

# Pingala Series

## EE3900: Linear Systems and Signal Processing

### Indian Institute of Technology Hyderabad

Ankit Saha  
AI21BTECH11004

28 Sep 2022

#### 1. JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

**Solution:** Download the following Python code that verifies the above

```
wget https://github.com/Ankit-Saha-2003/EE3900/
raw/main/Pingala/codes/1.py
```

Run the code by executing

```
python 1.py
```

It turns out that everything apart from the last equation is true

#### 2. PINGALA SERIES

2.1 The *one sided* Z-transform of  $x(n)$  is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n) \quad (2.2)$$

$$x(0) = x(1) = 1, n \geq 0 \quad (2.3)$$

Generate a stem plot for  $x(n)$ .

**Solution:** Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit-Saha-2003/
EE3900/raw/main/Pingala/codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

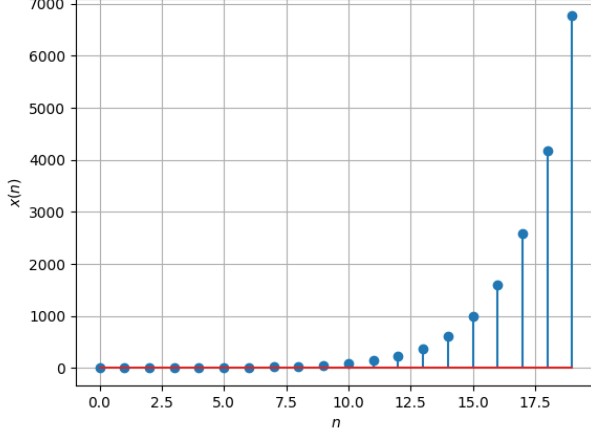


Fig. 2.2. Plot of  $x(n)$

### 2.3 Find $X^+(z)$

**Solution:**

We have

$$x(n+2) = x(n+1) + x(n) \quad n \geq 0 \quad (2.4)$$

On taking the one-sided Z-transform on both sides of the equation, we get

$$\mathcal{Z}^+ \{x(n+2)\} = \mathcal{Z}^+ \{x(n+1) + x(n)\} \quad (2.5)$$

Since, the one-sided Z-transform is a linear transformation

$$\Rightarrow \mathcal{Z}^+ \{x(n+2)\} = \mathcal{Z}^+ \{x(n+1)\} + \mathcal{Z}^+ \{x(n)\} \quad (2.6)$$

$$\Rightarrow \sum_{n=0}^{\infty} x(n+2)z^{-n} = \sum_{n=0}^{\infty} x(n+1)z^{-n} + \sum_{n=0}^{\infty} x(n)z^{-n} \quad (2.7)$$

But

$$\sum_{n=0}^{\infty} x(n+2)z^{-n} = z^2 \sum_{k=2}^{\infty} x(k)z^{-k} \quad (2.8)$$

$$= z^2 (X^+(z) - x(0) - x(1)z^{-1}) \quad (2.9)$$

$$= z^2 X^+(z) - z^2 - z \quad (2.10)$$

$$\sum_{n=0}^{\infty} x(n+1)z^{-n} = z \sum_{k=1}^{\infty} x(k)z^{-k} \quad (2.11)$$

$$= z (X^+(z) - x(0)) \quad (2.12)$$

$$= z X^+(z) - z \quad (2.13)$$

Thus

$$\Rightarrow z^2 X^+(z) - z^2 - z = z X^+(z) - z + X^+(z) \quad (2.14)$$

$$\Rightarrow (z^2 - z - 1) X^+(z) = z^2 \quad (2.15)$$

$$\Rightarrow X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.16)$$

$$\therefore X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad |z| > \max(|\alpha|, |\beta|) \quad (2.17)$$

### 2.4 Find $x(n)$

**Solution:** Using partial fraction decomposition

$$X^+(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left( \frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right) \quad (2.18)$$

We know that

$$\frac{1}{1 - \alpha z^{-1}} \stackrel{\mathcal{Z}}{\rightleftharpoons} \alpha^n u(n) \quad (2.19)$$

$$\frac{1}{1 - \beta z^{-1}} \stackrel{\mathcal{Z}}{\rightleftharpoons} \beta^n u(n) \quad (2.20)$$

$$zF(z) \stackrel{\mathcal{Z}}{\rightleftharpoons} f(n+1) \quad (2.21)$$

Since we are dealing with the unit step function here, the Z-transform and the one-sided Z-transform are both the same

$$x(n) = \frac{1}{\alpha - \beta} (\alpha^{n+1} u(n+1) - \beta^{n+1} u(n+1)) \quad (2.22)$$

Since at  $n = -1$ ,  $x(n)$  is zero anyway, we can write

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \quad (2.23)$$

### 2.5 Sketch

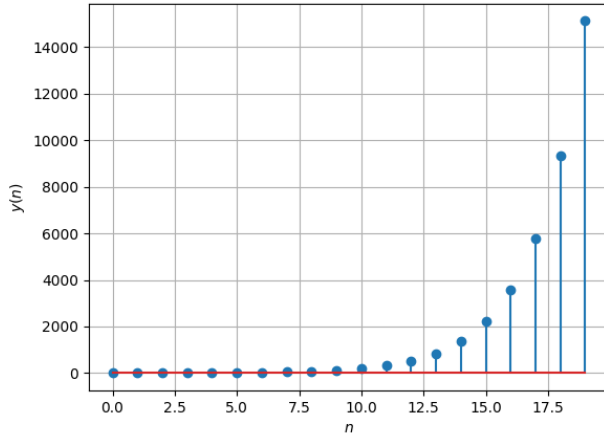
$$y(n) = x(n-1) + x(n+1) \quad n \geq 0 \quad (2.24)$$

**Solution:** Download the following Python code that plots Fig. 2.5

```
wget https://github.com/Ankit-Saha-2003/EE3900/raw/main/Pingala/codes/2.5.py
```

Run the code by executing

```
python 2.5.py
```

Fig. 2.5. Plot of  $y(n)$ 2.6 Find  $Y^+(z)$ **Solution:** We have

$$y(n) = x(n-1) + x(n+1) \quad n \geq 0 \quad (2.25)$$

On taking the one-sided Z-transform on both sides of the equation, we get

$$\mathcal{Z}^+ \{y(n)\} = \mathcal{Z}^+ \{x(n-1) + x(n+1)\} \quad (2.26)$$

$$= \mathcal{Z}^+ \{x(n+1)\} + \mathcal{Z}^+ \{x(n-1)\} \quad (2.27)$$

$$= \sum_{n=0}^{\infty} x(n+1)z^{-n} + \sum_{n=0}^{\infty} x(n-1)z^{-n} \quad (2.28)$$

But

$$\sum_{n=0}^{\infty} x(n-1)z^{-n} = z^{-1} \sum_{k=-1}^{\infty} x(k)z^{-k} \quad (2.29)$$

$$= z^{-1} (X^+(z) + x(-1)z) \quad (2.30)$$

$$= z^{-1} X^+(z) \quad (2.31)$$

Thus

$$Y^+(z) = zX^+(z) - z + z^{-1}X^+(z) \quad (2.32)$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \quad (2.33)$$

$$= \frac{z + z^{-1} - z + 1 + z^{-1}}{1 - z^{-1} - z^{-2}} \quad (2.34)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad |z| > \max(|\alpha|, |\beta|) \quad (2.35)$$

2.7 Find  $y(n)$ **Solution:**

$$y(n) = x(n+1) + x(n-1) \quad n \geq 0 \quad (2.36)$$

$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} u(n+1) + \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n-1) \quad (2.37)$$

We have

$$y(n) = 0 \quad \forall n < 0 \quad (2.38)$$

$$y(0) = x(1) + x(-1) = 1 + 0 = 1 \quad (2.39)$$

For  $n > 0$ 

$$y(n) = \frac{\alpha^{n+2} - \beta^{n+2} + \alpha^n - \beta^n}{\alpha - \beta} \quad (2.40)$$

Since  $\alpha\beta = -1$ 

$$y(n) = \frac{\alpha^{n+2} - (\alpha\beta)\alpha^n - \beta^{n+2} + (\alpha\beta)\beta^n}{\alpha - \beta} \quad (2.41)$$

$$= \frac{\alpha^{n+1}(\alpha - \beta) + \beta^{n+1}(\alpha - \beta)}{\alpha - \beta} \quad (2.42)$$

$$= \alpha^{n+1} + \beta^{n+1} \quad (2.43)$$

If we plug in  $n = 0$  in this equation, we get

$$y(0) = \alpha + \beta = 1 \quad (2.44)$$

which is consistent with what we obtained earlier

Therefore

$$y(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (2.45)$$

## 3. POWER OF THE Z TRANSFORM

## 3.1 Show that

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \quad (3.1)$$

**Solution:** We have obtained that  $a_n = x(n-1)$ 

$$\sum_{k=1}^n a_k = \sum_{k=1}^n x(k-1) \quad (3.2)$$

$$= \sum_{k=0}^{n-1} x(k) \quad (3.3)$$

Also

$$x(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \quad (3.4)$$

$$u(n-1-k) = \begin{cases} 1 & k \leq n-1 \\ 0 & k > n-1 \end{cases} \quad (3.5)$$

Thus

$$\sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \quad (3.6)$$

$$= x(n) * u(n-1) \quad (3.7)$$

3.2 Show that

$$a_{n+2} - 1 \quad n \geq 1 \quad (3.8)$$

can be expressed as

$$(x(n+1) - 1)u(n) \quad (3.9)$$

**Solution:**

$$a_{n+2} - 1 \quad n \geq 1 \quad (3.10)$$

can be written as

$$(a_{n+2} - 1)u(n-1) \quad (3.11)$$

Since the system defined by  $a$  is time-invariant and  $a_n$  is just  $x(n)$  shifted forward by one unit, we can shift this backward by one unit to get the expression in terms of  $x$

$$a_{n+2} \Rightarrow x(n+1) \quad (3.12)$$

$$u(n-1) \Rightarrow u(n) \quad (3.13)$$

Therefore, the above expression can be expressed as

$$(x(n+1) - 1)u(n) \quad (3.14)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+ (10) \quad (3.15)$$

**Solution:**

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}} \quad k \rightarrow k+1 \quad (3.16)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \quad (3.17)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad \because a_{k+1} = x(k) \quad (3.18)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} x(k)10^{-k} \quad (3.19)$$

$$= \frac{1}{10} X^+(10) \quad (3.20)$$

3.4 Show that

$$\alpha^n + \beta^n \quad n \geq 1 \quad (3.21)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.22)$$

and find  $W(z)$

**Solution:**

$$\alpha^n + \beta^n \quad n \geq 1 \quad (3.23)$$

can be written as

$$(\alpha^n + \beta^n)u(n-1) \quad (3.24)$$

On shifting this backward by one unit, we get

$$(\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.25)$$

by time-invariance

Now

$$w(n) = \alpha \cdot \alpha^n u(n) + \beta \cdot \beta^n u(n) \quad (3.26)$$

We know that

$$\alpha^n u(n) \stackrel{Z}{\rightleftharpoons} \frac{1}{1 - \alpha z^{-1}} \quad |z| > |\alpha| \quad (3.27)$$

$$\beta^n u(n) \stackrel{Z}{\rightleftharpoons} \frac{1}{1 - \beta z^{-1}} \quad |z| > |\beta| \quad (3.28)$$

Therefore

$$W(z) = \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \quad |z| > \max(|\alpha|, |\beta|) \quad (3.29)$$

$$= \frac{\alpha(1 - \beta z^{-1}) + \beta(1 - \alpha z^{-1})}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad (3.30)$$

$$= \frac{\alpha + \beta - 2\alpha\beta z^{-1}}{1 - (\alpha + \beta)z^{-1} + \alpha\beta z^{-2}} \quad (3.31)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.32)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+ (10) \quad (3.33)$$

**Solution:** Note that

$$b_n = a_{n-1} + a_{n+1} \quad n \geq 2, b_1 = 1 \quad (3.34)$$

$$= x(n-2) + x(n) \quad (3.35)$$

$$= y(n-1) \quad n \geq 2 \quad (3.36)$$

At  $n = 1$ ,  $y(0) = 1$  anyway, thus

$$b_n = y(n-1) \quad n \geq 1 \quad (3.37)$$

Now

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}} \quad k \rightarrow k+1 \quad (3.38)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \quad (3.39)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad \because b_{k+1} = y(k) \quad (3.40)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \quad (3.41)$$

$$= \frac{1}{10} Y^+(10) \quad (3.42)$$

3.6 Solve the JEE 2019 problem.

**Solution:** We have shown that

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.43)$$

On taking the Z-transform on both sides of the equation

$$\mathcal{Z} \left\{ \sum_{k=1}^n a_k \right\} = X(z) (z^{-1} U(z)) \quad (3.44)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \quad (3.45)$$

$$= \frac{1}{z^{-1}} \left( \frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right) \quad (3.46)$$

On taking the inverse Z-transform on both sides of this equation, we get

$$\sum_{k=1}^n a_k = x(n+1)u(n) - u(n+1) \quad (3.47)$$

For  $n \geq 1$

$$\sum_{k=1}^n a_k = x(n+1) - 1 \quad (3.48)$$

$$= a_{n+2} - 1 \quad n \geq 1 \quad (3.49)$$

We have proved that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^+(10) \quad (3.50)$$

$$= \frac{1}{10} \frac{1}{(1 - 10^{-1} - 10^{-2})} \quad (3.51)$$

$$= \frac{1}{10} \frac{100}{100 - 10 - 1} \quad (3.52)$$

$$= \frac{10}{89} \quad (3.53)$$

We have also shown that

$$b_n = y(n-1) \quad n \geq 1 \quad (3.54)$$

But

$$y(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.55)$$

Thus

$$b_n = (\alpha^n + \beta^n)u(n-1) \quad n \geq 1 \quad (3.56)$$

$$= \alpha^n + \beta^n \quad n \geq 1 \quad (3.57)$$

We have also proved that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \quad (3.58)$$

$$= \frac{1}{10} \frac{1 + 2(10)^{-1}}{(1 - 10^{-1} - 10^{-2})} \quad (3.59)$$

$$= \frac{1}{10} \frac{100 + 20}{100 - 10 - 1} \quad (3.60)$$

$$= \frac{12}{89} \quad (3.61)$$

Therefore, all of the options except the last option are correct