into the difference equation, we get:

$$A_{n+1} = \begin{cases} A_n + h\sqrt{1 - x_n^2} + \frac{h^2}{2} \left( -\frac{x_n}{\sqrt{1 - x_n^2}} \right) & 0 \le x_n \le \frac{1}{2}, \\ A_n + h\sqrt{1 - (x_n - 1)^2} + \frac{h^2}{2} \left( -\frac{(x_n - 1)}{\sqrt{1 - (x_n - 1)^2}} \right) & \frac{1}{2} \le x_n \le 1. \end{cases} \tag{44}$$

The value of $x_{n+1}$ is updated as:

$$x_{n+1} = x_n + h. \tag{45}$$

**Computational Area:**

Using the trapezoidal rule with a small step size $h$, the computational area is: 1.2284

**Theoretical Area:**
Using the analytical approach, the theoretical area is: $\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.22837$

By substituting the values of $x_k$, $f(x_k)$, and $g(x_k)$ into the trapezoidal rule, we compute the area iteratively.