The Transformer: A Mathematical Deep Dive for Language Translation

1. ☑ Tokenization + Embedding
Let's assume these embeddings for simplicity (real models learn these):

Token.	Embedding
"I"	[1.0, 0.0]
"love"	[0.0, 1.0]
"you"	[1.0, 1.0]

→ X= [1.0, 0.0] [0.0, 1.0] [1.0, 1.0]

2. ☑ Positional Encoding
For simplicity, skip real sin/cos positional encodings and assume:

Position	Encoding
0	[0.1, 0.0]
1	[0.0, 0.1]
2	[0.1, 0.1]

So after addition: --> X+PE = [1.1, 0.0] [0.0, 1.1] [1.1, 1.1]

☑ Encoder: Self-Attention Calculation

Step 1: Initialization of Wq, Wk & Wv

WQ matrix
WQ is defined as [1 0] [0 1].

WV matrix
WV is defined as [1 1] [0 1].

WK matrix
WK is defined as [1 0] [1 1].

Step 2: Compute Q, K, V

Q = X*Wq = [1.1, 0.0] [0.0, 1.1] [1.1, 1.1] * [1, 0] [0, 1] = [1.1, 0.0] [0.0, 1.1] [1.1, 1.1]

K = X*Wk = [1.1, 0.0] [0.0, 1.1] [1.1, 1.1] * [1, 0] [1, 1] = [1.1, 0.0] [1.1, 1.1] [2.2, 1.1]

V = X*Wv = [1.1, 0.0] [0.0, 1.1] [1.1, 1.1] * [1, 1] [0, 1] = [1.1, 1.1] [0.0, 1.1] [1.1, 2.2]

Step 3: Calculation of Attention Scores

Compute scores = Q.KT/√2

Q.KT =
[1.1*1.1+0.0*1.1, 1.1*0.0+0.0*1.1 ,1.1*1.1+0.0*2.2]
[0.0*1.1+1.1*1.1, 0.0*0.0+1.1*1.1 ,0.0*1.1+1.1*2.2]
[1.1*1.1+1.1*1.1, 1.1*0.0+1.1*1.1, 1.1*1.1+1.1*2.2]

= [[1.21, 0.00, 1.21], [1.21, 1.21, 2.42], [2.42, 1.21, 3.63]]

Scaled scores Q.KT/√2 = [[0.86, 0.0, 0.86], [0.86, 0.86, 1.71], [1.71, 0.86, 2.57]]

Step 4: Apply Softmax (row-wise)

Softmax for row 1: softmax([0.86, 0.00, 0.86]) ≈ [0.42, 0.16, 0.42]
Softmax for row 2: softmax([0.86, 0.86, 1.71]) ≈ [0.27, 0.27, 0.46]
Softmax for row 3: softmax([1.71, 0.86, 2.57]) ≈ [0.25, 0.11, 0.64]

Step 5: Multiply with V

Attention output = softmax scores x V

0.42 ∗ 1.1 + 0.16 ∗ 1.1 + 0.42 ∗ 2.2, 0.42 ∗ 0.0 + 0.16 ∗ 1.1 + 0.42 ∗ 1.1
0.27 ∗ 1.1 + 0.27 ∗ 1.1 + 0.46 ∗ 2.2, 0.27 ∗ 0.0 + 0.27 ∗ 1.1 + 0.46 ∗ 1.1
0.25 ∗ 1.1 + 0.11 ∗ 1.1 + 0.64 ∗ 2.2, 0.25 ∗ 0.0 + 0.11 ∗ 1.1 + 0.64 ∗ 1.1

= [1.563, 0.635] [1.606, 0.803] [1.804, 0.825]

3. ☑ Encoder Feedforward Network (FFN)

attention outputs from the encoder are: [1.563, 0.635] [1.606, 0.803] [1.804, 0.825]

Each attention output (per token) is passed through a position-wise feedforward network:

FFN(x)=max(0, xW1 + b1) W2+b 2

Let's define:

d_model = 2
d_ff = 4

W1 = [1, 0, 1, 0] [0, 1, 0, 1]

W2 = [1, 0] [0, 1] [1, 0] [0, 1]

For Simplicity lets consider bias =0:

b1 = [0, 0, 0, 0]
b2 = [0, 0]

For Token 1: x = [1.563, 0.635]

Step 1: Linear Layer 1 xW1 =[1.563*1 + 0.635*0, 1.563*0+0.635*1, 1.563*1+0.635*0, 1.563*0+0.635*1] = [1.563, 0.635, 1.563, 0.635]

Step 2: ReLU Activation ReLU(xW1) = [1.563, 0.635, 1.563, 0.635]

Step 3: Linear Layer 2 xW2 =[1.563 + 1.563, 0.635 + 0.635] =[3.126, 1.270]

For Token 2: x = [1.606, 0.803]

Step 1: Linear Layer 1 xW1 = [1.606, 0.803, 1.606, 0.803]

Step 2: ReLU Activation ReLU(xW1) = [1.606, 0.803, 1.606, 0.803]

Step 3: Linear Layer 2 xW2 =[1.606+ 1.606, 0.803+ 0.803] =[3.212, 1.606]

For Token 3: x = [1.804, 0.825]

Step 1: Linear Layer 1 xW1 = [1.804, 0.825, 1.804, 0.825]

Step 2: ReLU Activation ReLU(xW1) = [1.804, 0.825, 1.804, 0.825]

Step 3: Linear Layer 2 xW2 = [1.804 + 1.804, 0.825 + 0.825] =[3.608, 1.650]

4. ☑ Add & Norm (Residual Connection)

We need to add FFN output to the attention output and apply layer normalization.

Residual Sum = Attention Output + FFN Output

[1.563, 0.635] [1.606, 0.803] [1.804, 0.825] + [3.126, 1.270] [3.212, 1.606] [3.608, 1.650] = [4.689, 1.905] [4.818, 2.409] [5.412, 2.475]

5. ☑ Batch Normalisation

LayerNorm is applied per token (per row), across the features (axis=-1).
For token vector x = [x1, x2]

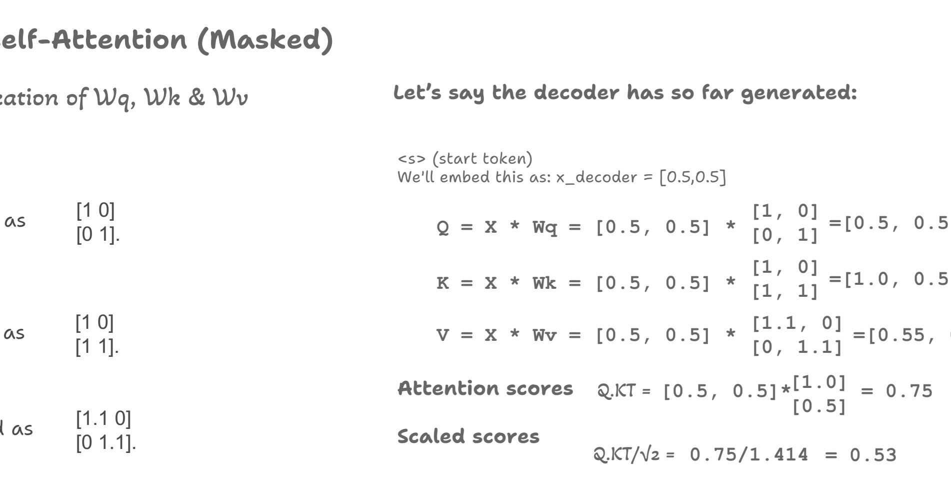
Mean(μ) = (x1 + x1) / 2

$$\text{Variance}(\sigma) = \sqrt{\frac{(x1 - \mu)2 + (x1 - \mu)2}{2} + \varepsilon}$$

LayerNorm(x) = (x - μ) / σ

☑ Decoder: Step-by-step calculation

Transformer Decoder Process Flow



☑ 1. Decoder Self-Attention (Masked)

Step 1: Initialization of Wq, Wk & Wv

WQ is defined as [1 0] [0 1].

WK is defined as [1 0] [1 1].

WV is defined as [1.1 0] [0 1.1].

Let's say the decoder has so far generated:

<s> (start token)
We'll embed this as: x_decoder = [0.5,0.5]

Q = X * Wq = [0.5, 0.5] * [1, 0] [0, 1] =[0.5, 0.5]

K = X * Wk = [0.5, 0.5] * [1, 0] [1, 1] =[1.0, 0.5]

V = X * Wv = [0.5, 0.5] * [1.1, 0] [0, 1.1] =[0.55, 0.55]

Attention scores Q.KT = [0.5, 0.5]*[1.0] [0.5] = 0.75

Scaled scores Q.KT/√2 = 0.75/1.414 = 0.53

Applying Softmax :
only 1 token, so it becomes 1.0

Attention Output :
1.0 * [0.55, 0.55] = [0.55, 0.55]

☑ 2. Add & Norm

Residual connection:
x + attention output
= [0.5 + 0.55, 0.5 + 0.55]

Output of Decoder Self-Attention = [1.05, 1.05]

☑ 3. Cross-Attention with Encoder Output

We now attend to encoder outputs:

Decoder Q(Q_d) = [1.05, 1.05]

Step 1: Attention scores
Score 1: = 1.05 * 4.689 + 1.05 * 1.905 = 4.92345 + 2.00025 = 6.9237
Score 2: = 1.05 * 4.818 + 1.05 * 2.409 = 5.0589 + 2.52945 = 7.58835
Score 3: = 1.05 * 5.412 + 1.05 * 2.475 = 5.6826 + 2.59875 = 8.28135

Step 2: Scale
[6.9237/1.414, 7.58835/1.414, 8.28135/1.414]
≈ [4.895, 5.363, 5.854]

Step 3: Softmax
[0.1923, 0.3064, 0.5013]

Step 4: Weighted Sum of V_enc
[0.1923, 0.3064, 0.5013] * [4.689, 1.905] [4.818, 2.409] [5.412, 2.475]
= [0.9018 + 1.4765 + 2.7135, 0.3667 +0.7383 + 1.2408]
= [5.0918, 2.3458]

4. ☑ Decoder Feedforward Network (FFN)

The FFN is applied independently per token: FFN(x)=max(0, xW1 + b1) W2+b2

Let's define:

Input dim = 2
Hidden dim = 4
Output dim = 2

W1 = [0.2, 0.3, 0.5, -0.4] [0.6, -0.1, 0.2, 0.3]

W2 = [0.4, -0.2] [0.3, 0.5] [-0.1, 0.2] [0.6, 0.3]

b1 = [0.1, 0.1, 0.1, 0.1]
b2 = [0.05, -0.05]

Step 1: First Linear Layer (x * W1 + b1)

z1 = [5.0918, 2.3458] * [0.2, 0.3, 0.5, -0.4] [0.6, -0.1, 0.2, 0.3] = [5.0918*0.2 , 2.3458*0.6] [5.0918*0.3 , 2.3458*(-0.1)] [5.0918*0.5 , 2.3458*0.2] [5.0918*(-0.4) , 2.3458*0.3]

= [2.42584, 1.29296, 3.01506, -1.33298]

Now z1 + b1 = [2.42584, 1.29296, 3.01506, -1.33298] + [0.1, 0.1, 0.1, 0.1]
= [2.52584, 1.39296, 3.11506, -1.23298]

Step 2: ReLU Activation

Relu(z1 + b1)= [2.52584, 1.39296, 3.11506, 0.0]

Step 3: Second Linear Layer (a1 * W2 + b2)

z2 = [2.52584, 1.39296, 3.11506, 0.0] * [0.4, -0.2] [0.3, 0.5] [-0.1, 0.2] [0.6, 0.3]
= [1.11672, 0.81432]

Now z2 + b2 = [1.11672, 0.81432] + [0.05, -0.05] = [1.16672, 0.76432]

5. ☑ Final Output + Linear → Softmax

Assume vocab size is 3 (just for demo), and projection matrix:

W_vocab =
[0.8, 0.2] → Token "Le"
[0.1, 0.5] → Token "chat"
[-0.2, 0.4] → Token "mange"

logits = W_vocab * [1.16672, 0.76432]T = [1.16672*0.8 + 0.76432*0.2] [1.16672*0.1 + 0.76432*0.5] [1.16672*(-0.2) + 0.76432*0.4] = [1.08623] [0.49883] [0.07238]

Now apply softmax:

exp(1.08623) ≈ 2.962
exp(0.49883) ≈ 1.647
exp(0.07238) ≈ 1.075

Probabilities:

"Le" → 2.962 / 5.684 ≈ 0.521
"chat" → 1.647 / 5.684 ≈ 0.290
"mange" → 1.075 / 5.684 ≈ 0.189

☑ Predicted Token: "Le" with 52.1% confidence ☑

☑ 1. Decoding Next Token

Step 1: Initialization of Wq, Wk & Wv

WQ is defined as [1 0] [0 1].

WK is defined as [1 0] [1 1].

WV is defined as [1.1 0] [0 1.1].

Let's say the decoder has so far generated: ["<s>", "Le"]

We'll embed this as: X_decoder = [0.5, 0.5] [0.6, 0.4]

Q = X * Wq = [0.5, 0.5] [0.6, 0.4] * [1, 0] [0, 1] = [0.5, 0.5] [0.6, 0.4]

V = X * Wv = [0.5, 0.5] [0.6, 0.4] * [1.1, 0] [0, 1.1] = [0.55, 0.55] [0.66, 0.44]

K = X * Wk = [0.5, 0.5] [0.6, 0.4] * [1, 0] [1, 1] = [1.0, 0.5] [1.0, 0.4]

Scaled scores (Q.KT/√2)

Scaled scores = [0.8 / 1.414, 0.76 / 1.414]
≈ [0.5656, 0.5373]

Applying Softmax :

exp(0.5656) ≈ 1.760 & exp(0.5373) ≈ 1.711
Sum = 1.760 + 1.711 = 3.471

Attention weights = [1.760/3.471, 1.711/3.471]
= [0.507, 0.493]

Attention scores(Q.KT)

We'll compute attention for token2 attending over K0 and K1

Dot products of Q2 with each Ki:

$Q_2 \cdot K_0$ = 0.6*1.0 + 0.4*0.5 = 0.6 + 0.2 = 0.8

$Q_2 \cdot K_1$ = 0.6*1.0 + 0.4*0.4 = 0.6 + 0.16 = 0.76

Attention Output : Softmax(Q.KT/√2) * V

= [0.507, 0.493] ∗ [0.55, 0.55] [0.66, 0.44] = [0.60423, 0.49577]

☑ 2. Add & Norm

Residual connection: x + attention output = [0.6 + 0.604, 0.4 + 0.495]

Output of Decoder Self-Attention = [1.204, 0.895]

☑ 3. Cross-Attention with Encoder Output

We now attend to encoder outputs:

Decoder Q(Q_d) =[1.204, 0.895]

Step 1: Attention scores
Score 1: = 1.204 * 4.689 + 0.895 * 1.905 = 5.642 + 1.707 = 7.349
Score 2: = 1.204 * 4.818 + 0.895 * 2.409 = 5.798 + 2.158 = 7.956
Score 3: = 1.204 * 5.412 + 0.895 * 2.475 = 6.513 + 2.217 = 8.73

Step 2: Scale
[7.349/1.414, 7.956/1.414, 8.73/1.414]
≈ [4.895, 5.363, 5.854]

Step 3: Softmax
[0.147, 0.269, 0.584]

Step 4: Weighted Sum of V_enc
[0.147, 0.269, 0.584] * [4.689, 1.905] [4.818, 2.409] [5.412, 2.475]
= [2.738 + 1.296 + 3.160, 0.28 +0.6480 + 1.4454] = [7.194, 2.3734]

4. ☑ Decoder Feedforward Network (FFN)

The FFN is applied independently per token: FFN(x)=max(0, xW1 + b1) W2+b2

W1 = [0.2, 0.3, 0.5, -0.4] [0.6, -0.1, 0.2, 0.3]

W2 = [0.4, -0.2] [0.3, 0.5] [-0.1, 0.2] [0.6, 0.3]

Step 1: First Linear Layer (x * W1 + b1)

z1 = [7.194, 2.3734] * [0.2, 0.3, 0.5, -0.4] [0.6, -0.1, 0.2, 0.3] = [2.42584, 1.29296, 4.07168, -2.16558]

Now z1 + b1 = [2.42584, 1.29296, 4.07168, -2.16558] + [0.1, 0.1, 0.1, 0.1]
= [2.52584, 1.39296, 4.08168, -2.06558]

Step 2: ReLU Activation

Relu(z1 + b1) = [2.52584, 1.39296, 4.08168, 0.0]

Step 3: Second Linear Layer (a1 * W2 + b2)

z2 = [2.52584, 1.39296, 4.08168, 0.0] * [0.4, -0.2] [0.3, 0.5] [-0.1, 0.2] [0.6, 0.3] = [1.02006, 1.00765]

Now z2 + b2 = [1.02006, 1.00765] + [0.05, -0.05] = [1.07006, 0.95765]

5. ☑ Final Output + Linear → Softmax

Assume vocab size is 3 (just for demo), and projection matrix:

W_vocab =
[0.8, 0.2] → Token "Le"
[0.1, 0.5] → Token "chat"
[-0.2, 0.4] → Token "mange"

W_vocab * [1.07006, 0.95765]T = [1.07006*0.8 + 0.95765*0.2] [1.07006*0.1 + 0.95765*0.5] [1.07006*(-0.2) + 0.95765*0.4] = [1.047578, 0.585831, 0.169048]

Now apply softmax and getting probabilities:

logits =
"Le" → ≈ 0.489
"chat" → ≈ 0.308
"mange" → ≈ 0.203

☑ Predicted Token: "Le" with 52.1% confidence ☑