Random Numbers

AI1110: Probability and Random Variables

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I. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

I.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the C source code by executing the following command

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.1.c
```

Complie and run the C program by executing the following

```
cc -lm 1.1.c
./a.out
```

I.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{I.1}$$

Solution: Download the following Python code that plots Fig. I.2

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.2.py
```

Run the code by executing

```
python 1.2.py
```

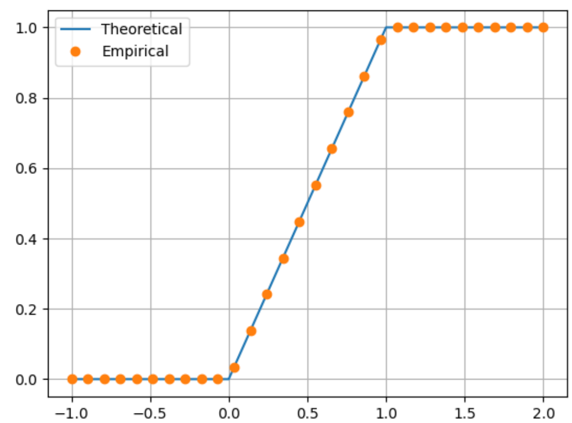

Fig. I.2. The CDF of U

I.3 Find a theoretical expression for $F_U(x)$
Solution: The PDF of U is given by

$$P_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{I.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} P_U(x)\ \mathrm{d}x \tag{I.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} P_U(x)\ \mathrm{d}x = \int_{-\infty}^{x} 0\ \mathrm{d}x = 0 \tag{I.4}$$

If c,

$$\int_{-\infty}^{x} P_U(x)\ \mathrm{d}x = \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{x} 1\ \mathrm{d}x \tag{I.5}$$

$$= 0 + x \tag{I.6}$$

$$= x \tag{I.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} P_U(x)\ \mathrm{d}x$$

$$= \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{1} 1\ \mathrm{d}x + \int_{1}^{x} 0\ \mathrm{d}x \tag{I.8}$$

$$\int_{-\infty}^{x} P_U(x)\ \mathrm{d}x = 0 + 1 + 0 \tag{I.9}$$
$$= 1 \tag{I.10}$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{I.11}$$

I.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{I.12}$$

and its variance as

$$\mathrm{Var}[U] = E[U - E[U]]^2 \tag{I.13}$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following command

```
wget https://github.com/Ankit-Saha-2003/
    AI1110/raw/main/Random-Numbers/
    codes/1.4.c
```

Complie and run the C program by executing the following

```
cc -lm 1.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \tag{I.14}$$
$$\mu_{\text{the}} = 0.500000 \tag{I.15}$$
$$\sigma^2_{\text{emp}} = 0.083301 \tag{I.16}$$
$$\sigma^2_{\text{the}} = 0.083333 \tag{I.17}$$

I.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \tag{I.18}$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x\ \mathrm{d}F_U(x) \tag{I.19}$$

On differentiating the CDF of U, we get

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \tag{I.20}$$

$$\therefore E[U] = \int_0^1 x\ \mathrm{d}x = \frac{1}{2} = 0.5 \tag{I.21}$$

Similarly,

$$\therefore E\left[U^2\right] = \int_0^1 x^2\ \mathrm{d}x = \frac{1}{3} \tag{I.22}$$

Now, the variance of U is given by

$$\mathrm{Var}[U] \tag{I.23}$$
$$= E[U - E[U]]^2 \tag{I.24}$$
$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \tag{I.25}$$

By linearity of expectation, we have

$$E\left[U^2\right] + E[-2UE[U]] + E\left[(E[U])^2\right] \tag{I.26}$$
$$= E\left[U^2\right] - 2E[U]E[U] + (E[U])^2 \tag{I.27}$$
$$= E\left[U^2\right] - (E[U])^2 \tag{I.28}$$
$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{I.29}$$
$$= \frac{1}{12} \approx 0.083333 \tag{I.30}$$