Lecture 9: Classification: Naive Bayes
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Columbia University

Jiayi Lily Ma

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1 Learning by example

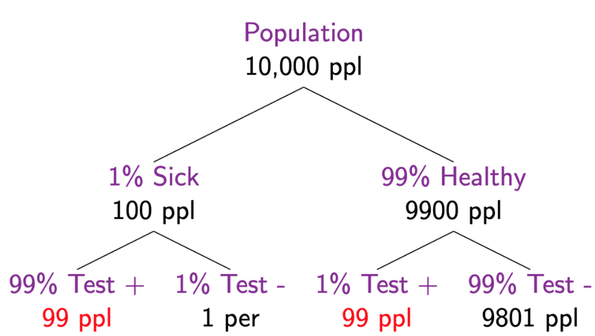
Spam vs Ham

- We can separate spam from non-spam emails by looking at various indicative things
- For example, words such as 'special offer' are highly indicative of spam, whereas words like 'supercomputing cluster' are much less indicative.
- A potential problem is with ubiquitous words like 'the'.
- It's hard to write down all rules, so we learn them from data instead.

2 Diagnoses a la Bayes

- You're testing for a rare disease:
  - 1% of the population is infected
- You have a highly sensitive and specific test:
  - 99% of sick patients test positive
  - 99% of healthy patients test negative
- Given that a patient tests positive, what is probability the patient is sick?

2.1 Method 1



So given that a patient tests positive (198 ppl), there is a 50% chance the patient is sick (99 ppl)!

2.2 Method 2

We know from the given that:

$$P(sick) = 0.01$$
$$P(healthy) = 0.99$$
$$P(+|sick) = 0.99$$
$$P(+|healthy) = 0.01$$

According to Bayes' Theorem (proven later):

$$P(sick|+) = \frac{P(+|sick)P(sick)}{P(+)} = \frac{P(+|sick)P(sick)}{P(sick)P(+|sick) + P(healthy)P(+|healthy)} = \frac{(0.99)(0.01)}{(0.01)(0.99) + (0.99)(0.01)} = \frac{1}{2}$$

3 Natural Frequencies a la Gigenrenzer

- Compared to the 1000 women who didn't have screening, the 1000 women with screening suffered 100 cases of false-positive negative results and 5 cases of unnecessary treatments.
- Thus, although the claim is that screening reduces the number of patients who died from breast cancer by 20% (5 deaths vs 4 deaths), there's a risk present for error for women with screening that is not present for those without.

4 Inverting Conditional Probabilities

Bayes' Theorem

We know that

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

because $P(x, y) = P(y, x)$

Divide to get the probability of y given x from the probability of x given y:

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

where $P(x) = \sum_{y \in \Omega_Y} P(x|y)P(y)$

5 (Super) Naive Bayes

Idea: use Bayes' Rule to build a one-word spam classifier:

$$P(spam|word) = \frac{P(word|spam)P(spam)}{P(word)}$$

Estimate each term with ratio of counts:

$$\hat{P}(word|spam) = \frac{\text{\# spam docs containing word}}{\text{\# spam docs}}$$
$$\hat{P}(word|ham) = \frac{\text{\# ham docs containing word}}{\text{\# ham docs}}$$
$$\hat{P}(spam) = \frac{\text{\# spam docs}}{\text{\# docs}}$$
$$\hat{P}(ham) = \frac{\text{\# ham docs}}{\text{\# docs}}$$

Examples from running bash script:

```
$ ./enron_naive_bayes.sh money
1500 spam examples
3672 ham examples
194 spam examples containing money
50 ham examples containing money

estimated P(spam) = .2900
estimated P(ham) = .7100
estimated P(money|spam) = .1293
estimated P(money|ham) = .0136

P(spam|money) = .7957
```

```
$ ./enron_naive_bayes.sh enron
1500 spam examples
3672 ham examples
0 spam examples containing enron
1478 ham examples containing enron

estimated P(spam) = .2900
estimated P(ham) = .7100
estimated P(enron|spam) = 0
estimated P(enron|ham) = .4025

P(spam|enron) = 0
```

Note that the probability of an email with the word 'enron' being classified as spam is 0 because none of the spam examples has 'enron'. This is probably not what we want.

A possible way around this is:

$$\hat{P}(word|spam) = \frac{n_{word,spam} + \alpha}{N_{spam} + \beta}$$

where the best α, β combination on the test data can be found through grid search.

6 Naive Bayes

- 'Naive' in the sense that all words in a document are seen as independent given the class label of the document.

- Represent each document by a binary vector x where $x_j = 1$ if the j-th word appears in the document ($x_j = 0$ otherwise).
- If we model each words as *independent* Bernoulli random variable, the probability of observing document $\vec{x}$ of class c is:

$$P(\vec{x}|c) = \prod_i \theta_{jc}^{x_j}(1-\theta_{jc})^{1-x_j}$$

where θ_{jc} is the probability that the j-th word occurs in a document of class c.

- Using this likelihood in Bayes' Rule and taking the logarithm, we have:

$$logP(c|\vec{x}) = log\frac{P(\vec{x}|c)P(c)}{P(\vec{x})} = \sum_j x_j log(\frac{\theta_{jc}}{1-\theta_{jc}}) + \sum_j log(1-\theta_{jc}) + log\frac{\theta_c}{P(\vec{x})}$$

where θ_j is the probability of observing a document of class c and $logP(\vec{x}|c)$ is calculated as follows:

$$logP(\vec{x}|c) = \sum_j log[\theta_{jcj}^{x}(1-\theta_{jc})^{1-x_j}] = \sum_j x_j log\theta_{jc} + (1-x_j)log(1-\theta_{jc}) = \sum_j x_j log(\frac{\theta_{jc}}{1-\theta_{jc}}) + \sum_j log(1-\theta_{jc})$$

Note that the second term $\sum_j log(1-\theta_{jc}$ is a constant because it doesn't depend on $\vec{x}$. It can be interpreted as the base rate of class c for an empty document. Recall that θ_{jc} is the probability that the j-th word occurs in a document of class c.

We can remove $P(\vec{x})$ by calculating the log-odds:

$$log\frac{P(1|\vec{x})}{P(0|\vec{x})} = \sum_j x_j \underbrace{log\frac{\theta_{j1}(1-\theta_{j0})}{\theta_{j0}(1-\theta_{j1})}}_{w_j} + \underbrace{\sum_j log\frac{1-\theta_{j1}}{1-\theta_{j0}} + log\frac{\theta_1}{\theta_0}}_{w_0}$$

which gives a linear classifier of the form $\vec{w} \cdot \vec{x} + w_0$

We train by counting words and documents within classes to estimate θ_{jc} and θ_c:

$$\hat{\theta}_{jc} = \frac{n_{jc}}{n_c}$$

$$\hat{\theta}_c = \frac{n_c}{n}$$

and then calculate the weights $\hat{w}_j$ and bias $\hat{w}_0$:

$$\hat{w}_j = log\frac{\hat{\theta}_{j1}(1-\hat{\theta}_{j0})}{\hat{\theta}_{j0}(1-\hat{\theta}_{j1})}$$

$$\hat{w}_0 = \sum_j log\frac{1-\hat{\theta}_{j1}}{1-\hat{\theta}_{j0}} + log\frac{\hat{\theta}_1}{\hat{\theta}_0}$$

We predict by adding the weights of the words that appear in the document to the bias term.

7 Logistic Regression

Form of classifier:

$$log\frac{p}{1-p} = \vec{w}\cdot\vec{x}$$

Solve for p:

$$\frac{p}{1-p} = e^{\vec{w}\cdot\vec{x}}$$

$$p = (1-p)e^{\vec{w}\cdot\vec{x}}$$

$$p(1+e^{\vec{w}\cdot\vec{x}}) = e^{\vec{w}\cdot\vec{x}}$$

$$p = \frac{e^{\vec{w}\cdot\vec{x}}}{1+e^{\vec{w}\cdot\vec{x}}}\cdot\frac{e^{-\vec{w}\cdot\vec{x}}}{e^{-\vec{w}\cdot\vec{x}}} = \frac{1}{1+e^{-\vec{w}\cdot\vec{x}}} = P(\vec{x}|\vec{w})$$

The probability of seeing data $(\vec{x}_i, y_i)$ is:

$$L = \prod_i p_i^{y_i}(1-p_i)^{1-y_i}$$

Note that $y_i \in 0,1$ and $p_i = P(\vec{x_i}|\vec{w})$.

Take the log of L:

$$\ell = logL = \sum_i y_i log(p_i) + (1-y_i)log(1-p_i) = \sum_i y_i log\frac{p_i}{1-p_i} + log(1-p_i) = \sum_i y_i(\vec{w}\cdot\vec{x_i}) - log(1+e^{\vec{x}\cdot\vec{x_i}})$$

Note that $log(1-p_i) = log(1-\frac{1}{1+e^{-\vec{w}\cdot\vec{x}}}) = log(\frac{1}{1+e^{\vec{w}\cdot\vec{x}}}) = -log(1+e^{\vec{w}\cdot\vec{x}})$.

We want to find the $\vec{w}$ that gives the highest likelihood to the data that we have i.e. the $\vec{w}$ that maximizes $P(D|\vec{w})$. This is equivalent to finding a $\vec{w}$ that maximizes $logP(D|\vec{w})$. Note that this log probability is ℓ, what we found previously. So we take the derivative of ℓ with respect to $\vec{w}$:

$$\frac{d\ell}{dw_j} = \sum_j y_i x_{ij} - \frac{1}{1+e^{\vec{w}\cdot\vec{x}_i}}e^{\vec{w}\cdot\vec{x}_i}x_{ij} = \sum_j y_i x_{ij} - p_i x_{ij} = \sum_j (y_i - p_i)x_{ij}$$

note that $p_i = \frac{1}{1+e^{\vec{w}\cdot\vec{x}_i}}e^{\vec{w}\cdot\vec{x}_i}$ (see previous calculation).
We use this gradient in gradient descent to find the best $\vec{w}$.

- Naive Bayes gives extreme estimates because it doesn't take the weight of other words into account.
- Logistic Regression, however, allows for the sharing of weights because p_i is calculated with $\vec{w}$.