Tutorial-4.

1) $f_X(x) = c x e^{-x/2} \quad , \quad x \geq 0$

$= 0 \quad , \quad x < 0$

(i) $\because$ $f_X(x)$ is pdf. then.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} c.x.e^{-x/2}\,dx = 1$$

$$\Rightarrow c\int_{0}^{\infty} x e^{-\frac{x}{2}}\,dx = 1$$

$$\Rightarrow c.\left[\left[x\int e^{-\frac{x}{2}}dx\right]_0^{\infty} - \int_0^{\infty}\left[\frac{d(x)}{dx}.\int e^{-\frac{x}{2}}dx\right]dx\right] = 1$$

$$\Rightarrow c\left[\left[-x.2.e^{-\frac{x}{2}}\right]_0^{\infty} + \int_0^{\infty} 2e^{-\frac{x}{2}}\,dx\right] = 1$$

$$\Rightarrow c\left[0 + 2\int_0^{\infty} e^{-\frac{x}{2}}\,dx\right] = 1$$

$$\Rightarrow 2c.2\left[-e^{-\frac{x}{2}}\right]_0^{\infty} = 1$$

$$\Rightarrow 2c.2.[0+1] = 1$$

$$\Rightarrow c = \frac{1}{4}$$

(ii) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$\therefore F_X(x) = 0 \quad , \quad x < 0$

$= \int_{-\infty}^{x} \frac{1}{4}.t.e^{-\frac{t}{2}}\,dt \quad x \geq 0$

$$= \frac{1}{4}\int_{0}^{x} t.e^{-\frac{t}{2}}\,dt$$

$$= \frac{1}{4}.\int_0^x t\,e^{-\frac{t}{2}}\,dt$$

$$= \frac{1}{4}\left[\left[t\int e^{-\frac{t}{2}}dt\right]_0^x - \int_0^x\left[\frac{d}{dt}(t).\int e^{-\frac{t}{2}}dt\right]dt\right]$$

$$= \frac{1}{4}\left[\left[2t.\left[-e^{-\frac{t}{2}}\right]\right]_0^x + 2\int_0^x e^{-\frac{t}{2}}\,dt\right]$$

$$= \frac{1}{4}\left[-2x\,e^{-\frac{x}{2}} - 4\left[e^{-\frac{x}{2}}-1\right]\right]$$

$$= \frac{1}{2}\left[-x\,e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + 2\right]$$

$$\therefore F_x(x) = 0 \quad , x<0$$
$$= \frac{1}{2}\left[-x\,e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + 2\right] \quad , x \geq 0$$

(iii) $E(X) = \int_{-\infty}^{\infty} x.f_x(x)\,dx$

$$= \int_0^{\infty} x.f_x(x)\,dx$$

$$= \frac{1}{4}\int_0^{\infty} x.\ x.e^{-\frac{x}{2}}\,dx \quad \left[\because f_x(x) = \frac{1}{4}x e^{-\frac{x}{2}} \quad x\geq 0 \atop = 0 \quad x<0\right.$$

$$= \frac{1}{4}\int_0^{\infty} x^2 e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\left[\left[x^2\int e^{-\frac{x}{2}}dx\right]_0^{\infty} - \int_0^{\infty}\left[\frac{d}{dx}(x^2)\int e^{-\frac{x}{2}}dx\right]dx\right]$$

$$= \frac{1}{4}\left[\left[x^2\, 2\left[-e^{\frac{x}{2}}\right]\right]_0^{\alpha} + 4\int_0^{\alpha} x.e^{-\frac{x}{2}}\,dx\right]$$

$$= \frac{1}{4} \times 4 \int_0^{\alpha} x.e^{-\frac{x}{2}}\,dx$$

$$= \left[-2x.e^{-\frac{x}{2}}\right]_0^{\alpha} + 2\int_0^{\alpha} e^{-\frac{x}{2}}\,dx$$

$$= 2\times\left[-2.e^{-\frac{x}{2}}\right]_0^{\alpha}$$

$$= 4.$$

$$E(x^2) = \frac{1}{4}\int_0^{\alpha} x^2.x e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\int_0^{\alpha} x^3 e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\left[\left[-2x^3 e^{-\frac{x}{2}}\right]_0^{\alpha} - \int_0^{\alpha}\left\{3x^2.\int e^{-\frac{x}{2}}\,dx\right\}dx\right]$$

$$= \frac{1}{4}\left[3\times 2\int_0^{\alpha} x^2 e^{-\frac{x}{2}}\,dx\right]$$

$$= \frac{1}{4}\times 6.\ 4\times E(x)$$

$$= 6\times 4$$

$$= 24$$

$$\therefore V(x) = E(x^2) - \{E(x)\}^2 = 24 - (4)^2$$

$$= 24 - 16$$

$$= 8$$

$S.D.(X) = \sqrt{V(X)} = \sqrt{8} = 2\sqrt{2}$

iv) For continuous random variable, if μ is median, then,

$$F_X(\mu) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left[-\mu e^{-\frac{\mu}{2}} - 2e^{-\frac{\mu}{2}} + 2\right] = \frac{1}{2}$$

$$\Rightarrow -\mu e^{-\frac{\mu}{2}} - 2e^{-\frac{\mu}{2}} + 2 = 1$$

$$\Rightarrow (\mu+2)\, e^{-\frac{\mu}{2}} = 1$$

$$\Rightarrow (\mu+2) = e^{\frac{\mu}{2}}$$

$$\Rightarrow (\mu+2) - e^{\frac{\mu}{2}} = 0$$

Solve this eqn numerically. [use Newton Raphson with interval $[-6, 6]$]

2) X: The distance between the dart's impact point and the center of the target.

$$P(X \le x) = c\pi x^2, \quad 0 \le x \le 25$$
$$= 1, \quad x > 25$$

(i) CDF of X is

$$F_X(x) = 0 \quad x < 0$$
$$= c\pi x^2 \quad 0 \le x \le 25$$
$$= 1 \quad x > 25$$

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(c\pi x^2) = 2c\pi x$$ when $0 \le x \le 25$

$$\therefore f_X(x) = 2c\pi x \quad 0 \le x \le 25$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{25} 2c\pi x\,dx = 1$$

$$\Rightarrow 2c\pi\left[\frac{x^2}{2}\right]_0^{25} = 1$$

$$\Rightarrow c = \frac{1}{(25)^2\pi} = \frac{1}{625\pi}$$

(ii) P.DF of X is $f_x(x) = \frac{2}{625\cancel{\pi}}\cancel{\pi}x \qquad 0 \le x \le 25$

$= 0 \qquad$ otherwise

(iii) $E(X) = \int_{-\infty}^{\infty} x f_x(x)\,dx$

$$= \int_0^{25} x \cdot \frac{2}{625\cancel{\pi}}\cancel{\pi}x\,dx$$

$$= \frac{2}{625\cancel{\pi}}\cancel{\pi}\left[\frac{x^3}{3}\right]_0^{25}$$

$$= \frac{25 \times 2 \times \cancel{\pi}}{3}$$

$$= \frac{50\cancel{\pi}}{3}$$

(iv) $P(X \leq 10 \mid X \geq 5) = \dfrac{P(5 \leq X \leq 10)}{P(X \geq 5)}$

$$= \frac{P(5 \leq X \leq 10)}{1 - P(X < 5)}$$

$$= \frac{\int_5^{10} f_X(x)\,dx}{1 - \int_0^5 f_X(x)\,dx}$$

$$= \frac{\int_5^{10} \frac{2}{625} x\,dx}{1 - \int_0^5 \frac{2}{625} x\,dx}$$

$$= \frac{\frac{2}{625}\left[\frac{x^2}{2}\right]_5^{10}}{1 - \frac{2}{625}\left[\frac{x^2}{2}\right]_0^5}$$

$$= \frac{\frac{1}{625}(10^2 - 5^2)}{1 - \frac{1}{625} \times 25}$$

$$= \frac{10^2 - 5^2}{625 - 25}$$

$$= \frac{5 \times 15}{600}$$

$$= \frac{1}{8}$$

(v) $10 \times \int_0^r x \cdot \frac{2}{625} x\,dx + 1 \times \int_r^{2r} x \cdot \frac{2}{625} x\,dx + 0 \times \int_{2r}^{25} x \cdot \frac{2}{625} x\,dx = 1 + 0.25$

$\Rightarrow \frac{2}{625} \times \frac{1}{3}\left[10r^3 + 8r^3 - r^3\right] = 1.25$

$\Rightarrow r^3 = \dfrac{3 \times 625 \times 1.25}{2 \times 17}$

3)

$$f_X(x) = \frac{x(6+x)}{3(3+x)^2}, \quad 0 < x \le 3$$

$$= \frac{9(3+2x)}{x^2(3+x)^2}, \quad x > 3$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^3 \frac{x(6+x)}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{x^2(3+x)^2}\,dx$$

$$= \int_0^3 \frac{x^2+6x}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{(3x+x^2)^2}\,dx$$

$$= \int_0^3 \frac{x^2+6x+9-9}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{(3x+x^2)^2}\,dx$$

$$= \int_0^3 \frac{1}{3}\,dx - 3\int_0^3 \frac{1}{(3+x)^2}\,dx + \int_3^{\infty} \frac{9\,d(3x+x^2)}{(3x+x^2)^2}$$

$$= \left[\frac{x}{3}\right]_0^3 + 3\left[\frac{1}{(3+x)}\right]_0^3 - \left[\frac{9}{(3x+x^2)}\right]_3^{\infty}$$

$$= 1 + \frac{1}{2} - 1 + \frac{9}{18}$$

$$= 1$$

$\therefore f_X(x)$ is a pdf.

4) $f_x(x) = \theta^2 x e^{-\theta x}, \quad x > 0$

$= 0, \quad x \leq 0$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_0^{\infty} \theta^2 x e^{-\theta x}\,dx$$

$$= \theta^2 \int_0^{\infty} x e^{-\theta x}\,dx$$

$$= \theta^2 \left\{ \left[-\frac{x.e^{-\theta x}}{\theta} \right]_0^{\infty} + \frac{1}{\theta} \int_0^{\infty} e^{-\theta x}\,dx \right]$$

$$= \theta^2 \times \frac{1}{\theta} \cdot \left[-e^{-\theta x} \right]_0^{\infty}$$

$$= \frac{1}{\theta^2} \times \theta^2$$

$$= 1.$$

$\therefore f_x(x)$ is a pdf.

$$F_x(x) = \int_{-\infty}^{x} f_x(t)\,dt$$

$$= \int_0^{x} \theta^2 t e^{-\theta t}\,dt$$

$$= \theta^2 \int_0^{x} t e^{-\theta t}\,dt$$

$$= \theta^2 \left\{ \left[\frac{t e^{-\theta t}}{-\theta} \right]_0^{x} + \frac{1}{\theta} \int_0^{x} e^{-\theta t}\,dt \right\}$$

$$= \theta^2 \left[-\frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta^2} \left[e^{-\theta x} - 1 \right] \right]$$

$$= 1 - e^{-\theta x} - x\theta e^{-\theta x}$$

$$= 1 - e^{-\theta x}(1 + x\theta)$$

$$P(X \geq 1) = 1 - P(X < 1)$$
$$= 1 - [1 - e^{-\theta}(\theta+1)]$$
$$= e^{-\theta}(\theta+1)$$

5) i) $F(x) = 0 \quad x \leq 0$
$$= \frac{x}{2} \quad 0 \leq x < 1$$
$$= \frac{1}{2} \quad 1 \leq x < 2$$
$$= \frac{x}{4} \quad 2 \leq x < 4$$
$$= 1 \quad x \geq 4$$

$F(x)$ is right continuous and non decreasing
$F(-\infty) = 0$
and $F(\infty) = 1$

$\therefore F(x)$ is C.D.F.

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2} \quad \text{when } 0 \leq x < 1$$

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{x}{4}\right) = \frac{1}{4} \quad 2 \leq x < 4$$

$$\therefore f_X(x) = \frac{1}{2} \quad 0 \leq x < 1$$
$$= \frac{1}{4} \quad 2 \leq x < 4$$
$$= 0 \quad \text{otherwise}$$

(ii) $F(x) = 0, \quad x < -\theta$

$= \frac{1}{2}\left(\frac{x}{\theta}+1\right), \quad |x| \le \theta$

$= 1, \quad x > \theta$

$F(x)$ is right continuous and non decreasing.

Also, $F(-\infty) = 0$

$F(\infty) = 1$

$\therefore F(x)$ is a CDF

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{x}{\theta}+1\right)\right) = \frac{1}{2\theta} \quad \text{when } |x| \le \theta$$

$\therefore f_x(x) = \frac{1}{2\theta} \quad |x| \le \theta$

$= 0 \quad$ otherwise

(iii) $F(x) = 0 \quad x < 1$

$= \frac{(x-1)^2}{8} \quad 1 \le x < 3$

$= 1 \quad x \ge 3$

$F(x)$ is right continuous everywhere except at $x=3$ ~~and~~ and of $F(x)$ is non decreasing. Also $F(-\infty) = 0$ and $F(\infty) = 1$.

$\therefore F(x)$ is a CDF. At $x=3$, $F(3-0) - F(3) = P(X=3)$

$\Rightarrow P(X=3) = \frac{1}{2}$

$$\frac{d}{dx}(F(x)) = \frac{8(x-1)}{64} = \frac{(x-1)}{4} \quad 1 \le x < 3$$

$\therefore f_x(x) = \frac{x-1}{4} \quad 1 \le x < 3$

$= \frac{1}{2} \quad x = 3$

$= 0$ elsewhere

6)

$$f(x) = \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})\,(1+x^2)^m}, \quad -\alpha < x < \alpha, \quad m \geq 1.$$

$$E(x^{2r}) = \int_{-\alpha}^{\alpha} x^{2r} \cdot f(x)\, dx$$

$$= \int_{-\alpha}^{\alpha} x^{2r} \cdot \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})\,(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{-\alpha}^{\alpha} \frac{x^{2r}}{(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \cdot 2\int_{0}^{\alpha} \frac{x^{2r}}{(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{0}^{\alpha} \frac{x^{2r-1}}{(1+x^2)^m}\, d(x^2)$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{0}^{\alpha} \frac{(x^2)^{(r+\frac{1}{2})-1}}{(1+x^2)^m}\, d(x^2)$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})}\, \beta\left(r+\frac{1}{2},\, m-r-\frac{1}{2}\right) \quad \left[\because \beta(m,n) = \int_{0}^{\alpha} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx\right]$$

7) ~~X~~ X has a symmetric distribution about a.

$\therefore\ P(X \geq a+x) = P(X \leq a-x)$

$\Rightarrow\ 1 - F_X(a+x) = F_X(a-x)$

$\Rightarrow\ -\frac{d}{dx} F_X(a+x) = \frac{d}{dx} F_X(a-x)$

$\Rightarrow\ f_X(a+x) = f_X(a-x)$

Now $E[X-a] = \int_{-\alpha}^{\alpha} (x-a) f_X(x)\, dx$

$= \int_{-\alpha}^{a} (x-a) f_X(x)\, dx + \int_{a}^{\alpha} (x-a) f_X(x)\, dx$

$= I_1 + I_2$

~~Now but~~

Now, $I_1 = \int_{-\alpha}^{a} (x-a) f_X(x)\, dx$

Let $x - a = z$

$\Rightarrow dx = dz$

$\therefore\ I_1 = \int_{-\alpha}^{0} z \cdot f_X(z+a)\, dz$

$I_2 = \int_{a}^{\alpha} (x-a) f_X(x)\, dx$

Let, $x - a = -z$

$dx = -dz$

$\therefore\ I_2 = +\int_{0}^{-\alpha} z \cdot f_X(a-z)\, dz$

$= -\int_{-\alpha}^{0} z \cdot f_X(a-z)\, dz = -\int_{-\alpha}^{0} z \cdot f_X(a+z)\, dz$

$= -I_1$

$\therefore \quad E(X-c) = I_1 - I_1$

$= 0$

$\Rightarrow E(X) - c = 0$

$\Rightarrow E(X) = c$

8) (i) $\int_0^{\infty} [1 - F(x)]\,dx - \int_{-\infty}^{0} F(x)\,dx$

$$= \left[[1-F(x)].\int 1.dx\right]_0^{\infty} - \int_0^{\infty}\left[\frac{d}{dx}[1-F(x)].\int 1.dx\right]dx$$

$$- \left[F(x).\int 1.dx\right]_{-\infty}^{0} + \int_{-\infty}^{0}\left[\frac{d}{dx}(F(x)).\int 1.dx\right]dx$$

$$= \left[x.(1-F(x))\right]_0^{\infty} + \int_0^{\infty} x.f_X(x)\,dx - \left[xF(x)\right]_{-\infty}^{0} + \int_{-\infty}^{0} x f_X(x)\,dx$$

$$= \left[x(1-F(x))\right]_0^{\infty} - \left[xF(-x)\right]_0^{\infty} + \int_{-\infty}^{\infty} x.f_X(x)\,dx$$

$$= \left[x(1-F(x) - F(-x))\right]_0^{\infty} + E(x)$$

Given that as $x \to \infty$ $\quad x(1-F(x) - F(-x)) \to 0$

$$\Rightarrow \int_0^{\infty}(1-F(x))\,dx - \int_{-\infty}^{0} F(x)\,dx = E(X)$$

(ii) when X is non negative,

then, $F(x) = 0$ as $x < 0$

Then $\int_{-\infty}^{0} F(x)\,dx = 0$

$\therefore \quad E(X) = \int_0^{\infty} (1 - F(x))\, dx$

9) $F(x) = 1 - 0.8\, e^{-x}, \quad x \geq 0$

$\quad = 0 \quad , \quad x < 0$

$f_X(x) = \frac{d}{dx}(F(x)) = 0.8 . e^{-x}$

$\therefore f_X(x) = 0 \quad , x < 0$

$= 1 - 0.8 \quad , x = 0$

$= 0.2 \quad , x = 0$

$= 0.8 . e^{-x}, \quad x > 0$

$E(X) = \int_{-\infty}^{0} x . f_X(x)\, dx + 0 \times P(X=0) + \int_0^{\infty} x . f_X(x)\, dx$

$= \int 0 + 0 \times 0.2 + \int_0^{\infty} x . 0.8 \times e^{-x}\, dx$

$= (0.8) \int_0^{\infty} x e^{-x}\, dx$

$= 0.8 \left[\left[-x e^{-x} \right]_0^{\infty} + \int_0^{\infty} e^{-x}\, dx \right]$

$= 0.8 \times \left[- e^{-x} \right]_0^{\infty}$

$= 0.8$

10) $f(x) = \frac{1}{2} \quad -1 \le x \le 1$

$= 0$ otherwise

$Y = \max(X, 0)$

$\therefore \; Y = 0 \qquad x \le 0$

$= x$ when $x \geq 0$

Now, $F_X(x) = \int_{-\infty}^{x} \cancel{f(x)\,dx} \; \cancel{\int_{-\infty}^{x} f(t)\,dt}$ w.r.t.

$= \cancel{\int_{-1}^{x} \frac{1}{2}\,dt}$

$= \cancel{\frac{x+1}{2}}$

$F(y) = P(Y \le y) = \int_{-\infty}^{y} f(x)\,dx$

$= \int_{-\infty}^{y} \frac{1}{2}\,dx$

$= \frac{1}{2}(y+1) \qquad -1 \le y \le 1$

$F(y) = \frac{1}{2}(y+1) \quad -1 \le y \le 1$

$= 0$ elsewhere

11) $f_x(x) = \frac{1}{2a} e^{-\frac{|x-\mu|}{a}}$, $-\infty < x < \infty$, $a > 0$, $-\infty < \mu < \infty$

$$\int_{-\infty}^{\infty} f_x(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx$$

$$= \int_{-\infty}^{\mu} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx + \int_{\mu}^{\infty} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx$$

$$= \frac{1}{2a}\int_{-\infty}^{\mu} e^{\frac{(x-\mu)}{a}} dx + \int_{\mu}^{\infty} \frac{1}{2a} e^{-\frac{(x-\mu)}{a}} dx$$

$$= \frac{1}{2a} \cdot a \left[e^{\frac{x-\mu}{a}} \right]_{-\infty}^{\mu} - \frac{1}{2a} \cdot a \left[e^{-\frac{(x-\mu)}{a}} \right]_{\mu}^{\infty}$$

$$= \frac{1}{2a} \cdot a + \frac{1}{2a} \cdot a = 1$$

$\therefore f_x(x)$ is pdf.

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_x(x)\, dx$$

$$= \frac{1}{2a}\int_{-\infty}^{\mu} e^{tx} e^{\frac{(x-\mu)}{a}} dx + \frac{1}{2a}\int_{\mu}^{\infty} e^{tx} e^{-\frac{(x-\mu)}{a}} dx$$

$$= \frac{1}{2a}\left[\int_{-\infty}^{\mu} e^{(t+\frac{1}{a})x - \frac{\mu}{a}} dx + \int_{\mu}^{\infty} e^{(t-\frac{1}{a})x + \frac{\mu}{a}} dx \right]$$

$$= \frac{1}{2a}\left\{ \left[\frac{a}{(at+1)} e^{(t+\frac{1}{a})x - \frac{\mu}{a}} \right]_{-\infty}^{\mu} + \left[\frac{a}{at-1} e^{(t-\frac{1}{a})x + \frac{\mu}{a}} \right]_{\mu}^{\infty} \right\}$$

$$= \frac{1}{2a}\left[\frac{a}{(at+1)} e^{t\mu} + \frac{a}{(at-1)} e^{\mu t} \right]$$

$$= \frac{1}{2} e^{\mu t} \left[\frac{1}{at+1} + \frac{1}{(at-1)} \right]$$

$$= \frac{1}{2} \cdot e^{\mu t} \cdot \frac{2\alpha t}{(\alpha^2 t^2 - 1)}$$

$$= \frac{\alpha t\, e^{\mu t}}{(\alpha^2 t^2 - 1)}$$

12) $f_x(x) = \frac{1}{2}\left[1 - \frac{|x-3|}{2}\right], \quad 1 < x < 5$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_1^5 \frac{1}{2}\left(1 - \frac{|x-3|}{2}\right) dx$$

$$= \frac{1}{2}\int_1^3 \left[1 + \frac{(x-3)}{2}\right] dx + \frac{1}{2}\int_3^5 \left(1 - \frac{(x-3)}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_1^3 \left(\frac{x}{2} - \frac{1}{2}\right) dx + \int_3^5 \left(\frac{5}{2} - \frac{x}{2}\right) dx\right]$$

$$= \frac{1}{2}\left\{\left[\frac{x^2}{4} - \frac{x}{2}\right]_1^3 + \left[\frac{5x}{2} - \frac{x^2}{4}\right]_3^5\right\}$$

$$= \frac{1}{2}\left[\frac{9}{4} - \frac{3}{2} - \frac{1}{4} + \frac{1}{2} + \frac{25}{2} - \frac{25}{4} - \frac{15}{2} + \frac{9}{4}\right]$$

$$= \frac{1}{2} \times 2$$

$$= 1.$$

$\therefore f_x(x)$ is a ~~prob~~ PDF

$$E(x) = \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx$$

$$= \frac{1}{2}\int_1^5 x \cdot \left[1 - \frac{|x-3|}{2}\right] dx$$

$$= \frac{1}{2}\left\{ \int_1^3 x.\left(1 + \frac{x-3}{2}\right) dx + \int_3^5 x\left(1 - \frac{x-3}{2}\right) dx \right\}$$

$$= \frac{1}{2}\left\{ \int_1^3 \left(\frac{x^2}{2} - \frac{x}{2}\right) dx + \int_3^5 \left(\frac{5x}{2} - \frac{x^2}{2}\right) dx \right\}$$

$$= \frac{1}{2}\left\{ \left[\frac{x^3}{6} - \frac{x^2}{4}\right]_1^3 + \left[\frac{5x^2}{4} - \frac{x^3}{6}\right]_3^5 \right\}$$

$$= \frac{1}{2}\left[\frac{27}{6} - \frac{9}{4} - \frac{1}{6} + \frac{1}{4} + \frac{125}{4} - \frac{125}{6} - \frac{45}{4} + \frac{27}{6}\right]$$

$$= \frac{1}{2} \times 6$$

$$= 3$$

For median,

$$\int_1^x \frac{1}{2}\left[1 - \frac{|t-3|}{2}\right] dt = \frac{1}{2}$$

$$\Rightarrow \int_1^3 \frac{1}{2}\left[1 + \frac{(t-3)}{2}\right] dt + \int_3^x \frac{1}{2}\left[1 - \frac{(t-3)}{2}\right] dt = \frac{1}{2}$$

$$\Rightarrow \int_1^3 \left(\frac{t}{2} - \frac{1}{2}\right) dt + \int_3^x \left(\frac{5}{2} - \frac{t}{2}\right) dt = 1$$

$$\Rightarrow \left[\frac{t^2}{4} - \frac{t}{2}\right]_1^3 + \left[\frac{5t}{2} - \frac{t^2}{4}\right]_3^x = 1$$

$$\Rightarrow \left[\frac{9}{4} - \frac{3}{2} - \frac{1}{4} + \frac{1}{2}\right] + \left[\frac{5x}{2} - \frac{x^2}{4} - \frac{15}{2} + \frac{9}{4}\right] = 1$$

$$\Rightarrow 1 + \left[\frac{10x - x^2 - 21}{4}\right] = 1$$

$$\Rightarrow x^2 - 10x + 21 = 0$$

$$\Rightarrow (x-3)(x-7) = 0$$

$\Rightarrow x = 3.$ $\therefore$ Median is $x = 3$

$$E(X^2) = \int_1^5 \frac{x^2}{2}\left[1 - \frac{|x-3|}{2}\right] dx$$

$$= \frac{1}{2}\int_1^3 x^2\left(1 + \frac{x-3}{2}\right) dx + \frac{1}{2}\int_3^5 x^2\left(1 - \frac{x-3}{2}\right) dx$$

$$= \frac{1}{2}\int_1^3 \left(\frac{x^3}{2} - \frac{x^2}{2}\right) dx + \frac{1}{2}\int_3^5 \left[\frac{5x^2}{2} - \frac{x^3}{2}\right] dx$$

$$= \frac{1}{2}\left\{\left[\frac{x^4}{8} - \frac{x^3}{6}\right]_1^3 + \left[\frac{5x^3}{6} - \frac{x^4}{8}\right]_3^5\right\}$$

$$= \frac{1}{2}\left[\frac{81}{8} - \frac{27}{6} - \frac{1}{8} + \frac{1}{6} + \frac{625}{6} - \frac{625}{8} - \frac{135}{8} + \frac{81}{8}\right]$$

$$= \frac{599}{48}$$

$$V(X) = \frac{599}{48} - \{E(X)\}^2 = \frac{599}{48} - 9$$

$$= \frac{167}{48}$$

<u>p-th quantile</u>

$$F_X(x_p) = p$$

$$\Rightarrow \int_1^{x_p} \frac{1}{2}\left[1 - \frac{|x-3|}{2}\right] dx = p$$

$$\Rightarrow \frac{1}{2}\int_1^3 \left(1 + \frac{x-3}{2}\right) dx + \frac{1}{2}\int_3^{x_p} \left(1 - \frac{x-3}{2}\right) dx = p$$

$$\Rightarrow \frac{1}{2}\int_1^3 \left(\frac{x}{2} - \frac{1}{2}\right) dx + \frac{1}{2}\int_3^{x_p} \left(\frac{5}{2} - \frac{x}{2}\right) dx = p$$

$$\Rightarrow \frac{1}{2} \times \left[\frac{x^2}{4} - \frac{x}{2}\right]_1^3 + \frac{1}{2}\cdot\left[\frac{5x}{2} - \frac{x^2}{4}\right]_3^{x_p} = p$$

$$\Rightarrow \frac{1}{2}\left[\frac{9}{4}-\frac{3}{2}-\frac{1}{4}+\frac{1}{2}\right]+\frac{1}{2}\left[\frac{5x_p}{2}-\frac{x_p^2}{4}-\frac{15}{2}+\frac{9}{4}\right]=p$$

$$\Rightarrow \frac{1}{2}+\frac{1}{2}\left[\frac{10x_p-x_p^2-21}{4}\right]=p$$

$$\Rightarrow 4+10x_p-x_p^2-21=8p$$

$$\Rightarrow x_p^2-10x_p+17-8p=0$$

Solving this equation for given p gives the p th quantile.

13) $$f_X(x)=\frac{K}{\beta}\left[1-\frac{(x-\alpha)^2}{\beta^2}\right] \quad (\alpha-\beta)<x<(\alpha+\beta)$$

where $-\alpha<\alpha<\alpha$, $\beta>0$

$$\int_{-\alpha}^{\alpha} f_X(x) = \int_{-\alpha}^{\alpha}\frac{K}{\beta}\left[1-\frac{(x-\alpha)^2}{\beta^2}\right]dx=1$$

$$\Rightarrow \frac{K}{\beta}\int_{\alpha-\beta}^{\alpha+\beta}\left[1-\frac{(x-\alpha)^2}{\beta^2}\right]dx=1$$

$$\Rightarrow \frac{K}{\beta}\left[x-\frac{(x-\alpha)^3}{3\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)}=1$$

$$\Rightarrow \frac{K}{\beta}\left[(\alpha+\beta)-\frac{\beta^3}{3\beta^2}-(\alpha-\beta)-\frac{\beta^3}{3\beta^2}\right]=1$$

$$\Rightarrow \frac{K}{\beta}\left[2\beta-\frac{\beta}{3}-\frac{\beta}{3}\right]=1$$

$$\Rightarrow K\left[\frac{4}{3}\right]=1$$

$$\Rightarrow K=\frac{3}{4}$$

$$E(X) = \int_{-\alpha}^{\alpha} x \cdot f_X(x)\, dx$$

$$= \int_{\alpha-\beta}^{\alpha+\beta} x \cdot \frac{3}{4\beta}\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta} \int_{\alpha-\beta}^{\alpha+\beta} x \left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta} \int_{\alpha-\beta}^{\alpha+\beta} \left[x - \frac{x^3 - 2\alpha x^2 + \alpha^2 x}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta}\left[\frac{x^2}{2} - \frac{x^4}{4\beta^2} + \frac{2\alpha x^3}{3\beta^2} - \frac{\alpha^2 x^2}{2\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)}$$

Median: $\int_{\alpha-\beta}^{x} \frac{3}{4\beta}\left(1 - \frac{(t-\alpha)^2}{\beta^2}\right) dt = \frac{1}{2}$

$$\Rightarrow \frac{3}{4\beta}\left[t - \frac{(t-\alpha)^3}{3\beta^2}\right]_{(\alpha-\beta)}^{x} = \frac{1}{2}$$

$$\Rightarrow x - \frac{(x-\alpha)^3}{3\beta^2} - (\alpha-\beta) - \frac{\beta^3}{3\beta^2} = \frac{2\beta}{3}$$

$$\Rightarrow x - \frac{(x-\alpha)^3}{3\beta^2} - \alpha + \beta - \frac{\beta}{3} = \frac{2\beta}{3}$$

$$\Rightarrow x - \frac{(x-\alpha)^2}{3\beta^2} = \alpha$$

$$\Rightarrow (x-\alpha)\left[1 - \frac{(x-\alpha)}{3\beta^2}\right] = 0$$

$\therefore x = \alpha$ is a median.

<u>Variance</u>

$$E(X^2) = \int_{-\alpha}^{\alpha} x^2 f_X(x)\,dx$$

$$= \int_{\alpha-\beta}^{\alpha+\beta} x^2 \cdot \frac{3}{4\beta}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx$$

$$= \frac{3}{4\beta}\int_{\alpha-\beta}^{\alpha+\beta}\left(x^2 - \frac{x^4 - 2x^3\alpha + \alpha^2 x^2}{\beta^2}\right) dx$$

$$= \frac{3}{4\beta}\left[\frac{x^3}{3} - \frac{x^5}{5\beta^2} + \frac{2x^4\alpha}{4\beta^2} - \frac{\alpha^2 x^3}{3\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)}$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

<u>P-th quantile</u>

$$F(x_p) = p$$

$$\Rightarrow \frac{3}{4\beta}\int_{\alpha-\beta}^{x_p}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx = p$$

$$\Rightarrow \int_{\alpha-\beta}^{x_p}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx = \frac{4\beta p}{3}$$

$$\Rightarrow \left[x - \frac{(x-\alpha)^3}{3\beta^2} \right]_{(\alpha-\beta)}^{x_p} = \frac{4\beta p}{3}$$

$$\Rightarrow x_p - \frac{(x_p-\alpha)^3}{3\beta^2} - (\alpha-\beta) - \frac{\beta^3}{3\beta^2} = \frac{4\beta p}{3}$$

$$\Rightarrow \frac{3\beta^2 x_p - (x_p-\alpha)^3}{3\beta^2} - \alpha + \beta - \frac{\beta}{3} = \frac{4\beta}{3} p$$

Solving this eauation gives the p-th auantile