Digital Signal Processing

EE3900: Linear Systems and Signal Processing

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1. Software Installation

Install the necessary packages by running the following commands

```
sudo dnf up
sudo dnf install libffi−devel libsndfile python3−
    scipy python3−numpy python3−matplotlib
python −m pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There is a lot of background noise and the key strokes are audible. This noise is represented by the large blue and red regions spread from 440 Hz to beyond 18.9 kHz. The key tones are represented by the yellow lines that are present in the lower regions between 440 Hz and 5.1 kHz.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the python code for the reduction of noise by executing the following command

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /2.3.py
```

Run the code by executing

```
python 2.3.py
```

Play the newly created audio file by executing

```
aplay Sound_With_Reduced_Noise.wav
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_Reduced_Noise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The noise has been reduced considerably and the key strokes are not audible anymore. The blue region is restricted between 440 Hz and 5.1 kHz and there are no signals beyond this range.

3. Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$

3.2 Let

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$
Solution: Download the following Python code that plots Fig. 3.2.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /3.2.py
```

Run the code by executing

```
python 3.2.py
```

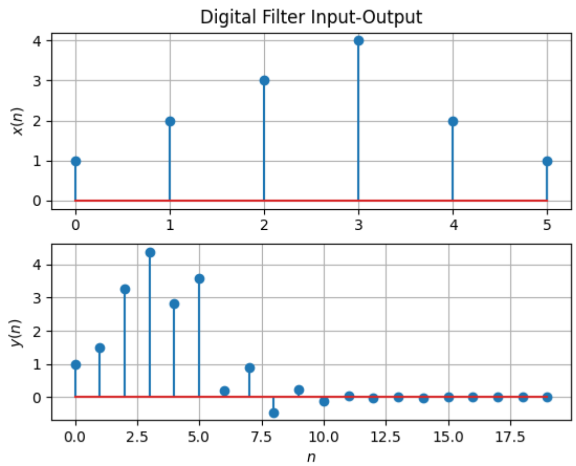

Fig. 3.2. The sketches of $x(n)$ and $y(n)$

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution:

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

Substitute $n-1=m$

$$\mathcal{Z}\{x(n-1)\} = \sum_{m=-\infty}^{\infty} x(m)z^{-(m+1)} \tag{4.5}$$

$$= z^{-1}\sum_{m=-\infty}^{\infty} x(m)z^{-m} \tag{4.6}$$

$$= z^{-1}\mathcal{Z}\{x(m)\} \tag{4.7}$$

$$= z^{-1}X(z) \tag{4.8}$$

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.9}$$

$$= \sum_{m=-\infty}^{\infty} x(m)z^{-(m+k)} \tag{4.10}$$

$$= z^{-k}\sum_{m=-\infty}^{\infty} x(m)z^{-m} \tag{4.11}$$

$$= z^{-k}X(z) \tag{4.12}$$

4.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.13}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution:

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{4.14}$$

On applying the Z-transform on both sides of the equation, we get

$$\mathcal{Z}\left\{y(n) + \frac{1}{2}y(n-1)\right\} = \mathcal{Z}\{x(n) + x(n-2)\} \tag{4.15}$$

Since we are assuming that the Z-transform is a linear operation,

$$\mathcal{Z}\{y(n)\} + \frac{1}{2}\mathcal{Z}\{y(n-1)\} = \mathcal{Z}\{x(n)\} + \mathcal{Z}\{x(n-2)\} \tag{4.16}$$

$$\implies Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.17}$$

$$\implies Y(z)\left(1 + \frac{1}{2}z^{-1}\right) = X(z)(1 + z^{-2}) \tag{4.18}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.19}$$

4.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.21}$$

is

$$U(z) = \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{4.22}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.23}$$

$$= \delta(0)z^{-0} \tag{4.24}$$

$$= 1 \tag{4.25}$$

$$\mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.26}$$

$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n \tag{4.27}$$

This is the sum of an infinite geometric progression with first term 1 and common ratio z^{-1}. The sum converges when

$$\left|z^{-1}\right| < 1 \iff |z| > 1 \tag{4.28}$$

Therefore,

$$U(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{4.29}$$

4.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.30}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.31}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \tag{4.32}$$

This is the sum of an infinite geometric progression with first term 1 and common ratio az^{-1}. The sum converges when

$$\left|az^{-1}\right| < 1 \iff |z| > |a| \tag{4.33}$$

Therefore,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.34}$$

4.5 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.35}$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete-Time Fourier Transform* (DTFT) of $x(n)$

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.36}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.37}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.38}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.39}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.40}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.41}$$

Download the following Python code that plots Fig. 4.5.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /4.5.py
```

Run the code by executing

```
python 4.5.py
```

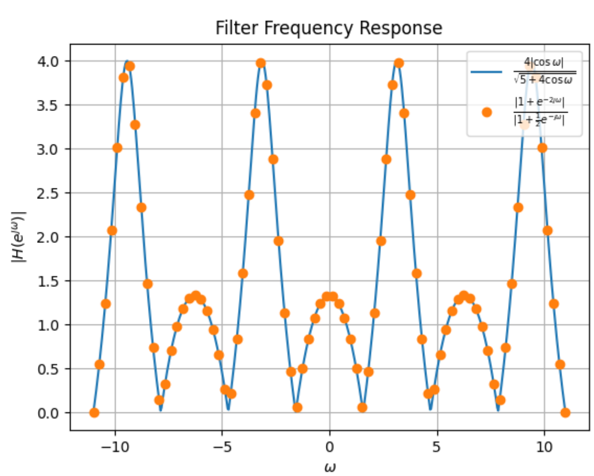

Fig. 4.5. The plot of the magnitude of the discrete-time Fourier transform of $x(n)$

From the plot, it is clear that the magnitude of the discrete-time Fourier transform of $x(n)$ is symmetric about $x = 0$ (even function) and is periodic with a period of 2π.

Also, it attains a maximum value of 4 at

$$x = (2n+1)\pi, \quad n \in \mathbb{Z} \tag{4.42}$$

and a minimum of 0 at

$$x = (2m+1)\frac{\pi}{2}, \quad m \in \mathbb{Z} \tag{4.43}$$

5. Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2)
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

From (4.30),

$$\frac{1}{1-az^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} a^n u(n) \quad |z| > |a| \tag{5.4}$$

$$\implies \frac{1}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \tag{5.5}$$

$$\implies \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.6}$$

Since the Z-transform is a linear operator, for $|z| > \frac{1}{2}$

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.7}$$

Therefore,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

5.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: Download the following Python code that plots Fig. 5.2.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /5.2.py
```

Run the code by executing

```
python 5.2.py
```

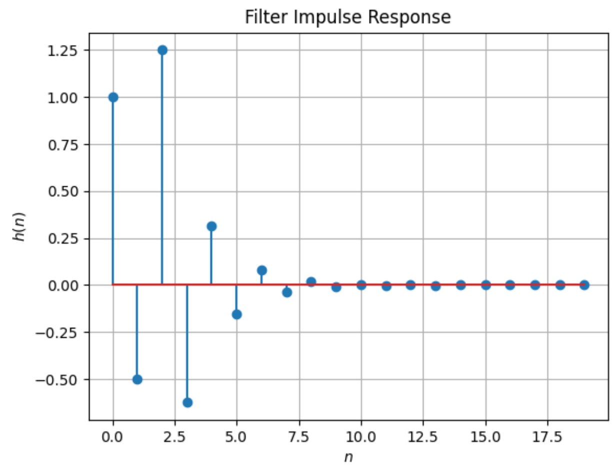

Fig. 5.2. The plot of $h(n)$

From the plot, it is clear that the sequence is convergent to 0, which implies that it is bounded as well.

5.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.9}$$

Is the system defined by (3.2) stable for the impulse response in (5.1)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \tag{5.11}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.12}$$

$$= \frac{4}{3} < \infty \tag{5.13}$$

Therefore, the system is stable.

5.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \tag{5.14}$$

This is the definition of $h(n)$
Solution:

$$h(0) = 1 \tag{5.15}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.16}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.17}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.18}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{3}{2} \tag{5.19}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.20}$$

$$h(3) = \frac{3}{2}\left(-\frac{1}{2}\right) \tag{5.21}$$

$$h(4) = \frac{3}{2}\left(-\frac{1}{2}\right)^2 \tag{5.22}$$

$$\vdots \tag{5.23}$$

$$h(n) = \frac{3}{2}\left(-\frac{1}{2}\right)^{n-2} \tag{5.24}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{3}{2}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.25}$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \tag{5.26}$$

Download the following Python code that plots Fig. 5.4.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /5.4.py
```

Run the code by executing

```
python 5.4.py
```

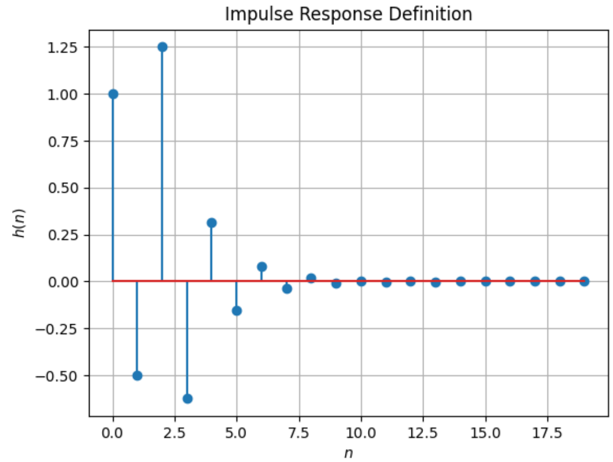

Fig. 5.4. The plot of $h(n)$ from its definition

5.5 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.27}$$

Comment. The operation in (5.27) is known as *convolution*
Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.28}$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \tag{5.29}$$

since $x(k) = 0$ for $k < 0$ and $k > 5$
Download the following Python code that plots Fig. 5.5.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /5.5.py
```

Run the code by executing

```
python 5.5.py
```

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we can conclude that

$$y(n) = x(n) * h(n) \tag{5.30}$$

5.6 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.31}$$

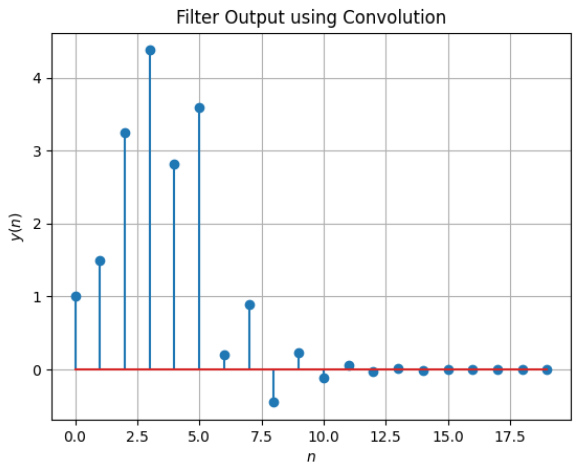

Fig. 5.5. Plot of the convolution of $x(n)$ and $h(n)$

Solution: We know that

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.32)$$

Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \quad (5.33)$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \quad (5.34)$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \quad (5.35)$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.36)$$

$$\implies x(n) * h(n) = h(n) * x(n) \quad (5.37)$$

Therefore, convolution is commutative.

6. DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N} \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$
Solution: Download the following Python code that plots Fig. 6.1.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /6.1.py
```

Run the code by executing

```
python 6.1.py
```

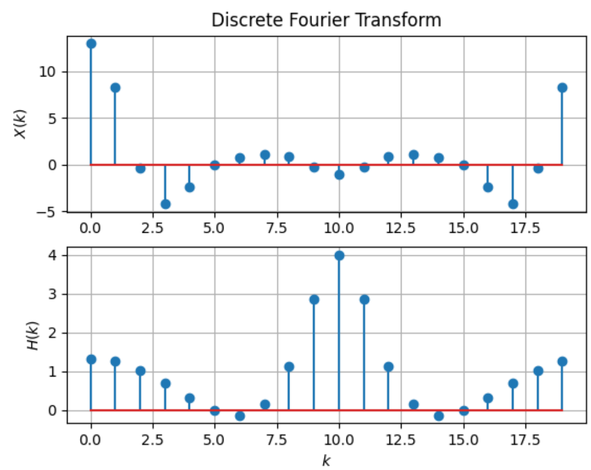

Fig. 6.1. Plots of the real parts of the discrete Fourier transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

Solution: Download the following Python code that plots Fig. 6.2.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /6.2.py
```

Run the code by executing

```
python 6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\jmath 2\pi kn/N} \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

Solution: Download the following Python code that plots Fig. 6.3.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /6.3.py
```

Run the code by executing

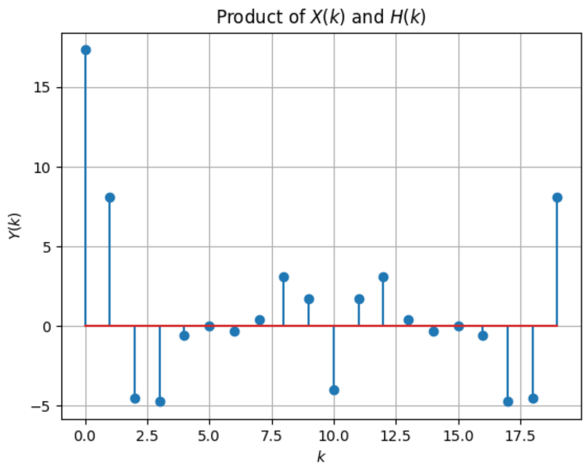

Fig. 6.2. Plot of $Y(k)$

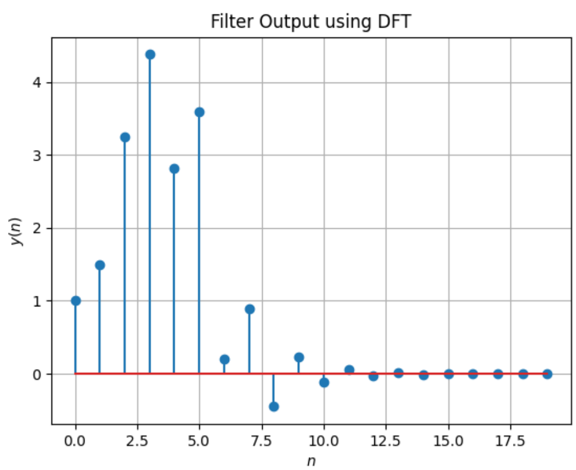

Fig. 6.3. Plot of the inverse discrete Fourier transform of $Y(k)$

```
python 6.3.py
```

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we conclude that

$$y(n) = x(n) * h(n) \tag{6.4}$$

$$\iff Y(k) = X(k)H(k) \tag{6.5}$$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the following Python code that plots Fig. 6.4.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /6.4.py
```

Run the code by executing

```
python 6.4.py
```

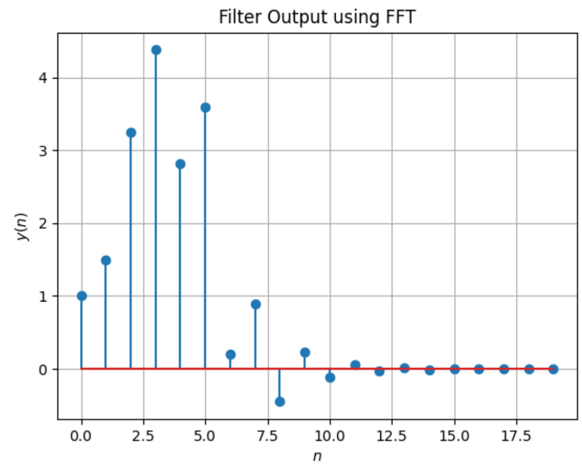

Fig. 6.4. Plot of $y(n)$ by fast Fourier transform

The plot is exactly the same as that obtained in Fig. 3.2.

6.5 Wherever possible, express all the above equations as matrix equations.
Solution:

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.6}$$

$$\mathbf{h} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.7}$$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{6.8}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N-1} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \tag{6.9}$$

The convolution can be written using a Toeplitz matrix.
Consider the DFT matrix

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.10}$$

where $\omega = e^{-\jmath 2\pi/N}$ is the N^{th} root of unity
Then the discrete Fourier transforms of $\mathbf{x}$ and $\mathbf{h}$ are given by

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{6.11}$$

$$\mathbf{H} = \mathbf{W}\mathbf{h} \tag{6.12}$$

$\mathbf{Y}$ is then given by

$$\mathbf{Y} = \mathbf{X} \circ \mathbf{H} \tag{6.13}$$

where $\circ$ denotes the Hadamard product (element-wise multiplication)
But $\mathbf{Y}$ is the discrete Fourier transform of the filter output $\mathbf{y}$

$$\mathbf{Y} = \mathbf{W}\mathbf{y} \tag{6.14}$$

Thus,

$$\begin{aligned} \mathbf{W}\mathbf{y} &= \mathbf{X} \circ \mathbf{H} && (6.15) \\ \implies \mathbf{y} &= \mathbf{W}^{-1} (\mathbf{X} \circ \mathbf{H}) && (6.16) \\ &= \mathbf{W}^{-1} (\mathbf{W}\mathbf{x} \circ \mathbf{W}\mathbf{h}) && (6.17) \end{aligned}$$

This is the inverse discrete Fourier transform of $\mathbf{Y}$