Multidimensional RV

For a set

PMF $P_{X_1, X_2, \ldots X_N}(x_{k_1}, x_{k_2}, \ldots x_{k_N}) = P(X_1 = x_{k_1}, X_2 = x_{k_2}, \ldots X_N = x_{k_N})$ ← discrete

CDF $F_{X_1, X_2 \ldots X_N}(x_1, x_2 \ldots x_N) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_N \leq x_N)$

PDF $$f_{X_1, X_2 \ldots X_N}(x_1, x_2, \ldots x_N) = \frac{\partial^N}{\partial x_1 \partial x_2 \ldots \partial x_N} F_{X_1, X_2 \ldots X_N}(x_1, x_2 \ldots x_N)$$

$f_x = \int f_{xy} dy$

Above Notation seems cumbersome for large N

Introduce : let $X = [X_1, X_2, \ldots X_N]^T$ be a col. vector with N RV & similarly $x = [x_1, x_2, \ldots x_N]^T$. Then PMF, CDF & PDF are expressed as $P_X(x), F_X(x)$ & $f_X(x)$

- Marginal PMF, CDF, PDF

$$P_{X_1, X_2, \ldots X_M}(x_{k_1}, x_{k_2} \ldots x_{k_M}) = \sum_{k_{M+1}} \sum_{k_{M+2}} \ldots \sum_{k_N} P_{X_1, X_2, \ldots X_N}(x_{k_1}, x_{k_2} \ldots x_{k_N})$$

$$\underset{\infty\, 4}{F_{X_1, X_2 \ldots X_M}}(x_1, x_2 \ldots x_M) = \underset{10}{F_{X_1, X_2 \ldots X_N}}(x_1, x_2 \ldots x_M, \infty, \infty, \ldots \infty)$$

$$f_{X_1, X_2, \ldots X_M}(x_1, x_2 \ldots x_M) = \int\int_{-\infty}^{\infty} \ldots \int f_{X_1, X_2 \ldots X_N}(x_1, x_2 \ldots x_N)\, dx_{M+1}\, dx_{M+2} \ldots dx_N$$

Conditional PMF & PDF

$$\underset{1-4}{P_{X_1 \ldots X_M}} / \underset{5-10}{X_{M+1} \ldots X_N}(x_{k_1}, \ldots x_{k_N} / x_{k_{M+1}} \ldots x_{k_N}) = \frac{P(X_1 = x_{k_1}, X_2 \ldots X_N = x_{k_N})}{P(X_{M+1} = x_{k_{M+1}}, X_{M+2} = \ldots)}$$

Expectation involving multiple RV

For a R Vector $X = [X_1 \text{ — } X_N]^T$ correlation matrix is defined as $R_{XX} = E[XX^T]$

i.e., the $(i,j)^{th}$ element of $N \times N$ matrix R_{XX} is $E[X_i X_j]$

Similarly, the covariance matrix C_{XX} is defined as

$$C_{XX} = E\left[(\underbrace{X-\mu}_{\text{vector}})(X-\mu)^T\right]$$

i.e., $(i,j)^{th}$ element of C_{XX} is $Cov(X_i, X_j)$ $\begin{bmatrix} Var(X_1) & cov(X_1,X_2) \\ & Var(X_2) \end{bmatrix}$

$X = \begin{Bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{Bmatrix}$

$E(XX^T) = \begin{bmatrix} E(X_1^2) & E(X_1X_2) \\ E(X_2X_1) & E(X_2^2) \end{bmatrix}$

Proof

Theorem: The matrix R_{XX}, C_{XX} are symmetric & +ve definite

A matrix $Z_{n\times n}$ is (+ve) definite $\Leftrightarrow x^T Z x > 0 \ \forall \ x \in \mathbb{R}^n \setminus \{0\}$

* Gaussian (Normal) RV in multiple dimension

(why only Gaussian not others??)

@ There are certain phenomenon which are evolving with time ⇒ millions of R.V.

we need collection of RV ⇒ Name → R.P.

Stochastic Process (Random Process)

Consider a system that evolves randomly in time. For eg.

(at every time → RV)

1. Stock market index
2. The inventory in a warehouse
3. The queue of customers at service station
4. water level in a reservoir
5. The state of machines

These systems may be observed at discrete time points $n = 0, 1, \ldots$

i.e., every sec, every min, hour, day, week

X_n = the state of system at time n

or X_n = NSE index at end of the n^{th} day

= # of unsold car on a dealer's lot at the starting of n^{th} day

We say that $\{X_n, n \geq 0\}$ is a discrete-time stochastic process describing the system

These systems may be observed continuously in time

$X(t)$ = the state of system at time t

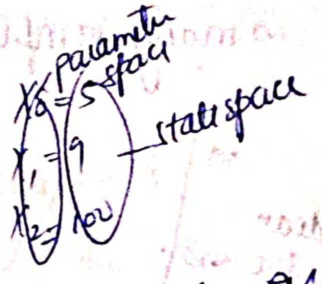
$X(t)$ = no. of failed machines in a machine shop at the time 't'

or = the amount of money in a bank account at time t

⇒ we say $\{X(t), t \geq 0\}$ is a continuous time stochastic process

* A stochastic process is a collection of r.v. $\{X(t), t \in T\}$ indexed by the parameter t taking values in the parameter space T (which is usually subset of $\mathbb{R}$)

* The R.V. take values in the set S, called the state-space of the stochastic process



* Following 4 combinations are possible

1. Discrete-state, discrete-time stochastic process (semi-water)

(a) The no. of individuals in a country at the end of year t – stochastic process

$X_0 = 1.3b$
$X_1 = 1.35b$ (2014)
$X_2 = 1.4b$

1 year RV
Poly → RP

$\{X_t : t \in T\}$ with $T = \{0, 1, \dots\}$ & $S = \{0, 1, 2, \dots\}$

(b) A motor insurance company reviews the status of its customers yearly. Based on this, 3 discounts 5%, 10%, 15% are given depending on the accident record of the drivers.

X_t = % of discount at the end of year t

Then $T = \{0, 1, 2, \dots\}$ & $S = \{5, 10, 15\}$

2. Discrete state continuous time s.p.

(a) The no. of incoming calls X_t in an interval $[0, t]$ Then $T = \{t : 0 \le t \le \infty\}$

$S = \{0, 1, \dots\}$

(b) The no. of cars X_t parked at a commercial center in the time interval $[0, t]$ Again, T, S same

3. continuous-state, ~~discrete time~~ s.p.

(a) The share price of an asset at the close of trading on day t with $T = \{0, 1, \dots\}$

$S = \{x : 0 \le x < \infty\}$

4. continuous - state, cont. time s.p.

The value of NSE index at time t where $T = \{t : 0 \le t < \infty\}$, $S = \{x : 0 \le x < \infty\}$

* Let $\{X(t), t \in T\}$ be a stoch. process with state space S (parameter space T). Let $x : T \to S$ be a fn. We may think that $\{x(t) : t \in T\}$ as a possible evolution. The fn's x are called the sample paths (trajectory) of $\{X(t) : t \in T\}$.

S. P. Ghanmaja, IIT-D; V. G. Kulkarni - USA of the stochastic process.

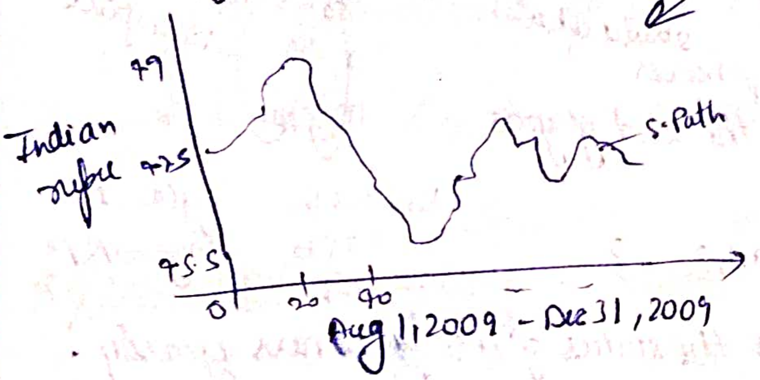
<u>Remark</u>: In general, the set of all possible sample paths, called the sample space of s.p., is ~~con~~ uncountable

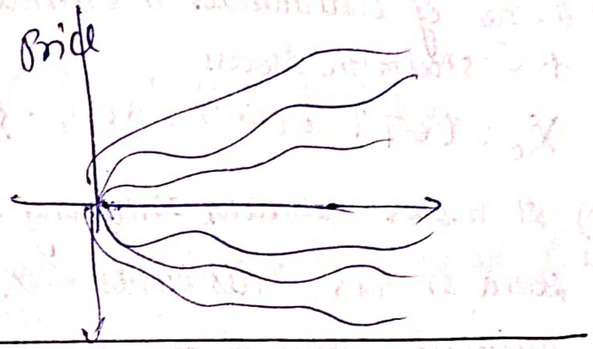
<u>Think</u>: Discrete-time s.p. with finite state space

⊙ ∞ many sample paths are possible

discrete
$\{X_t : t \in T\}$
$\{X(t) ; t \in T\}$
continuous

ignored!!
हर बच्चा स्कूल जा रहा है !! What





<u>some imp. char. of stoi. P.</u>

<u>Independent Increments</u>: If $\forall$ $t_0, t_1, t_2, \ldots t_n$ s.t. $t_0 < t_1 \ldots < t_n$ the R.V. $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots X_{t_n} - X_{t_{n-1}}$ are independent (or equivalently) $X_{t+\tau} - X_\tau$ is independent of X_s for $s < \tau$), then the process $\{X_t : t \in T\}$ is said to be a process with independent increments.

<u>Stationary increments</u>: A SP $\{X_t : t \in T\}$ is said to have stationary increment if $X_{t_2+\tau} - X_{t_1+\tau}$ has the same distribution as $X_{t_2} - X_{t_1}$ $\forall$ choices of t_1, t_2 & $\tau > 0$.

<u>Stationary process</u>: If for arbitrary $t_1, t_2, \ldots t_n$ s.t. $t_1 < t_2 < \ldots < t_n$ the Joint distribution of the vector RV $(X_{t_1}, X_{t_2}, \ldots X_{t_n})$ & $(X_{t_1+h}, X_{t_2+h}, X_{t_3+h}, \ldots X_{t_n+h})$ are the same for all $h > 0$. Then the stocha. P. $\{X_t : t \in T\}$ is said to be stationary SP.

of order n (or simply a stationary process). The SP $\{X_t : t \in T\}$ is said to be strong static SP or strictly stationary process if the above prop is satisfied $\forall n$

Markov Process

Consider a system that is modelled by a discrete time S.P. $\{X_n, n \geq 0\}$ with a countable state-space S, say $S = \{0, 1, \ldots\}$
Consider a fixed value of n that we ~~should~~ shall call 'the ⊕nt time' or just the '⊕nt'. Then X_n is called the ⊕nt (state) of the system, $\{X_0, X_1, X_2, \ldots X_{n-1}\}$ is called past of the system & $\{X_{n+1}, X_{n+2}, \ldots\}$ is called future of system. If $X_n = i$ & $X_{n+1} = j$. It is said that the system has jumped (or made a transition) from state i to state j from time n to n+1

Markov Prop.

If the ⊕nt state of the system is known, the future of the system is independent of the past. Or in other words: the ⊕nt state of the system contains all the relevant inf. needed to predict the future of the system in a probabilistic sense.

Defn: Let $\{X_t : t \geq 0\}$ be a stochastic process defined over a probability space $(\Omega, \mathcal{F}, P)$ & with state space $(\mathbb{R}, \mathcal{B})$ we say that $\{X_t : t \geq 0\}$ is a Markov process if for any $0 \leq t_1 < t_2 \ldots < t_n$ and for any $B \in \mathcal{B}$

$$P(X_{t_n} \in B \mid X_{t_1}, \ldots X_{t_{n-1}}) = P(X_{t_n} \in B \mid X_{t_{n-1}})$$

Remark: Any stochastic Process which has independent increments is a Markov process

Eg: (i) X_n: no. of heads in the first n tosses
(ii) Let $Y_0, Y_1, \ldots, Y_n$ be non(-ve) independent & identically distributed R.V

If Y is non-ve
X_n is ↑ing

The sequence $\{X_n ; n \geq 0\}$ with

$$X_0 = Y_0$$

$$X_n = X_0 + Y_1 + \dots + Y_n, \quad n \geq 1$$ is a Markov process

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0) =$$

$$\frac{P(X_{n+1} = j, X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0)}{P(X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0)}$$

$$= \frac{P(X_0 = i_0)\, P(Y_1 = i_1 - i_0) \dots P(Y_n = i - i_{n-1})\, P(Y_{n+1} = j - i)}{P(X_0 = i_0)\, P(Y_1 = i_1 - i_0) \dots P(Y_n = i - i_{n-1})}$$

$$= P(Y_{n+1} = j - i) = P(X_{n+1} = j \mid X_n = i)$$

$\boxed{X_1 = X_0 + Y_1}$
$X_n = X_{n-1} + Y_n$

- A discrete state Markov Process is known as Markov chain
  (NSE index, service centre)
  Time (Discrete)

$$\frac{X_{n+1} = j, X_n = i}{X_n = i}$$ ✗

$$P(Y_{n+1} = j - i)\, P(X_n = i)$$

Markov Chain

Based on T

Discrete Time Markov Chain (DTMC)

Continuous Time Markov chain (CTMC)

SP = $\{T, S\}$
value

$T = \{0, 1, \dots\}$
countable no of RV

$T = [0, \infty)$
uncountable no of RV
$X(t); t \geq 0$

DTMC

A sequence of RV $\{X_n\}_{n \in \mathbb{N}}$ with discrete state space is called a DTMC if it satisfies the condition

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) \quad \text{--- (1)}$$

$\forall\, n \in \mathbb{N}$ & $\forall\, i_0, i_1, \dots, i_{n-1}, i, j \in S$ with

$$P(X_0 = i_0, \dots, X_n = i_n) > 0$$

In other words, Eq^n (1) ⇒ that if we know the present state "$X_n = i$", the knowledge of past History "$X_{n-1}, X_{n-2}, \dots X_0$" has no influence on the probabilistic structure of the future state X

~~Remark~~ **Def^n: Transition Probability Matrix**

The matrix: $P = (p_{ij}) = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & \cdot & - & \\ & & - & \end{bmatrix}$

is called transition prob. matrix or stochastic matrix s.t.

$$\left.\begin{array}{l} p_{ij} \geq 0 \quad \forall i,j \in S \\ \sum_j p_{ij} = 1 \quad \forall i \in S \end{array}\right\} \text{—— *}$$

- p_{ij} is conditional prob. so $p_{ij} \geq 0$ Now,

$$\sum_{j \in S} p_{ij} = \sum_{j \in S} P(X_{n+1} = j / X_n = i)$$

$$= P(X_{n+1} \in S / X_n = i)$$

$$= 1 \quad \left(\text{since } X_{n+1} \in S \text{ with prob. } 1\right)$$

~~X_{n+1}~~ $S \to$ collection of all values

Eg: On any given day Kuldip is cheerful (C), Normal (N) or depressed (D). He will be C, N or D. If he is cheerful today, then he will be C, N or D tomorrow with p. 0.5, 0.4 & 0.1 resp. If is feeling so-so today then he will C, N, D tomorrow with p. 0.3, 0.4, 0.3 resp. If he is glum today, then he will be C, N, D with p = 0.2, 0.3, 0.5 resp.

Let X_n denote the mood of Kuldip on the n^{th} day. Then $\{X_n : n \geq 0\}$ is a 3 state discrete time Markov chain (state 0 = C, state 1 = N, state 2 = D) with transition prob. matrix

n×n

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

$T = \{0, 1, \ldots\}$
$S = \{C, N, D\}$

Q: Is a DTMC completely characterised by its trans. prob. matrix?

NO ______ . ______ Initial distribution is given ______ . ______

Def^n: Let $\underline{a_i = P(X_0 = i)}$, $i \in S$ & $a = [a_i]_{i \in S}$

particular row is upersum
" time

be a row vector representing the PMF of X_0 we say that a is the initial distribution of the DTMC.

Theorem: A DTMC $\{X_n : n \geq 0\}$ is completely described by its initial distribution a & the transition probability matrix P

Proof: DIY

$i_0, i_1 \ldots i_n$
$S = \{15 \ldots 110\}$

If you know all joint probability of all possible P&C
$\Rightarrow$ you know P

Hint: Show that the finite dimensional joint PMF $P(X_0 = i_0, X_1 = i_1, \ldots X_n = i_n)$ can be computed in terms of a & P

How to characterise a stochastic process

for both we have :

: CDF

If we give probability

This only helps when time is discrete

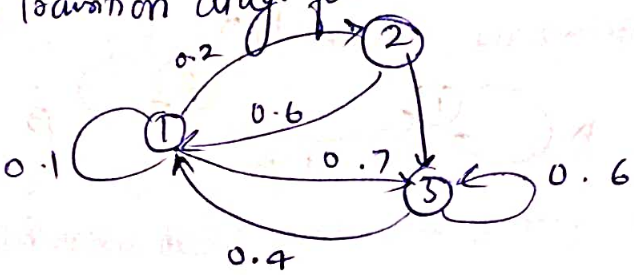
Eg : Consider a DTMC $\{X_n, n \geq 0\}$ on statespace $\{1,2,3\}$ with foll. transition probability matrix :

notation

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0 & 0.6 \end{bmatrix}$$

transition probabilities

The Transition diag. for above P is as follows :



~~confortable~~

Ex: Coming back to characterisation, we have shown that a DTMC $\{X_n, n \geq 0\}$ is completely characterised by initial distribution a & transition probability matrix P i.e, we can find

$$P(X_0 = i_0, X_1 = i_1, \ldots X_n = i_n) = a_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-2} i_{n-2}} \cdot p_{i_{n-1} i_n}$$ where

$$a = P(X_0 = i_0) \ \& \ p_{ij} = P(X_{n+1} = j / X_n = i)$$

NOTE that : $P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$ where $S = \{1,2, \ldots n\}$

$\boxed{a_i = P(X_0 = i)}$

$\boxed{a_{i_0} = P(X_0 = i_0)}$

$p_{ij} = P(X_{n+1} = j / X_n = i)$

$p = [p_{ij}]_{n \times n}$

$a = [a_{i_0} \ a_{i_1} \ \ldots \ a_{i_n}]$

$P(X_0 = i_0)$ $P(X_0 = i_n)$

Let $\{X_n, n \geq 0\}$ be a DTMC on state-space $S = \{1,2,3,4\}$ &

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.6 & 0.2 & 0.1 & 0.1 \end{pmatrix}$$

$a = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$

$a_1 = P(X_0 = 1) = 0.25$; $a_2 = P(X_0 = 2) = 0.25$; $a_3 = P(X_0 = 3) = 0.25$ ...

Further, $i_0 = 1,\ i_1 = 2,\ i_2 = 3,\ i_3 = 4$, for

$\Rightarrow P(X_0 = 1, X_1 = 3, X_2 = 1, X_3 = 4)$

$= P(X_0 = i_0, X_1 = i_2, X_2 = i_0, X_3 = i_3)$

$= a_{i_0}\, p_{i_0 i_2}\, p_{i_2 i_0}\, p_{i_0 i_3}$

$= a_1\, p_{13}\, p_{31}\, p_{14}$

$= 0.25 \times 0.3 \times 0.5 \times 0.4 = 0.015$

$S = \{i_0, i_1, \ldots\}$ (1 2 3 4)

Time homogeneity

Today sunny p. = Tom. sunny p.

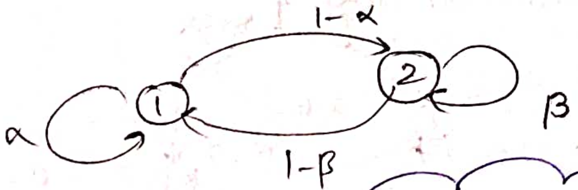
$S = \{1, 2\}$ 2 state DTMC

$n = 0$: $P(X_1 = 1 / X_0 = 1) = p_{11}$

$P(X_2 = 1 / X_1 = 1) = p_{11}$

Eg: 2 state DTMC, $S = \{1, 2\}$

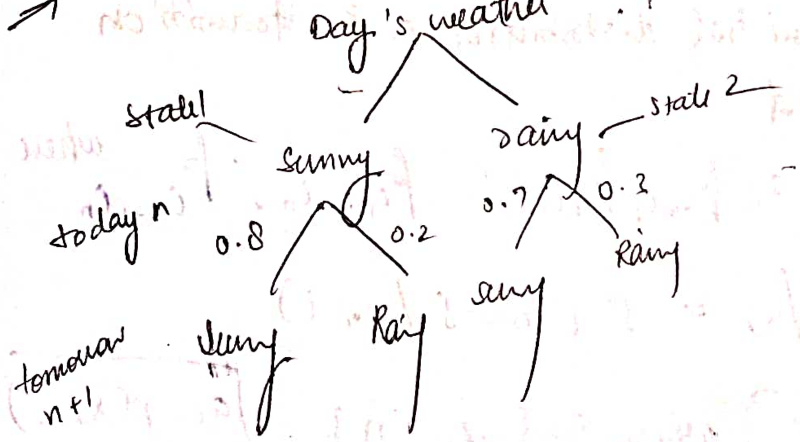
$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$



Real Research is I don't know, what I don't know

WAO!!

Eg: 2 state weather model $S = \{1, 2\}$



at any nth day if the day is sunny, then the chances are 80% that tomorrow is sunny

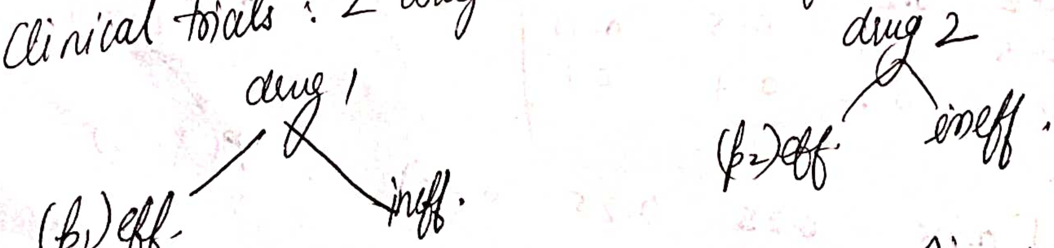
X_n = weather on day n

$\{X_n, n \geq 0\}$ is a DTMC on $S = \{1, 2\}$ with

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$$

This is a model of the weather at sunny place.

Eg: Clinical trials: 2 drugs are available for a disease



In practice values of p_1 & p_2 are unknown. Aim is to check:

$p_1 \geq p_2$ or $p_2 \geq p_1$

Play the winner Rule

Initial patient (patient 0)

either drug 1 or drug 2 at random

n th patient is given drug i $(i = 1, 2 \ldots)$

effective / ineffective

same drug is given to $(n+1)^{th}$ patient

eff. / ineff.

change the drug

$X_0 <^{1}_{2}$ $X_1 <^{1}_{2}$

patient 1

Thus we stick with a drug as long as its results are good; when we get bad result, we switch to the other drug

↓ Hence the name = Play the winner game

Let X_n be the drug (1 or 2 administered to nth patient. If the successive patients are chosen from a completely randomized pool, then

$$P(X_{n+1} = 1 / X_n = 1, X_{n-1} \ldots X_0) = P(\text{drug 1 is effective on the } n\text{th patient})$$
$$= p_1$$

i.e we can similarly obtain $P(X_{n+1} = j / X_n = i, \text{history}) \ \forall (i,j)$ combinations

Thus $\{X_n, n \geq 0\}$ is a DTMC & P is given by

$$P = \begin{bmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{bmatrix}$$

Eg: 2 state Machine:

Machine

up (working) / down (failed)

nth day

p_u / $1-p_u$; $1-p_d$ / p_d

$(n+1)$ th day up / down ; up / down

(Markov process is given as assumption) p_u does not depend on past

Let X_n be the state of machine on day n i.e, $X_n = 0$ (if down) or $X_n = 1$ (if up)

Then $\{X_n : n \geq 0\}$ is a DTMC on state space $S = \{0, 1\}$ with

$$P = \begin{bmatrix} p_d & 1-p_d \\ 1-p_u & p_u \end{bmatrix}$$

Eg: 2 machine workshop: Suppose a workshop has 2 identical machines as in above eg.

X_n = # of working machines on day n

Ques: Is $\{X_n : n \geq 0\}$ a DTMC?

Note that $S = \{0, 1, 2\}$ why?

- $P(X_{n+1} = 0 / X_n = 0, X_{n-1} \ldots X_0) = P(X_{n+1} = 0 /$ Both machines are down on day n, $X_{n-1} \ldots X_0)$

like all

$= P($Both machines are down on day $n+1$ / Both machines are down on day $n)$

$= p_d \, p_d$

(Given that both machines behave independently)

Similarly, we can verify that

$P(X_{n+1} = j / X_n = i, X_{n-1} \ldots X_0)$ depends only on i & j $\forall i, j \in S$

Thus $\{X_n, n \geq 0\}$ is a DTMC on S with

POISSON PROCESS — defined in terms of exp.

P(1st new laptop will work for more 5 yrs)

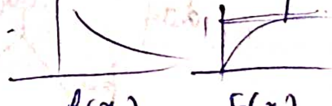
P(3 yr old —)

These are continuous-time stochastic process and are defined in terms of random variables with exponential distribution | memoryless prop.

Recap: Exponential distribution : A non-ve r.v. ($X \sim exp(\lambda)$) if

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & ; \text{ if } x < 0 \\ 1 - e^{-\lambda x} & ; \ x \geq 0 \end{cases}$$

service centre | trading



where $\lambda > 0$. The PDF of X is given as

$$f_X(x) = \lambda e^{-\lambda x} ; x \geq 0$$

$\lambda x = t$ $\lambda dx = dt$

$x = \frac{t}{\lambda}$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda^2} \int_0^\infty t e^{-t} dt$$

$$\frac{\Gamma 2}{\lambda} = \frac{1}{\lambda}$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda^3} \int_0^\infty t^2 e^{-t} dt = \frac{2}{\lambda^2}$$

- Memoryless prop.

Defn: A non-ve r.v. X is said to have memoryless prop. if

$$P(X > s+t / X > s) = P(X > t) \quad ; s, t \geq 0$$

$t+s$

__Illustration__ : If X represents the lifetime of a component (eg. hard drive), the memory less prop. says that the prob. that an s-year old hard ~~year~~ drive will last an additional t years is the same as the prob. that a new hard drive will last t years

$\Downarrow$

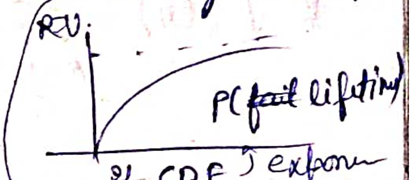
This ~~is~~ is hard drive has no memory that it has already been functioning for s years

__Mem. P.__ = A continuous non-ve RV has memoryless prop. iff it is an exp(λ)) R.V. with $\lambda > 0$

- __Probability of 1st failure__: Let $X_i \sim exp(\lambda_i)$, $i = 1, 2$ be 2 independent RV representing lifetime of 2 machines. Then

$$P(\text{Machine 1 fails before machine 2}) = P(\underset{\text{lifetime of M1}}{X_1} < X_2) = \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

(memoryless $\Leftrightarrow$ exp)



($\lambda \to$ same for same compon. otw diff.

__Q__: A running Track is 1 Km long. 2 runners start on it at same time. The speed of runner i is X_i, $i = 1, 2$. Suppose $X_i \sim exp(\lambda_i)$ and X_1 & X_2 are independent. The mean speed of runner 1 is 20 kmph & that of runner 2 is 22 kmph. What is the prob. that runner 1 wins the race?

$\lambda_1 = 1/20$, $\lambda_2 = 1/22$; $\left(P\left(\frac{1}{X_1} < \frac{1}{X_2}\right)\right) = P(X_1 > X_2) = \frac{1}{20\left(\frac{1}{20} + \frac{1}{22}\right)} = \frac{\cancel{20} \times 20}{22 + 20} = \frac{20}{42} = \frac{10}{21}$

- __Sum of iid exponentially distributed R.V.__

entire system is successful

room → lights
how many hours room is lighted

Let X_i, $i = 1, \cdots n$ be iid R.V.

Define $Z = X_1 + X_2 \cdots X_n$

Let X_i denotes the lifetime of ith component

Suppose we start by putting component 1 in use & when it fails replace it with component 2 and so on until all components fail. The replacement are instantaneous

Then $Z \xrightarrow{\text{exp}}$ lifetime of the system & $f_Z(z) = \lambda e^{-\lambda z} \frac{(\lambda z)^{n-1}}{(n-1)!}$ $z \geq 0$ &

If RV is discrete & following Memoryless prop. $\leftrightarrow$ geometrical

——— continuous ——— $\leftrightarrow$ Exponential

- **Memorylessness**: Only 2 kind of distributions are memoryless
  - geometric (of non(-ve) integers)
  - Exponential (of non(-ve) real no)

The geometric and exponential CDFs

Since CDFs are defined for any type of RV, it provides a convenient means for defining the relation b/w discrete & continuous R.V.

$\downarrow$

One such eg. is geometric & exponential

Let $X \sim Geo(p)$ i.e.,

$$P(X=k) = p_X(k) = (1-p)^{k-1} p \qquad k = 1, 2, \dots$$

$$\& \quad F_X(n) = \sum_{k=1}^{n} p(1-p)^{k-1} = \boxed{1-(1-p)^n} \longrightarrow p$$

$\cancel{P[(1-p)+(1-p)\dots}$ $\cancel{p(1-p)\ 1(1-(1-p)^n / 1-1+p}$

Further if $X \sim Exp(\lambda)$ i.e., $f_X(x) = \lambda e^{-\lambda x}$; $x > 0$ &

the CDF, $\boxed{F_X(x) = 1 - e^{-\lambda x}}$; $x > 0$ $\longrightarrow \lambda$

How to compare above 2 CDFs

Let us define $\delta = -\ln(1-p)/\lambda$ s.t.

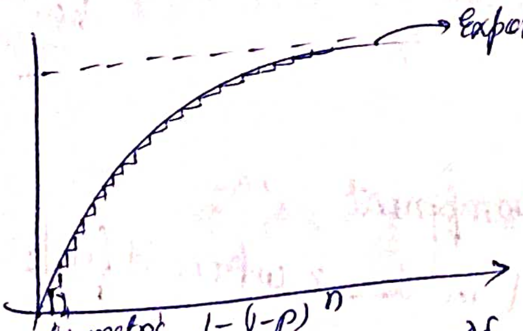
$$e^{-\lambda\delta} = 1-p$$

then $F_{exp}(n\delta) = F_{geo}(n)$, $n = 1, 2, \dots$

$$1 - e^{-\lambda n\delta} = 1-(1-p)^n$$

$$1 - (e^{-\lambda\delta})^n$$

$$= 1 - (1-p)^n$$



This relation in Geometric & Exponential CDF is plays imp.

Memoryless $\rightarrow$ Markov
(RV)

$$F_Z(z) = 1 - e^{-\lambda z} \sum_{r=0}^{n-1} \frac{(\lambda z)^r}{r!} \quad z \geq 0,$$

we say that Z ~~ft~~ follows Gamma (n, λ) distribution,

* Random sums of iid exponentially distributed R.V.

Let $\{X_i, i \geq 1\}$ be a sequence of iid R.V. & $X_i \sim exp(\lambda)$ (same) & N (stats) be a geometric R.V. $N \sim geo(p)$ independent of X's

Then $Z = \sum_{i=1}^{N} X_i \sim exp(\lambda p)$ ← Random sum follows exp with λp — p. that you get success in that exam

i=1 → Random sum

If sum random → exp.

If sum definite → Gamma

Eg: A machine is subject to series of Randomly occuring shocks. The time b/w 2 consecutive shocks are iid exp. RV. with common mean of 10 hrs. each shock results in breaking the machine with probability 0.3

What is the distribution of the lifetime of the machine.

Suppose Nth shock (when u are getting success) breaks the machine

Then N is G(0.3) RV. That is $P(N=k) = (0.7)^{k-1} 0.3 ; k \geq 1$

Let X_i be the time b/w (i-1)st & ith shock. Then

$\{X_i, i \geq 1\}$ is sequence of iid exp(0.1) RV

Then $Z = \sum_{i=1}^{N} X_i$ represents lifetime of machine & $Z \sim exp(0.03)$

with mean ~~33.33 ~~exp~~~~ 33.33 hours machine will at least work for 33 hrs.

Def^n: Poisson process is (is used for counting) frequently used in model for counting events occuring one at a time

① no. of births in a hospital

② no. of arrival at a service system (assuming one by one things are happening)

③ the no. of calls made

④ the no. of accidents ... of a road

Let $\{X_n : n \geq 1\}$ be a sequence of non(-ve) random variables representing inter-event times (time b/w shocks). Define $S_0 = 0, S_n = X_1 + X_2 + \dots + X_n, n \geq 1$

$S_0 = 0$
$S_1 = X_1$
$S_2 = X_1 + X_2$

Then S_n is the time of occurance of the nth event $n \geq 1$. Now, for $t \geq 0$, define

$$N(t) = \max\{n \geq 0 : S_n \leq t\}$$

2nd event

3 sec 2sec

$X_1 = 3$ $X_2 = 5$

$\{n \geq 0 \; ; \; S \leq t$
till S how many events occured
$N(t)=2$

X_3 is the time of occurence of 3rd event

Thus $N(t)$ is the no. of events that take place over the time interval $(0,t]$ & $\{N(t): t \geq 0\}$ is called counting process generated by $\{X_n : n \geq 1\}$

If X_1, X_2 --
geometrical
then
useful

$X \to S \to N$

- Poisson process is a spl. case of a counting process as defined below

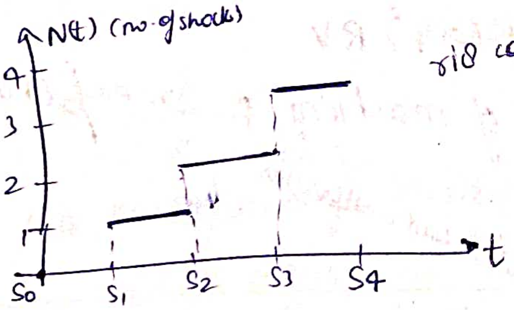
Defⁿ: The counting process $\{N(t): t \geq 0\}$ (s.p.) generated by $\{X_n : n \geq 1\}$ is called a Poisson process with parameter (a rate) λ if $\{X_n, n \geq 1\}$ is a sequence of iid $exp(\lambda)$ R.V.

$X_i \to S \to N$
$N(t) \to$ no.

Notation $\to PP(\lambda)$

Note that : $N(0)=0$ & the process jumps up by one at $t = S_n$, $n \geq 1$. Thus it has piecewise-constant sample paths.

P.P.
how many times earthquake?



Started with X_i ; exp.
$N(t) \to$ P.P

Theorem: Let $\{N(t): t \geq 0\}$ be a $PP(\lambda)$. Then

$$P(N(t) = k) = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!} \quad , k = 0, 1, \ldots$$

(P. that 10 shocks occur)
$(PP(\lambda)$ rate$)$

Pf: DIY

The above theorem states that for a fixed t, $N(t)$ is a poisson R.V. with parameter λt, denoted as $P(\lambda t)$.

$\{N(t): t \geq 0\}$ is called Poisson process.

$\to$ A justification, why

eg: Arrival at a Post office. suppose customers arrive at a Post office according to a PP with rate 10 per hour. Compute the distribution ... during an 8-hour day ... customers who use the post office

$N(t)$ be the no. of arrivals over $(0,t]$. No. of cust = 0, 1, ...

We see that the arrival process is $PP(\lambda)$ with $\lambda = 10$ per hour. Hence,

$N(8) \sim P(\lambda \cdot 8) = P(80)$ Thus $P(N(8) = k) = \dfrac{e^{-80}(80)^k}{k!}$ $k = 0, 1, \ldots$

($N(t) \sim$ Poisson with parameter λt)

compute the expected no. of customers who use the post office during an 8-hours day

since, $N(8) \sim P(80)$, $E(N(8)) = 80$ people are expected to visit.

Theorem: A poisson process has stationary & indep. increment.

Pf: DIY

Another characterisation

Theorem: A stochastic process $\{N(t) : t \geq 0\}$ (continuous) is a $PP(\lambda)$ iff

(i) it has stationary & independent increments (not bothering about $X_1, X_2, \ldots$)

(ii) $N(t) \sim P(\lambda t)$, $\forall\, t \geq 0$.

of this is true

Discrete → DTMC the CTMP → P.P

$X \to \exp$

$\downarrow$

S_n Poisson

$\downarrow$

$N(t)$

Pf: