Fourier Series

EE3900: Linear Systems and Signal Processing

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1. Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$
Solution: Download the following Python code that plots Fig. 1.1.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/1.1.py
```

Run the code by executing

```
python 1.1.py
```

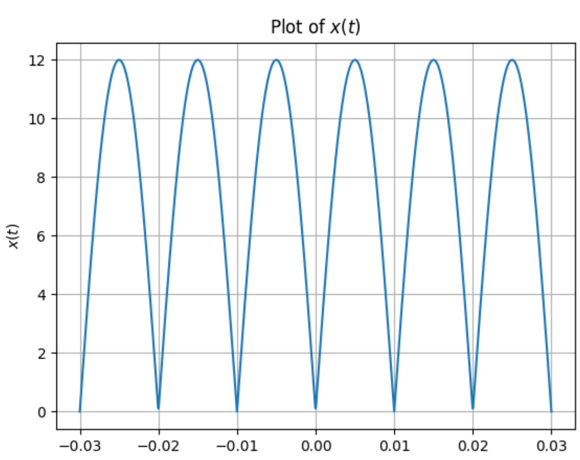

Fig. 1.1. Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period
Solution: Since $x(t)$ is the absolute value of a sinusoidal function, it is periodic, which is also evident from the plot

Consider $x(t + \frac{1}{2f_0})$

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$
$$= A_0 \left|\sin\left(2\pi f_0 t + \pi f_0\right)\right| \tag{1.3}$$
$$= A_0 \left|(-1)^{f_0} \sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$
$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.5}$$
$$= x(t) \tag{1.6}$$

Therefore, $x(t)$ is periodic with period $\frac{1}{2f_0}$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:

$$x(t) e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi (n-k) f_0 t} \tag{2.3}$$

$$\implies \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt \tag{2.4}$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt = \begin{cases} \frac{1}{f_0} & k = n \\ 0 & k \neq n \end{cases} \tag{2.5}$$
$$= \frac{1}{f_0} \delta(n-k) \tag{2.6}$$

$$\sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi(n-k)f_0 t}\, \mathrm{d}t = \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0}\delta(n-k) \tag{2.7}$$

$$= \frac{1}{f_0} c_n * \delta(n) \tag{2.8}$$

$$= \frac{1}{f_0} c_n \tag{2.9}$$

Therefore

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.10}$$

2.2 Find c_k for (1.1)
Solution:

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.11}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{0} A_0 \left(-\sin(2\pi f_0 t)\right) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\sin(2\pi f_0 t)\right) e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.12}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 u)\, e^{\jmath 2\pi k f_0 u}\, \mathrm{d}t + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t)\, e^{-\jmath 2\pi k f_0 t}\, \mathrm{d}t \tag{2.13}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \left(e^{\jmath 2\pi k f_0 t} + e^{-\jmath 2\pi k f_0 t}\right)\, \mathrm{d}t \tag{2.14}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} 2\sin(2\pi f_0 t)\cos(2\pi k f_0 t)\, \mathrm{d}t \tag{2.15}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left\{\sin(2\pi(1+k)f_0 t) + \sin(2\pi(1-k)f_0 t)\right\}\, \mathrm{d}t \tag{2.16}$$

$$= f_0 A_0 \left[-\frac{\cos(2\pi(1+k)f_0 t)}{2\pi(1+k)f_0} - \frac{\cos(2\pi(1-k)f_0 t)}{2\pi(1-k)f_0}\right]_0^{\frac{1}{2f_0}} \tag{2.17}$$

$$= \frac{f_0 A_0}{2\pi f_0}\left[\frac{1-(-1)^{1+k}}{1+k} + \frac{1-(-1)^{1-k}}{1-k}\right] \tag{2.18}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{2\pi}\left[\frac{1}{1+k} + \frac{1}{1-k}\right] \tag{2.19}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{\pi(1-k^2)} \tag{2.20}$$

Therefore

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases} \tag{2.21}$$

2.3 Verify (1.1) using Python
Solution: Download the following Python code that plots Fig. 2.3.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/2.3.py
```

Run the code by executing

```
python 2.3.py
```

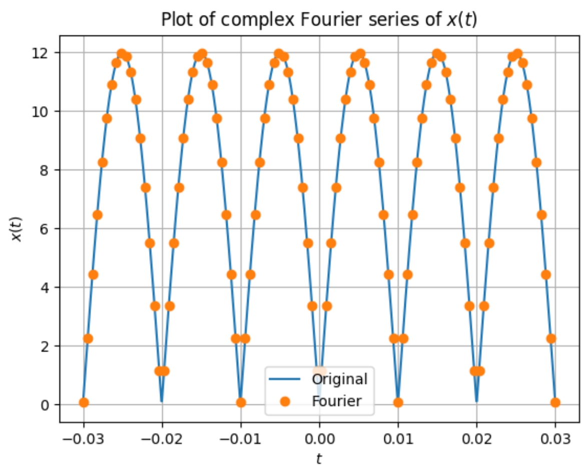

Fig. 2.3. Plot of $x(t)$ along with its complex Fourier series expansion

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty}\left(a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)\right) \tag{2.22}$$

and obtain the formulae for a_k and b_k
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.23}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.24}$$

Thus

$$x(t) = c_0 + \sum_{k=1}^{\infty}(c_k + c_{-k})\cos(2\pi k f_0 t) + \sum_{k=1}^{\infty} \jmath\,(c_k - c_{-k})\sin(2\pi k f_0 t) \tag{2.25}$$

Therefore

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.26}$$

$$b_k = \jmath(c_k - c_{-k}) \quad k \geq 0 \tag{2.27}$$

2.5 Find a_k and b_k for (1.1)
Solution:

$$a_0 = c_0 = \frac{2A_0}{\pi} \tag{2.28}$$

For $k > 0$, if k is odd

$$a_k = 0 + 0 = 0 \tag{2.29}$$

and if k is even

$$a_k = \frac{2A_0}{\pi(1-k^2)} + \frac{2A_0}{\pi(1-k^2)} = \frac{4A_0}{\pi(1-k^2)} \tag{2.30}$$

For odd or even k, $c_k = c_{-k}$ always

$$b_k = 0 \quad \forall k \geq 0 \tag{2.31}$$

Therefore

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k = 2m, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{2.32}$$

$$b_k = 0 \qquad k \geq 0 \tag{2.33}$$

2.6 Verify (2.22) using Python
Solution: Download the following Python code that plots Fig. 2.6.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/2.6.py
```

Run the code by executing

```
python 2.6.py
```

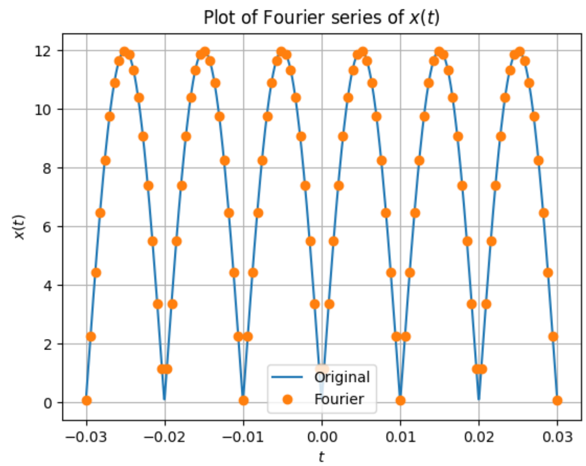

Fig. 2.6. Plot of $x(t)$ along with its Fourier series expansion

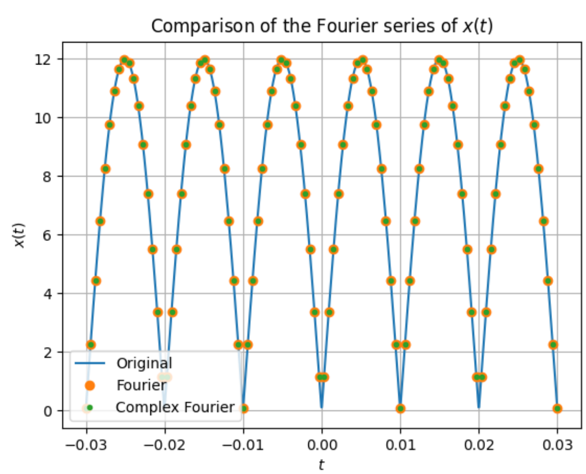

Fig. 2.6. Comparison of the Fourier series of $x(t)$