Random Numbers

AI1110: Probability and Random Variables

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1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.1.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 1.1.c
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: Download the following Python code that plots Fig. 1.2

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.2.py
```

Run the code by executing

```
python 1.2.py
```

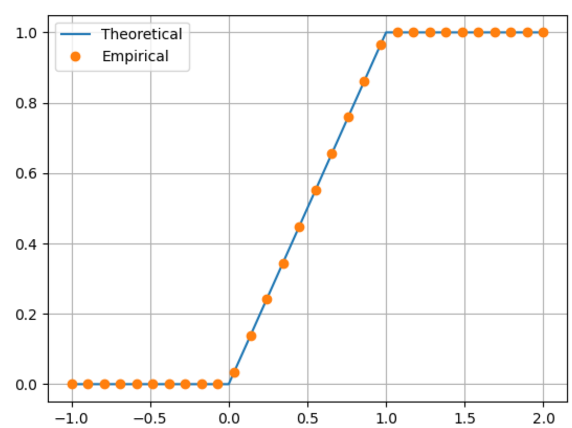

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$

Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, \mathrm{d}x \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{x} 0\, \mathrm{d}x = 0 \tag{1.4}$$

If c,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{x} 1\, \mathrm{d}x \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x$$
$$= \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{1} 1\ \mathrm{d}x + \int_{1}^{x} 0\ \mathrm{d}x \quad (1.8)$$
$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = 0 + 1 + 0 \quad (1.9)$$
$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \quad (1.12)$$

and its variance as

$$\mathrm{Var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.4.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 1.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \quad (1.14)$$
$$\mu_{\text{the}} = 0.500000 \quad (1.15)$$
$$\sigma^2_{\text{emp}} = 0.083301 \quad (1.16)$$
$$\sigma^2_{\text{the}} = 0.083333 \quad (1.17)$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \quad (1.18)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x\ \mathrm{d}F_U(x) \quad (1.19)$$

On differentiating the CDF of U, we get

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \quad (1.20)$$

$$\therefore E[U] = \int_{0}^{1} x\ \mathrm{d}x = \frac{1}{2} = 0.5 \quad (1.21)$$

Similarly,

$$\therefore E\left[U^2\right] = \int_{0}^{1} x^2\ \mathrm{d}x = \frac{1}{3} \quad (1.22)$$

Now, the variance of U is given by

$$\mathrm{Var}[U] \quad (1.23)$$
$$= E[U - E[U]]^2 \quad (1.24)$$
$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \quad (1.25)$$

By linearity of expectation, we have

$$E\left[U^2\right] + E[-2UE[U]] + E\left[(E[U])^2\right] \quad (1.26)$$
$$= E\left[U^2\right] - 2E[U]E[U] + (E[U])^2 \quad (1.27)$$
$$= E\left[U^2\right] - (E[U])^2 \quad (1.28)$$
$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.29)$$
$$= \frac{1}{12} \approx 0.083333 \quad (1.30)$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/2.1.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

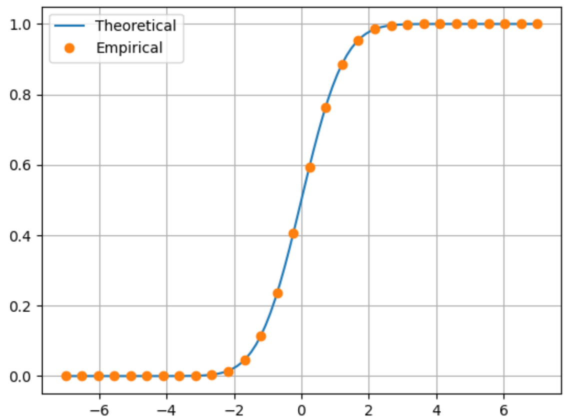

Fig. 2.2. The CDF of X

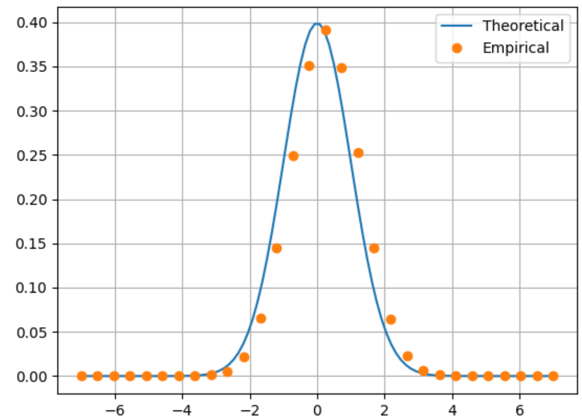

Fig. 2.3. The PDF of X

Compile and run the C program by executing the following

```
cc -lm 2.1.c
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

Every CDF is monotone increasing and right-continuous. Furthermore,

$$\lim_{x\to-\infty} F_X(x) = 0 \qquad \lim_{x\to\infty} F_X(x) = 1 \tag{2.2}$$

Thus, every CDF is bounded between 0 and 1 and hence, convergent.
In this case, the CDF is also left-continuous. Therefore, X is a continuous random variable.

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \tag{2.3}$$

What properties does the PDF have?
Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/2.3.py
```

Run the code by executing

```
python 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x)\ \mathrm{d}x = 1 \tag{2.4}$$

In this case, the PDF is symmetric about $x = 0$

2.4 Find the mean and variance of X by writing a C program
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/2.4.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 2.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.000294 \tag{2.5}$$
$$\mu_{\text{the}} = 0.000000 \tag{2.6}$$
$$\sigma^2_{\text{emp}} = 0.999560 \tag{2.7}$$
$$\sigma^2_{\text{the}} = 1.000000 \tag{2.8}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.9}$$

repeat the above exercise theoretically
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x \tag{2.10}$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.11}$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.12}$$
$$\implies g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \tag{2.13}$$
$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.14}$$
$$= -g(x) \tag{2.15}$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) \mathrm{d}x = 0 \tag{2.16}$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) \mathrm{d}x \tag{2.17}$$
$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.18}$$
$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.19}$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.20}$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \right)\Bigg|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.21}$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int \exp(-t) \mathrm{d}t \tag{2.22}$$
$$= -\exp(-t) \tag{2.23}$$
$$= -\exp\left(-\frac{x^2}{2}\right) \tag{2.24}$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right)\Bigg|_0^{\infty} = 0 - 0 = 0 \tag{2.25}$$
$$\because \lim_{x\to\infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x\to\infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \tag{2.26}$$

as exponential function grows much faster than a polynomial function
Also,

$$\int_0^{\infty} -\exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.27}$$
$$\xleftrightarrow{x=t\sqrt{2}} \int_0^{\infty} -\exp(-t^2) \mathrm{d}t \sqrt{2} \tag{2.28}$$
$$= -\sqrt{2} \int_0^{\infty} \exp(-t^2) \mathrm{d}t \tag{2.29}$$
$$= -\sqrt{2} \frac{\sqrt{\pi}}{2} \tag{2.30}$$
$$= -\sqrt{\frac{\pi}{2}} \tag{2.31}$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}}\right) \tag{2.32}$$
$$= 1 \tag{2.33}$$
$$\therefore \mathrm{Var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.34}$$
$$= 1 - 0 \tag{2.35}$$
$$= 1 \tag{2.36}$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{3.1}$$

and plot its CDF
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/3.1.c
```

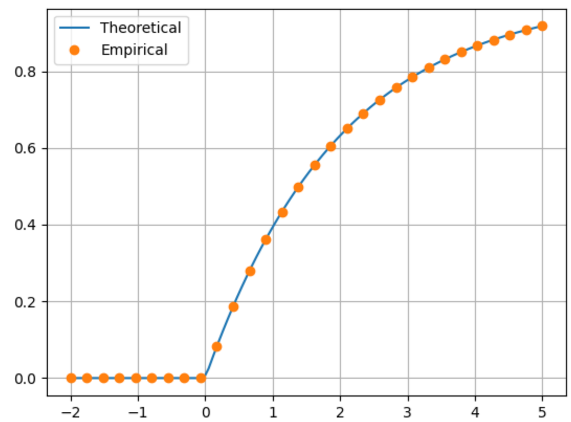

Fig. 3.1. The CDF of V

Compile and run the C program by executing the following

```
cc -lm 3.1.c
./a.out
```

Download the following Python code that plots Fig. 3.1

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/3.1.py
```

Run the code by executing

```
python 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$
Solution: We have

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

Now,

$$0 \le 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \ge 0 \tag{3.8}$$
$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \tag{3.9}$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{3.10}$$

4. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/4.1.c
```

Compile and run the C program by executing the following

```
cc -lm 4.1.c
./a.out
```

4.2 Find the CDF of T
Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$
Therefore, if $t \ge 2$, then $U_1 + U_2 \le t$ is always true and if $t < 0$, then $U_1 + U_2 \le t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \le t \implies U_2 \le t - x \tag{4.3}$$

If $0 \le t \le 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \le t - x)\, p_{U_1}(x) \mathrm{d}x \tag{4.4}$$
$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x) \mathrm{d}x \tag{4.5}$$

$$0 \le x \le t \implies 0 \le t - x \le t \le 1 \tag{4.6}$$
$$\implies F_{U_2}(t-x) = t - x \tag{4.7}$$

$$F_T(t) = \int_0^t (t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.8}$$
$$= tx - \frac{x^2}{2}\Big|_0^t \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$

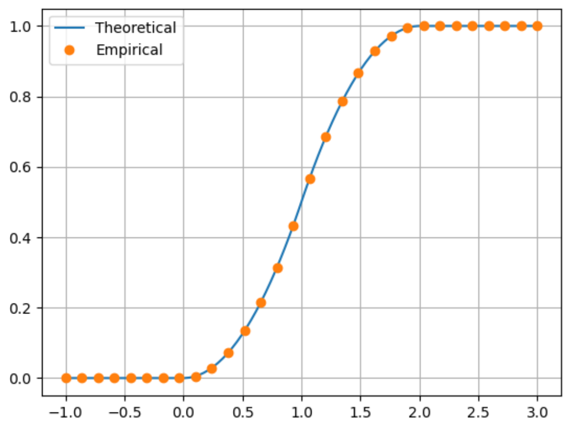

Fig. 4.2. The CDF of T

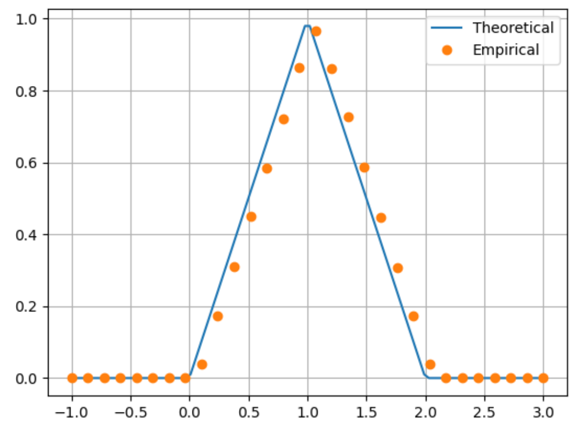

Fig. 4.3. The PDF of T

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.11}$$

$$0 \leq x \leq t-1 \implies 1 \leq t-x \leq t \tag{4.12}$$

$$t-1 \leq x \leq 1 \implies 0 < t-1 \leq t-x \leq 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1\mathrm{d}x + \int_{t-1}^1 (t-x)\mathrm{d}x \tag{4.14}$$

$$= t-1+t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$

$$= t-1+2t-t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.16}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.18}$$

4.3 Find the PDF of T

Solution: The PDF of T is given by

$$p_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) \tag{4.19}$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.20}$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution: The theoretical expressions for the CDF and PDF have been found in problems 4.2 and 4.3 respectively

4.5 Verify your results through a plot

Solution: Download the following Python codes that plot Fig. 4.2 and Fig. 4.3

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/4.2.py
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/4.3.py
```

Run the codes by executing

```
python 4.2.py
python 4.3.py
```

5. Maximal Likelihood

5.1 Generate equiprobable $X \in \{-1, 1\}$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/5.1.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 5.1.c
./a.out
```

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{-1, 1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/5.2.c
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 5.2.c
./a.out
```

5.3 Plot Y

Solution: Download the following Python code that plots Fig. 5.3

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/5.3.py
```

Run the code by executing

```
python 5.3.py
```

5.4 Guess how to estimate X from Y

Solution:

$$X = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

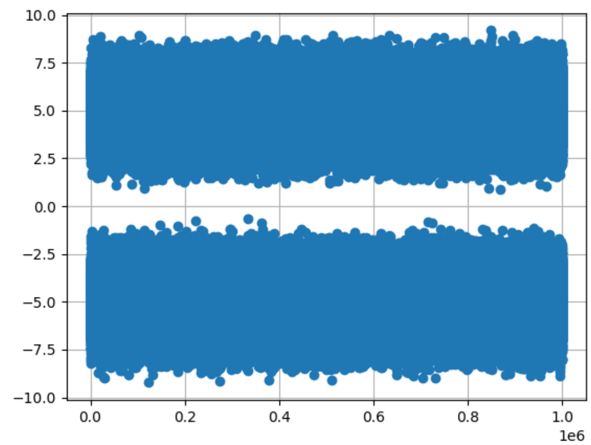

Fig. 5.3. Plot of Y

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution:

$$\begin{aligned} \Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) && (5.5) \\ &= \Pr\left(A + N < 0\right) && (5.6) \\ &= \Pr\left(N < -5\right) && (5.7) \\ &= 1 - \Pr\left(N > -5\right) && (5.8) \\ &= 1 - Q(-5) && (5.9) \\ &= Q(5) && (5.10) \end{aligned}$$

where $Q(x) = \Pr\left(N > x\right)$ is the Q-function

$$Q(x) = 1 - Q(-x) \qquad \forall x \in \mathbb{R} \tag{5.11}$$

$$\begin{aligned} \Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) && (5.12) \\ &= \Pr\left(-A + N > 0\right) && (5.13) \\ &= \Pr\left(N > 5\right) && (5.14) \\ &= Q(5) && (5.15) \end{aligned}$$

5.6 Find P_e assuming that X has equiprobable symbols

Solution:

$$P_e = \Pr\left(X = -1\right) P_{e|0} + \Pr\left(X = 1\right) P_{e|1} \tag{5.16}$$

Since X has equiprobable symbols, $\Pr\left(X = -1\right) = \Pr\left(X = 1\right) = \frac{1}{2}$

$$\begin{aligned} P_e &= \frac{1}{2}Q(5) + \frac{1}{2}Q(5) && (5.17) \\ &= Q(5) && (5.18) \end{aligned}$$

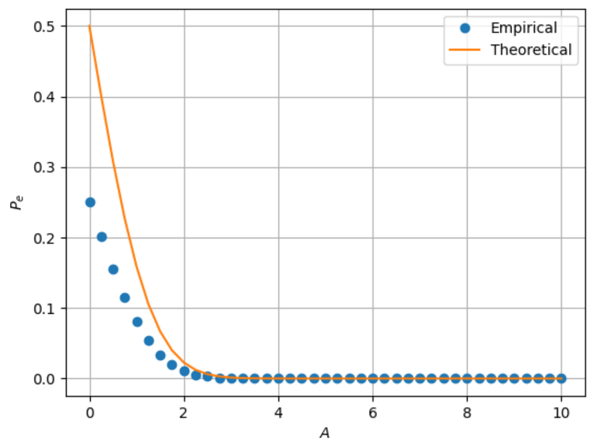

Fig. 5.7. Plot of P_e

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB
Solution: Download the following Python code that plots Fig. 5.7

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/5.7.py
```

Run the code by executing

```
python 5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e
Solution:

$$X = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \tag{5.19}$$

$$\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (5.20)\\
&= \Pr\left(Y < \delta | X = 1\right) && (5.21)\\
&= \Pr\left(A + N < \delta\right) && (5.22)\\
&= \Pr\left(N < \delta - 5\right) && (5.23)\\
&= Q(5 - \delta) && (5.24)\\
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) && (5.25)\\
&= \Pr\left(Y > \delta | X = -1\right) && (5.26)\\
&= \Pr\left(-A + N > \delta\right) && (5.27)\\
&= \Pr\left(N > \delta + 5\right) && (5.28)\\
&= Q(5 + \delta) && (5.29)\\
& && (5.30)
\end{aligned}$$

Now, P_e is given by

$$\begin{aligned}
P_e &= \Pr\left(X = -1\right) P_{e|0} + \Pr\left(X = 1\right) P_{e|1} && (5.31)\\
&= \frac{1}{2} Q(5 - \delta) + \frac{1}{2} Q(5 + \delta) && (5.32)\\
&= \frac{Q(5 - \delta) + Q(5 + \delta)}{2} && (5.33)\\
&= g(\delta) && (5.34)
\end{aligned}$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{Q'(5 + \delta) - Q'(5 - \delta)}{2} \tag{5.35}$$

Recall the definition of $Q(x)$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{5.36}$$

$$\implies Q'(x) = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{5.37}$$

Thus,

$$g'(\delta) = \frac{\exp\left(-\frac{(5-\delta)^2}{2}\right) - \exp\left(-\frac{(5+\delta)^2}{2}\right)}{2\sqrt{2\pi}} \tag{5.38}$$

$$\begin{aligned}
g'(\delta) = 0 &\implies (5 - \delta)^2 = (5 + \delta)^2 && (5.39)\\
&\implies |5 - \delta| = |5 + \delta| && (5.40)\\
&\implies \delta = 0 && (5.41)
\end{aligned}$$

$$\begin{aligned}
g''(\delta) = \frac{(5 - \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(5 - \delta)^2}{2}\right) \\
+ \frac{(5 + \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(5 + \delta)^2}{2}\right) && (5.42)
\end{aligned}$$

$$g''(0) = \frac{5}{\sqrt{2\pi}} \exp\left(-\frac{25}{2}\right) > 0 \tag{5.43}$$

Therefore, $\hat{\delta} = 0$ is a minima and it is what minimizes P_e

5.9 Repeat the above exercise when

$$p_X(-1) = p \tag{5.44}$$

Solution:

$$\begin{aligned}
P_e &= p_X(1) P_{e|0} + p_X(-1) P_{e|1} && (5.45)\\
&= (1 - p) Q(5 - \delta) + p Q(5 + \delta) && (5.46)\\
&= g(\delta) && (5.47)
\end{aligned}$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{(1-p)\exp\left(-\frac{(5-\delta)^2}{2}\right) - p\exp\left(-\frac{(5+\delta)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.48}$$

$g'(\delta) = 0$ when

$$(1-p)\exp\left(-\frac{(5-\delta)^2}{2}\right) = p\exp\left(-\frac{(5+\delta)^2}{2}\right) \tag{5.49}$$

$$\implies \exp\left(\frac{(5+\delta)^2-(5-\delta)^2}{2}\right) = \frac{p}{1-p} \tag{5.50}$$

$$\implies \exp(10\delta) = \frac{p}{1-p} \tag{5.51}$$

$$\therefore \hat{\delta} = \frac{1}{10}\ln\frac{p}{1-p} \tag{5.52}$$

5.10 Repeat the above exercise using the MAP criterion
Solution: The PDF of $X|Y$ is given by

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \tag{5.53}$$

Assuming X has equiprobable symbols, $p_X(x) = \frac{1}{2}, \quad x = -1, 1$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \tag{5.54}$$

$$= \frac{1}{2}\Pr(-A+N=y) + \frac{1}{2}\Pr(A+N=y) \tag{5.55}$$

$$= \frac{p_N(y+5)+p_N(y-5)}{2} \tag{5.56}$$

$$= \frac{\exp\left(-\frac{(y+5)^2}{2}\right) + \exp\left(-\frac{(y-5)^2}{2}\right)}{2\sqrt{2\pi}} \tag{5.57}$$

(5.58)

Now,

$$p_{X|Y}(1|y) = \frac{\Pr(A+N=y)\,p_X(1)}{p_Y(y)} \tag{5.59}$$

$$= \frac{p_N(y-5)p_X(1)}{p_Y(y)} \tag{5.60}$$

$$= \frac{\exp\left(-\frac{(y-5)^2}{2}\right)}{2\left(\exp\left(-\frac{(y+5)^2}{2}\right) + \exp\left(-\frac{(y-5)^2}{2}\right)\right)} \tag{5.61}$$

$$= \frac{1}{2\left(1+\exp\left(\frac{(y-5)^2-(y+5)^2}{2}\right)\right)} \tag{5.62}$$

$$= \frac{1}{2(1+\exp(-10y))} \tag{5.63}$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{1}{2(1+\exp(10y))} \tag{5.64}$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \tag{5.65}$$

$$\iff \frac{1}{2(1+\exp(-10y))} > \frac{1}{2(1+\exp(10y))} \tag{5.66}$$

$$\iff \exp(-10y) < \exp(10y) \tag{5.67}$$

$$\iff y > 0 \tag{5.68}$$

And $p_{X|Y}(1|y) < p_{X|Y}(-1|y) \iff y < 0$
Therefore, $X = 1$ is more probable than $X = -1$ when $Y > 0$ and vice versa
Consider now a general Bernoulli random variable X with $p_X(-1) = p, p_X(1) = 1-p$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \tag{5.69}$$

$$= p\,p_N(y+5) + (1-p)p_N(y-5) \tag{5.70}$$

$$= \frac{p\exp\left(-\frac{(y+5)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-5)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.71}$$

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_Y(y)} \tag{5.72}$$

$$= \frac{(1-p)\exp\left(-\frac{(y-5)^2}{2}\right)}{p\exp\left(-\frac{(y+5)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-5)^2}{2}\right)} \tag{5.73}$$

$$= \frac{1-p}{1-p+p\exp(-10y)} \tag{5.74}$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)\exp(10y)} \tag{5.75}$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \tag{5.76}$$

$$\iff \frac{1-p}{1-p+p\exp(-10y)} > \frac{p}{p+(1-p)\exp(10y)} \tag{5.77}$$

$$\iff 1 + \frac{p}{1-p}\exp(-10y) < 1 + \frac{1-p}{p}\exp(10y) \tag{5.78}$$

$$\iff \exp(20y) > \left(\frac{p}{1-p}\right)^2 \tag{5.79}$$

$$\iff y > \frac{1}{10}\ln\frac{p}{1-p} = \hat{\delta} \tag{5.80}$$

and

$$p_{X|Y}(1|y) < p_{X|Y}(-1|y) \tag{5.81}$$

$$\iff \frac{1-p}{1-p+p\exp(-10y)} < \frac{p}{p+(1-p)\exp(10y)} \tag{5.82}$$

$$\iff 1 + \frac{p}{1-p}\exp(-10y) > 1 + \frac{1-p}{p}\exp(10y) \tag{5.83}$$

$$\iff \exp(20y) < \left(\frac{p}{1-p}\right)^2 \tag{5.84}$$

$$\iff y < \frac{1}{10}\ln\frac{p}{1-p} = \hat{\delta} \tag{5.85}$$

Therefore, $X = 1$ is more probable than $X = -1$ when $Y > \hat{\delta}$ and vice versa