

AI1110

Assignment 12

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Outline

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Papoulis Exercise 4-35

Poisson Theorem

If $n \rightarrow \infty$ and $p \rightarrow 0$ such that $np \rightarrow \lambda$ then

$$\frac{n!}{k!(n-k)!} p^k q^{n-k} \xrightarrow{n \rightarrow \infty} e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, \dots \quad (1)$$

Reasoning as in (1), show that, if

$$k_1 + k_2 + k_3 = n \quad p_1 + p_2 + p_3 = 1 \quad k_1 p_1 \ll 1 \quad k_2 p_2 \ll 1 \quad (2)$$

then

$$\frac{n!}{k_1! k_2! k_3!} \simeq \frac{n^{k_1+k_2}}{k_1! k_2!} \quad p_3^{k_3} \simeq e^{-n(p_1+p_2)} \quad (3)$$

Proving the first part

From (2), we have $k_3 = n - k_1 - k_2$

Thus,

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n!}{k_1!k_2!(n - k_1 - k_2)!} \quad (4)$$

$$= \frac{n(n-1) \cdots (n - k_1 - k_2 + 1)}{k_1!k_2!} \quad (5)$$

$$= \frac{n(n-1) \cdots (n - k_1 - k_2 + 1)}{n^{k_1+k_2}} \frac{n^{k_1+k_2}}{k_1!k_2!} \quad (6)$$

$$= \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k_1 + k_2 - 1}{n}\right) \frac{n^{k_1+k_2}}{k_1!k_2!} \quad (7)$$

$$= \left(\prod_{m=0}^{k_1+k_2-1} \left(1 - \frac{m}{n}\right) \right) \frac{n^{k_1+k_2}}{k_1!k_2!} \quad (8)$$

Now, if we assume that $k_1 + k_2$ is finite, i.e.,

$$\text{as } n \rightarrow \infty, k_1 + k_2 \ll n \quad (9)$$

Then the finite product

$$\prod_{m=0}^{k_1+k_2-1} \left(1 - \frac{m}{n}\right) = 1 + O\left(\frac{1}{n}\right) \quad (10)$$

tends to unity as $n \rightarrow \infty$

Therefore,

$$\frac{n!}{k_1!k_2!k_3!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!} \quad \square \quad (11)$$

Proving the second part

From (2), we have $p_3 = 1 - p_1 - p_2$

$$p_3^{k_3} = (1 - p_1 - p_2)^{k_3} \quad (12)$$

We have already assumed that $k_1 \ll n$ and $k_2 \ll n$ in (9)

Assume $p_1 \ll 1$ and $p_2 \ll 1$ satisfying $k_1 p_1 \ll 1$, $k_2 p_2 \ll 1$

$$\therefore p_1 + p_2 \ll 1 \quad (13)$$

$$\implies (1 - p_1 - p_2)^{k_3} \simeq (e^{-(p_1 + p_2)})^{k_3} \quad (14)$$

$$= e^{-k_3(p_1 + p_2)} \quad (15)$$

Also, $k_1 + k_2 \ll n \implies k_3 \simeq n$ since $k_1 + k_2 + k_3 = n$

Therefore,

$$p_3^{k_3} \simeq e^{-n(p_1 + p_2)} \quad \square \quad (16)$$

Follow-up

Random Poisson points in non-overlapping intervals

Use these results to justify, for non-overlapping intervals t_a and t_b ,

$$\Pr(k_a \text{ in } t_a, k_b \text{ in } t_b) = e^{-\lambda t_a} \frac{(\lambda t_a)^{k_a}}{k_a!} e^{-\lambda t_b} \frac{(\lambda t_b)^{k_b}}{k_b!} \quad (17)$$

where $\Pr(k \text{ in } t)$ denotes the probability that k of n randomly placed points in the interval $(-\frac{T}{2}, \frac{T}{2})$ will lie in an interval of length t and $\lambda = \frac{n}{T}$ is constant

Justification

Let $k_1 = k_a$, $k_2 = k_b$ and $k_3 = n - k_a - k_b$ denote the number of points lying in t_a , t_b and outside both t_a and t_b respectively out of a total of n randomly placed points in $(-\frac{T}{2}, \frac{T}{2})$

$$k_1 + k_2 + k_3 = n \quad (18)$$

Let p_1 , p_2 and p_3 denote the probabilities that an arbitrary point lies in t_a , t_b and outside both t_a and t_b respectively

$$p_1 + p_2 + p_3 = 1 \quad (19)$$

Assuming small intervals, i.e., $t_a, t_b \ll T$

$$k_1 p_1 \ll 1 \quad k_2 p_2 \ll 1 \quad (20)$$

The conditions have thus been met and we can now use the previously proven results

$$\Pr(k_a \text{ in } t_a, k_b \text{ in } t_b) = \frac{n!}{k_a!k_b!(n-k_a-k_b)!} p_1^{k_a} p_2^{k_b} p_3^{n-k_a-k_b} \quad (21)$$

$$= \frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \quad (22)$$

$$\simeq \frac{n^{k_1+k_2}}{k_1!k_2!} p_1^{k_1} p_2^{k_2} e^{-n(p_1+p_2)} \quad (23)$$

$$= e^{-np_1} \frac{(np_1)^{k_1}}{k_1!} e^{-np_2} \frac{(np_2)^{k_2}}{k_2!} \quad (24)$$

Now, $p_1 = \frac{t_a}{T} \implies np_1 = \frac{n}{T} t_a = \lambda t_a$

Similarly, $np_2 = \lambda t_b$

Substituting back, we get the desired result. \square