Digital Signal Processing

EE3900: Linear Systems and Signal Processing

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1. Software Installation

Install the necessary packages by running the following commands

```
sudo dnf up
sudo dnf install libffi−dev libsndfile1 python3−
    scipy python3−numpy python3−matplotlib
pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There is a lot of background noise and the key strokes are audible. This noise is represented by the large blue and red regions spread from 440 Hz to beyond 18.9 kHz. The key tones are represented by the yellow lines that are present in the lower regions between 440 Hz and 5.1 kHz.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the python code for the reduction of noise by executing the following command

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /2.3.py
```

Run the code by executing

```
python 2.3.py
```

Play the newly created audio file by executing

```
aplay Sound_With_Reduced_Noise.wav
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_Reduced_Noise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The noise has been reduced considerably and the key strokes are not audible anymore. The blue region is restricted between 440 Hz and 5.1 kHz and there are no signals beyond this range.

3. Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$
Solution: Download the following Python code that plots Fig. 3.2.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /3.2.py
```

Run the code by executing

```
python 3.2.py
```

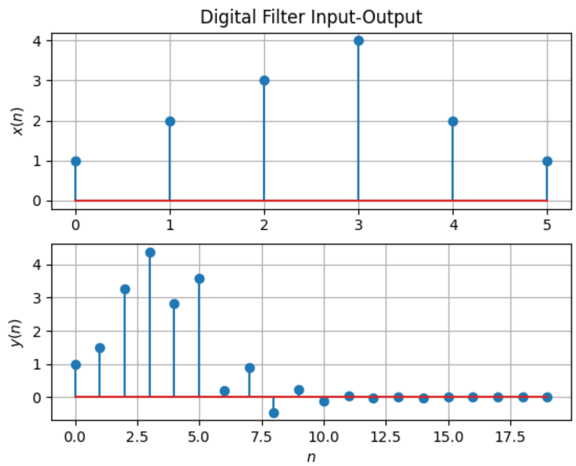

Fig. 3.2. The sketches of $x(n)$ and $y(n)$

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution:

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.4}$$

Substitute $n - 1 = m$

$$\mathcal{Z}\{x(n-1)\} = \sum_{m=-\infty}^{\infty} x(m) z^{-(m+1)} \tag{4.5}$$

$$= z^{-1} \sum_{m=-\infty}^{\infty} x(m) z^{-m} \tag{4.6}$$

$$= z^{-1} \mathcal{Z}\{x(m)\} \tag{4.7}$$

$$= z^{-1} X(z) \tag{4.8}$$

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n} \tag{4.9}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)} \tag{4.10}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-m} \tag{4.11}$$

$$= z^{-k} X(z) \tag{4.12}$$

4.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.13}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution:

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \tag{4.14}$$

On applying the Z-transform on both sides of the equation, we get

$$\mathcal{Z}\left\{y(n) + \frac{1}{2} y(n-1)\right\} = \mathcal{Z}\{x(n) + x(n-2)\} \tag{4.15}$$

Since we are assuming that the Z-transform is a linear operation,

$$\mathcal{Z}\{y(n)\} + \frac{1}{2}\mathcal{Z}\{y(n-1)\} = \mathcal{Z}\{x(n)\} + \mathcal{Z}\{x(n-2)\} \tag{4.16}$$

$$\implies Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.17}$$

$$\implies Y(z)\left(1 + \frac{1}{2} z^{-1}\right) = X(z)(1 + z^{-2}) \tag{4.18}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.19}$$

4.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.21}$$

is

$$U(z) = \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{4.22}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.23}$$

$$= \delta(0) z^{-0} \tag{4.24}$$

$$= 1 \tag{4.25}$$

$$\mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.26}$$

$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n \tag{4.27}$$

This is the sum of an infinite geometric progression with first term 1 and common ratio z^{-1}. The sum converges when

$$\left|z^{-1}\right| < 1 \iff |z| > 1 \tag{4.28}$$

Therefore,

$$U(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{4.29}$$

4.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.30}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.31}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \tag{4.32}$$

This is the sum of an infinite geometric progression with first term 1 and common ratio az^{-1}. The sum converges when

$$\left|az^{-1}\right| < 1 \iff |z| > |a| \tag{4.33}$$

Therefore,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.34}$$

4.5 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.35}$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete-Time Fourier Transform* (DTFT) of $x(n)$

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.36}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.37}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.38}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.39}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.40}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.41}$$

Download the following Python code that plots Fig. 4.5.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Assignment_1/codes
    /4.5.py
```

Run the code by executing

```
python 4.5.py
```

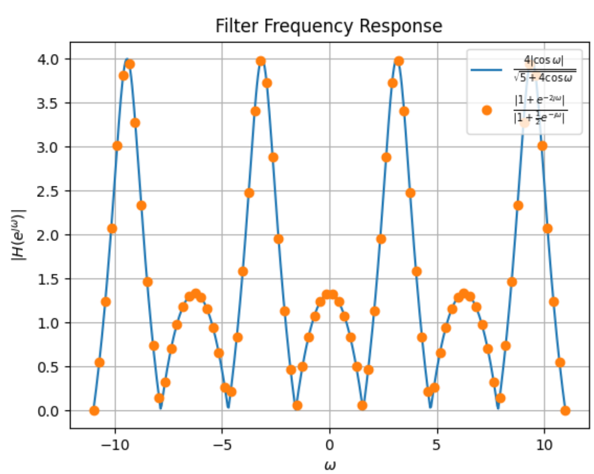

Fig. 4.5. The plot of the magnitude of the discrete-time Fourier transform of $x(n)$

From the plot, it is clear that the magnitude of the discrete-time Fourier transform of $x(n)$ is symmetric about $x = 0$ and is periodic with a period of 2π.

Also, it attains a maximum value of 4 at

$$x = (2n + 1)\pi, \quad n \in \mathbb{Z} \tag{4.42}$$

and a minimum of 0 at

$$x = (2m + 1)\frac{\pi}{2}, \quad m \in \mathbb{Z} \tag{4.43}$$

5. Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2)
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$= \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.3}$$

From (4.30),

$$\frac{1}{1 - az^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} a^n u(n) \quad |z| > |a| \tag{5.4}$$

$$\implies \frac{1}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \tag{5.5}$$

$$\implies \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.6}$$

Since the Z-transform is a linear operator, for $|z| > \frac{1}{2}$

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.7}$$

Therefore,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

5.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: Download the following Python code that plots Fig. 5.2.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Assignment_1/codes
    /5.2.py
```

Run the code by executing

```
python 5.2.py
```

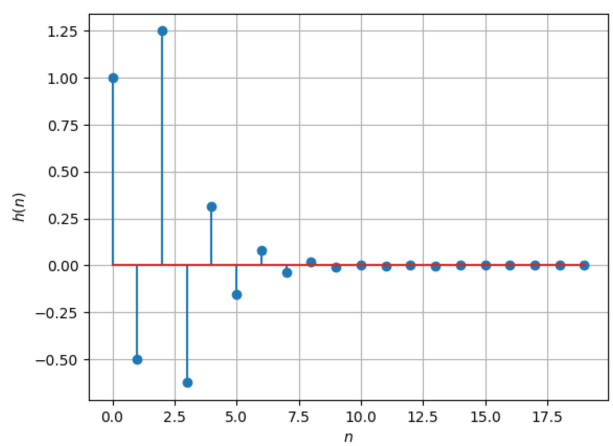

Fig. 5.2. The plot of $h(n)$