GRAVITATION

THE DISCOVERY OF THE LAW OF GRAVITATION

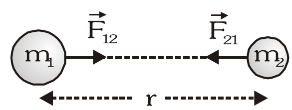
The way the law of universal gravitation was discovered is often considered the paradigm of modern scientific technique. The major steps involved were

- The hypothesis about planetary motion given by **Nicolaus Copernicus (1473–1543)**.
- The careful experimental measurements of the positions of the planets and the Sun by **Tycho Brahe (1546–1601)**.
- Analysis of the data and the formulation of empirical laws by **Johannes Kepler (1571–1630)**.
- The development of a general theory by **Isaac Newton (1642–1727)**.

NEWTON'S LAW OF GRAVITATION

It states that every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

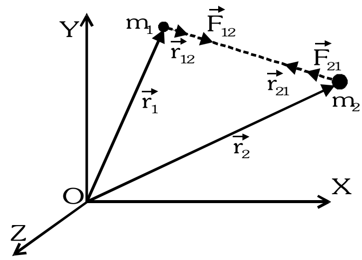
$F \propto m_1 m_2$ and $F \propto \dfrac{1}{r^2}$ so $F \propto \dfrac{m_1 m_2}{r^2}$



$\therefore\ F = \dfrac{Gm_1 m_2}{r^2}$ [G = Universal gravitational constant]

Note : This formula is only applicable for spherical symmetric masses or point masses.

Vector form of Newton's law of Gravitation :



Let $\vec{r}_{12}$ = Displacement vector from m_1 to m_2

$\vec{r}_{21}$ = Displacement vector from m_2 to m_1

$\vec{F}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{F}_{12}$ = Gravitational force exerted on m_1 by m_2

$$\vec{F}_{12} = -\frac{Gm_1 m_2}{r_{21}^2}\hat{r}_{21} = -\frac{Gm_1 m_2}{r_{21}^3}\vec{r}_{21}$$

Negative sign shows that :

(i) The direction of $\vec{F}_{12}$ is opposite to that $\hat{r}_{21}$

(ii) The gravitational force is attractive in nature

Similarly $\vec{F}_{21} = -\dfrac{Gm_1 m_2}{r_{12}^2}\hat{r}_{12}$ or $\vec{F}_{21} = -\dfrac{Gm_1 m_2}{r_{12}^3}\vec{r}_{12}$ $\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

The gravitational force between two bodies are equal in magnitude and opposite in direction.

GRAVITATIONAL CONSTANT "G"

- Gravitational constant is a scalar quantity.
- **UNIT : S I :** G = 6.67×10^{-11} N–m^2/kg^2 **CGS :** 6.67×10^{-8} dyne–cm^2/g^2 **Dimensions :** $[M^{-1}L^3T^{-2}]$
- Its value is same throughout the universe, G does not depend on the nature and size of the bodies, it also does not depend upon nature of the medium between the bodies.
- Its value was first find out by the scientist **"Henry Cavendish"** with the help of "Torsion Balance" experiment.
- Value of G is small therefore gravitational force is weaker than electrostatic and nuclear forces.

Example

Two particles of masses 1 kg and 2 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial acceleration of heavier particle.

Solution

Force exerted by one particle on another $F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.34\times10^{-10}\,N$

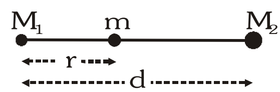
Acceleration of heavier particle $= \frac{F}{m_2} = \frac{5.3\times10^{-10}}{2} = 2.67\times10^{-10}\,ms^{-2}$

This example shows that gravitation is very weak but only this force keep bind our solar system and also this universe of all galaxies and other interstellar system.

Example

Two stationary particles of masses M_1 and M_2 are at a distance 'd' apart. A third particle lying on the line joining the particles, experiences no resultant gravitational forces. What is the distance of this particle from M_1.

Solution



The force on m towards M_1 is $F_1 = \frac{GM_1m}{r^2}$

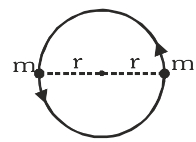
The force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

According to question net force on m is zero i.e. $F_1 = F_2$

$$\Rightarrow \frac{GM_1m}{r^2} = \frac{GM_2m}{(d-r)^2} \Rightarrow \left(\frac{d-r}{r}\right)^2 = \frac{M_2}{M_1} \Rightarrow \frac{d}{r}-1 = \frac{\sqrt{M_2}}{\sqrt{M_1}} \Rightarrow r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right]$$

Example

Two particles of equal mass (m) each move in a circle of radius (r) under the action of their mutual gravitational attraction. Find the speed of each particle.



Solution

For motion of particle $\frac{mv^2}{r} = \frac{Gmm}{(2r)^2} \Rightarrow v^2 = \frac{Gm}{4r} \Rightarrow v = \frac{1}{2}\sqrt{\frac{Gm}{r}}$

Example

Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side 'a'. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining their original separation 'a'. Determine the initial velocity that should be given to each particle and time period of circular motion.

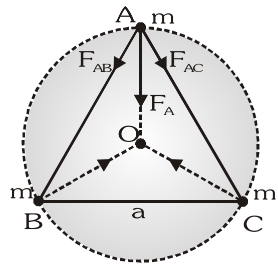
Solution

The resultant force on particle at A due to other two particles is

$$F_A = \sqrt{F_{AB}^2 + F_{AC}^2 + 2F_{AB}F_{AC}\cos60°} = \sqrt{3}\frac{Gm^2}{a^2} \quad ...(i) \quad \left[\because F_{AB} = F_{AC} = \frac{Gm^2}{a^2}\right]$$

Radius of the circle $r = \frac{2}{3} \times a\sin 60° = \frac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v, so that F acts as the centripetal force,



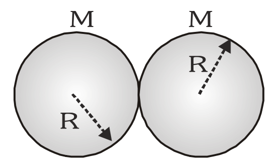
Now $\frac{mv^2}{r} = \sqrt{3}\frac{mv^2}{a}$ ...(ii)

From (i) and (ii) $\sqrt{3}\frac{mv^2}{a} = \frac{Gm^2\sqrt{3}}{a^2} \Rightarrow v = \sqrt{\frac{Gm}{a}}$

Time period $T = \frac{2\pi r}{v} = \frac{2\pi a}{\sqrt{3}}\sqrt{\frac{a}{Gm}} = 2\pi\sqrt{\frac{a^3}{3Gm}}$

Example

Two solid sphere of same size of a metal are placed in contact by touching each other. Prove that the gravitational force acting between them is directly proportional to the fourth power of their radius.



Solution

The weights of the spheres may be assumed to be concentrated at their centres.

So $F = \frac{G\left[\frac{4}{3}\pi R^3\rho\right] \times \left[\frac{4}{3}\pi R^3\rho\right]}{(2R)^2} = \frac{4}{9}(G\pi^2\rho^2)R^4 \quad \therefore F \propto R^4$

Example

A mass (M) is split into two parts (m) and (M–m). Which are then separated by a certain distance. What ratio $\frac{m}{M}$ will maximise the gravitational force between the parts ?

Solution

If r is the distance between m and (M − m), the gravitational force will be $F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(mM - m^2)$

For F to be maximum $\frac{dF}{dm} = 0$ as M and r are constant, i.e. $\frac{d}{dm}\left[\frac{G}{r^2}(mM - m^2)\right] = 0$ i.e. M − 2m = 0 $\left[\because \frac{G}{r^2} \neq 0\right]$

or $\frac{m}{M} = \frac{1}{2}$, i.e., the force will be maximum when the two parts are equal.

Example

A thin rod of mass M and length L is bent in a semicircle as shown in figure. (a) What is its gravitational force (both magnitude and direction) on a particle with mass m at O, the centre of curvature? (b) What would be the force on m if the rod is, in the form of a complete circle?

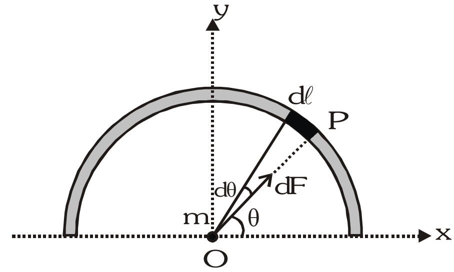
Solution

(a) Considering an element of rod of length $d\ell$ as shown in figure and treating it as a point of mass (M/L) $d\ell$ situated at a distance R from P, the gravitational force due to this element on the particle will be

$$dF = \frac{Gm(M/L)(Rd\theta)}{R^2} \text{ along OP [as } d\ell = Rd\theta\text{]}$$

So the component of this force along x and y–axis will be

$$dF_x = dF\cos\theta = \frac{GmM\cos\theta\,d\theta}{LR}\;;\; dF_y = dF\sin\theta = \frac{GmM\sin\theta\,d\theta}{LR}$$



So that $$F_x = \frac{GmM}{LR}\int_0^{\pi}\cos\theta\,d\theta = \frac{GmM}{LR}\left[\sin\theta\right]_0^{\pi} = 0$$

and $$F_y = \frac{GmM}{LR}\int_0^{\pi}\sin\theta\,d\theta = \frac{GmM}{LR}\left[-\cos\theta\right]_0^{\pi} = \frac{2\pi GmM}{L^2}\left[\text{as } R = \frac{L}{\pi}\right]$$

So $$F = \sqrt{F_x^2 + F_y^2} = F_y = \frac{2\pi GmM}{L^2}\quad [\text{as } F_x \text{ is zero}]$$

i.e., the resultant force is along the y–axis and has magnitude ($2\pi GmM/L^2$)

(b) If the rod was bent into a complete circle,

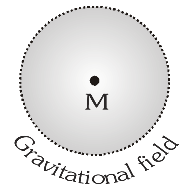
$$F_x = \frac{GmM}{LR}\int_0^{2\pi}\cos\theta\,d\theta = 0 \text{ and also } F_y = \frac{GmM}{LR}\int_0^{2\pi}\sin\theta\,d\theta = 0$$

i.e, the resultant force on m at O due to the ring is zero.

GOLDEN KEY POINT

- Gravitational force is always attractive.
- Gravitational forces are developed in form of action and reaction pair. Hence obey Newton's third law of motion.
- It is independent of nature of medium in between two masses and presence or absence of other bodies.
- Gravitational forces are central forces as they act along the line joining the centres of two bodies.
- The gravitational forces are conservative forces so work done by gravitational force does not depends upon path.
- If any particle moves along a closed path under the action of gravitational force then the work done by this force is always zero.
- Gravitational force is weakest force of nature.
- Force developed between any two masses is called gravitational force and force between Earth and any body is called gravity force.
- The total gravitational force on one particle due to number of particles is the resultant of forces of attraction exerted on the given particle due to individual particles i.e. $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots\ldots\ldots$ It means the principle of superposition is valid.
- Gravitational force holds good over a wide range of distances. It is found the true from interplanetary distances to interatomic distances.
- It is a two body interaction i.e. gravitational force between two particles is independent of the presence or absence of other particles.
- A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its centre.

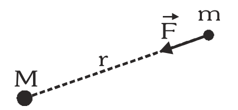
GRAVITATIONAL FIELD



The gravitational field is the space around a mass or an assembly of masses over which it can exert gravitational forces on other masses.

Theoretically speaking, the gravitational field extends up to infinity. However, in actual practice, the gravitational field may become too weak to be measured beyond a particular distance.

Gravitational Field Intensity [I_g or E_g]

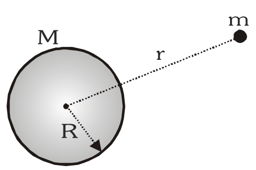


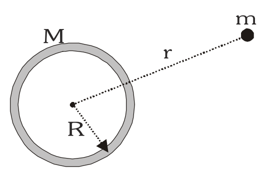
Gravitational force acting per unit mass at any point in the gravitational field is called Gravitational field intensity.

$$I_g = \left(\frac{GMm}{r^2}\Big/m\right) = \frac{GM}{r^2} \qquad \text{Vector form} : \vec{I}_g = \frac{\vec{F}}{m} \quad \text{or} \quad \vec{I}_g = -\frac{GM}{r^2}\tilde{r}$$

GOLDEN KEY POINT

- Gravitational field intensity is a vector quantity having dimension [LT^{-2}] and unit N/kg.
- As by definition $\vec{I}_g = \frac{\vec{F}}{m}$ i.e. $\vec{F} = m\vec{I}_g$ so force on a point mass (m) is multiplication of intensity of field and mass of point mass.
- **Solid Sphere** / **Spherical shell**

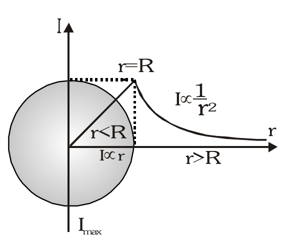




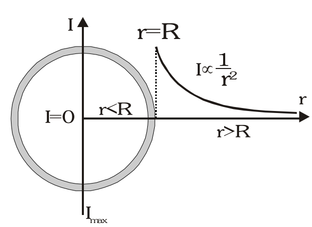
Solid Sphere		Spherical shell	
• $r > R$	$I_{out} = \frac{GM}{r^2}$	• $r > R$	$I_{out} = \frac{GM}{r^2}$
• $r = R$	$I_{sur} = \frac{GM}{R^2}$	• $r = R$	$I_{sur} = \frac{GM}{R^2}$
• $r < R$	$I_{inside} = \frac{GMr}{R^3} = \frac{4}{3}\pi G r \rho$	• $r < R$	$I_{inside} = 0$

- **Graph between (I_g) and (r)**

For Solid Sphere



For Spherical shell



GRAVITY

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is Earth then the gravitation is called 'gravity'. Hence, gravity is the force by which Earth attracts a body towards its centre. It is a special case of gravitation.

Gravitation near Earth's surface

Let us assume that Earth is a uniform sphere of mass M and radius R. The magnitude of the gravitational force from Earth on a particle of mass m, located outside Earth at a distance r from Earth is centre, is $F = \frac{GMm}{r^2}$

Now according to Newton's second laws $F = ma_g$

Therefore $a_g = \frac{GM}{r^2}$ ...(i)

At the surface of Earth, acceleration due to gravity $g = \frac{GM}{R^2} = 9.8\ m/s^2$

However any a_g value measured at a given location will differ from the a_g value calculated with equation (i) due to three reasons

(i) Earth's mass is not distributed uniformly.

(ii) Earth is not a perfect sphere and

(iii) Earth rotates

GOLDEN KEY POINT

- In form of density $g = \frac{GM_e}{R_e^2} = \frac{G}{R_e^2} \times \frac{4}{3}\pi R_e^3 \times \rho \quad \therefore g = \frac{4}{3}\pi G R_e \rho$

  If ρ is constant then $g \propto R_e$

- If M is constant $\therefore\ g \propto \frac{1}{R^2}$ % variation in 'g' (upto 5%) $\frac{\Delta g}{g} = -2\left(\frac{\Delta R_e}{R_e}\right)$

- If mass (M) and radius (R) of a planet, if small change is occurs in (M) and (R) then

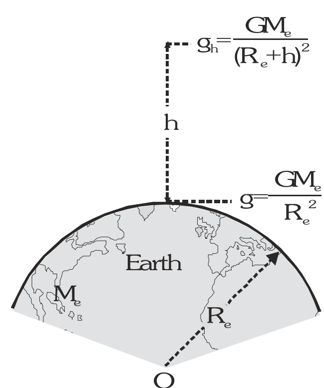
  by $g = \frac{GM}{R^2}$; $\frac{\Delta g}{g} = \frac{\Delta M}{M} - 2\frac{\Delta R_e}{R_e}$

  If R is constant $\frac{\Delta g}{g} = \frac{\Delta M}{M}$ and if M is constant $\frac{\Delta g}{g} = -2\left(\frac{\Delta R_e}{R_e}\right)$

VARIATION IN ACCELERATION DUE GRAVITY

Due to Altitude (height)

From diagram



$$\frac{g_h}{g} = \frac{R_e^2}{(R_e+h)^2} = \frac{R_e^2}{R_e^2\left[1+\frac{h}{R_e}\right]^2} = \left(1+\frac{h}{R_e}\right)^{-2}$$

By binomial expansion $\left(1+\frac{h}{R_e}\right)^{-2} \simeq \left(1-\frac{2h}{R_e}\right)$

[If h << R_e, then higher power terms are negligible] $\therefore\ g_h = g\left[1-\frac{2h}{R_e}\right]$

Example

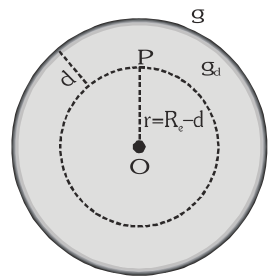
Two equal masses m and m are hung from a balance whose scale pans differ in vertical height by 'h'. Determine the error in weighing in terms of density of the Earth ρ.

Solution

$$g_h = g\left[1-\frac{2h}{R_e}\right] \quad , \; W_2 - W_1 = mg_2 - mg_1 = 2mg\left[\frac{h_1}{R_e}-\frac{h_2}{R_e}\right] = 2m\frac{GM}{R_e^2}\times\frac{h}{R_e}\left[\because g=\frac{GM}{R_e^2}\;\&\; h_1-h_2=h\right]$$

$$\text{Error in weighing} = W_2 - W_1 = 2mG\frac{4}{3}\pi R_e^3\rho\frac{h}{R_e^3} = \frac{8\pi}{3}Gm\rho h$$

Due to depth : Assuming density of Earth remains same throughout.



At Earth surface : $g = \frac{4}{3}\pi G R_e \rho$ ...(i) At depth d inside the Earth :

For point P only mass of the inner sphere is effective $g_d = \frac{GM'}{r^2}$

$$g_d = \frac{G}{r^2}\times\frac{Mr^3}{R_e^3} = \frac{GM}{R_e^2}\times\frac{r}{R_e} = \frac{GM}{R_e^2}\times\frac{R_e-d}{R_e}$$

Mass of sphere of radius r = M'

$$M' = \frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi r^3 \times \frac{M}{4/3\pi R^3} = M' = \frac{M}{R^3}r^3$$

$$g_d = g\left[1-\frac{d}{R_e}\right] \quad \text{valid for any depth}$$

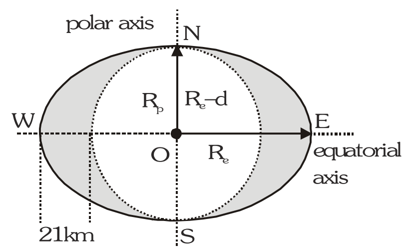
Example

At which depth from Earth surface, acceleration due to gravity is decreased by 1%

Solution

$$\frac{\Delta g_d}{g} = \frac{d}{R_e} \Rightarrow \frac{1}{100} = \frac{d}{6400} \quad \therefore \quad d = 64 \text{ km}$$

Due to shape of the Earth

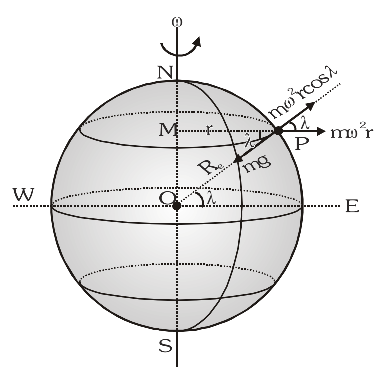


From diagram

$$R_p < R_e \; (R_e = R_p + 21 \text{ km}) \; g_p = \frac{GM_e}{R_p^2} \;\&\; g_e = \frac{GM_e}{(R_p+21)^2} \quad \therefore g_e < g_p$$

$\Rightarrow$ by putting the values $g_p - g_e = 0.02$ m/s^2

Due to Rotation of the Earth



Net force on particle at P $mg' = mg - mr\omega^2\cos\lambda$

$g' = g - r\omega^2\cos\lambda$ from ΔOMP $\quad r = R_e\cos\lambda$

Substituting value of r $\quad g' = g - R_e\omega^2\cos^2\lambda$

If latitude angle $\lambda = 0$. It means at equator. $g'_{min} = g_e = g - R_e\omega^2$

If latitude angle $\lambda = 90$. it means at poles. $g'_{max.} = g_p = g \Rightarrow g_p > g_e$

Change in "g" only due to rotation $\Delta g_{rot.} = g_p - g_e = 0.03$ m/s^2

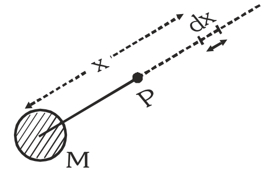
$\Delta g_{total} = g_p - g_e = (0.05 \text{ m/s}^2) \rightarrow$
- 0.02 m/s^2 (due to shape)
- 0.03 m/s^2 (due to rotation)

If rotation of Earth suddenly stops then acceleration due to gravity is increases at all places on Earth except the poles.

GRAVITATIONAL POTENTIAL

Gravitational field around a material body can be described not only by gravitational intensity $\vec{I}_g$, but also by a scalar function, the gravitational potential V. Gravitational potential is the amount of work done in bringing a body of unit mass from infinity to that point without changing its kinetic energy. $V = \frac{W}{m}$

Gravitational force on unit mass at (A) will be $= \frac{GM(1)}{x^2} = \frac{GM}{x^2}$



Work done by this force, through the distance (AB) $dW = Fdx = \frac{GM}{x^2} . dx$

Total work done in bringing the body of unit mass from infinity to point (P) $W = \int_{\infty}^{r} \frac{GM}{x^2} dx = -\left(\frac{GM}{x}\right)_{\infty}^{r} = -\frac{GM}{r}$

This work done is the measure of gravitational potential at point (P) $\therefore V_P = -\frac{GM}{r}$

- If $r = \infty$ then $v_\infty = 0$. Hence gravitational potential is maximum at infinite (as it is negative quantity at point P)
- If $r = R_e$ at the surface of Earth $V_S = -\frac{GM_e}{R_e}$

GOLDEN KEY POINTS

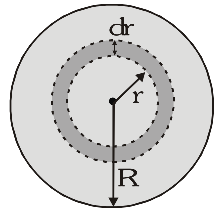
- Gravitational potential is a scalar quantity and its unit and its dimensions are J/kg and $[L^2T^{-2}]$
- $V = -\frac{W}{m} = -\int \frac{\vec{F}.\vec{dr}}{m}$ $\left[\text{As} \frac{\vec{F}}{m} = \vec{I}\right]$, $V = -\int \vec{I}.\vec{dr} \Rightarrow dV = -\vec{I} . dr \therefore I = -\frac{dV}{dr}$ = –ve potential gradient

	Solid Sphere	Hollow Spherical shell
Case I	r > R (outside the sphere)	r > R (outside the sphere)
	$V_{out} = -\frac{GM}{r}$	$V_{out} = -\frac{GM}{r}$
Case II	r = R (on the surface)	r = R (on the surface)
	$V_{surface} = -\frac{GM}{R}$	$V_{surface} = -\frac{GM}{R}$
Case III	r < R (inside the sphere)	r < R (Inside the sphere)
	$V_{in} = -\frac{GM}{2R^3}[3R^2 - r^2]$	Potential every where is same and equal to its value at the surface $V_{in} = -\frac{GM}{R}$

It is clear that the potential |V| will be maximum at the centre (r = 0)

$|V_{centre}| = \frac{3}{2}\frac{GM}{R}$, $V_{centre} = \frac{3}{2}V_{surface}$

Gravitational Self Energy of a Uniform Sphere



Consider a sphere of radius R and mass M uniformly distributed. Consider a stage of formation at which the radius of the spherical core is r. Its mass will be $\frac{4}{3}\pi r^3\rho$, where, ρ is the density of the sphere. Let, through an additional mass, the radius of the core be increased by dr in the form of spherical layer over the core.

The mass of this layer will be $4\pi r^2 dr.\rho$. The mutual gravitational potential energy of the above mass and the spherical core of radius r

$$dU = \frac{-G\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2 dr.\rho\right)}{r} = -\frac{16}{3}\pi^2\rho^2 G r^4 dr$$

Hence, total energy involved in the formation of the spherical body of radius R i.e. self energy

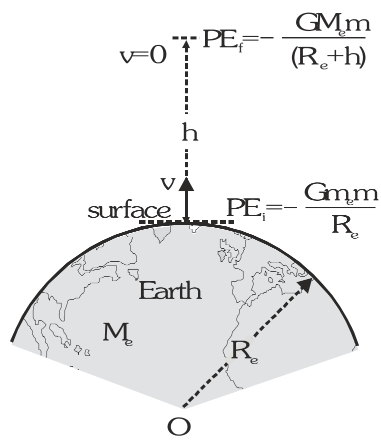
$$U = \int_0^R \frac{16}{3}\pi^2\rho^2 G r^4 dr = \frac{16}{15}\pi^2 G R^5 = \frac{3}{5}\frac{GM^2}{R}$$

GRAVITATIONAL POTENTIAL ENERGY

Work done by Gravitational force in shifting a mass from one place to another place. $W = U = -\frac{GMm}{r}$

(Here negative sign shows the boundness of the two bodies)

(Velocity (v) required to project a body till height "h")



by conservation energy $KE_i + PE_i = KE_f + PE_f$

$$\frac{1}{2}mv^2 + \left[-\frac{GM_e m}{R_e}\right] = 0 + \left[-\frac{GM_e m}{R_e + h}\right]$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{GM_e m}{R_e} - \frac{GM_e m}{R_e + h} = GM_e m\left[\frac{1}{R_e} - \frac{1}{R_e + h}\right]$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{GM_e mh}{R_e(R_e + h)} \Rightarrow v^2 = \frac{2GM_e h}{R_e^2\left(1 + \frac{h}{R_e}\right)} \Rightarrow v^2 = \frac{2gh}{1 + \frac{h}{R_e}}$$

Maximum height reached by the body projected by velocity "v" from the Earth surface $h = \frac{v^2 R_e}{2gR - v^2}$

If reference point is taken at Earth surface then U = mgh

ESCAPE VELOCITY (v_e)

It is the minimum velocity required for an object at Earth's surface so that it just escapes the Earth's gravitational field.

Consider a projectile of mass m, leaving the surface of a planet (or some other astronomical body or system), of radius R and mass M with esape speed v_e.

Projectile reaches inifinitely, it has no kinetic energy and no potential energy.

From conservation of mechanical energy $\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) = 0 + 0 \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$

Escape speed depends on :

(i) Mass (M) and size (R) of the planet

(ii) Position from where the particle is projected.

Escape speed does not depend on :

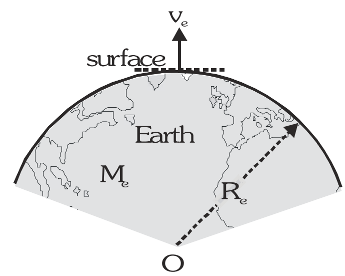
(i) Mass of the body which is projected (m)

(ii) Angle of projection.

If a body is thrown from Earth's surface with escape speed, it goes out of earth's gravitational field and never returns to the earth's surface.

Escape energy

Minimum energy given to a particle in form of kinetic energy so that it can just escape from Earth's gravitational field.



Magnitude of escape energy $= \dfrac{GMm}{R}$ (–ve of PE of Earth's surface)

Escape energy = Kinetic Energy $\Rightarrow \dfrac{GMm}{R} = \dfrac{1}{2}mv_e^2$

GOLDEN KEY POINT

- $v_e = \sqrt{\dfrac{2GM}{R}}$ (In form of mass) If M = constant $v_e \propto \dfrac{1}{\sqrt{R}}$
- $v_e = \sqrt{2gR}$ (In form of g) If g = constant $v_e \propto \sqrt{R}$
- $v_e = R\sqrt{\dfrac{8\pi G.\rho}{3}}$ (In form of density) If ρ = constant $v_e \propto R$
- Escape velocity does not depend on mass of body, angle of projection or direction of projection.

  $v_e \propto m^0$ and $v_e \propto \theta$
- Escape velocity at : Earth surface v_e = 11.2 km/s Moon surface v_e = 2.31 km/s
- Atmosphere on Moon is absent because root mean square velocity of gas particle is greater then escape velocity. $v_{rms} > v_e$

Example

A space–ship is launched into a circular orbit close to the Earth's surface. What additional speed should now be imparted to the spaceship so that orbit to overcome the gravitational pull of the Earth.

Solution

Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitation pull then

$$K = \frac{GMm}{2R}$$

Total kinetic energy $= \dfrac{GMm}{2R} + \Delta K = \dfrac{GMm}{2R} + \dfrac{GMm}{2R} = \dfrac{GMm}{R}$ then $\dfrac{1}{2}mv_2^2 = \dfrac{GMm}{R} \Rightarrow v_2 = \sqrt{\dfrac{2GM}{R}}$

But $v_1 = \sqrt{\dfrac{GM}{R}}$. So Additional velocity $= v_2 - v_1 = \sqrt{\dfrac{2GM}{R}} - \sqrt{\dfrac{GM}{R}} = \left(\sqrt{2}-1\right)\sqrt{\dfrac{GM}{R}}$

Example

If velocity given to an object from the surface of the Earth is n times the escape velocity then what will be the residual velocity at infinity.

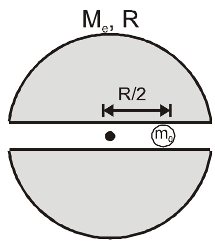
Solution

Let residual velocity be v then from energy conservation $\frac{1}{2}\, m\,(nv_e)^2 - \frac{GMm}{R} = \frac{1}{2} mv^2 + 0$

$$\Rightarrow v^2 = n^2 v_e^2 - \frac{2GM}{R} = n^2 v_e^2 - v_e^2 = (n^2 - 1)\, v_e^2 \Rightarrow v = \left(\sqrt{n^2 - 1}\right) v_e$$

Example

A very small groove is made in the earth, and a particle of mass m_0 is placed at $\frac{R}{2}$ distance from the centre. Find the escape speed of the particle from that place.



Solution

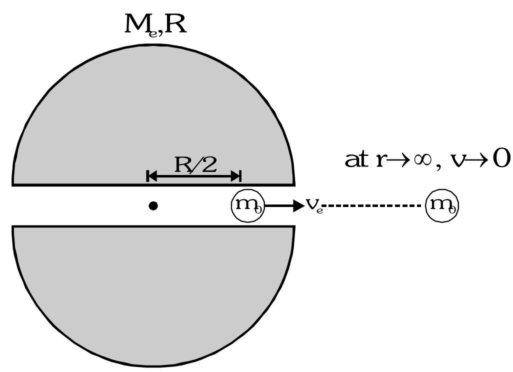
Suppose we project the particle with speed v_e, so that it just reaches at $(r \to \infty)$.

Applying energy conservation

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m_0 v_e^2 + m_0 \left[-\frac{GM_e}{2R^3} \left\{ 3R^2 - \left(\frac{R}{2}\right)^2 \right\} \right] = 0$$

$$\Rightarrow v_e = \sqrt{\frac{11 GM_e}{4R}}$$



KEPLER'S LAWS

Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

(a) First Law (Law of Orbits) :

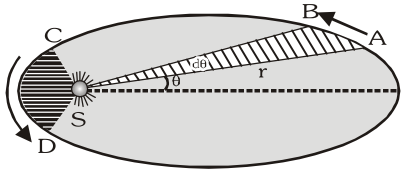
All planets move around the Sun in elliptical orbits, having the Sun at one focus of the orbit.

(b) Second Law (Law of Areas) :

A line joining any planet to the Sun sweeps out equal areas in equal times, that is, the areal speed of the planet remains constant.

According to the second law, when the planet is nearest to the Sun, then its speed is maximum and when it is farthest from the Sun, then its speed is minimum. In figure if a planet moves from A to B in a given time–interval, and from C to D in the same time–interval, then the areas ASB and CSD will be equal.

dA = area of the curved triangle SAB $= \frac{1}{2}(AB \times SA) = \frac{1}{2}(r\,d\theta \times r) = \frac{1}{2} r^2\, d\theta$



Thus, the instantaneous areal speed of the planet is $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega$ ...(i)

where ω is the angular speed of the planet.

Let J be the angular momentum of the planet about the Sun S and m the mass of the planet.

Then $J = I\omega = mr^2\,\omega$, ...(ii)

where I ($=mr^2$) is the instantaneous moment of inertia of the planet about the Sun S.

From eq. (i) and (ii), $\frac{dA}{dt} = \frac{J}{2m}$ ...(iii)

Now, the areal speed dA/dt of the planet is constant, according to Kepler's second law. Therefore, according to eq. (iii), the angular momentum J of the planet is also constant, that is, the angular momentum of the planet is conserved. Thus, Kepler's second law is equivalent to conservation of angular momentum.

(c) **Third Law : (Law of Periods) :** The square of the period of revolution (time of one complete revolution) of any planet around the Sun is directly proportional to the cube of the semi–major axis of its elliptical orbit.

Proof : If a and b are the semimajor and the semi–minor axes of the ellipse, then the area of the ellipse will be π ab. Hence if T be the period of revolution of the planet, then

$$T = \frac{\text{area of the ellipse}}{\text{areal speed}} = \frac{\pi ab}{J/2m} \quad \text{or} \quad T^2 = \frac{4\pi^2 m^2 a^2 b^2}{J^2}$$

Let ℓ be the semi–latus rectum of the elliptical orbit. Then $\ell = \frac{b^2}{a}$ $\therefore\ T^2 = \frac{4\pi^2 m^2 a^3 \ell}{J^2}$ or $T^2 \propto a^3$

As all the other quantities are constant. So it is clear through this rule that the farthest planet from the Sun has largest period of revolution. The period of revolution of the closest planet Mercury is 88 days, while that of the farthest dwarf planet Pluto is 248 years.

NEWTON'S CONCLUSIONS FROM KEPLER'S LAWS

Newton found that the orbits of most of the planets (except Mercury and Pluto) are nearly circular. According to Kepler's second law, the areal speed of a planet remains constant. This means that in a circular orbit the linear speed of the planet (v) will be constant. Since the planet is moving on a circular path; it is being acted upon by a centripetal force directed towards the centre (Sun). This force is given by $F = mv^2/r$ where m is the mass of the planet, v is its linear speed and r is the radius of its circular orbit. If T is the period of revolution of the planet, then $v = \frac{\text{Linear distance travellel in one revolution}}{\text{Period of revolution}} = \frac{2\pi r}{T}$

$\therefore\ F = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 m r}{T^2}$ But, for circular orbit, according to Kepler's third law,

$T^2 = Kr^3$, where K is some constant. $\therefore\ F = \frac{4\pi^2 m r}{K r^3} = \frac{4\pi^2}{K}\left(\frac{m}{r^2}\right)$ or $F \propto m/r^2$

Example

Two satellites S_1 and S_2 are revolving round a planet in coplanar and concentric circular orbit of radii R_1 and R_2 in the same direction respectively. Their respective periods of revolution are 1 hr and 8 hr. The radius of the orbit of satellite S_1 is equal to 10^4 km. Find the relative speed in kmph when they are closest.

Solution

By Kepler's 3rd law, $\dfrac{T^2}{R^3}$ = constant $\quad \therefore \dfrac{T_1^2}{R_1^3} = \dfrac{T_2^2}{R_2^3}$ or $\dfrac{1}{(10^4)^3} = \dfrac{64}{R_2^3}$ or $R_2 = 4 \quad 10^4$ km

Distance travelled in one revolution, $\quad S_1 = 2\pi R_1 = 2\pi \quad 10^4$ and $S_2 = 2\pi R_2 = 2\pi \quad 4 \quad 10^4$

$$v_1 = \frac{S_1}{t_1} = \frac{2\pi \times 10^4}{1} = 2\pi \times 10^4 \text{ kmph} \quad \text{and} \quad v_2 = \frac{S_2}{t_2} = \frac{2\pi \times 4 \times 10^4}{8} = \pi \times 10^4 \text{ kmph}$$

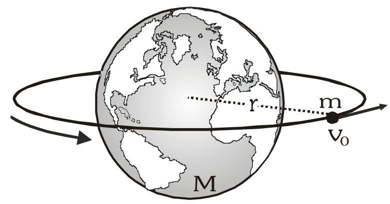
$\therefore$ Relative velocity $= v_1 - v_2 = 2\pi \quad 10^4 - \pi \quad 10^4 = \pi \quad 10^4$ kmph

SATELLITE MOTION

A light body revolving round a heavier body due to gravitational attraction, is called satellite. Earth is a satellite of the Sun while Moon is satellite of Earth.

Orbital velocity (v_0) : A satellite of mass m moving in an orbit of radius r with speed v_0 then required centripetal force is provided by gravitation.

$$F_{cp} = F_g \Rightarrow \frac{mv_0^2}{r} = \frac{GMm}{r^2} \Rightarrow v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R_e+h)}} \quad (r = R_e + h)$$



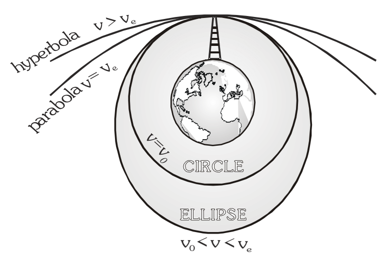
For a satellite very close to the Earth surface $h << R_e \; \therefore \; r = R_e$

$$v_0 = \sqrt{\frac{GM}{R_e}} = \sqrt{gR_e} = 8 \text{ km/s}$$

- If a body is taken at some height from Earth and given horizontal velocity of magnitude 8 km/sec then the body becomes satellite of Earth.
- v_o depends upon : Mass of planet, Radius of circular orbit of satellite, g (at planet), Density of planet
- If orbital velocity of a near by satellite becomes $\sqrt{2}\, v_o$ (or increased by 41.4%, or K.E. is doubled) then the satellite escapes from gravitational field of Earth.

Bound and Unbound Trajectories

Imagine a very tall tower on the surface of Earth where from a projectile is fired with a velocity v parallel to the surface of Earth. The trajectory of the projectile depends on its velocity.



Velocity	Trajectory
$v < \dfrac{v_0}{\sqrt{2}}$	Projectile does not orbit the Earth. It falls back on the Earth's surface.
$v = \dfrac{v_0}{\sqrt{2}}$	Projectile orbits the Earth in a circular path.

$\frac{v_0}{\sqrt{2}} < v < v_e$	Projectile orbits in an elliptical path.
$v = v_e$	Projectile does not orbit. It escapes the gravitational field of Earth in a parabolic path.
$v > v_e$	Projectile does not orbit. It escape the gravitational field of Earth in a hyperbolic path.

Time Period of a Satellite $T = \frac{2\pi r}{v_0} = \frac{2\pi r^{3/2}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{R\sqrt{g}} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow T^2 \propto r^3$ $(r = R + h)$

For Geostationary Satellite $T = 24$ hr, $h = 36{,}000$ km $\simeq 6\,R_e$ $(r \simeq 7\,R_e)$, $v_0 = 3.1$ km/s

For Near by satellite $v_0 = \sqrt{\frac{GM_e}{R_e}} \simeq 8$ km/s

$$T_{Ns} = 2\pi\sqrt{\frac{R_e}{g}} = 84 \text{ minute} = 1 \text{ hour } 24 \text{ minute} = 1.4 \text{ hr} = 5063 \text{ s}$$

In terms of density $T_{Ns} = \frac{2\pi(R_e)^{1/2}}{(G \times 4/3\,\pi R_e \times \rho)^{1/2}} = \sqrt{\frac{3\pi}{G\rho}}$

Time period of near by satellite only depends upon density of planet.

For Moon $h_m = 380{,}000$ km and $T_m = 27$ days

$$v_{om} = \frac{2\pi(R_e + h)}{T_m} = \frac{2\pi(386400 \times 10^3)}{27 \times 24 \times 60 \times 60} \simeq 1.04 \text{ km/sec.}$$

Energies of a Satellite

Kinetic energy $K.E. = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2}$

Potential energy $P.E. = -\frac{GMm}{r} = -mv_0^2 = -\frac{L^2}{mr^2}$

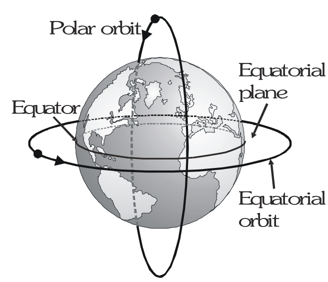
Total mechanical energy $T.E. = P.E. + K.E. = -\frac{mv_0^2}{2} = -\frac{GMm}{2r} = -\frac{L^2}{2mr^2}$

Essential Condition's for Satellite Motion

- Centre of satellite's orbit should coincide with centre of the Earth.
- Plane of orbit of satellite should pass through centre of the Earth.

Special Points about Geo-Stationary Satellite

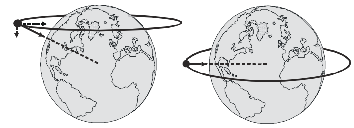
- It rotates in equatorial plane.
- Its height from Earth surface is 36000 km. ($\tilde{}\,6R_e$)
- Its angular velocity and time period should be same as that of Earth.
- Its rotating direction should be same as that of Earth (West to East).
- Its orbit is called parking orbit and its orbital velocity is 3.1 km./sec.



Polar Satellite (Sun – synchronous satellite)

It is that satellite which revolves in polar orbit around Earth. A polar orbit is that orbit whose angle of inclination with equatorial plane of Earth is 90 and a satellite in polar orbit will pass over both the north and south geographic poles once per orbit. Polar satellites are Sun–synchronous satellites.

The polar satellites are used for getting the cloud images, atmospheric data, ozone layer in the atmosphere and to detect the ozone hole over Antarctica.



Unstable orbit Stable orbit

BINDING ENERGY

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is bound or different parts of the system are bound to each other.

Binding energy of satellite (system)

$$B.E. = -T.E. \qquad B.E. = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2} \quad \text{Hence} B.E. = K.E. = -T.E. = \frac{-P.E.}{2}$$

Work done in Changing the Orbit of Satellite

$$W = \text{Change in mechanical energy of system} \quad \text{but } E = \frac{-GMm}{2r} \quad \text{so} \quad W = E_2 - E_1 = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Example

A satellite moves eastwards very near the surface of the Earth in equatorial plane with speed (v_0). Another satellite moves at the same height with the same speed in the equatorial plane but westwards. If R = radius of the Earth and ω be its angular speed of the Earth about its own axis. Then find the approximate difference in the two time period as observed on the Earth.

Solution

$$T_{west} = \frac{2\pi R}{v_0 + R\omega} \text{ and } T_{east} = \frac{2\pi R}{v_0 - R\omega} \Rightarrow \Delta T = T_{east} - T_{west} = 2\pi R\left[\frac{2R\omega}{v_0^2 - R^2\omega^2}\right] = \frac{4\pi\omega R^2}{v_0^2 - R^2\omega^2}$$

Example

A planet of mass m moves along an ellipse around the Sun of mass M so that its maximum and minimum distances from Sun are a and b respectively. Prove that the angular momentum L of this planet relative to the centre of the Sun is $L = m\sqrt{\frac{2GMab}{(a+b)}}$.

Solution

Angular momentum at maximum distance from Sun = Angular momentum at minimum distance from Sun

$$mv_1a = mv_2b \Rightarrow v_1 = \frac{v_2b}{a} \text{ by applying conservation of energy } \frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$$

$$\text{From above equations } v_1 = \sqrt{\frac{2GMb}{a(a+b)}} \text{ Angular momentum of planet } L = mv_1a = ma\sqrt{\frac{2GMb}{(a+b)a}}$$

Example

A body of mass m is placed on the surface of earth. Find work required to lift this body by a height

(i) $h = \frac{R_e}{1000}$ (ii) $h = R_e$

Solution :

(i) $h = \frac{R_e}{1000}$, as $h << R_e$, so we can apply $W_{ext} = mgh$; $W_{ext} = (m)\left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{1000}\right) = \frac{GM_e m}{1000 R_e}$

(ii) $h = R_e$, in this case h is not very less than R_e, so we cannot apply $\Delta U = mgh$

$$W_{ext} = U_f - U_i = m(V_f - V_i)\ ;\quad W_{ext} = m\left[\left(-\frac{GM_e}{R_e + R_e}\right) - \left(-\frac{GM_e}{R_e}\right)\right]\ ;\ W_{ext} = -\frac{GM_e m}{2R_e}$$

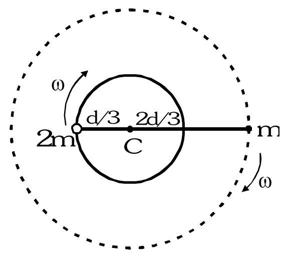
Example

In a double star, two stars (one of mass m and the other of 2m) distant d apart rotate about their common centre of mass. Deduce an expression of the period of revolution. Show that the ratio of their angular momenta about the centre of mass is the same as the ratio of their kinetic energies.

Solution :

The centre of mass C will be at distances d/3 and 2d/3 from the masses 2m and m respectively. Both the stars rotate round C in their respective orbits with the same angular velocity ω. The gravitational force acting on each star due to the other supplies the necessary centripetal force.

For rotation of the smaller star, the centripetal force $\left[m\left(\frac{2d}{3}\right)\omega^2\right]$ is provided by gravitational force.



$$\therefore \qquad \frac{G(2m)m}{d^2} = m\left(\frac{2d}{3}\right)\omega^2 \quad \text{or } \omega = \sqrt{\left(\frac{3Gm}{d^3}\right)}$$

Therefore, the period of revolution is given by $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{d^3}{3Gm}\right)}$

The ratio of the angular momenta is $\frac{(I\omega)_{big}}{(I\omega)_{small}} = \frac{I_{big}}{I_{small}} = \frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2} = \frac{1}{2}$ since ω is same for both.

The ratio of their kinetic energies is $\frac{\left(\frac{1}{2}I\omega^2\right)_{big}}{\left(\frac{1}{2}I\omega^2\right)_{small}} = \frac{I_{big}}{I_{small}} = \frac{1}{2}$,

which is the same as the ratio of their angular momenta.

WEIGHTLESSNESS

When the weight of a body (either true or apparent) becomes zero, the body is said to be in the state of weightlessness. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface then

True weight $= mg_h = \frac{mGM}{(R+h)^2} = \frac{mg}{\left(1+\frac{h}{R}\right)^2}$ and Apparent weight $= m(g_h - a)$

But $a = \frac{v_0^2}{r} = \frac{GM}{r^2} = \frac{GM}{(R+h)^2} = g_h \Rightarrow$ Apparent weight $= m(g_h - g_h) = 0$

Note : The condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

Condition of weightlessness on Earth surface

If apparent weight of body is zero then angular speed of Earth can be calculated as $mg' = mg - mR_e\omega^2\cos^2\lambda$

$$0 = mg - mR_e\omega^2\cos^2\lambda \Rightarrow \omega = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R_e}}$$

SOME WORKED OUT EXAMPLES

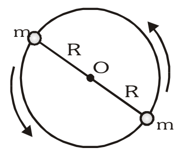
Example#1

Two particles of equal mass m go round a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(A) $\frac{1}{2R}\sqrt{\frac{1}{Gm}}$ (B) $\sqrt{\frac{Gm}{2R}}$ (C) $\frac{1}{2}\sqrt{\frac{Gm}{R}}$ (D) $\sqrt{\frac{4Gm}{R}}$

Solution **Ans. (C)**

Centripetal force provided by the gravitational force of attraction



between two particles i.e. $\frac{mv^2}{R} = \frac{Gm \times m}{(2R)^2} \Rightarrow v = \frac{1}{2}\sqrt{\frac{Gm}{R}}$

Example#2

The escape velocity for a planet is v_e. A particle starts from rest at a large distance from the planet, reaches the planet only under gravitational attraction, and passes through a smooth tunnel through its centre. Its speed at the centre of the planet will be-

(A) $\sqrt{1.5}v_e$ (B) $\frac{v_e}{\sqrt{2}}$ (C) v_e (D) zero

Solution **Ans. (A)**

From mechanical energy conservation $0 + 0 = \frac{1}{2}mv^2 - \frac{3GMm}{2R} \Rightarrow v = \sqrt{\frac{3GM}{R}} = \sqrt{1.5}\ v_e$

Example#3

A particle is projected vertically upwards the surface of the earth (radius R_e) with a speed equal to one fourth of escape velocity. What is the maximum height attained by it from the surface of the earth ?

(A) $\frac{16}{15}R_e$ (B) $\frac{R_e}{15}$ (C) $\frac{4}{15}R_e$ (D) None of these

Solution **Ans. (B)**

From conservation of mechanical energy $\frac{1}{2}mv^2 = \frac{GMm}{R_e} - \frac{GMm}{R}$

Where R = maximum distance from centre of the earth Also $v = \frac{1}{4}v_e = \frac{1}{4}\sqrt{\frac{2GM}{R_e}}$

$$\Rightarrow \frac{1}{2}m \times \frac{1}{16} \times \frac{2GM}{R_e} = \frac{GMm}{R_e} - \frac{GMm}{R} \Rightarrow R = \frac{16}{15}R_e \quad \Rightarrow h = R - R_e = \frac{R_e}{15}$$

Example#4

A mass 6 10^{24} kg (= mass of earth) is to be compressed in a sphere in such a way that the escape velocity from its surface is 3 10^8 m/s (equal to that of light). What should be the radius of the sphere?

(A) 9 mm (B) 8 mm (C) 7 mm (D) 6 mm

Solution **Ans. (A)**

$$\text{As, } v_e = \sqrt{\left(\frac{2GM}{R}\right)},\ R = \left(\frac{2GM}{v_e^2}\right),\ \therefore R = \frac{2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{(3 \times 10^8)^2} = 9 \times 10^{-3}\,m = 9\ mm$$

Example#5

Calculate the mass of the sun if the mean radius of the earth's orbit is $1.5\ \ 10^{8}$ km and G = $6.67\ \ 10^{-11}$ N m²/kg².

(A) $M \simeq 2 \times 10^{30}$ kg (B) $M \simeq 3 \times 10^{30}$ kg (C) $M \simeq 2 \times 10^{15}$ kg (D) $M \simeq 3 \times 10^{15}$ kg

Solution

Ans. (A)

In case of orbital motion as $v = \sqrt{\left(\frac{GM}{r}\right)}$ so $T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}}$, i.e., $M = \frac{4\pi^2 r^3}{GT^2}$

$$M = \frac{4 \times \pi^2 \times (1.5 \times 10^{11})^3}{6.67 \times 10^{-11} \times (3.15 \times 10^7)^2}$$ [as T = 1 year = $3.15\ \ 10^{7}$s] i.e., $M \simeq 2 \times 10^{30}$ kg

Example#6

Gravitational potential difference between a point on surface of planet and another point 10 m above is 4 J/kg. Considering gravitational field to be uniform, how much work is done in moving a mass of 2 kg from the surface to a point 5m above the surface?

(A) 4 J (B) 5 J (C) 6 J (D) 7 J

Solution

Ans. (A)

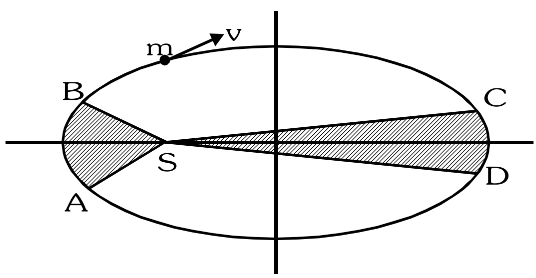
Gravitational field $g = -\frac{\Delta V}{\Delta x} = -\left(\frac{-4}{10}\right) = \frac{4}{10} J/kg\ m$

Work done in moving a mass of 2 kg from the surface to a point 5 m above the surface,

$W = mgh = (2kg)\left(\frac{4}{10}\frac{J}{kgm}\right)(5m) = 4J$

Example#7

The figure shows elliptical orbit of a planet m about the sun S. The shaded area SCD is twice the shaded area SAB. If t_1 be the time for the planet to move from C to D and t_2 is the time to move from A to B, then :



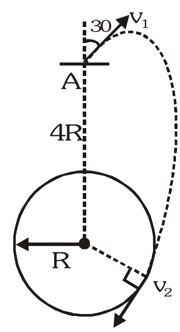
(A) $t_1 = t_2$ (B) $t_1 = 8t_2$ (C) $t_1 = 4t_2$ (D) $t_1 = 2t_2$

Solution

Ans. (D)

From Kepler's law : Areal velocity = constant so $\frac{\text{Area SCD}}{t_1} = \frac{\text{Area SAB}}{t_2} \Rightarrow t_1 = 2t_2$

Example#8



A particle is projected from point A, that is at a distance 4R from the centre of the Earth, with speed v_1 in a direction making 30 with the line joining the centre of the Earth and point A, as shown. Find the speed v_1 of particle (in m/s) if particle passes grazing the surface of the earth. Consider gravitational interaction only between these two.

(use $\frac{GM}{R} = 6.4\ \ 10^{7}$ m²/s²)

(A) $\frac{8000}{\sqrt{2}}$ (B) 800 (C) $800\sqrt{2}$ (D) None of these

Solution **Ans. (A)**

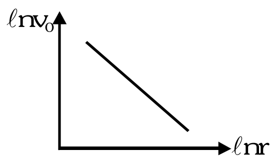
Conserving angular momentum : $m(v_1 \cos 60)4R = mv_2R \Rightarrow \frac{v_2}{v_1} = 2.$

Conserving energy of the system : $-\frac{GMm}{4R} + \frac{1}{2}mv_1^2 = -\frac{GMm}{R} + \frac{1}{2}mv_2^2$

$$\Rightarrow \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{3}{4}\frac{GM}{R} \Rightarrow v_1^2 = \frac{1}{2}\frac{GM}{R} \Rightarrow v_1 = \frac{1}{\sqrt{2}}\sqrt{64 \times 10^6} = \frac{8000}{\sqrt{2}} m/s$$

Example#9

If the law of gravitation be such that the force of attraction between two particles vary inversely as the $5/2^{th}$ power of their separation, then the graph of orbital velocity v_0 plotted against the distance r of a satellite from the earth's centre on a log-log scale is shown alongside. The slope of line will be-



(A) $-\frac{5}{4}$ (B) $-\frac{5}{2}$ (C) $-\frac{3}{4}$ (D) -1

Solution **Ans. (C)**

$$\frac{mv_0^2}{r} = \frac{GMm}{r^{5/2}} \Rightarrow v_0 = \frac{\sqrt{GM}}{r^{3/4}} \Rightarrow \ell n v_0 = \ell n\sqrt{GM} - \frac{3}{4}\ell n r$$

Example#10

Two point objects of masses m and 4m are at rest at an infinite separation. They move towards each other under mutual gravitational attraction. If G is the universal gravitational constant, then at a separation r

(A) the total mechanical energy of the two objects is zero (B) their relative velocity is $\sqrt{\frac{10Gm}{r}}$

(C) the total kinetic energy of the objects is $\frac{4Gm^2}{r}$ (D) their relative velocity is zero

Solution **Ans. (A,B,C)**

By applying law of conservation of momentum m $\rightarrow v_1$ $v_2 \leftarrow$ 4m

$mv_1 - 4mv_2 = 0 \Rightarrow v_1 = 4v_2$

By applying conservation of energy $\frac{1}{2}mv_1^2 + \frac{1}{2}4mv_2^2 = \frac{Gm4m}{r} \Rightarrow 10mv_2^2 = \frac{G4m^2}{r} \Rightarrow v_2 = 2\sqrt{\frac{Gm}{10r}}$

$\therefore$ Total kinetic energy $= \frac{4Gm^2}{r}$; Relative velocity for the particle $\Rightarrow v_{rel} = |\vec{v}_1 - \vec{v}_2| = 5v_2 = \sqrt{\frac{10Gm}{r}}$

Or

Mechanical energy of sysem = 0 = constant. By using reduced mass concept

$$\frac{1}{2}\mu v_{rel}^2 = \frac{Gm(4m)}{r} \text{ where } \mu = \frac{(m)(4m)}{m+4m} = \frac{4}{5}m \Rightarrow v_{rel} = \sqrt{\frac{10Gm}{r}}$$

Also total KE of system $= \frac{G(m)(4m)}{r} = \frac{4Gm^2}{r}$

Example#11

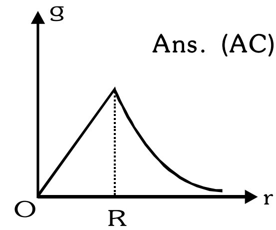
Which of the following statements are true about acceleration due to gravity?

(A) 'g' decreases in moving away from the centre if $r > R$

(B) 'g' decreases in moving away from the centre if $r < R$

(C) 'g' is zero at the centre of earth

(D) 'g' decreases if earth stops rotating on its axis



Solution **Ans. (AC)**

Variation of g with distance

variation of g with ω : $g' = g - \omega^2 R\cos^2\lambda$

If $\omega=0$ then g will not change at poles where $\cos\lambda = 0$.

Example#12

An astronaut, inside an earth satellite experiences weightlessness because:

(A) he is falling freely

(B) no external force is acting on him

(C) no reaction is exerted by floor of the satellite

(D) he is far away from the earth surface

Solution **Ans. (AC)**

As astronaut's acceleration = g so he is falling freely. Also no reaction is exerted by the floor of the satellite.

Example#13

If a satellite orbits as close to the earth's surface as possible

(A) its speed is maximum

(B) time period of its revolution is minimum

(C) the total energy of the 'earth plus satellite' system is minimum

(D) the total energy of the 'earth plus satellite' system is maximum

Solution **Ans. (ABC)**

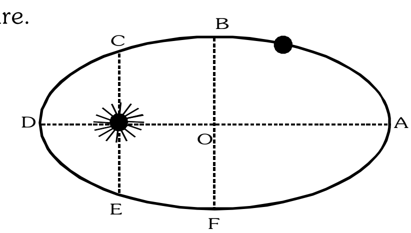
For (A) : orbital speed $v_0 = \sqrt{\frac{GM}{r}}$

For (B) : Time period of revolution $T^2 \propto r^3$

For (C/D): Total energy $= -\frac{GMm}{2r}$

Example#14

A planet is revolving around the sun is an elliptical orbit as shown in figure. Select correct alternative(s)



(A) Its total energy is negative at D.

(B) Its angular momentum is constant

(C) Net torque on planet about sun is zero

(D) Linear momentum of the planet is conserved

Solution **Ans. (ABC)**

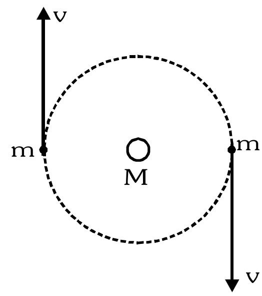
For (A) : For bounded system, the total energy is always negative.

For (B) : For central force field, angular momentum is always conserved

For (C) :For central force field, torque = 0.

For (D) : In presence of external force, linear momentum is not conserved.

Example#15 to 17



A triple star system consists of two stars, each of mass m, in the same circular orbit about central star with mass $M = 2 \quad 10^{30}$ kg. The two outer stars always lie at opposite ends of a diameter of their common circular orbit. The radius of the circular orbit is $r = 10^{11}$ m and the orbital period of each star is $1.6 \quad 10^{7}$ s. [Take $\pi^2 = 10$ and $G = \frac{20}{3} \quad 10^{-11}\ Nm^2kg^{-2}$]

15. The mass m of the outer stars is

(A) $\frac{16}{15}\times 10^{30}\,kg$ (B) $\frac{11}{8}\times 10^{30}\,kg$ (C) $\frac{15}{16}\times 10^{30}\,kg$ (D) $\frac{8}{11}\times 10^{30}\,kg$

16. The orbital velocity of each star is

(A) $\frac{5}{4}\sqrt{10}\times 10^{3}$ m/s (B) $\frac{5}{4}\sqrt{10}\times 10^{5}$ m/s (C) $\frac{5}{4}\sqrt{10}\times 10^{2}$ m/s (D) $\frac{5}{4}\sqrt{10}\times 10^{4}$ m/s

17. The total mechanical energy of the system is

(A) $-\frac{1375}{64}\times 10^{35}\,J$ (B) $-\frac{1375}{64}\times 10^{38}\,J$ (C) $-\frac{1375}{64}\times 10^{34}\,J$ (D) $-\frac{1375}{64}\times 10^{37}\,J$

Solution

15. Ans. (B)

F_{mm} = Gravitational force between two outer stars $= \frac{Gm^2}{4r}$

F_{mM} = Gravitational force between central star and outer star $= \frac{GmM}{r^2}$

For circular motion of outer star, $\frac{mv^2}{r} = F_{mm} + F_{mM} \quad \therefore v^2 = \frac{G(m+4M)}{4r}$

T = period of orbital motion $= \frac{2\pi r}{v} \quad \therefore m = \frac{16\pi^2 r^3}{GT^2} - 4M = \left(\frac{150}{16} - 8\right)10^{30} = \frac{11}{8}\times 10^{30}$ kg

16. Ans. (D)

$$T = \frac{2\pi r}{v} \Rightarrow v = \frac{2\pi r}{T} = \frac{(2)(\sqrt{10})(10^{11})}{1.6\times 10^{7}} = \frac{5}{4}\sqrt{10}\times 10^{4} \text{ m/s}$$

17. Ans. (B)

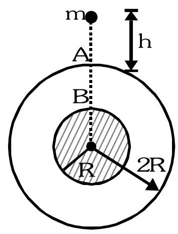
Total mechanical energy = K.E. + P.E.

$$= 2\left(\frac{1}{2}mv^2\right) - \frac{2GMm}{r} - \frac{Gm^2}{2r} = m\left[\frac{G(4M+m)}{4r} - \frac{2GM}{r} - \frac{Gm}{2r}\right] = -\frac{Gm}{r}\left[M + \frac{m}{4}\right]$$

$$= -\left(\frac{20}{3}\times 10^{-11}\right)\left(\frac{11}{8}\times 10^{30}\right)\times \frac{1}{10^{11}}\left(2\times 10^{30} + \frac{11}{32}\times 10^{30}\right) = -\frac{1375}{64}\times 10^{38}\ \text{J}$$

Example#18 to 20

A solid sphere of mass M and radius R is surrounded by a spherical shell of same mass M and radius 2R as shown. A small particle of mass m is released from rest from a height h (<<R) above the shell. There is a hole in the shell.



18. In what time will it enter the hole at A :–

(A) $2\sqrt{\frac{hR^2}{GM}}$ (B) $\sqrt{\frac{2hR^2}{GM}}$ (C) $\sqrt{\frac{hR^2}{GM}}$ (D) none of these

19. What time will it take to move from A to B ?

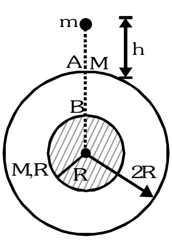
(A) $= \frac{4R^2}{\sqrt{GMR}}$ (B) $> \frac{4R^2}{\sqrt{GMR}}$ (C) $< \frac{4R^2}{\sqrt{GMR}}$ (D) none of these

20. With what approximate speed will it collide at B ?

(A) $\sqrt{\frac{2GM}{R}}$ (B) $\sqrt{\frac{GM}{2R}}$ (C) $\sqrt{\frac{3GM}{2R}}$ (D) $\sqrt{\frac{GM}{R}}$

Solution

18. Ans. (A)



$$a = \frac{GM}{(2R)^2} + \frac{GM}{(2R)^2} = \frac{GM}{2R^2} \quad \therefore t = \sqrt{\frac{2\times h}{a}} = \sqrt{\frac{2\times h\times 2R^2}{GM}} = 2\sqrt{\frac{hR^2}{GM}}$$

19. Ans. (C)

Given that (h<<R), so the velocity at A' is also zero.

We can see here that the acceleration always increases from 2R to R and its value must be greater than

$$a = \frac{GM}{4R^2}(\text{at A}) \quad \therefore t < \frac{v}{a} \Rightarrow t < \sqrt{\frac{GM}{R}} \times \frac{4R^2}{GM} \Rightarrow t < \frac{4R^2}{\sqrt{GMR}}$$

20. Ans. (D)

Given that (h<<R), so the velocity at A' is also zero.

Loss in PE = gain in KE $\quad \therefore \frac{GMm}{2R} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{GM}{R}}$

Example#21

Imagine a light planet revolving around a very massive star in a circular orbit of radius R. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then match the following

	Column I		Column II
(A)	Time period of revolution is proportional to	(P)	R^0
(B)	Kinetic energy of planet is proportional to	(Q)	$R^{7/4}$
(C)	Orbital velocity of planet is proportional to	(R)	$R^{-1/2}$
(D)	Total mechanical energy of planet is proportional to	(S)	$R^{-3/2}$
		(T)	$R^{-3/4}$

Solution **Ans. (A) Q (B) S(C) T (D) S**

According to question $F = \frac{C}{R^{5/2}}$ where C is a constant so $\frac{mv^2}{R} = \frac{C}{R^{5/2}} \Rightarrow mv^2 \propto \frac{1}{R^{3/2}} \Rightarrow KE \propto R^{-3/2}$

Also $v \propto R^{-3/4}$ and $PE \propto R^{-3/2}$. Total mechanical energy = KE + PE $\propto R^{-3/2}$

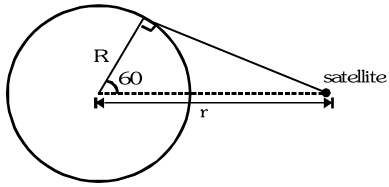
Time period $T = \frac{2\pi R}{v} \propto R^{7/4}$

Example#22

A satellite is launched in the equatorial plane in such a way that it can transmit signals upto 60 latitude on the earth. The orbital velocity of the satellite is found to be $\sqrt{\frac{GM}{\alpha R}}$. Find the value of α.

Solution **Ans. 2**

$r\cos 60 = R \Rightarrow r = 2R$ orbital velocity $v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{2R}} \Rightarrow \alpha = 2$



Example#23

An artificial satellite (mass m) of a planet (mass M) revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming the force of resistance on satellite to depend on velocity as $F=av^2$ where 'a' is a constant, calculate how long the satellite will stay in the space before it falls onto the planet's surface.

Solution

Air resistance $F = -av^2$, where orbital velocity $v = \sqrt{\frac{GM}{r}}$

r = the distance of the satellite from planet's centre $\Rightarrow F = -\frac{GMa}{r}$

The work done by the resistance force $dW = Fdx = Fvdt = \frac{GMa}{r}\sqrt{\frac{GM}{r}}dt = \frac{(GM)^{3/2}a}{r^{3/2}}dt$ ....(i)

The loss of energy of the satellite = dE $\therefore \frac{dE}{dr} = \frac{d}{dr}\left[-\frac{GM\,m}{2r}\right] = \frac{GMm}{2r^2} \Rightarrow dE = \frac{GMm}{2r^2}dr$...(ii)

Since dE = – dW (work energy theorem) $-\frac{GMm}{2r^2}dr = \frac{(GM)^{3/2}}{r^{3/2}}dt$

$$\Rightarrow t = -\frac{m}{2a\sqrt{GM}}\int_{nR}^{R}\frac{dr}{\sqrt{r}} = \frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{GM}} = \left(\sqrt{n}-1\right)\frac{m}{a\sqrt{gR}}$$