Linear Regression

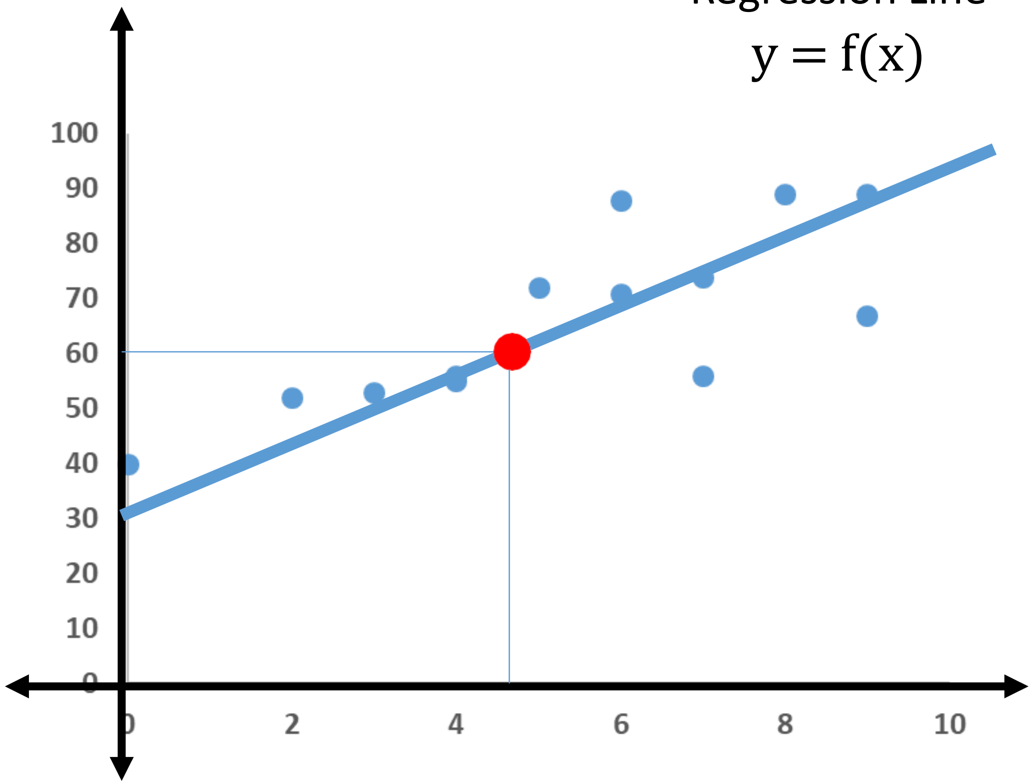
Simple Linear Regression

Regression Analysis

- Statistical process for estimating the relationships among variables
- The predictor is a continuous variable
- Relationship between a dependent variable and one or more independent variables (or 'predictors')
- Can also be used to infer causal relationships between dependent and independent variables.

Regression Line

$$y = f(x)$$



Predicting Continuous Value

What will be the stock price in future? Should I buy it now?

Start

REGRESSION

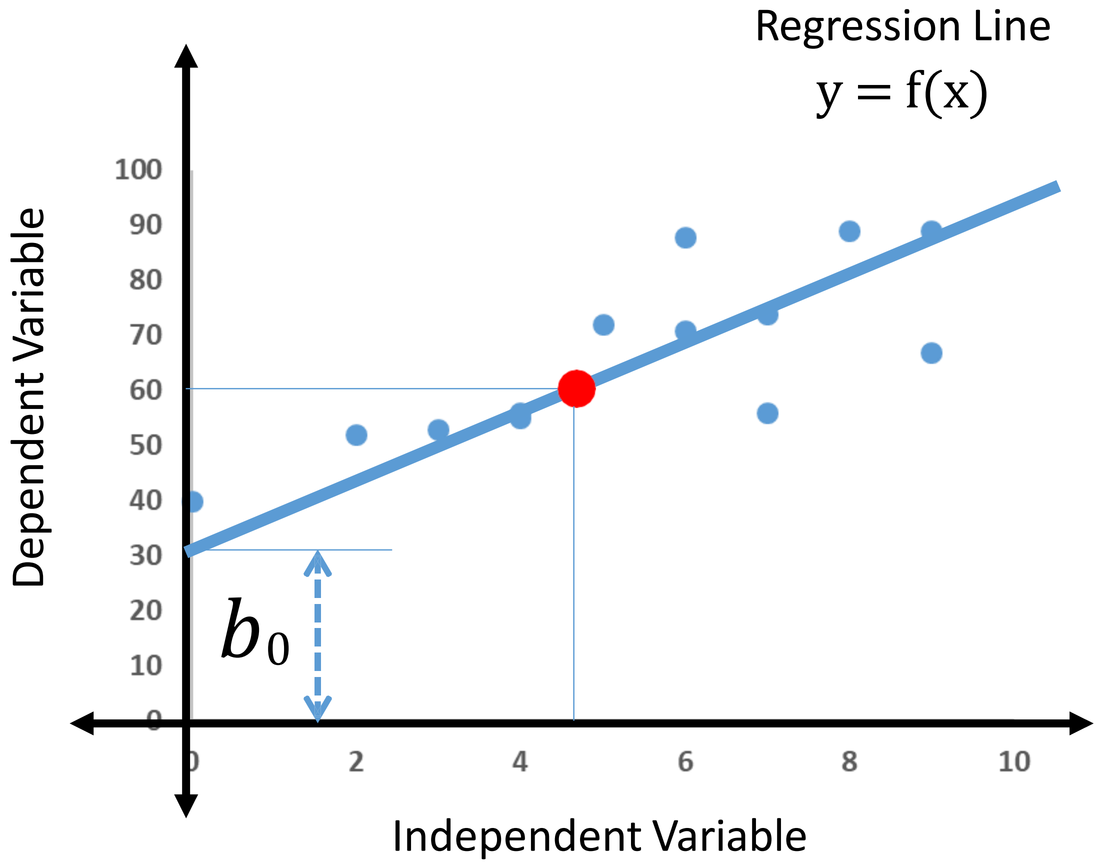
Ordinal Regression	Data in Rank Order categories
Poisson Regression	Predicting Event Counts
Fast Forest Quantile Regression	Predicting a Distribution
Linear Regression	Fast Training, Linear Model
Bayesian Linear Regression	Linear Model, Small datasets
Neural Network Regression	Accuracy, Long Training Time
Decision Forest Regression	Accuracy, Fast Training
Boosted Decision Tree Regression	Accuracy, Fast Training, large Memory

Simple Linear Regression

Simple Regression :

$$y = b_0 + \quad b_1 x$$

Only one Dependent
Only one Independent



Simple Linear Regression

Hrs Studied (X)	Marks (Y)
0	40
2	52
3	53
4	55
4	56
5	72
6	71
6	88
7	56
7	74
8	89
9	67
9	89
5.38	66.31
Mean	

X – Mean (A)	Y – Mean (B)	A^2	A*B
-5.38	-26.31	28.99	141.66
-3.38	-14.31	11.46	48.43
-2.38	-13.31	5.69	31.73
-1.38	-11.31	1.92	15.66
-1.38	-10.31	1.92	14.27
-0.38	5.69	0.15	-2.19
0.62	4.69	0.38	2.89
0.62	21.69	0.38	13.35
1.62	-10.31	2.61	-16.65
1.62	7.69	2.61	12.43
2.62	22.69	6.84	59.35
3.62	0.69	13.07	2.50
3.62	22.69	13.07	82.04
		89.08	405.46
		Sum	

$$y = b_0 + b_1 x$$

$$b_1 = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sum (X - \bar{X})^2}$$

$$= 405.46 / 89.08$$

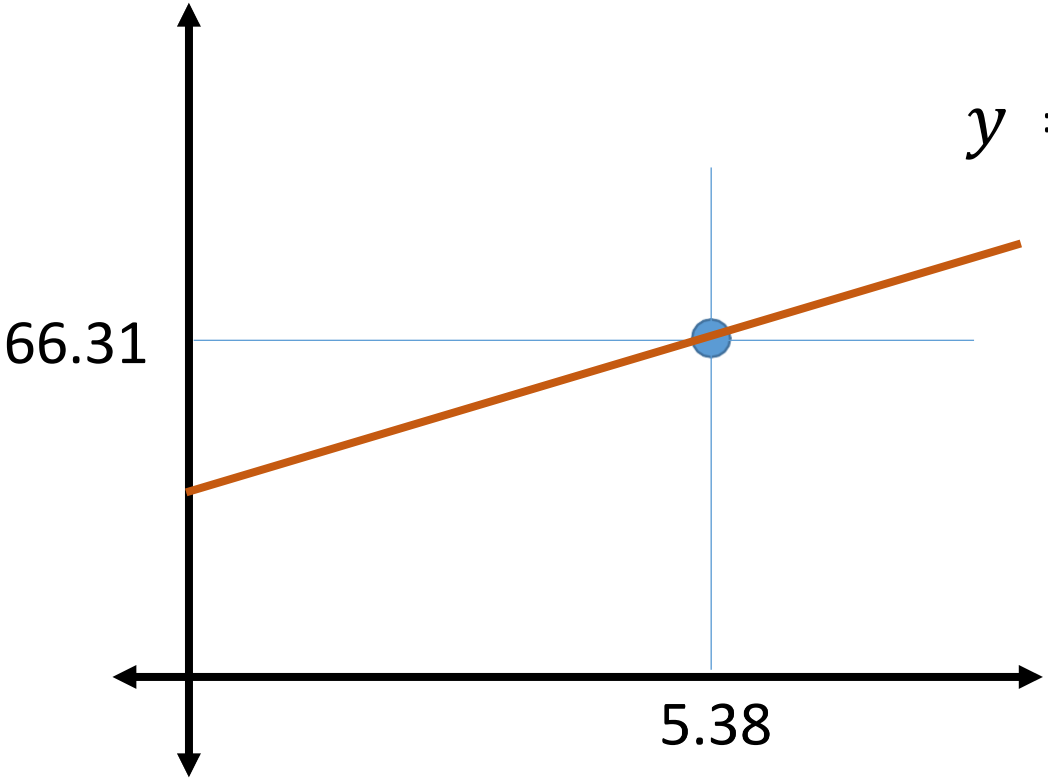
$$= 4.55$$

Simple Linear Regression

Hrs Studied (X)	Marks (Y)
0	40
2	52
3	53
4	55
4	56
5	72
6	71
6	88
7	56
7	74
8	89
9	67
9	89
5.38	66.31
Mean	

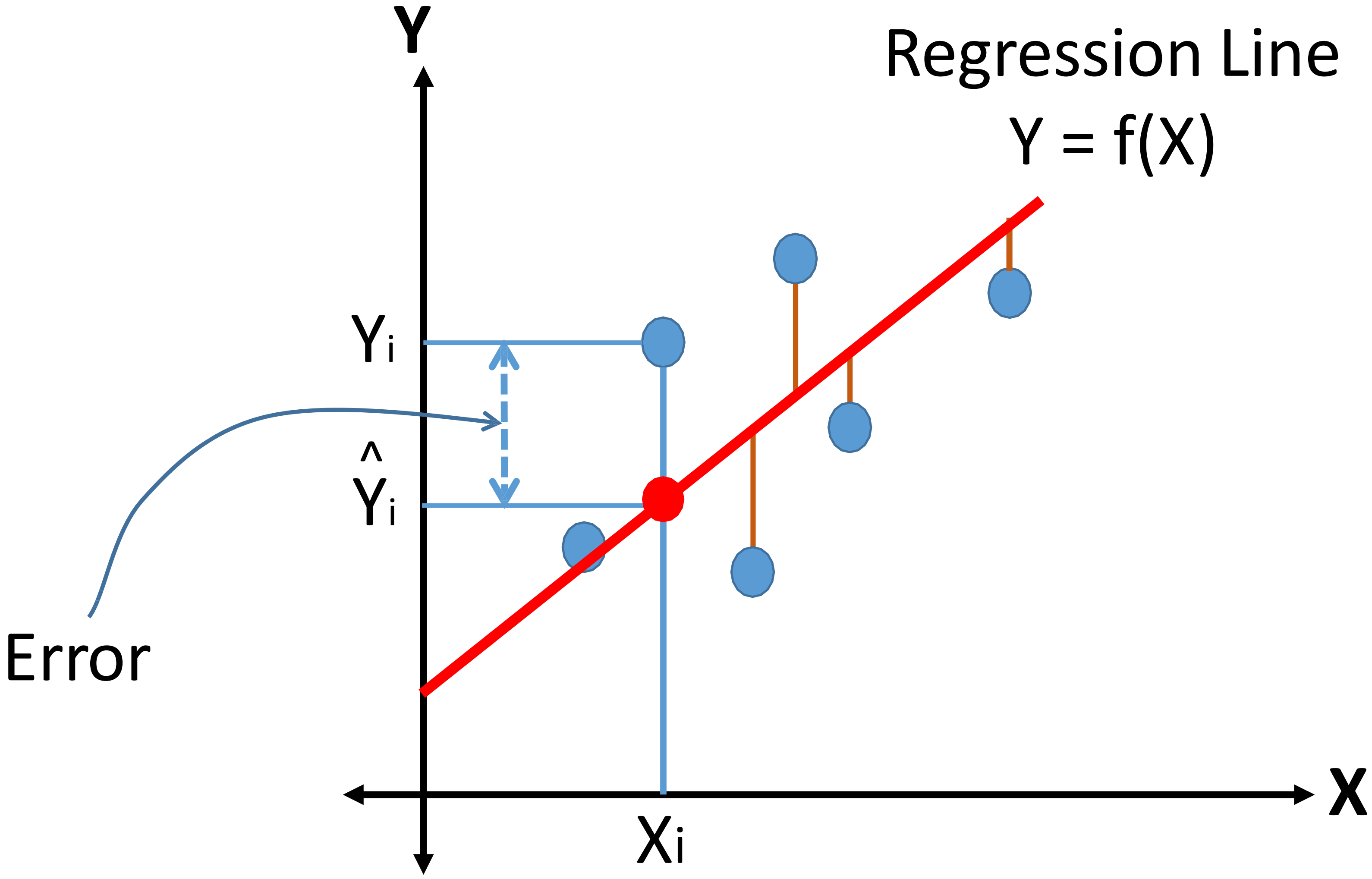
$$y = b_0 + b_1 x \qquad b_1 = 4.55 \qquad b_0 = 41.8$$

$$y = 41.8 + 4.55\,x$$



Simple Linear Regression – OLS (Ordinary Least Square) & Error Terms

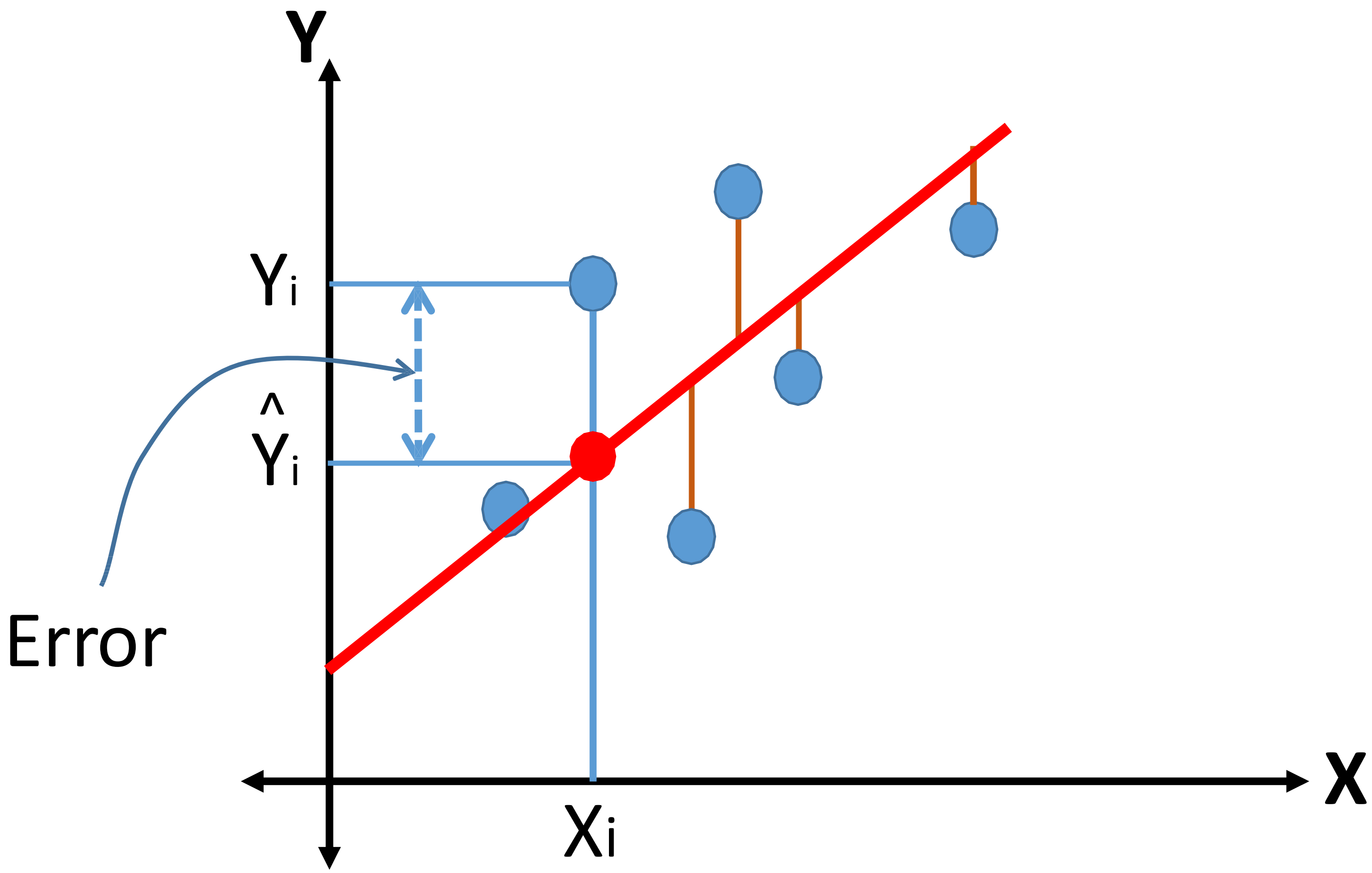
Ordinary Least Square



Minimum

$$\sum_{i=1}^{n} (yi - \hat{y}i)^2$$

Mean Absolute Error



$$\text{MAE} = \frac{1}{n}\sum_{i=0}^{n} |y_i - \hat{y}_i|$$

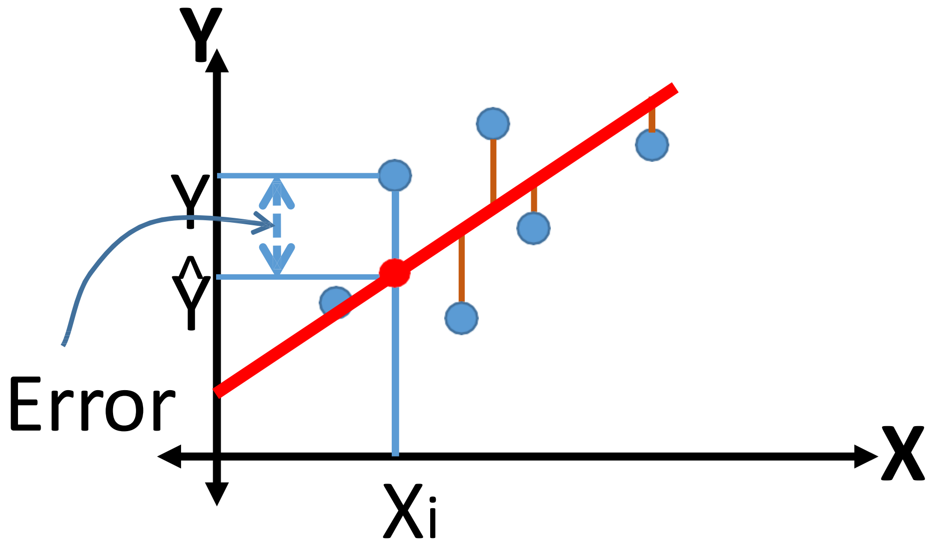
Mean absolute error (MAE) is a quantity used to measure how close forecasts or predictions are to the eventual outcomes.

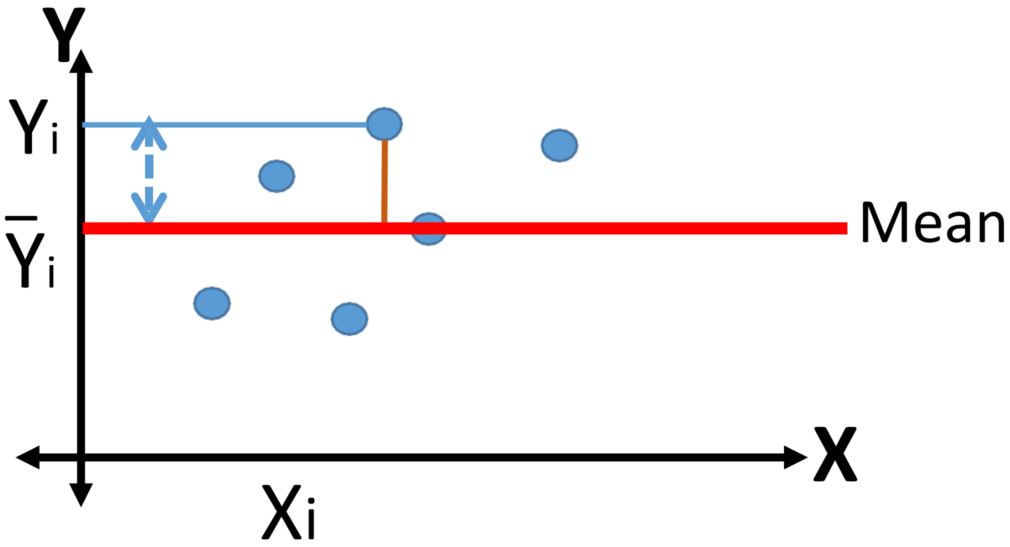
Root Mean Squared Error

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=0}^{n}(y_i - \hat{y}_i)^2}$$

- Very commonly used and makes for an excellent general purpose error metric for numerical predictions.
- Compared to the similar Mean Absolute Error, RMSE amplifies and severely punishes large errors.

Relative Absolute Error



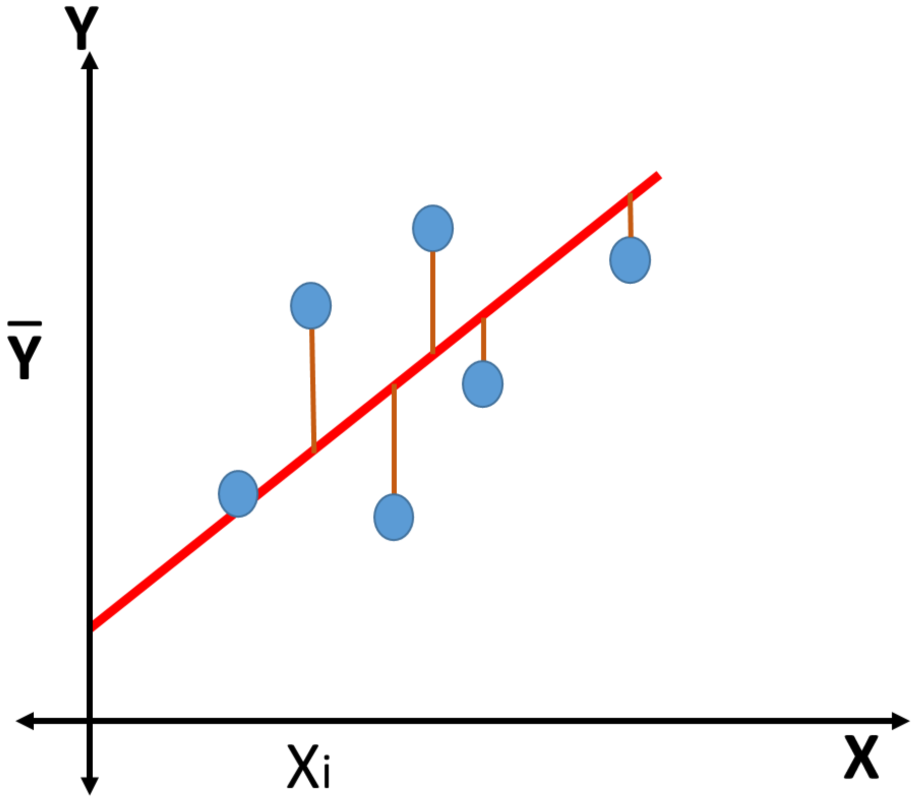


$$RAE = \frac{\sum_{i=1}^{n} |yi - \hat{y}i|}{\sum_{i=1}^{n} |yi - \bar{y}i|}$$

Simple Linear Regression – R Squared or Coefficient of Determination

Coefficient of Determination

How much (what %) of variation in Y is described by the variation in X?



R-Square With an Example

Hrs Studied (X)	Marks (Y)
0	40
2	52
3	53
4	55
4	56
5	72
6	71
6	88
7	56
7	74
8	89
9	67
9	89
5.38	66.31
Mean	

$Y = 41.8 + 4.55X$

Predicted Marks $\hat{Y}$
41.80
50.90
55.45
60.00
60.00
64.55
69.10
69.10
73.65
73.65
78.20
82.75
82.75

$(Y - \bar{Y})^2$	$(\hat{Y} - \bar{Y})^2$
692.22	600.74
204.78	237.47
177.16	117.94
127.92	39.82
106.30	39.82
32.38	3.10
22.00	7.78
470.46	7.78
106.30	53.88
59.14	53.88
514.84	141.37
0.48	270.27
514.84	270.27
3028.77	1844.12
SST	SSR

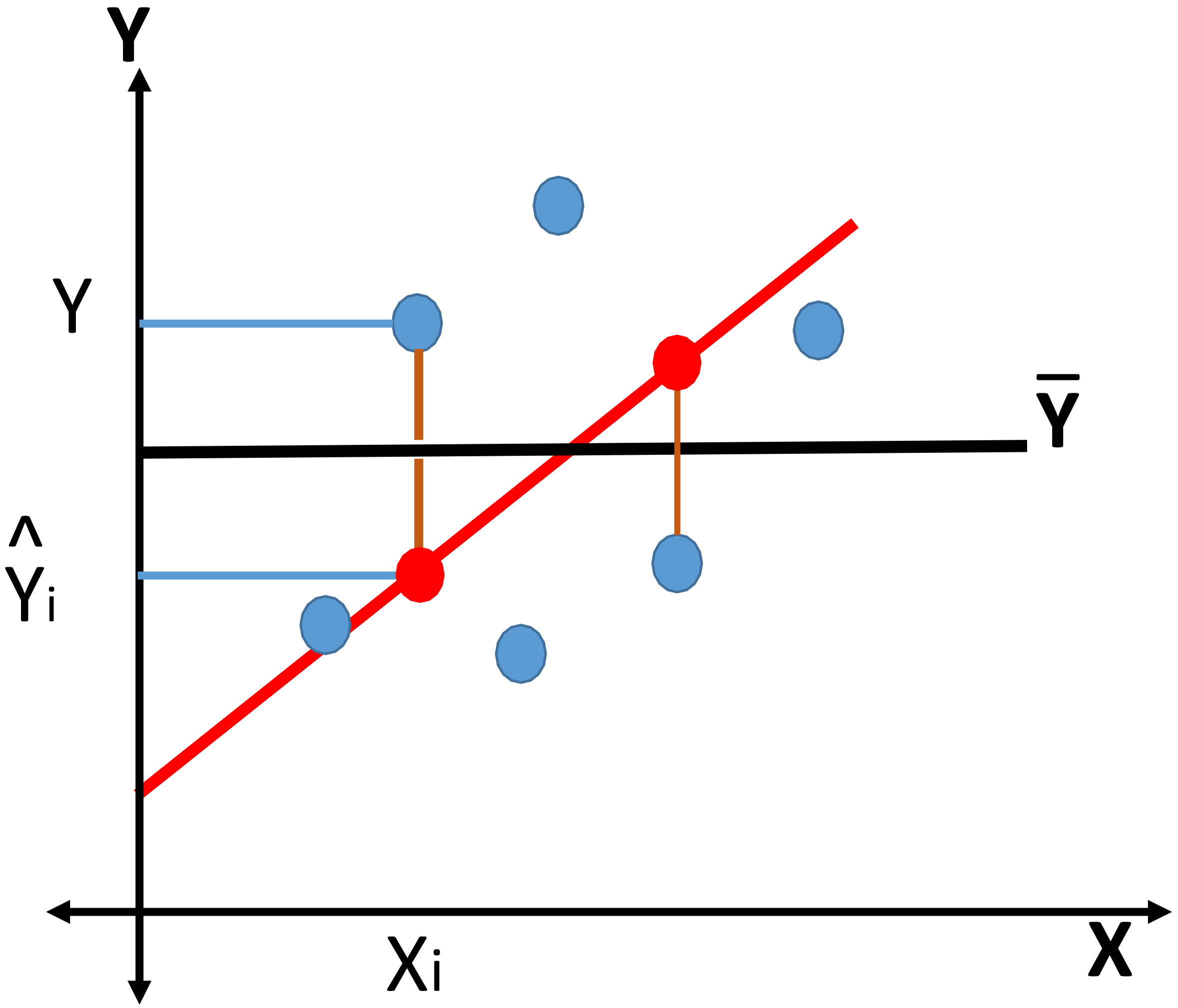
Coefficient of Determination

Sum of Squares Due to Regression

$$SSR = \sum_{i=1}^{n} (\boxed{\hat{y}i} - \bar{y}i)^2$$

Total Sum of Squares

$$SST = \sum_{i=1}^{n} (\boxed{yi} - \bar{y}i)^2$$



R-Square With an Example

Hrs Studied (X)	Marks (Y)
0	40
2	52
3	53
4	55
4	56
5	72
6	71
6	88
7	56
7	74
8	89
9	67
9	89
5.38	66.31
Mean	

$$R^2 = SSR/SST$$

$$= 1844.12/3028.77$$

$$= 0.60886$$

Higher the value → Variation in Y is explained by variation in X.

$(Y - \bar{Y})^2$	$(\hat{Y} - \bar{Y})^2$
692.22	600.74
204.78	237.47
177.16	117.94
127.92	39.82
106.30	39.82
32.38	3.10
22.00	7.78
470.46	7.78
106.30	53.88
59.14	53.88
514.84	141.37
0.48	270.27
514.84	270.27
3028.77	1844.12
SST	SSR

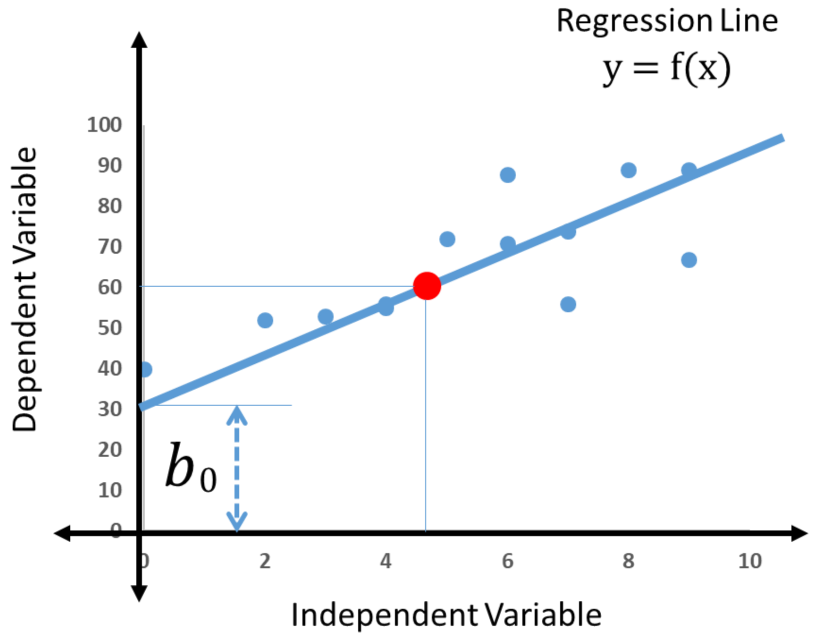
Multiple Linear Regression

Multiple Linear Regression

Simple Regression :

$$y = b_0 + b_1 x$$

Only one Dependent
Only one Independent



Multiple Linear Regression :

$$y = b_0 + b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$$

Multiple Linear Regression

Hrs Studied (X1)	Hrs Slept (X2)	Marks (Y)
0	8	40
2	8	52
3	7.5	53
4	7	55
4	9	56
5	8.5	72
6	9	71
6	7	88
7	6	56
7	7	74
8	9	89
9	6	67
9	9	89

$$y = b_0 + b_1 x_1 + b_2 x_2$$

Dependent Variable

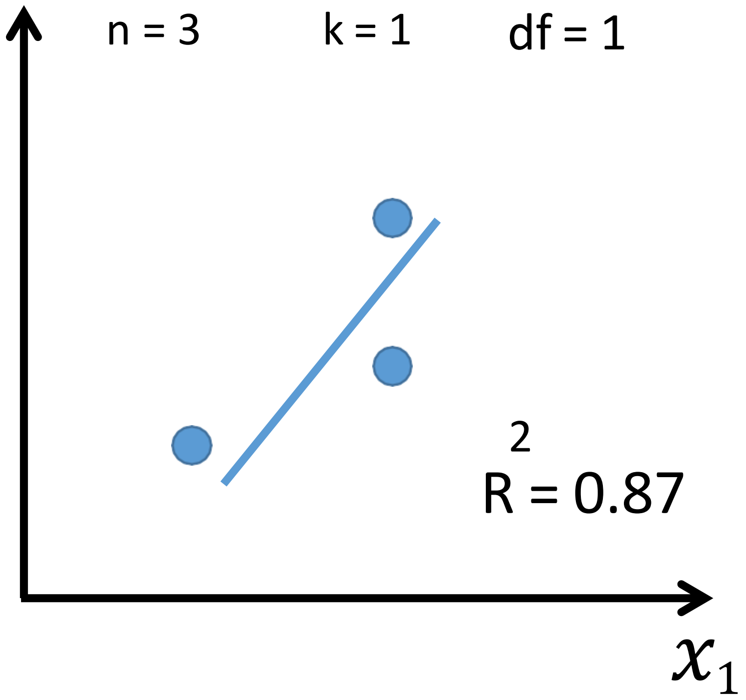
Marks Obtained

Independent Variable

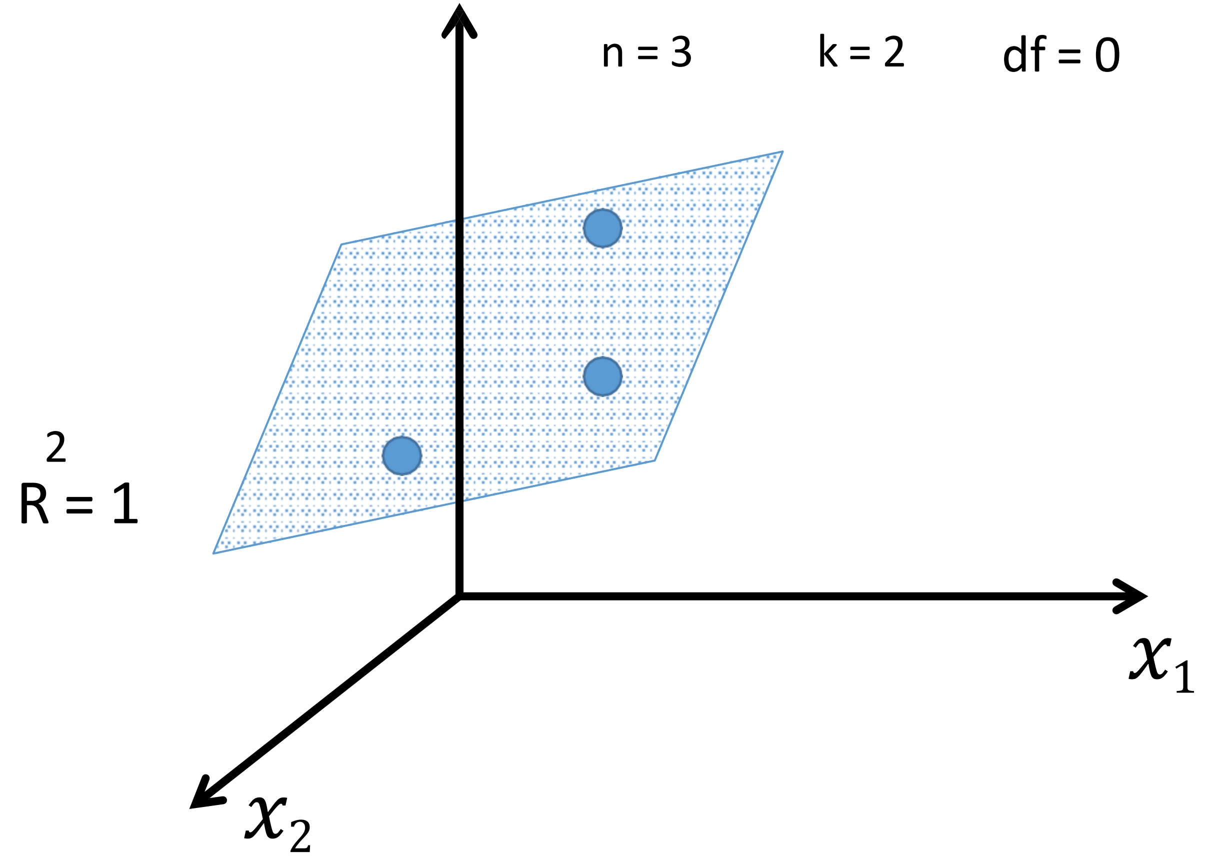
Hrs Studied
Hrs Slept

Degrees of Freedom (n – p – 1)

$$y = b_0 + b_1 x_1$$



$$y = b_0 + b_1 x_1 + b_2 x_2$$



Adjusted R-Squared

$$\overline{R}^2 = 1 - \frac{(1 - R^2) * (n - 1)}{n - p - 1}$$

$R \quad = \; Sample\ R{-}Squared$

p $\quad =$ Number of independent variables

n $\quad =$ sample size or number of observations

Assumptions of Multiple Linear Regression

Relationship Among Variables

- Linear Relationship
- Multicollinearity
- No Auto-Correlation
- Endogeneity

Behaviour of Data

- Sample Size
- Normality
- Homoscedasticity

Multiple Linear Regression – Degree of Freedom

Degrees of Freedom in Statistics

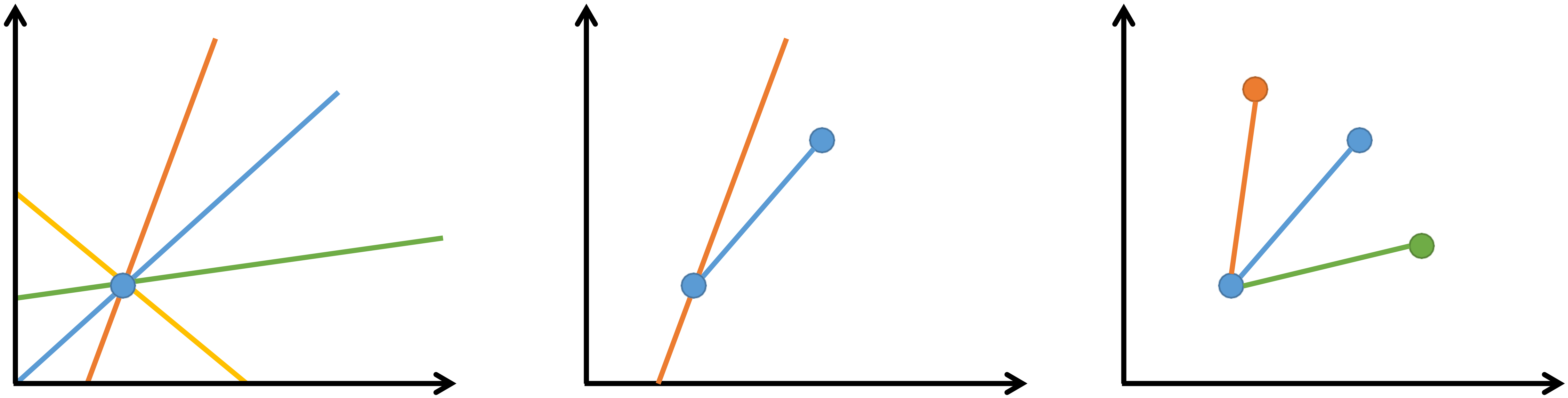
The number of values in the final calculation of a statistic that are free to vary.

OR

The minimum number of independent coordinates that can specify the position of the system completely.

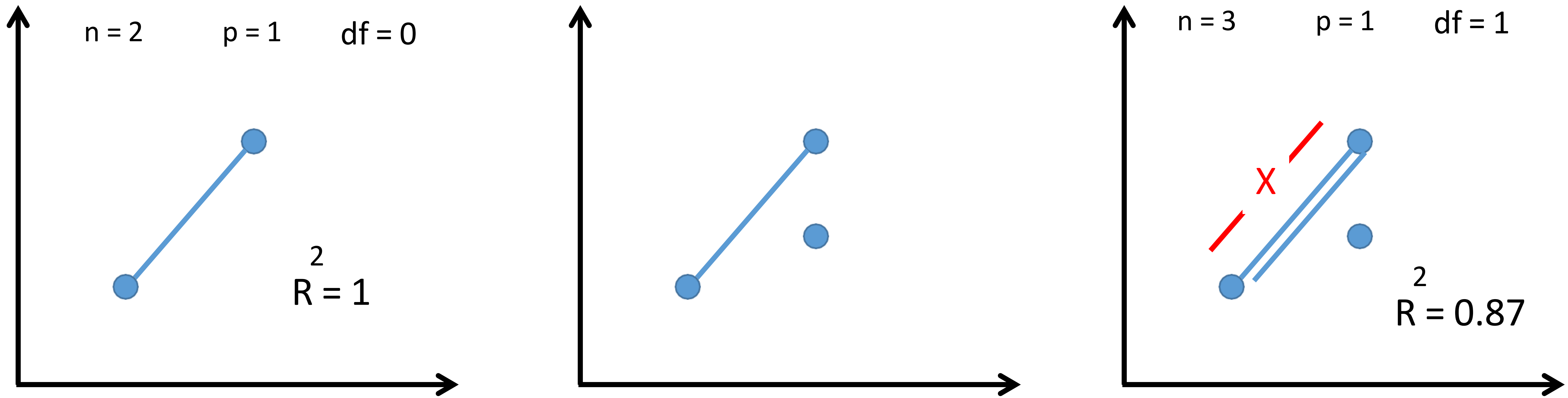
$$\mathrm{df} = \underbrace{\mathrm{n}}_{\text{Number of Observations}} - \underbrace{\mathrm{p}}_{\text{Number of variables}} - 1$$

Degrees of Freedom in Statistics



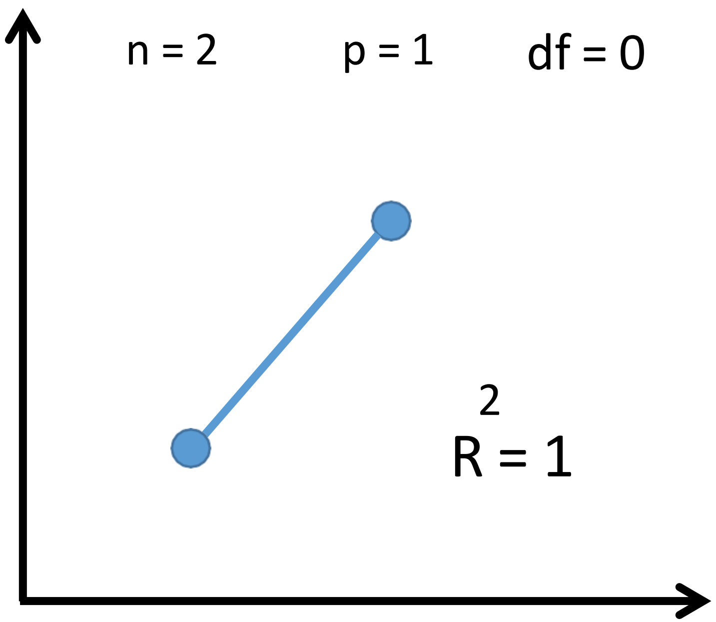
Degrees of Freedom in Statistics (n – p – 1)

$$y = b_0 + \ b_1 x_1$$



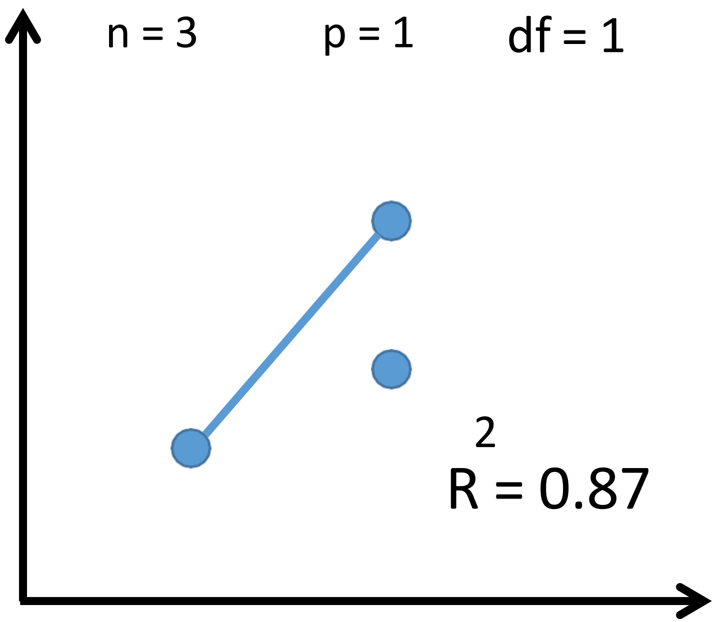
Degrees of Freedom in Statistics (n – p – 1)

$$y = b_0 + \ b_1 x_1$$



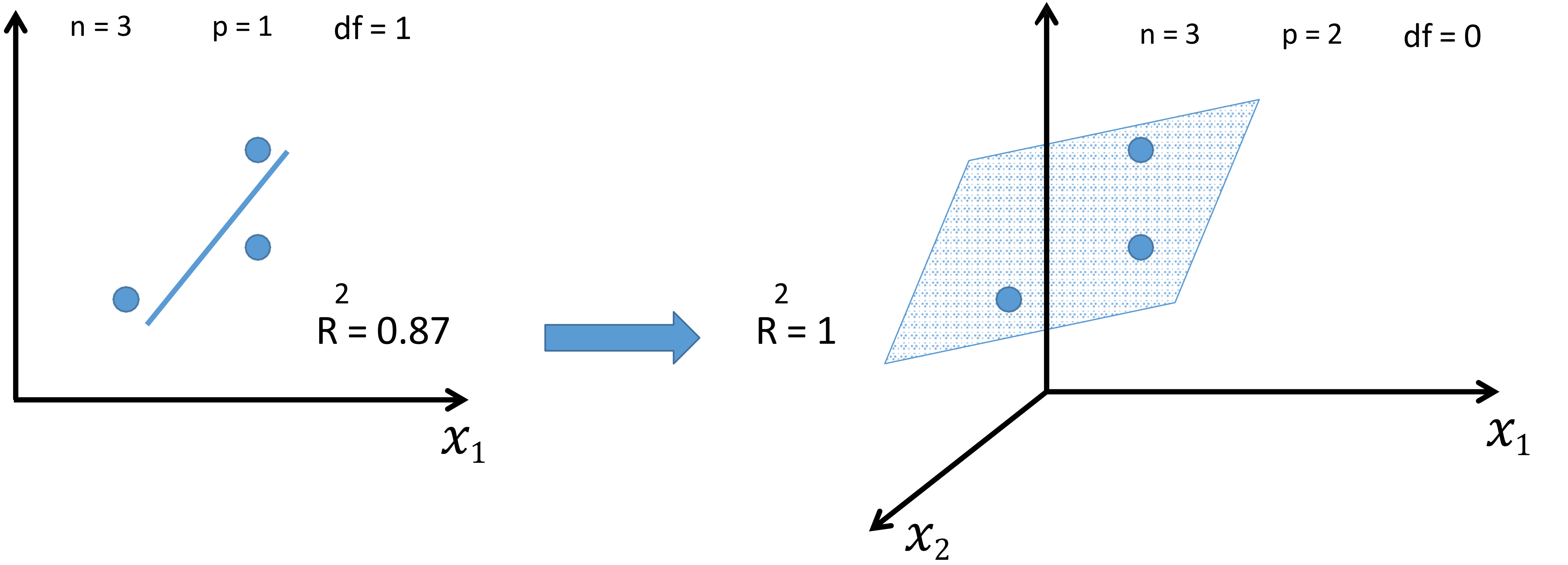
df = 0 → 1

$R^2 = 1 \rightarrow 0.87$



Degrees of Freedom in Statistics (n – p – 1)

$$y = b_0 + \; b_1 x_1 \qquad\qquad y = b_0 + \; b_1 x_1 + b_2 x_2$$



Adjusted R-Squared

$$\bar{R}^2 = 1 - \left[\frac{(1 - R^2) * (n - 1)}{n - p - 1} \right]$$

$R \quad = \; Sample\ R{-}Squared$

p $\quad =$ Number of independent variables

n $\quad =$ sample size or number of observations

Adjusted R-Squared

Lower value of
Adjusted R-Squared

If the R-Squared does not increase significantly.

$$\overline{R}^2 = 1 - \left[\frac{(1-R^2)*(n-1)}{n-p-1}\right]$$

Increase in this term

Lower Denominator due to higher value of p.

$R \;\; = \textit{Sample R−Squared}$

p $\;\; =$ Number of independent variables

n $\;\; =$ sample size or number of observations

Adjusted R-Squared

N	p	R-Squared	Adjusted R-Squared
50	10	0.80	0.75
50	12	0.82	0.76
50	15	0.83	0.75
50	20	0.84	0.73

$$\bar{R}^2 = 1 - \frac{(1 - R^2) * (n - 1)}{n - p - 1}$$

Multiple Linear Regression – Assumptions

Assumptions of Multiple Linear Regression

Relationship Among Variables

- Linear Relationship
- No Multicollinearity
- No Auto-Correlation
- Endogeneity

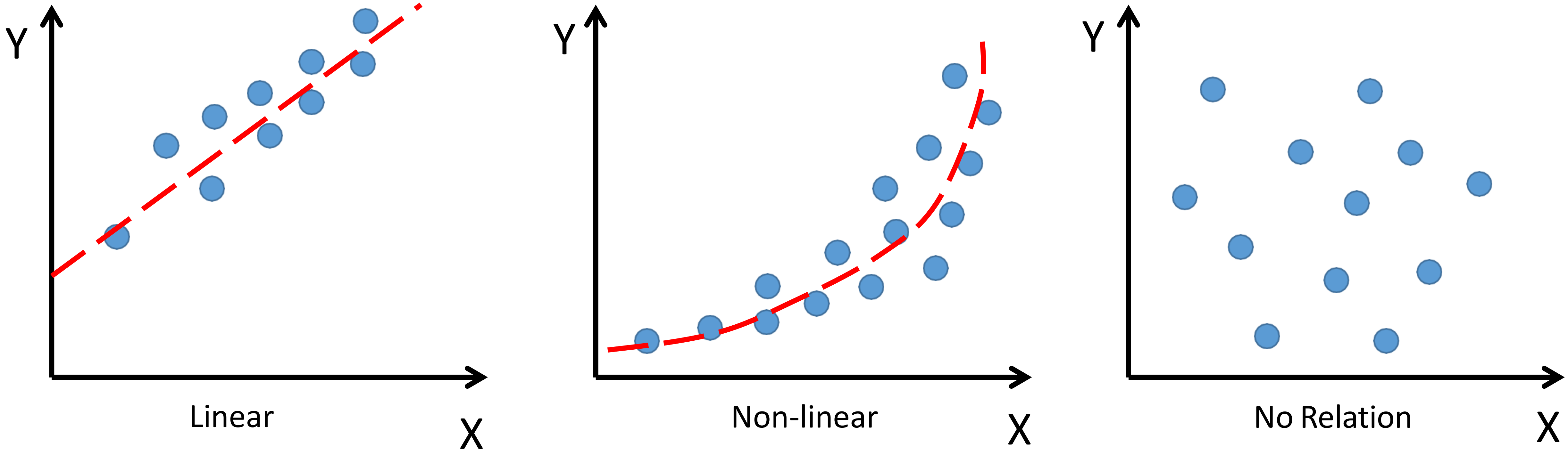
Behaviour of Data

- Sample Size
- Normality
- Homoscedasticity

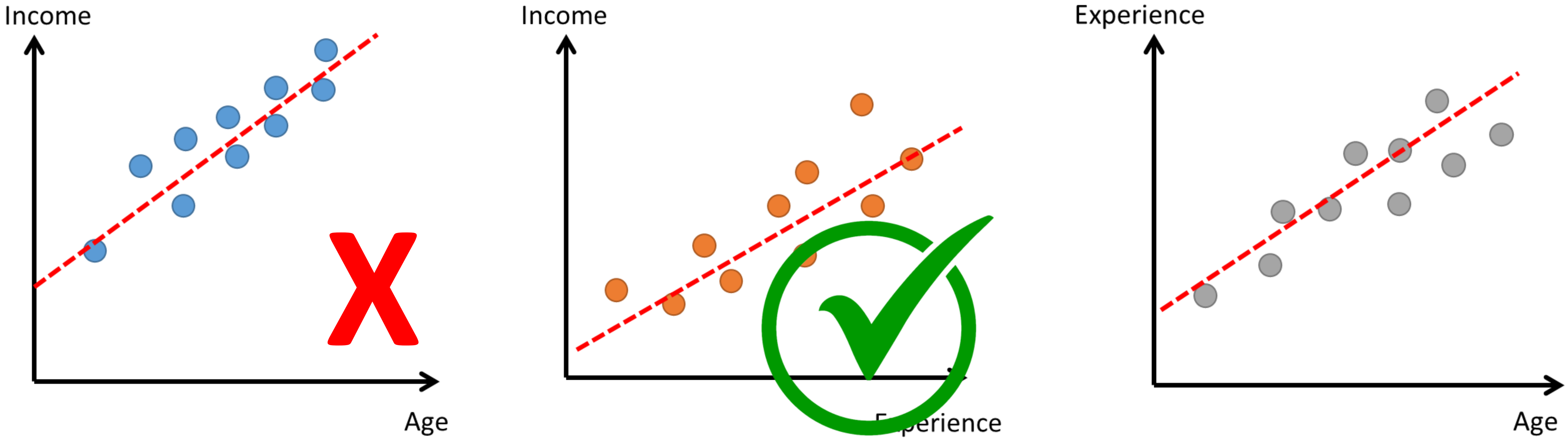
Linear Relationship

- Dependent and Independent Features have linear relationship
- Can be Positive or Negative correlation
- Can be checked using Pearson Correlation Coefficient as well as visualisation

Linear Relationship

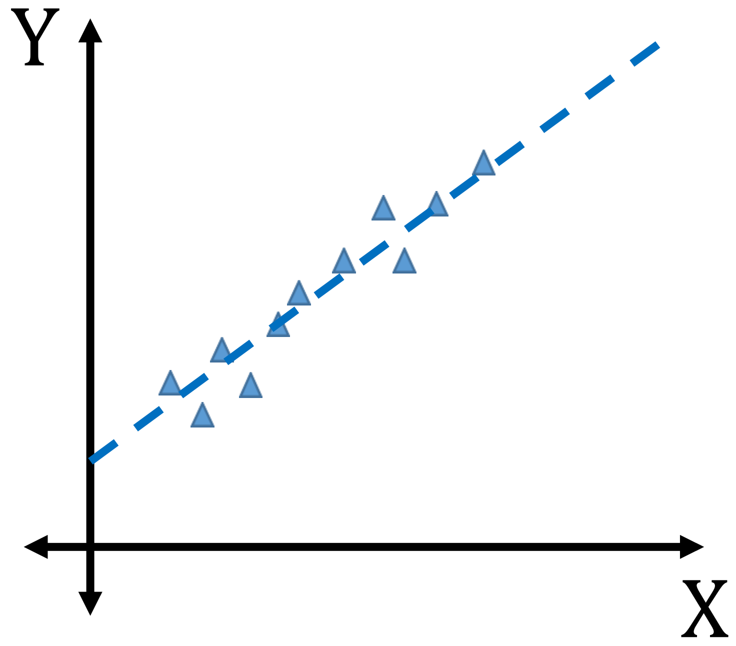


No Multicollinearity



Statistically Correlated

- Strength of the correlation – Coefficient of Correlation
- Direction of correlation – Sign of the Coefficient



Pearson Correlation Coefficient

$$r = \frac{\sum (x - \bar{x}) * (y - \bar{y})}{(N - 1) * \sigma_x * \sigma_y}$$

Correlation Coefficient Matrix

Age	Experience	Education Received	Salary
32	8	6	$ 8,000
40	15	8	$ 12,000
35	6	8	$ 10,000

	Age	Experience	Education Received	Salary
Age	1	0.9	0.2	0.7
Experience	0.9	1	0.15	0.72
Education Received	0.2	0.15	1	0.85
Salary	0.7	0.72	0.85	1

Auto-Correlation

The value of one record for the same variable or feature is dependent on the value from the same column but of different record.

X1	X2	Y
Value-11	Value-21	Y1
Value-12	Value-22	Y2
Value-13	Value-23	Y3

Auto-Correlation

Measure of Autocorrelation

$$\text{ACF, } \rho_k = \frac{\sum_{t=k+1}^{T} (x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}$$

Lagged Data

Number of time Units or Lag

Autocorrelation for lag of 1.

$\boxed{k = 1}$

$$\text{ACF, } \rho_1 = \frac{(x_2 - \bar{x})(x_1 - \bar{x})}{(x_2 - \bar{x})^2}$$

Day	Temperature
1	32
2	33
3	33
4	36
5	33
6	37

$$\text{ACF, } \rho_1 = \frac{(33 - 34)(32 - 34)}{(33 - 34)^2}$$

Autocorrelation

$\boxed{k = 4}$

$$\frac{\sum_{t=k+1}^{T}(x_t-\bar{x})(x_{t-k}-\bar{x})}{\sum_{t=1}^{T}(x_t-\bar{x})^2}$$

Day	Temperature
1	32
2	33
3	33
4	36
5	33
6	37
7	.....
8	......

Autocorrelation

t0	t-1	t-2	t-3	t-4
8	NaN	NaN	NaN	NaN
14	8	NaN	NaN	NaN
36	14	8	NaN	NaN
56	36	14	8	NaN
84	56	36	14	8
94	84	56	36	14
106	94	84	56	36
110	106	94	84	56
93	110	106	94	84
67	93	110	106	94
35	67	93	110	106
37	35	67	93	110
36	37	35	67	93
34	36	37	35	67
28	34	36	37	35
39	28	34	36	37
17	39	28	34	36

Sliding Window Approach

t0	t-1	t-2	t-3	t-4
8	NaN	NaN	NaN	NaN
14	8	NaN	NaN	NaN
36	14	8	NaN	NaN
56	36	14	8	NaN
84	56	36	14	8
94	84	56	36	14
106	94	84	56	36
110	106	94	84	56
93	110	106	94	84
67	93	110	106	94
35	67	93	110	106
37	35	67	93	110
36	37	35	67	93
34	36	37	35	67
28	34	36	37	35
39	28	34	36	37
17	39	28	34	36

8	14	36	56	84	94	106	110	93	67

pandas.shift( )

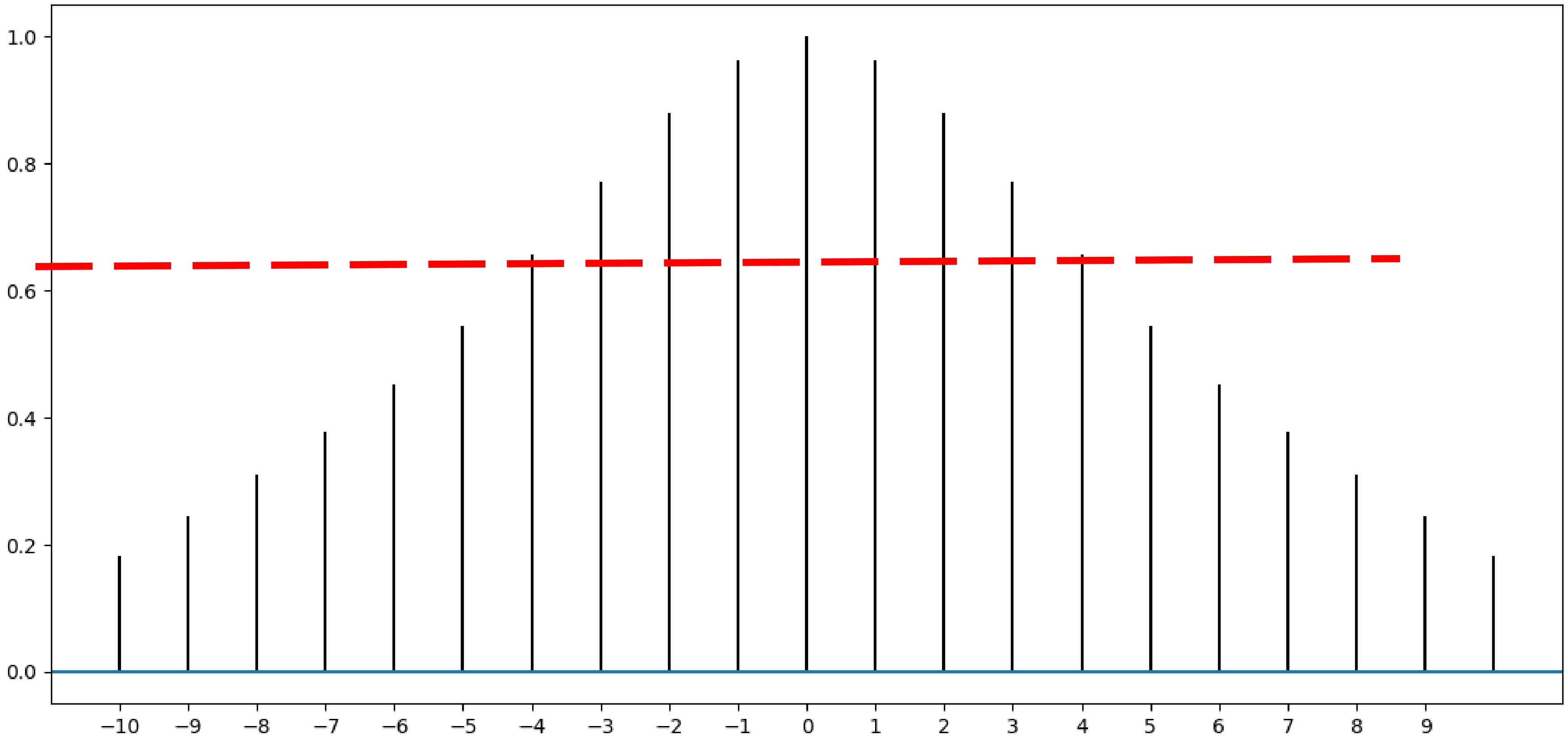
Autocorrelation Function

t0	t-1	t-2	t-3	t-4
8	NaN	NaN	NaN	NaN
14	8	NaN	NaN	NaN
36	14	8	NaN	NaN
56	36	14	8	NaN
84	56	36	14	8
94	84	56	36	14
106	94	84	56	36
110	106	94	84	56
93	110	106	94	84
67	93	110	106	94
35	67	93	110	106
37	35	67	93	110
36	37	35	67	93
34	36	37	35	67
28	34	36	37	35
39	28	34	36	37
17	39	28	34	36

$$\frac{\sum_{t=k+1}^{T} (x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}$$

Autocorrelation Function (ACF)

pyplot.acorr( )



Assumptions of Multiple Linear Regression

Relationship Among Variables

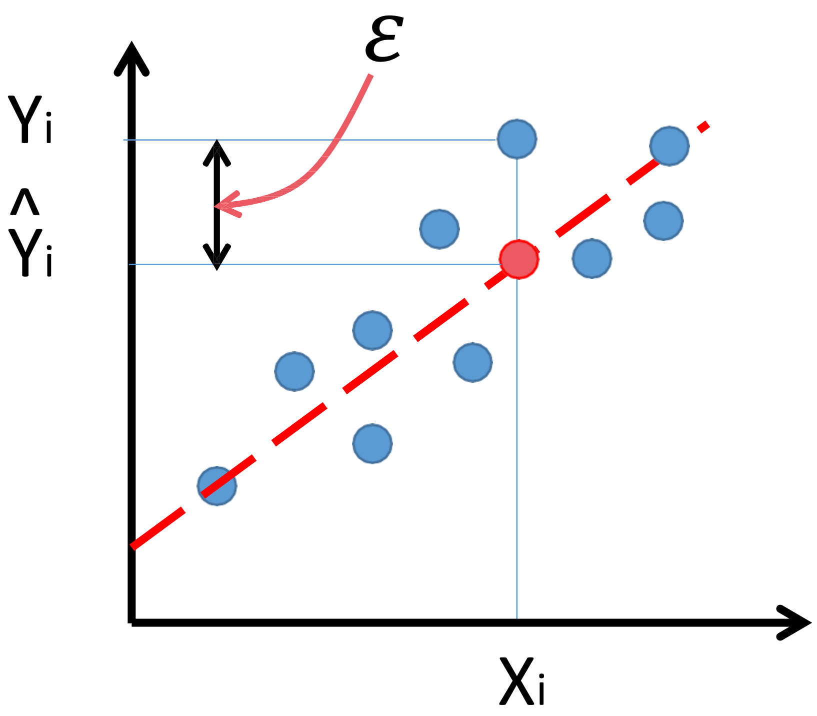
- Linear Relationship
- No Multicollinearity
- No Auto-Correlation
- Endogeneity

Behaviour of Data

- Sample Size
- Normality
- Homoscedasticity

Endogeneity

- Situations in which an explanatory/independent variable is correlated with the error term.



$$y = b_0 + b_1 x_1$$

$$y_i = \hat{y}_i + \varepsilon$$

$$y_i = b_0 + b_1 x_1 + \varepsilon$$

$$\varepsilon = f(x) \quad ??$$

Endogeneity

Wheelbase	Length	Engine Size	Horsepower	Price
88.6	168.8	130	~~111~~	
94.5	171.2	152	~~154~~	
99.8	176.6	109	~~102~~	
99.4	176.6	136	~~115~~	

$$\text{Price} = 8215 + 11.5*\text{Wheelbase} + 7.8*\text{Length} + 2.8*\text{EngineSize} + \cancel{3.2*\text{HP}}$$

Omitted Variable Bias

Wheelbase	Length	Engine Size	Horsepower	Price
88.6	168.8	130	111	
94.5	171.2	152	154	
99.8	176.6	109	102	
99.4	176.6	136	115	

$$\text{Price} = 8215 + 11.5{*}\text{Wheelbase} + 7.8{*}\text{Length} + 2.8{*}\text{EngineSize} + \varepsilon$$

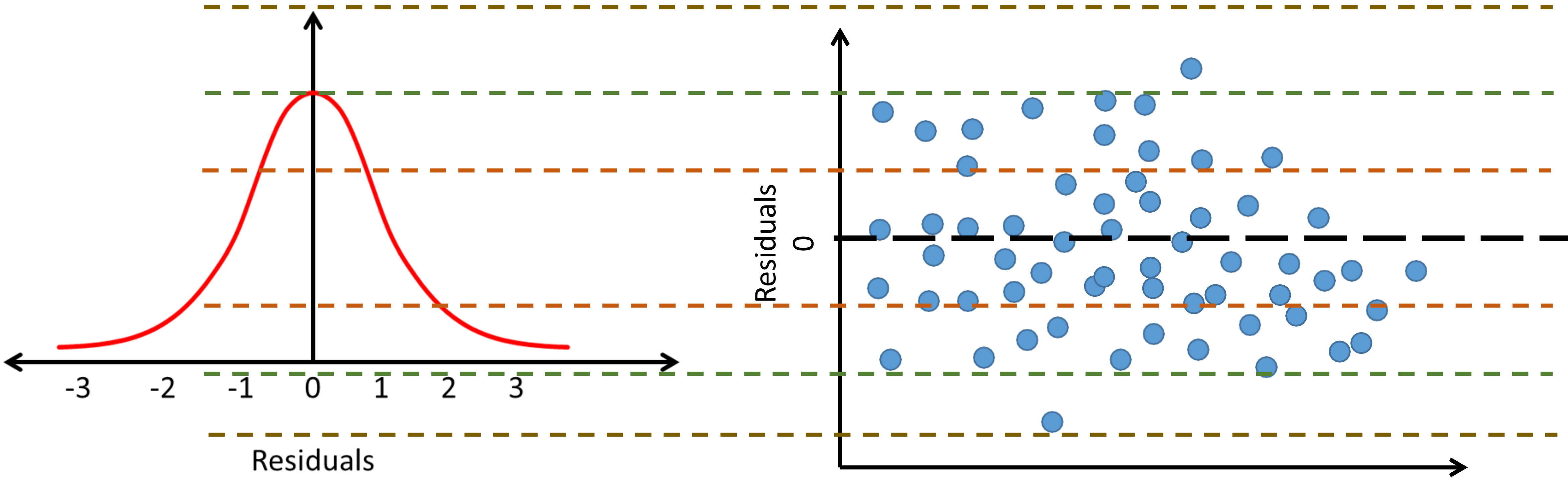
Price ~ EngineSize ~ Horsepower ~ error

QuickFix

- Start with All the variable

- Remove unwanted ones using Adjusted R-Squared or feature selection methods

Normality of Residuals



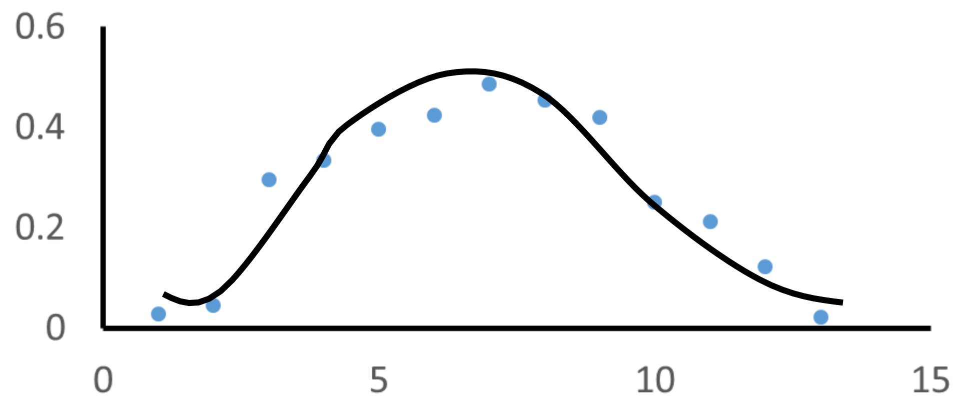
Normality of Residuals

Hrs Studied (X)	Marks (Y)
0	40
2	52
3	53
4	55
4	56
5	72
6	71
6	88
7	56
7	74
8	89
9	67
9	89

Marks predicted
41.80
50.90
55.45
60.00
60.00
64.55
69.10
69.10
73.65
73.65
78.20
82.75
82.75

Residuals
-1.80
1.10
-2.45
-5.00
-4.00
7.45
1.90
18.9
-17.65
0.35
10.8
-15.75
6.25

$$y = 41.8 + 4.55\,x$$



Residuals or Errors should be normally distributed.

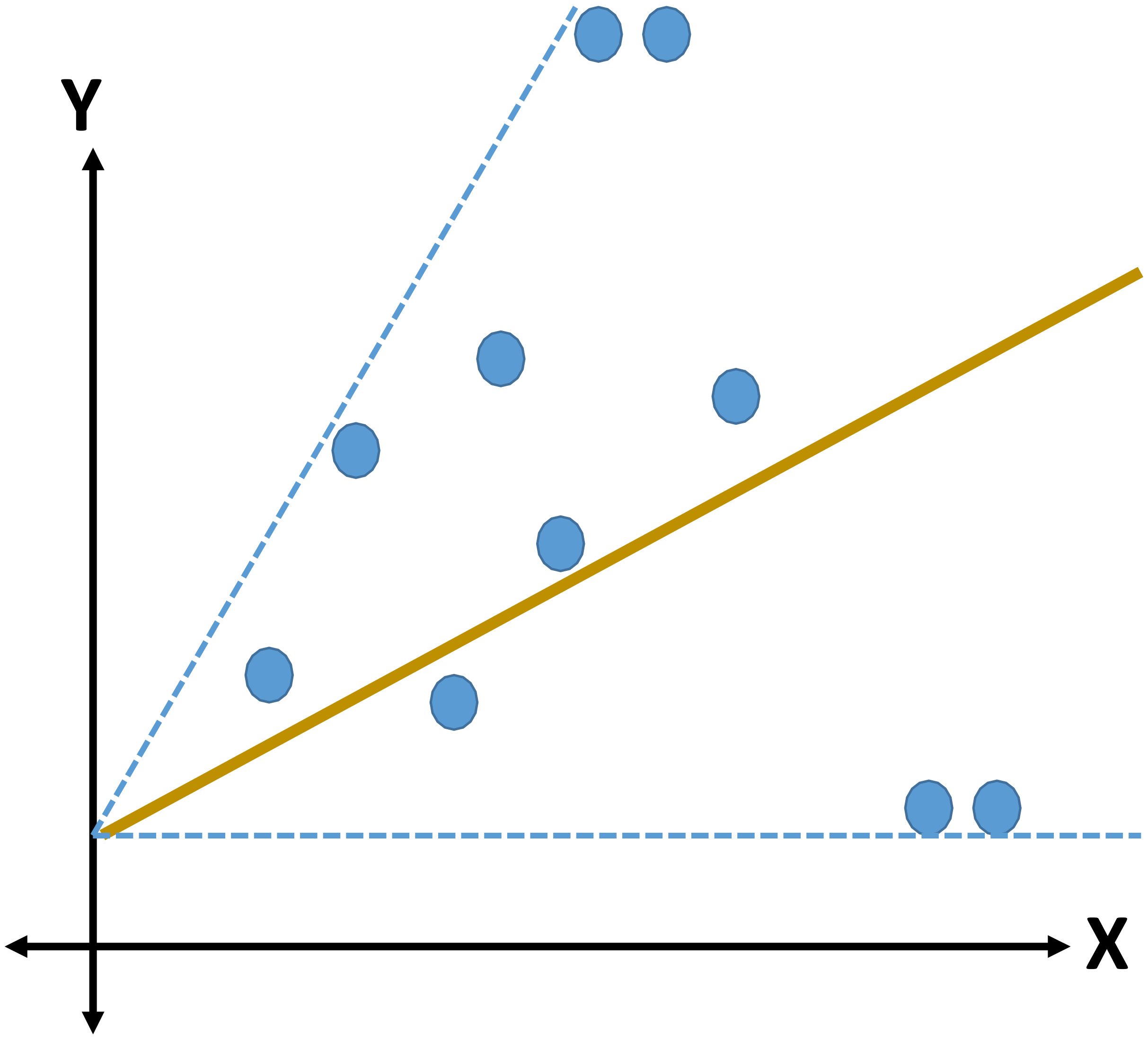
Homoscedasticity

HomoScedasticity

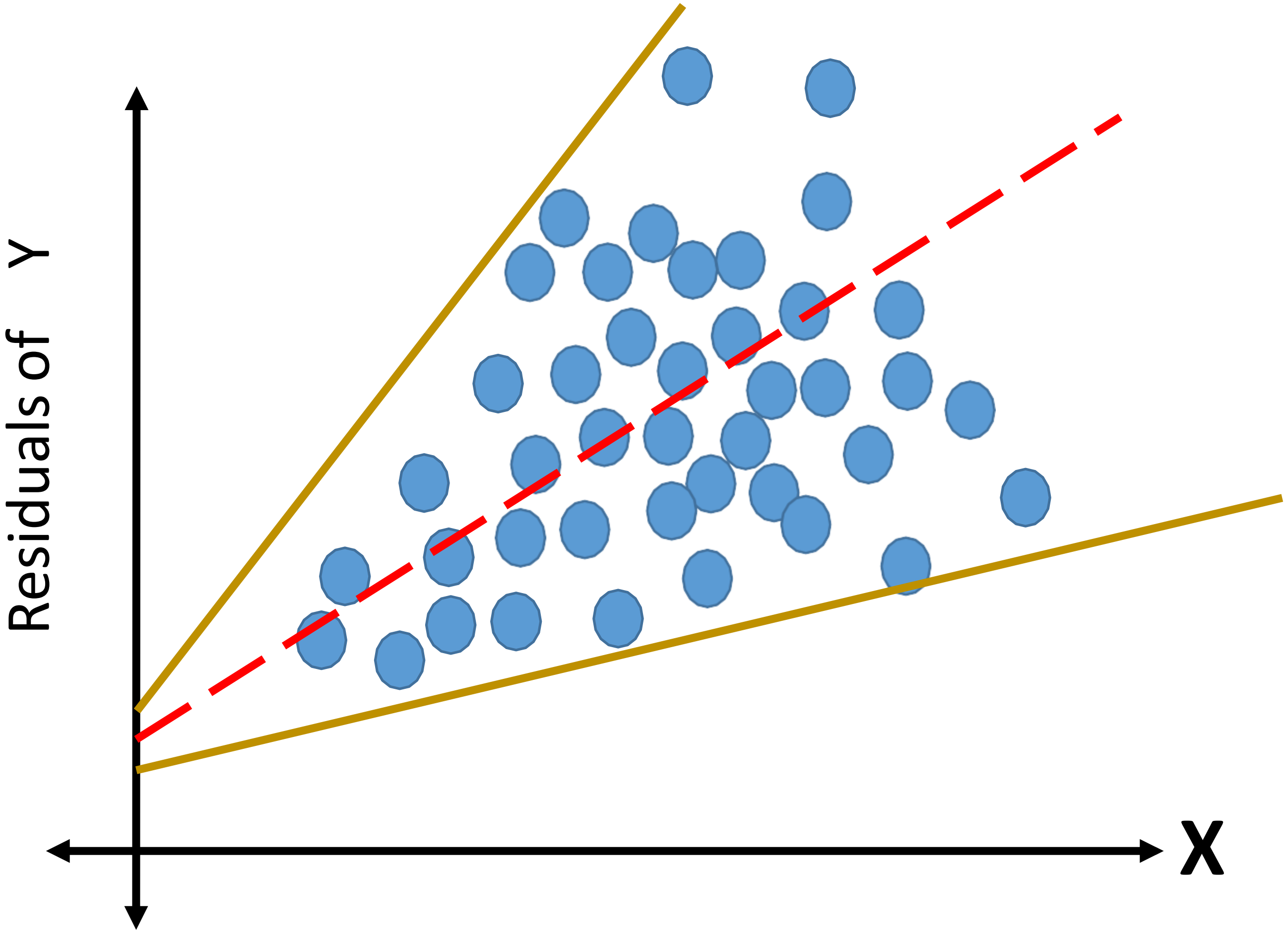
Same

Variance/Spread

Ordinary Least Square



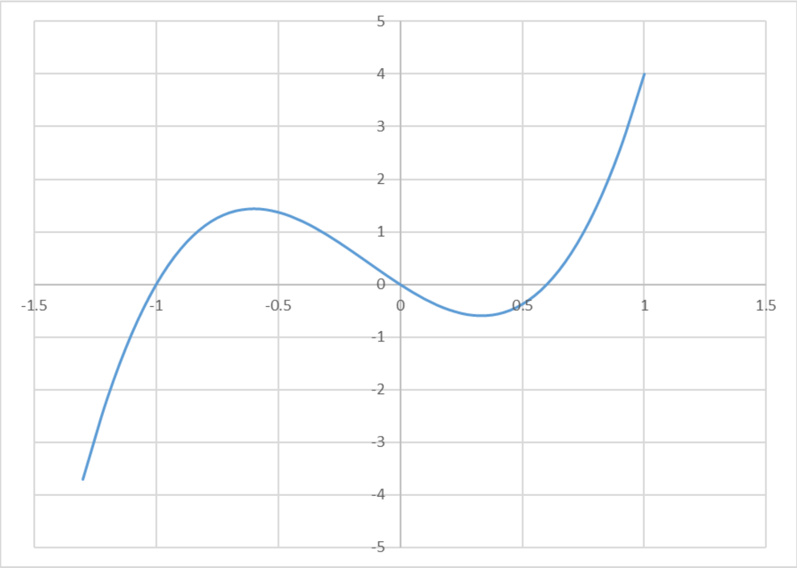
Heteroscedasticity



Variance should follow Homoscedasticity

Remedies

- Rebuild the model with new predictors
- Look for outliers
- Variable transformation using Log or Power transformation
- Consider Polynomial or other regression algorithm



Multiple Linear Regression – Dummy Variable Trap

Multiple Linear Regression

Hrs Studied (X1)	Hrs Slept (X2)	Marks (Y)
0	8	40
2	8	52
3	7.5	53
4	7	55
4	9	56
5	8.5	72
6	9	71
6	7	88
7	6	56
7	7	74
8	9	89
9	6	67
9	9	89

$$y = b_0 + b_1 x_1 + b_2 x_2$$

Dependent Variable

Marks Obtained

Independent Variable

Hrs Studied
Hrs Slept

Dummy Variable Trap

Hrs Studied (X1)	Hrs Slept (X2)	Math (X3)	Science (X4)	~~Art (X5)~~	Marks (Y)
0	8	1	0	0	40
2	8	0	1	0	52
3	7.5	0	0	1	53
4	7	1	0	0	55
4	9	1	0	0	56
5	8.5	1	0	0	72
6	9	0	1	0	71
6	7	0	0	1	88
7	6	0	0	1	56
7	7	0	1	0	74
8	9	0	1	0	89
9	6	1	0	0	67
9	9	0	0	1	89

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4$$