Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t) e^{-st} \, \mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 μC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 μC

2.2. Draw the circuit using latex-tikz
Solution:

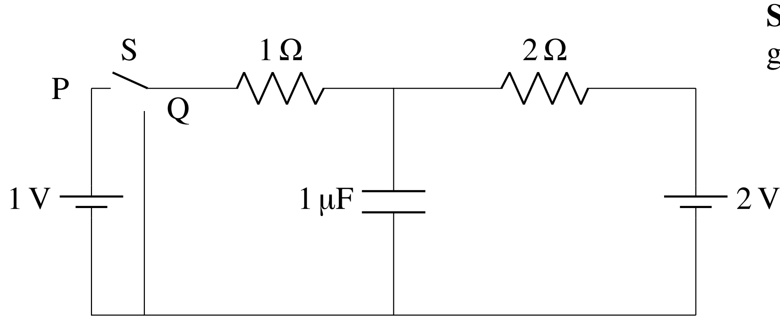
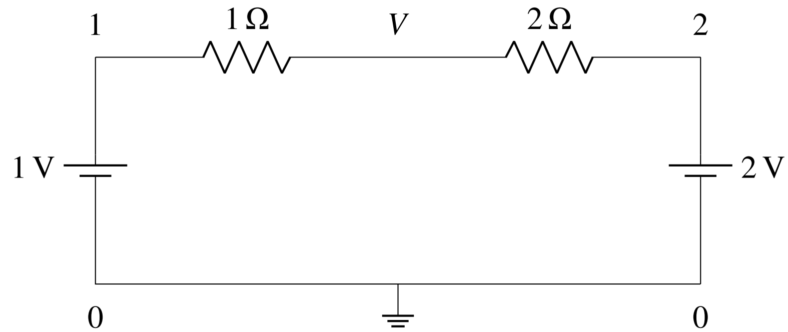


Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero



By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC
Solution: The Laplace transform of $u(t)$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-st} \, \mathrm{d}t \tag{2.4}$$

$$= \int_{0}^{\infty} e^{-st} \, \mathrm{d}t \tag{2.5}$$

$$= \lim_{R \to \infty} \frac{1 - e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC
Therefore

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find the ROC

Solution: The Laplace transform of $e^{-at}u(t)$ for $a > 0$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,\mathrm{d}t \tag{2.9}$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\,\mathrm{d}t \tag{2.10}$$

$$= \lim_{R\to\infty} \frac{1-e^{-(s+a)R}}{s+a} \tag{2.11}$$

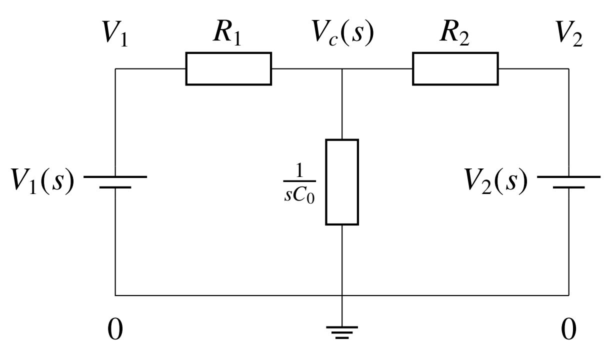
This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC

Therefore

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2



where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.13}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_c(s)$

Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$

$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.17}$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.18}$$

$$\implies V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.19}$$

$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.20}$$

2.7. Find $v_c(t)$. Plot using Python.

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right), \Re(s) > 0 \tag{2.21}$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{2.22}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{2.23}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\mathrm{F}$

$$v_c(t) = \frac{4}{3}\left(1 - \mathrm{e}^{-\frac{3}{2}\times 10^6 \mathrm{t}}\right)\mathrm{u(t)}\,\mathrm{V} \tag{2.24}$$

2.8. Verify your result using ngspice

3. Initial Conditions

3.1. Find q_2 in Fig. 2.2

Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

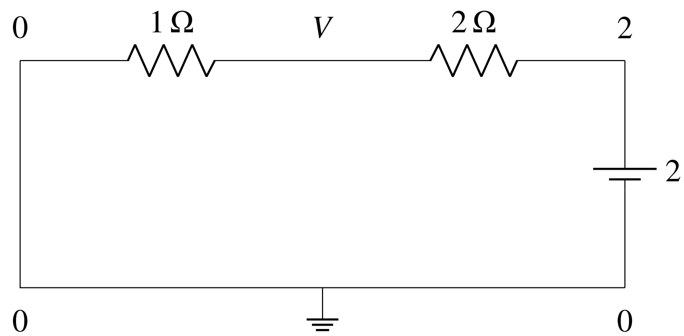
By Kirchoff's junction law, we get

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \tag{3.1}$$

$$\implies V = \frac{2}{3}\,\mathrm{V} \tag{3.2}$$

$$\implies q_2 = CV = \frac{2}{3}\,\mu\mathrm{C} \tag{3.3}$$

3.2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables



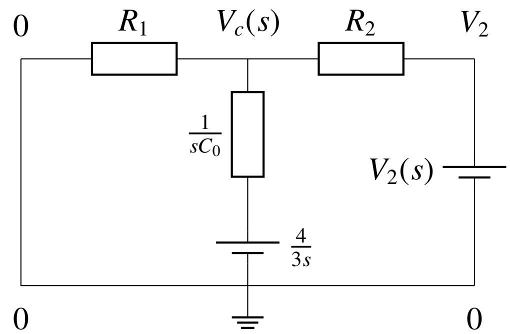


Fig. 3.2. Circuit diagram in s-domain

R_1, R_2, C_0 for the passive elements. Use latex-tikz

Solution:

The battery $\frac{4}{3s}$ corresponds to the intial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3. Find $V_c(s)$

Solution: By Kirchoff's junction law, we get

$$\frac{V_c - 0}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.4)$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_2}{R_2} + \frac{4}{3}C_0 \quad (3.5)$$

$$\implies V_c(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (3.6)$$

$$= \frac{\frac{2}{R_2C_0} + \frac{4}{3}s}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \quad (3.7)$$

3.4. Find $v_c(t)$. Plot using Python

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3}\left(\frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) \quad (3.8)$$

for $\Re(s) > 0$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \quad (3.9)$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu$F

$$v_c(t) = \frac{4}{3}e^{-\frac{3}{2}\times 10^6 t}u(t) + \frac{2}{3}\left(1 - e^{-\frac{3}{2}\times 10^6 t}\right)u(t) \quad (3.10)$$

$$= \frac{2}{3}\left(1 + \mathrm{e}^{-\frac{3}{2}\times 10^6 \mathrm{t}}\right)\mathrm{u(t)}\,\mathrm{V} \quad (3.11)$$

3.5. Verify your result using ngspice

3.6. Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$

Solution: At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3}\,\mathrm{V} \quad (3.12)$$

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t \to 0^+} v_c(t) = \frac{4}{3}\,\mathrm{V} \quad (3.13)$$

$$v_c(\infty) = \lim_{t \to \infty} v_c(t) = \frac{2}{3}\,\mathrm{V} \quad (3.14)$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \quad (3.15)$$

where $q(t)$ is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (3.16)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.17}$$

$$\Longrightarrow Q(s) = C_0 V_c(s) \tag{3.18}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.19}$$

$$\Longrightarrow \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.20}$$

which is the same equation as the one we obtained from Fig. 3.2