SIMPLE CLOSED CURVES ON SURFACES

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1. *Introduction*

Let M be a surface (i.e. a connected compact 2-manifold) with boundary ∂M and some base-point $x_0 \in M$. In this paper we state an algorithm for determining in a finite number of steps whether a given element of the fundamental group $\pi_1(M, x_0)$ can be represented by a simple curve. Algorithms have been given by Reinhart [3] and Zieschang [5, 6]; one relies on non-euclidean geometry and the other involves the problem of equivalence of elements of a free group. The method here is based on *winding numbers* as defined by Reinhart [2, 4] and generalized in the author's Ph.D. thesis. We merely announce those results which are needed to describe the algorithm, and refer to [1] for the proofs. Also we restrict attention to orientable surfaces, although a similar algorithm (using winding numbers in Z_2) works in the non-orientable case.

2. *Winding numbers*

Suppose that M is smooth, say C^∞. A *regular curve* γ in M is the image of a C^1-map $f: S^1 \to M$ such that the tangent map $Tf: TS^1 \to TM$ is an embedding on each fibre. Assume $\partial M \neq \emptyset$ so that M admits continuous fields of non-zero tangent vectors, and let $X: M \to TM$ be such a field. When M is given a Riemannian metric and an orientation it is meaningful to refer to the angle $\theta(z)$ ($0 \leqslant \theta(z) < 2\pi$) turned on going from $Xf(z)$ to $(Tf)_z(1)$ in the tangent plane to M at $f(z)$. The function θ defines a continuous map $h: S^1 \to S^1$ by $z \to \exp i\theta(z)$, and the homotopy class $\{h\} \in \pi_1(S^1) \cong Z$ is called the *winding number of* γ *with respect to* X and denoted by $\omega_X(\gamma)$. It can be loosely described as the total change in angle between the tangent to γ at x and the X-vector at x as x varies round γ, divided by 2π.

A closed curve on M is *direct* if and only if it contains no loop (i.e. image of an arc $z_1 z_2 \subset S^1$ with $f(z_1) = f(z_2)$) which is nullhomotopic as a closed curve.

LEMMA. *Let M be as above, with X a continuous non-vanishing vector field on M. Let γ_1, γ_2 be direct regular curves which are homotopic but not homotopic to a point. Then $\omega_X(\gamma_1) = \omega_X(\gamma_2)$. (Note that the directness condition is essential, since for example the addition of a small nullhomotopic loop to a curve alters its winding number by ± 1.)*

Let $c \in \pi_1(M, x_0)$ and let γ be a direct regular curve representing c: it is easy to show such a γ always exists. If $c \neq 1$ define the *winding number of* c to be $\omega_X(\gamma)$, and denote it by $\omega_X(c)$. By the lemma this depends only on c and not on γ. If $c = 1$ we conventionally define $\omega_X(c) = 1$.

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The function $\omega_X : \pi_1(M, x_0) \to Z : c \to \omega_X(c)$ is not a homomorphism. For example, it is easy to verify that if M is a punctured torus with $\pi_1(M, x_0)$ a free group on two standard generators a, b then $\omega_X(aba^{-1}b^{-1}) \neq 0$ for any X. It is the way in which ω_X differs from a homomorphism that is used to provide the algorithm below. A full understanding of ω_X would do much to illuminate the relation between the geometry of a surface and the algebraic structure of its fundamental group.

3. *The algorithm*

Let M be any orientable surface which is not a sphere. The group $\pi_1(M, x_0)$ may be presented in terms of generators a_i, b_i $(1 \leqslant i \leqslant g)$ and (if $\partial M \neq \emptyset$) s_j $(1 \leqslant j \leqslant r)$ (where g is the genus of M and r is the number of components of ∂M) with the defining relation

$$d = 1$$

where $$d \equiv a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \ldots a_g^{-1} b_g^{-1} s_1 \ldots s_r$$

if $\partial M \neq \emptyset$, or

$$d \equiv a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \ldots a_g^{-1} b_g^{-1}$$

if $\partial M = \emptyset$. Any $c \in \pi_1(M, x_0)$ can be represented (not uniquely) by a "word" $w = x_1 x_2 \ldots x_p$ in the "letters" $x_1, x_2, \ldots, x_p$ where each $x_m = a_i^{\pm 1}$, $b_i^{\pm 1}$ or $s_j^{\pm 1}$ $(1 \leqslant m \leqslant p)$. Write $c = [w]$. A word of the form $x_i x_{i+1} \ldots x_p x_1 \ldots x_{i-1}$ for some i $(1 \leqslant i \leqslant p)$ is a *cyclic permutation of* w. The identity element is represented by the empty word, but since $1 \in \pi_1(M, x_0)$ can clearly be represented by simple closed curves we restrict attention to elements $c \neq 1$.

Given a word w consider the following operations which may be applied to w if $p > 1$:

(i) removing x_m and x_{m+1} if $x_m^{-1} = x_{m+1}$ $(1 \leqslant m \leqslant p-1)$,

(ii) removing any subword of w which consists of more than half of some cyclic permutation $\tilde{d}$ of $d^{\pm 1}$ and replacing it by the inverse of its complement in $\tilde{d}$,

(iii) removing any subword of w which consists of exactly half of some cyclic permutation $\tilde{d}$ of d^{ε} $(\varepsilon = \pm 1)$ and does not contain a_1^{ε} (if $g > 0$) or s_1^{ε} (if $g = 0$), and replacing it by the inverse of its complement in $\tilde{d}$.

It is easy to check that if M is not a torus $(g = 1, r = 0)$ a sequence of the operations (i)–(iii) together with cyclic permutations of the letters will eventually reduce w to a word W with the property that no further operations (i)–(iii) can be applied to W or to any cyclic permutation of W. We call W a *completely reduced* word obtained from the element $c = [w]$.

A word w represents not only an element of $\pi_1(M, x_0)$ but also an element of the free group F generated by the a_i, b_i and s_j, which is the fundamental group of a surface M' obtained by removing an open disc from M disjoint from x_0. Denote by $[w]' \in \pi_1(M', x_0)$ the element represented by w, and write $\omega_X(w)$ for $\omega_{X|M'}([w]')$.

If $\tilde{w}$ is some cyclic permutation of $w = x_1 \dots x_p$ there are $(p-1)$ ways of expressing $\tilde{w}$ as uv where u, v are non-empty subwords of $\tilde{w}$. Call each of these a *division* of w, and write $w \sim u|v$. Regarding $u|v$ as the same as $v|u$ we have therefore $\frac{1}{2}p(p-1)$ possible divisions of w.

Now we can state the algorithm. If M is a sphere the problem is trivial, and if M is a torus it is easily proved that a non-trivial element $a^m b^n$ of $\pi_1(M, x_0) \cong Z \oplus Z$ contains simple representatives if and only if m, n are coprime. Hence assume M is not a sphere or a torus.

ALGORITHM. *Let $c \in \pi_1(M, x_0)$ and let W be a completely reduced word obtained from c. Then c has representatives which are simple closed curves if and only if W satisfies either of the following conditions*:

(1) *W is empty* (i.e. $c = 1$) *or contains only one letter*;

(2) *W contains at least two letters, $W \neq t^k$ for any $k \geqslant 2$, and for every division $W \sim u|v$ the equation*

$$\omega_X(uv^{-1}) - \omega_X(u) - \omega_X(v^{-1}) = 0$$

is satisfied, where X is any continuous non-vanishing vector field on M'.

4. *A formula for the winding number*

In order to verify (2) it is of course necessary to be able to calculate $\omega_X(u)$, etc. We state a formula for this which is derived in [1], and observe that (as would be expected) the vector field X and the smoothness of M are irrelevant to (2).

A word $w = x_1 \dots x_p$ such that $x_p^{-1} \neq x_1$ and (if $p \geqslant 2$) $x_m^{-1} \neq x_{m+1}$ $(1 \leqslant m \leqslant p-1)$ is *cyclically reduced*. Let $w = x_1 \dots x_p$ be cyclically reduced, let y denote any particular one of the letters $a_i^{\pm 1}$, $b_i^{\pm 1}$ or $s_j^{\pm 1}$, and let $N(w, y)$ be the total number of values of m for which $x_m = y$. Write

$$\bar{A}(w) = \sum_{i=1}^{g} N(w, a_i^{-1}),$$

$$B(w) = \sum_{i=1}^{g} N(w, b_i),$$

$$\bar{S}(w) = \sum_{j=1}^{r} N(w, s_j^{-1}) \quad (\partial M \neq \varnothing)$$

$$= 0 \qquad\qquad (\partial M = \varnothing).$$

Thus $\bar{A}(w)$ is just the number of letters in w which are inverses of a_i's, and so on.

Let $\mathscr{R}$ denote the following ordering of the letters $a_i^{\pm 1}$, etc.:

$$a_1, b_1^{-1}, a_1^{-1}, b_1, a_2, b_2^{-1}, \dots, a_g^{-1}, b_g, s_1, s_1^{-1}, s_2, \dots, s_r^{-1}.$$

If $p \geqslant 2$ let $P(w)$ be the number of values of m for which ${x_m}^{-1}$ appears *before* x_{m+1} in the ordering $\mathscr{R}$ (where x_{p+1} means x_1). Write

$$T(w) = P(w) - (\bar{A}(w) + B(w) + \bar{S}(w)),$$

and define $T(w) = 0$ if $p = 1$. Finally, let $\phi_X : F \to Z$ be the homomorphism defined by $\phi_X(a_i) = \omega_X(a_i)$, $\phi_X(b_i) = \omega_X(b_i)$ $(1 \leqslant i \leqslant g)$, $\phi_X(s_j) = \omega_X(s_j)$ $(1 \leqslant j \leqslant r)$ and write $\phi_X(w)$ for $\phi_X([w]')$. Then the *formula for* ω_X is

$$\omega_X(w) = \phi_X(w) + T(w).$$

Now in the application of the algorithm the word W will automatically be cyclically reduced since it is completely reduced, although u, v^{-1} and uv^{-1} will not necessarily be cyclically reduced. However, any word w can be converted into a cyclically reduced word $\bar{w}$ by a sequence of applications of operation (i) together with cyclic permutations of the letters, and the elements $[w]'$ and $[\bar{w}]'$ will be conjugate in $\pi_1(M', x_0)$. From the fact that there is a diffeomorphism $M' \to M'$ isotopic to the identity that induces an automorphism of $\pi_1(M', x_0)$ taking $[w]'$ to $[\bar{w}]'$, it follows easily that $\omega_X(w) = \omega_X(\bar{w})$. Since ϕ_X is a homomorphism we have $\phi_X(\bar{w}) = \phi_X(w)$ for all w, and so also $\phi_X(\overline{uv^{-1}}) = \phi_X(\bar{u}) + \phi_X(\overline{v^{-1}})$. Hence

$$\omega_X(uv^{-1}) - \omega_X(u) - \omega_X(v^{-1}) = T(\overline{uv^{-1}}) - T(\bar{u}) - T(\overline{v^{-1}}),$$

and the right-hand side is independent of X. Therefore condition (2) of the algorithm can be replaced by the following condition:

(2′) *W contains at least two letters, $W \neq t^k$ for any $k \geqslant 2$, and for every division $W \sim u|v$ the equation*

$$T(\overline{uv^{-1}}) - T(\bar{u}) - T(\overline{v^{-1}}) = 0$$

is satisfied.

5. *Some examples*

1. $g \geqslant 2$, $c = [a_1 a_2]$. Take $W = a_1 a_2$ which is already completely reduced. The unique division is $W \sim a_1|a_2$. We have

$$\bar{A}(a_1 {a_2}^{-1}) = 1,\ B(a_1 {a_2}^{-1}) = \bar{S}(a_1 {a_2}^{-1}) = 0;\ P(a_1 {a_2}^{-1}) = 1,$$

so
$$T(a_1 {a_2}^{-1}) = 1 - (1+0+0) = 0 = T(a_1) + T({a_2}^{-1}).$$

Therefore c does contain representatives which are simple curves.

2. $g \geqslant 2$, $c = [a_1 {a_2}^{-1}]$. Here

$$\bar{A}(a_1 a_2) = 0 = B(a_1 a_2) = \bar{S}(a_1 a_2),\ P(a_1 a_2) = 1$$

so
$$T(a_1 a_2) = 1 \neq T(a_1) + T(a_2).$$

Therefore c contains no simple curves.

3. $g \geqslant 2$, $c = [b_1{}^{-1} b_2 a_1{}^{-1} b_2 a_2{}^{-1} b_1 a_2{}^{-1} a_1 b_1]$. A completely reduced word obtained from c is $W = b_2 a_1{}^{-1} b_2 a_2{}^{-1} b_1 a_2{}^{-1} a_1$. Consider the division

$$W \sim a_1 b_2 a_1{}^{-1} b_2 a_2{}^{-1} | b_1 a_2{}^{-1} = u|v,$$

say. In this case $\bar{u} = u$, $\overline{v^{-1}} = v^{-1}$ but $\overline{uv^{-1}} = a_1 b_2 a_1{}^{-1} b_2 b_1{}^{-1}$.

$$\bar{A}(\bar{u}) = 2,\ B(\bar{u}) = 2;\ P(\bar{u}) = 3;\ T(\bar{u}) = 3-(2+2) = -1$$

$$\bar{A}(\overline{v^{-1}}) = 0,\ B(\overline{v^{-1}}) = 0;\ P(\overline{v^{-1}}) = 1;\ T(\overline{v^{-1}}) = 1-(0+0) = 1$$

$$\bar{A}(\overline{uv^{-1}}) = 1,\ B(\overline{uv^{-1}}) = 2;\ P(\overline{uv^{-1}}) = 2;\ T(\overline{uv^{-1}}) = 2-(2+1) = -1$$

and so $T(\overline{uv^{-1}}) - T(\bar{u}) - T(\overline{v^{-1}}) = -1 \neq 0$. Hence c contains no simple curves.

4. $g = 0$, $r \geqslant 1$. Here M is a "disc with holes". The fundamental group is generated by $s_1, \ldots, s_r$ with the defining relation $d = 0$ where $d \equiv s_1 s_2 \ldots s_r$, the ordering $\mathscr{R}$ is just $s_1, s_1{}^{-1} s_2, \ldots, s_r{}^{-1}$, and $T = P - \bar{S}$. Consider for example $c = [s_1 s_2 s_3]$ when $r \geqslant 6$. There are three divisions of $W = s_1 s_2 s_3$, namely $u|v = s_1|s_2 s_3$, $s_2|s_3 s_1$ or $s_3|s_1 s_2$, and in each case $T(\bar{u}) = 0$, $T(\overline{v^{-1}}) = 1-2 = -1$ and $T(\overline{uv^{-1}}) = 1-2 = -1$. Hence c contains simple curves. However, $c = [s_1 s_3 s_2]$ does not contain simple curves since

$$T(s_2) = 0,\ T(s_3{}^{-1} s_1{}^{-1}) = 1-2 = -1$$

but

$$T(s_2 s_3{}^{-1} s_1{}^{-1}) = 2-2 = 0.$$

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