11.16.3.7

EE24BTECH11004 - Ankit Jainar

Question: Three coins are tossed once. Find the probability of getting exactly two tails.

Theoretical Solution

The sample space for tossing three coins is:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \tag{1}$$

The total number of outcomes is:

$$|S| = 8 \tag{2}$$

The favorable outcomes (exactly two tails) are:

$$A = \{HTT, THT, TTH\} \tag{3}$$

The number of favorable outcomes is:

$$|A| = 3 \tag{4}$$

The probability of getting exactly two tails is:

$$P(A) = \frac{|A|}{|S|} = \frac{3}{8} = 0.375 \tag{5}$$

Theoretical Solution Using Z-Transform

To solve the problem using the Z-transform, we first define the random variable X to represent the number of tails. The probability mass function (PMF) for a single coin toss is:

$$P(X = 0) = 0.5, \quad P(X = 1) = 0.5. \tag{6}$$

For three independent coin tosses, the generating function using the Z-transform is:

$$G(z) = (0.5 + 0.5z)^3. \tag{7}$$

Expanding $G(z)$ gives the following:

$$G(z) = 0.125 + 0.375z + 0.375z^2 + 0.125z^3. \tag{8}$$

Thus, the probability mass function of the number of tails Y is:

$$P(Y = k) = \begin{cases} 0.125, & k = 0, \\ 0.375, & k = 1, \\ 0.375, & k = 2, \\ 0.125, & k = 3. \end{cases} \tag{9}$$

The probability of getting exactly two tails is:

$$P(\text{exactly two tails}) = P(Y = 2) = 0.375. \tag{10}$$

INTRODUCTION

This task involves simulating the random tossing of three coins using a C program, compiling it into a shared object (.so) file, and using Python to process the results and generate a probability distribution plot.

C CODE DESCRIPTION

The C program generates random samples for the coin tosses, where the outcomes are categorized based on the number of tails. The program uses the `rand()` function to simulate the random tosses and increments a counter for each outcome with exactly two tails.

PYTHON CODE DESCRIPTION

The Python code performs the following:

1) Loads the shared object file generated from the C program using the `ctypes` library.
2) Simulates a specified number of random coin tosses (e.g., 1,000,000 trials).
3) Calculates the probability of getting exactly two tails using the formula:

$$P(\text{exactly two tails}) = \frac{\text{frequency of exactly two tails}}{\text{total trials}} \tag{11}$$

4) Plots the probability distribution using `matplotlib`.

GRAPHICAL OUTPUT

The Python code generates a bar chart where:

- The x-axis represents the outcomes: "0 tails", "1 tail", "2 tails", and "3 tails".
- The y-axis represents the probabilities, ranging from 0 to 1.
- The bar height for "2 tails" corresponds to the probability $P(A) = 0.375$.

Probability Mass Function (PMF): The PMF represents the probability of each individual outcome in the sample space S. For the coin toss:

$$S = \{0 \text{ tails}, 1 \text{ tail}, 2 \text{ tails}, 3 \text{ tails}\}, \tag{12}$$

The PMF is extracted by looking at the coefficients of powers of z in the expansion of $G(z)$. This corresponds to the probability of obtaining k tails, where $k = 0, 1, 2, 3$. From the expansion:

$$G(z) = 0.125 + 0.375z + 0.375z^2 + 0.125z^3,$$

we can directly read off the probabilities:

$$P(X = 0) = 0.125, \quad P(X = 1) = 0.375, \quad P(X = 2) = 0.375, \quad P(X = 3) = 0.125.$$

Thus, the PMF is:

$$P(X = k) = \begin{cases} 0.125 & \text{for } k = 0, \\ 0.375 & \text{for } k = 1, \\ 0.375 & \text{for } k = 2, \\ 0.125 & \text{for } k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

CDF (Cumulative Distribution Function)

The CDF is obtained by summing up the probabilities from the PMF up to a given value k. The CDF, $F(x)$, is defined as:

$$F(x) = P(X \leq x) = \sum_{k \in S, k \leq x} P(X = k).$$

We calculate the CDF at the points $x = 0, 1, 2, 3$:
- For $x = 0$ (0 tails):

$$F(0) = P(X \leq 0) = P(X = 0) = 0.125.$$

- For $x = 1$ (1 tail):

$$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.125 + 0.375 = 0.5.$$

- For $x = 2$ (2 tails):

$$F(2) = P(X \leq 2) = P(X = 0)+P(X = 1)+P(X = 2) = 0.125+0.375+0.375 = 0.875.$$

- For $x = 3$ (3 tails):

$$F(3) = P(X \leq 3) = P(X = 0)+P(X = 1)+P(X = 2)+P(X = 3) = 0.125+0.375+0.375+$$

Thus, the CDF is:

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ 0.125 & \text{for } 0 \leq x < 1, \\ 0.5 & \text{for } 1 \leq x < 2, \\ 0.875 & \text{for } 2 \leq x < 3, \\ 1 & \text{for } x \geq 3. \end{cases}$$

Simulation Process

We simulate the tossing of three coins using the following steps:
1) The sample space consists of outcomes in the set:

$$S = \{0 \text{ tails}, 1 \text{ tail}, 2 \text{ tails}, 3 \text{ tails}\}. \tag{13}$$

2) For each simulated toss, a random integer X is generated such that:

$$X \in \{0, 1, 2, 3\}, \tag{14}$$

using a random number generator function based on binomial trials.
3) The number of occurrences of each outcome is tracked over N trials, where N is the total number of simulations.
4) Both the PMF and CDF are computed:
   - **PMF**: The frequency of each outcome is divided by the total number of trials to comput the probabilities.
   - **CDF**: The cumulative probabilities are calculated as the running total of the PMF values.

Calculation of Probabilities

Probability of Exactly Two Tails (PMF): The probability of exactly two tails is computed as:

$$P(\text{exactly two tails}) = \frac{3}{8} = 0.375. \tag{15}$$

Cumulative Probability (CDF): The cumulative probability of outcomes up to a given value is:

$$F(x) = \begin{cases} P(0 \text{ tails}), & x = 0 \text{ tails}, \\ P(0 \text{ tails}) + P(1 \text{ tail}), & x = 1 \text{ tail}, \\ P(0 \text{ tails}) + P(1 \text{ tail}) + P(2 \text{ tails}), & x = 2 \text{ tails}, \\ 1, & x = 3 \text{ tails}. \end{cases} \tag{16}$$

For the coin toss:

$$F(0 \text{ tails}) = 0.125, \quad F(1 \text{ tail}) = 0.5, \quad F(2 \text{ tails}) = 0.875, \quad F(3 \text{ tails}) = 1. \tag{17}$$

Probability of Selecting $X \notin S$: Since all outcomes belong to the set $S = \{0, 1, 2, 3\}$, the probability of selecting $X \notin S$ is:

$$P(X \notin S) = 0. \tag{18}$$

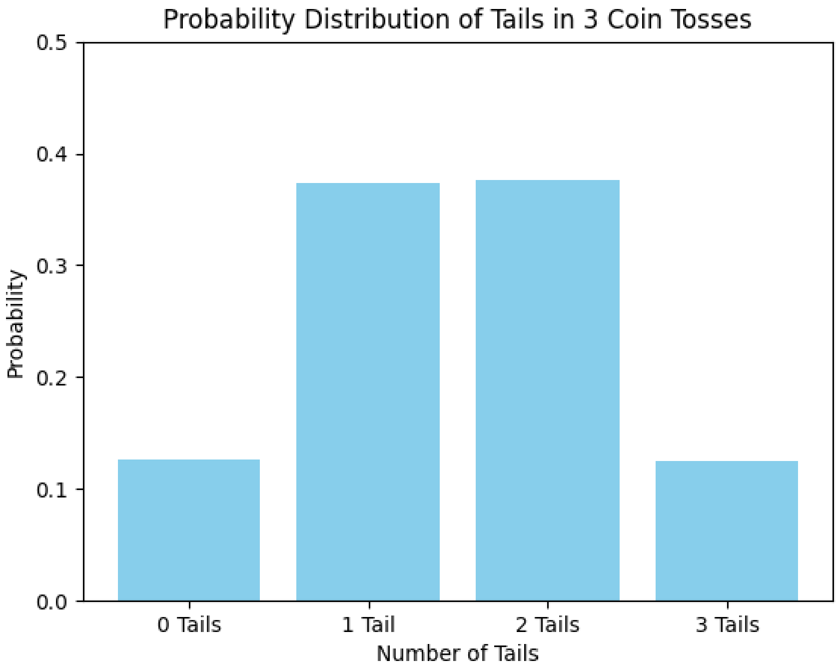
Output Representation

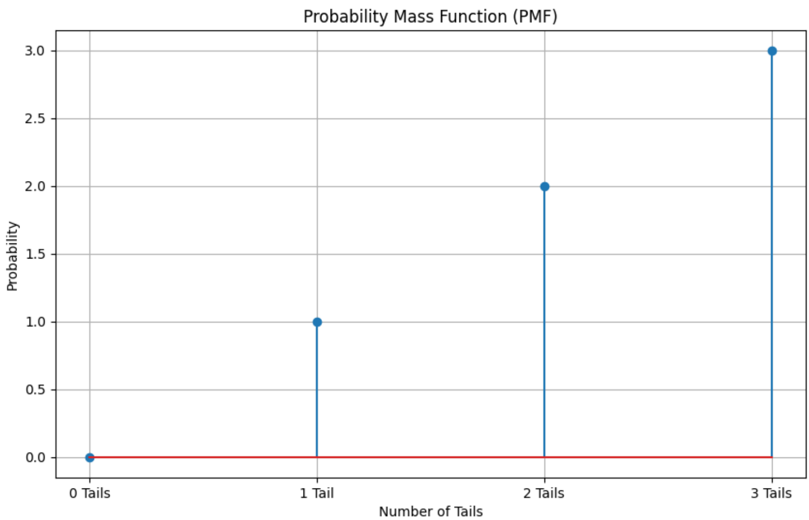
The computed probabilities are represented in two forms:

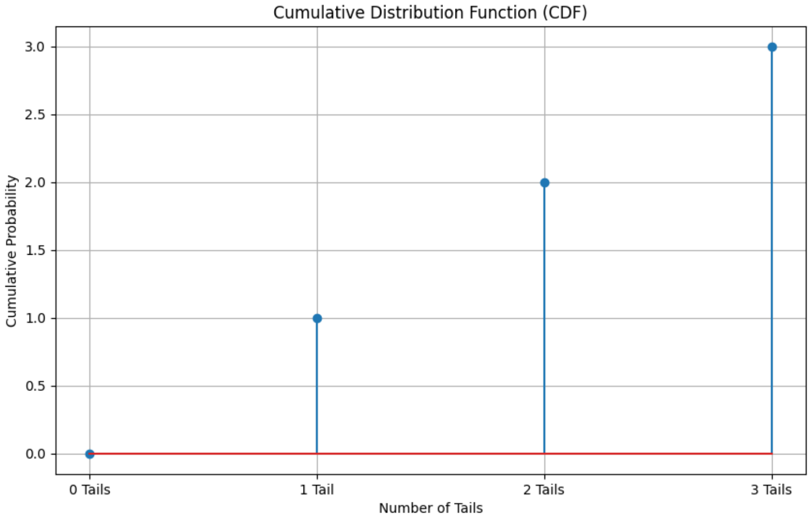
- PMF: The probabilities of each outcome (0 tails, 1 tail, 2 tails, 3 tails).
- CDF: The cumulative probabilities up to each outcome.

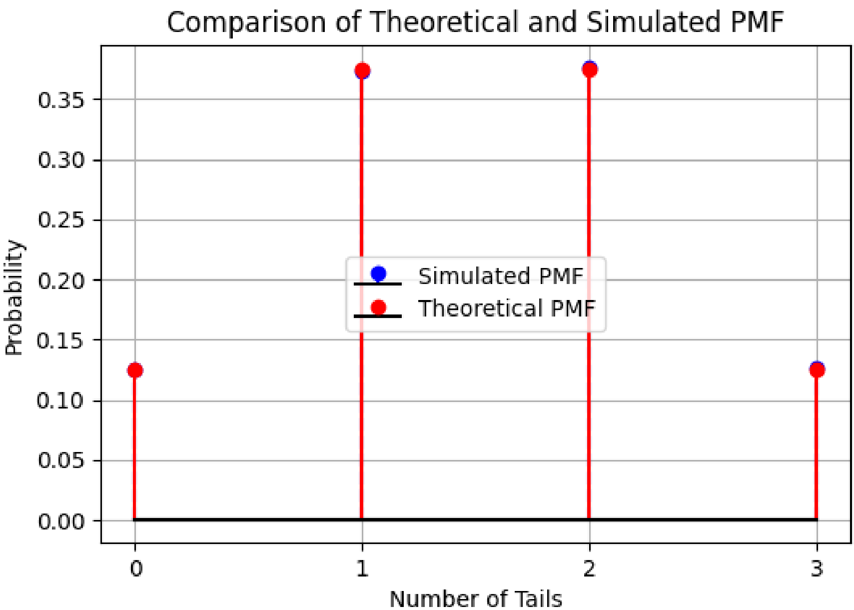
Conclusion

This task demonstrates the integration of C and Python for simulating and visualizing a probabilistic experiment. The probability of getting exactly two tails from tossing three coins is calculated as **0.375**, matching the theoretical value.


Probability Distribution of Tails in 3 Coin Tosses
Probability
0.0
0.1
0.2
0.3
0.4
0.5
0 Tails
1 Tail
2 Tails
3 Tails
Number of Tails


Probability Mass Function (PMF)
Probability
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0 Tails
1 Tail
2 Tails
3 Tails
Number of Tails


Cumulative Distribution Function (CDF)
Cumulative Probability
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0 Tails
1 Tail
2 Tails
3 Tails
Number of Tails


Comparison of Theoretical and Simulated PMF
Probability
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Simulated PMF
Theoretical PMF
0
1
2
3
Number of Tails