AI1110
Assignment 15

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Outline

Papoulis Exercise 9-21

Show that if $X(t)$ is a stationary process with derivative $X'(t)$, then for a given t the random variables $X(t)$ and $X'(t)$ are orthogonal and uncorrelated.

Proof

We know that the mean of a stationary process is constant, i.e.,

$$E\left[X(t)\right] = \eta_x(t) = \eta \tag{1}$$

The mean of $X'(t)$ is thus given by

$$\begin{align} E\left[X'(t)\right] &= \eta_{x'}(t) \tag{2} \\ &= L[\eta_x(t)] \tag{3} \\ &= \eta'_x(t) \tag{4} \\ &= 0 \tag{5} \end{align}$$

where L is the differentiator

$$L[X(t)] = X'(t) \tag{6}$$

Since the mean of $X'(t)$ is constant, $X'(t)$ is also a stationary process

The autocorrelation of a wide sense stationary process $X(t)$ depends only on $\tau = t_1 - t_2$

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right] \quad (7)$$
$$= E\left[X(t_2 + \tau)X(t_2)\right] \quad (8)$$

This is independent of t_2, i.e.,

$$\implies R_{xx}(t + \tau, t) = E\left[X(t + \tau)X(t)\right] = R_{xx}(\tau) \qquad \forall t \quad (9)$$

Now,

$$R_{xx}(-\tau) = E\left[X(t - \tau)X(t)\right] \quad (10)$$
$$= E\left[X(t)X(t - \tau)\right] \quad (11)$$
$$= E\left[X((t - \tau) + \tau)X(t - \tau)\right] \quad (12)$$
$$= E\left[X(t' + \tau)X(t')\right] \qquad \text{where } t' = t - \tau \quad (13)$$
$$= R_{xx}(\tau) \qquad \text{from 9} \quad (14)$$

Similarly for jointly wide sense stationary processes $X(t)$ and $X'(t)$,

$$R_{xx'}(t+\tau, t) = E\left[X(t+\tau)X'(t)\right] \tag{15}$$

$$= R_{xx'}(\tau) \qquad \forall t \tag{16}$$

For a given t, $X(t)$ and $X'(t)$ are orthogonal if

$$R_{xx'}(t, t) = 0 \tag{17}$$

$$\implies R_{xx'}(t+0, t) = 0 \tag{18}$$

$$\implies R_{xx'}(0) = 0 \qquad \because \tau = t - t = 0 \tag{19}$$

Now, for any linear system $Y(t) = L[X(t)]$,

$$R_{xy}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] \tag{20}$$

Since, differentiators are also linear systems, we have

$$R_{xx'}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] \tag{21}$$

For $\tau = t_1 - t_2$

$$R_{xx'}(\tau) = L[R_{xx}(\tau)] = R'_{xx}(\tau) \tag{22}$$

Thus, $R_{xx'}(0) = R'_{xx}(0)$
We have already obtained that

$$R_{xx}(\tau) = R_{xx}(-\tau) \qquad \forall\tau \tag{23}$$

On differentiating with respect to τ, we get

$$R'_{xx}(\tau) = -R'_{xx}(-\tau) \tag{24}$$
$$\implies R'_{xx}(\tau) + R'_{xx}(-\tau) = 0 \tag{25}$$

Substituting $\tau = 0$, we get

$$R'_{xx}(0) = 0 \tag{26}$$
$$\implies R_{xx'}(0) = 0 \tag{27}$$

Therefore,
$X(t)$ and $X'(t)$ are orthogonal

Now, the cross-covariance of jointly wide sense stationary processes $X(t)$ and $X'(t)$ only depends on $\tau = t_1 - t_2$ and is given by

$$C_{xx'}(t_1, t_2) = R_{xx'}(t_1, t_2) - \eta_x(t_1)\eta_{x'}(t_2) \tag{28}$$
$$\implies C_{xx'}(\tau) = R_{xx'}(\tau) - \eta_x\eta_{x'} \tag{29}$$

For a given t, $t_1 = t_2 = t$ and $\tau = t_1 - t_2 = 0$

$$C_{xx'}(0) = R_{xx'}(0) - \eta \cdot 0 \tag{30}$$
$$= 0 - 0 \tag{31}$$
$$= 0 \tag{32}$$

Therefore,

$X(t)$ and $X'(t)$ are uncorrelated