Fourier Series
EE3900: Linear Systems and Signal Processing
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1. Periodic Function

Let

$$x(t) = A_0 \left| \sin \left(2\pi f_0 t \right) \right| \tag{1.1}$$

1.1 Plot $x(t)$

Solution: Download the following Python code that plots Fig. 1.1.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/1.1.py
```
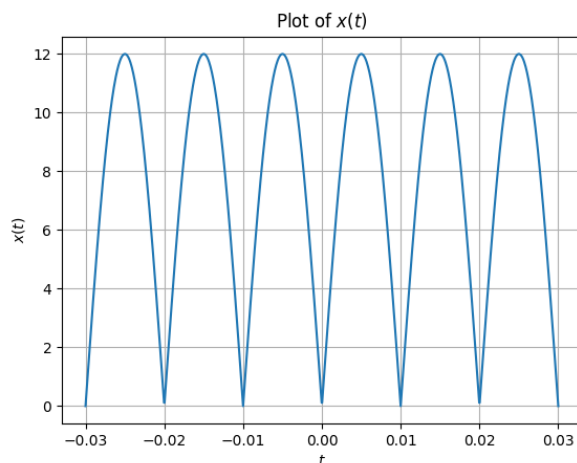
Run the code by executing

```
python 1.1.py
```


Fig. 1.1. Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period
Solution: Since $x(t)$ is the absolute value of a sinusoidal function, it is periodic, which is also evident from the plot

Consider $x(t + \frac{1}{2f_0})$

$$x \left(t + \frac{1}{2f_0} \right) = A_0 \left| \sin \left(2\pi f_0 \left(t + \frac{1}{2f_0} \right) \right) \right| \tag{1.2}$$

$$= A_0 \left| \sin \left(2\pi f_0 t + \pi f_0 \right) \right| \tag{1.3}$$

$$= A_0 \left| (-1)^{f_0} \sin \left(2\pi f_0 t \right) \right| \tag{1.4}$$

$$= A_0 \left| \sin \left(2\pi f_0 t \right) \right| \tag{1.5}$$

$$= x(t) \tag{1.6}$$

Therefore, $x(t)$ is periodic with period $\frac{1}{2f_0}$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t} \, \mathrm{d}t \tag{2.2}$$

Solution:

$$x(t) e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi (n-k) f_0 t} \tag{2.3}$$

$$\implies \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t} \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t} \, \mathrm{d}t \tag{2.4}$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t} \, \mathrm{d}t = \begin{cases} \frac{1}{f_0} & k = n \\ 0 & k \neq n \end{cases} \tag{2.5}$$

$$= \frac{1}{f_0} \delta(n - k) \tag{2.6}$$

$$\sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi(n-k)f_0 t}\, \mathrm{d}t = \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0}\delta(n-k) \tag{2.7}$$

$$= \frac{1}{f_0} c_n * \delta(n) \tag{2.8}$$

$$= \frac{1}{f_0} c_n \tag{2.9}$$

Therefore

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.10}$$

2.2 Find c_k for (1.1)

Solution:

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.11}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{0} A_0 \left(-\sin\left(2\pi f_0 t\right)\right) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t$$

$$+ f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\sin\left(2\pi f_0 t\right)\right) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.12}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 u\right) e^{\jmath 2\pi k f_0 u}\, \mathrm{d}t$$

$$+ f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.13}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right)\left(e^{\jmath 2\pi k f_0 t} + e^{-\jmath 2\pi k f_0 t}\right) \mathrm{d}t \tag{2.14}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} 2\sin\left(2\pi f_0 t\right)\cos\left(2\pi k f_0 t\right)\, \mathrm{d}t \tag{2.15}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left\{\sin\left(2\pi(1+k)f_0 t\right) + \sin\left(2\pi(1-k)f_0 t\right)\right\}\, \mathrm{d}t \tag{2.16}$$

$$= f_0 A_0 \left[-\frac{\cos\left(2\pi(1+k)f_0 t\right)}{2\pi(1+k)f_0} - \frac{\cos\left(2\pi(1-k)f_0 t\right)}{2\pi(1-k)f_0}\right]_{0}^{\frac{1}{2f_0}} \tag{2.17}$$

$$= \frac{f_0 A_0}{2\pi f_0}\left[\frac{1-(-1)^{1+k}}{1+k} + \frac{1-(-1)^{1-k}}{1-k}\right] \tag{2.18}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{2\pi}\left[\frac{1}{1+k} + \frac{1}{1-k}\right] \tag{2.19}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{\pi(1-k^2)} \tag{2.20}$$

Therefore

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases} \tag{2.21}$$

2.3 Verify (1.1) using Python

Solution: Download the following Python code that plots Fig. 3.8.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/2.3.py
```
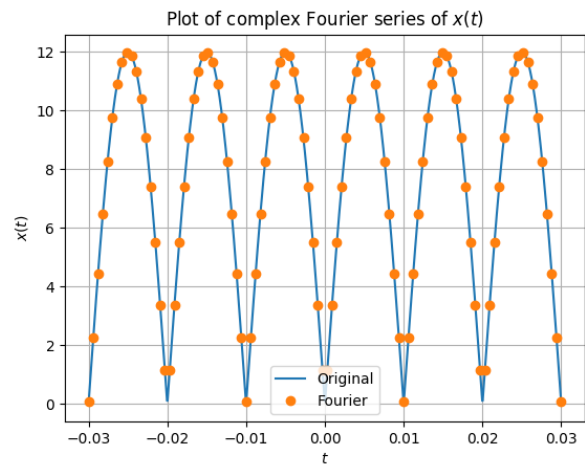
Run the code by executing

```
python 2.3.py
```


Fig. 2.3. Plot of $x(t)$ along with its complex Fourier series expansion

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos\left(2\pi k f_0 t\right) + b_k \sin\left(2\pi k f_0 t\right)\right) \tag{2.22}$$

and obtain the formulae for a_k and b_k

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.23}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.24}$$

Thus

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right)\cos\left(2\pi k f_0 t\right)$$

$$+ \sum_{k=1}^{\infty} \jmath\left(c_k - c_{-k}\right)\sin\left(2\pi k f_0 t\right) \tag{2.25}$$

Therefore

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.26)$$

$$b_k = \jmath(c_k - c_{-k}) \quad k \geq 0 \quad (2.27)$$

2.5 Find a_k and b_k for (1.1)
Solution:

$$a_0 = c_0 = \frac{2A_0}{\pi} \quad (2.28)$$

For $k > 0$, if k is odd

$$a_k = 0 + 0 = 0 \quad (2.29)$$

and if k is even

$$a_k = \frac{2A_0}{\pi(1 - k^2)} + \frac{2A_0}{\pi(1 - k^2)} = \frac{4A_0}{\pi(1 - k^2)} \quad (2.30)$$

For odd or even k, $c_k = c_{-k}$ always

$$b_k = 0 \quad \forall k \geq 0 \quad (2.31)$$

Therefore

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k = 2m, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad (2.32)$$

$$b_k = 0 \quad k \geq 0 \quad (2.33)$$

2.6 Verify (2.22) using Python
Solution: Download the following Python code that plots Fig. 2.6.

wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/2.6.py

Run the code by executing

python 2.6.py

3. FOURIER TRANSFORM

3.1

$$\delta(t) = 0 \qquad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of $g(t)$ is

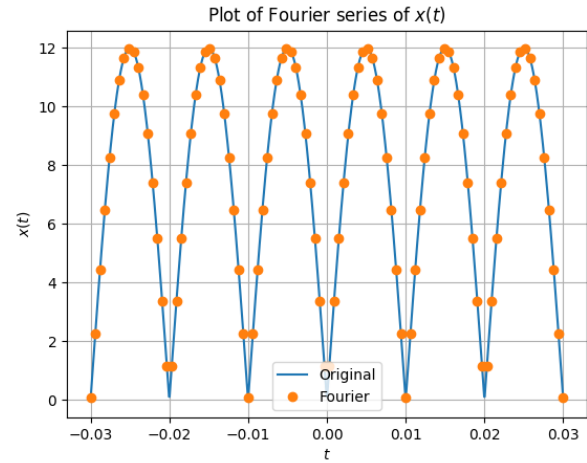$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \quad (3.3)$$

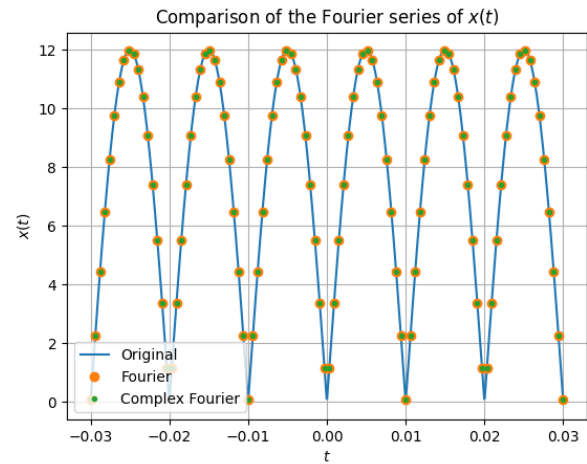Fig. 2.6. Plot of $x(t)$ along with its Fourier series expansion

Fig. 2.6. Comparison of the Fourier series of $x(t)$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi ft_0} \quad (3.4)$$

Solution:

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi ft} dt \quad (3.5)$$

$$= \int_{-\infty}^{\infty} g(u) e^{-j2\pi f(u+t_0)} du \quad (3.6)$$

$$= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(u) e^{-j2\pi fu} du \quad (3.7)$$

$$= G(f) e^{-j2\pi ft_0} \quad (3.8)$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.9)$$

Solution:

$$G(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\,dt \qquad (3.10)$$

But

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\,df \qquad (3.11)$$

$$= \int_{-\infty}^{\infty} G(u)e^{j2\pi ut}\,du \qquad (3.12)$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(u)e^{-j2\pi uf}\,du \qquad (3.13)$$

$$= \mathcal{F}\{G(t)\} \qquad (3.14)$$

3.5 Find the Fourier transform of $\delta(t)$
   Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\,dt \qquad (3.15)$$

$$= e^{-j2\pi ft}\Big|_{t=0} \qquad (3.16)$$

$$= 1 \qquad (3.17)$$

3.6 Find the Fourier transform of $e^{-j2\pi f_0 t}$
   Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \qquad (3.18)$$

$$\implies \delta(t - f_0) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f f_0} \qquad (3.19)$$

$$\implies e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(-f - f_0) \qquad (3.20)$$

$$\therefore e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(f + f_0) \qquad (3.21)$$

3.7 Find the Fourier transform of $\cos(2\pi f_0 t)$
   Solution:

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \qquad (3.22)$$

$$\implies \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \qquad (3.23)$$

3.8 Find the Fourier transform of $x(t)$ and plot it. Verify using Python
   Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \qquad (3.24)$$

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left\{e^{j2\pi k f_0 t}\right\} \qquad (3.25)$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \qquad (3.26)$$

$$= \frac{2A_0}{\pi} \sum_{k \text{ is even}} \frac{\delta(f - k f_0)}{1 - k^2} \qquad (3.27)$$

Download the following Python code that plots Fig. **??**.

```
wget https://github.com/Ankit−Saha−2003/
     EE3900/raw/main/Fourier/codes/3.8.py
```

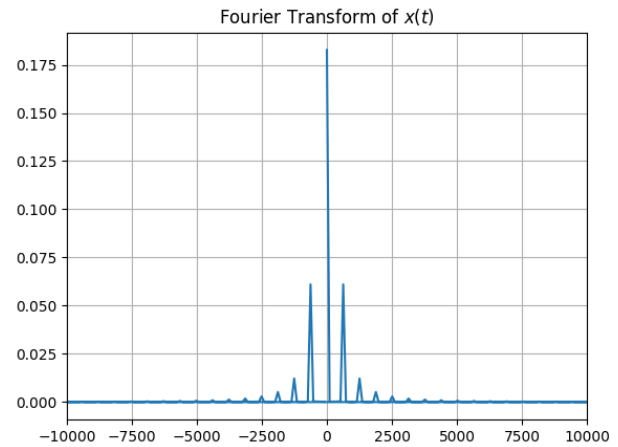Run the code by executing

```
python 3.8.py
```


Fig. 3.8. Plot of the Fourier transform of $x(t)$

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(f) \qquad (3.28)$$

Verify using Python
   Solution:

$$\text{rect}(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (3.29)$$

Its Fourier transform is given by

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect}(t)\,e^{-j2\pi ft}\,dt \qquad (3.30)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft}\,dt \qquad (3.31)$$

$$= \frac{e^{-j\pi f} - e^{j\pi f}}{-j2\pi f} \qquad (3.32)$$

$$= \frac{\sin \pi f}{\pi f} \qquad (3.33)$$

$$= \text{sinc}(f) \qquad (3.34)$$

Download the following Python code that plots Fig. 3.9.

```
wget https://github.com/Ankit−Saha−2003/
     EE3900/raw/main/Fourier/codes/3.9.py
```

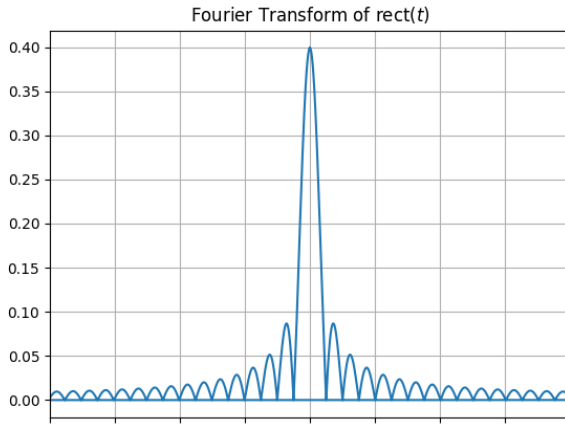Run the code by executing

```
python 3.9.py
```

Fourier Transform of rect(t)

Fig. 3.9. Plot of the Fourier transform of $\text{rect}(t)$

3.10 Find the Fourier transform of $\text{sinc}(t)$. Verify using Python

Solution:

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(f) \tag{3.35}$$

$$\implies \text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(-f) \tag{3.36}$$

$$= \text{rect}(f) \tag{3.37}$$

Download the following Python code that plots Fig. 3.10.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/3.10.py
```

Run the code by executing

```
python 3.10.py
```

4. FILTER

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5 V

Solution: Since we want a DC output, the filter we need is a low-pass filter that only lets the zero frequency component pass through, i.e., the amplitude of a frequency components with a frequency higher than the cutoff frequency f_c has to be zero

We can use a rectangular filter for this purpose

$$H(f) = k \, \text{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} k & |f| \le f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$
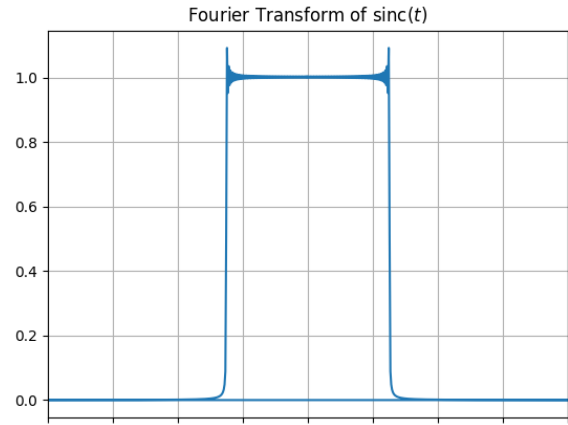
Fourier Transform of sinc(t)

Fig. 3.10. Plot of the Fourier transform of $\text{sinc}(t)$

Now

$$H(0) = \frac{Y(0)}{X(0)} \tag{4.2}$$

where $Y(k)$ and $X(k)$ are the Fourier transforms of the output 5 V DC and the input signal respectively

$$k = \frac{5}{\frac{2A_0}{\pi}} = \frac{5\pi}{2A_0} \tag{4.3}$$

$$\therefore H(f) = \frac{5\pi}{2A_0} \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.4}$$

4.2 Find $h(t)$

Solution:

$$\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(f) \tag{4.5}$$

$$\implies \text{sinc}(2f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2f_c} \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.6}$$

$$\implies \frac{5\pi}{2A_0} 2f_c \text{sinc}(2f_c t) \xleftrightarrow{\mathcal{F}} \frac{5\pi}{2A_0} \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.7}$$

$$\therefore h(t) = \frac{5\pi f_c}{A_0} \text{sinc}(2f_c t) \tag{4.8}$$

4.3 Verify your result using through convolution

Solution: Download the following Python code that plots Fig. 4.3.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/4.3.py
```

Run the code by executing
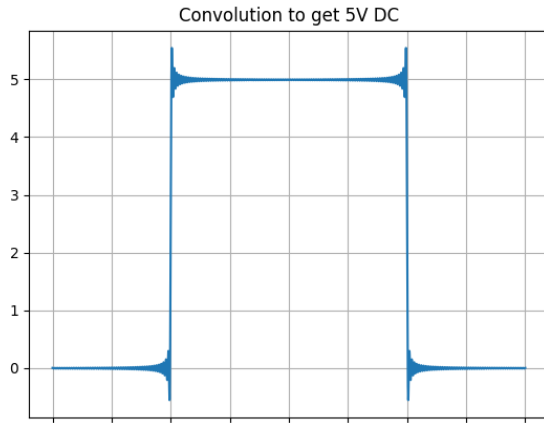
```
python 4.3.py
```

Fig. 4.3. Plot of the convolution of $x(t)$ and $h(t)$

5. Filter Design

5.1 Design a Butterworth filter for $H(f)$

Solution: The transfer function of a Butterworth filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}} \quad (5.1)$$

where n is the order of the filter and f_c is the cutoff frequency

Let the passband and stopband frequency thresholds be $50\,\text{Hz}$ and $100\,\text{Hz}$ and their corresponding attenuations be $-1\,\text{dB}$ and $-5\,\text{dB}$ respectively

$$A_p = 10 \log_{10} \left|H_n(f_p)\right|^2 \quad (5.2)$$

$$= -10 \log_{10} \left(1 + \left(\frac{f_p}{f_c}\right)^{2n}\right) \quad (5.3)$$

$$A_s = -10 \log_{10} \left(1 + \left(\frac{f_s}{f_c}\right)^{2n}\right) \quad (5.4)$$

$$\implies n = \frac{\log\left(\frac{10^{-\frac{A_p}{10}}-1}{10^{-\frac{A_s}{10}}-1}\right)}{2 \log\left(\frac{f_p}{f_s}\right)} \approx 1.53 \quad (5.5)$$

Hence, we choose a 2nd order Butterworth filter with

$$f_c = \frac{f_p}{\left(10^{-\frac{A_p}{10}} - 1\right)^{\frac{1}{2n}}} \approx 77.74\,\text{Hz} \quad (5.6)$$

5.2 Design a Chebyschev filter for $H(f)$

Solution: The transfer function of a Chebyshev filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1 + \epsilon^2 T_n^2\left(\frac{f}{f_c}\right)}} \quad (5.7)$$

where ϵ is the ripple factor, f_c is the cutoff frequency and T_n is a Chebyshev polynomial of the nth order

Assuming the same parameters as before along with a ripple of $0.1\,\text{dB}$, we get

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \approx 0.15 \quad (5.8)$$

Also, assume that $f_c = f_p \implies \frac{f_s}{f_c} > 1$

$$A_s = -10 \log_{10}\left(1 + \epsilon^2 T_n^2\left(\frac{f_s}{f_c}\right)\right) \quad (5.9)$$

$$\implies T_n\left(\frac{f_s}{f_c}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \quad (5.10)$$

$$\implies \cosh\left(n \cosh^{-1}\left(\frac{f_s}{f_c}\right)\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \quad (5.11)$$

Thus

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_c}\right)} \approx 2.26 \quad (5.12)$$

Hence, we choose a 3rd order Chebyshev filter

5.3 Design a circuit for your Butterworth filter

Solution: Using the table of normalized Butterworth coefficients, we can see that for a 2nd order Butterworth filter

$$C_1 = 1.4142\,\text{F} \quad (5.13)$$

$$L_2 = 1.4142\,\text{H} \quad (5.14)$$

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 2.89\,\text{mF} \quad (5.15)$$

$$L_2' = \frac{L_2}{2\pi f_c} = 2.89\,\text{mH} \quad (5.16)$$

5.4 Design a circuit for your Chebyschev filter

Solution: Using the table of normalized Chebyshev coefficients, we can see that for a 3rd order Chebyshev filter

$$C_1 = 1.4328\,\text{F} \quad (5.17)$$

$$L_2 = 1.5937\,\text{H} \quad (5.18)$$

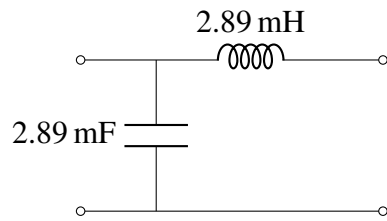$$C_3 = 1.4328\,\text{F} \quad (5.19)$$

Fig. 5.3. 2nd order Butterworth filter circuit

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 4.56\,\text{mF} \qquad (5.20)$$

$$L_2' = \frac{L_2}{2\pi f_c} = 5.07\,\text{mH} \qquad (5.21)$$

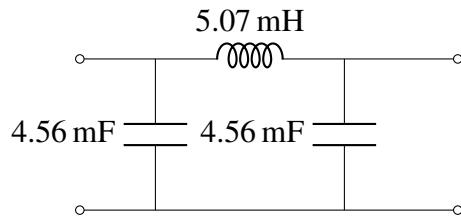$$C_3' = \frac{C_3}{2\pi f_c} = 4.56\,\text{mF} \qquad (5.22)$$

Fig. 5.4. 3rd order Chebyshev filter circuit