8.2.2

EE24BTECH11004 - Ankit Jainar

Question: Find the area bounded by the curves:

$$(x-1)^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1 \tag{1}$$

Theoretical Solution:
The curves are two circles:

- Circle 1: $(x-1)^2 + y^2 = 1$, centered at $(1, 0)$ with radius 1.
- Circle 2: $x^2 + y^2 = 1$, centered at $(0, 0)$ with radius 1.

Step 1: Finding the Points of Intersection

The points of intersection occur where the two circles overlap. To find the intersection points, we solve the system of equations:

$$(x-1)^2 + y^2 = 1 \quad \text{(eq 1)} \tag{2}$$

$$x^2 + y^2 = 1 \quad \text{(eq 2)} \tag{3}$$

Expanding and simplifying these equations:

$$\text{From Circle 1:} \quad x^2 - 2x + 1 + y^2 = 1 \tag{4}$$

$$\text{From Circle 2:} \quad x^2 + y^2 = 1 \tag{5}$$

Subtracting the second equation from the first:

$$(x^2 - 2x + 1 + y^2) - (x^2 + y^2) = 1 - 1 \tag{6}$$

$$-2x + 1 = 0 \quad \implies \quad x = \frac{1}{2} \tag{7}$$

Substitute $x = \frac{1}{2}$ into the equation of Circle 2:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1 \tag{8}$$

$$\frac{1}{4} + y^2 = 1 \tag{9}$$

$$y^2 = \frac{3}{4} \quad \implies \quad y = \pm\frac{\sqrt{3}}{2} \tag{10}$$

Thus, the points of intersection are:

$$P_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad P_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{11}$$

Step 2: Theoretical Area Derivation

The area between the two circles is symmetric about the x-axis. Therefore, we calculate the area of the upper region and multiply it by 2.

Integral Setup:

The area is given by:

$$A = 2 \int_{x=0.5}^{x=1} \left[\sqrt{1-(x-1)^2} - \sqrt{1-x^2} \right] dx \tag{12}$$

Here:

- $\sqrt{1-(x-1)^2}$ represents the upper semicircle of Circle 1.
- $\sqrt{1-x^2}$ represents the upper semicircle of Circle 2.

Simplify the Expressions:

Expanding $1-(x-1)^2$:

$$\begin{aligned} 1-(x-1)^2 &= 1-(x^2-2x+1) \qquad (13)\\ &= 2x-x^2 \qquad (14) \end{aligned}$$

Thus, the integral becomes:

$$A = 2 \int_{x=0.5}^{x=1} \left[\sqrt{2x-x^2} - \sqrt{1-x^2} \right] dx \tag{15}$$

Numerical Computation:

The integral can be solved numerically using software tools or approximations. The exact value of the area can be found using such methods.

Matrix Representation: The system of equations can be represented in matrix form as:

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \tag{16}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x \\ y^2 \\ c \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{17}$$

Solve for $\boldsymbol{x}$ using Gaussian elimination or matrix inversion:

$$\boldsymbol{x} = \boldsymbol{A}^{-1} \cdot \boldsymbol{b} \tag{18}$$

Step 2: Area Calculation: The area between the two circles is given by:

$$A = 2 \int_{x=0.5}^{x=1} \left[\sqrt{1-(x-1)^2} - \sqrt{1-x^2} \right] dx \tag{19}$$

Using matrix-based numerical integration, the integral is discretized as:

$$A \approx 2 \cdot h \cdot \sum_{k=0}^{N-1} \left[f(x_k) - g(x_k) \right] \tag{20}$$

where:

$$f(x_k) = \sqrt{1-(x_k-1)^2}, \quad g(x_k) = \sqrt{1-x_k^2} \tag{21}$$

Here, x_k are discretized points in the range $[0.5, 1]$, and h is the step size.

Step 3: Numerical Integration: By substituting the values of x_k and performing matrix-based summation, the approximate area is calculated. The integral can also be solved symbolically for exact results.