6.5.2.4

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Question

Find the minimum and maximum values of the function:

$$f(x) = |\sin(4x) + 3|$$

We need to determine the global extrema (minimum and maximum) of $f(x)$, which involves both mathematical analysis and numerical computation.

Theoretical Method

The given function is:

$$f(x) = |\sin(4x) + 3|.$$

Let us define $g(x) = \sin(4x) + 3$, so that $f(x) = |g(x)|$. To find the extrema of $f(x)$, we first locate the critical points of $g(x)$, where its derivative is zero.

The derivative of $g(x)$ is:

$$g'(x) = \frac{d}{dx}[\sin(4x) + 3] = 4\cos(4x).$$

Setting $g'(x) = 0$, we find:

$$\cos(4x) = 0.$$

The cosine function is zero when:

$$4x = \frac{\pi}{2} + n\pi \quad \text{where } n \in \mathbb{Z}.$$

This gives:

$$x = \frac{\pi}{8} + \frac{n\pi}{4}, \quad n \in \mathbb{Z}$$

Theoretical Method

To analyze the behavior of $f(x)$, consider the function $g(x)$ over one period. Since $g(x)$ is periodic, the extrema of $f(x)$ repeat in each period. Substituting $x = \frac{\pi}{8} + \frac{n\pi}{4}$ into $g(x) = \sin(4x) + 3$, we calculate:

$$g(x)_{\text{max}} = \sin(4x) + 3 \quad (\text{when } \sin(4x) = 1),$$
$$g(x)_{\text{min}} = \sin(4x) + 3 \quad (\text{when } \sin(4x) = -1).$$

Thus:

$$f(x)_{\text{max}} = |3 + 1| = 4, \quad f(x)_{\text{min}} = |3 - 1| = 2.$$

The **maximum value** of $f(x)$ is **4** and the **minimum value** is **2**.

Computational Method

For computational analysis, we numerically determine the extrema using gradient-based optimization techniques.

Numerical Gradient Computation
The gradient of $f(x)$ is approximated using the central difference formula:

$$f'(x) \approx \frac{f(x+\delta) - f(x-\delta)}{2\delta},$$

where δ is a **small step size**. This approach avoids issues with symbolic differentiation and is used in numerical algorithms.

Finding Minima and Maxima
Gradient Descent: For minima, we iteratively move in the direction of the negative gradient:

$$x_{n+1} = x_n - \mu f'(x_n),$$

where μ is the **learning rate**.

Gradient Ascent: For maxima, we move in the direction of the positive gradient:

$$x_{n+1} = x_n + \mu f'(x_n).$$

Using an initial guess $x = 0$, step size $\mu = 0.01$, and convergence threshold 10^{-5}, the algorithm converges to the following results:

$$\text{Minimum Value:} \quad f(x_{\text{min}}) = 2 \quad \text{at } x_{\text{min}} = 0.785,$$
$$\text{Maximum Value:} \quad f(x_{\text{max}}) = 4 \quad \text{at } x_{\text{max}} = 0.392.$$

Comparison: The theoretical and computational results match, confirming the accuracy of the method.

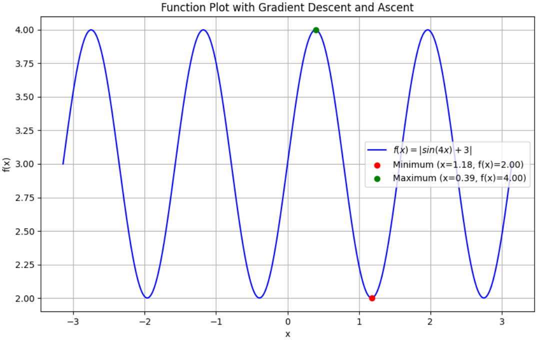

Figure: Graphical Comparison of $f(x)$ Extrema

Conclusion I

The **maximum value** of $f(x)$ is 4.
The **minimum value** of $f(x)$ is 2.