

AI1110

Assignment 16

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Outline

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Papoulis Exercise 11-7

Show that if $R_{xx}(\tau) = e^{-c|\tau|}$, then the Karhunen-Loeve expansion of $X(t)$ in the interval $(-a, a)$ is the sum

$$\hat{X}(t) = \sum_{n=1}^{\infty} (\beta_n b_n \cos \omega_n t + \beta'_n b'_n \sin \omega'_n t) \quad (1)$$

where

$$\tan a\omega_n = \frac{c}{\omega_n} \quad \cot a\omega'_n = -\frac{c}{\omega'_n} \quad (2)$$

$$\beta_n = (a + c\lambda_n)^{-\frac{1}{2}} \quad \beta'_n = (a - c\lambda'_n)^{-\frac{1}{2}} \quad (3)$$

$$E[b_n^2] = \lambda_n = \frac{2c}{c^2 + \omega_n^2} \quad E[b_n'^2] = \lambda'_n = \frac{2c}{c^2 + \omega_n'^2} \quad (4)$$

Proof

The Karhunen-Loeve expansion of a process $X(t)$ is given by

$$\hat{X}(t) = \sum_{n=1}^{\infty} c_n \varphi_n(t) \quad 0 < t < T \quad (5)$$

where $\varphi_n(t)$ is a set of orthonormal functions in the interval $(0, T)$

$$\int_0^T \varphi_n(t) \varphi_m(t) dt = \delta[n - m] \quad (6)$$

and the coefficients c_n are random variables given by

$$c_n = \int_0^T X(t) \varphi_n(t) dt \quad (7)$$

Now, $R_{xx}(\tau) = e^{-c|\tau|} \implies R_{xx}(t_1, t_2) = e^{-c|t_1 - t_2|}$ where $\tau = t_1 - t_2$

We now form the integral equation for a general $\varphi_n \triangleq \varphi$

$$\int_{-a}^a R_{xx}(t_1, t_2) \varphi(t_2) dt_2 = \lambda \varphi(t_1) \quad (8)$$

$$\implies \int_{-a}^a e^{-c|t_1 - t_2|} \varphi(t_2) dt_2 = \lambda \varphi(t_1) \quad (9)$$

$$\implies \int_{-a}^{t_1} e^{-c(t_1 - t_2)} \varphi(t_2) dt_2 + \int_{t_1}^a e^{c(t_1 - t_2)} \varphi(t_2) dt_2 = \lambda \varphi(t_1) \quad (10)$$

On differentiating with respect to t_1 , we get

$$\frac{d}{dt_1} \int_{-a}^{t_1} e^{-c(t_1 - t_2)} \varphi(t_2) dt_2 = \frac{dt_1}{dt_1} e^{-c(t_1 - t_1)} \varphi(t_1) + \int_{-a}^{t_1} \frac{\partial}{\partial t_1} (e^{-c(t_1 - t_2)} \varphi(t_2)) dt_2 \quad (11)$$

$$= \varphi(t_1) - c \int_{-a}^{t_1} e^{-c(t_1 - t_2)} \varphi(t_2) dt_2 \quad (12)$$

Similarly,

$$\frac{d}{dt_1} \int_{t_1}^a e^{c(t_1-t_2)} \varphi(t_2) dt_2 = -\varphi(t_1) + c \int_{t_1}^a e^{c(t_1-t_2)} \varphi(t_2) dt_2 \quad (13)$$

Thus,

$$\lambda \varphi'(t_1) = -c \int_{-a}^{t_1} e^{-c(t_1-t_2)} \varphi(t_2) dt_2 + c \int_{t_1}^a e^{c(t_1-t_2)} \varphi(t_2) dt_2 \quad (14)$$

$$\begin{aligned} \Rightarrow \lambda \varphi''(t_1) = & -c \left(\varphi(t_1) - c \int_{-a}^{t_1} e^{-c(t_1-t_2)} \varphi(t_2) dt_2 \right) \\ & + c \left(-\varphi(t_1) + c \int_{t_1}^a e^{c(t_1-t_2)} \varphi(t_2) dt_2 \right) \end{aligned} \quad (15)$$

$$\Rightarrow \lambda \varphi''(t_1) = -2c\varphi(t_1) + c^2 \left(\int_{-a}^{t_1} e^{-c(t_1-t_2)} \varphi(t_2) dt_2 + \int_{t_1}^a e^{c(t_1-t_2)} \varphi(t_2) dt_2 \right) \quad (16)$$

Therefore, we have obtained the differential equation

$$\lambda \varphi''(t_1) = -2c\varphi(t_1) + c^2\lambda\varphi(t_1) \quad (17)$$

$$\implies \varphi''(t_1) = -\left(\frac{2c}{\lambda} - c^2\right)\varphi(t_1) \quad (18)$$

$$= -\omega^2\varphi(t_1) \quad (19)$$

Assuming $2c - \lambda c^2 \geq 0$, the solutions to this differential equation are $\beta \cos(\omega t_1 + \theta)$ (corresponding to λ) and $\beta' \cos(\omega' t_1 + \theta')$ (corresponding to λ') for arbitrary constants β , θ and β' , θ'

$$\frac{2c}{\lambda} - c^2 = \omega^2 \quad (20)$$

$$\implies \lambda = \frac{2c}{c^2 + \omega^2} \quad (21)$$

Similarly,

$$\lambda' = \frac{2c}{c^2 + \omega'^2} \quad (22)$$

Assuming $X(t)$ to be wide sense stationary, we can always shift the origin such that $\varphi(t)$ is of the form $\beta \cos \omega t$ without affecting the autocorrelation. Substituting this in the integral equation, we get

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)} \beta \cos \omega t_2 dt_2 + \int_{t_1}^a e^{c(t_1-t_2)} \beta \cos \omega t_2 dt_2 = \lambda \beta \cos \omega t_1 \quad (23)$$

Using integration by parts,

$$I = \int e^{ct_2} \cos \omega t_2 dt_2 \quad (24)$$

$$= \cos \omega t_2 \frac{e^{ct_2}}{c} + \omega \int \sin \omega t_2 \frac{e^{ct_2}}{c} dt_2 \quad (25)$$

$$= \cos \omega t_2 \frac{e^{ct_2}}{c} + \frac{\omega}{c^2} \left(\sin \omega t_2 \frac{e^{ct_2}}{c} - \omega \int e^{ct_2} \cos \omega t_2 dt_2 \right) \quad (26)$$

$$= \cos \omega t_2 \frac{e^{ct_2}}{c} + \frac{\omega}{c^2} \left(\sin \omega t_2 \frac{e^{ct_2}}{c} - \omega I \right) \quad (27)$$

$$I = e^{ct_2} \frac{c \cos \omega t_2 + \omega \sin \omega t_2}{c^2 + \omega^2} \quad (28)$$

Thus,

$$\begin{aligned} \beta e^{-ct_1} \int_{-a}^{t_1} e^{ct_2} \cos \omega t_2 dt_2 &= \frac{\beta}{c^2 + \omega^2} [c \cos \omega t_1 + \omega \sin \omega t_1 \\ &\quad + e^{-c(a+t_1)} (-c \cos \omega a + \omega \sin \omega a)] \quad (29) \end{aligned}$$

$$\begin{aligned} \beta e^{ct_1} \int_{t_1}^a e^{-ct_2} \cos \omega t_2 dt_2 &= \frac{\beta}{c^2 + \omega^2} [c \cos \omega t_1 - \omega \sin \omega t_1 \\ &\quad + e^{-c(a-t_1)} (-c \cos \omega a + \omega \sin \omega a)] \quad (30) \end{aligned}$$

Therefore,

$$\lambda \beta \cos \omega t_1 = \frac{2\beta c \cos \omega t_1}{c^2 + \omega^2} + \frac{\beta e^{-ac}(-c \cos \omega a + \omega \sin \omega a)}{c^2 + \omega^2} (e^{ct_1} + e^{-ct_1}) \quad \forall t_1 \quad (31)$$

Comparing both sides of the equation, we get

$$\lambda = \frac{2c}{c^2 + \omega^2} \quad (32)$$

which is consistent with what we obtained earlier.
and

$$-c \cos \omega a + \omega \sin \omega a = 0 \quad (33)$$

$$\implies \tan a\omega = \frac{c}{\omega} \quad (34)$$

Performing a similar computation, we also get

$$\cot a\omega' = -\frac{c}{\omega} \quad (35)$$

Since, $\varphi(t)$ is an orthonormal function, we have to normalize it using the condition

$$\int_{-a}^a \varphi_n(t) \varphi_m(t) dt = \delta[n - m] \quad (36)$$

Putting $m = n$

$$\int_{-a}^a \varphi_n^2(t) dt = \delta[0] = 1 \quad (37)$$

$$\Rightarrow \int_{-a}^a \beta^2 \cos^2 \omega t dt = 1 \quad (38)$$

$$\Rightarrow \beta^2 \int_0^a 2 \cos^2 \omega t dt = 1 \quad (39)$$

$$\Rightarrow \beta^2 \int_0^a (1 + \cos 2\omega t) dt = 1 \quad (40)$$

$$\Rightarrow \beta^2 \left(a + \frac{\sin 2\omega a}{2\omega} \right) = 1 \quad (41)$$

But $\tan a\omega = \frac{c}{\omega} \implies \sin 2a\omega = \frac{2c\omega}{c^2 + \omega^2}$

$$\beta^2 \left(a + \frac{c}{c^2 + \omega^2} \right) = 1 \quad (42)$$

We have found out that $\lambda = \frac{2c}{c^2 + \omega^2}$ Therefore,

$$\beta^2 \left(a + \frac{\lambda}{2} \right) = 1 \quad (43)$$

$$\implies \beta = \left(a + \frac{\lambda}{2} \right)^{-\frac{1}{2}} \quad (44)$$

Performing a similar computation,

$$\beta' = \left(a - \frac{\lambda}{2} \right)^{-\frac{1}{2}} \quad (45)$$

Practical Applications

Brownian Motion

Brownian motion is the random motion of particles suspended in a fluid medium. It can be mathematically modelled using Karhunen-Loeve expansion. If the stochastic random variable $W(t)$ denotes the Brownian motion (Wiener process), then $W(t)$ is given by

$$W(t) = \sqrt{2} \sum_{i \geq 1} \frac{2}{(2i-1)\pi} \sin\left(\left(i - \frac{1}{2}\right)\pi t\right) Z_i \quad (46)$$

where $\{Z_i\}$ are independent and identically distributed Gaussian processes with mean zero and variance one.