AI1110
Assignment 15

Ankit Saha
AI21BTECH11004

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Outline

Papoulis Exercise 9-21

Show that if $X(t)$ is a stationary process with derivative $X'(t)$, then for a given t the random variables $X(t)$ and $X'(t)$ are orthogonal and uncorrelated.

Assumptions

1. The derivative $X'(t)$ of a wide sense stationary process $X(t)$ is also wide sense stationary, i.e., its mean is constant and its autocorrelation depends only on $\tau = t_1 - t_2$

$$E\left[X'(t+\tau)X'(t)\right] = R_{X'X'}(\tau) \qquad \forall t \tag{1}$$

2. $X(t)$ and $X'(t)$ are jointly wide sense stationary, i.e., they are both wide sense stationary and their cross-correlation depends only on $\tau = t_1 - t_2$

$$E\left[X(t+\tau)X'(t)\right] = R_{XX'}(\tau) \qquad \forall t \tag{2}$$

Proof

We know that the mean of a stationary process is constant, i.e.,

$$E\left[X(t)\right] = \eta_x(t) = \eta \tag{3}$$

The mean of $X'(t)$ is thus given by

$$\begin{aligned} E\left[X'(t)\right] &= \eta_{x'}(t) && (4) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} E\left[X(t)\right] && (5) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \eta && (6) \\ &= 0 && (7) \end{aligned}$$

The autocorrelation of a wide sense stationary process $X(t)$ depends only on $\tau = t_1 - t_2$

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right] \tag{8}$$
$$= E\left[X(t_2 + \tau)X(t_2)\right] \tag{9}$$

This is independent of t_2, i.e.,

$$\implies R_{xx}(t + \tau, t) = E\left[X(t + \tau)X(t)\right] = R_{xx}(\tau) \qquad \forall t \tag{10}$$

Now,

$$R_{xx}(-\tau) = E\left[X(t - \tau)X(t)\right] \tag{11}$$
$$= E\left[X(t)X(t - \tau)\right] \tag{12}$$
$$= E\left[X((t - \tau) + \tau)X(t - \tau)\right] \tag{13}$$
$$= E\left[X(t' + \tau)X(t')\right] \qquad \text{where } t' = t - \tau \tag{14}$$
$$= R_{xx}(\tau) \qquad \text{from 9} \tag{15}$$

On differentiating with respect to τ, we get

$$
\begin{align}
R'_{xx}(\tau) &= -R'_{xx}(-\tau) \tag{16}\\
\implies R'_{xx}(\tau) + R'_{xx}(-\tau) &= 0 \tag{17}
\end{align}
$$

Substituting $\tau = 0$, we get

$$
R'_{xx}(0) = 0 \tag{18}
$$

For jointly wide sense stationary processes $X(t)$ and $X'(t)$,

$$
\begin{align}
R_{xx'}(t+\tau, t) &= E\left[X(t+\tau)X'(t)\right] \tag{19}\\
&= R_{xx'}(\tau) \qquad \forall t \tag{20}
\end{align}
$$

For a given t, $X(t)$ and $X'(t)$ are orthogonal if

$$
\begin{align}
R_{xx'}(t, t) &= 0 \tag{21}\\
\implies R_{xx'}(t+0, t) &= 0 \tag{22}\\
\implies R_{xx'}(0) &= 0 \qquad \because \tau = t - t = 0 \tag{23}
\end{align}
$$

Now,

$$R_{xx'}(t_1, t_2) = E\left[X(t_1)X'(t_2)\right] \quad (24)$$

$$= \frac{\partial}{\partial t_2} E\left[X(t_1)X(t_2)\right] \quad (25)$$

$$= \frac{\partial}{\partial t_2} R_{xx}(t_1, t_2) \quad (26)$$

For $\tau = t_1 - t_2$

$$R_{xx'}(\tau) = \frac{\mathrm{d}R_{xx}(\tau)}{\mathrm{d}\tau}\frac{\partial \tau}{\partial t_2} = R'_{xx}(\tau)\frac{\partial \tau}{\partial t_2} = -R'_{xx}(\tau) \quad (27)$$

Thus,

$$R_{xx'}(0) = -R'_{xx}(0) = 0 \quad (28)$$

Therefore,

$X(t)$ and $X'(t)$ are orthogonal

Now, the cross-covariance of jointly wide sense stationary processes $X(t)$ and $X'(t)$ only depends on $\tau = t_1 - t_2$ and is given by

$$C_{xx'}(t_1, t_2) = R_{xx'}(t_1, t_2) - \eta_x(t_1)\eta_{x'}(t_2) \tag{29}$$
$$\implies C_{xx'}(\tau) = R_{xx'}(\tau) - \eta_x\eta_{x'} \tag{30}$$

For a given t, $t_1 = t_2 = t$ and $\tau = t_1 - t_2 = 0$

$$C_{xx'}(0) = R_{xx'}(0) - \eta \cdot 0 \tag{31}$$
$$= 0 - 0 \tag{32}$$
$$= 0 \tag{33}$$

Therefore,

$X(t)$ and $X'(t)$ are uncorrelated