Indian Institute of Technology Patna
Department of Mathematics
MA225: Probability Theory and Random Process
B.Tech. 2nd year

Tutorial Sheet-1

1. (i) Let $P(A) = 1/3, P(B) = 1/4$, can events A and B be disjoint? Explain.
(ii) Show that if $A \cap B = \{\phi\}$, then $P(A) \leq P(\bar{B})$.

2. Let two events A and B be such that $B \subset A$. Then show that (i)$P(A \cap \bar{B}) = P(A) - P(B)$ (ii) $P(B) \leq P(A)$.

3. Show that(a) $\overline{\bar{A} \cup \bar{B}} \cup \overline{\bar{A} \cup B} = A$
(b) $(A \cup B) \cap \overline{(A \cap B)} = (A \cap \bar{B}) \cup (B \cap \bar{A})$.

4. Show that If $A = \{2 \leq x \leq 5\}$ and $B = \{3 \leq x \leq 6\}$, find $(A \cup B), (A \cap B)$ and $(A \cup B) \cap \overline{(A \cap B)}$.

5. Show that (a) If $P(A) = P(B) = P(A \cap B)$, then $P((A \cap \bar{B}) \cup (B \cap \bar{A})) = 0$; (b)$P(A) = P(B) = 1$, then $P(A \cap B) = 1$.

6. Event A and B are such that $P(A \cup B) = \frac{3}{4}, P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{2}{3}$, show that $P(B) = \frac{2}{3}$ and $P(A \cap \bar{B}) = \frac{1}{12}$.

7. Each coefficient in equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have (a) Real Root (b) Complex Root.

8. Prove the Bonferroni inequality:
For some arbitrary events $A_1, A_2, \ldots, A_n$ we have $P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$.

9. Suppose that there are n students in a class room and assume that $n \leq 365$. Also let no student has birthday on 29th February. What is the probability that at least two students share the same birthday.

10. Suppose that the population of a certain city is 40% male and 60% female. Suppose also that 50% of the males and 30% of the females smoke. Find the probability that a smoker is male.

11. Let two fair coin are tossed once. (i) Find the probability that both coins show head given that the first shows a head. (ii) What is the probability that the both are heads given that at least on of them is a head.

12. Find the minimum number of times a die has to be thrown such that the probability of no six is less than $\frac{1}{2}$.

13. Why does it pay to bet consistently on seeing 6 at least once in 4 throws of a die, but not seeing a double six at least once in 24 throws with two die?

14. A problem is given to three students A, B and C whose chance of solving it are $1/2, 3/4$ and $1/4$ respectively. What is the probability that the problem is solved if all of them try independently?

15. Consider two boxes, one containing 1 black and 1 white marble, the other, 2 black and 1 white marble. A box is selected at random and a marble is drawn at random from the selected box. What is the probability that the marble is black?

16. Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that: (i) none of the men selects his own hat (ii) exactly k of the men select their own hats? (iii) Evaluate part (i) when $N = 3$ and $N = 4$ (iv) Discuss the case when N approaches infinity.

17. A box contains m white balls and n black balls. Balls are drawn at random one at time without replacement. Find the probability of encountering a white ball by the kth draw.

18. Two players A and B draw balls one at time alternatively from a box containing m white and n black balls. Suppose the player who picks the first white ball wins the game. What is the probability that the player who starts the game will win?

19. A box contain n identical balls numbered 1 through n. Suppose k balls are drawn in succession. (i) What is the probability that m is the largest number drawn ? (ii) What is the probability that the largest number drawn is less than or equal to m ?

20. A box contains m white and n black balls. Suppose k balls are drawn. Find the probability of drawing at least one white ball ?

Tutorial sheet 1.

1)

(i) $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$

Since A and B are the only events in S.

we have, ~~A∩f~~ $S = A \cup B$

If $A \cap B = \phi$, then.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B)$$
$$= \frac{1}{3} + \frac{1}{4}$$
$$= \frac{7}{12}$$

but $P(A \cup B) = P(S) = 1$.

So. $A \cap B \neq \phi$.

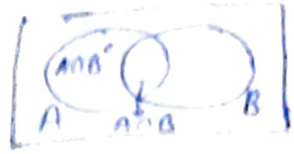
(ii) Given that $A \cap B = \phi$

~~Also~~, Then. $P(A \cap B) = 0$

$$\Rightarrow P((A \cap B)^c) = 1$$
$$\Rightarrow P(A^c \cup B^c) = 1$$
$$\Rightarrow P(A^c) + P(B^c) - P(A^c \cap B^c) = 1$$
$$\Rightarrow P(B^c) = 1 - P(A^c) + P(A^c \cap B^c)$$
$$\Rightarrow P(B^c) = P(A) + P(A^c \cap B^c)$$
$$\therefore P(B^c) \geq P(A).$$

2)

i) $A = (A \cap B^c) \cup (A \cap B)$



Then $P(A) = P(A \cap B^c) + P(A \cap B)$ $\quad$ [$\because (A \cap B^c)$ and $(A \cap B)$ are disjoint]

$\Rightarrow P(A) = P(A \cap B^c) + P(B)$ $\quad$ [$\because B \subset A \Rightarrow A \cap B = B$]

$\Rightarrow P(A \cap B^c) = P(A) - P(B)$

ii) From (i) we have

$P(A) - P(B) = P(A \cap B^c) \geq 0$

$\Rightarrow P(A) \geq P(B)$

3)

a) $(A^c \cup B)^c \cup (A^c \cup B)^c$

$= ((A \cap B)^c)^c \cup ((A^c)^c \cap B^c)$

$= (A \cap B) \cup (A \cap B^c)$

$= A$

b) $(A \cup B) \cap (A \cap B)^c$

$= (A \cup B) \cap (A^c \cup B^c)$

$= \{(A \cup B) \cap A^c\} \cup ((A \cup B) \cap B^c)$

$= (A^c \cap B) \cup (A \cap B^c)$

$= (A \cap B^c) \cup (B \cap A^c)$

4) $A = \{x : 2 \le x \le 5\}$ $B = \{x : 3 \le x \le 6\}$

$A \cup B = \{x : 2 \le x \le 6\}$

$A \cap B = \{x : 3 \le x \le 5\}$

$$(A \cup B) \cap (A \cap B)^c = (A \cap B^c) \cup (B \cap A^c)$$
$$= \{x : \cancel{x}\ 2 \le x < 3\} \cup \{x : 5 < x \le 6\}$$
$$= \{x : x \in [2,3) \cup (5,6]\}$$

5)

a) $P(A) = P(B) = P(A \cap B)$

$P((A \cap B^c) \cup (B \cap A^c))$

$$= P(((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c))$$
$$= P((A \cup B) \cap (A^c \cup B^c))$$
$$= P(A \cup B) + P(A^c \cup B^c) - P((A \cup B) \cup (A^c \cup B^c))$$
$$\cancel{= P(A) + P(B) - P(A \cap B}$$
$$= P(A \cup B) + P(A^c) + P(B^c) - P(A^c \cap B^c) - P(S)$$
$$= P(A \cup B) + P(A^c) + P(B^c) - 1 + P(A \cup B) - P(S)$$
$$= P(A \cup B) + P(A^c) + P(B^c) - 2$$
$$= 2P(A \cup B) + P(A^c) + P(B^c) - 2$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$$= P(A)$$

$$\therefore P((A \cap B^c) \cup (B \cap A^c)) = 2P(A) + P(A^c) + P(B^c) - 2$$
$$= 2P(A) + 1 - P(A) + 1 - P(B) - 2$$
$$= 0$$

b) $P(A) = P(B) = 1$

Now, $A \subseteq A \cup B$

$\therefore P(A) \leq P(A \cup B)$

$\Rightarrow 1 \leq P(A \cup B) \leq 1$

$\therefore P(A \cup B) = 1.$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 1 = 1 + 1 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 1.$

6) $B \subset C$

$\Rightarrow B \cap A \subset C \cap A$

$\Rightarrow P(B \cap A) \leq P(C \cap A)$

$$\Rightarrow \frac{P(B \cap A)}{P(A)} \leq \frac{P(C \cap A)}{P(A)}$$

$\Rightarrow P(B|A) \leq P(C|A)$

7) $P(A \cup B) = \frac{3}{4}$

$P(A \cap B) = \frac{1}{4}$

$P(A^c) = \frac{2}{3}$

$\Rightarrow 1 - P(A) = \frac{2}{3}$

$\Rightarrow P(A) = \frac{1}{3}$

$\therefore P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$= \frac{3}{4} - \frac{1}{3} + \frac{1}{4}$

$= \frac{2}{3}$

$P(A \cap B^c) = P(A) - P(A \cap B)$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{1}{12}$

8) $ax^2 + bx + c = 0$ — (1)

The discriminant of the quadratic equation (1) is

$D = b^2 - 4ac$

Now, eq^n (1) has real roots if $D \geq 0$

and complex roots if $D < 0$

a) Since the coefficients of eq^n (1) ~~can~~ is be determined by throwing an ordinary die, then a, b and c must be from the set $\{1, 2, 3, 4, 5, 6\}$

Let A be the event that $D \geq 0$

and B " " " " $D < 0$

Now, for $D \geq 0$,

$b^2 \geq 4ac$

If $b = 1$, then $1 \geq 4ac$ So, ~~no such pair~~ there is no such outcome

If $b = 2$,

$\Rightarrow 4 \geq 4ac$

$\Rightarrow ac \leq 1$

Then the coefficients for which $D \geq 0$ are $\{(1,1,1)\}$

If $b = 3$,

$\Rightarrow 9 \geq 4ac$

Then the coefficients for which $D \geq 0$ are $\{(1,3,1), (2,3,1), (1,3,2)\}$

If $b = 4$,

$\Rightarrow 4 \geq ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,4,1), (2,4,1), (3,4,1), (4,4,1), (2,4,2), (1,4,4), (1,4,2), (1,4,3)\}$

If $b = 5$,

$\Rightarrow 25 \geq 4ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,5,1), (2,5,1), (3,5,1), (4,5,1), (5,5,1), (6,5,1), (2,5,2), (1,5,2), (1,5,3), (1,5,4), (1,5,5), (1,5,6), (2,5,3), (3,5,2)\}$

If $b=6$,

$\ 9 \geq ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,6,1), (2,6,1), (3,6,1), (4,6,1), (5,6,1), (6,6,1), (2,6,2), (1,6,2), (1,6,3), (1,6,4), (1,6,5), (1,6,6), (2,6,3), (2,6,4), (3,6,3), (3,6,2), (4,6,2)\}$

$\therefore P(A) = \dfrac{1+3+8+14+17}{6^3} = \dfrac{43}{216}$

(b) $P(B) = 1 - P(A)$

$= 1 - \dfrac{43}{216}$

$= \dfrac{173}{216}$

9) We use induction on n.

Base case for $n=2$ $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$

$\Rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

$= \sum_{i=1}^{2} P(A_i) - (2-1)$

Induction hyp. Assume that the statement is true for n.

Induction step.

$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \cap A_{n+1}) \geq P(A_1 \cap A_2 \cap \cdots \cap A_n) + P(A_{n+1}) - 1$

$\geq \sum_{i=1}^{n} P(A_i) - (n-1) - 1 + P(A_{n+1})$

$= \sum_{i=1}^{n+1} P(A_i) - \{(n+1) - 1\}$

$\therefore$ The statement is true for $n+1$.

$\therefore$ The statement is true.

10) Let A be the event that no students share the same birthday. Then A^c is our required event.

Now, $P(A) = \dfrac{\binom{365}{n}}{(365)^n}$ [$\because |S| = 365^n$ and no students have same birthday so A can occur in $\binom{365}{n}$ ways]

$\therefore P(A^c) = 1 - \dfrac{\binom{365}{n}}{(365)^n}$

11) Let M be the event that a person is male

F " " " " " " Female

and S " " " " " " Smoker.

Given that, $P(M) = \frac{40}{100} = \frac{2}{5}$

and $P(F) = \frac{60}{100} = \frac{3}{5}$

and $P(S|M) = \frac{50}{100} = \frac{1}{2}$

and $P(S|F) = \frac{30}{100} = \frac{3}{10}$

Now, $P(S) = P(S \cap M) + P(S \cap F)$

$= P(M) \cdot P(S|M) + P(F) \cdot P(S|F)$

$= \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{3}{10}$

$= \frac{19}{50}$

$$P(M|S) = \frac{P(S\cap M)}{P(S)}$$

$$= P(S|M) \cdot \frac{P(M)}{P(S)}$$

$$= \frac{1}{2} \cdot \frac{\cancel{2}}{5} \cdot \frac{\cancel{50}\,10}{19}$$

$$= \frac{10}{19}$$

12>

(i) Let A be the event that both coins show head.

B " " " " the first coin shows a head.

$\therefore A = \{(H,H)\}$

$B = \{(H,H), (H,T)\}$

$\therefore$ ~~[illegible]~~ $A\cap B = \{(H,H)\}$

$$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{\frac{1}{2^2}}{\frac{2}{2^2}} = \frac{1}{2}$$

(ii) Let C be the event that at least one of them is head.

Then $C = \{(H,T), (H,H), (T,H)\}$

$\therefore C\cap A = \{(H,H)\}$

$$\therefore P(A|C) = \frac{P(C\cap A)}{P(C)} = \frac{\frac{1}{4}}{3/4} = \frac{1}{3}$$

13) Let a die is trown n number of times.

Let A be the event that no ~~6~~ six occurs.

~~Since no six occurs~~ ~~then the number of ways event~~

~~A occurs~~

Now, event A occurs in 5^n number of ways.

$$\therefore P(A) = \frac{5^n}{6^n}$$

For, $P(A) < \frac{1}{2}$

we have, $\left(\frac{5}{6}\right)^n < \frac{1}{2}$

For, $n = 1$, $\frac{5}{6} > \frac{1}{2}$

for $n = 2$, $\frac{25}{36} > \frac{1}{2}$

for $n = 3$, $\frac{125}{216} > \frac{1}{2}$

for $n = 4$, $\frac{5^4}{6^4} < \frac{1}{2}$

So, the die has to be thrown 4 times, ~~[illegible]~~.

14) Let A be the event that 6 doesnot occur in 4 throws of a die. Our required event is A^c.

The total number of outcomes for throwing a die 4 times is $= 6^4$

The total number of ways event A occurs is $= 5^4$.

$\therefore P(A) = \frac{5^4}{6^4}$

$\therefore P(A^c) = 1 - \frac{5^4}{6^4} = 0.517$

Now let B be the event that no double six occurs in 24 throws with two die.

~~Now,~~ The total number of ~~ou~~ outcome for throwing two die 24 times is $= (6^2)^{24}$

the number of ways event B occurs is $= (6^2-1)^{24} = (35)^{24}$ ~~...~~

[The total number of outcomes for throwing two die is $6^2 = 36$ and double six occurs in only one of them. So no double six occurs in $(6^2-1) = 35$ ways]

$\therefore P(B) = \frac{(35)^{24}}{(36)^{24}}$

$\therefore P(B^c) = 1 - \frac{(35)^{24}}{(36)^{24}}$

$= 0.491$

So, $P(A^c) > P(B^c)$

$\therefore$ The chance of seeing 6 at least once in 4 throws of a die is higher than seeing a double six at least once in 24 throws ~~o~~ with two die.

$\therefore$ The former one is suitable ~~for~~ ~~to~~ to bet.

15) Let E_1 be the event that A solves the problem

E_2 " " " " B " " "

E_3 " " " " C " " "

$\therefore P(E_1) = \frac{1}{2}$ $P(E_2) = \frac{3}{4}$ and $P(E_3) = \frac{1}{4}$.

Now, it is given that the events E_1, E_2 and E_3 are independent

Then, $P(E_1 \cup E_2 \cup E_3) = 1 - P(E_1^c \cap E_2^c \cap E_3^c)$

$= 1 - P(E_1^c) \cdot P(E_2^c) \cdot P(E_3^c)$

$= 1 - (1 - \frac{1}{2}) \cdot (1 - \frac{3}{4}) (1 - \frac{1}{4})$

$= 1 - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4}$

$= 1 - \frac{3}{32}$

$= \frac{29}{32}$

$\boxed{\begin{array}{l} \text{If } E_1 \text{ \& } E_2 \text{ are independent events then} \\ P(E_1^c \cap E_2^c) = 1 - P(E_1 \cup E_2) \\ = 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2) \\ = (1 - P(E_1)) \cdot (1 - P(E_2)) \\ = P(E_1^c) \cdot P(E_2^c) \end{array}}$

16) Let A_1 and A_2 be the two boxes where A_1 contains 1 black and 1 white marble and A_2 contains 2 black and 1 white marble.

Let B be the event that the selected marble is black

and W " " " " " " " white.

Now, $P(A_1) = P(A_2) = \frac{1}{2}$.

Now, $P(B|A_1) = \frac{1}{2}$

and $P(B|A_2) = \frac{2}{3}$

$$\therefore \; P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3}$$

$$= \frac{7}{12}$$

17) i) Let E be the event that none of the men selects their own hat.

We use principle of inclusion - exclusion.

Let A_i be the event that i-th man selects his own hat.

then $|A_i| = (N-1)! \quad \forall \; 1 \leq i \leq N$

$|A_i \cap A_j| = (N-2)! \quad$ for $i \neq j$

$|A_1 \cap A_2 \cap \cdots \cap A_N| = 1.$

$$\therefore \; |E| = N! - \binom{N}{1} \cdot (N-1)! + \binom{N}{2} \cdot (N-2)! - \cdots + (-1)^N \, 1$$

$$= N! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^N \frac{1}{N!} \right)$$

$$\therefore \; P(E) = \frac{N! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^N \frac{1}{N!} \right)}{N!}$$

$$= \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^N \frac{1}{N!} \right)$$

$$= \sum_{i=2}^{N} (-1)^i \frac{1}{i!}$$

ii) Let E_2 be the event that exactly k of the men select their own hats.

$$\therefore |E_2| = \binom{N}{K} \cdot \left[(N-K)! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{(N-K)} \cdot \frac{1}{(N-K)!} \right) \right]$$

$$\therefore P(E_2) = \frac{\binom{N}{K} \cdot (N-K)! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{N-K} \cdot \frac{1}{(N-K)!} \right)}{N!}$$

$$= \frac{1}{K!} \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{N-K} \frac{1}{(N-K)!} \right)$$

$$= \frac{1}{K!} \sum_{i=2}^{N-K} (-1)^i \cdot \frac{1}{i!}$$

(iii)

for $N=3$,

$$P(E) = \sum_{i=2}^{3} (-1)^i \cdot \frac{1}{i!}$$

$$= \frac{1}{2!} - \frac{1}{3!}$$

$$= \frac{1}{3}$$

For $N=4$,

$$P(E) = \sum_{i=2}^{4} (-1)^i \frac{1}{i!}$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

$$= \frac{3}{8}$$

iv) We can ~~a~~ rewrite the expression $\sum_{i=2}^{N} (-1)^i \frac{1}{i!}$ as $\sum_{i=0}^{N} (-1)^i \frac{1}{i!}$

$$\therefore \quad P(E) = \sum_{i=0}^{N} (-1)^i \frac{1}{i!}$$

As $N \to \infty$, $\sum_{i=0}^{N} (-1)^i \frac{1}{i!} \to e^{-1}$

$$\therefore \quad P(E) \to e^{-1} \text{ as } N \to \infty$$

18) Let W_K be the event that encountering a white ball by the K.th draw. and B_i be the event of drawing i black balls followed by a white ball.

$$\therefore \quad W_K = B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_{K-1}$$

The events B_i, B_j for $i \neq j$ are mutually exclusive.

Now, $$P(B_0) = \frac{m}{m+n}$$

$$P(B_1) = \frac{n}{(m+n)} \cdot \frac{m}{(m+n-1)}$$

$$P(B_2) = \frac{n}{m+n} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{m}{(m+n-2)}$$

$\vdots$

$$P(B_{K-1}) = \frac{n}{(m+n)} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{(n-2)}{(m+n-2)} \cdots \frac{(n-K+2)}{(m+n-K+2)} \cdot \frac{m}{(m+n-K+1)}$$

$$\therefore \quad P(W_K) = P(B_0) + P(B_1) + \cdots + P(B_{K-1})$$

$$= \frac{m}{m+n} + \frac{n}{(m+n)} \cdot \frac{m}{(m+n-1)} + \cdots + \frac{n(n-1)\cdots(n-K+2) \cdot m}{(m+n)(m+n-1)\cdots(m+n-K+1)}$$

$$= \frac{n! \cdot m}{(m+n)!} \sum_{i=0}^{K-1} \frac{(m+n-(i+1))!}{(n-i)!}$$

19) Let us consider that player A starts the game.

Let W_i be the event that a white ball is drawn by A at i-th draw

Then $P(A \text{ wins}) = P(W_1) + P(W_2) + P(W_3) + \ldots$

Now, $P(W_1) = \frac{m}{m+n}$

For W_2, A draws a black ball, then B draws a black ball and then A draw a white ball.

$\therefore \; P(W_2) = \frac{n}{m+n} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{m}{(m+n-2)}$

$\therefore \; P(W_k) = \frac{n}{(m+n)} \cdot \frac{(n-1)}{(m+n-1)} \cdots \frac{n-2k+3}{m+n-2k+3} \cdot \frac{m}{m+n-2k+2}$

The process will terminate after all the balls are drawn from the box.

$\therefore \; P(A \text{ wins}) = \frac{m}{m+n} + \frac{n(n-1).m}{(m+n)(m+n-1)(m+n-2)} + \frac{n(n-1)(n-2).(n-3).m}{(m+n)(m+n-1)(m+n-2)(m+n-3)} + \ldots$

2 c)

i) If m is the largest number drawn then remaining $K-1$ balls are drawn from the set of $\{1, 2, \ldots, m-1\}$ balls.

$$\therefore P(m \text{ is the largest number}) = \frac{\binom{m-1}{K-1}}{\binom{n}{K}}$$

ii) Since the largest number drawn is ~~from~~ less or equal to m, the largest number must be from the set $\{1, 2, \ldots, m\}$. Therefore, the event ~~that~~ of ~~largest~~ selecting K balls such that largest number drawn is ~~same~~ less or equal to m is same as selecting K balls from the set $\{1, 2, \ldots, m\}$.

$$\therefore P(\text{largest number drawn is} \le m) = \frac{\binom{m}{K}}{\binom{n}{K}}$$

21) Let A be the event that ~~all the K balls~~ none of the K balls ~~are~~ ~~out~~ is white.

$\therefore$ The required event is A^c.

Now, $$P(A) = \frac{\binom{n}{K}}{\binom{m+n}{K}}$$

$$\therefore P(A^c) = 1 - \frac{\binom{n}{K}}{\binom{m+n}{K}}$$

22) Let W be the event that the ball drawn from A is white.

Now, consider the following events.

W_0 = No ~~zero~~ white balls is ~~are~~ transferred from B to A

W_1 = one " " " B to A

W_2 = two " balls are " B to A.

i)

$\therefore P(W) = P(W|W_0).P(W_0) + P(W|W_1) P(W_1) + P(W|W_2).P(W_2)$

Now, $P(W_0) = \frac{\binom{8}{2}}{\binom{12}{2}}$, $P(W_1) = \frac{\binom{4}{1}.\binom{8}{1}}{\binom{12}{2}}$, $P(W_2) = \frac{\binom{4}{2}}{\binom{12}{2}}$

and $P(W|W_0) = \frac{\binom{6}{1}}{\binom{13}{1}}$ $P(W|W_1) = \frac{\binom{7}{1}}{\binom{13}{1}}$ $P(W|W_2) = \frac{\binom{8}{1}}{\binom{13}{1}}$

$\therefore P(W) = \frac{6}{13}.\frac{8\times7}{12\times11} + \frac{7}{13}.\frac{4\times8\times2}{12\times11} + \frac{8}{13}.\frac{4\times3}{12\times11}$

$= \frac{20}{39}$

ii) Let E be the event that at least one white ball was transferred to A.

$\therefore E = W_1 \cup W_2$

$\therefore P(E|W) = P(W_1|W) + P(W_2|W)$ [$\because W_1$ and W_2 are mutually exclusive]

$= \frac{P(W|W_1).P(W_1) + P(W|W_2).P(W_2)}{P(W)}$

$= \frac{\frac{7}{13}.\frac{4\times8\times2}{12\times11} + \frac{8}{13}.\frac{4\times3}{12\times11}}{\frac{20}{39}} = \frac{34}{35}$

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Tutorial Sheet-2

1. Let A and B be two events. Then verify that following statements are equivalent.
(*i*) Events A and B are independent. (*ii*) Events A^c and B are independent. (*iii*) Events A and B^c are independent. (*iv*) Events A^c and B^c are independent.

2. Consider events A and B such that $P(A) = p_1 > 0$ and $P(B) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(B|A) \geq 1 - \left[\frac{1-p_2}{p_1}\right]$.

3. Consider a random experiment consisting of three identical coins one of which is fair and other two are biased with probabilities 1/4 and 3/4 respectively for turning up head. One Coin is taken up at random and tossed twice. If a head appears both times, show that the probability that the fair coin was chosen is 2/7.

4. A box contains three white balls w_1, w_2, w_3 and two red balls r_1 and r_2. We remove at random two balls in succession. What is the probability that the first removed ball is white and the second is red? (3/10).

5. Rain is forecast half the time in a certain region during a given time period. We estimate that the weather forecasts are accurate two times out of three. Mr. X goes out every day and he really fears being caught in the rain without an umbrella. Consequently, he always carries his umbrella if rain is forecast. Moreover, he even carries his umbrella one time out of three if rain is not forecast. Calculate the probability that it is raining and Mr. X does not have his umbrella. (1/9)

6. A commuter has two vehicles, one being a compact car and the other one a minivan. Three times out of four, he uses the compact car to go to work and the remainder of the time he uses the minivan. When he uses the compact car (respectively, the minivan), he gets home before $5:30$ pm 75% (resp. 60%) of the time. Calculate the probability that
(a) he gets home before $5:30$ pm on a given day (b) he used the compact car if he did not get home before $5:30$ pm (c) he uses the minivan and he gets home after $5:30$pm (d) he gets home before $5:30$pm on two (independent) consecutive days and he does not use the same vehicle on these two days. $(0.712, 0.652, 0.1, 0.169)$

7. Five percent of the patients suffering from a certain disease are selected to undergo a new treatment that is believed to increase the recovery rate from 30% to 50%. A person is randomly selected from these patients after the completion of the treatment and found to have recovered. What is the probability that the patient received the new treatment? (0.08)

8. Consider all families with exactly two children. Also let each child has a 50-50 chance of being a boy. Let the events be: $A_1 =$ both male and female child are represented among the children $A_2 =$ at most one child is a girl
(a) Are A_1 and A_2^c incompatible events (b) Are A_1 and A_2^c independent events (c) We also suppose that the probability that the third child of an arbitrary family is a boy is equal to 11/20 if the first two children are boys, to 2/5 if the first two children are girls, and to 1/2 in the other cases. Knowing that the third child of a given family is a boy what is the probability that the first two are also boys? $(Y, N, 0.282)$

9. Suppose box 1 contains a white balls and b black balls, and box 2 contains c white balls and d black balls. One ball of unknown color is transferred from the first box into the second one and then a ball is drawn from the later. What is the probability that it will be a white ball?

10. A certain test for a particular cancer is known to be 95% accurate. A person submits to the test and results are positive. Suppose that the person comes from a population of $100,000$ where 2000 people suffer from that disease. What can be concluded about the probability that the person under test has that particular cancer? (0.278)

11. Four roads leads away from a jail. A prisoner has escaped from the jail and selects a road at random. If road I is selected, the probability of escaping is $1/8$; if road II is selected the probability of success is $1/6$; if road III is selected the probability of escaping is $1/4$ and if road IV is selected the probability of success is $9/10$. (i) Find the probability that the prisoner will succeed in escaping. (ii) If the prisoner succeeds, what is the probability that prisoner escaped by using road IV ? By using road I? $((i)173/480(ii)108/173, 15/173)$

12. A biased coin is tossed till a head appears for the first time (assume that p and q $(p + q = 1)$ are the probability of getting head and tail respectively in a single trial). What is the probability that the number of required tosses is odd? $(1/(2 - p))$

13. In examining a past records of a corporation's account balances, an auditor finds that 15% of them have contained errors. Of those balances in error, 60% were regarded as unusual values based on historical figures. Of all the account balances, 20% were unusual values. If the figure for a particular balance appears unusual on this basis, what is the probability that it is in error? (0.45)

14. A stock market analyst examined the prospects of the share of a large number of corporations. When the performance of these stocks was investigated one year later, it turned out that 25% performed much better than the market average, 25% much worse and the remaining 50% about the same as the average. Forty percent of the stocks that turned out to do much better than the market were rated 'good buy' by the analyst, as were 20% of those that did about as well the market and 10% of those that did much worse. What is the probability that a stock rated a 'good buy' by the analyst performed much better than the market average? (0.444)

15. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. (2/5)

16. There are n socks, 3 of which are red, in a drawer. What is the value of n if when 2 of the socks are chosen randomly, the probability that they both are red is 1/2? (4)

17. Let E and F be mutually exclusive events in the sample space of an experiment. Suppose experiment is repeated until either E or F occurs. Show that the probability of the event E occurs before event F is $P(E)/(P(E) + P(F))$.

18. Box 1 contains 1 white and 999 red balls. Box 2 contains 999 white and 1 red balls. A ball is drawn from a randomly selected box. If the ball is red what is the probability that it came from box 1? (0.999)

19. Box 1 contains 1000 bulbs of which 10% are defective. Box 2 contains 2000 bulbs of which 5% are defective.Two bulbs are drawn from a randomly selected box. (i) Find the probability that both bulbs are defective? (ii) Assuming that both are defective, find the probability that they came from box 1? $(0.006, 0.8)$

20. We have two coins. The first is fair and the second two headed. We pick one of the coins at random, we toss it twice and heads shows both times. Finds the probability that the coin picked is fair. (1/5)

Tutorial 2

1)
(i) ⇒ (ii) If A and B are independent

then $P(A\cap B) = P(A).P(B)$

$P(A^c\cap B) = ?$

$P(B) = P(A\cap B) + P(A^c\cap B)$

$= P(A).P(B) + P(A^c\cap B)$

$\therefore P(A^c\cap B) = P(A^c).P(B)$

(ii) ⇒ (iii) $P(A^c\cap B) = P(A^c).\ P(B)$

$P(A\cap B^c) = ?$

~~$P(A) =$~~ ~~$P(A)$~~

$P(A^c\cup B) = P(A^c) + P(B) - P(A^c\cap B)$

$= P(A^c) + P(B) - P(A^c).P(B)$

$= P(A^c) + P(B).P(A)$

$= 1 - P(A) + P(B).P(A)$

⇒ $= 1 - P(A).P(B^c)$

$\therefore P(A).P(B^c) = P(A^c\cap B)$

(iii) ⇒ (iv) $P(A^c\cap B^c) = P(A^c).P(B^c)$

~~Pofonus~~ $P(A^c\cup B) = 1 - P(A\cap B^c)$

$= 1 - P(A).\ P(B^c)$

Now $P(A^c\cap B^c) + P(B)$ ~~$P(A\cap B^c)$~~ $= P(A^c\cup B)$

$P(A^c\cap B^c) = 1 - P(B) - P(A).P(B^c)$

$= P(A^c).P(B^c)$

2)

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$P(A\cap B) = P(A) + P(B) - P(A\cup B)$$

$$= p_1 + p_2 - P(A\cup B) \geq p_1 + p_2 - 1$$

$$\therefore P(B|A) \geq \frac{(p_1 + p_2 - 1)}{p_1}$$

$$\geq 1 - \left[\frac{(1-p_2)}{p_1}\right]$$

3)

E_1 = head appears both time.

E_2 = The coin which was chosen is the fair coin

$P(E_2|E_1) = ?$

$P(E_1|E_2) = \frac{1}{4}$

E_3 = The chosen coin is biased with $\frac{1}{4}$ prob of head

E_4 = ... $\frac{3}{4}$

$$\therefore P(E_1|E_3) = \frac{1}{16} \qquad P(E_1|E_4) = \frac{9}{16}$$

$$P(E_1) = P(E_1|E_2).P(E_2) + P(E_1|E_3).P(E_3) + P(E_1|E_4).P(E_4)$$

$$= \frac{1}{3}\cdot\left[\frac{1}{4} + \frac{1}{16} + \frac{9}{16}\right] = \frac{1}{3}\times\frac{\cancel{14}\,7}{\cancel{16}\,8} = \frac{1}{3}\times\frac{7}{8}$$

$$P(E_2) = \frac{1}{3}$$

$$\therefore P(E_2|E_1) = \frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{3}\times\frac{7}{8}} = \frac{2}{7}$$

4)

E_1 = A white ball is drawn at the first draw

E_2 = A red ball is drawn.

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 | E_1)$$

$$= \frac{3}{5} \times \frac{2}{4}$$

$$= \frac{3}{10}$$

5)

E_1 = Rain is forecast

E_2 = Weather forecast is accurate

E_3 = Mr. X is carrying umbrella

$$P(E_1) = \frac{1}{2}$$

$$P(E_3 | E_1^c) = \frac{1}{3}$$

$$P(E_3 | E_1) = 1$$

$$P(E_3^c | E_1^c) = \frac{2}{3}$$

$$P(E_3) = P(E_1) \cdot P(E_3 | E_1) + P(E_1^c) \cdot P(E_3 | E_1^c)$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\therefore P(E_3^c \cap E_2^c \cap E_1^c) = P(E_1^c) \cdot P(E_2^c | E_1^c) \cdot P(E_3^c | E_2^c \cap E_1^c)$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3}$$

$$= \frac{1}{9}$$

6

C = Commuter uses compact car

M = " " Minivan

H = " gets home before 5 30 pm

$P(C) = \frac{3}{4}$ $\qquad P(H|C) = \frac{75}{100} = \frac{3}{4}$ $\qquad P(H|M) = \frac{60}{100} = \frac{3}{5}$

$P(M) = \frac{1}{4}$

a) $P(H) = P(C) . P(H|C) + P(M) . P(H|M)$

$= \frac{3}{4} . \frac{3}{4} + \frac{1}{4} . \frac{3}{5}$

$= \frac{\cancel{16}9}{16} + \frac{3}{20}$

$= \frac{57}{80}$

b) $P(C|H^c) = \frac{P(C) . P(H^c|C)}{P(H^c)}$

$= \frac{\frac{3}{4} . \frac{1}{4}}{\frac{23}{80}}$

$= \frac{15}{23}$

c) $P(M|H^c) = \frac{P(M) . P(H^c/M)}{P(H^c)}$

$= \frac{\frac{1}{4} . \cancel{\frac{9}{4}} \frac{2}{5}}{\frac{23}{80}}$

$= \frac{8}{23}$

$P(M \cap H^c) = P(M|H^c) . P(H^c) = \frac{8}{80}$

d) $2 \times P(H \cap C) . P(H \cap M)$

$= \cancel{2} \times \frac{3}{\cancel{4}_2} \times \frac{3}{4} \times \frac{3}{5} \times \frac{1}{4}$

$= \frac{27}{160}$

7) A = person selected for new treatment

C = person is recovered

$P(C|A) = \frac{1}{2}$ $\quad P(A) = \frac{5}{100}$ $\quad \therefore P(C) = \frac{1}{2} \times \frac{5}{100} + \frac{3}{10} \times \frac{95}{100}$

$P(C|A^c) = \frac{3}{10}$ $\quad P(A^c) = \frac{95}{100}$

$$\therefore P(A|C) = \frac{P(A) \cdot P(C|A)}{P(C)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{100}}{\frac{1}{2} \cdot \frac{5}{100} + \frac{3}{10} \cdot \frac{95}{100}}$$

$$= 0.08$$

8) A_1 = both male and female child are represented among the children.

A_2 = At most one child is a girl.

a) $P(A_1) = \frac{2}{4} = \frac{1}{2}$

$P(A_2) = \frac{3}{4}$

$P(A_1 \cap A_2^c) = 0$

$\therefore A_1$ and A_2^c are incompatible.

b) $P(A_1) \cdot P(A_2^c) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$\therefore P(A_1) \cdot P(A_2^c) \neq P(A_1 \cap A_2^c)$

c) ~~For Third child is a~~

T = Third child is a boy.

$P(T|A_2^c) = \frac{2}{5}$ $\quad P(T|A_1) = \frac{1}{2}$ $\quad P(T|\text{both child are boy}) = \frac{11}{20}$

$P(T)\ \frac{11}{20}\times\frac{1}{4}+\frac{2}{5}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{2}$

$= \frac{39}{80}$

$P(F|T) = \frac{P(F)\ P(T|F)}{P(T)} = \frac{\frac{1}{4}\times\frac{11}{20}}{\frac{39}{80}} = \frac{11}{39}$

9) E_1 = A white ball is drawn from the first box

E_2 = A black ball " " " ~~second~~ first box

A = A ball is drawn from the second box.

$P(A) = P(E_1).P(A|E_1) + P(E_2).\ P(A|E_2)$

$= \frac{a}{a+b}\cdot\frac{c+1}{c+d+1} + \frac{b}{a+b}\cdot\frac{c}{c+d+1}$

$= \frac{c(a+b)+a}{(a+b)(c+d+1)}$

10) A = Test is positive

B = The person is form infected people

$P(A|B) = \frac{95}{100}$

$P(B) = \frac{2000}{100000} = \frac{2}{100}$

$P(A) = P(B).P(A|B) + P(B^c).\ P(A|B^c)$

$= \frac{2}{100}\cdot\frac{95}{100} + \frac{98}{100}\cdot\frac{5}{100}$

$= \frac{680}{(100)^2}$

$$P(B|A) = \frac{P(B).P(A/B)}{P(A)}$$

$$= \frac{\frac{2}{100} \cdot \frac{95}{100}}{\frac{680}{(100)^2}}$$

$$= \frac{190}{680}$$

$$= \frac{19}{68}$$

11) E_1 = prisoner chose Road I

E_2 = " " II

E_3 = " " III

E_4 = " " IV

E = Prisoner's success.

$P(E|E_1) = \frac{1}{8}$ $\quad P(E|E_2) = \frac{1}{6}$ $\quad P(E|E_3) = \frac{1}{4}$

$P(E|E_4) = \frac{9}{10}$.

$P(E_i) = \frac{1}{4} \quad \forall\, i = \{1,2,3,4\}$

(i) $\therefore P(E) = P(E_1).P(E|E_1) + P(E_2).P(E|E_2) + P(E_3) \cdot P(E|E_3) + P(E_4).P(E|E_4)$

$$= \frac{1}{4}\left[\frac{1}{8} + \frac{1}{6} + \frac{1}{4} + \frac{9}{10}\right]$$

$$= \frac{173}{480}$$

(ii) $P(E_4|E) = \dfrac{P(E|E_4) \cdot P(E_4)}{P(E)}$

$= \dfrac{\frac{9}{10} \cdot \frac{1}{4}}{\frac{173}{480}}$

$= \dfrac{108}{173}$

$P(E_1|E) = \dfrac{P(E|E_1) \cdot P(E_1)}{P(E)}$

$= \dfrac{\frac{1}{8} \cdot \frac{1}{4}}{\frac{173}{480}}$

$= \dfrac{15}{173}$

12) A = Number of tosses is odd

$P(A) = p + q^2 p + q^4 p + q^6 p + \cdots$

$= p[1 + q^2 + q^4 + q^6 + \cdots]$

$= p \cdot \dfrac{1}{1-q^2}$

$= \dfrac{p}{(1+q)(1-q)}$

$= \dfrac{1}{2-p}$

13) E = Account balance have error

U = ~~unusual~~ unsusual values

$P(E) = \dfrac{15}{100}$ $\quad P(U) = \dfrac{20}{100}$ $\quad P(U|E) = \dfrac{60}{100}$

$P(U \cap E) = \dfrac{60}{100} \cdot P(E)$

$= \dfrac{60}{100} \cdot \dfrac{15}{100}$

$P(E|U) = \dfrac{P(U \cap E)}{P(U)}$

$= \dfrac{\frac{60}{100} \cdot \frac{15}{100}}{\frac{20}{100}} = \dfrac{45}{100} = 0.45$

14)

B = Better than market avg

W = worse " " "

A = same as " "

E = "Good buy" rated by Analyst

$P(B) = \frac{25}{100} = \frac{1}{4}$ $\quad P(W) = \frac{25}{100} = \frac{1}{4}$ $\quad P(A) = \frac{50}{100} = \frac{1}{2}$

$P(E|B) = \frac{40}{100} = \frac{4}{10}$ $\quad P(E|W) = \frac{10}{100} = \frac{1}{10}$ $\quad P(E|A) = \frac{20}{100} = \frac{2}{10}$

$$\therefore P(E) = P(B).P(E|B) + P(W).P(E|W) + P(A).P(E|A)$$

$$= \frac{1}{4}\cdot\frac{4}{10} + \frac{1}{4}\cdot\frac{1}{10} + \frac{1}{2}\cdot\frac{2}{10}$$

$$= \frac{4+1+4}{40}$$

$$= \frac{9}{40}$$

$$P(B|E) = \frac{P(B).P(E|B)}{P(E)}$$

$$= \frac{\frac{1}{4}\cdot\frac{4}{10}}{\frac{9}{40}}$$

$$= \frac{4}{9}$$

15)

E_1 = The sum is 5

E_2 = " " " 7

E = ~~The~~ ~~first~~ 5 occurs first

$P(E_1) = \frac{4}{36}$

$P(E_2) = \frac{6}{36}$

$P(E_1 \cup E_2) = \frac{10}{36}$

$\therefore P(E_1^c \cap E_2^c) = \frac{26}{36}$

Let D be the event that ~~was~~ ~~have~~ neither 5 nor 7 occurs

$\therefore D = E_1^c \cap E_2^c$

$P(E) = P(E_1) + P(D)\cdot P(E_1) + P(D)\, P(D)\, P(E_1) + \dots$

$$= \frac{4}{36} + \left(\frac{26}{36}\right)\cdot\frac{4}{36} + \left(\frac{26}{36}\right)^2\cdot\frac{4}{36} + \dots$$

$$= \left(\frac{4}{36}\right)\left[1 + \frac{26}{36} + \left(\frac{26}{36}\right)^2 + \dots\right]$$

$$= \left(\frac{4}{36}\right)\cdot\left[\frac{1}{1-\left(\frac{26}{36}\right)}\right]$$

$$= \frac{4}{10}$$

$$= \frac{2}{5}$$

16) # ways to select two socks from n socks $= \binom{n}{2}$

" " " ~~both~~ two red socks from 3 red socks $= \binom{3}{2}$

$\therefore \frac{\binom{3}{2}}{\binom{n}{2}} = \frac{1}{2} \Rightarrow n^2 - n - 12 = 0 \Rightarrow (n-4)(n+3) = 0$

$\Rightarrow n = 4.$

17) Since E and F are mutually exclusive, we have

$$P(E \cup F) = P(E) + P(F)$$

Now, let D be the event that event E occurs before event F.

$$\therefore P(D) = P(E) + P(E^c \cap F^c) \cdot P(E) + \{P(E^c \cap F^c)\}^2 P(E) + \{P(E^c \cap F^c)\}^3 P(E) + \cdots$$

$$= P(E) \cdot \left[1 + P(E^c \cap F^c) + \{P(E^c \cap F^c)\}^2 + \{P(E^c \cap F^c)\}^3 + \cdots\right]$$

$$= P(E)\left[\frac{1}{1 - P(E^c \cap F^c)}\right]$$

$$= \frac{P(E)}{P(E \cup F)}$$

$$= \frac{P(E)}{P(E) + P(F)}$$

18) B_1 = Box 1 is selected

B_2 = Box 2 " "

R = The ball is red

$$P(R|B_1) = \frac{999}{1000} \qquad P(R|B_2) = \frac{1}{1000}$$

$$P(B_1) = P(B_2) = \frac{1}{2}$$

$$P(R) = P(R|B_1) \cdot P(B_1) + P(R|B_2) \cdot P(B_2) = \frac{1}{2}.$$

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)}$$

$$= 0.999$$

19) $B_1 =$ Box 1 is selected

$B_2 =$ Box 2 is selected

$D =$ The bulb is defective

$E =$ ~~the~~ selecting two bulbs from a box is defective

$$P(B_1) = P(B_2) = \frac{1}{2}$$

$$P(D|B_1) = \frac{10}{100}$$

$$P(D|B_2) = \frac{5}{100}$$

$$P(E|B_1) = \frac{10}{100} \cdot \frac{10}{100} = \frac{100}{(100)^2} = \frac{1}{100}$$

$$P(E|B_2) = \frac{5}{100} \cdot \frac{5}{100} = \frac{1}{400}$$

(i) $$P(E) = P(B_1) \cdot P(E|B_1) + P(B_2) \cdot P(E|B_2)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{100} + \frac{1}{400}\right]$$

$$= \frac{5}{800}$$

(ii) $$P(B_1|E) = \frac{P(B_1) \cdot P(E|B_1)}{P(E)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{100}}{\frac{5}{800}}$$

$$= \frac{4}{5}$$

20) C_1 = Fair coin

C_2 = Biased coin

H = ~~pr~~ head appears both time

$P(C_1) = P(C_2) = \frac{1}{2}$

$P(H|C_1) = \frac{1}{4}$

$P(H) = P(C_1).P(H/C_1) + P(C_2).P(H/C_2)$

$= \frac{1}{2}\cdot\left[\frac{1}{4}+1\right]$

$= \frac{5}{8}$

$P(H|C_2) = 1$

$$P(C_1|H) = \frac{P(C_1).P(H|C_1)}{P(H)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{4}}{\frac{5}{8}}$$

$$= \frac{1}{5}$$

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Tutorial Sheet-3

1. Consider the following probability mass function:
$P(X = 0) = 0, P(X = 1) = k, P(X = 2) = 2k = P(X = 3), P(X = 4) = 3k, P(X = 5) = k^2, P(X = 6) = 2k^2, P(X = 7) = 7k^2 + k$. Find (i) k (ii) Evaluate $P(X < 6), P(X \geq 6), P(0 < X < 5)$ (iii) the corresponding distribution function.

2. Let X be an random variable (RV) such that $E|X| < \infty$. Show that $E|X - c|$ is minimized if we choose c to be the median of the distribution of X.

3. Consider a question paper with two problems only. The first one has 3 possible answers and the second one has 5(for each problem only one answer is correct). Answers are chosen at random for each of the two problems. Evaluate EX and $V(X)$ where X is the RV representing the right answers.

4. Let X be an RV with PMF $P(X = -2) = P(X = 0) = 1/4, P(X = 1) = 1/3, P(X = 2) = 1/6$. Find median of X. Also find quantile of order 0.2.

5. A coin is tossed until a head appears. Find the expected number of tosses required to obtain the first head.

6. A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials required to open the door, (i) if unsuccessful keys are not eliminated from further selection, (ii) if they are eliminated.

7. (i) Comment on 'Variance of a $B(n, p)$ distribution can be more than its mean'.
(ii) Let the RV X has a $B(6, 0.5)$ distribution. Check which outcome is most likely.
(iii) Ten percent of the articles produced by a certain machine are defective. If 10 independent articles fabricated by this machine are taken at random what is the probability that exactly two of them are defective?
(vi) A device is made up of five independent components and it will operate if at least 4 of its 5 components are active. Each components operates with probability 0.95. Find the probability that a device taken at random operates.
(v) During a war 1 ship out of 9 was sunk on an average in making a certain voyage. Find the probability that exactly 3 out of a convoy of 6 ships would arrive safely.
(vi) Find the probability $P(X \geq 1 | X \leq 1)$ where $X \sim P(5)$.

8. The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates at least 4 passed the examinations?

9. An unprepared student appears a true false examination consisting of 10 questions and randomly guesses answers.
(i) Find the probability that he guesses correctly at least five times.
(ii) Find the probability that he guesses correctly nine times.
(iii) Find the smallest n so that probability of guessing at least n correct answers is less than 1/2.

10. A certain company produces bulbs of which 10% are defective. What is the probability of getting exactly 3 defective in a sample of 10 bulbs selected at random. Compute the probability using Poisson approximations and also compare your result with that obtained from binomial distribution.

11. Derive a recurrence formula formula to find the the k^{th} central moments of a $B(n, p)$ distribution. Also discuss the case of $P(\lambda)$ distribution.

12. Consider a binomial distribution with 4 independent trials where it is known that probabilities of 1 and 2 successes are $2/3$ and $1/3$ respectively. Find the parameter p of the distribution. Also find the mean and variance of the distribution.

13. A coin is biased so that a head is thrice as likely to appear as a tail. Suppose that the coin is tossed 5 times, find the probability of getting (i) at least 3 heads, (ii) at most 3 heads, (iii) exactly 3 tails.

14. In a certain factory producing razor blades, there is 1% for any blades to be defective. The blades are in packets of 10. In a consignment of 1000 packets, calculate the approximate number of packets containing (i) no defective blades (ii) one defective blades (iii) at most two defective blades (iv) at least two defective blades.

15. Suppose that a trainee soldier shoots a target according to a geometric distribution. If the probability that a target is shot in any one trial is 0.8, find the probability that odd number of trials are needed. What is the probability that even number of trials will be needed.

16. In a company 3% defective components are produced. Find the probability that at least 6 components are to be examined in order to get 3 defective.

17. One per thousand of population is subject to certain kinds of accident each year. Given that an insurance company has insured 5000 persons from the population, find the probability that at most 2 persons will incur this accident.

18. A certain airline company having observed that 5% of the persons making reservations on a flight do not show up for the flight, sells 100 seats on a plane that has 95 seats. What is the probability that there will a seat available for every person who shows up for the flight?

19. A pair of die is rolled 50 times. Find the probability of getting a double six at least three times.

20. Suppose number of accidents occurring on a highway each day is a random variable having a $P(3)$ distribution.
(i) Find the probability that three or more accidents occur today.
(ii) Repeat part (i) under the assumption that at least one accident had already occurred today.

21. Suppose that an airplane engine will fail, when in flight, with probability $1 - p$ independently from engine to engine. Suppose that airplane will make a successful flight if at least 50% of engines remain operative. For what values of p is a four engine plane preferable to a two engine plane?

22. Let $X \sim Geo(p)$ distribution. Find the probability that X is even. Also find the probability that X is odd.

Tutorial sheet 3.

1) $P(X=0)=0$, $P(X=1)=K$, $P(X=2)=2K=P(X=3)$, $P(X=4)=3K$

$P(X=5)=K^2$, $P(X=6)=2K^2$, $P(X=7)=7K^2+K$

(i) $\sum_{K=-\infty}^{\infty} P(X=K) = 1$

$\Rightarrow \sum_{K=0}^{7} P(X=K) = 1$

$\Rightarrow P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7) = 1$

$\Rightarrow 0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$

$\Rightarrow 10K^2+9K-1=0$

$\Rightarrow (10K-1)(K+1)=0$

$\therefore K = \frac{1}{10}, -1$

Now $K \neq -1$ as otherwise $P(X=1)<0$.

$\therefore K = \frac{1}{10}$

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$= K+2K+2K+3K+K^2$

$= K^2+8K$

$= K(K+8) = \frac{81}{100}$

$P(X\geq 6) = 1-P(X<6) = 1-\frac{81}{100} = \frac{19}{100}$

$$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$$

$$= K+2K+2K+3K$$

$$= 8K$$

$$= \frac{8}{10}$$

(iii) C.D.F is

$$F(x) = 0 \qquad x \leq 0$$

$$= 0 \qquad 0 \leq x < 1$$

$$= 0+K \qquad 1 \leq x < 2$$

$$= K \qquad 1 \leq x < 2$$

$$= 0+K+2K \qquad 2 \leq x < 3$$

$$= 3K \qquad 2 \leq x < 3$$

$$= 0+K+2K+2K \qquad 3 \leq x < 4$$

$$= 5K \qquad 3 \leq x < 4$$

$$= 5K+3K \qquad 4 \leq x < 5$$

$$= 8K \qquad 4 \leq x < 5$$

$$= 8K+K^2 \qquad 5 \leq x < 6$$

$$= 8K+3K^2 \qquad 6 \leq x < 7$$

$$= 1 \qquad x \geq 7$$

2) Median: A real number μ is said to be median of a random variable (discrete) X, if $P(X \leq \mu) \geq \frac{1}{2}$ and $P(X \geq \mu) \geq \frac{1}{2}$

Thus, from the condition ~~[illegible]~~ $P(X \leq \mu) \geq \frac{1}{2}$, we have

$F(\mu) \geq \frac{1}{2}$ [where $F(x)$ is c.d.f. of X]

If $c > \mu$,

$$E(|x-c|) = \sum_{x=-\alpha}^{\alpha} |x-c|\, p(x) \qquad \text{where } p(x) \text{ is the pmf of } x.$$

$$= \sum_{x=-\alpha}^{c} (c-x)\, p(x) + \sum_{x=c}^{\alpha} (x-c)\, p(x)$$

$$= \sum_{x=-\alpha}^{\mu} (c-x)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\alpha} (x-c)\, p(x) - \sum_{x=\mu}^{c} (x-c)\, p(x)$$

$$= \sum_{x=-\alpha}^{\mu} (c-\mu+\mu-x)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\alpha} (x-\mu+\mu-c)\, p(x) + \sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= \sum_{x=-\alpha}^{\mu} \left[(c-\mu)\, p(x) + (\mu-x)\, p(x)\right] + 2\sum_{x=\mu}^{c} (c-x)\, p(x) + \sum_{x=\mu}^{\alpha} \left[(x-\mu)\cdot p(x) + (\mu-c)\cdot p(x)\right]$$

$$= (c-\mu) \sum_{x=-\alpha}^{\mu} p(x) + \sum_{x=-\alpha}^{\mu} (\mu-x)\, p(x) + \sum_{x=\mu}^{\alpha} (x-\mu)\, p(x) + (\mu-c) \sum_{x=\mu}^{\alpha} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= (c-\mu) \sum_{x=-\alpha}^{\mu} p(x) + \sum_{x=-\alpha}^{\alpha} |x-\mu|\, p(x) + (\mu-c) \sum_{x=\mu}^{\alpha} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= (c-\mu) \sum_{x=-\alpha}^{\mu} p(x) + E(|x-\mu|) + (\mu-c) \sum_{x=\mu}^{\alpha} p(x) + 2\sum_{x=\mu}^{c} (c-x)\, p(x)$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (\mu-c)\sum_{x=\mu}^{\infty} p(x) + 2\sum_{x=\mu}^{c}(c-x)p(x)$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (\mu-c)\left[1-\sum_{x=-\infty}^{\mu} p(x)\right] + 2\sum_{x=\mu}^{c}(c-x)p(x)$$

$$\left[\because \sum_{x=-\infty}^{\infty} p(x) = 1 \Rightarrow \sum_{x=\mu}^{\infty} p(x) = 1-\sum_{x=-\infty}^{\mu} p(x)\right]$$

$$= E(|x-\mu|) + (c-\mu)\sum_{x=-\infty}^{\mu} p(x) + (c-\mu)\left[\sum_{x=-\infty}^{\mu} p(x) - 1\right] + 2\sum_{x=\mu}^{c}(c-x)p(x)$$

$$= E(|x-\mu|) + (c-\mu)\left[2\sum_{x=-\infty}^{\mu} p(x) - 1\right] + 2\sum_{x=\mu}^{c}(c-x)p(x)$$

$$= E(|x-\mu|) + (c-\mu)\left[2F(\mu)-1\right] + 2\sum_{x=\mu}^{c}(c-x)p(x) \quad \text{——— (1)}$$

$$\left[\because F(\mu) = P(X \leq \mu) = \sum_{x=-\infty}^{\mu} p(x)\right]$$

Since μ is median, then $F(\mu) \geq \frac{1}{2}$

$\Rightarrow 2F(\mu)-1 \geq 0$

~~For~~

$\therefore$ From (1) we have,

$$E(|x-c|) \geq E(|x-\mu|) + 2\sum_{x=\mu}^{c}(c-x)p(x)$$

Since $c > \mu$, then $\sum_{x=\mu}^{c}(c-x)p(x) \geq 0$

$$\therefore E(|x-c|) \geq E(|x-\mu|)$$

Similarly for $c < \mu$,

$$E(|X-c|) \geq E(|X-\mu|) + 2\sum_{x=c}^{\mu}(x-c)\,p(x)$$

$$\therefore E(|X-c|) \geq E(|X-\mu|)$$

Thus, $E(|X-c|) \geq E(|X-\mu|)$

3) X: Number of right answers.

Then X can take values 0, 1 and 2. ~~because~~ ~~there is either~~

~~none~~ ~~denotes~~

$$P(X=0) = \frac{2}{3} \times \frac{4}{5} \qquad P(X=1) = \frac{2}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{4}{5} = \frac{6}{15}$$

$$P(X=2) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

$$\therefore E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$$

$$= 1 \cdot \frac{6}{15} + 2 \cdot \frac{1}{15} = \frac{8}{15}$$

$$E(X^2) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 4 \cdot P(X=2)$$

$$= 1 \cdot \frac{6}{15} + 4 \cdot \frac{1}{15} = \frac{10}{15}$$

$$\therefore V(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{10}{15} - \frac{64}{225}$$

$$= \frac{86}{225}$$

4) $P(X=-2)=\frac{1}{4}$ $P(X=0)=\frac{1}{4}$, $P(X=1)=\frac{1}{3}$, $P(X=2)=\frac{1}{6}$

C.D.F of X.

$$
\begin{aligned}
F(x) &= 0 && x<-2 \\
&= \frac{1}{4} && -2\le x<0 \\
&= \frac{1}{2} && 0\le x<1 \\
&= \frac{5}{6} && 1\le x<2 \\
&= 1 && x\ge 2
\end{aligned}
$$

Now, $P(0)\ge\frac{1}{2}$

and $P(X\ge 0) = 1-P(X<0) = 1-F(0)+P(X=0)$

$$= 1+\frac{1}{4}-\frac{1}{2}$$

$$= 1-\frac{1}{4} = \frac{3}{4} \ge \frac{1}{2}$$

$\therefore$ 0 is a median.

x is a quantile of order 0.2 if $P(X\le x)\ge 0.2$

$P(X\ge x)\ge 0.8$

$\therefore x=-2.$

5) X : Number of tosses required ~~before~~ for first head

$R_x = \{ 1, 2, \quad \}$

$P(X=1) = p$ where p is the probability for getting a head.

$P(X=2) = qp$ where $q = 1-p$

$P(X=3) = q^2 p$

$P(X=k) = q^{k-1} p$

$$E(X) = \sum_{k=-\infty}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k \cdot q^{k-1} p$$

$$= p \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

$$= p \sum_{k=1}^{\infty} \frac{d}{dq}(q^k)$$

$$= p \cdot \frac{d}{dq}\left[\sum_{k=1}^{\infty} q^k\right]$$

$$= p \cdot \frac{d}{dq}\left[\frac{1}{1-q}\right]$$

$$= p \cdot \frac{1}{(1-q)^2} \quad \left[\because q<1 \Rightarrow \sum_{k=1}^{\infty} q^k = \frac{1}{1-q}\right]$$

$$= \frac{1}{(1-q)}$$

$$= \frac{1}{p}$$

6) X : Number of trials required to open the door.

R_x

(i) $R_x = \{1, 2, \quad \}$

$P(x=1) = \frac{1}{n}$

$P(x=2) = \frac{(n-1)}{n} \cdot \frac{1}{n}$

$P(x=k) = \left\{\frac{(n-1)}{n}\right\}^{k-1} \cdot \frac{1}{n}$

$$E(x) = \sum_{k=1}^{\infty} k \cdot P(x=k)$$

$$= \sum_{k=1}^{\infty} k \cdot \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} k \cdot \left(\frac{n-1}{n}\right)^{k-1}$$

$$= \frac{1}{n} \cdot \frac{1}{\left(1-\frac{n-1}{n}\right)^2}$$

$$= n.$$

(ii)

$$E(x^2) = \sum_{k=1}^{\infty} k^2 \cdot P(x=k)$$

$$= \sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1}$$

$$= \frac{1}{n} \cdot \frac{1+\frac{n-1}{n}}{\left(1-\frac{n-1}{n}\right)^3}$$

$$= (2n-1) \cdot n.$$

$V(x) = E(x^2) - \{E(x)\}^2 = n(2n-1) - n^2$

$= n \cdot (n-1)$

To evaluate

$\sum_{k=1}^{\infty} k^2 \left(\frac{n-1}{n}\right)^{k-1}$

Let $x = \frac{n-1}{n}$

$\therefore \sum_{k=1}^{\infty} k^2 x^{k-1}$

$= \sum_{k=1}^{\infty} \frac{d}{dx}(k x^k)$

$= \frac{d}{dx}\left(\sum_{k=1}^{\infty} k x^k\right)$

$= \frac{d}{dx}\left(x \cdot \sum_{k=1}^{\infty} k x^{k-1}\right)$

$= \frac{d}{dx}\left(x \cdot \frac{1}{(1-x)^2}\right)$

$= \frac{1+x}{(1-x)^3}$

(ii) If the keys are eleminated then the key we select in a turn will be excluded. ~~from the~~

$$R_x = \{1, 2, \quad , n\}$$

$$P(x=1) = \frac{1}{n}$$

$$P(x=2) = \frac{(n-1)}{n} \cdot \frac{1}{(n-1)} = \frac{1}{n}$$

$$P(x=3) = \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{1}{(n-2)} = \frac{1}{n}$$

$$P(x=n) = \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{(n-3)}{(n-2)} \cdots \frac{1}{2} \cdot 1 = \frac{1}{n}.$$

$$E(x) = \sum_{k=1}^{n} k \cdot P(x=k)$$

$$= \sum_{k=1}^{n} k \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{n} k$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \left(\frac{n+1}{2}\right)$$

$$E(x^2) = \sum_{k=1}^{n} k^2 P(x=k)$$

$$= \frac{1}{n} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$V(x) = E(x^2) - \{E(x)\}^2$

$= \frac{(2n+1)(n+1)}{6} - \left(\frac{n+1}{2}\right)^2$

7)

i) $X \sim B(n,p)$

then $E(x) = np$ and $V(x) = npq$

Now $q \leq 1 \Rightarrow V(x) \leq E(x)$

So, variance can not be greater than mean.

ii) The most likely outcome is corresponding to the mode of X

Since $X \sim B(n,p)$, the mode of $x = \lfloor (n+1)p \rfloor$

$\therefore$ mode of $X = \lfloor (6+1) \times 0.5 \rfloor$

$= \lfloor 7 \times 0.5 \rfloor$

$= 3$

(iii) X: number of defective articles.

probability that a article is defective, is $= \frac{10}{100} = 0.1$

$X \sim B(10, 0.1)$

$P(X=2) = \binom{10}{2} \cdot (0.1)^2 \cdot (0.9)^8$

(iv) X: Number of defective articles in the sample

$X \sim B(20, p)$

~~Here~~ So, pmf of $X = \binom{20}{x} p^x \cdot (1-p)^{20-x}$

For $p = 0.25$, $P(X=10) = \binom{20}{10} (0.25)^{10} (1-0.25)^{10}$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \binom{20}{0} \cdot (0.25)^0 \cdot (0.75)^{20} - \binom{20}{1} \cdot (0.25)^1 \cdot (0.75)^{19}$

For poisson approximation, $\lambda = 20 \times 0.25 = 5$

$\therefore P(X=10) = e^{-5} \cdot \frac{5^{10}}{10!}$

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-5} \cdot \frac{5^0}{0!} - e^{-5} \cdot \frac{5^1}{1!}$

(v) X : Number of active components

$X \sim B(5, 0.95)$

$P(X \geq 4) = P(X=4) + P(X=5)$

$= \binom{5}{4} (0.95)^4 \cdot 0.05 + \binom{5}{5} \cdot (0.95)^5 \cdot (0.05)^0$

(vi) Probability that the ship will arrive safely is $= p = \frac{8}{9}$

X : Number of ships arrive safely.

$X \sim B(6, \frac{8}{9})$

$P(X=3) = \binom{6}{3} \cdot \left(\frac{8}{9}\right)^3 \cdot \left(\frac{1}{9}\right)^3$

(vii) Probability that the vessel will arive safely is $= p = \frac{97}{100}$

$= 0.97$

$X \sim B(10, 0.97)$

$\therefore P(X=6) = \binom{10}{6} \cdot (0.97)^6 \cdot (0.03)^4$

$P(x \geq 6) = P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$= \binom{10}{6} (0.97)^6 (0.03)^4 + \binom{10}{7} \cdot (0.97)^7 (0.03)^3 + \binom{10}{8} \cdot (0.97)^8 (0.03)^2$

$+ \binom{10}{9} (0.97)^9 (0.03)$

$+ \binom{10}{10} (0.97)^{10}$

(viii) $X \sim P(5)$

$\therefore$ p.m.f. of x = $p(x) = \dfrac{e^{-5} \cdot 5^x}{x!}$

Now $P(X \geq 1 \mid X \leq 1) = \dfrac{P(X=1)}{P(X \leq 1)}$

$= \dfrac{e^{-5} \cdot \frac{5^1}{1!}}{e^{-5} \frac{5^0}{0!} + e^{-5} \frac{5^1}{1!}}$

$= \dfrac{5}{6}$

8) Probability that a candidate will pass $= \dfrac{60}{100} = 0.6$ the examination

X: Number of candidates passed the examination

$X \sim B(6, 0.6)$

$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$

$= \binom{6}{4} \cdot (0.6)^4 \cdot (0.4)^2 + \binom{6}{5} \cdot (0.6)^5 \cdot (0.4) + \binom{6}{6} \cdot (0.6)^6$

9) X : Number of correct guesses.

$X \sim B(10, 0.5)$. The probability that a guess is correct is $= 0.5$

(i) $P(X \geq 5) = 1 - P(X < 5)$

$= 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3) - P(X=4)$

$= 1 - \sum_{K=0}^{4} \binom{10}{K} \cdot (0.5)^K (0.5)^{10-K}$

(ii) $P(X=9) = \binom{10}{9} \cdot (0.5)^9 \cdot (0.5)$

(iii) $P(X \geq n) < \frac{1}{2}$

$\Rightarrow 1 - P(X < n) < \frac{1}{2}$

$\Rightarrow P(X < n) > \frac{1}{2}$

For $n=6$, we have $P(X<6) = \sum_{K=0}^{5} P(X=K) > \frac{1}{2}$

$\therefore n=6$ is the smallest.

10) Probability that a product is defective is $\frac{10}{100} = 0.1$

X : Number of defective bulbs in the sample

$X \sim B(10, 0.1)$

$P(X=3) = \binom{10}{3} (0.1)^3 (0.9)^7$

Now $\lambda = np = 10 \cdot 0.1 = 1$

$P(X=3) = \frac{e^{-1} \cdot 1^3}{3!}$

11) Probability of getting a Tv set 0.5

X: Number of requests for TV set

~~Pof~~ $X \sim B(5, 0.5)$

(i) $P(X \geq 4) = P(X=4) + P(X=5)$

$= \binom{5}{4} (0.5)^4 \cdot (0.5) + \binom{5}{5} \cdot (0.5)^5 = \frac{3}{16}$

(ii) $P(X \leq 3) = 1 - P(X > 3)$

$= 1 - \frac{3}{16}$

$= \frac{13}{16}$

(iii) $3C = R \cdot \left[1 \, P(X=1) + 2 \, P(X=2) + 3 \, P(X=3) \right]$

$= 1 \cdot R \cdot \binom{5}{1} (0.5)^5 + 2 \cdot R \cdot \binom{5}{2} \cdot (0.5)^5 + 3 \cdot R \cdot \binom{5}{3} \cdot (0.5)^5$

$R = \frac{96C}{73}$

12) To evaluate K-th central moment, we will discuss a new tool to determine the K-th moment of a random variable. The tool is called Moment Generating Function (MGF).

MGF of a random variable X is defined as

$M_X(t) = E(e^{tX})$

provided the expectation exists for some t satisfying $|t| < h$, $h > 0$.

Now, $M_X(t) = \sum_x e^{tx} \, p(x)$

$\frac{d}{dt}(M_X(t)) = \sum_x x \cdot e^{tx} \, p(x)$ — (1)

$$\frac{d}{dt}(M_x(t))\Big|_{t=0} = \sum_x x \cdot p(x) = E(X)$$

From (1),

$$\frac{d^2}{dt^2}(M_x(t)) = \sum_x x^2 e^{tx} p(x)$$

$$\frac{d^2}{dt^2}(M_x(t))\Big|_{t=0} = \sum_x x^2 p(x) = E(X^2)$$

Thus we can determine the k-th moment by differentiating MGF k times.

$$\frac{d^k}{dt^k}(M_x(t)) = E(X^k)$$

Now,

$$E\left((X - E(X))^k\right) = \sum_{i=0}^{k} \binom{k}{i} E(X^i) \cdot \{E(X)\}^{k-i}$$

$$= \sum_{i=0}^{k} \binom{k}{i} \cdot \frac{d^i}{dt^i}(M_x(t))\Big|_{t=0} \{E(X)\}^{k-i} \quad (2)$$

For binomial distribution,

$$M_x(t) = \sum_x e^{xt} \binom{n}{x} p^x q^{n-x} = (pe^t + q)^n$$

Using this MGF and (2), we can determine the k-th central moment.

Similarly for Poisson we can determine the k-th central moment.

13)

$X \sim B(4, p)$

$P(X=1) = \frac{2}{3}$

$\Rightarrow \binom{4}{1} p (1-p)^3 = \frac{2}{3}$ ⟵ ①

$P(X=2) = \frac{1}{3}$

$\binom{4}{2} p^2 (1-p)^2 = \frac{1}{3}$ ⟵ ②

② ÷ ① gives,

$$\frac{6 p^2 (1-p)^2}{4p (1-p)^3} = \frac{1}{2}$$

$\Rightarrow 3p = (1-p)$

$\Rightarrow p = \frac{1}{4}$

$E(X) = 4 \times \frac{1}{4} = 1$

$V(X) = 4 \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{4}$

14) X : Number of heads appeared in five tosses.

Probability that the head will appear $= p$

" " tail " " $= \frac{p}{3}$

$p + \frac{p}{3} = 1 \Rightarrow p = \frac{3}{4}$

$X \sim B\left(5, \frac{3}{4}\right)$

(i) $P(X \geq 3) = 1 - P(X<3)$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \binom{5}{0}\left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 - \binom{5}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 - \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2$

(ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \binom{5}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4 + \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

(iii) $P(X=3) = \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

15) Probability of success = P(getting 4) + P(getting 5)

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

X: Number of success in 9 throws.

$X \sim B(9, \frac{1}{3})$

(i) $E(X) = 9 \cdot \frac{1}{3} = 3$

$V(X) = 9 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right) = 2$

(ii) $P(X=2) = \binom{9}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^7$

(iii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{9}{0} \cdot \left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + \binom{9}{1} \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^8 + \binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7$

(iv) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \binom{9}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 - \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8$

16) Probability that the blade is defective = 0.01

X: Number of defective blades in packet of 10

$X \sim B(10, 0.01)$

(i) $P(X=0) = \binom{10}{0} \cdot (0.01)^0 (0.99)^{10}$

The ~~products~~ consignment of 1000 packets has approximately

$= 1000 \times P(X=0)$ number of packets containing no defective blade

(ii) $P(X=1) = \binom{10}{1} \cdot (0.01)^1 (0.99)^9$

$\therefore$ Number of packets containing one defective blade $= 1000 \times P(X=1)$

(iii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{10}{0} (0.01)^0 (0.99)^{10} + \binom{10}{1} \cdot (0.01)^1 (0.99)^9 + \binom{10}{2} (0.01)^2 (0.99)^8$

$\therefore$ Number of packets containing at most two defective blades $= 1000 \times P(X \leq 2)$

(iv) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$\therefore$ Number of packets containing at ~~most~~ least two defective blades $= 1000 \times P(X \geq 2)$

17) X: Number of trials before first target is shot

$X \sim Geo(0.8)$

$\therefore$ p.m.f of $X = P(X=k) = q^{k-1}p$, $p = 0.8$, $q = 0.2$

$\therefore P(X = odd) = P(X=1) + P(X=3) + P(X=5) + \ldots$

$= p + q^2 p + q^4 p + \ldots$

$= p(1 + q^2 + q^4 + \ldots)$

$= p \cdot \frac{1}{1-q^2}$

$= \frac{1}{(1+q)}$

$= \frac{1}{(2-p)}$

$P(X = even) = 1 - P(X = odd)$

$= 1 - \frac{1}{2-p}$

$= \frac{2-p-1}{2-p}$

$= \frac{1-p}{2-p}$

18) Probability that a product is defective $= \frac{3}{100} = 0.03$

X: Number of ~~trials~~ components to be examined to get 3 defectives

$X \sim NB(3, 0.03)$

$\therefore P(X=x) = \binom{x-1}{2} \cdot (0.03)^2 (0.97)^{x-3} \cdot (0.03)$

Now, $P(X \geq 6) = 1 - P(X < 6)$

$= 1 - P(X=3) - P(X=4) - P(X=5)$

$P(X \geq 6) = 1 - \binom{2}{2} (0.03)^3 - \binom{3}{2} (0.03)^3 (0.97)$
$- \binom{4}{2} (0.03)^3 (0.97)^2$

19) X : Number of shots for fourth hit ~~the target~~

Probability of hitting the target = 0.7

$X \sim NB(4, 0.7)$

$P(X = K) = \binom{K-1}{3} (0.7)^3 \, (0.3)^{K-4} \, (0.7)$

$\therefore P(X = 7) = \binom{6}{3} . (0.7)^4 (0.3)^3$

20) X : Number of defective in the sample

Probability of a item is defective = $\frac{10}{100} = 0.1$

$\therefore X \sim B(10, 0.1)$

The machine will not stop when there is no defective product in sample.

$\therefore P(X = 0) = \binom{10}{0} . (0.1)^0 \, (0.9)^{10}$

21) Probability of a person getting into the accident $= \frac{1}{1000}$

Number of people insured $= 5000$

Using Poisson Approximation, we have,

$$\lambda = np = 5000 \times \frac{1}{1000} = 5$$

X: Number of people getting into the accident

$$X \sim P(5)$$

$$\therefore P(X = x) = e^{-5} \cdot \frac{5^x}{x!}$$

$$\therefore P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-5} \cdot \frac{5^0}{0!} + e^{-5} \cdot \frac{5^1}{1!} + e^{-5} \cdot \frac{5^2}{2!}$$

22) Probability of the person making reservation on flight doesn't show up is $= \frac{5}{100} = 0.05$

$\therefore$ Probability of the person making reservation on flight show up is $= 1 - 0.05 = 0.95$

X: Number of people show up for the flight.

$X \sim B(100, 0.95)$ $\therefore P(X = x) = \binom{100}{x} \cdot (0.95)^x \cdot (0.05)^{100-x}$

~~For~~ Everyone who shows up for flight will ~~show up~~ get a seat if number of people show up for flight is less ~~of~~ or equal to 95.

$$\therefore P(X \leq 95) = 1 - P(X > 95)$$

$$= 1 - P(X=96) - P(X=97) - P(X=98) - P(X=99) - P(X=100).$$

23) Probability of getting double six by rolling a pair of die

$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

X: Number of times double six occur. in 50.

The pair of die is rolled 50 times.

$\therefore X \sim B\left(50, \frac{1}{36}\right)$

$\therefore P(X=x) = \binom{50}{x} \cdot \left(\frac{1}{36}\right)^{x} \left(\frac{35}{36}\right)^{50-x}$

Probability for getting a double six at least three times

$= P(X \geq 3) = 1 - P(X < 3)$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \binom{30}{0}\left(\frac{1}{36}\right)^{0}\left(\frac{35}{36}\right)^{50} - \binom{30}{1}\left(\frac{1}{36}\right)^{1}\left(\frac{35}{36}\right)^{49} - \binom{30}{2}\left(\frac{1}{36}\right)^{2}\left(\frac{35}{36}\right)^{48}$

24) X: Number of accidents occurring on a highway each day.

Given that $X \sim P(3)$

$\Rightarrow P(X=x) = e^{-3} \cdot \frac{3^{x}}{x!}$

(i) ~~P(X≥3)~~ Probability of three or more accident occur.

today $= P(X \geq 3) = 1 - P(X<3)$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - e^{-3}\frac{3^{0}}{0!} - e^{-3} \cdot \frac{3^{1}}{1!} - e^{-3} \cdot \frac{3^{2}}{2!}$

(ii) Given that ~~one occ~~ at least one accident had already occurred, the probability of three or more accident is

$= P(X \geq 3 \mid X \geq 1) = \frac{P(X \geq 3)}{P(X \geq 1)}$

determine $P(X \geq 1)$ similarly

25) Probability of a engine will fail = $1-p$

$\therefore$ Probability of a engine will operate = p

X: Number of engines are operative for four engine plane

Y: Number of engines are operative for two engine plane.

$X \sim B(4, p)$

$Y \sim B(2, p)$

Now for a successful flight at least 50% of the engines remain operative

$\therefore$ For a ~~4~~ four engine plane we need 2 ~~engine~~ at least 2 engine remain operative and for two engine plane we need at least 1 engine remain operative

$\therefore P(X \geq 2)$ is the probability of at least two engines are operative for four engine plane.

$P(Y \geq 1)$ .. - - one engine is operative for two engine plane

Therefore $P(X \geq 2) > P(Y \geq 1)$

$\Rightarrow 1 - P(X<2) > 1 - P(Y<1)$

$\Rightarrow 1 - P(X=0) - P(X=1) > 1 - P(Y=0)$

$\Rightarrow \binom{4}{0} . p^0 (1-p)^4 + \binom{4}{1} p . (1-p)^3 < \binom{2}{0} . p^0 (1-p)^2$ [$\because (1-p) \neq 0$]

$\Rightarrow (1-p)^2 + 4 . p . (1-p) < 1$

$\Rightarrow 1 - 2p + p^2 + 4p - 4p^2 < 1$

$\Rightarrow \quad 2p - 3p^2 \leq 0$

$\Rightarrow \quad p \geq \frac{2}{3} \quad [\because p \neq 0]$

26) Probability of a unit is defective $= \frac{5}{100} = 0.05$

X: Number of defective units in the sample of 15 units

$X \sim B(15, 0.05)$

$\therefore P(X = x) = \binom{15}{x} (0.05)^x (0.95)^{15-x}$

$\therefore$ Probability of 5 items defective $= \binom{15}{5} \cdot (0.05)^5 (0.95)^{10}$

27) Probability of a diode failure is 0.03.

X: Number of ~~diode~~ diode failure in the circuit.

$\therefore X \sim B(200, 0.03)$

$\therefore$ Mean number of failures among the diode $= 200 \times 0.03$
$= 6$

Variance $= 200 \times (0.03) \times (1 - 0.03)$

$= 200 \times 0.03 \times 0.97$

$= 5.82$

The probability ~~of~~ that the board will work

$= P(X = 0) = \binom{200}{0} \cdot (0.03)^0 \cdot (0.97)^{200}$

28) $X \sim Geo(p)$

$\therefore P(X=k) = (1-p)^{k-1} \cdot p$

$= q^{k-1} p \qquad (q = 1-p)$

$P(X = \text{even}) = P(X=2) + P(X=4) + P(X=6) + \ldots$

$= qp + q^3 p + q^5 p + \ldots$

$= p\left[q + q^3 + q^5 + \ldots\right]$

$= p \cdot \dfrac{q}{(1-q^2)}$

$= p \cdot \dfrac{q}{p(1+q)}$

$= \dfrac{q}{1+q}$

$P(X = \text{odd}) = 1 - P(X = \text{even})$

$= 1 - \dfrac{q}{1+q}$

$= \dfrac{1}{1+q}$

29) The probability of ~~the soldier~~ a shot hit the target is $= 0.7$

X: Number of shots required for the first hit

~~then~~ ~~Since~~ $X \sim Geo(0.7)$

$\therefore P(X=n) = (1-0.7)^{n-1} \cdot (0.7)$

$= (0.3)^{n-1} (0.7)$

(i) $P(X=10) = (0.3)^9 \cdot (0.7)$

(ii) ~~P(~~ The Probability of the target would be hit ~~on~~ in less than 4 shots $= P(X<4)$

$= P(X=1) + P(X=2) + P(X=3)$

$= (0.7) + (0.3).(0.7) + (0.3)^2(0.7)$

$= (0.7) \times [1 + (0.3) + (0.3)^2]$

$= (0.7) \times \frac{1-(0.3)^3}{1-(0.3)}$

$= 1 - (0.3)^3$

(iii) Probability that the target would be hit in an even number of shots $= P(X = \text{even})$

$= P(X=2) + P(X=4) + \cdots$

$= \frac{0.3}{1+0.3}$ [Using problem no 28] $q = 0.3$

The average number of shots needed to hit the target is $= E(X)$.

$= \sum_{n\geq 1}^{\infty} n \cdot q^{n-1} p$

$= p \sum_{n\geq 1}^{\infty} n \cdot q^{n-1}$

$= p \cdot \frac{1}{(1-q)^2}$ $\left[\because \sum_{n\geq 1}^{\infty} n\, q^{n-1} = \frac{1}{(1-q)^2} \text{ (see prob. no. 6 or 5)}\right]$

$= \frac{1}{p} = \frac{1}{0.7}$

30) Same as 12.

31) Probability of a person will believe a rumor = 0.10

X: Number of person ~~heard~~ used to heard the rumor ~~before~~ only and the last one believe it.

$X \sim Geo(0.10)$

$\therefore P(X=x) = q^{x-1} p$ where $p = 0.10$
$q = 1-p = 0.90$

$\therefore$ The probability that the sixth person to hear the rumor is the first one to believe the rumor

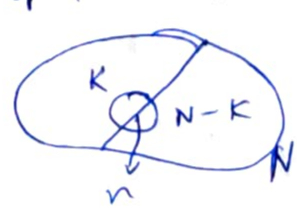
$= \cancel{P(X=6)}\ P(X=6)$

$= (0.90)^5 . (0.10)$

32) The random variable for this problem follows hypergeometric distribution.

Hypergeometric distribution: Given a N number of items, there are special type of K objects such that if any of the K object is choosen is a success. Choosing any of the remaining $(N-K)$ element is a failure. Let a sample of size n is taken from the items.

X: Number of succes in the sample of size n.



$R_X = \{0, 1, 2, \ldots, n\}$

$P(X=x) = \dfrac{\binom{K}{x} . \binom{N-K}{n-x}}{\binom{N}{n}}$ $X \sim HG(N, K, n)$

In this problem

X: Number of narcotic tablets are selected in 3 samples

$X \sim HG(15, 6, 3)$

Probability that the person will be caught $= P(X=1) + P(X=2) + P(X=3)$

$$= \frac{\binom{6}{1}\cdot\binom{9}{2}}{\binom{15}{3}} + \frac{\binom{6}{2}\cdot\binom{9}{1}}{\binom{15}{3}} + \frac{\binom{6}{3}\binom{9}{0}}{\binom{15}{3}}$$

33) Probability that the salesman will make his sale to a family $= \frac{1}{10}$

X: Number of ~~family~~ families the salesman contacted till the first sale

$X \sim Geo\left(\frac{1}{10}\right)$

$$\therefore P(X=n) = \left(\frac{9}{10}\right)^{n-1}\left(\frac{1}{10}\right)$$

(i) Probability that ~~I~~ he will make his first sale to the fourth ~~m~~ family $= P(X=4)$

$$= \left(\frac{9}{10}\right)^{3}\cdot\left(\frac{1}{10}\right)$$

(ii) If he is still waiting to make his first sale after visited 10 families then all the 10 ~~families~~ attempts are failure $\therefore$ P(failure after calling 10 families) $= \left(\frac{9}{10}\right)^{10}$

34) X: Number of defective bulbs in the sample.

Probability that a ~~bad~~ bulb is defective $= \frac{300}{10000}$

$= \frac{3}{100}$

$X \sim B\left(30, \frac{3}{100}\right)$

$\therefore P(X=x) = \binom{30}{x} \cdot \left(\frac{3}{100}\right)^{x} \cdot \left(\frac{97}{100}\right)^{30-x}$

Now, probability that at least one defective bulb in sample

$= P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - \binom{30}{0}\left(\frac{3}{100}\right)^{0}\left(\frac{97}{100}\right)^{30}$

35) Probability of contacting the disease $= p = \frac{1}{6}$

X: Number of mice are ~~ex~~ inoculated ~~bef~~ untill 2 mice have contacted the disease

$X \sim NB\left(2, \frac{1}{6}\right)$

Then p

$\therefore P(X=x) = \binom{x-1}{1} p \cdot (1-p)^{x-2} \cdot p$

$= \binom{x-1}{1} p^2 (1-p)^{x-2}$

$\therefore$ Probability that 8 mice are required $= P(X=8)$

$= \binom{7}{1} \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$

[putting $x=8$ and $p=\frac{1}{6}$]

Q6) The probability that all the coins are same

$= P($All three are head$) + P($All three are tail$)$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$

$= \frac{1}{4}$

$\therefore$ Probability that a odd one occur $= 1 - \frac{1}{4} = \frac{3}{4}$

X: Number of tosses till the ~~fair~~ odd one occurs

$X \sim Geo\left(\frac{3}{4}\right)$

$\therefore P(X=x) = \left(\frac{1}{4}\right)^{x-1}\left(\frac{3}{4}\right)$

$\therefore$ Probability that fewer than 4 tosses are needed

$= P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$$= \left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)$$

$$= \left(\frac{3}{4}\right)\left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2\right]$$

$$= \left(\frac{3}{4}\right) \times \frac{\left(1 - \left(\frac{1}{4}\right)^3\right)}{\left(1 - \frac{1}{4}\right)}$$

$$= 1 - \left(\frac{1}{4}\right)^3$$

37) Probability of refusal $= \frac{20}{100} = 0.2$

X: Number of people interviewed till ~~the~~ the first refusal

$\therefore X \sim Geo(0.2)$

$\therefore P(X = x) = (1 - 0.2)^{x-1}(0.2)$

$= (0.8)^{x-1}(0.2)$

~~50~~ Probability that 50 people were interviewed before first refusal $= P(X = 51) = (0.8)^{50} \cdot (0.2)$

~~Ans~~ (i) The probability of occurrance of this event is very low.

(ii) Expected number of people interviewed before first refused $= E(X)$

$= \frac{1}{0.2}$ [See problem number 29 for the expectation]

$= 5$

38) Probability that a product is defective $= \frac{5}{100} = 0.05$

X: Number of items are to be examined in order to get 2 defective.

$\therefore X \sim NB(2, 0.05)$

$P(X=x) = \binom{x-1}{1}(0.05)^1 \cdot (0.95)^{x-2} \; (0.05)$

$= \binom{x-1}{1}(0.05)^2 (0.95)^{x-2}$

The probability that at least 4 items are to be examined

$= P(X \geq 4)$

$= 1 - P(X<4)$

$= 1 - P(X=2) - P(X=3)$

$= 1 - \binom{1}{1}(0.05)^2 (0.95)^0 - \binom{2}{1}(0.05)^2 (0.95)$

39) Probability that a lot is defective $= 1 - 0.9$
$= 0.1$

X: Number of lots need to produce to obtain 3 defective lot

$X \sim NB(3, 0.1)$

$\therefore P(X=x) = \binom{x-1}{2}(0.1)^2 \cdot (0.9)^{x-3} \cdot (0.1)$

$= \binom{x-1}{2}(0.1)^3 (0.9)^{x-3}$

Probability that 20 lots will be produced in order to obtain

3rd defective lot = $P(X=20)$

$$=\binom{19}{2}(0.1)^3(0.9)^{17}$$

$$E(X)=\sum_{x=3}^{\infty} x.\binom{x-1}{2}(0.1)^3(0.9)^{x-3}$$

$$=(0.1)^3\sum_{x=3}^{\infty} x.\binom{x-1}{2}(0.9)^{x-3}$$

$$=(0.1)^3\sum_{x=3}^{\infty}\frac{x!}{2!\,(x-3)!}(0.9)^{x-3}$$

$$=(0.1)^3.3.\sum_{x=3}^{\infty}\frac{x!}{3!\,(x-3)!}(0.9)^{x-3}$$

$$=3.(0.1)^3\sum_{x=3}^{\infty}\binom{x}{3}(0.9)^{x-3}$$

$$\because \sum_{t=0}^{\infty}\binom{t+r-1}{r-1}a^t=\frac{1}{(1-a)^r}$$

$$=3.(0.1)^3\left[\frac{1}{(1-(0.9))^{4}}\right]$$

$$=\frac{3}{(0.1)}=30$$

~~Befor~~

$V(X) = \dfrac{3 \times 0.9}{(0.1)^2}$ [variance of negative binomial $NB(r,p)$ $= \dfrac{rq}{p^2}$]

$= 270$

40) ~~Numbe~~. Total number of items = 20.

X: Number of defective items in the sample of six items.

$\therefore X \sim HG(20, 5, 6)$

$$P(X=x) = \frac{\binom{5}{x}\binom{15}{6-x}}{\binom{20}{6}}$$

Probability that the shipment will be accepted ~~is~~

$= P(X \leq 1)$

$= P(X=0) + P(X=1)$

$$= \frac{\binom{5}{0}\binom{15}{6}}{\binom{20}{6}} + \frac{\binom{5}{1}\binom{15}{5}}{\binom{20}{6}}$$

41) Probability of success at a location = 0.25

(i) Probability that driller drills 10 locations and find one success $= \binom{10}{1} \cdot (0.25)(0.75)^9$

(ii) Probability that the driller go bankrupt $= (0.75)^{10} \cdot (0.25)$

42) X: Number of defective batteries in the lot.

$X \sim B(75, 0.001)$

~~P(X=0)~~

(i) Probability that the lot is accepted $= P(X=0) = \binom{75}{0}(0.001)^0 \cdot (0.999)^{75}$

(ii) Probability that the lot is rejected on 20th test $= (0.999)^{19} \cdot (0.001)$

(iii) ~~Probability that the lot is rejected in 10 or less trials~~

~~= Probability that the lot is~~ .

Y: Number of tests till the first failure.

$Y \sim Geo(0.001)$

$P(Y=x) = (0.999)^{x-1}(0.001)$

Probability that the lot is rejected in 10 or less trials

$= P(X \leq 10)$

$= P(X=1) + P(X=2) + P(X=3) + \dots + P(X=10)$

43) X: Number of defective in the ~~lot~~ sample of 6 from the 40% lot.

Y: Number of defective in the sample of 6 from 60% lot.

$X \sim HG(20, 5, 6)$

$P(X=x) = \dfrac{\binom{5}{x}\binom{15}{6-x}}{\binom{20}{6}}$

$Y \sim HG(20, 2, 6)$

$P(Y=y) = \dfrac{\binom{2}{y}\binom{18}{6-y}}{\binom{20}{6}}$

Probability that the first lot is accepted $= P(X=0)$

$= \dfrac{\binom{5}{0}\binom{15}{6}}{\binom{20}{6}}$

~~P~~ Probability that the second lot is accepted = $P(Y=0)$

$$= \frac{\binom{8}{0}\binom{18}{6}}{\binom{20}{6}}$$

$\therefore$ Probability that the lot is accepted = $P(X=0)+P(Y=0)$

$$= \frac{\binom{15}{6}+\binom{18}{6}}{\binom{20}{6}}$$

$\therefore$ Probability that the lot is rejected $= 1-\frac{\binom{15}{6}+\binom{18}{6}}{\binom{20}{6}}$

44) X: Number of births in a family untill the second daughter is born.

Probability of male child = 0.5

$\therefore$ probability of ~~girl~~ female child = 0.5

$X \sim NB(2, 0.5)$

$$P(X=x) = \binom{x-1}{1}(0.5)(0.5)^{x-2}\cdot(0.5)$$

$$= \binom{x-1}{1}(0.5)^x$$

$\therefore$ Probability that the sixth child in the family is the second daughter = $P(X=6) = \binom{5}{1}(0.5)^6$

$$= 5\times(0.5)^6$$

45) Probability that the item is defective $= \frac{3}{10}$

Probability that the item is non defective $= \frac{7}{10}$

X: Number of defective items in the sample of 10 items.

$$X \sim B\left(10, \frac{3}{10}\right)$$

$$P(X = x) = \binom{10}{x} \left(\frac{3}{10}\right)^x \left(\frac{7}{10}\right)^{10-x}$$

(i) Probability that not more than one defective will be obtained $= P(X=0) + P(X=1)$

$$= \binom{10}{0} \left(\frac{3}{10}\right)^0 \left(\frac{7}{10}\right)^{10} + \binom{10}{1} \left(\frac{3}{10}\right) \left(\frac{7}{10}\right)^9$$

(ii) Poisson approximation. $\lambda = np = 10 \times \frac{3}{10} = 3$

$\therefore X \sim P(3)$

$\therefore P(X = x) = e^{-3} \frac{3^x}{x!}$

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MA225: Probability Theory and Random Process
B.Tech. 2nd year

Tutorial Sheet-4

1. A continuous random variable has probability density function (PDF) defined as $f_X(x) = cxe^{-\frac{x}{2}}$, $x \geq 0$ and $f_X(x) = 0$, $x < 0$.(i) Find the constant c, (ii) What is the CDF of X. (iii) Find the mean, variance and standard deviation of X.(iv) Where is the median of X located.

2. In a dart game the player wins at a circular target having a radius of 25 centimeters. Let X be the distance (in centimeters) between the dart's impact point and the center of the target. Suppose that $P(X \leq x) = c\pi x^2$, $0 \leq x \leq 25$ and $= 1$, $x > 25$, where c is a constant.Evaluate (i) the constant c (ii) the PDF of X (iii) the mean of X (iv) the probability $P(X \leq 10|X \geq 5)$ (v) It costs 1\$ to throw a dart and the player wins 10\$ if $X \leq r$, 1\$ if $r < X \leq 2r$, 0\$ if $2r < X \leq 25$. For what values of r is the average gain of the player equal to 0.25\$.

3. Show that the function defined as $f_X(x) = \frac{x(6+x)}{3(3+x)^2}$, $0 < x \leq 3$ and $= \frac{9(3+2x)}{x^2(3+x)^2}$, $x > 3$ is a probability density function (PDF).

4. Does the function $\theta^2 xe^{-\theta x},\ x > 0$, and $= 0,\ x \leq 0,\ \theta > 0$ defines a probability density function? If yes, find the corresponding distribution function and also evaluate $P(X \geq 1)$.

5. Are the following functions distribution functions. If so, find the corresponding PDF/PMF.
(i) $F(x) = 0,\ x \leq 0, = x/2,\ 0 \leq x < 1, = 1/2,\ 1 \leq x < 2,\ \ = x/4,\ 2 \leq x < 4,\ \ = 1,\ x \geq 4.$
(ii) $F(x) = 0,\ x < -\theta, = \frac{1}{2}(x/\theta + 1),\ |x| \leq \theta,\ \ = 1, x > \theta$
(iii) $F(x) = 0,\ x < 1, = \frac{(x-1)^2}{8},\ 1 \leq x < 3, = 1,\ x \geq 3.$

6. Let X be an RV with pdf $f(x) = \frac{\Gamma(m)}{\Gamma(1/2)\Gamma(m-\frac{1}{2})(1+x^2)^m},\quad -\infty < x < \infty,\ \ m \geq 1$. Evaluate $E(X^{2r})$ whenever it exists.

7. Let $f(x)$ be the density function of the RV X. Suppose that X has symmetric distribution about a. Show that the mean of X is a itself.

8. (i) Let X be a continuous random variable with density function function $f(x)$ and distribution function $F(x)$. Then Show that $E(X) = \int_0^{\infty}[1 - F(x)]dx - \int_{-\infty}^{0} F(x)dx$ provided $x\{1 - F(x) - F(-x)\} \to 0$ as $x \to \infty$.
(ii) When X is a nonnegative RV, then $E(X) = \int_0^{\infty}[1 - F(x)]dx$.

9. Let X be a RV with Distribution Function $F(x) = 1 - 0.8e^{-x},\ x \geq 0$ and $F(x) = 0,\ x < 0$. Find EX.

10. Let X be an RV with density function $f(x) = 1/2,\ \ -1 \leq x \leq 1$, and $= 0$ otherwise. Find the distribution function of $\max(X, 0)$.

11. Find the moment generating function for the density function $\frac{1}{2a}e^{-\frac{|x-\mu|}{a}},\ \ -\infty < x < \infty, a > 0,\ \ -\infty < \mu < \infty$. Check whether or not it is a density function.

12. Let $f_X(x) = \frac{1}{2}[1 - \frac{|x-3|}{2}]$, $1 < x < 5$. Check that $f_X(x)$ is a PDF. Find mean, median, variance and p^{th} quantile of X.

13. Let $f_X(x) = \frac{k}{\beta}[1 - \frac{(x-\alpha)^2}{\beta^2}]$, $(\alpha - \beta) < x < (\alpha + \beta)$ where $-\infty < \alpha < \infty$,$\beta > 0$. Find the value of k so that $f_X(x)$ is a PDF. Find mean, median, variance and p^{th} quantile of X. Also evaluate $E(|X - \alpha|)$

Tutorial-4.

1) $f_X(x) = c\,x\,e^{-x/2}$, $x \geq 0$

$= 0$, $x < 0$

(i) $\because$ $f_X(x)$ is pdf. then.

$$\int_{-\alpha}^{\alpha} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\alpha} c.\,x.\,e^{-x/2}\,dx = 1$$

$$\Rightarrow c\int_{0}^{\alpha} x\,e^{-\frac{x}{2}}\,dx = 1$$

$$\Rightarrow c.\left[\left[x\int e^{-\frac{x}{2}}\,dx\right]_0^{\alpha} - \int_0^{\alpha}\left[\frac{d(x)}{dx}.\int e^{-\frac{x}{2}}\,dx\right]dx\right] = 1$$

$$\Rightarrow c\left[\left[-x.\,2.\,e^{-\frac{x}{2}}\right]_0^{\alpha} + \int_0^{\alpha} 2\,e^{-\frac{x}{2}}\,dx\right] = 1$$

$$\Rightarrow c\left[0 + 2\int_0^{\alpha} e^{-\frac{x}{2}}\,dx\right] = 1$$

$$\Rightarrow 2c\,.2\left[-e^{-\frac{x}{2}}\right]_0^{\alpha} = 1$$

$$\Rightarrow 2c.\,2.\,[0+1] = 1$$

$$\Rightarrow c = \frac{1}{4}$$

(ii) $F_X(x) = \int_{-\alpha}^{x} f_X(t)\,dt$

$\therefore F_X(x) = 0$, $x < 0$

$= \int_{-0}^{x} \frac{1}{4}.\,t.\,e^{-\frac{t}{2}}\,dt$ $x \geq 0$

$$= \frac{1}{4}\int_{0}^{x} t \cdot e^{-\frac{t}{2}}\, dt$$

$$= \frac{1}{4} \cdot \int_0^x t\, e^{-\frac{t}{2}}\, dt$$

$$= \frac{1}{4}\left[\left[t \int e^{-\frac{t}{2}} dt\right]_0^x - \int_0^x \left[\frac{d}{dt}(t) \cdot \int e^{-\frac{t}{2}} dt\right] dt\right]$$

$$= \frac{1}{4}\left[\left[2t \cdot \left[-e^{-\frac{t}{2}}\right]\right]_0^x + 2\int_0^x e^{-\frac{t}{2}} dt\right]$$

$$= \frac{1}{4}\left[-2x\, e^{-\frac{x}{2}} - 4\left[e^{-\frac{x}{2}} - 1\right]\right]$$

$$= \frac{1}{2}\left[-x e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + 2\right]$$

$$\therefore F_X(x) = 0 \quad , x < 0$$
$$= \frac{1}{2}\left[-x e^{-\frac{x}{2}} - 2e^{-\frac{x}{2}} + 2\right] \quad , x \geq 0$$

(iii) $$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^{\infty} x \cdot f_X(x)\, dx$$

$$= \frac{1}{4}\int_0^{\infty} x \cdot x \cdot e^{-\frac{x}{2}} dx \quad \left[\because f_X(x) = \frac{1}{4} x e^{-\frac{x}{2}} \quad x \geq 0; \quad = 0 \quad x < 0\right]$$

$$= \frac{1}{4}\int_0^{\infty} x^2 e^{-\frac{x}{2}} dx$$

$$= \frac{1}{4}\left[\left[x^2 \int e^{-\frac{x}{2}} dx\right]_0^{\infty} - \int_0^{\infty}\left[\frac{d}{dx}(x^2) \int e^{-\frac{x}{2}} dx\right] dx\right]$$

$$= \frac{1}{4}\left[\left[x^2\, 2\left[-e^{\frac{x}{2}}\right]\right]_0^{\infty} + 4\int_0^{\infty} x.e^{-\frac{x}{2}}\,dx\right]$$

$$= \cancel{\frac{1}{4}} \times 4 \int_0^{\infty} x.e^{-\frac{x}{2}}\,dx$$

$$= \left[-2x.e^{-\frac{x}{2}}\right]_0^{\infty} + 2\int_0^{\infty} e^{-\frac{x}{2}}\,dx$$

$$= \cancel{4}\, 2 \times \left[-2.e^{-\frac{x}{2}}\right]_0^{\infty}$$

$$= 4.$$

$$E(x^2) = \frac{1}{4}\int_0^{\infty} x^2.x e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\int_0^{\infty} x^3 e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\left[\left[-2x^3 e^{-\frac{x}{2}}\right]_0^{\infty} - \int_0^{\infty}\left\{3x^2.\int e^{-\frac{x}{2}}\,dx\right\}dx\right]$$

$$= \frac{1}{4}\left[3 \times 2 \int_0^{\infty} x^2 e^{-\frac{x}{2}}\,dx\right]$$

$$= \frac{1}{4} \times 6.\ 4 \times E(x)$$

$$= 6 \times 4$$

$$= 24$$

$$\therefore V(x) = E(x^2) - \{E(x)\}^2 = 24 - (4)^2$$

$$= 24 - 16$$

$$= 8$$

$S.D.(X) = \sqrt{V(X)} = \sqrt{8} = 2\sqrt{2}$

iv) For continuous random variable, if μ is median, then,

$$F_X(\mu) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left[-\mu e^{-\frac{\mu}{2}} - 2e^{-\frac{\mu}{2}} + 2\right] = \frac{1}{2}$$

$$\Rightarrow -\mu e^{-\frac{\mu}{2}} - 2e^{-\frac{\mu}{2}} + 2 = 1$$

$$\Rightarrow (\mu+2)\, e^{-\frac{\mu}{2}} = 1$$

$$\Rightarrow (\mu+2) = e^{\frac{\mu}{2}}$$

$$\Rightarrow (\mu+2) - e^{\frac{\mu}{2}} = 0$$

Solve this eqn numerically. [use Newton Raphson with interval [-6,6]]

2) X: The distance between the dart's impact point and the center of the target.

$$P(X \leq x) = c\pi x^2, \quad 0 \leq x \leq 25$$
$$= 1, \quad x > 25$$

(i) CDF of X is

$$F_X(x) = 0 \quad x < 0$$
$$= c\pi x^2 \quad 0 \leq x \leq 25$$
$$= 1 \quad x > 25$$

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(c\pi x^2) = 2c\pi x \quad \text{when } 0 \leq x \leq 25$$

$$\therefore f_X(x) = 2c\pi x \quad 0 \leq x \leq 25$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \int_{-\infty}^{\infty} f_x(x)\, dx = 1$$

$$\Rightarrow \int_{0}^{25} 2c\pi x\, dx = 1$$

$$\Rightarrow 2c\pi \left[\frac{x^2}{2}\right]_0^{25} = 1$$

$$\Rightarrow c = \frac{1}{(25)^2 \pi}$$

$$= \frac{1}{625\pi}$$

(ii) P.D.F of X is $f_x(x) = \frac{2}{625\cancel{\pi}} \cancel{\pi} x \qquad 0 \le x \le 25$

$= 0 \qquad$ otherwise

(iii) $$E(X) = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$= \int_0^{25} x \cdot \frac{2}{625\cancel{\pi}} \cancel{\pi} x\, dx$$

$$= \frac{2}{625\cancel{\pi}} \left[\frac{x^3}{3}\right]_0^{25}$$

$$= \frac{25 \times 2 \times \cancel{\pi}}{3}$$

$$= \frac{50\cancel{\pi}}{3}$$

(iv) $P(X \le 10 \mid X \ge 5) = \dfrac{P(5 \le X \le 10)}{P(X \ge 5)}$

$$= \frac{P(5 \le X \le 10)}{1 - P(X < 5)}$$

$$= \frac{\int_5^{10} f_X(x)\,dx}{1 - \int_0^5 f_X(x)\,dx}$$

$$= \frac{\int_5^{10} \frac{2}{625} x\,dx}{1 - \int_0^5 \frac{2}{625} x\,dx}$$

$$= \frac{\frac{2}{625}\left[\frac{x^2}{2}\right]_5^{10}}{1 - \frac{2}{625}\left[\frac{x^2}{2}\right]_0^5}$$

$$= \frac{\frac{1}{625}(10^2 - 5^2)}{1 - \frac{1}{625} \times 25}$$

$$= \frac{10^2 - 5^2}{625 - 25}$$

$$= \frac{5 \times 15}{600}$$

$$= \frac{1}{8}$$

(v) $10 \times \int_0^r x \cdot \frac{2}{625} x\,dx + 1 \times \int_r^{2r} x \cdot \frac{2}{625} x\,dx + 0 \times \int_{2r}^{25} x \cdot \frac{2}{625} x\,dx = 1 + 0.25$

$\Rightarrow \frac{2}{625} \times \frac{1}{3}\left[10r^3 + 8r^3 - r^3\right] = 1.25$

$\Rightarrow r^3 = \frac{3 \times 625 \times 1.25}{2 \times 17}$

3)

$$f_x(x) = \frac{x(6+x)}{3(3+x)^2}, \quad 0 < x \le 3$$

$$= \frac{9(3+2x)}{x^2(3+x)^2}, \quad x > 3$$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_0^3 \frac{x(6+x)}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{x^2(3+x)^2}\,dx$$

$$= \int_0^3 \frac{x^2+6x}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{(3x+x^2)^2}\,dx$$

$$= \int_0^3 \frac{x^2+6x+9-9}{3(3+x)^2}\,dx + \int_3^{\infty} \frac{9(3+2x)}{(3x+x^2)^2}\,dx$$

$$= \int_0^3 \frac{1}{3}\,dx - 3\int_0^3 \frac{1}{(3+x)^2}\,dx + \int_3^{\infty} \frac{9\,d(3x+x^2)}{(3x+x^2)^2}$$

$$= \left[\frac{x}{3}\right]_0^3 + 3\left[\frac{1}{(3+x)}\right]_0^3 - \left[\frac{9}{(3x+x^2)}\right]_3^{\infty}$$

$$= 1 + \frac{1}{2} - 1 + \frac{9}{18}$$

$$= 1$$

$\therefore f_x(x)$ is a pdf.

4) $f_X(x) = \theta^2 x e^{-\theta x}, \quad x > 0$

$\quad\quad = 0 \quad , \quad x \le 0$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \theta^2 x e^{-\theta x}\,dx$$

$$= \theta^2 \int_0^{\infty} x e^{-\theta x}\,dx$$

$$= \theta^2 \left\{ \left[-\frac{x.e^{-\theta x}}{\theta} \right]_0^{\infty} + \frac{1}{\theta} \int_0^{\infty} e^{-\theta x}\,dx \right]$$

$$= \theta^2 \times \frac{1}{\theta} \cdot \left[-e^{-\theta x} \right]_0^{\infty}$$

$$= \frac{1}{\theta^2} \times \theta^2$$

$$= 1.$$

$\therefore f_X(x)$ is a pdf.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_0^{x} \theta^2 t e^{-\theta t}\,dt$$

$$= \theta^2 \int_0^{x} t e^{-\theta t}\,dt$$

$$= \theta^2 \left\{ \left[\frac{t e^{-\theta t}}{-\theta} \right]_0^{x} + \frac{1}{\theta} \int_0^{x} e^{-\theta t}\,dt \right\}$$

$$= \theta^2 \left[-\frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta^2} \left[e^{-\theta x} - 1 \right] \right]$$

$$= 1 - e^{-\theta x} - x\theta e^{-\theta x}$$

$$= 1 - e^{-\theta x}(1 + x\theta)$$

$$P(X \geq 1) = 1 - P(X < 1)$$
$$= 1 - \left[1 - e^{-\theta}(\theta+1)\right]$$
$$= e^{-\theta}(\theta+1)$$

5) i) $F(x) = 0 \quad x \leq 0$

$= \frac{x}{2} \quad 0 \leq x < 1$

$= \frac{1}{2} \quad 1 \leq x < 2$

$= \frac{x}{4} \quad 2 \leq x < 4$

$= 1 \quad x \geq 4$

$F(x)$ is right continuous and non decreasing

$F(-\infty) = 0$

and $F(\infty) = 1$

$\therefore F(x)$ is C.D.F.

$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$ when $0 \leq x < 1$

$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{x}{4}\right) = \frac{1}{4}$ $2 \leq x < 4$

$\therefore f_X(x) = \frac{1}{2} \quad 0 \leq x < 1$

$= \frac{1}{4} \quad 2 \leq x < 4$

$= 0$ otherwise

(ii) $F(x) = 0, \quad x < -\theta$

$= \frac{1}{2}\left(\frac{x}{\theta}+1\right), \quad |x| \leq \theta$

$= 1, \quad x > \theta$

$F(x)$ is right continuous and non decreasing.

Also, $F(-\infty) = 0$

$F(\infty) = 1$

$\therefore F(x)$ is a CDF

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{x}{\theta}+1\right)\right) = \frac{1}{2\theta} \quad \text{when } |x| \leq \theta$$

$\therefore f_x(x) = \frac{1}{2\theta} \quad |x| \leq \theta$

$= 0 \quad$ otherwise

(iii) $F(x) = 0 \quad x < 1$

$= \frac{(x-1)^2}{8} \quad 1 \leq x < 3$

$= 1 \quad x \geq 3$

$F(x)$ is right continuous everywhere except at $x=3$ ~~and~~ and $F(x)$ is non decreasing. Also $F(-\infty) = 0$ and $F(\infty) = 1$.

$\therefore F(x)$ is a CDF. At $x=3$, $F(3-0) - F(3) = P(X=3)$

$\Rightarrow P(X=3) = \frac{1}{2}$

$$\frac{d}{dx}(F(x)) = \frac{8(x-1)}{64} = \frac{(x-1)}{4} \quad 1 \leq x < 3$$

$\therefore f_x(x) = \frac{x-1}{4} \quad 1 \leq x < 3$

$= \frac{1}{2} \quad x = 3 \quad = 0$ elsewhere.

6)

$$f(x) = \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})\,(1+x^2)^m}, \quad -\infty < x < \infty, \quad m \geq 1.$$

$$E(x^{2r}) = \int_{-\infty}^{\infty} x^{2r} \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^{2r} \cdot \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})\,(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{-\infty}^{\infty} \frac{x^{2r}}{(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \cdot 2\int_{0}^{\infty} \frac{x^{2r}}{(1+x^2)^m}\, dx$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{0}^{\infty} \frac{x^{2r-1}}{(1+x^2)^m}\, d(x^2)$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \int_{0}^{\infty} \frac{(x^2)^{(r+\frac{1}{2})-1}}{(1+x^2)^m}\, d(x^2)$$

$$= \frac{\Gamma(m)}{\Gamma(1/2)\,\Gamma(m-\frac{1}{2})} \beta\left(r+\frac{1}{2},\, m-r-\frac{1}{2}\right) \quad \left[\because \beta(m,n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx\right.$$

7) ~~X~~ X has a symmetric distribution about a.

$\therefore\ P(X \geq a+x) = P(X \leq a-x)$

$\Rightarrow\ 1 - F_X(a+x) = F_X(a-x)$

$\Rightarrow\ -\frac{d}{dx} F_X(a+x) = \frac{d}{dx} F_X(a-x)$

$\Rightarrow\ f_X(a+x) = f_X(a-x)$

Now $E[X-a] = \int_{-\infty}^{\infty} (x-a) f_X(x)\, dx$

$= \int_{-\infty}^{a} (x-a) f_X(x)\, dx + \int_{a}^{\infty} (x-a) f_X(x)\, dx$

$= I_1 + I_2$

~~Now, but~~

Now, $I_1 = \int_{-\infty}^{a} (x-a) f_X(x)\, dx$

Let $x - a = z$

$dx = dz$

$\therefore\ I_1 = \int_{-\infty}^{0} z \cdot f_X(z+a)\, dz$

$I_2 = \int_{a}^{\infty} (x-a) f_X(x)\, dx$

Let, $x - a = -z$

$dx = -dz$

$\therefore\ I_2 = +\int_{0}^{-\infty} z \cdot f_X(a-z)\, dz$

$= -\int_{-\infty}^{0} z \cdot f_X(a-z)\, dz = -\int_{-\infty}^{0} z \cdot f_X(a+z)\, dz$

$= -I_1$

$\therefore \quad E(X-c) = I_1 - I_1$

$= 0$

$\Rightarrow E(X) - c = 0$

$\Rightarrow E(X) = c$

8) (i) $\int_0^{\infty} [1-F(x)]\,dx - \int_{-\infty}^{0} F(x)\,dx$

$$= \left[[1-F(x)].\int 1.dx\right]_0^{\infty} - \int_0^{\infty}\left[\frac{d}{dx}[1-F(x)].\int 1.dx\right]dx$$

$$- \left[F(x).\int 1.dx\right]_{-\infty}^{0} + \int_{-\infty}^{0}\left[\frac{d}{dx}(F(x)).\int 1.dx\right]dx$$

$$= \left[x.(1-F(x))\right]_0^{\infty} + \int_0^{\infty} x.f_X(x)\,dx - \left[x F(x)\right]_{-\infty}^{0} + \int_{-\infty}^{0} x f_X(x)\,dx$$

$$= \left[x(1-F(x))\right]_0^{\infty} - \left[x F(-x)\right]_0^{\infty} + \int_{-\infty}^{\infty} x.f_X(x)\,dx$$

$$= \left[x(1-F(x) - F(-x))\right]_0^{\infty} + E(x)$$

Given that as $x \to \infty$ $\quad x(1-F(x) - F(-x)) \to 0$

$$\Rightarrow \int_0^{\infty} (1-F(x))\,dx - \int_{-\infty}^{0} F(x)\,dx = E(X)$$

(ii) when X is non negative,

then, $F(x) = 0$ as $x < 0$

Then $\int_{-\infty}^{0} F(x)\,dx = 0$

$\therefore \quad E(X) = \int_{0}^{\infty} (1 - F(x))\, dx$

9) $\quad F(x) = 1 - 0.8\, e^{-x}, \quad x \geq 0$

$\quad = 0 \quad , \quad x < 0$

$f_X(x) = \frac{d}{dx}(F(x)) = 0.8 . e^{-x}$

$\therefore f_X(x) = 0 \quad , x < 0$

$= 1 - 0.8 \quad , x = 0$

$= 0.2 \quad , x = 0$

$= 0.8 . e^{-x}, \quad x > 0$

$$E(X) = \int_{-\infty}^{0} x . f_X(x)\, dx + 0 \times P(X=0) + \int_{0}^{\infty} x . f_X(x)\, dx$$

$$= 0 + 0 \times 0.2 + \int_{0}^{\infty} x . 0.8 \times e^{-x}\, dx$$

$$= (0.8) \int_{0}^{\infty} x e^{-x}\, dx$$

$$= 0.8 \left[\left[-x e^{-x} \right]_{0}^{\infty} + \int_{0}^{\infty} e^{-x}\, dx \right]$$

$$= 0.8 \times \left[- e^{-x} \right]_{0}^{\infty}$$

$$= 0.8$$

10) $f(x) = \frac{1}{2} \quad -1 \leq x \leq 1$

$\quad = 0 \quad$ otherwise

$Y = \max(X, 0)$

$\therefore \; Y = 0 \quad x \leq 0$

$\quad = x \;$ when $\; x \geq 0$

Now,

$$F(y) = P(Y \leq y) = \int_{-\infty}^{y} f(x)\,dx$$

$$= \int_{-\infty}^{y} \frac{1}{2}\,dx$$

$$= \frac{1}{2}(y+1) \qquad -1 \leq y \leq 1$$

$$F(y) = \frac{1}{2}(y+1) \quad -1 \leq y \leq 1$$

$$= 0 \quad \text{elsewhere}$$

11) $f_x(x) = \frac{1}{2a} e^{-\frac{|x-\mu|}{a}}$, $-\infty < x < \infty)$, $a > 0$, $-\infty < \mu < \infty$

$$\int_{-\infty}^{\infty} f_x(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx$$

$$= \int_{-\infty}^{\mu} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx + \int_{\mu}^{\infty} \frac{1}{2a} e^{-\frac{|x-\mu|}{a}} dx$$

$$= \frac{1}{2a}\int_{-\infty}^{\mu} e^{\frac{(x-\mu)}{a}} dx + \int_{\mu}^{\infty} \frac{1}{2a} e^{-\frac{(x-\mu)}{a}} dx$$

$$= \frac{1}{2a} \cdot a \left[e^{\frac{x-\mu}{a}} \right]_{-\infty}^{\mu} + -\frac{1}{2a} \cdot a \left[e^{-\frac{(x-\mu)}{a}} \right]_{\mu}^{\infty}$$

$$= \frac{1}{2a} \cdot a + \frac{1}{2a} \cdot a = 1$$

$\therefore f_x(x)$ is pdf.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_x(x)\, dx$$

$$= \frac{1}{2a}\int_{-\infty}^{\mu} e^{tx} e^{\frac{(x-\mu)}{a}} dx + \frac{1}{2a}\int_{\mu}^{\infty} e^{tx} e^{-\frac{(x-\mu)}{a}} dx$$

$$= \frac{1}{2a}\left[\int_{-\infty}^{\mu} e^{(t+\frac{1}{a})x - \frac{\mu}{a}} dx + \int_{\mu}^{\infty} e^{(t-\frac{1}{a})x + \frac{\mu}{a}} dx \right]$$

$$= \frac{1}{2a}\left\{ \left[\frac{a}{(at+1)} e^{(t+\frac{1}{a})x - \frac{\mu}{a}} \right]_{-\infty}^{\mu} + \left[\frac{a}{at-1} \cdot e^{(t-\frac{1}{a})x + \frac{\mu}{a}} \right]_{\mu}^{\infty} \right\}$$

$$= \frac{1}{2a}\left[\frac{a}{(at+1)} e^{t\mu} + \frac{a}{(at-1)} e^{\mu t} \right]$$

$$= \frac{1}{2} e^{\mu t} \left[\frac{1}{at+1} + \frac{1}{(at-1)} \right]$$

$$= \frac{1}{2} \cdot e^{bt} \cdot \frac{2at}{(a^2t^2 - 1)}$$

$$= \frac{at\, e^{bt}}{(a^2t^2 - 1)}$$

12) $f_x(x) = \frac{1}{2}\left[1 - \frac{|x-3|}{2}\right], \quad 1 < x < 5$

$$\int_{-\alpha}^{\alpha} f_x(x)\, dx = \int_1^5 \frac{1}{2}\left(1 - \frac{|x-3|}{2}\right) dx$$

$$= \frac{1}{2}\int_1^3 \left[1 + \frac{(x-3)}{2}\right] dx + \frac{1}{2}\int_3^5 \left(1 - \frac{(x-3)}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_1^3 \left(\frac{x}{2} - \frac{1}{2}\right) dx + \int_3^5 \left(\frac{5}{2} - \frac{x}{2}\right) dx\right]$$

$$= \frac{1}{2}\left\{\left[\frac{x^2}{4} - \frac{x}{2}\right]_1^3 + \left[\frac{5x}{2} - \frac{x^2}{4}\right]_3^5\right\}$$

$$= \frac{1}{2}\left[\frac{9}{4} - \frac{3}{2} - \frac{1}{4} + \frac{1}{2} + \frac{25}{2} - \frac{25}{4} - \frac{15}{2} + \frac{9}{4}\right]$$

$$= \frac{1}{2} \times 2$$

$$= 1.$$

$\therefore f_x(x)$ is a ~~prob~~ PDF

$$E(x) = \int_{-\alpha}^{\alpha} x \cdot f_x(x)\, dx$$

$$= \frac{1}{2}\int_1^5 x \cdot \left[1 - \frac{|x-3|}{2}\right] dx$$

$$= \frac{1}{2}\left\{ \int_1^3 x.\left(1+\frac{x-3}{2}\right)dx + \int_3^5 x\left(1-\frac{x-3}{2}\right)dx \right\}$$

$$= \frac{1}{2}\left\{ \int_1^3 \left(\frac{x^2}{2}-\frac{x}{2}\right)dx + \int_3^5 \left(\frac{5x}{2}-\frac{x^2}{2}\right)dx \right\}$$

$$= \frac{1}{2}\left\{ \left[\frac{x^3}{6}-\frac{x^2}{4}\right]_1^3 + \left[\frac{5x^2}{4}-\frac{x^3}{6}\right]_3^5 \right\}$$

$$= \frac{1}{2}\left[\frac{27}{6}-\frac{9}{4}-\frac{1}{6}+\frac{1}{4}+\frac{125}{4}-\frac{125}{6}-\frac{45}{4}+\frac{27}{6}\right]$$

$$= \frac{1}{2}\times 6$$

$$= 3$$

For median,

$$\int_1^x \frac{1}{2}\left[1-\frac{|t-3|}{2}\right]dt = \frac{1}{2}$$

$$\Rightarrow \int_1^3 \frac{1}{2}\left[1+\frac{(t-3)}{2}\right]dt + \int_3^x \frac{1}{2}\left[1-\frac{(t-3)}{2}\right]dt = \frac{1}{2}$$

$$\Rightarrow \int_1^3 \left(\frac{t}{2}-\frac{1}{2}\right)dt + \int_3^x \left(\frac{5}{2}-\frac{t}{2}\right)dt = 1$$

$$\Rightarrow \left[\frac{t^2}{4}-\frac{t}{2}\right]_1^3 + \left[\frac{5t}{2}-\frac{t^2}{4}\right]_3^x = 1$$

$$\Rightarrow \left[\frac{9}{4}-\frac{3}{2}-\frac{1}{4}+\frac{1}{2}\right] + \left[\frac{5x}{2}-\frac{x^2}{4}-\frac{15}{2}+\frac{9}{4}\right] = 1$$

$$\Rightarrow 1 + \left[\frac{10x-x^2-21}{4}\right] = 1$$

$$\Rightarrow x^2-10x+21=0$$

$$\Rightarrow (x-3)(x-7)=0$$

$$\Rightarrow x=3. \quad \therefore \text{Median is } x=3$$

$$E(X^2) = \int_1^5 \frac{x^2}{2}\left[1 - \frac{|x-3|}{2}\right] dx$$

$$= \frac{1}{2}\int_1^3 x^2\left(1 + \frac{x-3}{2}\right) dx + \frac{1}{2}\int_3^5 x^2\left(1 - \frac{x-3}{2}\right) dx$$

$$= \frac{1}{2}\int_1^3 \left(\frac{x^3}{2} - \frac{x^2}{2}\right) dx + \frac{1}{2}\int_3^5 \left[\frac{5x^2}{2} - \frac{x^3}{2}\right] dx$$

$$= \frac{1}{2}\left\{\left[\frac{x^4}{8} - \frac{x^3}{6}\right]_1^3 + \left[\frac{5x^3}{6} - \frac{x^4}{8}\right]_3^5\right\}$$

$$= \frac{1}{2}\left[\frac{81}{8} - \frac{27}{6} - \frac{1}{8} + \frac{1}{6} + \frac{625}{6} - \frac{625}{8} - \frac{135}{8} + \frac{81}{8}\right]$$

$$= \frac{599}{48}$$

$$V(X) = \frac{599}{48} - \{E(X)\}^2 = \frac{599}{48} - 9$$

$$= \frac{167}{48}$$

p-th quantile

$$F_X(x_p) = p$$

$$\Rightarrow \int_1^{x_p} \frac{1}{2}\left[1 - \frac{|x-3|}{2}\right] dx = p$$

$$\Rightarrow \frac{1}{2}\int_1^3 \left(1 + \frac{x-3}{2}\right) dx + \frac{1}{2}\int_3^{x_p} \left(1 - \frac{x-3}{2}\right) dx = p$$

$$\Rightarrow \frac{1}{2}\int_1^3 \left(\frac{x}{2} - \frac{1}{2}\right) dx + \frac{1}{2}\int_3^{x_p} \left(\frac{5}{2} - \frac{x}{2}\right) dx = p$$

$$\Rightarrow \frac{1}{2} \times \left[\frac{x^2}{4} - \frac{x}{2}\right]_1^3 + \frac{1}{2}\cdot\left[\frac{5x}{2} - \frac{x^2}{4}\right]_3^{x_p} = p$$

$\Rightarrow \frac{1}{2}\left[\frac{9}{4} - \frac{3}{2} - \frac{1}{4} + \frac{1}{2}\right] + \frac{1}{2}\left[\frac{5x_p}{2} - \frac{x_p^2}{4} - \frac{15}{2} + \frac{9}{4}\right] = p$

$\Rightarrow \frac{1}{2} + \frac{1}{2}\left[\frac{10x_p - x_p^2 - 21}{4}\right] = p$

$\Rightarrow 4 + 10x_p - x_p^2 - 21 = 8p$

$\Rightarrow x_p^2 - 10x_p + 17 - 8p = 0$

Solving this equation for given p gives the p th quantile.

13) $f_X(x) = \frac{K}{\beta}\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] \quad (\alpha - \beta) < x < (\alpha + \beta)$

where $-\infty < \alpha < \infty$, $\beta > 0$

$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{K}{\beta}\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx = 1$

$\Rightarrow \frac{K}{\beta}\int_{\alpha-\beta}^{\alpha+\beta}\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx = 1$

$\Rightarrow \frac{K}{\beta}\left[x - \frac{(x-\alpha)^3}{3\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)} = 1$

$\Rightarrow \frac{K}{\beta}\left[(\alpha+\beta) - \frac{\beta^3}{3\beta^2} - (\alpha-\beta) - \frac{\beta^3}{3\beta^2}\right] = 1$

$\Rightarrow \frac{K}{\beta}\left[2\beta - \frac{\beta}{3} - \frac{\beta}{3}\right] = 1$

$\Rightarrow K\left[\frac{4}{3}\right] = 1$

$\Rightarrow K = \frac{3}{4}$

$$E(X) = \int_{-\alpha}^{\alpha} x \cdot f_X(x)\, dx$$

$$= \int_{\alpha-\beta}^{\alpha+\beta} x \cdot \frac{3}{4\beta}\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta}\int_{\alpha-\beta}^{\alpha+\beta} x\left[1 - \frac{(x-\alpha)^2}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta}\int_{\alpha-\beta}^{\alpha+\beta}\left[x - \frac{x^3 - 2\alpha x^2 + \alpha^2 x}{\beta^2}\right] dx$$

$$= \frac{3}{4\beta}\left[\frac{x^2}{2} - \frac{x^4}{4\beta^2} + \frac{2\alpha x^3}{3\beta^2} - \frac{\alpha^2 x^2}{2\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)}$$

<u>Median:</u>

$$\int_{\alpha-\beta}^{x} \frac{3}{4\beta}\left(1 - \frac{(t-\alpha)^2}{\beta^2}\right) dt = \frac{1}{2}$$

$$\Rightarrow \frac{3}{4\beta}\left[t - \frac{(t-\alpha)^3}{3\beta^2}\right]_{(\alpha-\beta)}^{x} = \frac{1}{2}$$

$$\Rightarrow x - \frac{(x-\alpha)^3}{3\beta^2} - (\alpha-\beta) - \frac{\beta^3}{3\beta^2} = \frac{2\beta}{3}$$

$$\Rightarrow x - \frac{(x-\alpha)^3}{3\beta^2} - \alpha + \beta - \frac{\beta}{3} = \frac{2\beta}{3}$$

$$\Rightarrow x - \frac{(x-\alpha)^2}{3\beta^2} = \alpha$$

$$\Rightarrow (x-\alpha)\left[1 - \frac{(x-\alpha)}{3\beta^2}\right] = 0$$

$\therefore$ $x = \alpha$ is a median.

<u>Variance</u>

$$E(X^2) = \int_{-\alpha}^{\alpha} x^2 f_x(x)\,dx$$

$$= \int_{\alpha-\beta}^{\alpha+\beta} x^2 \cdot \frac{3}{4\beta}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx$$

$$= \frac{3}{4\beta}\int_{\alpha-\beta}^{\alpha+\beta}\left(x^2 - \frac{x^4 - 2x^3\alpha + \alpha^2 x^2}{\beta^2}\right) dx$$

$$= \frac{3}{4\beta}\left[\frac{x^3}{3} - \frac{x^5}{5\beta^2} + \frac{2x^4\alpha}{4\beta^2} - \frac{\alpha^2 x^3}{3\beta^2}\right]_{(\alpha-\beta)}^{(\alpha+\beta)}$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

<u>P-th quantile</u>

$$F(x_p) = p$$

$$\Rightarrow \frac{3}{4\beta}\int_{\alpha-\beta}^{x_p}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx = p$$

$$\Rightarrow \int_{\alpha-\beta}^{x_p}\left(1 - \frac{(x-\alpha)^2}{\beta^2}\right) dx = \frac{4\beta p}{3}$$

$$\Rightarrow \left[x - \frac{(x-\alpha)^3}{3\beta^2} \right]_{(\alpha-\beta)}^{x_p} = \frac{4\beta p}{3}$$

$$\Rightarrow x_p - \frac{(x_p-\alpha)^3}{3\beta^2} - (\alpha-\beta) - \frac{\beta^3}{3\beta^2} = \frac{4\beta p}{3}$$

$$\Rightarrow \frac{3\beta^2 x_p - (x_p-\alpha)^3}{3\beta^2} - \alpha + \beta - \frac{\beta}{3} = \frac{4\beta}{3} p$$

Solving this equation gives the p-th quantile