3.2.1.2

EE24BTECH11004 - Ankit Jainar

Question: 5 pencils and 7 pens together cost Rs.50, whereas 7 pencils and 5 pens together cost Rs.46. Find the cost of one pencil and that of one pen.

Solution:

Let the cost of one pencil be denoted by x and the cost of one pen by y. The situation can be described using the following system of linear equations:

$$5x + 7y = 50, \tag{1}$$

$$7x + 5y = 46. \tag{2}$$

I. Theoretical Solution

We solve the above equations using elimination:

- Multiply equation (1) by 5 and equation (2) by 7.
- Subtract the resulting equations to eliminate y and solve for x.
- Substitute the value of x back into either equation to find y.

Performing these steps:

$$x = 3, \quad y = 5.$$

II. Numerical Method:

III. LU Decomposition to Solve the System

We now solve the system of equations using LU decomposition.

A. Matrix Form

The system of equations can be expressed in matrix form as:

$$\begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ 46 \end{bmatrix}. \tag{1}$$

Here, the coefficient matrix is:

$$A = \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 50 \\ 46 \end{bmatrix}. \tag{2}$$

B. Step 1: Decomposing A into L and U

The matrix A can be decomposed into:

$$A = L \cdot U, \tag{3}$$

where:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{7}{5} & 1 \end{bmatrix}, \tag{4}$$

$$U = \begin{bmatrix} 5 & 7 \\ 0 & -\frac{14}{5} \end{bmatrix}. \tag{5}$$

Step 2: LU Factorization Using Update Equations

Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

Step-by-Step Procedure: 1. **Initialization:** - Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.

2. **Iterative Update:** - For each pivot $k = 1, 2, \dots, n$: - Compute the entries of U using the first update equation. - Compute the entries of L using the second update equation.

3. **Result:** - After completing the iterations, the matrix $\mathbf{A}$ is decomposed into $\mathbf{L} \cdot \mathbf{U}$, where $\mathbf{L}$ is a lower triangular matrix with ones on the diagonal, and $\mathbf{U}$ is an upper triangular matrix.

1. Update for $U_{k,j}$ (Entries of U)

For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad \text{for } j \geq k.$$

This equation computes the elements of the upper triangular matrix $\mathbf{U}$ by eliminating the lower triangular portion of the matrix.

2. Update for $L_{i,k}$ (Entries of L)

For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad \text{for } i > k.$$

This equation computes the elements of the lower triangular matrix $\mathbf{L}$, where each entry in the column is determined by the values in the rows above it.

LU Factorization of Matrix A

We decompose A as:

$$A = LU,$$

where L is a lower triangular matrix and U is an upper triangular matrix. For the given example, we calculate L and U as follows:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 2 \\ 0 & -\frac{13}{3} \end{bmatrix}.$$

C. Step 2: Forward Substitution

The system $A\vec{x} = \vec{b}$ is transformed into $L \cdot U \cdot \vec{x} = \vec{b}$. Let $\vec{y}$ satisfy $L\vec{y} = \vec{b}$:

$$\begin{bmatrix} 1 & 0 \\ \frac{7}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 46 \end{bmatrix}. \tag{6}$$

Using forward substitution:

$$y_1 = 50, \tag{7}$$

$$\frac{7}{5}y_1 + y_2 = 46 \implies y_2 = 46 - \frac{7}{5}(50) = -24. \tag{8}$$

Thus:

$$\vec{y} = \begin{bmatrix} 50 \\ -24 \end{bmatrix}. \tag{9}$$

D. Step 3: Back Substitution

Next, solve $U\vec{x} = \vec{y}$:

$$\begin{bmatrix} 5 & 7 \\ 0 & -\frac{14}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ -24 \end{bmatrix}. \tag{10}$$

Using back substitution:

$$-\frac{14}{5}y = -24 \implies y = 5, \tag{11}$$

$$5x + 7(5) = 50 \implies x = 3. \tag{12}$$

E. Updated Equation:

$$A\vec{x} = L \cdot U \cdot \vec{x} = \vec{b}, \tag{13}$$

$$A = L \cdot U, \tag{14}$$

$$L \cdot U \cdot \vec{x} = \vec{b}, \tag{15}$$

$$U \cdot \vec{x} = \vec{y}, \tag{16}$$

$$L \cdot \vec{y} = \vec{b}. \tag{17}$$

F. Final Answer

The cost of one pencil is Rs.3, and the cost of one pen is Rs.5.

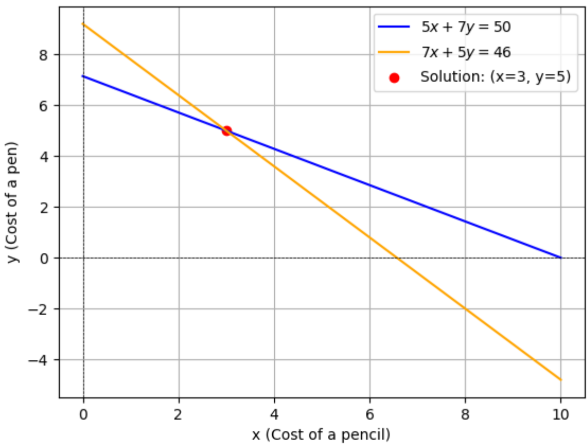

Fig. 1. Graphical Representation of the Solution