AI1110
Assignment 16

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Outline

1. Question
2. Proof

Papoulis Exercise 11-7

Show that if $R_{xx}(\tau) = e^{-c|\tau|}$, then the Karhunen-Loeve expansion of $X(t)$ in the interval $(-a, a)$ is the sum

$$\hat{X}(t) = \sum_{n=1}^{\infty} \left(\beta_n b_n \cos \omega_n t + \beta'_n b'_n \sin \omega'_n t\right) \tag{1}$$

where

$$\tan a\omega_n = \frac{c}{\omega_n} \qquad \cot a\omega'_n = -\frac{c}{\omega'_n} \tag{2}$$

$$\beta_n = (a + c\lambda_n)^{-\frac{1}{2}} \qquad \beta'_n = (a - c\lambda'_n)^{-\frac{1}{2}} \tag{3}$$

$$E\left[b_n^2\right] = \lambda_n = \frac{2c}{c^2 + \omega_n^2} \qquad E\left[b_n'^2\right] = \lambda'_n = \frac{2c}{c^2 + \omega_n'^2} \tag{4}$$

Proof

The Karhunen-Loeve expansion of a process $X(t)$ is given by

$$\hat{X}(t) = \sum_{n=1}^{\infty} c_n \varphi_n(t) \qquad 0 < t < T \tag{5}$$

where $\varphi_n(t)$ is a set of orthonormal functions in the interval $(0, T)$

$$\int_0^T \varphi_n(t)\varphi_m(t)\mathrm{d}t = \delta[n-m] \tag{6}$$

and the coefficients c_n are random variables given by

$$c_n = \int_0^T X(t)\varphi_n(t)\mathrm{d}t \tag{7}$$

Now, $R_{xx}(\tau) = e^{-c|\tau|} \implies R_{xx}(t_1, t_2) = e^{-c|t_1 - t_2|}$ where $\tau = t_1 - t_2$
We now form the integral equation for a general $\varphi_n \triangleq \varphi$

$$\int_{-a}^{a} R_{xx}(t_1, t_2)\varphi(t_2)\mathrm{d}t_2 = \lambda\varphi(t_1) \tag{8}$$

$$\implies \int_{-a}^{a} e^{-c|t_1 - t_2|}\varphi(t_2)\mathrm{d}t_2 = \lambda\varphi(t_1) \tag{9}$$

$$\implies \int_{-a}^{t_1} e^{-c(t_1 - t_2)}\varphi(t_2)\mathrm{d}t_2 + \int_{t_1}^{a} e^{c(t_1 - t_2)}\varphi(t_2)\mathrm{d}t_2 = \lambda\varphi(t_1) \tag{10}$$

On differentiating with respect to t_1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\int_{-a}^{t_1} e^{-c(t_1 - t_2)}\varphi(t_2)\mathrm{d}t_2 = \frac{\mathrm{d}t_1}{\mathrm{d}t_1}e^{-c(t_1 - t_1)}\varphi(t_1) + \int_{-a}^{t_1} \frac{\partial}{\partial t_1}\left(e^{-c(t_1 - t_2)}\varphi(t_2)\right)\mathrm{d}t_2 \tag{11}$$

$$= \varphi(t_1) - c\int_{-a}^{t_1} e^{-c(t_1 - t_2)}\varphi(t_2)\mathrm{d}t_2 \tag{12}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\int_{t_1}^{a} e^{c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2 = -\varphi(t_1) + c\int_{t_1}^{a} e^{c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2 \tag{13}$$

Thus,

$$\lambda\varphi'(t_1) = -c\int_{-a}^{t_1} e^{-c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2 + c\int_{t_1}^{a} e^{c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2 \tag{14}$$

$$\begin{aligned}\implies \lambda\varphi''(t_1) = -c\left(\varphi(t_1) - c\int_{-a}^{t_1} e^{-c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2\right)\\ + c\left(-\varphi(t_1) + c\int_{t_1}^{a} e^{c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2\right)\end{aligned} \tag{15}$$

$$\implies \lambda\varphi''(t_1) = -2c\varphi(t_1) + c^2\left(\int_{-a}^{t_1} e^{-c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\varphi(t_2)\mathrm{d}t_2\right) \tag{16}$$

Therefore, we have obtained the differential equation

$$\lambda\varphi''(t_1) = -2c\varphi(t_1) + c^2\lambda\varphi(t_1) \tag{17}$$

$$\implies \varphi''(t_1) = -\left(\frac{2c}{\lambda} - c^2\right)\varphi(t_1) \tag{18}$$

$$= -\omega^2\varphi(t_1) \tag{19}$$

Assuming $2c - \lambda c^2 \geq 0$, the solutions to this differential equation are $\beta\cos(\omega t_1 + \theta)$ (corresponding to λ) and $\beta'\cos(\omega' t_1 + \theta')$ (corresponding to λ') for arbitrary constants β, θ and β', θ'

$$\frac{2c}{\lambda} - c^2 = \omega^2 \tag{20}$$

$$\implies \lambda = \frac{2c}{c^2 + \omega^2} \tag{21}$$

Similarly,

$$\lambda' = \frac{2c}{c^2 + \omega'^2} \tag{22}$$

Assuming $X(t)$ to be wide sense stationary, we can always shift the origin such that $\varphi(t)$ is of the form $\beta\cos\omega t$ without affecting the autocorrelation. Substituting this in the integral equation, we get

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)}\beta\cos\omega t_2 \mathrm{d}t_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\beta\cos\omega t_2 \mathrm{d}t_2 = \lambda\beta\cos\omega t_1 \tag{23}$$

Using integration by parts,

$$\mathcal{I} = \int e^{ct_2}\cos\omega t_2 \mathrm{d}t_2 \tag{24}$$

$$= \cos\omega t_2 \frac{e^{ct_2}}{c} + \omega\int \sin\omega t_2 \frac{e^{ct_2}}{c}\mathrm{d}t_2 \tag{25}$$

$$= \cos\omega t_2 \frac{e^{ct_2}}{c} + \frac{\omega}{c^2}\left(\sin\omega t_2 \frac{e^{ct_2}}{c} - \omega\int e^{ct_2}\cos\omega t_2 \mathrm{d}t_2\right) \tag{26}$$

$$= \cos\omega t_2 \frac{e^{ct_2}}{c} + \frac{\omega}{c^2}\left(\sin\omega t_2 \frac{e^{ct_2}}{c} - \omega\mathcal{I}\right) \tag{27}$$

$$\mathcal{I} = e^{ct_2} \frac{c \cos \omega t_2 + \omega \sin \omega t_2}{c^2 + \omega^2} \tag{28}$$

Thus,

$$\beta e^{-ct_1} \int_{-a}^{t_1} e^{ct_2} \cos \omega t_2 \mathrm{d} t_2 = \frac{\beta}{c^2 + \omega^2} [c \cos \omega t_1 + \omega \sin \omega t_1 + e^{-c(a+t_1)} (-c \cos \omega a + \omega \sin \omega a)] \tag{29}$$

$$\beta e^{ct_1} \int_{t_1}^{a} e^{-ct_2} \cos \omega t_2 \mathrm{d} t_2 = \frac{\beta}{c^2 + \omega^2} [c \cos \omega t_1 - \omega \sin \omega t_1 + e^{-c(a-t_1)} (-c \cos \omega a + \omega \sin \omega a)] \tag{30}$$

Therefore,

$$\lambda\beta\cos\omega t_1 = \frac{2\beta c\cos\omega t_1}{c^2+\omega^2} + \frac{\beta e^{-ac}(-c\cos\omega a+\omega\sin\omega a)}{c^2+\omega^2}(e^{ct_1}+e^{-ct_1}) \qquad \forall t_1 \quad (31)$$

Comparing both sides of the equation, we get

$$\lambda = \frac{2c}{c^2+\omega^2} \quad (32)$$

which is consistent with what we obtained earlier.
and

$$-c\cos\omega a + \omega\sin\omega a = 0 \quad (33)$$

$$\implies \tan a\omega = \frac{c}{\omega} \quad (34)$$

Performing a similar computation, we also get

$$\cot a\omega' = -\frac{c}{\omega} \quad (35)$$

Since, $\varphi(t)$ is an orthonormal function, we have to normalize it using the condition

$$\int_{-a}^{a} \varphi_n(t)\varphi_m(t)\mathrm{d}t = \delta[n-m] \tag{36}$$

Putting $m = n$

$$\int_{-a}^{a} \varphi_n^2(t)\mathrm{d}t = \delta[0] = 1 \tag{37}$$

$$\implies \int_{-a}^{a} \beta^2 \cos^2 \omega t \mathrm{d}t = 1 \tag{38}$$

$$\implies \beta^2 \int_{0}^{a} 2\cos^2 \omega t \mathrm{d}t = 1 \tag{39}$$

$$\implies \beta^2 \int_{0}^{a} (1 + \cos 2\omega t)\mathrm{d}t = 1 \tag{40}$$

$$\implies \beta^2 \left(a + \frac{\sin 2\omega a}{2\omega}\right) = 1 \tag{41}$$

But $\tan a\omega = \frac{c}{\omega} \implies \sin 2a\omega = \frac{2c\omega}{c^2+\omega^2}$

$$\beta^2 \left(a + \frac{c}{c^2 + \omega^2}\right) = 1 \tag{42}$$

We have found out that $\lambda = \frac{2c}{c^2+\omega^2}$ Therefore,

$$\beta^2 \left(a + \frac{\lambda}{2}\right) = 1 \tag{43}$$

$$\implies \beta = \left(a + \frac{\lambda}{2}\right)^{-\frac{1}{2}} \tag{44}$$

Performing a similar computation,

$$\beta' = \left(a - \frac{\lambda}{2}\right)^{-\frac{1}{2}} \tag{45}$$