

Random Numbers

AI1110: Probability and Random Variables

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30 Jun 2022

1. UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/1.1.c
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 1.1.c
./a.out
```

- 1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: Download the following Python code that plots Fig. 1.2

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/1.2.py
```

Run the code by executing

```
python 1.2.py
```

- 1.3 Find a theoretical expression for $F_U(x)$

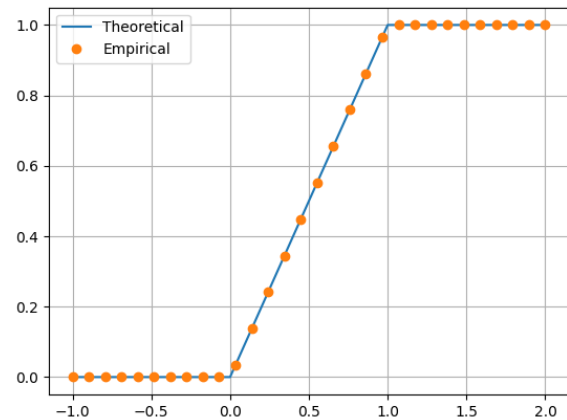


Fig. 1.2. The CDF of U

Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

If $x < 0$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^x 0 dx = 0 \quad (1.4)$$

If c ,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.5)$$

$$= 0 + x \quad (1.6)$$

$$= x \quad (1.7)$$

If $x > 1$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \quad (1.8)$$

$$\int_{-\infty}^x p_U(x) dx = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.12)$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/1.4.c
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 1.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \quad (1.14)$$

$$\mu_{\text{the}} = 0.500000 \quad (1.15)$$

$$\sigma_{\text{emp}}^2 = 0.083301 \quad (1.16)$$

$$\sigma_{\text{the}}^2 = 0.083333 \quad (1.17)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.18)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.19)$$

On differentiating the CDF of U , we get

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \quad (1.20)$$

$$\therefore E[U] = \int_0^1 x dx = \frac{1}{2} = 0.5 \quad (1.21)$$

Similarly,

$$\therefore E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.22)$$

Now, the variance of U is given by

$$\text{Var}[U] \quad (1.23)$$

$$= E[U - E[U]]^2 \quad (1.24)$$

$$= E[U^2 - 2UE[U] + (E[U])^2] \quad (1.25)$$

By linearity of expectation, we have

$$E[U^2] + E[-2UE[U]] + E[(E[U])^2] \quad (1.26)$$

$$= E[U^2] - 2E[U]E[U] + (E[U])^2 \quad (1.27)$$

$$= E[U^2] - (E[U])^2 \quad (1.28)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.29)$$

$$= \frac{1}{12} \approx 0.083333 \quad (1.30)$$

2. CENTRAL LIMIT THEOREM

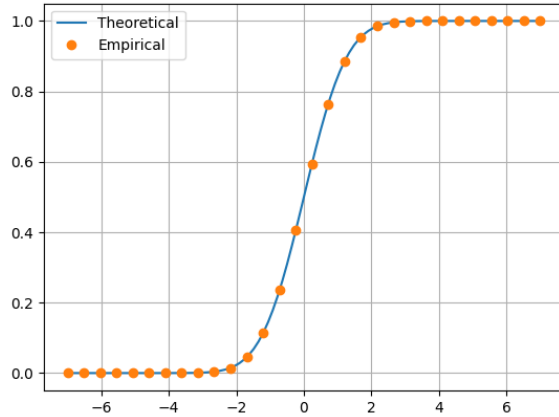
2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/2.1.c
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/header.h
```

Fig. 2.2. The CDF of X

Compile and run the C program by executing the following

```
cc -lm 2.1.c
./a.out
```

- 2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

Every CDF is monotone increasing and right-continuous. Furthermore,

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 \quad \lim_{x \rightarrow \infty} F_X(x) = 1 \quad (2.2)$$

Thus, every CDF is bounded between 0 and 1 and hence, convergent.

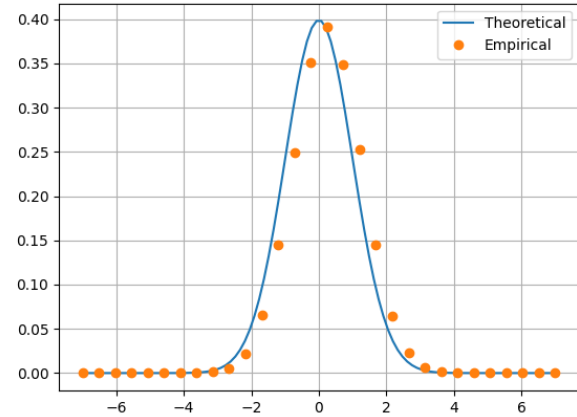
In this case, the CDF is also left-continuous. Therefore, X is a continuous random variable.

- 2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.3)$$

What properties does the PDF have?

Solution: Download the following Python code that plots Fig. 2.2

Fig. 2.3. The PDF of X

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/2.3.py
```

Run the code by executing

```
python 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x) dx = 1 \quad (2.4)$$

In this case, the PDF is symmetric about $x = 0$

- 2.4 Find the mean and variance of X by writing a C program

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/2.4.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 2.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.000294 \quad (2.5)$$

$$\mu_{\text{the}} = 0.000000 \quad (2.6)$$

$$\sigma_{\text{emp}}^2 = 0.999560 \quad (2.7)$$

$$\sigma_{\text{the}}^2 = 1.000000 \quad (2.8)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.9)$$

repeat the above exercise theoretically

Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.10)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.12)$$

$$\Rightarrow g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \quad (2.13)$$

$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.14)$$

$$= -g(x) \quad (2.15)$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) dx = 0 \quad (2.16)$$

Now,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.17)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.18)$$

$$= 2 \int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.19)$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function

Using integration by parts,

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.20)$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) dx \right) \Big|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_0^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.21)$$

Substitute $t = \frac{x^2}{2} \Rightarrow dt = x dx$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \quad (2.22)$$

$$= -\exp(-t) \quad (2.23)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.24)$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right) \Big|_0^{\infty} = 0 - 0 = 0 \quad (2.25)$$

$$\therefore \lim_{x \rightarrow \infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x \rightarrow \infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \quad (2.26)$$

as exponential function grows much faster than a polynomial function

Also,

$$\int_0^{\infty} -\exp\left(-\frac{x^2}{2}\right) dx \quad (2.27)$$

$$\xleftrightarrow{x=t\sqrt{2}} \int_0^{\infty} -\exp(-t^2) dt \sqrt{2} \quad (2.28)$$

$$= -\sqrt{2} \int_0^{\infty} \exp(-t^2) dt \quad (2.29)$$

$$= -\sqrt{2} \frac{\sqrt{\pi}}{2} \quad (2.30)$$

$$= -\sqrt{\frac{\pi}{2}} \quad (2.31)$$

Therefore,

$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}} \right) \quad (2.32)$$

$$= 1 \quad (2.33)$$

$$\therefore \text{Var}[X] = E[X^2] - (E[X])^2 \quad (2.34)$$

$$= 1 - 0 \quad (2.35)$$

$$= 1 \quad (2.36)$$

3. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/3.1.c
```

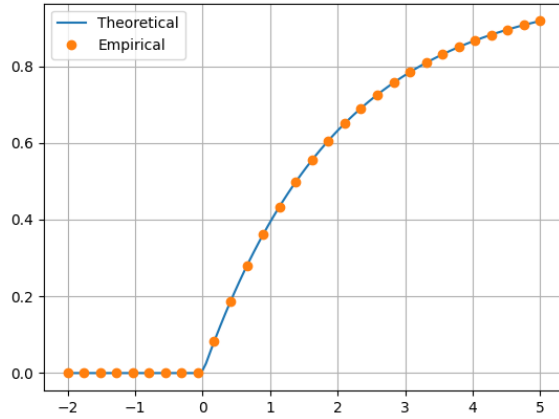


Fig. 3.1. The CDF of V

Compile and run the C program by executing the following

```
cc -lm 3.1.c
./a.out
```

Download the following Python code that plots Fig. 3.1

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/3.1.py
```

Run the code by executing

```
python 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$

Solution: We have

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(\ln(1 - U) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

Now,

$$0 \leq 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \geq 0 \quad (3.8)$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \quad (3.9)$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (3.10)$$

4. TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/4.1.c
```

Compile and run the C program by executing the following

```
cc -lm 4.1.c
./a.out
```

4.2 Find the CDF of T

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) = \Pr(U_1 + U_2 \leq t) \quad (4.2)$$

Since $U_1, U_2 \in [0, 1] \implies U_1 + U_2 \in [0, 2]$
Therefore, if $t \geq 2$, then $U_1 + U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.

Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \quad (4.3)$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr(U_2 \leq t - x) p_{U_1}(x) dx \quad (4.4)$$

$$= \int_0^t F_{U_2}(t - x) p_{U_1}(x) dx \quad (4.5)$$

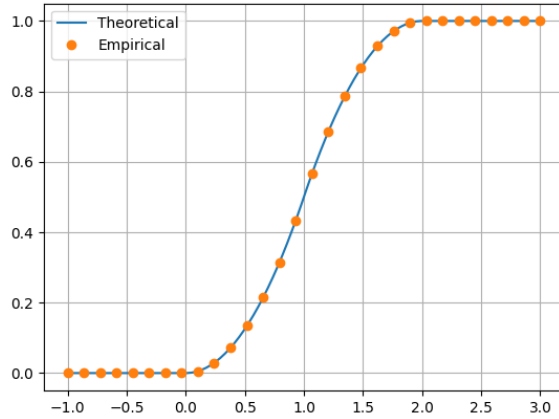
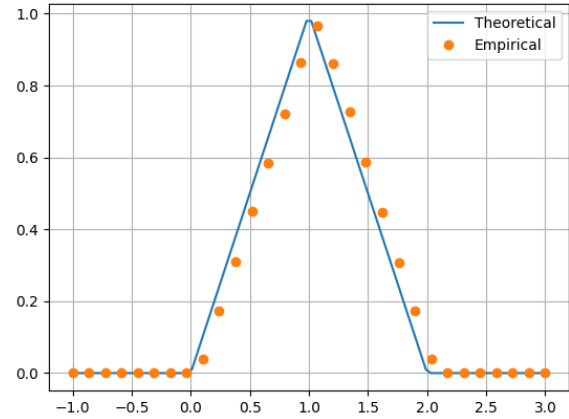
$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \quad (4.6)$$

$$\implies F_{U_2}(t - x) = t - x \quad (4.7)$$

$$F_T(t) = \int_0^t (t - x) \cdot 1 \cdot dx \quad (4.8)$$

$$= tx - \frac{x^2}{2} \Big|_0^t \quad (4.9)$$

$$= \frac{t^2}{2} \quad (4.10)$$

Fig. 4.2. The CDF of T Fig. 4.3. The PDF of T

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot dx \quad (4.11)$$

$$0 \leq x \leq t-1 \implies 1 \leq t-x \leq t \quad (4.12)$$

$$t-1 \leq x \leq 1 \implies 0 < t-1 \leq t-x \leq 1 \quad (4.13)$$

$$F_T(t) = \int_0^{t-1} 1 dx + \int_{t-1}^1 (t-x) dx \quad (4.14)$$

$$= t-1 + t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \quad (4.15)$$

$$= t-1 + 2t-t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \quad (4.16)$$

$$= -\frac{t^2}{2} + 2t - 1 \quad (4.17)$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.18)$$

4.3 Find the PDF of T

Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \quad (4.19)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.20)$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution: The theoretical expressions for the CDF and PDF have been found in problems 4.2 and 4.3 respectively

4.5 Verify your results through a plot

Solution: Download the following Python codes that plot Fig. 4.2 and Fig. 4.3

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/4.2.py
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/4.3.py
```

Run the codes by executing

```
python 4.2.py
python 4.3.py
```

5. MAXIMUM LIKELIHOOD

5.1 Generate equiprobable $X \in \{-1, 1\}$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/5.1.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 5.1.c
./a.out
```

5.2 Generate

$$Y = AX + N \quad (5.1)$$

where $A = 5$ dB, $X \in \{-1, 1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/5.2.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 5.2.c
./a.out
```

5.3 Plot Y

Solution: Download the following Python code that plots Fig. 5.3

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/5.3.py
```

Run the code by executing

```
python 5.3.py
```

5.4 Guess how to estimate X from Y

Solution:

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \quad (5.2)$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.3)$$

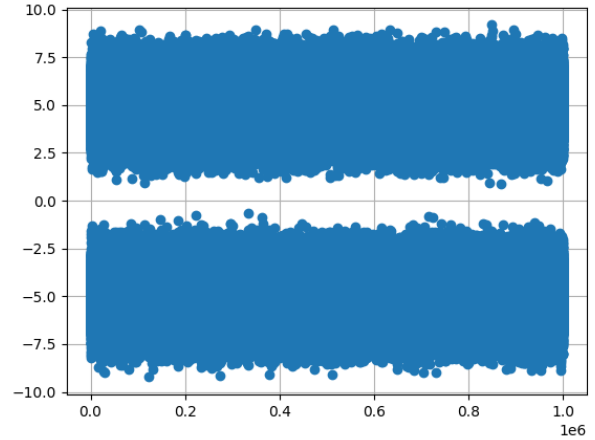


Fig. 5.3. Plot of Y

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.4)$$

Solution:

$$\Pr(\hat{X} = -1|X = 1) = \Pr(Y < 0|X = 1) \quad (5.5)$$

$$= \Pr(A + N < 0) \quad (5.6)$$

$$= \Pr(N < -A) \quad (5.7)$$

$$= 1 - \Pr(N > -A) \quad (5.8)$$

$$= 1 - Q(-A) \quad (5.9)$$

$$= Q(A) \quad (5.10)$$

where $Q(x) = \Pr(N > x)$ is the Q-function

$$Q(x) = 1 - Q(-x) \quad \forall x \in \mathbb{R} \quad (5.11)$$

$$\Pr(\hat{X} = 1|X = -1) = \Pr(Y > 0|X = -1) \quad (5.12)$$

$$= \Pr(-A + N > 0) \quad (5.13)$$

$$= \Pr(N > A) \quad (5.14)$$

$$= Q(A) \quad (5.15)$$

5.6 Find P_e assuming that X has equiprobable symbols

Solution:

$$P_e = \Pr(X = -1) P_{e|0} + \Pr(X = 1) P_{e|1} \quad (5.16)$$

Since X has equiprobable symbols, $\Pr(X = -1) = \Pr(X = 1) = \frac{1}{2}$

$$P_e = \frac{1}{2} Q(A) + \frac{1}{2} Q(A) \quad (5.17)$$

$$= Q(A) \quad (5.18)$$

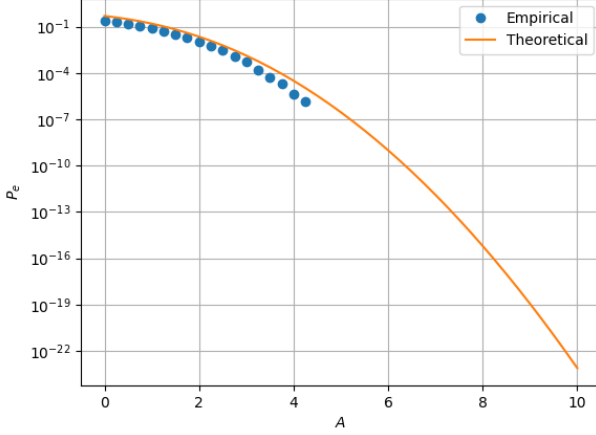


Fig. 5.7. Plot of P_e

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB

Solution: Download the following Python code that plots Fig. 5.7

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/5.7.py
```

Run the code by executing

```
python 5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y . Find the value of δ that minimizes the theoretical P_e

Solution:

$$X = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \quad (5.19)$$

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.20)$$

$$= \Pr(Y < \delta | X = 1) \quad (5.21)$$

$$= \Pr(A + N < \delta) \quad (5.22)$$

$$= \Pr(N < \delta - A) \quad (5.23)$$

$$= Q(A - \delta) \quad (5.24)$$

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.25)$$

$$= \Pr(Y > \delta | X = -1) \quad (5.26)$$

$$= \Pr(-A + N > \delta) \quad (5.27)$$

$$= \Pr(N > \delta + A) \quad (5.28)$$

$$= Q(A + \delta) \quad (5.29)$$

$$(5.30)$$

Now, P_e is given by

$$P_e = \Pr(X = -1) P_{e|0} + \Pr(X = 1) P_{e|1} \quad (5.31)$$

$$= \frac{1}{2} Q(A - \delta) + \frac{1}{2} Q(A + \delta) \quad (5.32)$$

$$= \frac{Q(A - \delta) + Q(A + \delta)}{2} \quad (5.33)$$

$$= g(\delta) \quad (5.34)$$

On differentiating g with respect to δ , we get

$$g'(\delta) = \frac{Q'(A + \delta) - Q'(A - \delta)}{2} \quad (5.35)$$

Recall the definition of $Q(x)$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad (5.36)$$

$$\Rightarrow Q'(x) = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (5.37)$$

Thus,

$$g'(\delta) = \frac{\exp\left(-\frac{(A-\delta)^2}{2}\right) - \exp\left(-\frac{(A+\delta)^2}{2}\right)}{2\sqrt{2\pi}} \quad (5.38)$$

$$g'(\delta) = 0 \Rightarrow (A - \delta)^2 = (A + \delta)^2 \quad (5.39)$$

$$\Rightarrow |A - \delta| = |A + \delta| \quad (5.40)$$

$$\Rightarrow \delta = 0 \quad (5.41)$$

$$g''(\delta) = \frac{(A - \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(A - \delta)^2}{2}\right) + \frac{(A + \delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(A + \delta)^2}{2}\right) \quad (5.42)$$

$$g''(0) = \frac{A}{\sqrt{2\pi}} \exp\left(-\frac{A^2}{2}\right) > 0 \quad (\because A > 0) \quad (5.43)$$

Therefore, $\hat{\delta} = 0$ is a minima and it is what minimizes P_e

5.9 Repeat the above exercise when

$$p_X(-1) = p \quad (5.44)$$

Solution:

$$P_e = p_X(1)P_{e|0} + p_X(-1)P_{e|1} \quad (5.45)$$

$$= (1 - p)Q(A - \delta) + pQ(A + \delta) \quad (5.46)$$

$$= g(\delta) \quad (5.47)$$

On differentiating g with respect to δ , we get

$$g'(\delta) = \frac{(1-p)\exp\left(-\frac{(A-\delta)^2}{2}\right) - p\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\sqrt{2\pi}} \quad (5.48)$$

$g'(\delta) = 0$ when

$$(1-p)\exp\left(-\frac{(A-\delta)^2}{2}\right) = p\exp\left(-\frac{(A+\delta)^2}{2}\right) \quad (5.49)$$

$$\Rightarrow \exp\left(\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = \frac{p}{1-p} \quad (5.50)$$

$$\Rightarrow \exp(2A\delta) = \frac{p}{1-p} \quad (5.51)$$

$$\therefore \hat{\delta} = \frac{1}{2A} \ln \frac{p}{1-p} \quad (5.52)$$

5.10 Repeat the above exercise using the MAP criterion

Solution: The PDF of $X|Y$ is given by

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \quad (5.53)$$

Assuming X has equiprobable symbols,
 $p_X(x) = \frac{1}{2}, \quad x = -1, 1$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \quad (5.54)$$

$$= \frac{1}{2} \Pr(-A + N = y) + \frac{1}{2} \Pr(A + N = y) \quad (5.55)$$

$$= \frac{p_N(y+A) + p_N(y-A)}{2} \quad (5.56)$$

$$= \frac{\exp\left(-\frac{(y+A)^2}{2}\right) + \exp\left(-\frac{(y-A)^2}{2}\right)}{2\sqrt{2\pi}} \quad (5.57)$$

$$(5.58)$$

Now,

$$p_{X|Y}(1|y) = \frac{\Pr(A + N = y) p_X(1)}{p_Y(y)} \quad (5.59)$$

$$= \frac{p_N(y-A)p_X(1)}{p_Y(y)} \quad (5.60)$$

$$= \frac{\exp\left(-\frac{(y-A)^2}{2}\right)}{2\left(\exp\left(-\frac{(y+A)^2}{2}\right) + \exp\left(-\frac{(y-A)^2}{2}\right)\right)} \quad (5.61)$$

$$= \frac{1}{2\left(1 + \exp\left(\frac{(y-A)^2 - (y+A)^2}{2}\right)\right)} \quad (5.62)$$

$$= \frac{1}{2(1 + \exp(-2Ay))} \quad (5.63)$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{1}{2(1 + \exp(2Ay))} \quad (5.64)$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \quad (5.65)$$

$$\Leftrightarrow \frac{1}{2(1 + \exp(-2Ay))} > \frac{1}{2(1 + \exp(2Ay))} \quad (5.66)$$

$$\Leftrightarrow \exp(-2Ay) < \exp(2Ay) \quad (5.67)$$

$$\Leftrightarrow y > 0 \quad (5.68)$$

And $p_{X|Y}(1|y) < p_{X|Y}(-1|y) \Leftrightarrow y < 0$

Therefore, $X = 1$ is more probable than $X = -1$ when $Y > 0$ and vice versa

Consider now a general Bernoulli random variable X with $p_X(-1) = p, p_X(1) = 1 - p$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \quad (5.69)$$

$$= pp_N(y+A) + (1-p)p_N(y-A) \quad (5.70)$$

$$= \frac{p\exp\left(-\frac{(y+A)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-A)^2}{2}\right)}{\sqrt{2\pi}} \quad (5.71)$$

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_Y(y)} \quad (5.72)$$

$$= \frac{(1-p)\exp\left(-\frac{(y-A)^2}{2}\right)}{p\exp\left(-\frac{(y+A)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-A)^2}{2}\right)} \quad (5.73)$$

$$= \frac{1-p}{1-p + p\exp(-2Ay)} \quad (5.74)$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{p}{p + (1 - p) \exp(2Ay)} \quad (5.75)$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \quad (5.76)$$

$$\Leftrightarrow \frac{1 - p}{1 - p + p \exp(-2Ay)} > \frac{p}{p + (1 - p) \exp(2Ay)} \quad (5.77)$$

$$\Leftrightarrow 1 + \frac{p}{1 - p} \exp(-2Ay) < 1 + \frac{1 - p}{p} \exp(2Ay) \quad (5.78)$$

$$\Leftrightarrow \exp(4Ay) > \left(\frac{p}{1 - p} \right)^2 \quad (5.79)$$

$$\Leftrightarrow y > \frac{1}{2A} \ln \frac{p}{1 - p} = \hat{\delta} \quad (5.80)$$

and

$$p_{X|Y}(1|y) < p_{X|Y}(-1|y) \quad (5.81)$$

$$\Leftrightarrow \frac{1 - p}{1 - p + p \exp(-2Ay)} < \frac{p}{p + (1 - p) \exp(2Ay)} \quad (5.82)$$

$$\Leftrightarrow 1 + \frac{p}{1 - p} \exp(-2Ay) > 1 + \frac{1 - p}{p} \exp(2Ay) \quad (5.83)$$

$$\Leftrightarrow \exp(4Ay) < \left(\frac{p}{1 - p} \right)^2 \quad (5.84)$$

$$\Leftrightarrow y < \frac{1}{2A} \ln \frac{p}{1 - p} = \hat{\delta} \quad (5.85)$$

Therefore, $X = 1$ is more probable than $X = -1$ when $Y > \hat{\delta}$ and vice versa

6. GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/6.1.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

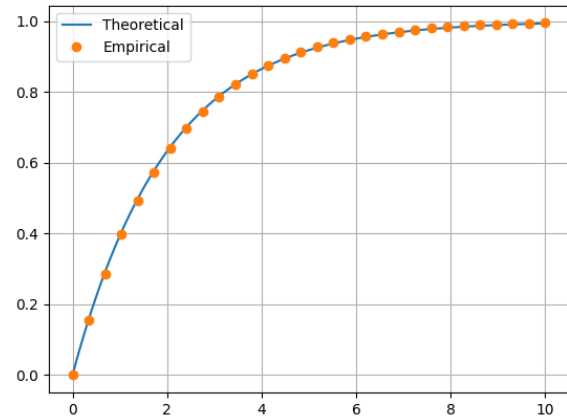


Fig. 6.1. CDF of V

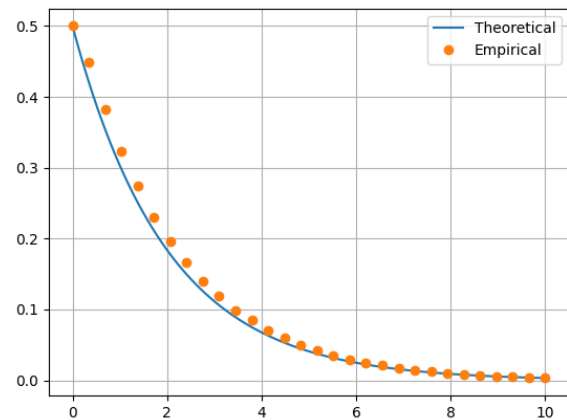


Fig. 6.1. PDF of V

Compile and run the C program by executing the following

```
cc -lm 6.1.c
./a.out
```

Download the following Python code that plots Fig. 6.1

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/6.1.py
```

Run the code by executing

```
python 6.1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α

Solution: Let $R \geq 0, \Theta \in [0, 2\pi]$

$$X_1 = R \cos \Theta \quad (6.3)$$

$$X_2 = R \sin \Theta \quad (6.4)$$

such that $V = X_1^2 + X_2^2 = R^2$

The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (6.5)$$

$$= \begin{pmatrix} \cos \Theta & -R \sin \Theta \\ \sin \Theta & R \cos \Theta \end{pmatrix} \quad (6.6)$$

$$\Rightarrow |\mathbf{J}| = R \cos^2 \Theta + R \sin^2 \Theta = R \quad (6.7)$$

Then,

$$p_{R,\Theta}(r, \theta) = p_{X_1, X_2}(x_1, x_2) |\mathbf{J}| \quad (6.8)$$

$$= p_{X_1, X_2}(x_1, x_2) r \quad (6.9)$$

$$= r p_{X_1}(x_1) p_{X_2}(x_2) \quad (6.10)$$

since X_1 and X_2 are independent

$$p_{R,\Theta}(r, \theta) = r \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2}\right) \quad (6.11)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{(x_1 + x_2)^2}{2}\right) \quad (6.12)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \quad (6.13)$$

The marginal distribution of R is given by

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r, \theta) d\theta \quad (6.14)$$

$$= \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \int_0^{2\pi} d\theta \quad (6.15)$$

$$= r \exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \quad (6.16)$$

The CDF of R is thus given by

$$F_R(r) = \Pr(R \leq r) \quad (6.17)$$

$$= \int_0^r p_R(r) dr \quad (6.18)$$

$$= \int_0^r r \exp\left(-\frac{r^2}{2}\right) dr \quad (6.19)$$

$$= - \int_0^r \exp\left(-\frac{r^2}{2}\right) (-r dr) \quad (6.20)$$

$$= - \int_0^r \exp\left(-\frac{r^2}{2}\right) d\left(-\frac{r^2}{2}\right) \quad (6.21)$$

$$= - \exp\left(-\frac{r^2}{2}\right) \Big|_0^r \quad (6.22)$$

$$= 1 - \exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \quad (6.23)$$

Now, $V = R^2$, hence the CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \quad (6.24)$$

$$= \Pr(R^2 \leq x) \quad (6.25)$$

$$= \Pr(|R| \leq \sqrt{x}) \quad (6.26)$$

$$= \Pr(R \leq \sqrt{x}) \quad (\because R \geq 0) \quad (6.27)$$

$$= F_R(\sqrt{x}) \quad (6.28)$$

$$= 1 - \exp\left(-\frac{x}{2}\right) \quad \text{for } x \geq 0 \quad (6.29)$$

And the PDF of V is given by

$$p_V(x) = \frac{d}{dx} F_V(x) \quad (6.30)$$

$$= \frac{d}{dx} \left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (6.31)$$

$$= \frac{1}{2} \exp\left(-\frac{x}{2}\right) \quad (6.32)$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.33)$$

$$p_V(x) = \begin{cases} \frac{1}{2} \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.34)$$

$$\therefore \alpha = \frac{1}{2} \quad (6.35)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.36)$$

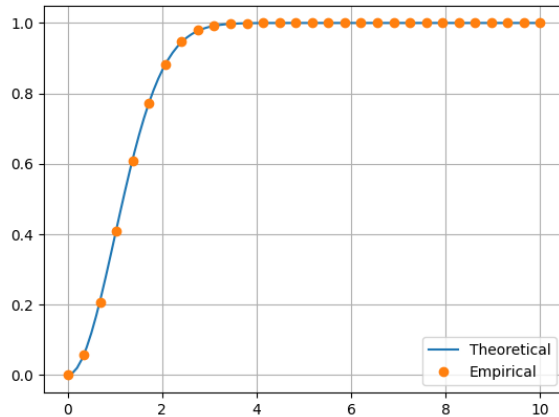


Fig. 6.3. CDF of A

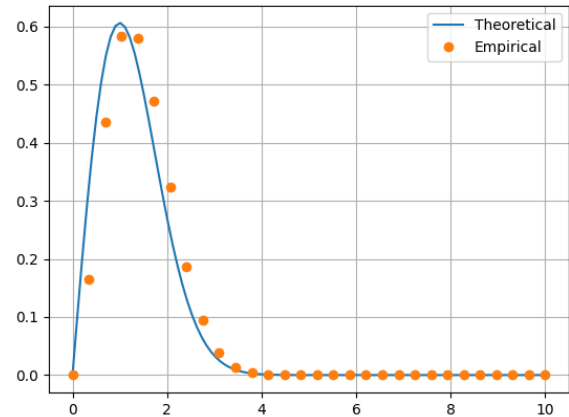


Fig. 6.3. PDF of A

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/6.3.c
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 6.3.c
./a.out
```

Download the following Python code that plots Fig. 6.3

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/6.3.py
```

Run the code by executing

```
python 6.3.py
```

The CDF of A for $x \geq 0$ is given by

$$F_A(x) = \Pr(A \leq x) \quad (6.37)$$

$$= \Pr(\sqrt{V} \leq x) \quad (6.38)$$

$$= \Pr(V \leq x^2) \quad (6.39)$$

$$= F_V(x^2) \quad (6.40)$$

$$= 1 - \exp\left(-\frac{x^2}{2}\right) \quad (6.41)$$

The PDF of A is given by

$$p_A(x) = \frac{d}{dx} F_A(x) \quad (6.42)$$

$$= \frac{d}{dx} \left(1 - \exp\left(-\frac{x^2}{2}\right)\right) \quad (6.43)$$

$$= x \exp\left(-\frac{x^2}{2}\right) \quad (6.44)$$

Therefore,

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.45)$$

$$p_A(x) = \begin{cases} x \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.46)$$

7. CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in \{-1, 1\}$ for $0 \leq \gamma \leq 10$ dB

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/7.1.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 7.1.c
./a.out
```

Download the following Python code that plots Fig. 7.4

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/7.1.py
```

Run the code by executing

```
python 7.1.py
```

7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

Solution:

$$P_e(N) = \Pr(\hat{X} = -1 | X = 1) \quad (7.3)$$

$$= \Pr(Y < 0 | X = 1) \quad (7.4)$$

$$= \Pr(A + N < 0) \quad (7.5)$$

$$= \Pr(A < -N) \quad (7.6)$$

$$= F_A(-N) \quad (7.7)$$

For Gaussian random variables with $\sigma = 1$, we found the CDF of A to be

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.8)$$

For Gaussian random variable Y with a general σ ,

$$Y = \sigma X \quad (7.9)$$

$$\Rightarrow X = \frac{Y}{\sigma} \quad (7.10)$$

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.11)$$

$$\Rightarrow p_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.12)$$

$$E[A^2] = \int_0^\infty x^2 \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \quad (7.13)$$

$$\text{Substitute } -\frac{x^2}{2\sigma^2} = u \Rightarrow du = -\frac{x}{\sigma^2} dx$$

$$E[A^2] = \int_0^\infty 2\sigma^2 u \exp(u) du \quad (7.14)$$

$$= 2\sigma^2 (u \exp(u) - \exp(u)) \Big|_0^\infty \quad (7.15)$$

$$= 2\sigma^2 \quad (7.16)$$

$$= \gamma \quad (7.17)$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.18)$$

7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^\infty g(x) p_X(x) dx \quad (7.19)$$

Find $P_e = E[P_e(N)]$

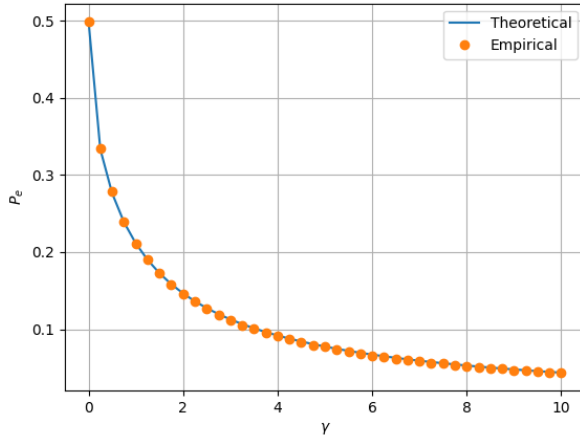
Solution:

$$P_e = \int_{-\infty}^0 P_e(x) p_N(x) dx \quad (7.20)$$

$$= \int_{-\infty}^0 \left(1 - \exp\left(-\frac{x^2}{\gamma}\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (7.21)$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-x^2 \left(\frac{1}{2} + \frac{1}{\gamma}\right)\right) dx \quad (7.22)$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-\frac{(2+\gamma)x^2}{2\gamma}\right) dx \quad (7.23)$$

Fig. 7.4. Plot of P_e

Now,

$$\int_{-\infty}^0 \exp\left(-\frac{x^2}{2a^2}\right) dx = \int_0^{\infty} \exp\left(-\frac{x^2}{2a^2}\right) dx \quad (7.24)$$

$$= a \sqrt{\frac{\pi}{2}} \quad (7.25)$$

Therefore,

$$P_e = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\gamma}{\gamma+2}} \quad (7.26)$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma}{\gamma+2}} \quad (7.27)$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ . Comment.

Solution: Download the following Python code that plots Fig. 7.4

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/7.1.py
```

Run the code by executing

```
python 7.1.py
```

The graphs in both cases coincide and thus

$$\Pr(\hat{X} = -1|X = 1) = E[P_e(N)] \quad (7.28)$$

8. TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1) \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/8.1.c
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/header.h
```

Compile and run the C program by executing the following

```
cc -lm 8.1.c
./a.out
```

Download the following Python code that plots Fig. 8.3

```
wget https://github.com/Ankit-Saha-2003/
AI1110/raw/main/Random-Numbers/
codes/8.1.py
```

Run the code by executing

```
python 8.1.py
```

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1

Solution: Let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (8.5)$$

Then, the line $y_2 - y_1 = 0$ seems to separate the two data sets. Thus,

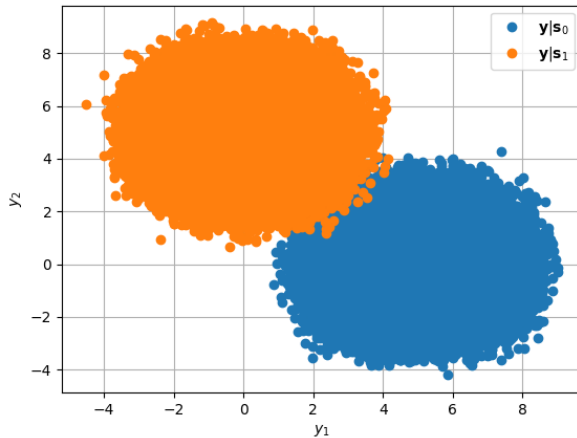
$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & y_2 - y_1 < 0 \\ \mathbf{s}_1 & y_2 - y_1 > 0 \end{cases} \quad (8.6)$$

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.7)$$

with respect to the SNR from 0 to 10 dB

Solution: Download the C source code to generate the data by executing the following commands

Fig. 8.1. Plot of $y|s_0$ and $y|s_1$

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/8.3.c
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/header.h
```

Compile and run the C program by executing the following

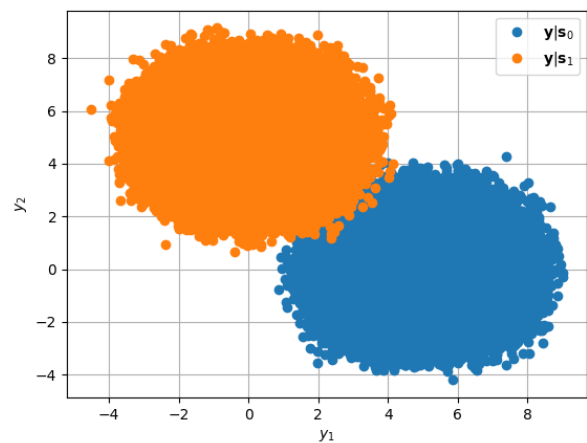
```
cc -lm 8.3.c
./a.out
```

Download the following Python code that plots Fig. ??

```
wget https://github.com/Ankit-Saha-2003/
  AI1110/raw/main/Random-Numbers/
  codes/8.3.py
```

Run the code by executing

```
python 8.3.py
```

Fig. 8.3. Plot of P_e