6.5.2.4

EE24BTECH11004 - Ankit Jainar

Question:
Find the minimum and maximum values of the given function:

$$f(x) = |\sin(4x) + 3|$$

Theoretical Method We analyze the function theoretically to find its critical points. Let:

$$f(x) = |g(x)|, \quad g(x) = \sin(4x) + 3 \tag{0.1}$$

The critical points of $g(x)$ occur where $g'(x) = 0$. Differentiating $g(x)$:

$$g'(x) = 4\cos(4x) \tag{0.2}$$

Setting $g'(x) = 0$, we find:

$$\cos(4x) = 0 \implies 4x = \frac{\pi}{2} + n\pi \implies x = \frac{\pi}{8} + \frac{n\pi}{4}, \ n \in \mathbb{Z} \tag{0.3}$$

For these x-values, we calculate $g(x)$ to find the maximum and minimum values of $|g(x)|$:

$$g(x) = \sin(4x) + 3, \quad f(x) = |g(x)| \tag{0.4}$$

At the critical points, evaluate $f(x)$ directly to determine the local maximum and minimum values. The function $f(x)$ achieves its minimum value at $f(x) = 3$ and maximum value at $f(x) = 4$.
Computational Method We numerically compute the minima and maxima of the function using gradient-based methods. The approach is divided into two cases:

Minima

To find the minima, we use **gradient descent**, which iteratively updates the variable x according to the following rule:

$$x_{n+1} = x_n - \mu f'(x_n) \tag{0.5}$$

Here, μ is the step size (learning rate), and $f'(x)$ is the derivative of the function. This process iteratively reduces the function value, converging to a local minimum.

Maxima

To find the maxima, we use **gradient ascent**, which is analogous to gradient descent but involves moving in the opposite direction of the gradient:

$$x_{n+1} = x_n + \mu f'(x_n) \tag{0.6}$$

By updating x in the direction of the gradient, the function value increases iteratively, converging to a local maximum.

Numerical Gradient Computation

The gradient $f'(x)$ is computed numerically using the central difference approximation:

$$f'(x) \approx \frac{f(x+\delta) - f(x-\delta)}{2\delta} \tag{0.7}$$

Here, δ is a small perturbation used to approximate the derivative.

Algorithm Steps:

- **Initialize:** Start with an initial guess for x, a step size (μ), and a threshold for convergence.
- **Update:** Compute $f'(x_n)$ and update x_n using the update rule.
- **Convergence:** Stop when $|f'(x_n)| <$ threshold.

Results: Using an initial guess $x = 0$, step size $\mu = 0.01$, and threshold $1e - 5$, the numerical method yields:

$$x_{\text{min}} = 0.785398, \ f(x_{\text{min}}) = 3.000000 \tag{0.8}$$

$$x_{\text{max}} = 0.392699, \ f(x_{\text{max}}) = 4.000000 \tag{0.9}$$

Thus, the maximum value of $f(x)$ is 4, and the minimum value of $f(x)$ is 3. These values match the theoretical results.