Tutorial sheet 3.

1) $P(X=0)=0$, $P(X=1)=K$, $P(X=2)=2K=P(X=3)$, $P(X=4)=3K$

$P(X=5)=K^2$, $P(X=6)=2K^2$, $P(X=7)=7K^2+K$

(i) $\sum_{K=-\alpha}^{\alpha} P(X=K) = 1$

$\Rightarrow \sum_{K=0}^{7} P(X=K) = 1$

$\Rightarrow P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7) = 1$

$\Rightarrow 0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$

$\Rightarrow 10K^2+9K-1=0$

$\Rightarrow (10K-1)(K+1)=0$

$\therefore K = \frac{1}{10}, -1$

Now $K \neq -1$ as otherwise $P(X=1)<0$.

$\therefore K = \frac{1}{10}$

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$= K+2K+2K+3K+K^2$

$= K^2+8K$

$= K(K+8) = \frac{81}{100}$

$P(X\geq 6) = 1-P(X<6) = 1-\frac{81}{100} = \frac{19}{100}$

$$P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= K + 2K + 2K + 3K$$

$$= 8K$$

$$= \frac{8}{10}$$

(iii) C.D.F is

$$
\begin{aligned}
F(x) &= 0 && x \le 0 \\
&= 0 && 0 \le x < 1 \\
&= 0 + K && 1 \le x < 2 \\
&= K && 1 \le x < 2 \\
&= 0 + K + 2K && 2 \le x < 3 \\
&= 3K && 2 \le x < 3 \\
&= 0 + K + 2K + 2K && 3 \le x < 4 \\
&= 5K && 3 \le x < 4 \\
&= 5K + 3K && 4 \le x < 5 \\
&= 8K && 4 \le x < 5 \\
&= 8K + K^2 && 5 \le x < 6 \\
&= 8K + 3K^2 && 6 \le x < 7 \\
&= 1 && x \ge 7
\end{aligned}
$$

2) Median: A real number μ is said to be median of a random variable (discrete) X, if $P(X \le \mu) \ge \frac{1}{2}$ and $P(X \ge \mu) \ge \frac{1}{2}$

Thus, from the condition $P(X \le \mu) \ge \frac{1}{2}$, we have

$F(\mu) \ge \frac{1}{2}$ [where $F(x)$ is c.d.f. of X]

If $c > \mu$,

$$E(|X-c|) = \sum_{x=-\infty}^{\infty} |x-c| \, p(x)$$ where $p(x)$ is the pmf of X.

$$= \sum_{x=-\infty}^{c} (c-x)\, p(x) + \sum_{x=c}^{\infty} (x-c)\, p(x)$$

$$= \sum_{x=-\infty}^{\mu} (c-x)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\infty} (x-c)\, p(x) - \sum_{x=\mu}^{c} (x-c)\, p(x)$$

$$= \sum_{x=-\infty}^{\mu} (c-\mu+\mu-x)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\infty} (x-\mu+\mu-c)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= \sum_{x=-\infty}^{\mu} \left[(c-\mu)\, p(x) + (\mu-x)\, p(x)\right] + 2\sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\infty} \left[(x-\mu)\cdot p(x) + (\mu-c)\cdot p(x)\right]$$

$$= (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + \sum_{x=-\infty}^{\mu} (\mu-x)\, p(x) + \sum_{x=\mu}^{\infty} (x-\mu)\, p(x) + (\mu-c)\sum_{x=\mu}^{\infty} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + \sum_{x=-\infty}^{\infty} |x-\mu|\, p(x) + (\mu-c)\sum_{x=\mu}^{\infty} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + E(|X-\mu|) + (\mu-c)\sum_{x=\mu}^{\infty} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (\mu-c)\sum_{x=\mu}^{\infty} p(x) + 2\sum_{x=\mu}^{c}(c-x)\,p(x)$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (\mu-c)\left[1-\sum_{x=-\infty}^{\mu} p(x)\right] + 2\sum_{x=\mu}^{c}(c-x)\,p(x)$$

$$\left[\because \sum_{x=-\infty}^{\infty} p(x) = 1 \quad \Rightarrow \sum_{x=\mu}^{\infty} p(x) = 1 - \sum_{x=-\infty}^{\mu} p(x)\right]$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (c-\mu)\left[\sum_{x=-\infty}^{\mu} p(x) - 1\right] + 2\sum_{x=\mu}^{c}(c-x)\,p(x)$$

$$= E(|x-\mu|) + (c-\mu)\left[2\sum_{x=-\infty}^{\mu} p(x) - 1\right] + 2\sum_{x=\mu}^{c}(c-x)\,p(x)$$

$$= E(|x-\mu|) + (c-\mu)\left[2F(\mu) - 1\right] + 2\sum_{x=\mu}^{c}(c-x)\,p(x) \quad \text{—— (1)}$$

$$\left[\because F(\mu) = P(X \le \mu) = \sum_{x=-\infty}^{\mu} p(x)\right]$$

Since μ is median, then $F(\mu) \geq \frac{1}{2}$

$\Rightarrow 2F(\mu) - 1 \geq 0$

~~For~~

$\therefore$ From (1) we have,

$$E(|x-c|) \geq E(|x-\mu|) + 2\sum_{x=\mu}^{c}(c-x)\,p(x)$$

Since $c > \mu$, then $\sum_{x=\mu}^{c}(c-x)\,p(x) \geq 0$

$$\therefore E(|x-c|) \geq E(|x-\mu|)$$

Similarly for $c < \mu$,

$$E(|X-c|) \geq E(|X-\mu|) + 2\sum_{x=c}^{\mu} (x-c)\, p(x)$$

$$\therefore E(|X-c|) \geq E(|X-\mu|)$$

Thus, $E(|X-c|) \geq E(|X-\mu|)$

3) X: Number of right answers.

Then X can take values 0, 1 and 2. ~~because there is either~~ ~~zero~~ ~~denotes~~

$$P(X=0) = \frac{2}{3} \times \frac{4}{5} \qquad P(X=1) = \frac{2}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{4}{5} = \frac{6}{15}$$

$$P(X=2) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

$$\therefore E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$$

$$= 1 \cdot \frac{6}{15} + 2 \cdot \frac{1}{15} = \frac{8}{15}$$

$$E(X^2) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 4 \cdot P(X=2)$$

$$= 1 \cdot \frac{6}{15} + 4 \cdot \frac{1}{15} = \frac{10}{15}$$

$$\therefore V(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{10}{15} - \frac{64}{225}$$

$$= \frac{86}{225}$$

4) $P(X=-2)=\frac{1}{4}$ $\quad P(X=0)=\frac{1}{4}$, $P(X=1)=\frac{1}{3}$, $P(X=2)=\frac{1}{6}$

C.D.F of X.

$$F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{4} & -2 \le x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{5}{6} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Now, $P(0) \geq \frac{1}{2}$

and $P(X \geq 0) = 1 - P(X<0) = 1 - F(0) + P(X=0)$

$$= 1 + \frac{1}{4} - \frac{1}{2}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \geq \frac{1}{2}$$

$\therefore$ 0 is a median.

x is a quantile of order 0.2 if $P(X \leq x) \geq 0.2$

$P(X \geq x) \geq 0.8$

$\therefore x = -2.$

5)

X : Number of tosses required ~~before~~ for first head

$R_x = \{ 1, 2, \quad \}$

$P(X=1) = p$ where p is the probability for getting a head.

$P(X=2) = qp$ where $q = 1-p$

$P(X=3) = q^2 p$

$P(X=k) = q^{k-1} p$

$$E(X) = \sum_{k=-\infty}^{\infty} k . P(X=k) = \sum_{k=1}^{\infty} k . P(X=k) = \sum_{k=1}^{\infty} k . q^{k-1} p$$

$$= p \sum_{k=1}^{\infty} k . q^{k-1}$$

$$= p \sum_{k=1}^{\infty} \frac{d}{dq} (q^k)$$

$$= p . \frac{d}{dq} \left[\sum_{k=1}^{\infty} q^k \right]$$

$$= p . \frac{d}{dq} \left[\frac{1}{1-q} \right]$$

$$= p . \frac{1}{(1-q)^2} \quad \left[\because q < 1 \Rightarrow \sum_{k=1}^{\infty} q^k = \frac{1}{1-q} \right]$$

$$= \frac{1}{(1-q)}$$

$$= \frac{1}{p}$$

6)

X : Number of trials required to open the door.

R_x

(i) $R_x = \{1, 2, \quad\quad \}$

$P(x=1) = \frac{1}{n}$

$P(x=2) = \frac{(n-1)}{n} \cdot \frac{1}{n}$

$P(x=k) = \left\{\frac{(n-1)}{n}\right\}^{k-1} \cdot \frac{1}{n}$

$$E(x) = \sum_{k=1}^{\infty} k \cdot P(x=k)$$

$$= \sum_{k=1}^{\infty} k \cdot \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} k \cdot \left(\frac{n-1}{n}\right)^{k-1}$$

$$= \frac{1}{n} \cdot \frac{1}{\left(1-\frac{n-1}{n}\right)^2}$$

$$= n.$$

(ii)

$$E(x^2) = \sum_{k=1}^{\infty} k^2 \cdot P(x=k)$$

$$= \sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1}$$

$$= \frac{1}{n} \cdot \frac{1+\frac{n-1}{n}}{\left(1-\frac{n-1}{n}\right)^3}$$

$$= (2n-1) \cdot n.$$

$$\therefore V(x) = E(x^2) - \{E(x)\}^2 = n \cdot (2n-1) - n^2$$
$$= n \cdot (n-1)$$

To evaluate

$$\sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1}$$

Let $x = \frac{n-1}{n}$

$$\therefore \sum_{k=1}^{\infty} k^2 x^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{d}{dx}(k\, x^k)$$

$$= \frac{d}{dx}\left(\sum_{k=1}^{\infty} k\, x^k\right)$$

$$= \frac{d}{dx}\left(x \cdot \sum_{k=1}^{\infty} k\, x^{k-1}\right)$$

$$= \frac{d}{dx}\left(x \cdot \frac{1}{(1-x)^2}\right)$$

$$= \frac{1+x}{(1-x)^3}$$

(ii) If the keys are eliminated then the key we select in a turn will be excluded. ~~from the~~

$$R_X = \{1, 2, \quad , n\}$$

$$P(X=1) = \frac{1}{n}$$

$$P(X=2) = \frac{(n-1)}{n} \cdot \frac{1}{(n-1)} = \frac{1}{n}$$

$$P(X=3) = \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{1}{(n-2)} = \frac{1}{n}$$

$$P(X=n) = \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{(n-3)}{(n-2)} \cdots \frac{1}{2} \cdot 1 = \frac{1}{n}.$$

$$E(X) = \sum_{k=1}^{n} k \cdot P(X=k)$$

$$= \sum_{k=1}^{n} k \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{n} k$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \left(\frac{n+1}{2}\right)$$

$$E(X^2) = \sum_{k=1}^{n} k^2 P(X=k)$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$V(X) = E(X^2) - \{E(X)\}^2$

$= \frac{(2n+1)(n+1)}{6} - \left(\frac{n+1}{2}\right)^2$

7)

i) $X \sim B(n,p)$

then $E(X) = np$ and $V(X) = npq$

Now $q \leq 1 \Rightarrow V(X) \leq E(X)$

So, variance can not be greater than mean.

ii) The most likely outcome is corresponding to the mode of X

Since $X \sim B(n,p)$, the mode of $X = \lfloor (n+1)p \rfloor$

$\therefore$ mode of $X = \lfloor (6+1) \times 0.5 \rfloor$

$= [7 \times 0.5]$

$= 3$

(iii) X: number of defective articles.

probability that a article is defective, is $= \frac{10}{100} = 0.1$

$X \sim B(10, 0.1)$

$P(X=2) = \binom{10}{2} \cdot (0.1)^2 \cdot (0.9)^8$

(iv) X: Number of defective articles in the sample

$X \sim B(20, p)$

~~For~~ So, pmf of $X = \binom{20}{x} p^x \cdot (1-p)^{20-x}$

For $p = 0.25$, $P(X=10) = \binom{20}{10} (0.25)^{10} (1-0.25)^{10}$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \binom{20}{0} \cdot (0.25)^0 \cdot (0.75)^{20} - \binom{20}{1} \cdot (0.25)^1 \cdot (0.75)^{19}$

For poisson approximation, $\lambda = 20 \times 0.25 = 5$

$\therefore P(X=10) = e^{-5} \cdot \frac{5^{10}}{10!}$

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-5} \cdot \frac{5^0}{0!} - e^{-5} \cdot \frac{5^1}{1!}$

(v) X : Number of active components

$X \sim B(5, 0.95)$

$P(X \geq 4) = P(X=4) + P(X=5)$

$= \binom{5}{4}(0.95)^4 \cdot 0.05 + \binom{5}{5} \cdot (0.95)^5 \cdot (0.05)^0$

(vi) Probability that the ship will arrive safely is $= p = \frac{8}{9}$

X : Number of ships arrive safely.

$X \sim B(6, \frac{8}{9})$

$P(X=3) = \binom{6}{3} \cdot \left(\frac{8}{9}\right)^3 \cdot \left(\frac{1}{9}\right)^3$

(vii) Probability that the vessel will arive safely is $= p = \frac{97}{100}$
$= 0.97$

$X \sim B(10, 0.97)$

$\therefore P(X=6) = \binom{10}{6} \cdot (0.97)^6 \cdot (0.03)^4$

$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= \binom{10}{6} (0.97)^6 (0.03)^4 + \binom{10}{7} \cdot (0.97)^7 (0.03)^3 + \binom{10}{8} \cdot (0.97)^8 (0.03)^2$

$+ \binom{10}{9} (0.97)^9 (0.03)$

$+ \binom{10}{10} (0.97)^{10}$

(viii) $X \sim P(5)$

$\therefore$ pmf of x = $p(x) = \dfrac{e^{-5} \cdot 5^x}{x!}$

Now $P(X \geq 1 \mid X \leq 1) = \dfrac{P(X=1)}{P(X \leq 1)}$

$= \dfrac{e^{-5} \cdot \frac{5^1}{1!}}{e^{-5} \frac{5^0}{0!} + e^{-5} \frac{5^1}{1!}}$

$= \dfrac{5}{6}$

8) Probability that a candidate will pass $= \dfrac{60}{100} = 0.6$

Probability that a candidate passed the examination

X: Number of candidates passed the examination

$X \sim B(6, 0.6)$

~~P(X)~~

$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$

$= \binom{6}{4} \cdot (0.6)^4 \cdot (0.4)^2 + \binom{6}{5} \cdot (0.6)^5 \cdot (0.4) + \binom{6}{6} \cdot (0.6)^6$

9) X : Number of correct guesses.

$X \sim B(10, 0.5)$. The probability that a guess is correct is $= 0.5$

(i) $P(X \geq 5) = 1 - P(X < 5)$

$= 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3) - P(X=4)$

$= 1 - \sum_{K=0}^{4} \binom{10}{K} \cdot (0.5)^K (0.5)^{10-K}$

(ii) $P(X=9) = \binom{10}{9} \cdot (0.5)^9 \cdot (0.5)$

(iii) $P(X \geq n) < \frac{1}{2}$

$\Rightarrow 1 - P(X < n) < \frac{1}{2}$

$\Rightarrow P(X < n) > \frac{1}{2}$

For $n = 6$, we have $P(X < 6) = \sum_{K=0}^{5} P(X = K) > \frac{1}{2}$

$\therefore n = 6$ is the smallest.

Probability that a product is defective is $\frac{10}{100} = 0.1$

10) Probability that a product is defective bulbs in the sample

X : Number of defective bulbs in the sample

$X \sim B(10, 0.1)$

$P(X=3) = \binom{10}{3} (0.1)^3 (0.9)^7$

Now $\lambda = np = 10 \cdot 0.1 = 1$

$P(X=3) = \frac{e^{-1} \cdot 1^3}{3!}$

11) Probability of getting a Tv set 0.5

X: Number of requests for TV set

~~Pr~~ $X \sim B(5, 0.5)$

(i) $P(X \geq 4) = P(X=4) + P(X=5)$

$$= \binom{5}{4} (0.5)^4 \cdot (0.5) + \binom{5}{5} \cdot (0.5)^5 = \frac{3}{16}$$

(ii) $P(X \leq 3) = 1 - P(X > 3)$

$$= 1 - \frac{3}{16}$$

$$= \frac{13}{16}$$

(iii) $3C = R \cdot \left[1 \; P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) \right]$

$$= 1 \cdot R \cdot \binom{5}{1} (0.5)^5 + 2 \cdot R \cdot \binom{5}{2} \cdot (0.5)^5 + 3 \cdot R \cdot \binom{5}{3} \cdot (0.5)^5$$

$$R = \frac{96C}{73}$$

12) To evaluate K-th central moment, we ~~f~~ will discuss a new tool to determine the K-th moment of a random variable. The tool is called Moment Generating Function (MGF).

MGF of a random variable X is defined as

$M_X(t) = E(e^{tX})$

provided the expectation exists for some t satisfying $|t| < h$, $h > 0$.

Now, $M_X(t) = \sum_x e^{tx} \cdot p(x)$ — (1)

$$\frac{d}{dt}(M_X(t)) = \sum_x x \cdot e^{tx} \, p(x)$$

$$\frac{d}{dt}(M_x(t))\Big|_{t=0} = \sum_x x \cdot p(x)$$
$$= E(X)$$

From (1)

$$\frac{d^2}{dt^2}(M_x(t)) = \sum_x x^2 e^{tx} p(x)$$

$$\frac{d^2}{dt^2}(M_x(t))\Big|_{t=0} = \sum_x x^2 p(x)$$
$$= E(X^2)$$

Thus we can determine the k-th moment by differentiating MGF k times.

$$\frac{d^k}{dt^k}(M_x(t)) = E(X^k)$$

Now, $$E((X-E(X))^k) = \sum_{i=0}^{k} \binom{k}{i} E(X^i) \cdot \{E(X)\}^{k-i}$$
$$= \sum_{i=0}^{k} \binom{k}{i} \cdot \frac{d^i}{dt^i}(M_x(t))\Big|_{t=0} \{E(X)\}^{k-i} \quad (2)$$

For binomial distribution,
$$M_x(t) = \sum_x e^{xt} \binom{n}{x} \cdot p^x q^{n-x}$$
$$= (pe^t + q)^n$$

Using this MGF and (2), we can determine the k-th central moment.

Similarly for Poisson we can determine the k-th central moment.

13) $X \sim B(4, p)$

$P(X=1) = \frac{2}{3}$

$\Rightarrow \binom{4}{1} p (1-p)^3 = \frac{2}{3}$ — ①

$P(X=2) = \frac{1}{3}$

$\binom{4}{2} p^2 (1-p)^2 = \frac{1}{3}$ — ②

② ÷ ① gives,

$$\frac{6 p^2 (1-p)^2}{4p (1-p)^3} = \frac{1}{2}$$

$\Rightarrow 3p = (1-p)$

$\Rightarrow p = \frac{1}{4}$

$E(X) = 4 \times \frac{1}{4} = 1$

$V(X) = 4 \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{4}$

14) X : Number of heads appeared in five tosses.

Probability that the head will appear $= p$

" " tail " " $= \frac{p}{3}$

$p + \frac{p}{3} = 1 \Rightarrow p = \frac{3}{4}$

$X \sim B\left(5, \frac{3}{4}\right)$

(i) $P(X \geq 3) = 1 - P(X<3)$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \binom{5}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 - \binom{5}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 - \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2$

(ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \binom{5}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4 + \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3$

$+ \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

(iii) $P(X=3) = \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

15) Probability of success = P(getting 4) + P(getting 5)

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

X: Number of success in 9 throws.

$X \sim B(9, \frac{1}{3})$

(i) $E(X) = 9 \cdot \frac{1}{3} = 3$

$V(X) = 9 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right) = 2$

(ii) $P(X=2) = \binom{9}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^7$

(iii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{9}{0} \cdot \left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + \binom{9}{1} \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^8 + \binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7$

(iv) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \binom{9}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 - \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8$

16) Probability that the blade is defective = 0.01

X: Number of defective blades in packet of 10

$X \sim B(10, 0.01)$

(i) $P(X=0) = \binom{10}{0} \cdot (0.01)^0 (0.99)^{10}$

The ~~probable~~ consignment of 1000 packets has approximately

$= 1000 * P(X=0)$ number of packets containing no defective blade

(ii) $P(X=1) = \binom{10}{1} \cdot (0.01)^1 (0.99)^9$

$\therefore$ Number of packets containing one defective blade $= 1000 \times P(X=1)$

(iii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{10}{0} (0.01)^0 (0.99)^{10} + \binom{10}{1} \cdot (0.01)^1 (0.99)^9 + \binom{10}{2} (0.01)^2 (0.99)^8$

$\therefore$ Number of packets containing at most two defective blades $= 1000 \times P(X \leq 2)$

(iv) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$\therefore$ Number of packets containing at ~~most~~ least two defective blades $= 1000 \times P(X \geq 2)$

17) X: Number of trials before first target is shot

$X \sim Geo(0.8)$

$\therefore$ p.m.f of $X = P(X=k) = q^{k-1}p$, $p = 0.8$, $q = 0.2$

$\therefore P(X = \text{odd}) = P(X=1) + P(X=3) + P(X=5) + \cdots$

$= p + q^2 p + q^4 p + \cdots$

$= p(1 + q^2 + q^4 + \cdots)$

$= p \cdot \frac{1}{1-q^2}$

$= \frac{1}{(1+q)}$

$= \frac{1}{(2-p)}$

$P(X = \text{even}) = 1 - P(X = \text{odd})$

$= 1 - \frac{1}{2-p}$

$= \frac{2-p-1}{2-p}$

$= \frac{1-p}{2-p}$

18) Probability that a product is defective $= \frac{3}{100} = 0.03$

X: Number of components to be examined to get 3 defectives

$X \sim NB(3, 0.03)$

$\therefore P(X=x) = \binom{x-1}{2} \cdot (0.03)^2 (0.97)^{x-3} \cdot (0.03)$

Now, $P(X \geq 6) = 1 - P(X < 6)$

$= 1 - P(X=3) - P(X=4) - P(X=5)$

$$P(X \geq 6) = 1 - \binom{2}{2}(0.03)^3 - \binom{3}{2}(0.03)^3(0.97) - \binom{4}{2}(0.03)^3(0.97)^2$$

19) X: Number of shots for fourth hit ~~the target~~

Probability of hitting the target = 0.7

$X \sim NB(4, 0.7)$

$$P(X=K) = \binom{K-1}{3}(0.7)^3 \cdot (0.3)^{K-4} \cdot (0.7)$$

$$\therefore P(X=7) = \binom{6}{3} \cdot (0.7)^4 (0.3)^3$$

20) X: Number of defective in the sample

Probability of a item is defective = $\frac{10}{100} = 0.1$

$\therefore X \sim B(10, 0.1)$

$\therefore$ ~~p(x=0)~~ The machine will not stop when there is no defective product in sample.

$$\therefore P(X=0) = \binom{10}{0} \cdot (0.1)^0 \cdot (0.9)^{10}$$

21) Probability of a person getting into the accident $= \frac{1}{1000}$

Number of people insured $= 5000$

Using Poisson Approximation, we have,

$$\lambda = np = 5000 \times \frac{1}{1000} = 5$$

X: Number of people getting into the accident

$X \sim P(5)$

$\therefore P(X = x) = e^{-5} \cdot \frac{5^x}{x!}$

$\therefore P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= e^{-5} \cdot \frac{5^0}{0!} + e^{-5} \cdot \frac{5^1}{1!} + e^{-5} \cdot \frac{5^2}{2!}$$

22) Probability of the person making reservation on flight doesn't show up is $= \frac{5}{100} = 0.05$

$\therefore$ Probability of the person making reservation on flight show up is $= 1 - 0.05 = 0.95$

X: Number of people show up for the flight.

$X \sim B(100, 0.95)$ $\therefore P(X = x) = \binom{100}{x} \cdot (0.95)^x \cdot (0.05)^{100-x}$

~~P(X~~ Everyone who shows up for flight will ~~show up~~ get a seat if number of people show up for flight is less ~~of~~ or equal to 95.

$\therefore P(X \leq 95) = 1 - P(X > 95)$

$$= 1 - P(X=96) - P(X=97) - P(X=98) - P(X=99) - P(X=100)$$

23) Probability of getting double six by rolling a pair of die

$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

X: Number of times double six occur. in 50.

The pair of die is rolled 50 times.

$$\therefore X \sim B\left(50, \frac{1}{36}\right)$$

$$\therefore P(X=x) = \binom{50}{x} \cdot \left(\frac{1}{36}\right)^{x} \left(\frac{35}{36}\right)^{50-x}$$

Probability for getting a double six at least three times

$$= P(X \geq 3) = 1 - P(X<3)$$

$$= 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - \binom{30}{0}\left(\frac{1}{36}\right)^{0}\left(\frac{35}{36}\right)^{50} - \binom{30}{1}\left(\frac{1}{36}\right)^{1}\left(\frac{35}{36}\right)^{49} - \binom{30}{2}\left(\frac{1}{36}\right)^{2}\left(\frac{35}{36}\right)^{48}$$

24) X: Number of accidents occurring on a highway each day.

Given that $X \sim P(3)$

$$\Rightarrow P(X=x) = e^{-3} \cdot \frac{3^{x}}{x!}$$

(i) Probability of three or more accident occur.

today $= P(X \geq 3) = 1 - P(X<3)$

$$= 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-3}\frac{3^{0}}{0!} - e^{-3} \cdot \frac{3^{1}}{1!} - e^{-3} \cdot \frac{3^{2}}{2!}$$

(ii) Given that at least one accident had already occurred the probability of three or more accident is

$$= P(X \geq 3 \mid X \geq 1) = \frac{P(X \geq 3)}{P(X \geq 1)}$$

determine $P(X \geq 1)$ similarly

25) Probability of a engine will fail $= 1-p$

$\therefore$ Probability of a engine will operate $= p$

X: Number of engines are operative for four engine plane

Y: Number of engines are operative for two engine plane.

$X \sim B(4, p)$

$Y \sim B(2, p)$

Now for a successful flight at least 50% of the engines remain operative

$\therefore$ For a ~~4~~ four engine plane we need 2 ~~engine~~ at least 2 engine remain operative and for two engine plane we need at least 1 engine remain operative

$\therefore P(X \geq 2)$ is the probability of at least two engines are operative for four engine plane.

$P(Y \geq 1)$ ... ... — one engine is operative for two engine plane

Therefore $P(X \geq 2) > P(Y \geq 1)$

$\Rightarrow 1 - P(X<2) > 1 - P(Y<1)$

$\Rightarrow 1 - P(X=0) - P(X=1) > 1 - P(Y=0)$

$\Rightarrow \binom{4}{0} \cdot p^0 \cdot (1-p)^4 + \binom{4}{1} p \cdot (1-p)^3 < \binom{2}{0} \cdot p^0 \cdot (1-p)^2$ [$\because (1-p) \neq 0$]

$\Rightarrow (1-p)^2 + 4 \cdot p \cdot (1-p) < 1$

$\Rightarrow 1 - 2p + p^2 + 4p - 4p^2 < 1$

$\Rightarrow \quad 2p - 3p^2 \leq 0$

$\Rightarrow \quad p \geq \frac{2}{3} \quad [\because p \neq 0]$

26) Probability of a unit is defective $= \frac{5}{100} = 0.05$

X: Number of defective units in the sample of 15 units

$X \sim B(15, 0.05)$

$\therefore P(X = x) = \binom{15}{x} (0.05)^x (0.95)^{15-x}$

$\therefore$ Probability of 5 items defective $= \binom{15}{5} \cdot (0.05)^5 (0.95)^{10}$

27) Probability of a diode failure is 0.03.

X: Number of diode failure in the circuit.

$\therefore X \sim B(200, 0.03)$

$\therefore$ Mean number of failures among the diode $= 200 \times 0.03$
$= 6$

Variance $= 200 \times (0.03) \times (1 - 0.03)$

$= 200 \times 0.03 \times 0.97$

$= 5.82$

The probability of that the board will work

$= P(X = 0) = \binom{200}{0} \cdot (0.03)^0 \cdot (0.97)^{200}$

28) $X \sim Geo(p)$

$\therefore P(X=k) = (1-p)^{k-1} \cdot p$

$= q^{k-1} p \qquad (q = 1-p)$

$P(X = \text{even}) = P(X=2) + P(X=4) + P(X=6) + \cdots$

$= qp + q^3 p + q^5 p + \cdots$

$= p\left[q + q^3 + q^5 + \cdots\right]$

$= p \cdot \dfrac{q}{(1-q^2)}$

$= p \cdot \dfrac{q}{p(1+q)}$

$= \dfrac{q}{1+q}$

$P(X = \text{odd}) = 1 - P(X = \text{even})$

$= 1 - \dfrac{q}{1+q}$

$= \dfrac{1}{1+q}$

29) The probability of ~~the soldier~~ a shot hit the target is $= 0.7$

X: Number of shots reqiured for the first hit

~~Then Say~~ $X \sim Geo(0.7)$

$\therefore P(X=n) = (1-0.7)^{n-1} \cdot (0.7)$

$= (0.3)^{n-1} (0.7)$

(i) $P(X=10) = (0.3)^9 \cdot (0.7)$

(ii) ~~P(~~ The Probability of the target would be hit ~~on~~ in less than 4 shots $= P(X<4)$

$= P(X=1) + P(X=2) + P(X=3)$ ~~+ P(X=4)~~

$= (0.7) + (0.3).(0.7) + (0.3)^2(0.7)$ ~~+ a~~

$= (0.7) \times [1 + (0.3) + (0.3)^2]$

$= (0.7) \times \dfrac{1-(0.3)^3}{1-(0.3)}$

$= 1-(0.3)^3$

(iii) Probability that the target would be hit in an even number of shots $= P(X = \text{even})$

$= P(X=2) + P(X=4) + \cdots$

$= \dfrac{0.3}{1+0.3}$ [Using problem no 28] $q = 0.3$

The average number of shots needed to hit the target is $= E(X)$.

$= \sum_{n\geq 1}^{\infty} n \cdot q^{n-1} p$

$= p \sum_{n\geq 1}^{\infty} n \cdot q^{n-1}$

$= p \cdot \dfrac{1}{(1-q)^2}$ $\left[\because \sum_{n\geq 1}^{\infty} n\, q^{n-1} = \dfrac{1}{(1-q)^2} \text{ (see prob. no. 6 or 5)}\right]$

$= \dfrac{1}{p} = \dfrac{1}{0.7}$

30) Same as 12.

31) Probability of a person will believe a rumor = 0.10

X: Number of person used to heard the rumor ~~before~~ only and the last one believe it.

$X \sim Geo(0.10)$

$\therefore P(X=x) = q^{x-1} p \quad$ where $p = 0.10$
$q = 1-p = 0.90$

$\therefore$ The probability that the sixth person to hear the rumor is the first one to believe

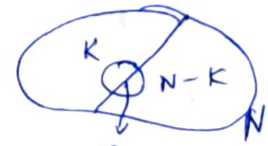
$= \cancel{P(X=6)}\ P(X=6)$

$= (0.90)^5 . (0.10)$

32) The random variable for this problem follows hypergeometric distribution.

Hypergeometric distribution: Given ~~a~~ N number of items, there are special type of K objects such that if any of the K object is choosen is ~~a~~ success. Chooseing any of the remaining (N−K) element is a failure. Let a sample of size n is taken from the items.

X: Number of succes in the sample of size n.



$R_X = \{0, 1, 2, \ldots, n\}$

$P(X=x) = \dfrac{\binom{K}{x} . \binom{N-K}{n-x}}{\binom{N}{n}}$

$X \sim HG(N, K, n)$

In this problem

X: Number of narcotic tablets are selected in 3 samples

$X \sim HG(15, 6, 3)$

Probability that the person will be caught $= P(X=1)$
$+ P(X=2) + P(X=3)$

$$= \frac{\binom{6}{1} \cdot \binom{9}{2}}{\binom{15}{3}} + \frac{\binom{6}{2} \cdot \binom{9}{1}}{\binom{15}{3}}$$
$$+ \frac{\binom{6}{3} \binom{9}{0}}{\binom{15}{3}}$$

33) Probability that the salesman will make his sale to a family $= \frac{1}{10}$

X: Number of ~~family~~ families the salesman contacted till the first sale

$X \sim Geo\left(\frac{1}{10}\right)$

$\therefore P(X=x) = \left(\frac{9}{10}\right)^{x-1} \left(\frac{1}{10}\right)$

(i) Probability that he will make his first sale to the fourth family $= P(X=4)$

$$= \left(\frac{9}{10}\right)^{3} \cdot \left(\frac{1}{10}\right)$$

(ii) ~~P~~ If he is still waiting to make his first sale after visited 10 families then all the 10 ~~families~~ ~~a~~ attempts are failure. $\therefore$ P(failure after calling 10 families) = $\left(\frac{9}{10}\right)^{10}$

34) X: Number of defective bulbs in the sample.

Probability that a ~~bad~~ bulb is defective = $\frac{300}{10000}$

$= \frac{3}{100}$

$X \sim B\left(30, \frac{3}{100}\right)$

$\therefore P(X=x) = \binom{30}{x} \cdot \left(\frac{3}{100}\right)^{x} \cdot \left(\frac{97}{100}\right)^{30-x}$

Now, probability that at least one defective bulb in sample

$= P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - \binom{30}{0}\left(\frac{3}{100}\right)^{0}\left(\frac{97}{100}\right)^{30}$

35) Probability of contacting the disease = $p = \frac{1}{6}$

X: Number of mice are ~~ax~~ inoculated ~~bef~~ untill 2 mice have contacted the disease

$X \sim NB\left(2, \frac{1}{6}\right)$

~~then p~~

$$\therefore P(X=x) = \binom{x-1}{1} p \cdot (1-p)^{x-2} \cdot p$$

$$= \binom{x-1}{1} p^2 (1-p)^{x-2}$$

$\therefore$ Probability that 8 mice are required $= P(X=8)$

$$= \binom{7}{1} \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

[putting $x=8$ and $p=\frac{1}{6}$]

Q 6) The probability that all the coins are same

$= P($All three are head$) + P($All three are tail$)$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{4}$$

$\therefore$ Probability that a odd one occur $= 1 - \frac{1}{4} = \frac{3}{4}$

X: Number of tosses till the ~~fair~~ odd one occurs

$X \sim Geo\left(\frac{3}{4}\right)$

$$\therefore P(X=x) = \left(\frac{1}{4}\right)^{x-1} \left(\frac{3}{4}\right)$$

$\therefore$ Probability that fewer than 4 tosses are needed

$= P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$$= \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)$$

$$= \left(\frac{3}{4}\right) \left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 \right]$$

$$= \left(\frac{3}{4}\right) \times \frac{\left(1 - \left(\frac{1}{4}\right)^3\right)}{\left(1 - \frac{1}{4}\right)}$$

$$= 1 - \left(\frac{1}{4}\right)^3$$

37) Probability of refusal $= \frac{20}{100} = 0.2$

X: Number of people interviewed till ~~the~~ the first refusal

$\therefore X \sim Geo(0.2)$

$$\therefore P(X = x) = (1 - 0.2)^{x-1} (0.2)$$
$$= (0.8)^{x-1} (0.2)$$

Probability that 50 people were interviewed before first refusal $= P(X = 51) = (0.8)^{50} \cdot (0.2)$

(i) The probability of occurrance of this event is very low.

(ii) Expected number of people interviewed before first refusal $= E(X)$

$$= \frac{1}{0.2}$$
$$= 5$$

[See problem number 29 for the expectation]

38) Probability that a product is defective $= \frac{5}{100} = 0.05$

X: Number of items are to be examined in order to get 2 defective.

$\therefore X \sim NB(2, 0.05)$

$$P(X=x) = \binom{x-1}{1}(0.05)^1 \cdot (0.95)^{x-2} (0.05)$$

$$= \binom{x-1}{1}(0.05)^2 (0.95)^{x-2}$$

The probability that at least 4 items are to be examined

$$= P(X \geq 4)$$

$$= 1 - P(X<4)$$

$$= 1 - P(X=2) - P(X=3)$$

$$= 1 - \binom{1}{1}(0.05)^2 (0.95)^0 - \binom{2}{1}(0.05)^2 (0.95)$$

39) Probability that a lot is defective $= 1 - 0.9$
$= 0.1$

X: Number of lots need to produce to obtain 3 defective lot

$X \sim NB(3, 0.1)$

$$\therefore P(X=x) = \binom{x-1}{2}(0.1)^2 \cdot (0.9)^{x-3} \cdot (0.1)$$

$$= \binom{x-1}{2}(0.1)^3 (0.9)^{x-3}$$

Probability that 20 lots will be produced in order to obtain

3rd defective bolt = $P(X=20)$

$$= \binom{19}{2} (0.1)^3 (0.9)^{17}$$

$$E(X) = \sum_{x=3}^{\infty} x \cdot \binom{x-1}{2} (0.1)^3 (0.9)^{x-3}$$

$$= (0.1)^3 \sum_{x=3}^{\infty} x \cdot \binom{x-1}{2} (0.9)^{x-3}$$

$$= (0.1)^3 \sum_{x=3}^{\infty} \frac{x!}{2!\,(x-3)!} (0.9)^{x-3}$$

$$= (0.1)^3 \cdot 3 \cdot \sum_{x=3}^{\infty} \frac{x!}{3!\,(x-3)!} (0.9)^{x-3}$$

$$= 3 \cdot (0.1)^3 \sum_{x=3}^{\infty} \binom{x}{3} (0.9)^{x-3}$$

$$\boxed{\because \sum_{t=0}^{\infty} \binom{t+r-1}{r-1} a^t = \frac{1}{(1-a)^r}}$$

$$= 3 \cdot (0.1)^3 \left[\frac{1}{(1-(0.9))^4} \right]$$

$$= \frac{3}{(0.1)} = 30$$

~~Sofor~~

$$V(x) = \frac{3 \times 0.9}{(0.1)^2} \quad \left[\text{variance of negative binomial } NB(r, p) = \frac{rq}{p^2} \right]$$

$$= 270$$

40) ~~Numbe~~. Total number of items = 20.

X: Number of defective items in the sample of six items.

$\therefore X \sim HG(20, 5, 6)$

$$P(X=x) = \frac{\binom{5}{x}\binom{15}{6-x}}{\binom{20}{6}}$$

Probability that the shipment will be accepted ~~if~~

$$= P(X \leq 1)$$

$$= P(X=0) + P(X=1)$$

$$= \frac{\binom{5}{0}\binom{15}{6}}{\binom{20}{6}} + \frac{\binom{5}{1}\binom{15}{5}}{\binom{20}{6}}$$

41) Probability of success at a location = 0.25

(i) Probability that driller drills 10 locations and find one success $= \binom{10}{1} \cdot (0.25)(0.75)^9$

(ii) Probability that the driller go bankrupt $= (0.75)^{10} \cdot (0.25)$

42) X: Number of defective batteries in the lot.

$X \sim B(75, 0.001)$

~~P(X=~~

(i) Probability that the lot is accepted $= P(X=0) = \binom{75}{0}(0.001)^0 . (0.999)^{75}$

(ii) Probability that the lot is rejected on 20th test $= (0.999)^{19} \cdot (0.001)$

(iii) ~~Probability that the lot is rejected in 10 or less trials~~

~~= Probability that the lot is~~ .

Y: Number of tests till the first failure.

$Y \sim Geo(0.001)$

$P(Y=x) = (0.999)^{x-1}(0.001)$

Probability that the lot is rejected in 10 or less trials

$= P(X \le 10)$

$= P(X=1) + P(X=2) + P(X=3) + \dots + P(X=10)$

43) X: Number of defective in the ~~lot~~ sample of 6 from the 40% lot.

Y: Number of defective in the sample of 6 from 60% lot.

$X \sim HG(20, 5, 6)$

$Y \sim HG(20, 2, 6)$

$$P(X=x) = \frac{\binom{5}{x}\binom{15}{6-x}}{\binom{20}{6}}$$

$$P(Y=y) = \frac{\binom{2}{y}\binom{18}{6-y}}{\binom{20}{6}}$$

Probability that the first lot is accepted $= P(X=0)$

$$= \frac{\binom{5}{0}\binom{15}{6}}{\binom{20}{6}}$$

~~P~~ Probability that the second lot is accepted $= P(Y=0)$

$$= \frac{\binom{2}{0}\binom{18}{6}}{\binom{20}{6}}$$

$\therefore$ Probability that the lot is accepted $= P(X=0) + P(Y=0)$

$$= \frac{\binom{15}{6} + \binom{18}{6}}{\binom{20}{6}}$$

$\therefore$ Probability that the lot is rejected $= 1 - \frac{\binom{15}{6} + \binom{18}{6}}{\binom{20}{6}}$

44) X: Number of births in a family untill the second daughter is born.

Probability of male child $= 0.5$

$\therefore$ probability of female ~~girl~~ child $= 0.5$

$X \sim NB(2, 0.5)$

$$P(X=x) = \binom{x-1}{1}(0.5)(0.5)^{x-2}\cdot(0.5)$$

$$= \binom{x-1}{1}(0.5)^x$$

$\therefore$ Probability that the sixth child in the family is the second daughter $= P(X=6) = \binom{5}{1}(0.5)^6$

$$= 5 \times (0.5)^6$$

45) Probability that the item is defective $= \frac{3}{10}$

Probability that the item is non defective $= \frac{7}{10}$

X: Number of defective items in the sample of 10 items.

$$X \sim B\left(10, \frac{3}{10}\right)$$

$$P(X=x) = \binom{10}{x}\left(\frac{3}{10}\right)^{x}\left(\frac{7}{10}\right)^{10-x}$$

(i) Probability that not more than one defective will be obtained $= P(X=0) + P(X=1)$

$$= \binom{10}{0}\left(\frac{3}{10}\right)^{0}\left(\frac{7}{10}\right)^{10-0} + \binom{10}{1}\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)^{9}$$

(ii) Poisson approximation. $\lambda = np = 10 \times \frac{3}{10} = 3$

$\therefore X \sim P(3)$

$\therefore P(X=x) = e^{-3}\frac{3^{x}}{x!}$