Random Numbers
AI1110: Probability and Random Variables
Indian Institute of Technology Hyderabad

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1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/1.1.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 1.1.c
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: Download the following Python code that plots Fig. 1.2

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/1.2.py
```

Run the code by executing

```
python 1.2.py
```
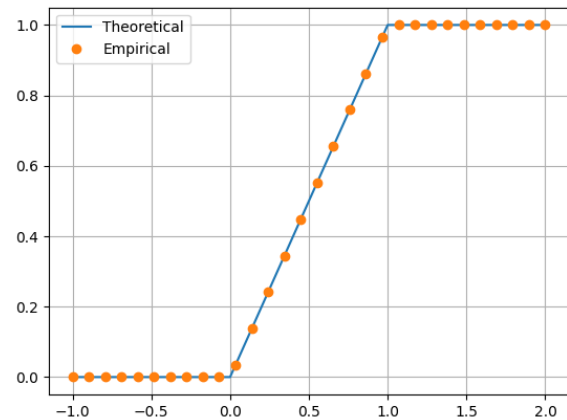
1.3 Find a theoretical expression for $F_U(x)$

Fig. 1.2. The CDF of U

Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, \mathrm{d}x \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{x} 0\, \mathrm{d}x = 0 \tag{1.4}$$

If c,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{x} 1\, \mathrm{d}x \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 1\, dx + \int_{1}^{x} 0\, dx \quad (1.8)$$

$$\int_{-\infty}^{x} p_U(x)\, dx = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \quad (1.12)$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/1.4.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 1.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \quad (1.14)$$
$$\mu_{\text{the}} = 0.500000 \quad (1.15)$$
$$\sigma^2_{\text{emp}} = 0.083301 \quad (1.16)$$
$$\sigma^2_{\text{the}} = 0.083333 \quad (1.17)$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.18)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x\, dF_U(x) \quad (1.19)$$

On differentiating the CDF of U, we get

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \quad (1.20)$$

$$\therefore E[U] = \int_{0}^{1} x\, dx = \frac{1}{2} = 0.5 \quad (1.21)$$

Similarly,

$$\therefore E\left[U^2\right] = \int_{0}^{1} x^2\, dx = \frac{1}{3} \quad (1.22)$$

Now, the variance of U is given by

$$\text{Var}[U] \quad (1.23)$$
$$= E[U - E[U]]^2 \quad (1.24)$$
$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \quad (1.25)$$

By linearity of expectation, we have

$$E\left[U^2\right] + E[-2UE[U]] + E\left[(E[U])^2\right] \quad (1.26)$$
$$= E\left[U^2\right] - 2E[U]E[U] + (E[U])^2 \quad (1.27)$$
$$= E\left[U^2\right] - (E[U])^2 \quad (1.28)$$
$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.29)$$
$$= \frac{1}{12} \approx 0.083333 \quad (1.30)$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/2.1.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```
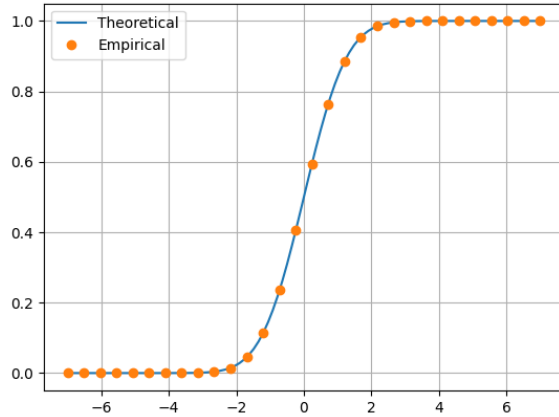
Fig. 2.2. The CDF of X

Fig. 2.3. The PDF of X

Compile and run the C program by executing the following

```
cc −lm 2.1.c
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

Every CDF is monotone increasing and right-continuous. Furthermore,

$$\lim_{x \to -\infty} F_X(x) = 0 \qquad \lim_{x \to \infty} F_X(x) = 1 \qquad (2.2)$$

Thus, every CDF is bounded between 0 and 1 and hence, convergent.
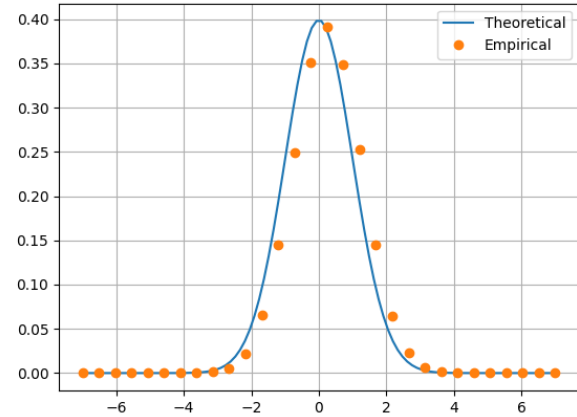
In this case, the CDF is also left-continuous. Therefore, X is a continuous random variable.

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \qquad (2.3)$$

What properties does the PDF have?
Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/2.3.py
```

Run the code by executing

```
python 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x) \, \mathrm{d}x = 1 \qquad (2.4)$$

In this case, the PDF is symmetric about $x = 0$

2.4 Find the mean and variance of X by writing a C program
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/2.4.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 2.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.000294 \qquad (2.5)$$

$$\mu_{\text{the}} = 0.000000 \qquad (2.6)$$

$$\sigma^2_{\text{emp}} = 0.999560 \qquad (2.7)$$

$$\sigma^2_{\text{the}} = 1.000000 \qquad (2.8)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.9)$$

repeat the above exercise theoretically
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \qquad (2.10)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.11)$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad (2.12)$$

$$\implies g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \qquad (2.13)$$

$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad (2.14)$$

$$= -g(x) \qquad (2.15)$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) dx = 0 \qquad (2.16)$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \qquad (2.17)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.18)$$

$$= 2 \int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.19)$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.20)$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) dx \right) \Bigg|_{0}^{\infty}$$

$$- \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.21)$$

Substitute $t = \frac{x^2}{2} \implies dt = x dx$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \qquad (2.22)$$

$$= -\exp(-t) \qquad (2.23)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \qquad (2.24)$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right) \Bigg|_{0}^{\infty} = 0 - 0 = 0 \quad (2.25)$$

$$\because \lim_{x \to \infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x \to \infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \quad (2.26)$$

as exponential function grows much faster than a polynomial function
Also,

$$\int_{0}^{\infty} -\exp\left(-\frac{x^2}{2}\right) dx \qquad (2.27)$$

$$\xleftrightarrow{x = t\sqrt{2}} \int_{0}^{\infty} -\exp(-t^2) dt \sqrt{2} \qquad (2.28)$$

$$= -\sqrt{2} \int_{0}^{\infty} \exp(-t^2) dt \qquad (2.29)$$

$$= -\sqrt{2} \frac{\sqrt{\pi}}{2} \qquad (2.30)$$

$$= -\sqrt{\frac{\pi}{2}} \qquad (2.31)$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}}\right) \qquad (2.32)$$

$$= 1 \qquad (2.33)$$

$$\therefore \text{Var}[X] = E\left[X^2\right] - (E[X])^2 \qquad (2.34)$$

$$= 1 - 0 \qquad (2.35)$$

$$= 1 \qquad (2.36)$$

3. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \qquad (3.1)$$

and plot its CDF
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/3.1.c
```
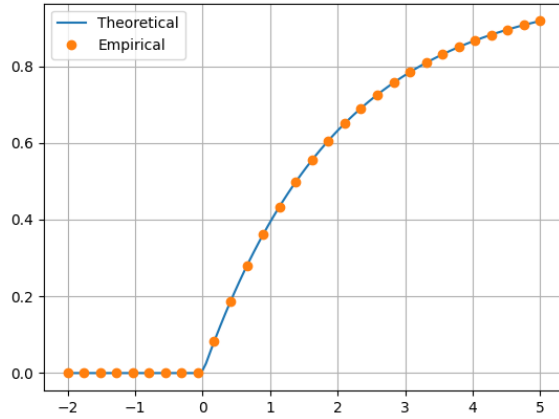
Fig. 3.1. The CDF of V

Compile and run the C program by executing the following

```
cc −lm 3.1.c
./a.out
```

Download the following Python code that plots Fig. 3.1

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/3.1.py
```

Run the code by executing

```
python 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$
   Solution: We have

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

Now,

$$0 \le 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \ge 0 \tag{3.8}$$
$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \tag{3.9}$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{3.10}$$

4. TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/4.1.c
```

Compile and run the C program by executing the following

```
cc −lm 4.1.c
./a.out
```

4.2 Find the CDF of T
   Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

Since $U_1, U_2 \in [0, 1] \implies U_1 + U_2 \in [0, 2]$
Therefore, if $t \ge 2$, then $U_1 + U_2 \le t$ is always true and if $t < 0$, then $U_1 + U_2 \le t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \le t \implies U_2 \le t - x \tag{4.3}$$

If $0 \le t \le 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \le t - x)\, p_{U_1}(x)\mathrm{d}x \tag{4.4}$$
$$= \int_0^t F_{U_2}(t - x)p_{U_1}(x)\mathrm{d}x \tag{4.5}$$
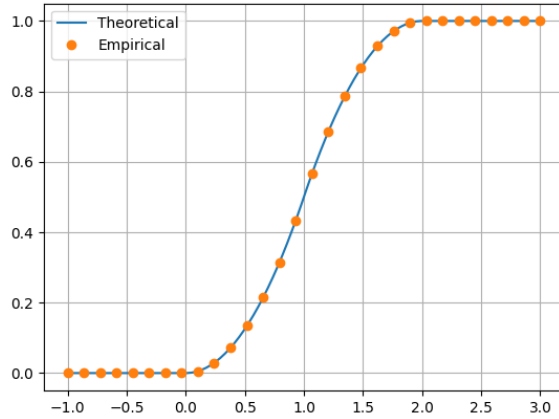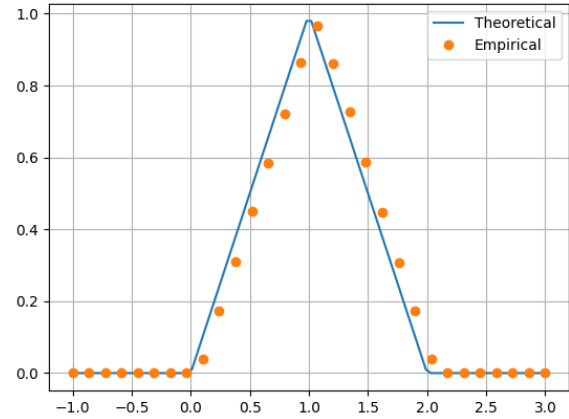
$$0 \le x \le t \implies 0 \le t - x \le t \le 1 \tag{4.6}$$
$$\implies F_{U_2}(t - x) = t - x \tag{4.7}$$

$$F_T(t) = \int_0^t (t - x) \cdot 1 \cdot \mathrm{d}x \tag{4.8}$$
$$= tx - \frac{x^2}{2}\Big|_0^t \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$

Fig. 4.2. The CDF of T

Fig. 4.3. The PDF of T

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t - x) \cdot 1 \cdot dx \qquad (4.11)$$

$$0 \leq x \leq t - 1 \implies 1 \leq t - x \leq t \qquad (4.12)$$

$$t - 1 \leq x \leq 1 \implies 0 < t - 1 \leq t - x \leq 1 \qquad (4.13)$$

$$F_T(t) = \int_0^{t-1} 1 dx + \int_{t-1}^1 (t - x) dx \qquad (4.14)$$

$$= t - 1 + t(1 - (t - 1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \qquad (4.15)$$

$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \qquad (4.16)$$

$$= -\frac{t^2}{2} + 2t - 1 \qquad (4.17)$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \dfrac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \qquad (4.18)$$

4.3 Find the PDF of T

Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \qquad (4.19)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \qquad (4.20)$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution: The theoretical expressions for the CDF and PDF have been found in problems 4.2 and 4.3 respectively

4.5 Verify your results through a plot

Solution: Download the following Python codes that plot Fig. 4.2 and Fig. 4.3

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/4.2.py
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/4.3.py
```

Run the codes by executing

```
python 4.2.py
python 4.3.py
```

5. Maximum Likelihood

5.1 Generate equiprobable $X \in \{-1, 1\}$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/5.1.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 5.1.c
./a.out
```

5.2 Generate

$$Y = AX + N \qquad (5.1)$$

where $A = 5$ dB, $X \in \{-1, 1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/5.2.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 5.2.c
./a.out
```

5.3 Plot Y

Solution: Download the following Python code that plots Fig. 5.3

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/5.3.py
```

Run the code by executing

```
python 5.3.py
```

5.4 Guess how to estimate X from Y

Solution:

$$X = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \qquad (5.2)$$
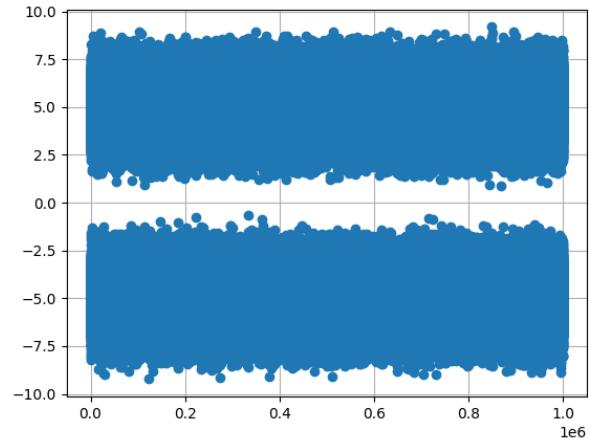
5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \qquad (5.3)$$

Fig. 5.3. Plot of Y

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \qquad (5.4)$$

Solution:

$$\Pr\left(\hat{X} = -1 | X = 1\right) = \Pr\left(Y < 0 | X = 1\right) \quad (5.5)$$
$$= \Pr\left(A + N < 0\right) \qquad (5.6)$$
$$= \Pr\left(N < -5\right) \qquad (5.7)$$
$$= 1 - \Pr\left(N > -5\right) \qquad (5.8)$$
$$= 1 - Q(-5) \qquad (5.9)$$
$$= Q(5) \qquad (5.10)$$

where $Q(x) = \Pr(N > x)$ is the Q-function

$$Q(x) = 1 - Q(-x) \qquad \forall x \in \mathbb{R} \qquad (5.11)$$

$$\Pr\left(\hat{X} = 1 | X = -1\right) = \Pr\left(Y > 0 | X = -1\right)$$
$$\qquad (5.12)$$
$$= \Pr\left(-A + N > 0\right) \quad (5.13)$$
$$= \Pr\left(N > 5\right) \qquad (5.14)$$
$$= Q(5) \qquad (5.15)$$

5.6 Find P_e assuming that X has equiprobable symbols

Solution:

$$P_e = \Pr(X = -1) P_{e|0} + \Pr(X = 1) P_{e|1} \quad (5.16)$$

Since X has equiprobable symbols, $\Pr(X = -1) = \Pr(X = 1) = \frac{1}{2}$

$$P_e = \frac{1}{2}Q(5) + \frac{1}{2}Q(5) \qquad (5.17)$$
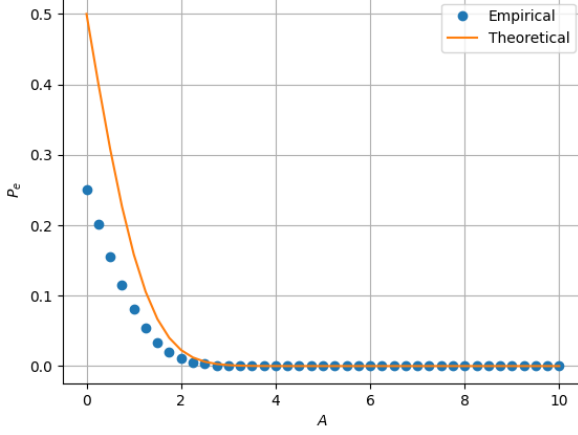$$= Q(5) \qquad (5.18)$$

Fig. 5.7. Plot of P_e

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB
Solution: Download the following Python code that plots Fig. 5.7

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/5.7.py
```

Run the code by executing

```
python 5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e
Solution:

$$X = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \quad (5.19)$$

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \quad (5.20)$$
$$= \Pr\left(Y < \delta | X = 1\right) \quad (5.21)$$
$$= \Pr\left(A + N < \delta\right) \quad (5.22)$$
$$= \Pr\left(N < \delta - 5\right) \quad (5.23)$$
$$= Q(5 - \delta) \quad (5.24)$$
$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \quad (5.25)$$
$$= \Pr\left(Y > \delta | X = -1\right) \quad (5.26)$$
$$= \Pr\left(-A + N > \delta\right) \quad (5.27)$$
$$= \Pr\left(N > \delta + 5\right) \quad (5.28)$$
$$= Q(5 + \delta) \quad (5.29)$$
$$\quad (5.30)$$

Now, P_e is given by

$$P_e = \Pr\left(X = -1\right)P_{e|0} + \Pr\left(X = 1\right)P_{e|1} \quad (5.31)$$
$$= \frac{1}{2}Q(5 - \delta) + \frac{1}{2}Q(5 + \delta) \quad (5.32)$$
$$= \frac{Q(5 - \delta) + Q(5 + \delta)}{2} \quad (5.33)$$
$$= g(\delta) \quad (5.34)$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{Q'(5 + \delta) - Q'(5 - \delta)}{2} \quad (5.35)$$

Recall the definition of $Q(x)$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (5.36)$$
$$\implies Q'(x) = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (5.37)$$

Thus,

$$g'(\delta) = \frac{\exp\left(-\frac{(5-\delta)^2}{2}\right) - \exp\left(-\frac{(5+\delta)^2}{2}\right)}{2\sqrt{2\pi}} \quad (5.38)$$

$$g'(\delta) = 0 \implies (5 - \delta)^2 = (5 + \delta)^2 \quad (5.39)$$
$$\implies |5 - \delta| = |5 + \delta| \quad (5.40)$$
$$\implies \delta = 0 \quad (5.41)$$

$$g''(\delta) = \frac{(5 - \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(5 - \delta)^2}{2}\right)$$
$$+ \frac{(5 + \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(5 + \delta)^2}{2}\right) \quad (5.42)$$

$$g''(0) = \frac{5}{\sqrt{2\pi}} \exp\left(-\frac{25}{2}\right) > 0 \quad (5.43)$$

Therefore, $\hat{\delta} = 0$ is a minima and it is what minimizes P_e

5.9 Repeat the above exercise when

$$p_X(-1) = p \quad (5.44)$$

Solution:

$$P_e = p_X(1)P_{e|0} + p_X(-1)P_{e|1} \quad (5.45)$$
$$= (1 - p)Q(5 - \delta) + pQ(5 + \delta) \quad (5.46)$$
$$= g(\delta) \quad (5.47)$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{(1-p)\exp\left(-\frac{(5-\delta)^2}{2}\right) - p\exp\left(-\frac{(5+\delta)^2}{2}\right)}{\sqrt{2\pi}}$$
(5.48)

$g'(\delta) = 0$ when

$$(1-p)\exp\left(-\frac{(5-\delta)^2}{2}\right) = p\exp\left(-\frac{(5+\delta)^2}{2}\right)$$
(5.49)

$$\implies \exp\left(\frac{(5+\delta)^2 - (5-\delta)^2}{2}\right) = \frac{p}{1-p}$$
(5.50)

$$\implies \exp(10\delta) = \frac{p}{1-p}$$
(5.51)

$$\therefore \hat{\delta} = \frac{1}{10}\ln\frac{p}{1-p}$$
(5.52)

5.10 Repeat the above exercise using the MAP criterion
Solution: The PDF of $X|Y$ is given by

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)}$$
(5.53)

Assuming X has equiprobable symbols, $p_X(x) = \frac{1}{2}, \quad x = -1, 1$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1)$$
(5.54)

$$= \frac{1}{2}\Pr(-A + N = y) + \frac{1}{2}\Pr(A + N = y)$$
(5.55)

$$= \frac{p_N(y+5) + p_N(y-5)}{2}$$
(5.56)

$$= \frac{\exp\left(-\frac{(y+5)^2}{2}\right) + \exp\left(-\frac{(y-5)^2}{2}\right)}{2\sqrt{2\pi}}$$
(5.57)

(5.58)

Now,

$$p_{X|Y}(1|y) = \frac{\Pr(A + N = y)\,p_X(1)}{p_Y(y)}$$
(5.59)

$$= \frac{p_N(y-5)p_X(1)}{p_Y(y)}$$
(5.60)

$$= \frac{\exp\left(-\frac{(y-5)^2}{2}\right)}{2\left(\exp\left(-\frac{(y+5)^2}{2}\right) + \exp\left(-\frac{(y-5)^2}{2}\right)\right)}$$
(5.61)

$$= \frac{1}{2\left(1 + \exp\left(\frac{(y-5)^2 - (y+5)^2}{2}\right)\right)}$$
(5.62)

$$= \frac{1}{2(1 + \exp(-10y))}$$
(5.63)

Similarly,

$$p_{X|Y}(-1|y) = \frac{1}{2(1 + \exp(10y))}$$
(5.64)

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y)$$
(5.65)

$$\iff \frac{1}{2(1 + \exp(-10y))} > \frac{1}{2(1 + \exp(10y))}$$
(5.66)

$$\iff \exp(-10y) < \exp(10y)$$
(5.67)

$$\iff y > 0$$
(5.68)

And $p_{X|Y}(1|y) < p_{X|Y}(-1|y) \iff y < 0$
Therefore, $X = 1$ is more probable than $X = -1$ when $Y > 0$ and vice versa
Consider now a general Bernoulli random variable X with $p_X(-1) = p, p_X(1) = 1 - p$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1)$$
(5.69)

$$= pp_N(y+5) + (1-p)p_N(y-5)$$
(5.70)

$$= \frac{p\exp\left(-\frac{(y+5)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-5)^2}{2}\right)}{\sqrt{2\pi}}$$
(5.71)

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_Y(y)}$$
(5.72)

$$= \frac{(1-p)\exp\left(-\frac{(y-5)^2}{2}\right)}{p\exp\left(-\frac{(y+5)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-5)^2}{2}\right)}$$
(5.73)

$$= \frac{1-p}{1 - p + p\exp(-10y)}$$
(5.74)

Similarly,

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)\exp(10y)} \quad (5.75)$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \quad (5.76)$$

$$\iff \frac{1-p}{1-p+p\exp(-10y)} > \frac{p}{p+(1-p)\exp(10y)} \quad (5.77)$$

$$\iff 1 + \frac{p}{1-p}\exp(-10y) < 1 + \frac{1-p}{p}\exp(10y) \quad (5.78)$$

$$\iff \exp(20y) > \left(\frac{p}{1-p}\right)^2 \quad (5.79)$$

$$\iff y > \frac{1}{10}\ln\frac{p}{1-p} = \hat{\delta} \quad (5.80)$$

and

$$p_{X|Y}(1|y) < p_{X|Y}(-1|y) \quad (5.81)$$

$$\iff \frac{1-p}{1-p+p\exp(-10y)} < \frac{p}{p+(1-p)\exp(10y)} \quad (5.82)$$

$$\iff 1 + \frac{p}{1-p}\exp(-10y) > 1 + \frac{1-p}{p}\exp(10y) \quad (5.83)$$

$$\iff \exp(20y) < \left(\frac{p}{1-p}\right)^2 \quad (5.84)$$

$$\iff y < \frac{1}{10}\ln\frac{p}{1-p} = \hat{\delta} \quad (5.85)$$

Therefore, $X = 1$ is more probable than $X = -1$ when $Y > \hat{\delta}$ and vice versa

6. GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/6.1.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```
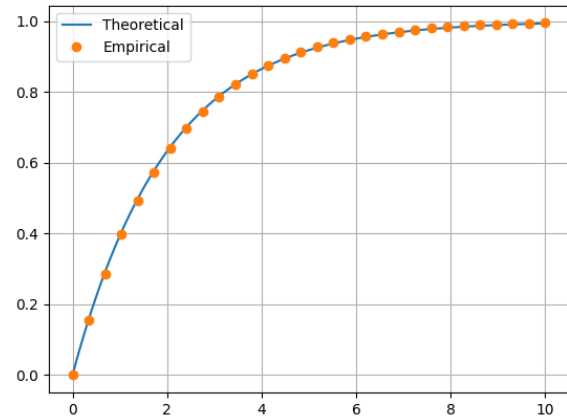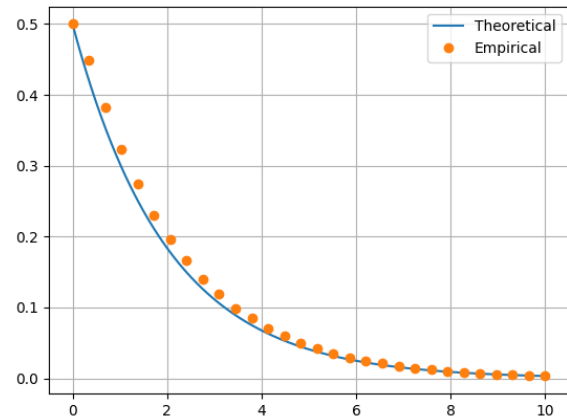

Fig. 6.1. CDF of V

Fig. 6.1. PDF of V

Compile and run the C program by executing the following

```
cc −lm 6.1.c
./a.out
```

Download the following Python code that plots Fig. 6.1

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/6.1.py
```

Run the code by executing

```
python 6.1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α

Solution: Let $R \geq 0, \Theta \in [0, 2\pi]$

$$X_1 = R \cos \Theta \quad (6.3)$$
$$X_2 = R \sin \Theta \quad (6.4)$$

such that $V = X_1^2 + X_2^2 = R^2$

The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (6.5)$$

$$= \begin{pmatrix} \cos \Theta & -R \sin \Theta \\ \sin \Theta & R \cos \Theta \end{pmatrix} \quad (6.6)$$

$$\implies |\mathbf{J}| = R \cos^2 \Theta + R \sin^2 \Theta = R \quad (6.7)$$

Then,

$$p_{R,\Theta}(r, \theta) = p_{X_1, X_2}(x_1, x_2) |\mathbf{J}| \quad (6.8)$$
$$= p_{X_1, X_2}(x_1, x_2) r \quad (6.9)$$
$$= r p_{X_1}(x_1) p_{X_2}(x_2) \quad (6.10)$$

since X_1 and X_2 are independent

$$p_{R,\Theta}(r, \theta) = r \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2}\right) \quad (6.11)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{(x_1 + x_2)^2}{2}\right) \quad (6.12)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \quad (6.13)$$

The marginal distribution of R is given by

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r, \theta) \, d\theta \quad (6.14)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \int_0^{2\pi} d\theta \quad (6.15)$$

$$= r \exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \quad (6.16)$$

The CDF of R is thus given by

$$F_R(r) = \Pr(R \leq r) \quad (6.17)$$

$$= \int_0^r p_R(r) dr \quad (6.18)$$

$$= \int_0^r r \exp\left(-\frac{r^2}{2}\right) dr \quad (6.19)$$

$$= -\int_0^r \exp\left(-\frac{r^2}{2}\right)(-r dr) \quad (6.20)$$

$$= -\int_0^r \exp\left(-\frac{r^2}{2}\right) d\left(-\frac{r^2}{2}\right) \quad (6.21)$$

$$= -\exp\left(-\frac{r^2}{2}\right)\Big|_0^r \quad (6.22)$$

$$= 1 - \exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \quad (6.23)$$

Now, $V = R^2$, hence the CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \quad (6.24)$$

$$= \Pr(R^2 \leq x) \quad (6.25)$$

$$= \Pr(|R| \leq \sqrt{x}) \quad (6.26)$$

$$= \Pr(R \leq \sqrt{x}) \quad (\because R \geq 0) \quad (6.27)$$

$$= p_R(\sqrt{x}) \quad (6.28)$$

$$= 1 - \exp\left(-\frac{x}{2}\right) \quad \text{for } x \geq 0 \quad (6.29)$$

And the PDF of V is given by

$$p_V(x) = \frac{d}{dx} F_V(x) \quad (6.30)$$

$$= \frac{d}{dx} F_V(x)\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (6.31)$$

$$= \frac{1}{2} \exp\left(-\frac{x}{2}\right) \quad (6.32)$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.33)$$

$$p_V(x) = \begin{cases} \frac{1}{2} \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.34)$$

$$\therefore \alpha = \frac{1}{2} \quad (6.35)$$

6.3 Plot the CDF and PDF of
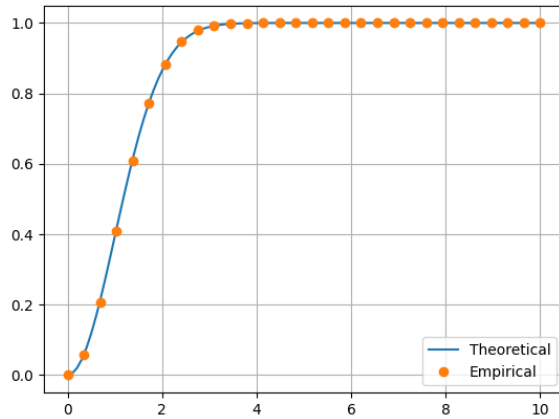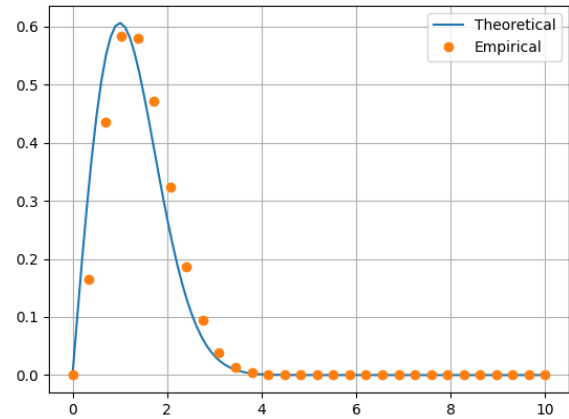
$$A = \sqrt{V} \quad (6.36)$$

Fig. 6.3. CDF of A

Fig. 6.3. PDF of A

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/6.3.c
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc −lm 6.3.c
./a.out
```

Download the following Python code that plots Fig. 6.3

```
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/6.3.py
```

Run the code by executing

```
python 6.3.py
```

The CDF of A for $x \geq 0$ is given by

$$F_A(x) = \Pr(A \leq x) \tag{6.37}$$

$$= \Pr\left(\sqrt{V} \leq x\right) \tag{6.38}$$

$$= \Pr\left(V \leq x^2\right) \tag{6.39}$$

$$= F_V(x^2) \tag{6.40}$$

$$= 1 - \exp\left(-\frac{x^2}{2}\right) \tag{6.41}$$

The PDF of A is given by

$$p_A(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_A(x) \tag{6.42}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \exp\left(-\frac{x^2}{2}\right)\right) \tag{6.43}$$

$$= x \exp\left(-\frac{x^2}{2}\right) \tag{6.44}$$

Therefore,

$$F_A(x) = \begin{cases} 1 - \exp\left(-\dfrac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.45}$$

$$p_A(x) = \begin{cases} x \exp\left(-\dfrac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.46}$$