Fourier Series
EE3900: Linear Systems and Signal Processing
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1. Periodic Function

Let

$$x(t) = A_0 \left| \sin\left(2\pi f_0 t\right) \right| \qquad (1.1)$$

1.1 Plot $x(t)$

Solution: Download the following Python code that plots Fig. 1.1.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/1.1.py
```

Run the code by executing
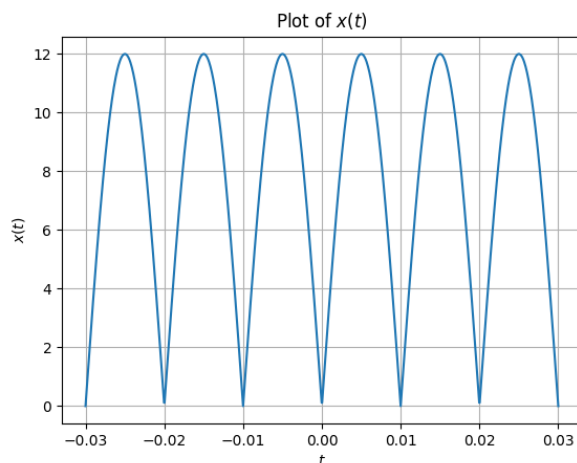
```
python 1.1.py
```


Fig. 1.1. Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period

Solution: Since $x(t)$ is the absolute value of a sinusoidal function, it is periodic, which is also evident from the plot

Consider $x(t + \frac{1}{2f_0})$

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left| \sin\left(2\pi f_0\left(t + \frac{1}{2f_0}\right)\right) \right| \qquad (1.2)$$

$$= A_0 \left| \sin\left(2\pi f_0 t + \pi f_0\right) \right| \qquad (1.3)$$

$$= A_0 \left| (-1)^{f_0} \sin\left(2\pi f_0 t\right) \right| \qquad (1.4)$$

$$= A_0 \left| \sin\left(2\pi f_0 t\right) \right| \qquad (1.5)$$

$$= x(t) \qquad (1.6)$$

Therefore, $x(t)$ is periodic with period $\frac{1}{2f_0}$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \qquad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t} \, \mathrm{d}t \qquad (2.2)$$

Solution:

$$x(t)e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi (n-k) f_0 t} \qquad (2.3)$$

$$\implies \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi n f_0 t} \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t} \, \mathrm{d}t \qquad (2.4)$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t} \, \mathrm{d}t = \begin{cases} \frac{1}{f_0} & k = n \\ 0 & k \neq n \end{cases} \qquad (2.5)$$

$$= \frac{1}{f_0} \delta(n - k) \qquad (2.6)$$

$$\sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi(n-k)f_0 t} \, dt = \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0} \delta(n-k)$$

$$(2.7)$$

$$= \frac{1}{f_0} c_n * \delta(n) \quad (2.8)$$

$$= \frac{1}{f_0} c_n \quad (2.9)$$

Therefore

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t} \, dt \quad (2.10)$$

2.2 Find c_k for (1.1)
Solution:

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin\left(2\pi f_0 t\right) \right| e^{-\jmath 2\pi k f_0 t} \, dt \quad (2.11)$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{0} A_0 \left(-\sin\left(2\pi f_0 t\right) \right) e^{-\jmath 2\pi k f_0 t} \, dt$$

$$+ f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\sin\left(2\pi f_0 t\right) \right) e^{-\jmath 2\pi k f_0 t} \, dt \quad (2.12)$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 u\right) e^{\jmath 2\pi k f_0 u} \, dt$$

$$+ f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) e^{-\jmath 2\pi k f_0 t} \, dt \quad (2.13)$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \left(e^{\jmath 2\pi k f_0 t} + e^{-\jmath 2\pi k f_0 t} \right) dt$$

$$(2.14)$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} 2 \sin\left(2\pi f_0 t\right) \cos\left(2\pi k f_0 t\right) \, dt$$

$$(2.15)$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left\{ \sin\left(2\pi(1+k)f_0 t\right) + \sin\left(2\pi(1-k)f_0 t\right) \right\} dt$$

$$(2.16)$$

$$= f_0 A_0 \left[-\frac{\cos\left(2\pi(1+k)f_0 t\right)}{2\pi(1+k)f_0} - \frac{\cos\left(2\pi(1-k)f_0 t\right)}{2\pi(1-k)f_0} \right]_{0}^{\frac{1}{2f_0}}$$

$$(2.17)$$

$$= \frac{f_0 A_0}{2\pi f_0} \left[\frac{1 - (-1)^{1+k}}{1+k} + \frac{1 - (-1)^{1-k}}{1-k} \right] \quad (2.18)$$

$$= \left(1 + (-1)^k \right) \frac{A_0}{2\pi} \left[\frac{1}{1+k} + \frac{1}{1-k} \right] \quad (2.19)$$

$$= \left(1 + (-1)^k \right) \frac{A_0}{\pi(1-k^2)} \quad (2.20)$$

Therefore

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases} \quad (2.21)$$

2.3 Verify (1.1) using Python
Solution: Download the following Python code that plots Fig. 2.3.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/2.3.py
```

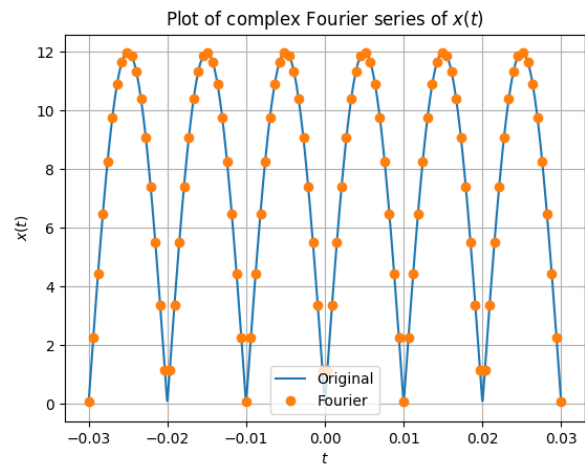Run the code by executing

```
python 2.3.py
```


Fig. 2.3. Plot of $x(t)$ along with its complex Fourier series expansion

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos\left(2\pi k f_0 t\right) + b_k \sin\left(2\pi k f_0 t\right) \right)$$

$$(2.22)$$

and obtain the formulae for a_k and b_k
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \quad (2.23)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \quad (2.24)$$

Thus

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k} \right) \cos\left(2\pi k f_0 t\right)$$

$$+ \sum_{k=1}^{\infty} \jmath \left(c_k - c_{-k} \right) \sin\left(2\pi k f_0 t\right) \quad (2.25)$$

Therefore

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.26)$$

$$b_k = \jmath(c_k - c_{-k}) \quad k \geq 0 \quad (2.27)$$

2.5 Find a_k and b_k for (1.1)

Solution:

$$a_0 = c_0 = \frac{2A_0}{\pi} \quad (2.28)$$

For $k > 0$, if k is odd

$$a_k = 0 + 0 = 0 \quad (2.29)$$

and if k is even

$$a_k = \frac{2A_0}{\pi(1 - k^2)} + \frac{2A_0}{\pi(1 - k^2)} = \frac{4A_0}{\pi(1 - k^2)} \quad (2.30)$$

For odd or even k, $c_k = c_{-k}$ always

$$b_k = 0 \quad \forall k \geq 0 \quad (2.31)$$

Therefore

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k = 2m, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad (2.32)$$

$$b_k = 0 \quad k \geq 0 \quad (2.33)$$

2.6 Verify (2.22) using Python

Solution: Download the following Python code that plots Fig. 2.6.

wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/2.6.py

Run the code by executing

python 2.6.py

3. FOURIER TRANSFORM

3.1

$$\delta(t) = 0 \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \quad (3.2)$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,\mathrm{d}t \quad (3.3)$$
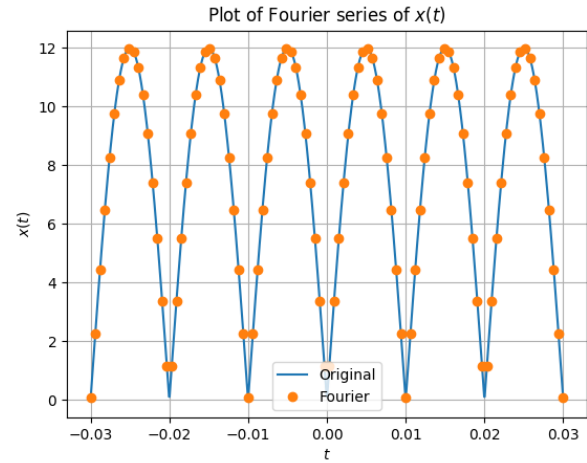

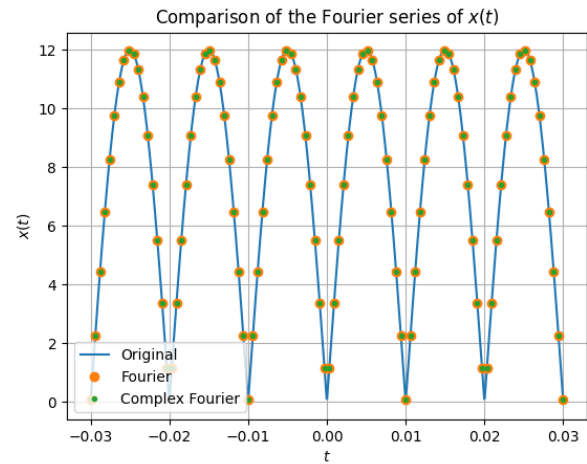Fig. 2.6. Plot of $x(t)$ along with its Fourier series expansion

Fig. 2.6. Comparison of the Fourier series of $x(t)$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f)e^{-j2\pi ft_0} \quad (3.4)$$

Solution:

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}\,\mathrm{d}t \quad (3.5)$$

$$= \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u+t_0)}\,\mathrm{d}u \quad (3.6)$$

$$= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(u)e^{-j2\pi fu}\,\mathrm{d}u \quad (3.7)$$

$$= G(f)e^{-j2\pi ft_0} \quad (3.8)$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.9)$$

Solution:

$$G(t) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\, dt \qquad (3.10)$$

But

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \qquad (3.11)$$

$$= \int_{-\infty}^{\infty} G(u)e^{j2\pi ut}\, du \qquad (3.12)$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(u)e^{-j2\pi uf}\, du \qquad (3.13)$$

$$= \mathcal{F}\{G(t)\} \qquad (3.14)$$

3.5 Find the Fourier transform of $\delta(t)$

Solution:

$$\delta(t) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\, dt \qquad (3.15)$$

$$= e^{-j2\pi ft}\Big|_{t=0} \qquad (3.16)$$

$$= 1 \qquad (3.17)$$

3.6 Find the Fourier transform of $e^{-j2\pi f_0 t}$

Solution:

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \qquad (3.18)$$

$$\implies \delta(t - f_0) \xrightarrow{\mathcal{F}} e^{-j2\pi f f_0} \qquad (3.19)$$

$$\implies e^{-j2\pi t f_0} \xrightarrow{\mathcal{F}} \delta(-f - f_0) \qquad (3.20)$$

$$\therefore e^{-j2\pi t f_0} \xrightarrow{\mathcal{F}} \delta(f + f_0) \qquad (3.21)$$

3.7 Find the Fourier transform of $\cos(2\pi f_0 t)$

Solution:

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \qquad (3.22)$$

$$\implies \cos(2\pi f_0 t) \xrightarrow{\mathcal{F}} \frac{\delta(f - f_0) + \delta(f + f_0)}{2}$$
$$(3.23)$$

3.8 Find the Fourier transform of $x(t)$ and plot it. Verify using Python

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \qquad (3.24)$$

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left\{e^{j2\pi k f_0 t}\right\} \qquad (3.25)$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \qquad (3.26)$$

$$= \frac{2A_0}{\pi} \sum_{k \text{ is even}} \frac{\delta(f - k f_0)}{1 - k^2} \qquad (3.27)$$