Probability theory :-

Recap of Set theory :-

A set is a well defined collection of objects, which are the elements of set.

If S is a set & x is an element of S, we write

$$x \in S.$$

If x is not an element of S, we write $x \notin S$.

If S contains infinitely many elements $x_1, x_2, \ldots$

$$S = \{x_1, x_2, \ldots\}$$

Complement of a set S, with respect to the universal set Ω, is the set $\{x \in \Omega \mid x \notin S\}$

Union of two set S & T

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

Intersection

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \dots = \{x \mid x \in S_n \text{ for some } n\}$$

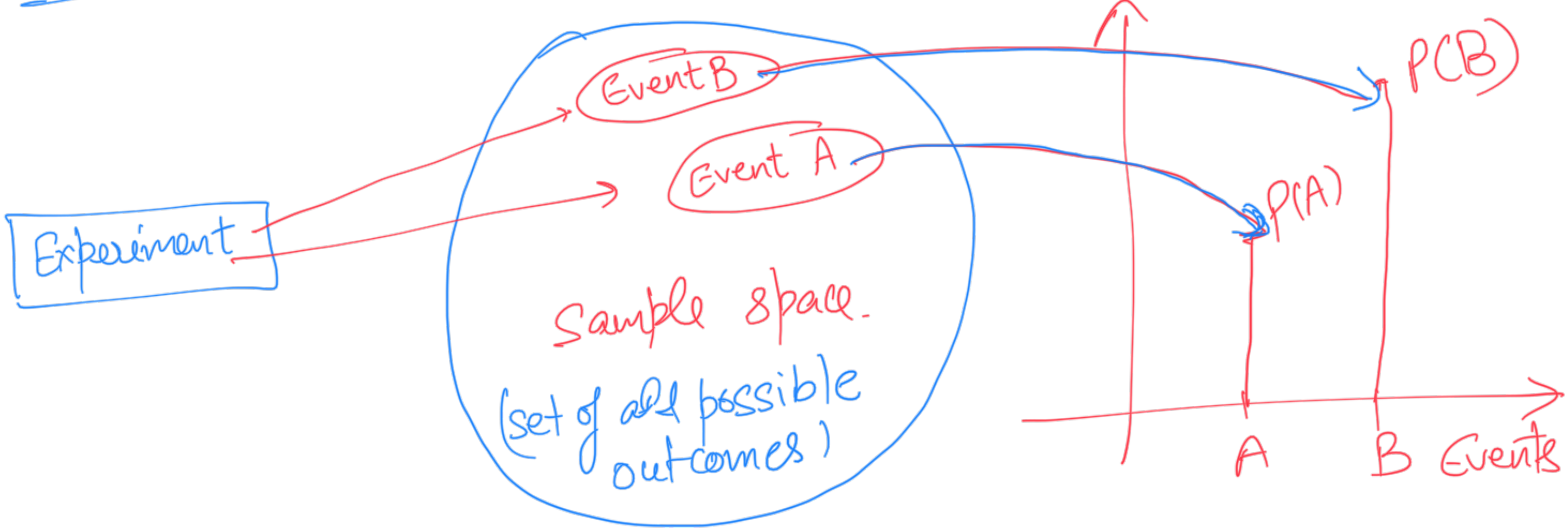
$$\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \dots = \{x \mid x \in S_n \text{ for all } n\}$$

De Morgan's law :-

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

Probabilistic models :-



Example :- Experiment: Rolling a die

Outcomes: 1, 2, 3, 4, 5, 6.

Sample space: $(\Omega) = \{1, 2, 3, 4, 5, 6\}$.

Event A :- Getting an odd number

" B :- " " even "

C :- " a prime number

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$ ✓

Every probabilistic model involves an underlying process. This process is called experiment.

It produces exactly one out of several possible outcomes.

Set of all possible outcomes is called sample space of that experiment (Ω)

A subset of the sample space is called an event.

Do it yourself :- Find an experiment where the sample space is infinite.

Definition :- The probability law assigns to every event A, a number P(A), called the probability of A, satisfying the following axioms:-

1. (Non negativity) $P(A) \geq 0$ for every event A.

2. (Additivity) If A & B are two disjoint events, then $P(A \cup B) = P(A) + P(B)$

3. (Normalization) The probability of entire sample space is 1, $P(\Omega) = 1$

Do it yourself: Using above, prove that

$$\boxed{P(\phi) = 0.}$$

Properties of Probability laws: Let $A, B, \& C$ be events from an experiment:

(1) If $A \subset B$ then $P(A) \leq P(B)$

(2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(3) $P(A \cup B) \leq P(A) + P(B)$

Discrete model:

Experiment: tossing a coin three times

$\Omega = \{HHH, HHT, HTH, HTT, TTH, THT, THH, TTT\}$.

$A = \{\text{exactly 2 heads occur}\} = \{HHT, HTH, THH\}$

$P(A) = ?$ $\left(\frac{3}{8}\right)$

<u>Example:-</u>

Conditional Probability :- Consider the following examples :-

(i) Rolling a die twice, you are told that sum of the two rolls is 9. What is the Probability that first roll was a '6'

(2) The medical report of a person about a decease is negative, how likely is it that the person has that descease.

Definition :- If all the outcomes are finitely many & equal likely.

$$P(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B} \quad \text{—} (*)$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad \text{—} (*)(*)$$

Example :- Toss a fair coin three times. Consider the events $A = \{\text{more heads than tails come up}\}$

$B = \{1^{st} \text{ toss is a head}\}$.

Sample space $\Omega = \{HHH, HHT, HTT, HTH, TTT, THH, TTH, THT\}$.

$B = \{HHH, HHT, HTH, HTT\}$

$$P(B) = \frac{4}{8} = \frac{1}{2}$$

we want to find $P(A|B)$?

$A \cap B = \{HHH, HHT, HTH\}$

$$P(A \cap B) = \frac{3}{8}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Example 2:- Radar Detection:- If an aircraft is present in a certain area, the radar detects it. It sends an alarm with probability 0.99. If no aircraft is there, it generates a alarm (false alarm) with probability 0.10. It is assumed that an aircraft is present with probability 0.05.

P(no aircraft present and a false alarm) $= ?$

P(aircraft present and no detection) $= ?$

Solution:- Let A & B be the events

$A = \{$an aircraft is present$\}$

$B = \{$the radar generates an alarm$\}$

$$P(A^c \cap B) = ?$$

$$P(A \cap B^c) = ?$$

Recap:-

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B) \cdot P(A|B)$$

generates alarm.

$$P(B \cap A^c) = P(A^c) \cdot P(B|A^c)$$

aircraft not present

$$= 0.95 * 0.1$$

$$= 0.095.$$

no detection.

$$P(B^c \cap A) = P(A)\, P(B^c | A)$$

aircraft is present

$$= 0.05 * 0.01$$

$$= 0.0005$$

Multiplication Rule :- Suppose an event A occurs if and only if each one of several events $A_1, A_2, \dots, A_n$ has occured, i.e.

$$A = A_1 \cap A_2 \cap \dots \cap A_n .$$

Definition :-

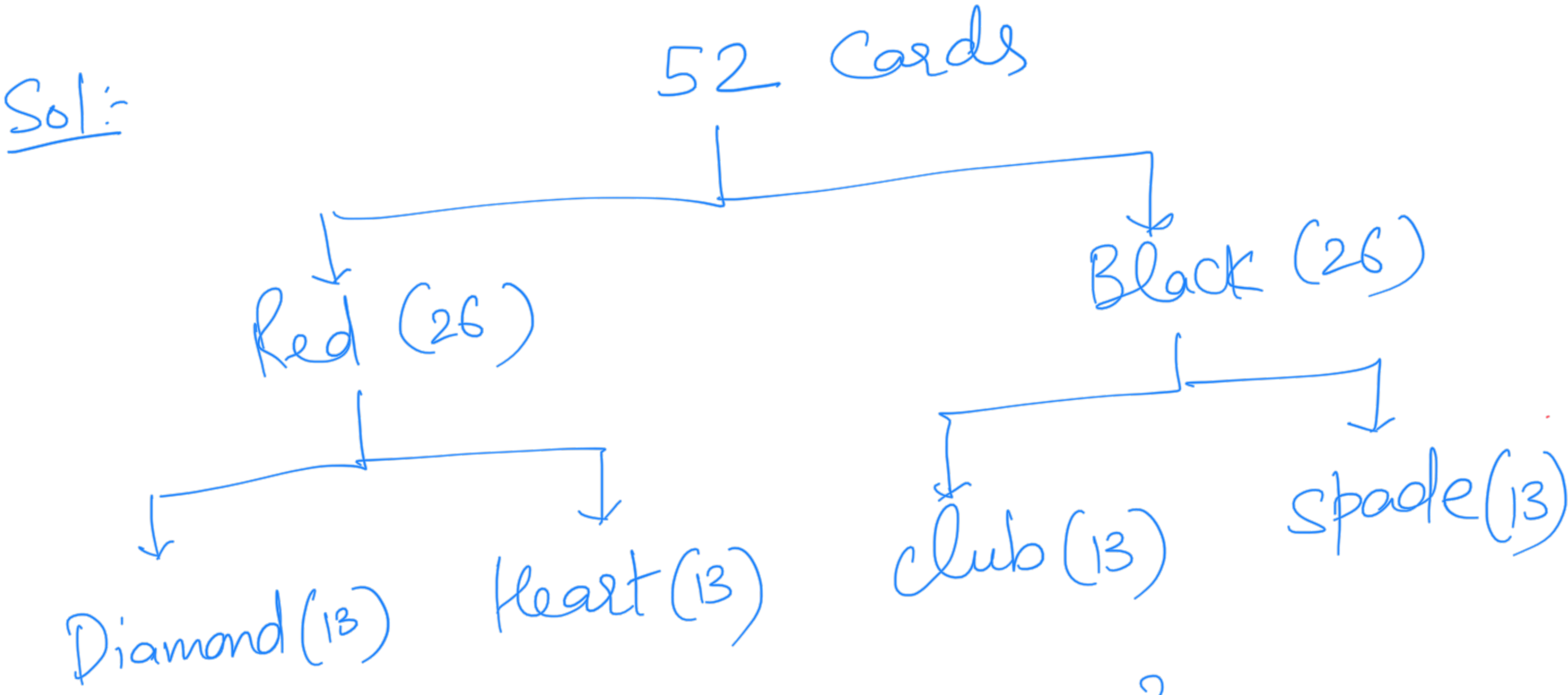
$$P(A) = P\left(\bigcap_{i=1}^{n} A_i\right)$$

$$= P(A_1) \cdot P(A_2 | A_1) \cdot P\left(A_3 \middle| A_1 \cap A_2\right) \cdot \dots \cdot P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right) \quad \text{— ★★★}$$

assuming that all the conditioning events have positive probability.

Example:- Three cards are drawn from a deck of 52 cards without replacement. What is the probability that none of these three cards is a "heart"?

Sol:-



Ace, King, Queen, Jack, 10, 9, ..., 2.

Sol:- Let $A_1 = \{1^{st}$ card is not heart$\}$

$A_2 = \{2^{nd}$ " " " $\}$

$A_3 = \{3^{rd}$ " " " " $\}$

$$P(A_1 \cap A_2 \cap A_3) = ?$$

$$= P(A_1)\, P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$$

Now $P(A_1) = P(1^{st} \text{ card is not a heart})$

$$= \frac{39}{52}$$

$$P(A_2|A_1) = P\left(2^{nd} \text{ card is not a heart} \,\middle|\, \begin{array}{l} 1^{st} \text{ card} \\ \text{is not a} \\ \text{heart} \end{array}\right)$$

$$= \frac{38}{51}$$

$$P(A_3|A_1 \cap A_2) = P\left(3^{rd} \text{ card is not a heart} \,\middle|\, \begin{array}{l} 1^{st} \text{ two card are} \\ \text{not hear} \end{array}\right)$$

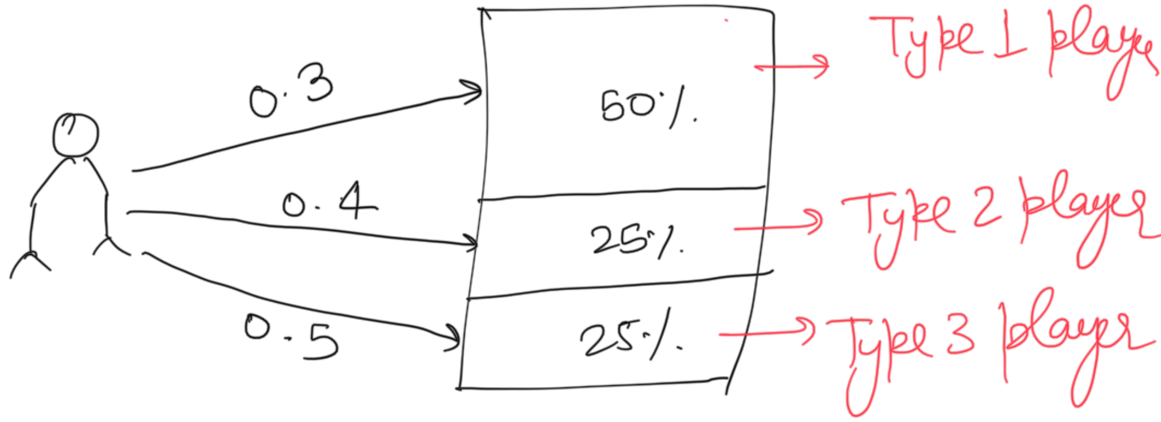
$$= \frac{37}{50}$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}}$$

Total probability theorem:- Let $A_1, A_2, \ldots A_n$ be disjoint events that form a partition of sample space. Assume that $P(A_i) > 0 \;\forall i$. Then for any event B, then

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \ldots + P(A_n) \cdot P(B|A_n)$$

Example:-



You are playing chess tournament. Your probability of winning a game is 0.3 against half of the players. The probability of winning is 0.4 against a quarter of the players. The probability of winning is 0.5 against a quarter of the players.

What is the probabity that you will win?

Solution:- A_1 = the event of playing with Type 1

A_2 = " " " " Type 2

A_3 = " " " " Type 3.

$P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{4}$, $P(A_3) = \frac{1}{4}$

Let B = event of winning.

$$P(B \mid A_1) = P(\text{you won} \mid \text{played against Type 1})$$

$$= 0.3$$

$$P(B|A_2) = 0.4$$

$$P(B|A_3) = 0.5$$

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

$$= \frac{1}{2} \times 0.3 + \frac{1}{4} \times 0.4 + \frac{1}{4} \times 0.5$$

$$= 0.375$$

Baye's Rule:- It is used for inference. There are a number of "causes" that may result in certain effect. We observe the effect & we wish to infer the cause.

Definition:- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space & assume that $P(A_i) > 0$. Then for any event B ($P(B) > 0$), we have

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B|A_i)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots + P(A_n) P(B|A_n)}$$

Example:- Relook at the previous example of chess tournament.

Suppose you win. What is the probability that you played against type 1 player?

(Do it yourself)

$$P(A_1 | B) = ?$$

Independence: Let A & B be two events.

The interesting case is, when the occurrence of B provides no information & does not alter the probability of occurrence of A

$$P(A|B) = P(A) \quad \text{--- (1)}$$

when Eq (1) holds, then the events A & B are said to be **independent**.

From conditional probability, we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{--- (A)}$$

putting in (1)

$$\boxed{P(A \cap B) = P(A) \cdot P(B)} \quad \text{--- (2)}$$

Eq^n (2) is adopted as the definition of independence of two events because it can be used even when $P(B) = 0$, in which case $P(A|B)$ is undefined.

Independence is a symmetric property:- If A is independent of B, then B is independent of A.

we say that "A & B are independent events"

Remark:- If two events are disjoint
$\Downarrow\!\!\!\!/$
they are independent
True or False?

Ans: FALSE

Two disjoint events A & B with $P(A)>0$, $P(B)>0$ &

$A \cap B = \phi$ ✓

$P(A \cap B) = 0$ ✓

$\because$ $P(A)$ & $P(B) > 0 \Rightarrow P(A) \cdot P(B) > 0$ ✓

For independence, $P(A \cap B) = P(A) \cdot P(B)$

which is not possible.

Example:- Consider an experiment involving two successive rolls of a 4-sided die. Assume that outcomes are equal likely.

$$\Omega = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{Bmatrix}$$

Consider the following two events:-

$A = \{1^{st}$ roll is $1\}$ $B = \{$sum of two rolls is $5\}$

Are A & B independent?

Solution:- we need to check $P(A \cap B) = P(A) \cdot P(B)$

$A \cap B = (1,4)$

$$P(A \cap B) = \frac{1}{16}$$

$$P(A) = \frac{4}{16}, \quad P(B) = \frac{4}{16}$$

$$\because P(A \cap B) = P(A) \cdot P(B) = \frac{1}{16}$$

$\Rightarrow$ A & B are independent.

Case II :- Let $A = \{$maximum of the two rolls is $2\}$

$B = \{$minimum " " " " $2\}$

Are A & B independent?

$A = \{(1,2)\ (2,1)\ (2,2)\}$

$B = \{(2,2)\ (2,3)\ (2,4), (3,2)\ (4,2)\}$

$P(A) = \frac{3}{16}$

$P(B) = \frac{5}{16}$

$A \cap B = \{(2,2)\}$

$P(A \cap B) = \frac{1}{16}$

$P(A \cap B) \neq P(A) \cdot P(B)$

$\Rightarrow$ A & B are <u>not</u> independent

<u>Conditional Independence</u>:- Given an event C, the events A & B are called conditionally independent if

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

Example:- Consider two independent fair coin tosses. (Assume outcomes are equal likely)

Let

$A = \{1^{st}$ toss is a head$\}$

$B = \{2^{nd}$ " " " "$\}$

$C = \{$two tosses have different results$\}$.

A & B are conditionally independent or not?

we need -

$P(A \cap B | C)$ $\quad$ $P(A|C) \cdot P(B|C)$

Try yourself -

Continuing the previous example :- $\Omega = \{HH, HT, TH, TT\}$

$$P(A|C) = \frac{1}{2}$$

$$P(B|C) = \frac{1}{2}$$

A = 1st toss is head
B = 2nd " " "
C = two tosses have different result.

$$P(A \cap B | C) = 0$$

$$\Rightarrow P(A \cap B | C) \neq P(A|C) \cdot P(B|C)$$

$\Rightarrow$ A & B are NOT conditionally independent

★ The independence of several events.

The events $A_1, A_2, \ldots, A_n$ are said to be independent if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$

product

for every subset S of $\{1, 2, \ldots, n\}$ ★

Example :- Let A_1, A_2 & A_3 be three ~~events~~. If

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

Pair wise independence

$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$$

happens, then A_1, A_2, & A_3 are said to be independent.

Example:- Consider the previous example of tossing two independent fair coins &

$A_1 = \{1^{st}$ toss is head$\}$ $\Omega = \{HH, HT, TH, TT\}$

$A_2 = \{2^{nd}$ " " " $\}$

$A_3 = \{$two tosses have different results$\}$

✓ A_1 & A_2 are independent.

$$P(A_3 | A_1) = \frac{P(A_3 \cap A_1)}{P(A_1)} = \frac{1/4}{2/4}$$

$$= \frac{1}{2} = P(A_3)$$

✓ $\Rightarrow$ A_1 & A_3 are independent.

$$P(A_3 | A_2) = \frac{P(A_3 \cap A_2)}{P(A_2)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$= P(A_3)$$

$\Rightarrow$ A_2 & A_3 are independent.

$\Rightarrow$ A_1, A_2 & A_3 are pairwise independent

Are they independent? NO

$$P(A_1 \cap A_2 \cap A_3) = 0$$

$$P(A_1) = \frac{1}{2}, \quad P(A_2) = \frac{1}{2}, \quad P(A_3) = \frac{1}{2}$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\,P(A_2)\,P(A_3)}$$

$\Rightarrow$ A_1, A_2 & A_3 are pairwise independent but they are not independent.

Example:- Consider two independent rolls of a fair six-sided die &

$A = \{1^{st}$ roll is $1, 2,$ or $3\}$

$B = \{1^{st}$ " " $3, 4,$ or $5\}$

$C = \{$the sum of two rolls is $9\}$.

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{4}{36}$$

$$\Omega = \left\{ \begin{matrix} (1,1) & (1,2) & \cdots & (1,6) \\ (2,1) & (2,2) & \cdots & (2,6) \\ \vdots & & & \\ (6,1) & (6,2) & \cdots & (6,6) \end{matrix} \right\}$$

$$A \cap B = \{ (3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6) \}$$

$$P(A \cap B) = \frac{6}{36}$$

✓ $P(A \cap B) \neq P(A) \cdot P(B)$

$$P(B \cap C) = \frac{3}{36}$$

✓ $P(B \cap C) \neq P(B) P(C)$

$$P(A \cap C) = \frac{1}{36}$$

✓ $P(A \cap C) \neq P(A) P(C)$

$$A \cap B \cap C = (3,6)$$

$$P(A \cap B \cap C) = \frac{1}{36}$$

$$P(A \cap B \cap C) = P(A) P(B) P(C)$$

$$\frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{36}$$

$\Rightarrow$ 1st Condition is satisfied, however they are not pairwise independent.

Independent trials & Binomial probabilities :

If an experiment involves a sequence of independent & identical stages, we say that we have a sequence of independent trials.

Suppose there are only two possible results at each stage, these trials are called Bernoulli trials.

MA225 (Probability and Random Processes):

INDIPENDENT TRAILS & BINOMIAL PROBABILITIES

→ If an expr. involves a seq. of independent & identical stages we say that we have a seq. of independent trails

→ suppose there are only 2 possible results at each stage these trails are called BERNOULLI TRAILS

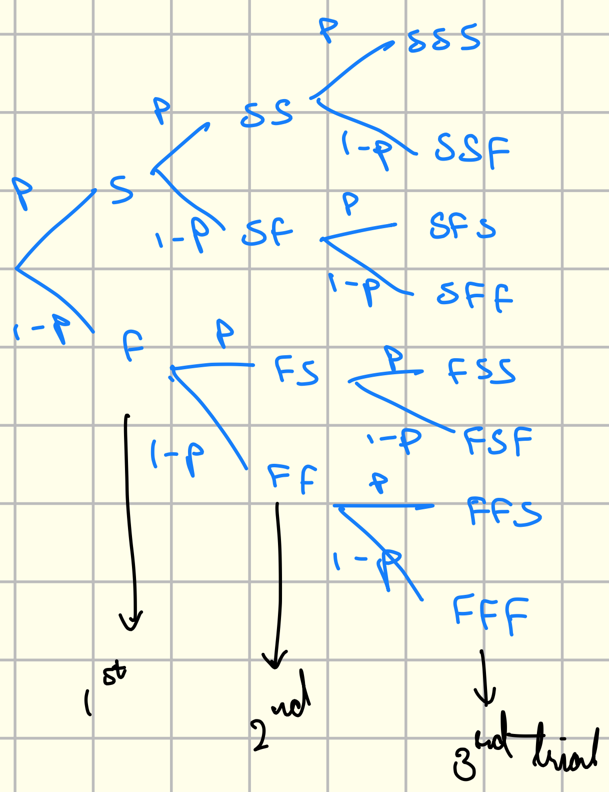
INDIPENDENT TRIALS AND BINOMIAL PROBABILITIES :

→ Let Prob. of success is p &

p " " failure : $1-p$

Success → S

Failure → F



$P(3 \text{ success}) = p^3$

$P\left(\begin{matrix}2 \text{ success} \,\&\\ 1 \text{ faure}\end{matrix}\right) = p^2(1-p)$

$P\left(\begin{matrix}1 \text{ sucess} \,\&\\ 2 \text{ failure}\end{matrix}\right) = p(1-p)^2$

$P(3 \text{ failure}) = (1-p)^3$

in 3 Trials

→ The prob. that we get k success in n trials is

$$\boxed{p(k) = \binom{n}{k} p^k (1-p)^{n-k}} \qquad \boxed{\binom{n}{k} = {}^nC_k}$$

→ The numbers $\binom{n}{k}$ are known as Binomial Coeff. the Prob. $p(k)$ are known as the Binonomial Prob.

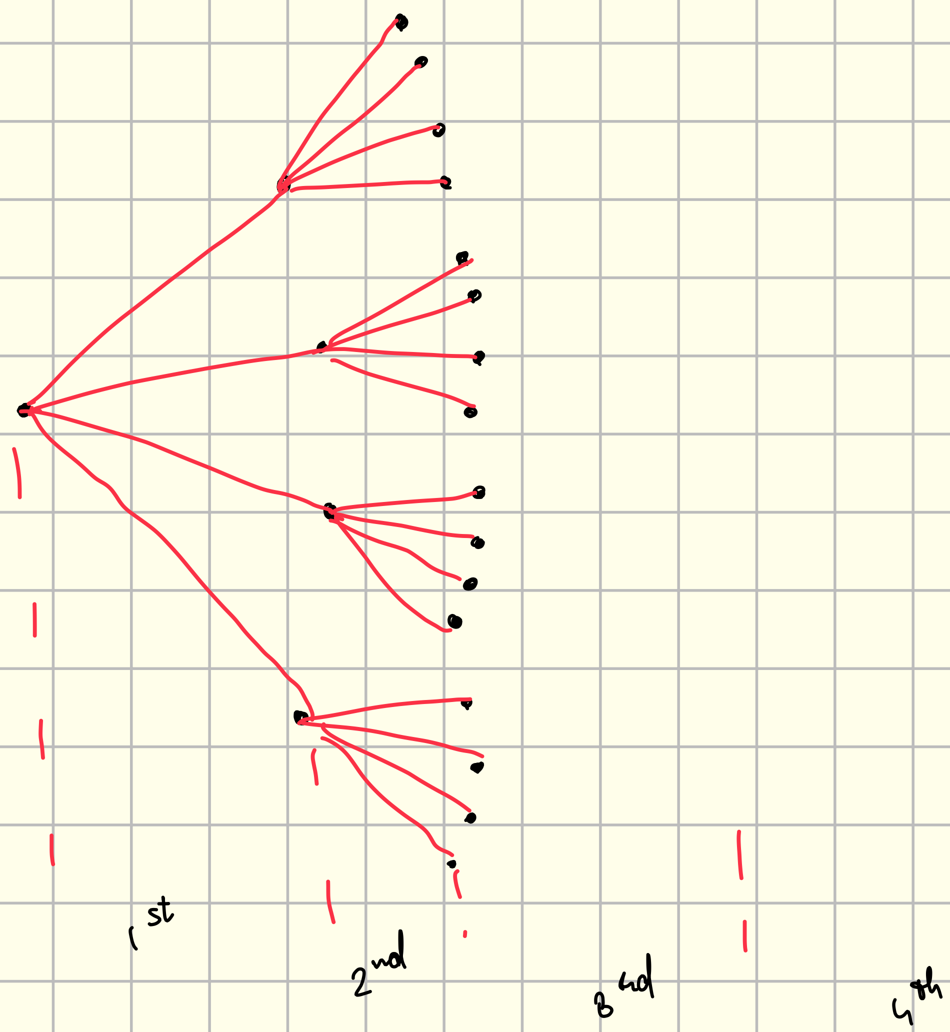
$$\boxed{\binom{n}{k} = {}^{n}C_{k} = \frac{n!}{k!\,(n-k)!}}$$

→ **Remark**: The binomial prob. $p(k)$ must add to 1 i.e

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} p^{k} (1-p)^{k} = 1} \quad \text{———} \; (*)$$

→ The eq.^{n} (*) is called the binomial formulas

COUNTING :



→ The first stage has n_1 possible
$\quad$ 2nd " n_2 "

3^{rd} " n_3 "

:

k " " n_k possible ways

the total no. of Possible values: $n_1 . n_2 . \ldots . n_k$

Example: A telephone number is a 7 digit seq. but the 1^{st} digit shldn't be 1 or 0 how many distinct telephone no.'s are there?

8 8 10 10 10 10 10 10 10 10

$\Rightarrow$ ans : $8^2 \times 10^8$

$\rightarrow$ $\boxed{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n}$

$\Rightarrow$ $p = \frac{1}{2}$ in $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

$\rightarrow$ Topics Covered till Now:

(1) sets

(2) Prob. models

(3) Conditional Prob. (Multiplication Theorem)

(4) Total Prob. Theo. & Bayes The.

(5) Independance

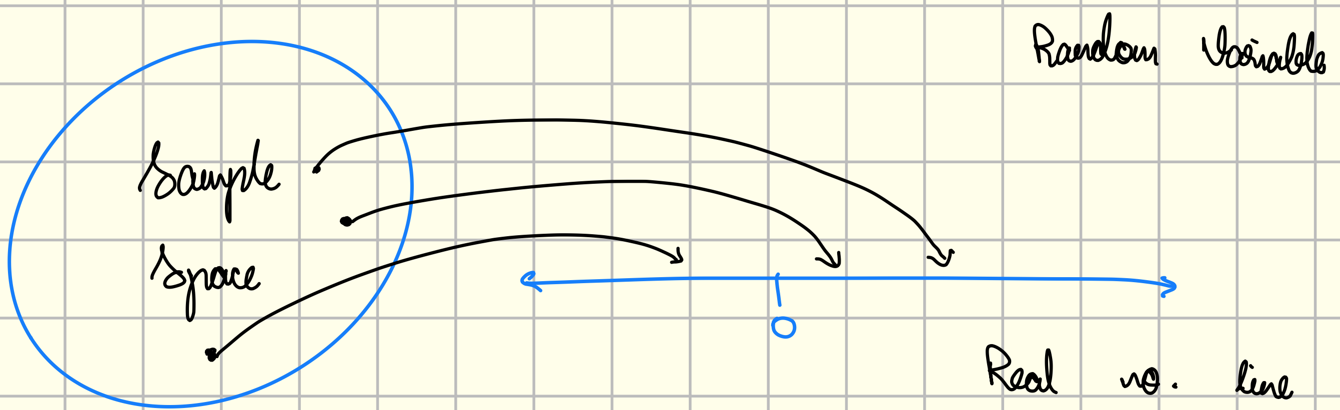
(6) Counting, P & C

RANDOM VARIABLE :

Experiment
↓
Outcomes

Numerical | Not Numerical

Ideas : Given an exp. & the possible outcomes, a r.v associates a particular no. with each outcome.



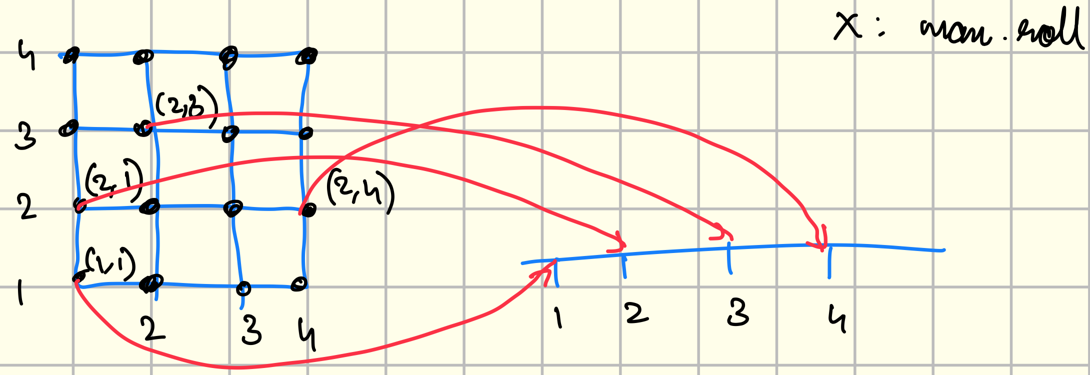
RANDOM VARIABLE :

→ A r.v is a real valued function of the experimental outcome i.e $X : \Omega \to \mathbb{R}$

X : r.v

Ω : sample space

$\mathbb{R}$: set of real no.s



$$X : \Omega \longrightarrow R$$

→ Example 1 : Experiment : rolling 2 '6' sided die

Outcomes : ~~(0,0)~~ (1,1) . . . . (6,6)

The following r.v can be defined

(i) sum of 2 rolls ($X = 2, 3, 4 \ldots\ldots 12$)

(ii) no. of 6 in the 2 rolls & so on. ($X = 0, 1, 2$)

→ what we have learned? ① r.v

② what values a r.v can have?

③ In which situation, r.v is taking what values?

RECALL: A r.v X ⇒ $X : \Omega \longrightarrow \mathbb{R}$

DEFINITION: A r.v is called discrete if its range is finite or countably infinite.

→ The r.v defined in ex. 1 are discrete r.v

DEFINITION : A r.v that can take an uncountably infinite number of values is NOT discrete r.v

EXAMPLE : Experiment : Chosing a point 'a' from the interval $[-1, 1]$

r.v is giving a^2 to the outcome a

Not discrete : we can get uncountably ∞ r.v

CONCEPTS RELATED TO DISCRETE R.V :

PROBABILITY MASS FUNCTION (PMF) :

→ If x is any possible value of r.v X the probability mass of X is the probability of event $\{X = x\}$ i.e Probability mass of X is

$$p_X(x) = P\{X = x\}$$

→ Example : Exp : Tossing a coin twice

R.V X : no. of heads obtained

$\Omega = \{HH, HT, TT, TH\}$

PMF of random variable X :

X = no. of heads, r.v

$x = 0, 1, 2$

$$p_X(x) = \begin{cases} 1/4 & x=0 & \{TT\} \\ 1/2 & x=1 & \{HT, TH\} \\ 1/4 & x=2 & \{HH\} \end{cases}$$

→ all are +ve & $\sum_x p_X(x) = 1$

→ Note that

$$\boxed{\sum_x p_X(x) = 1}$$

→ In the previous example of tossing coin 2-times, the PMF is

X : no. of heads

$x = 0, 1, 2$

$$p_X(x) = \begin{cases} 1/4 & x=0 \\ 1/2 & x=1 \\ 1/4 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$P(\text{at least one head}) = P(X=1) + P(X=2)$

$P(X>0) = 1/2 + 1/4 = 3/4$

BERNOULLI RANDOM VARIABLE :

→ when we want to deal with qualitative cases where there is a success / fail

X is said to be Bernoulli R.V if

$$X = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

we use this r.v

suppose Probability of success is p

failure is $1-p$

→ PMF is $\boxed{P_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}}$

→ Examples: ① A telephone at a given time can be either free or busy

② A person can be healthy / sick

BINOMIAL RANDOM VARIABLE:

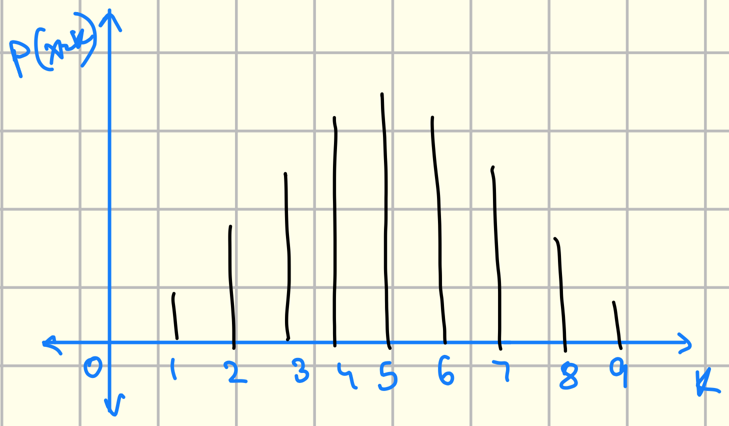
→ Let X be the no. of success in a n-trials. X is called Binomial R.V with parameters n & p the PMF of X is.

→ $$\boxed{P_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}} \quad k=0, 1, 2 \dots n$$

→ The Normalisation Property gives:

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1}$$

PLOT OF PMF OF BINOMIAL RANDOM VARIABLE:



→ we know that

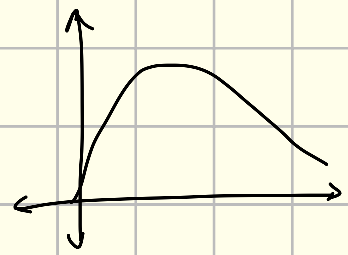
$$P_X(K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$n = 9$

$p = \frac{1}{2}$

→ put $K = 0, 1, 2 \ldots 9$ find $P_X(K)$ from eq.^{n} & plot.

→ when $n \to$ large, $p \to$ small } graph is left skewed



$n \to$ small, $p \to$ large } graph is right skewed

GEOMETRIC RANDOM VARIABLE:

→ The Geometric R.V is the no. x of trails to get the first success. Its PMF is given by

$X = 1, 2, 3 \ldots\ldots$

→ Suppose prob. of success is p.

$P(X=1) = p$

$P(X=2) = (1-p)p$

$\vdots$

$P(X=K) = (1-p)^{K-1} p$

} In the form of G.P

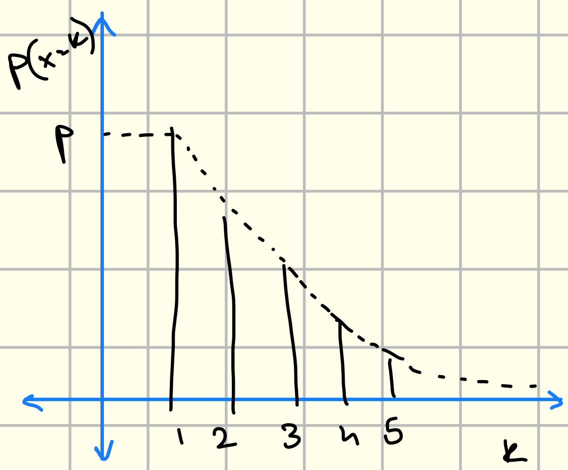
$$\Rightarrow \boxed{p_X(k) = P(X=k) = (1-p)^{k-1}p} \qquad k = 1, 2, 3 \ldots\ldots$$

$$\Rightarrow \sum_k p_X(k) = \sum_k (1-p)^{k-1}p = p + (1-p)p + \ldots\ldots$$

$$= p\left(1 + (1-p) + (1-p)^2 + \ldots\ldots\right)$$

$$= \frac{p}{1-1+p} = 1$$

$$\Rightarrow \boxed{\sum_k p_X(k) = \sum_k (1-p)^k p = 1}$$



$\Rightarrow$ graph is $\downarrow$ ing in values

$\because$ $p < 1$ & $(1-p) < 1$

THE POISSON RANDOM VARIABLE:

→ In a certain time Interval how many times your Event of Interest occurs

→ A Poisson r.v has the following PMF

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2 \ldots\ldots$$

$$\lambda > 0$$

→ Prove that: $\sum_k p_X(k) = \sum_k \frac{e^{-\lambda}\lambda^k}{k!} = 1$

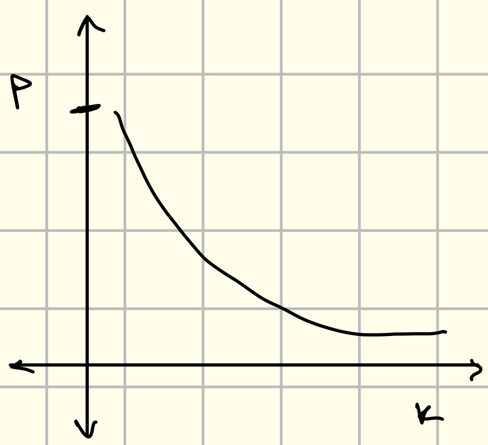
Proof : $\sum_k p_x(k) = \sum_k \frac{e^{-\lambda}\lambda^k}{k!}$

$$= e^{-\lambda} + e^{-\lambda}\lambda + \frac{e^{-\lambda}\lambda^2}{2!} + \cdots\cdots$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right) = \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = 1$$

$$\therefore \boxed{\sum_k p_x(k) = \sum_k \frac{e^{-\lambda}\lambda^k}{k!} = 1}$$

→ $\lambda = 0.5$



→ $\lambda = 2$

→ take a binomial r.v with very small p & very large n

→ <u>For Example</u>: Let x be the no. of typos in a book with a total of n words. Then x is binomial r.v.

The poisson PMF with parameter λ is a good approximation for a binomial PMF with parameter n & p i.e

$$\frac{e^{-\lambda}\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k} \quad \text{—— } (1)$$

$k = 0, 1, 2 \ldots n$

provided $\lambda = np$

→ Example: $n = 100$, $p = 0.01$, then probability of 5 success in 100 trials is

binomial → $\binom{100}{5}(0.01)^5(1-0.01)^{100-5}$

$= 0.0029$

→ Using Poisson PMF with $\lambda = np = 100 \times 0.01 = 1$

$$P(X=5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1}(1)^5}{5!} = 0.0030$$

Q) A software manufacturer knows that one out of 10 software games that the company markets will be a financial success. The manufacturer selects 10 new games to market what is the probability that exactly one game will be a financial success. what is the probability that atleast 2 games will be a success?

Sol: $p = P(\text{success}) = \frac{1}{10}$

$P(X=1) = {}^{10}C_1\, p^1 (1-p)^9$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - {}^{10}C_0 (1-p)^{10} - {}^{10}C_1 p^1 (1-p)^9$$

Q) In Pulse code modulation (PCM) a PCM word consists of a sequence of binary digits of 1's & 0's

a) Suppose the PCM word length is n bits. How many distinct words are there? 2^n

b) If each PCM word, three bits long, is equally likely to occur what is the prob. of a word with exactly two 1's occuring $\frac{{}^{3}C_2}{8} = \frac{3}{8}$

FUNCTION OF RANDOM VARIABLE:

→ Let X be a r.v if $Y = g(X)$ is a function of X, then Y is also a r.v, since it provides a numerical value for each possible outcome.

→ If X is discrete the Y is also discrete r.v. the PMF of Y is calculated from PMF of X

→ The PMF of Y is

$$\boxed{p_Y(y) = \sum_{\{x \mid g(x) = y\}} p_X(x)} \quad \text{—— } (2)$$

→ Now, we will find the PMF of $Y = g(X)$ as follows.

→ Example: Let X be a r.v with PMF

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

→ Let $Y = |X|$ ⇒ Y is a discrete r.v

Possible values of $Y = 0, 1, 2, 3, 4$

$P(Y=0) = P(X=0) = \frac{1}{9}$

$P(Y=1) = P(X=1) + P(X=-1) = \frac{2}{9}$

$P(Y=2) = P(X=2) + P(X=-2) = \frac{2}{9}$

$P(Y=3) = P(Y=3) + P(Y=-3) = 2/9$

$P(Y=4) = P(X=4) + P(X=-4) = \frac{2}{9}$

→ PMF Y is

$$p_Y(y) = \begin{cases} 1/9 & y = 0 \\ 2/9 & y = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

→ Homework: Find the PMF of $z = x^2$

Possible values of $z = 0, 1, 4, 9, 16$

$$p_z(z) = \begin{cases} 1/9 & z = 0 \\ 2/9 & z = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

EXPECTATION OR MEAN OF A RANDOM VARIABLE:

→ We define the expected value of a r.v X as follows:

$$\boxed{E[X] = \sum_{x} x \, p_X(x)}$$

where p_X is PMF of X

→ Example: Let X be a r.v with PMF

$$p_X(x) = \begin{cases} 1/9 & x \text{ is an integer from } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

find $E[X]$?

sol: Possible values of $X = -4, -3, -2, \ldots\ldots 0, 1, 2, \ldots 4$

$$E[X] = \left(-4 \times \frac{1}{9}\right) + \left(-3 \times \frac{1}{9}\right) + \cdots\cdots + \left(4 \times \frac{1}{9}\right) = 0$$

→ Expectation of r.v is 0

Remark: Average = $\frac{\text{sum of observation}}{}$

total no. of observations

If $x = 1,2,3,4,5$ with Probability $\frac{1}{5}$ each

$$E[x] = 1\times\frac{1}{5} + 2\times\frac{1}{5} + \dots\dots + 5\times\frac{1}{5} = 3$$

$$\text{Average}(1,2,3\dots5) = 3$$

In this case $E[x] =$ Average coz the event is equally likely

$$p_X(x) = \begin{cases} 1/2 & x=1 \\ 1/8 & x=2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{2} + 2\times\frac{1}{8} + 3\times\frac{1}{8} + 4\times\frac{1}{8} + 5\times\frac{1}{8} = \frac{1}{2} + \frac{\cancel{14}\,7}{\cancel{8}\,4} = \frac{9}{4}$$

$$\text{average} = \frac{5\times3}{5} = 3$$

Q) A coin is tossed 2 times, each with a prob. 3/4 for head

$x =$ no. of heads

Possible values of $x = 0,1,2$

$$p_X(x) = \begin{cases} \frac{1}{16} & x=0 \\ 3/8 & x=1 \\ 9/16 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{16}\times0 + \frac{3}{8}\times1 + \frac{9}{16}\times2$$

$$= \frac{3+9}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

VARIANCE :

→ Variance of random variable X is obtained as

$$\boxed{Var(X) = E[X - E[X]]^2} \quad — \text{③}$$

→ Example : Let X be a r.v with PMF

$$p_X(x) = \begin{cases} 1/9 & x \text{ is integer of } x \in [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

$E[X] = 0 \qquad E[X - E[X]]^2 = E[X - 0]^2 = E[X^2]$

$Y = X^2$

$$p_Y(y) = \begin{cases} 1/9 & y = 0 \\ 2/9 & y = 1, 4, 9, 16 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow E[X^2] = E[Y] = \frac{1}{9} \times 0 + \frac{2}{9} + \frac{8}{9} + \frac{18}{9} + \frac{32}{9}$$

$$= \frac{60}{9} = \frac{20}{3}$$

$\therefore$ variance $= E[X - E[X]]^2 = E[X^2] = \frac{20}{3}$

STANDARD DEVIATION:

→ used to match the unit $\begin{cases} \text{in var. unit}^2 \\ \text{in SD unit} \end{cases}$

→ It is a square root of variance & denoted by σ_X

$$\boxed{\sigma_X = \sqrt{Var(X)}}$$

Remark: To find variance

$\downarrow$ compute

$E[X]$

$\downarrow$ Compute

$E[(X - E[X])^2]$

for this we need PMF of $(X - E[X])^2$

Another method to compute variance will not require the PMF of $(X - E[X])^2$

EXPECTATION OF FUNCTION OF RANDOM VARIABLE:

$\rightarrow$ Let X be a r.v with PMF p_X & let $g(X)$ be the function of X then

$$\boxed{E[g(X)] = \sum_x g(x)\, p_X(x)} \quad \text{——— (1)}$$

$\rightarrow$ $Var(X) = E[(X - E[X])^2] = E[g(X)]$

$$= \sum_x g(x)\, p_X(x)$$

$$= \sum_x (x - E[X])^2 p_X(x) = \sum_x \left(x^2 + (E[X])^2 - 2xE[X]\right) p_X(x)$$

$$= \sum x^2 p_X(x) + \sum (E[X])^2 p_X(x) - 2E[X] \sum_x x\, p_X(x)$$

$$= E[x^2] + (E[x])^2 - 2(E[x])^2$$

$$= E[x^2] - (E[x])^2$$

$$\Rightarrow \boxed{\mathrm{Var}[x] = E\left[(x - E[x])^2\right] = E[x^2] - (E[x])^2}$$

PROPERTIES OF MEAN AND VARIANCE:

→ Let x be a r.v and let

$$Y = ax + b = g(x) \qquad (a, b \text{ are scalar})$$

$$E[Y] = E(ax+b) = \sum_x (ax+b)\, p_X(x)$$

$$= a \sum_x x\, p_X(x) + b \sum_x p_X(x)$$

$$= a E[x] + b$$

$$\therefore \boxed{E[Y] = a E[x] + b} \quad \text{—— } 2$$

$$\mathrm{Var}(Y) = E\left[(Y - E[Y])^2\right] = E[Y^2] - E[Y]^2$$

$$= E\left[(ax + \cancel{b} - aE[x] - \cancel{b})^2\right]$$

$$= a^2 E\left[(x - E[x])^2\right] = a^2 \mathrm{Var}(x)$$

$$\boxed{\mathrm{Var}(Y) = a^2 \mathrm{Var}(x)}$$

→ R.V ⟶ PMF ⟶ 4 types of r.v ⟶ functions of r.v ⟶ Mean & Variance

MEAN AND VARIANCE OF BERNOULLI R.V:

→ Let X be a r.v

$$p_X(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases} \qquad X = \begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$$

→ Mean : $E[X] = \sum x \, p_X(x)$

$$= p + (1-p)*0 = p$$

$$\Rightarrow \boxed{E[X] = p}$$

→ Variance : $\operatorname{Var}[X] = E[X^2] - (E[X])^2$

$$= \sum x^2 p_X(x) - \left(\sum x \, p_X(x)\right)^2$$

$$= (p+0) - (p)^2 = p - p^2 = p(1-p)$$

$$\boxed{\operatorname{Var}[X] = p(1-p)}$$

MEAN AND VARIANCE OF POISSON R.V

→ $p_X(k) = \dfrac{e^{-\lambda} \lambda^k}{k!}$ $\quad k = 0, 1, 2 \ldots$

→ $E[X] = \lambda e^{-\lambda} \left(1 + \lambda + \dfrac{\lambda^2}{2!} + \ldots \right)$

$$= \lambda \cancel{e^{-\lambda}} \cancel{e^{\lambda}} = \lambda$$

$$\Rightarrow \quad \boxed{E[X] = \lambda}$$

→ Variance : $E\left[(X - E(X))^2\right] = E\left[(X-\lambda)^2\right] = E[X^2] - E[X]^2$

$$= \sum \frac{k^2 e^{-\lambda} \lambda^k}{k!} - \lambda^2$$

$$= e^{-\lambda} \cdot \sum \frac{k^2 \lambda^k}{k!} - \lambda^2$$

$$= \cancel{e^{-\lambda}} \left(\lambda \cancel{e^{\lambda}} (\lambda - 1) \right) - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

$$e^x = \sum \frac{x^n}{n!}$$

$$e^x = \sum \frac{n x^{n-1}}{n!}$$

$$x e^x = \sum \frac{n x^n}{n!}$$

$$\Rightarrow \quad \boxed{Var(X) = \lambda}$$

MA225 (29/1/24).

Mean & Variance of Bernoulli R.V.

PMF: $\begin{cases} p & x=1 \\ 1-p & x=0. \end{cases}$

mean $= 0(1-p) + 1(p) = p.$

variance $= (1-p)^2 \cdot p + p^2(1-p)$
$= p(1-p)$

Mean & Variance of Poisson R.V.

$p_X(k) = \dfrac{e^{-\lambda}\lambda^k}{k!} \quad k = 0,1,2\ldots$

mean $= \sum_{k=1} \dfrac{e^{-\lambda}\lambda^k}{(k-1)!} = \lambda$

variance $= \cancel{\sum e^{-\lambda}\lambda^{k-\lambda}} \quad E(X^2) - [E(X)]^2$

$= \sum \dfrac{e^{-\lambda}\lambda^{2k}}{(2k-1)!} - \lambda^2 = \boxed{\lambda e^{-\lambda}\sin(\lambda) - \lambda^2}$

$\boxed{\text{variance} \neq \cancel{e^{\lambda} - \lambda^2}}$

Joint PMF of multiple random variables.

Consider two discrete R.V. X & Y associated with same experiment
Then joint PMF of X & Y is

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$

Ex- $X = 1,2,3$
$Y = 1,2,3$

$$p_{X,Y}(x,y) = \begin{cases} 1/9 & x=1, y=1 \\ 1/9 & x=2, y=1 \\ \vdots & \\ 1/9 & x=3, y=3 \end{cases}$$

~~PMF of X~~

$$\boxed{\cancel{PMF(X=x) = \sum_{Y} P(X=x, Y=y)}} \qquad \cancel{PMF(Y=y) = \sum_{X} P(X=x, Y=y)}.$$

$$\boxed{p_X(x) = \sum_{y} p_{X,Y}(x,y) \qquad p_Y(y) = \sum_{x} p_{X,Y}(x,y)}$$

Function of multiple random Variable.

Let $Z = g(X, Y)$

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

Example. $X = 1, 2, 3, 4$

$Y = 1, 2, 3, 4.$

$y \backslash x$	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

Find the PMF of $Z = X + 2Y$.

$$PMF_Z. \begin{cases} 1/20 + 1/20 \quad Z = 3 \quad (1,1) \; (2,1) \\ 1/20 + 1/20 \quad Z = 5 \quad (1,2) \; (3,1). \\ \vdots \end{cases}$$

$Z = 12$

$E[Y] = a\,E[X] + b$ — (2) provide Y is linear func^n of X.

example 8. consider the previous example of joint PMF given to us :-

$x = 1, 2, 3, 4$
$y = 1, 2, 3, 4$

$y \backslash x$	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

find the PMF of $Z = X + 2Y$? also find $E[Z]$

possible values of $z = 3, 4, 5, \ldots 12$

PMF of z $\quad p_Z(z) = \begin{cases} 1/20 & z = 3 \\ 1/20 & z = 4 \\ 2/20 & z = 5 \\ 2/20 & z = 6 \\ \vdots & z = 7 \\ \vdots & \end{cases}$

~~$E[Z] = E[X] + 2E[Y]$~~ $\qquad E(g(x)) = g(x)\,p_X(x)$

can we directly calculate the $E[Z]$ without calculating PMF of z ? yess

$E[Z] = E[X] + 2E[Y]$

we know $\quad p_X(x) = \begin{cases} 3/20 & x = 1 \\ 6/20 & x = 2 \\ 8/20 & x = 3 \\ 3/20 & x = 4 \end{cases}$

$$E[X] = \frac{3}{20} + \frac{12}{20} + \frac{24}{20} + \frac{12}{20} = \frac{51}{20}$$

$E[Y] = \frac{50}{20}$

$E[Z] = \frac{51}{20} + 2 \times \frac{50}{20} = \frac{151}{20}$

In general :- $X_1, \ldots X_n$ are random variable

$E[a_1 X_1 + a_2 X_2 + \ldots a_n X_n]$

$= a_1 E[X_1] + a_2 E[X_2] + \ldots + a_n E[X_n]$

where $a_1, a_2, \ldots a_n$ are constants.

Mean of binomial

Example :- Your CDA 101 class has 300 students, each student has probability 1/3 of getting an 'A', independent of any other student. Let X be the random variable ~~depend~~ defined as

$X =$ number of students that get an 'A'.

find $E[X]$.

$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$] tedious to solve

Let $X_1, X_2, \ldots X_{300}$ represents the students $1, 2, \ldots 300$ respectively. Define

$$X_i = \begin{cases} 1 & \text{getting A grade} \\ 0 & \text{otherwise} \end{cases}$$

Each X_i is a bernoulli random variable.

$$E[X_i] = p = 1/3$$

Illustration :- total grade = $X_1 + X_2 + \dots + X_{10}$

$= 1+0+0+1+0+0+0+1+1+0 = 4$

We have

$$X = X_1 + X_2 + \dots + X_{300}$$

Since X is the number of success in 300 independent trials, it is a binomial random variable

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_{300}]$$

$$= 300 \times \frac{1}{3} = 100$$

$$\boxed{E[X] = np}$$ Binomial random variable.

<u>Conditioning</u> Let X and Y be two random variables associated with same experiment. The conditional probability

$$p_{X|Y}(x|y) = P(X=x \mid Y=y) \quad -(1)$$

From conditional probab. definition

$$p_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\boxed{p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}}$$

Ques :- Prof. Anuj each of his student's question incorrectly with prob. 1/4 independent of other question.

Anuj may be asked $\xrightarrow{1/3}$ 0 question

Anuj may be asked $\xrightarrow{1/3}$ 1 question

Anuj may be asked $\xrightarrow{1/3}$ 2 question

X = number of question he is asked

Y = " " " he answered incorrectly

Joint PMF of X & Y

Possible values of $X = 0, 1, 2$

Possible values of $Y = 0, 1, 2$

CONDITIONAL EXPECTATION:

→ Let X & Y be 2 R.V with the same Exp. then

$$\boxed{E[X|Y=y] = \sum_x x \, p_{X|Y}(x|y)}$$

Recap: If $A_1, A_2, \dots, A_n$ be disjoint events that form a partition of the sample space with $P(A_i) > 0 \ \forall i$, then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i]}$$

$\Omega = \{HH, HT, TH, TT\}$

$A_1 = \{HH\}$

$A_2 = \{HT, TH\}$

$A_3 = \{TT\}$

MEAN OF GEOMETRIC R.V:

Example: Mr. Pawan is writing a code for some software. The Probability that it works correctly is P (independent of Previous attempts)

Let $X =$ no. of attempts until the code works correctly

$\Rightarrow X$ is a Geometric R.V with PMF

$$P(X=k) = p_X(k) = (1-p)^{k-1}p \qquad k=1,2\ldots\ldots$$

→ The mean of the r.v is X

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p \qquad k=1,2,3\ldots\ldots$$

→ The mean of r.v X

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p \longrightarrow p + 2(1-p)p + 3(1-p)^2p + \ldots\ldots$$

(which is difficult to evaluate)

$$p\left(1 + 2(1-p) + 3(1-p)^2 + \ldots\right)$$

$$= p\left(\frac{a}{1-r}\right)^1$$

$$= p\left(\frac{1}{1-(1-p)}\right)^1$$

$$= +p\left(\frac{+1}{1-(1-p)}\right)^2 = p\left(\frac{1}{p}\right)^2 = \frac{1}{p}$$

→ Define:

$A_1 = \{X=1\}$

$= \{$first attempt is a success$\}$

$A_2 = \{X>1\}$

$= \{$first attempt is a failure$\}$

from previous result we can write

$$E[X] = P(A_1) \cdot E[X|A_1] + P(A_2)E[X|A_2] \quad \text{——} (1)$$

$$= P(X=1)E[X|X=1] + P(X>1)E[X|X>1] \quad \text{——} (2)$$

$$= \underbrace{pE[X|X=1]}_{1^{st}\text{ attempt success}} + \underbrace{(1-p)E[X|X>1]}_{1^{st}\text{ attempt is failure}} \quad \text{——} (3)$$

$E[X|X=1] = ?$

→ If the first try is successful we have $X=1$

$$E[X|X=1] = 1 \quad \text{—— } ⊛$$

→ If the 1^{st} try fails $(X>1)$ we have wasted one try & we are back where we started. So the expected number of remaining tries is $E[X]$ &

$$E[X|X>1] = 1 + E[X] \quad \text{—— } ⊛$$

→ from eq. (3)

$$E[X] = p\,E[X|X=1] + (1-p)\,E[X|X>1] \quad \text{—— (4)}$$

$$= p \cdot 1 + (1-p)\,(1 + E[X])$$

$$= p \cdot 1 + (1-p)\,(1+E[X])$$

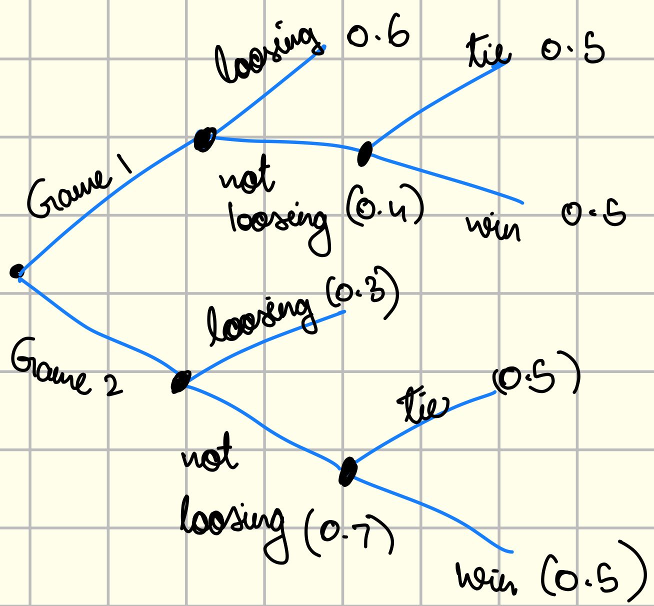
$$E[X] = p + (1+E[X]) - p(1+E[X])$$

$$\Rightarrow \boxed{E[X] = \frac{1}{p}} \quad \text{—— (5)}$$

Example: The MIT soccer teams has 2 games scheduled for one weekend it has a 0.4 probability of not loosing 1^{st} game & 0.7 prob. of not loosing the 2^{nd} game independent of 1^{st}. If it does not lose a particular game, it is equal to win or tie, independent of what happens in other games. The team will recieve 2 points for a win, 1 for a tie & 0 for a loss. Find the PMF of r.v X where X is the number of points that the team earns over the weekend

Sol: $X = 0, 1, 2, 3, 4$: no. of points the team earns over a weekend

PMF : $P(X = 0, 1, 2, 3, 4) = ?$



$P(X=0)$ = loosing both the games

$= 0.6 \times 0.3 = 0.18$

$P(X=1) = 0.6 \times 0.7 \times 0.5 +$
$0.3 \times 0.4 \times 0.5$
$= 0.27$

$P(X=2) = 0.6 \times 0.7 \times 0.5 +$

$P(X=3) = 0.4 \times 0.5 \times 0.7 \times 0.5 +$
$0.7 \times 0.5 \times 0.4 \times 0.5$
$= 0.14$

$0.3 \times 0.4 \times 0.8 +$
$0.4 \times 0.5 \times 0.7 \times 0.5$
$= 0.34$

$P(X=4) = 0.7 \times 0.5 \times 0.4 \times 0.5 = 0.07$

$P(X>4) = 0$

$$P_X(x) = \begin{cases} 0.18 & x=0 \\ 0.27 & x=1 \\ 0.34 & x=2 \\ 0.14 & x=3 \\ 0.07 & x=4 \\ 0 & \text{otherwise} \end{cases}$$

Question:

Fischer & Spassky play chess match in which the first player to win a game wins the match. After 10 successive draws, the match is declared drawn. Each game is won by Fischer with prob. 0.4, is won by Spassky with probability 0.3 & is a draw with 0.3 probability.

① What is the prob. that Fischer wins the match?

$$P = 0.4 + 0.3 \times 0.4 + (0.3)^2 \times 0.4 + (0.3)^3 \times 0.4 + \dots + (0.3)^9 \times 0.4$$

$$= 0.4 \times \frac{(1-0.3^{10})}{(1-0.3)} = 0.57$$

② what is the PMF of duration of match ?

X : no. of games played $\Rightarrow$ Possible values of $X =$ 1, 2, ..., 10

$$p_X(x) = \begin{cases} 0.4 + 0.3 & x = 1 \\ 0.3 \times 0.4 + 0.3 \times 0.3 & x = 2 \\ (0.3)^2 \times 0.4 + (0.3)^2 \times 0.3 & x = 3 \\ \vdots & \vdots \\ (0.3)^9 \times 0.4 + (0.3)^9 \times 0.3 + (0.3)^{10} & x = 10 \\ 0 & \text{otherwise} \end{cases}$$

INDEPENDANCE OF RANDOM VARIABLE:

→ Two r.v are independent if,

$$\boxed{\underbrace{p_{X,Y}(x,y)}_{\text{Joint PMF}} = p_X(x)\, p_Y(y) \qquad \forall\, x, y}$$

Result: If X & Y are Independent then

$$\boxed{E[XY] = E[X] \cdot E[Y]}$$

Proof: $E[XY] = \sum_x \sum_y xy\, p_{XY}(x,y)$

$= \sum_x \sum_y xy\, p_X(x) \cdot p_Y(y)$

$$= \sum_x x\, p_X(x) \sum_y y\, p_Y(y)$$ → Check when we can seperate the $\Sigma\Sigma$ symbols.

$$= E[X] \cdot E[Y]$$

$$\Rightarrow E[XY] = E[X]E[Y]$$

Result: If $X_1, X_2, \ldots\ldots X_n$ are independent r.v then

$$var(X_1 + X_2 + \cdots + X_n) = var(X_1) + var(X_2) + \cdots + var(X_n)$$

→ But In general if X_i's are not independent then,

$$var(X_1 + X_2 + \cdots + X_n) \neq var(X_1) + var(X_2) + \cdots + var(X_n)$$

Remark: Take 2 independent r.v X & Y.

$$\boxed{var(X+Y) = var(X) + var(Y)} \quad (5)$$

Q) Prove that $E[X^2] \geq (E[X])^2$ when do we have equality

$x^2 \geq x$

$x^2 p(x) \geq x\, p(x)$

$\sum x^2 p(x) \geq \sum x\, p(x)$

$E[X^2] \geq E[X]$

$var(X) \geq 0$

$E[(X - E[X])^2] \geq 0$

$E[X^2] - (E[X])^2 \geq 0$

$E[X^2] \geq (E[X])^2$

→ $E[X^2] = E[X]$ when $var(X) = 0$

Q) A coin is biased such that a head is 3 times as likely to occur as a tail. Find the expected no. of tails when this coin is tossed twice.

$P(H) = 3P(T) \Rightarrow P(T) = \frac{1}{4}$

$P(H) = \frac{3}{4}$

X: no. of tails
: 0, 1, 2

$\rightarrow E[X] = 0 \times \frac{9}{16} + 1 \times 2 \times \frac{1}{4} \times \frac{3}{4} + 2 \times \frac{1}{16}$

$2 \times \frac{1}{2} \times \frac{1}{2} + \frac{2 \times 1}{4}$

$= 2$

$= \frac{8}{16} = \frac{1}{2}$

X: no. of heads.

$\rightarrow E[X] = 0 \times \frac{9}{16} + 1 \times \cancel{2} \times \frac{1}{\cancel{4}_{2}} \times \frac{3}{4} + \cancel{2} \times \frac{9}{\cancel{16}_{8}}$

$= \frac{12}{8} = \frac{3}{2}$

Q) Suppose a antique jewellery dealer is interested in purchasing a gold necklass for which the Prob. are 0.22, 0.36, 0.28 & 0.14 respectively. that she will be able to sell it for profit of \$250, sell it for profit of \$150 sell it for break even, or sell it for a loss of \$150 what is the Expected value?

$$E[x] = 250\times 0.22 + 150\times 0.36 + 0\times 0.28 + (-150)\times 0.14$$

Q) A lot containing 7 components is sampled by a quality inspector this lot contains 4 good components of 3 defective components A sample of 3 is taken by the inspector. Find the expected value of no. of good components in this sample.

$X = 0, 1, 2, 3$

$$E[x] = 1\times \frac{{}^4C_1 \times {}^3C_2}{{}^7C_3} + 2\times \frac{{}^4C_2 \times {}^3C_1}{{}^7C_3} + 3\times \frac{{}^4C_3\times {}^3C_0}{{}^7C_3}$$

Q) An Industrial process manufactures items that can be classified as either defective (OR) not defective. The probability that item is defective is 0.1. An Experiment is conducted in which 5 items are drawn randomly from the process let X be the no. of defective in the sample of 5. what is the PMF of X

not defective $\leftarrow p = 0.9$ $\quad (1-p) = 0.1 \rightarrow$ defective

$X = 0, 1, 2, 3, 4, 5$

$$p_X(x) = \begin{cases} {}^5C_0\,(0.9)^5 & x = 0 \\ {}^5C_1\,(0.9)^4\,(0.1) & x = 1 \\ {}^5C_2\,(0.9)^3\,(0.1)^2 & x = 2 \\ {}^5C_3\,(0.9)^2\,(0.1)^3 & x = 3 \\ {}^5C_4\,(0.9)^1\,(0.1)^4 & x = 4 \\ {}^5C_5\,(0.1)^5 & x = 5 \end{cases}$$

CONTINUOUS RANDOM VARIABLE:

→ A random variable X is called continuous if there is a non-negative function f_X such that

$$\boxed{P(\underset{\substack{\downarrow \\ \text{subset of} \\ \text{real line}}}{X \in B}) = \int_B f_X(x)\,dx}$$

$$\boxed{P(a \le X \le b) = \int_a^b f_X(x)\,dx}$$ ⊛

→ The function $f_X(x)$ is called the probability density function (PDF) of X.

$$P(1 \le X \le 2) = \int_1^2 f_X(x)\,dx$$

$$P(2 \le X \le 3) = \int_2^3 f_X(x)\,dx$$

Ques 13:- An individual process manufactures items that can be classified as either defective or not defective. The probability that item is defective is 0.1. An experiment is conducted in which 5 items are drawn randomly from the process. Let X be the no. of defectives in this sample of 5. What is PMF of X?

$$p_X(x) = \begin{cases} {}^5C_0\, p^0 (1-p)^5 & x=0 \\ {}^5C_1\, p^1 (1-p)^4 & x=1 \\ {}^5C_2\, p^2 (1-p)^3 & x=2 \\ {}^5C_3\, p^3 (1-p)^2 & x=3 \\ {}^5C_4\, p^4 (1-p) & x=4 \\ {}^5C_5\, p^5 (1-p)^0 & x=5 \end{cases} \qquad \boxed{p = 0.1}$$

continuous random variables

A random variable X is called continuous if there is a non-negative function f_X such that

$$P(X \in \underbrace{B}_{\text{subset of real line}}) = \int_B f_X(x)\, dx$$

or

$$P(a \le X \le b) = \int_a^b f_X(x)\, dx$$

The function $f_X(x)$ is called the probability density function (PDF) of X.

$$\begin{cases} P(1 \le X \le 2) = \int_1^2 f_X(x)\, dx \\ P(2 \le X \le 3) = \int_2^3 f_X(x)\, dx \end{cases}$$

$P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b)$

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1} = P(-\infty < X < \infty)$$

This is known as normalization property

Uniform random variable

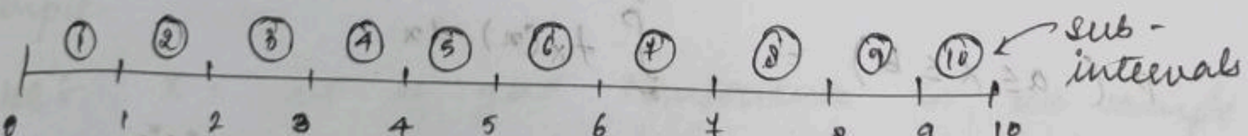
Consider a random variable X that takes values in an interval $[a,b]$. assuming that any two sub-intervals of same length have same probability. Such random variables are called uniform random variable.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

i) non-negative

ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_{a}^{b} \frac{1}{b-a}\,dx + \int_{b}^{\infty} 0\,dx = 1$

iii) suppose X takes values in $[0,10]$ $a=0$, $b=10$



$P(5 \le X \le 6) = 1/10$

$P(7 \le X \le 10) = 3/10$

$= \int_{5}^{6} f_X(x)\,dx = \int_{5}^{6} \frac{1}{10-0}\,dx = \frac{1}{10}$

$$\int_{7}^{10} f_X(x)\,dx = \int_{7}^{10} \frac{1}{10-0}\,dx = \frac{3}{10}.$$

The function $f_X(x)$ given in eq^n is PDF for random variable X.

<u>A PDF can take arbitrary large values -</u>

Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

non-ve

normalization property

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{0}^{1} \frac{1}{2\sqrt{x}}\,dx = 1$$

Note that $f_X(x)$ can be infinitely large.

<u>Expectation</u>: The expectation of a continuous random variable X is defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \quad \text{—— (1)}$$

If $g(X)$ is a func^n of continuous random variable X then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \quad \text{—— *}$$

for example $g(X) = x^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \quad \text{—— (2)}$$

$$Var(X) = E[X^2] - (E[X])^2 \quad \text{—(3)}$$

$$Var(X) = E[\underbrace{(X - E[X])^2}_{g(X)}]$$

$$Var(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx \quad \text{—(4)}$$

$$E[X] = \int_a^b \frac{x}{b-a}\, dx = \left.\frac{x^2}{2(b-a)}\right]_a^b = \frac{a+b}{2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{x^3}{3(b-a)} = \frac{b^2+a^2+ab}{3}$$

$$Var(X) = \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{(b-a)^2}{12}$$

Exponential Random Variable

An exponential random variable has PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{—(5)}$$

where λ is +ve

1. Non - ve

2. $\int_{-\infty}^{\infty} f_X(x)\, dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big]_{0}^{\infty} = 1$

A exponential random variable is used to model the amount of time until an incident of interest takes place. for eg :-

(1) Breaking down of some experiment equipment

(2) Burning out of a light bulb

(3) An accident occuring.

EXPONENTIAL RANDOM VARIABLE:

→ An Exponential R.V has PDF of the form

$$\boxed{f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}} \quad \text{—— (5)}$$

where λ is +ve

→ An exponential r.v is used to model the amount of time until an incident of interest takes place. For example:

1. Breaking down of some equipment
2. Burning out of a light bulb
3. An accident occuring.

→ In an exponential r.v.

$$P(X \geq a) = \int_a^{\infty} f_X(x)\,dx = \int_a^{\infty} \lambda e^{-\lambda x}\,dx = \frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}}\Big|_a^{\infty} = e^{-\lambda a}$$

$$\Rightarrow \boxed{P(X \geq a) = e^{-\lambda a}}$$

→ The mean & variance of X:

$$\int uv = u\int v - \int u' \int v$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \left[\frac{x e^{-\lambda x}}{-\cancel{\lambda}} - \frac{e^{-\lambda x}}{\lambda^{\cancel{2}}}\right]_0^{\infty} \cancel{\lambda} = \frac{1}{\lambda}$$

Var (X) =

$$E[(X-E[X])^2] = E[X^2] - (E[X])^2 = \int x^2 \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2} = \frac{2\cancel{\lambda}}{\lambda^2}\left(\frac{1}{\cancel{\lambda}}\right) - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

→ Q) The time until a small meteorite first lands anywhere in Sahara desert is modeled as an exponential random variable with mean of 10 days. The time is currently mid-night. What is the prob. that a meteorite first lands some time between 6 AM to 6 PM of the first day?

Sol:

$$E[X] = \frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$$

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{1/4}^{3/4} \lambda e^{-\lambda x} dx = \frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}}\Bigg|_{1/4}^{3/4} = (-1)\left(e^{-\frac{1}{10}\times\frac{3}{4}} - e^{-\frac{1}{10}\times\frac{1}{4}}\right)$$

$$= 0.048$$

CUMULATIVE DISTRIBUTION FUNCTION (CDF):

→ The CDF of a r.v X is denoted by F_X & provides the probability $P(X \le x)$ for every x, we have

$$F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} p_X(k) & ; \text{ if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & ; \text{ if } X \text{ is continuous} \end{cases}$$

→ Example: X is discrete if $X = 1, 2, 3, \dots\dots, 10$

$$F_X(5) = P(X \le 5) = p_X(1) + p_X(2) + p_X(3) + p_X(4) + p_X(5)$$

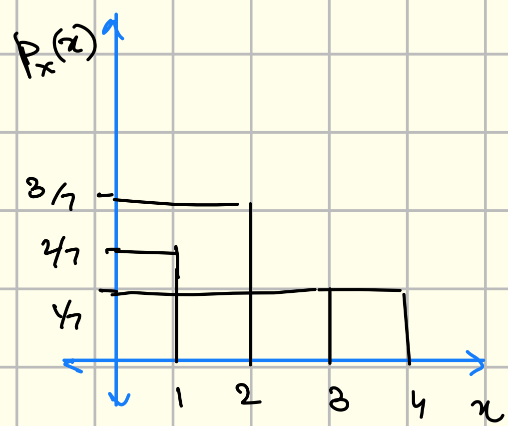
If X is continuous,

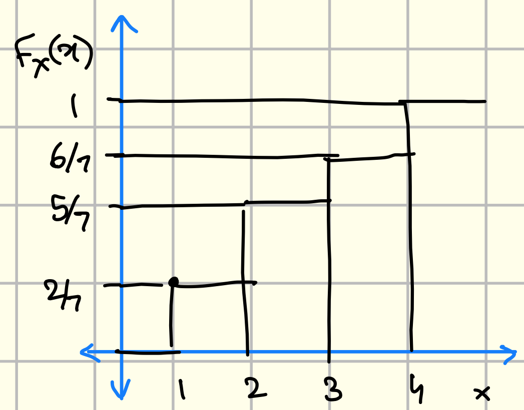
$$F_X(5) = P(X \le 5) = \int_{-\infty}^{5} f_X(t)\,dt$$

→ Let X be a discrete r.v. Let $X = 1, 2, 3, 4$

is it a PMF?

$$p_X(x) = \begin{cases} \frac{2}{7} & x = 1 \\ \frac{3}{7} & x = 2 \\ \frac{1}{7} & x = 3 \\ \frac{1}{7} & x = 4 \end{cases}$$



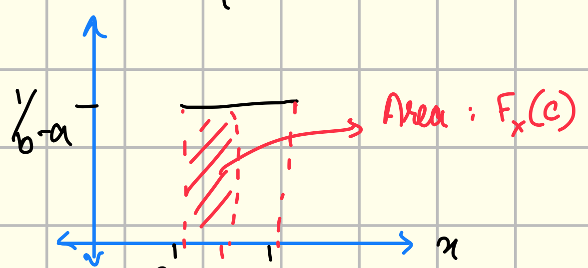


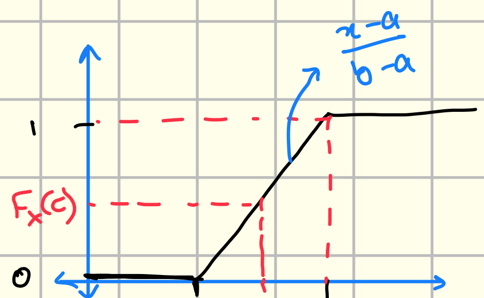
→ Let X be a continuous (Uniform) r.v with PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$x < a$
$x > b$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{b-a} = \frac{x}{b-a}\Big|_{-\infty}^{x}$$





$$\int_a^x \frac{x}{b-a} = \frac{x-a}{b-a}$$

a c b a c b

PROPERTIES OF CDF:

① If $x \leq y$ $F_X(x) \leq F_Y(y)$

② $F_X(x)$ tends to 0 as $x \to -\infty$ and tends to 1 as $x \to \infty$

③ If X is discrete, then $F_X(x)$ is a piecewise constant function

④ If X is continuous, then $F_X(x)$ is a continuous function

⑤ If X is continuous then

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt} \iff \boxed{f_X(t) = \frac{dF_X(x)}{dx}}$$

→ PDF → CDF

CDF → PDF

→ DIY: CDF ⟶ PMF

NORMAL RANDOM VARIABLE:

→ A continuous r.v X is said to be normal (or Gaussian) if it has a PDF of the form

$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \; -\infty < x < \infty}$$

→ where μ & σ are 2 scalar parameters with $\sigma > 0$

$$\int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad t = +\frac{(x-\mu)^2}{2\sigma^2} \Rightarrow (x-\mu) = \sqrt{2\sigma^2 t} = \sqrt{2t}\,\sigma$$

$$dt = +\frac{2(x-\mu)}{2\sigma^2} dx$$

$$\int \frac{+1}{\cancel{\sigma}\sqrt{2\pi}} e^{-t} \frac{\cancel{2}\sigma^{\cancel{2}}}{\cancel{2}(x-\mu)} dt \qquad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \int \frac{1}{\sqrt{2\pi}} \times \frac{\cancel{\sigma}}{\sqrt{2t}\,\cancel{\sigma}} e^{-t} dt \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt = \frac{\cancel{2}}{\cancel{2}\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \frac{1}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} = 1$$

$$\rightarrow E[x] = \int \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{2\sigma^2}{\sigma\sqrt{2\pi}(-2)} \int \left[\frac{-2(x-\mu)}{2\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} - \frac{2\mu}{2\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right] dx$$

$$= \frac{2\sigma^2}{2\sqrt{2\pi}(-2)} \int \frac{-2(x-\mu)}{2\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \frac{2\sigma^2}{\sigma\sqrt{2\pi}(-2)} \int \frac{-2\mu}{2\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \cancel{\frac{2\sigma^2}{2\sqrt{2\pi}(-2)} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big|_{-\infty}^{\infty}} \to 0 + \frac{\cancel{2\sigma^2}(\cancel{-2}\mu)}{(\cancel{-2})(\cancel{2\sigma^2})} \underbrace{\int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{1}$$

$$= 0 + \mu = \mu$$

$$\Rightarrow E[x] = \mu$$