Tutorial 2

1)

(i) ⇒ (ii) If A and B are independent

then $P(A\cap B) = P(A).P(B)$

$P(A^c\cap B) = ?$

$P(B) = P(A\cap B) + P(A^c\cap B)$

$= P(A).P(B) + P(A^c\cap B)$

$\therefore P(A^c\cap B) = P(A^c).P(B)$

(ii) ⇒ (iii) $P(A^c\cap B) = P(A^c).P(B)$

$P(A\cap B^c) = ?$

~~P(A) =~~ ~~P(A)~~

$P(A^c\cup B) = P(A^c) + P(B) - P(A^c\cap B)$

$= P(A^c) + P(B) - P(A^c).P(B)$

$= P(A^c) + P(B).P(A)$

$= 1 - P(A) + P(B).P(A)$

⇒ $= 1 - P(A).P(B^c)$

$\therefore P(A).P(B^c) = P(A\cap B^c)$

(iii) ⇒ (iv) $P(A\cap B^c) = P(A).P(B^c)$

~~P(A^c∪B)~~ $P(A^c\cup B) = 1 - P(A\cap B^c)$

$= 1 - P(A).P(B^c)$

Now $P(A^c\cap B^c) + \cancel{\ldots}\ P(B) = P(A^c\cup B)$

$P(A^c\cap B^c) = 1 - P(B) - P(A).P(B^c)$

$= P(A^c).P(B^c)$

2) $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= p_1 + p_2 - P(A \cup B) \geq p_1 + p_2 - 1$

$\therefore P(B|A) \geq \frac{(p_1 + p_2 - 1)}{p_1}$

$\geq 1 - \left[\frac{(1 - p_2)}{p_1}\right]$

3) E_1 = head appears both time.

E_2 = The coin which was chosen is the fair coin

$P(E_2|E_1) = ?$

$P(E_1|E_2) = \frac{1}{4}$

~~P(E_1) =~~

E_3 = The chosen coin is biased with $\frac{1}{4}$ prob of head

E_4 = ... $\frac{3}{4}$

$P(E_1|E_4) = \frac{9}{16}$

$\therefore P(E_1|E_3) = \frac{1}{16}$

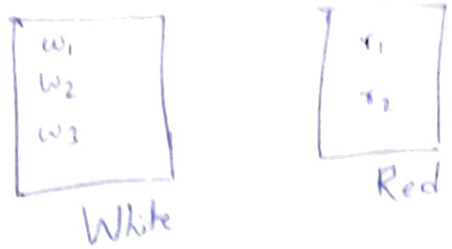
$P(E_1) = P(E_1|E_2).P(E_2) + P(E_1|E_3).P(E_3) + P(E_1|E_4).P(E_4)$

$= \frac{1}{3} \cdot \left[\frac{1}{4} + \frac{1}{16} + \frac{9}{16}\right] = \frac{1}{3} \times \frac{\cancel{14}\,7}{\cancel{16}\,8} = \frac{1}{3} \times \frac{7}{8}$

$P(E_2) = \frac{1}{3}$

$\therefore P(E_2|E_1) = \frac{\frac{1}{3} \times \frac{1}{4}}{\frac{1}{3} \times \frac{7}{8}} = \frac{2}{7}$

4)



E_1 = A white ball is drawn at the first draw

E_2 = A red ball is drawn.

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 | E_1)$$

$$= \frac{3}{5} \times \frac{2}{4}$$

$$= \frac{3}{10}$$

5) E_1 = Rain is forecast

E_2 = Weather forecast is accurate

E_3 = Mr. X is carrying umbrella

$$P(E_1) = \frac{1}{2}$$

$$P(E_3 | E_1^c) = \frac{1}{3}$$

$$P(E_3 | E_1) = 1$$

$$P(E_3^c | E_1^c) = \frac{2}{3}$$

$$P(E_3) = P(E_1) \cdot P(E_3 | E_1) + P(E_1^c) \cdot P(E_3 | E_1^c)$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\therefore P(E_3^c \cap E_2^c \cap E_1^c) = P(E_1^c) \cdot P(E_2^c | E_1^c) \cdot P(E_3^c | E_2^c \cap E_1^c)$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3}$$

$$= \frac{1}{9}$$

6

C = Commuter uses compact car

M = " " Minivan

H = " gets home before 5 30 pm

$P(C) = \frac{3}{4}$ $\quad P(H|C) = \frac{75}{100} = \frac{3}{4}$ $\quad P(H|M) = \frac{60}{100} = \frac{3}{5}$

$P(M) = \frac{1}{4}$

a) $P(H) = P(C).P(H|C) + P(M).P(H|M)$

$= \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{3}{5}$

$= \frac{9}{16} + \frac{3}{20}$

$= \frac{57}{80}$

b) $P(C|H^c) = \frac{P(C).P(H^c|C)}{P(H^c)}$

$= \frac{\frac{3}{4} \cdot \frac{1}{4}}{\frac{23}{80}}$

$= \frac{15}{23}$

c) $P(M|H^c) = \frac{P(M).P(H^c/M)}{P(H^c)}$

$= \frac{\frac{1}{4} \cdot \frac{2}{5}}{\frac{23}{80}}$

$= \frac{8}{23}$

$P(M \cap H^c) = P(M|H^c).P(H^c) = \frac{8}{80}$

d) $2 \times P(H \cap C).P(H \cap M)$

$= 2 \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{5} \times \frac{1}{4}$

$= \frac{27}{160}$

7) A = person selected for new treatment

C = person is recovered

$P(C|A) = \frac{1}{2}$ $\quad P(A) = \frac{5}{100}$

$P(C|A^c) = \frac{3}{10}$ $\quad P(A^c) = \frac{95}{100}$

$\therefore P(C) = \frac{1}{2} \times \frac{5}{100} + \frac{3}{10} \times \frac{95}{100}$

$$\therefore P(A|C) = \frac{P(A) \cdot P(C|A)}{P(C)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{100}}{\frac{1}{2} \cdot \frac{5}{100} + \frac{3}{10} \cdot \frac{95}{100}}$$

$$= 0.08$$

8) A_1 = both male and female child are represented among the children.

A_2 = At most one child is a girl.

a) $P(A_1) = \frac{2}{4} = \frac{1}{2}$

$P(A_2) = \frac{3}{4}$

$P(A_1 \cap A_2^c) = 0$

$\therefore A_1$ and A_2^c are incompatible.

b) $P(A_1) \cdot P(A_2^c) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$\therefore P(A_1) \cdot P(A_2^c) \neq P(A_1 \cap A_2^c)$

c) ~~For Third child is a~~

T = Third child is a boy.

$P(T|A_2^c) = \frac{2}{5}$ $\quad P(T|A_1) = \frac{1}{2}$ $\quad P(T|\text{both child are boy}) = \frac{11}{20}$

$P(T) = \frac{11}{20} \times \frac{1}{4} + \frac{2}{5} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{2}$

$= \frac{39}{80}$

$$P(F|T) = \frac{P(F)\ P(T|F)}{P(T)} = \frac{\frac{1}{4} \times \frac{11}{20}}{\frac{39}{80}} = \frac{11}{39}$$

9) E_1 = A white ball is drawn from the first box

E_2 = A black ball " " " " ~~second~~ first box

A = A ball is drawn from the second box.

$$P(A) = P(E_1) . P(A|E_1) + P(E_2) . P(A|E_2)$$

$$= \frac{a}{a+b} \cdot \frac{c+1}{c+d+1} + \frac{b}{a+b} \cdot \frac{c}{c+d+1}$$

$$= \frac{c(a+b)+a}{(a+b)(c+d+1)}$$

10) A = Test is positive

B = The person is form infected people

$$P(A|B) = \frac{95}{100}$$

$$P(B) = \frac{2000}{100000} = \frac{2}{100}$$

$$P(A) = P(B) . P(A|B) + P(B^c) . P(A|B^c)$$

$$= \frac{2}{100} \cdot \frac{95}{100} + \frac{98}{100} \cdot \frac{5}{100}$$

$$= \frac{680}{(100)^2}$$

$$P(B|A) = \frac{P(B) \cdot P(A/B)}{P(A)}$$

$$= \frac{\frac{2}{100} \cdot \frac{95}{100}}{\frac{680}{(100)^2}}$$

$$= \frac{190}{680}$$

$$= \frac{19}{68}$$

11) E_1 = prisoner chose Road I

E_2 = " " " II

E_3 = " " " III

E_4 = " " " IV

E = Prisoner's success.

$P(E|E_1) = \frac{1}{8}$ $\quad P(E|E_2) = \frac{1}{6}$ $\quad P(E|E_3) = \frac{1}{4}$

$P(E|E_4) = \frac{9}{10}$.

$P(E_i) = \frac{1}{4} \quad \forall\, i = \{1,2,3,4\}$

(i) $\therefore P(E) = P(E_1) \cdot P(E|E_1) + P(E_2) \cdot P(E|E_2) + P(E_3) \cdot P(E|E_3) + P(E_4) \cdot P(E|E_4)$

$$= \frac{1}{4}\left[\frac{1}{8} + \frac{1}{6} + \frac{1}{4} + \frac{9}{10}\right]$$

$$= \frac{173}{480}$$

(ii) $P(E_4|E) = \dfrac{P(E|E_4) \cdot P(E_4)}{P(E)}$

$= \dfrac{\frac{9}{10} \cdot \frac{1}{4}}{\frac{173}{480}}$

$= \dfrac{108}{173}$

$P(E_1|E) = \dfrac{P(E|E_1) \cdot P(E_1)}{P(E)}$

$= \dfrac{\frac{1}{8} \cdot \frac{1}{4}}{\frac{173}{480}}$

$= \dfrac{15}{173}$

12) A = Number of tosses is odd

$P(A) = p + q^2 p + q^4 p + q^6 p + \cdots$

$= p[1 + q^2 + q^4 + q^6 + \cdots]$

$= p \cdot \dfrac{1}{1-q^2}$

$= \dfrac{p}{(1+q)(1-q)}$

$= \dfrac{1}{2-p}$

13) E = Account balance have error

U = ~~unusual~~ unsusual values

$P(E) = \dfrac{15}{100}$ $\quad P(U) = \dfrac{20}{100}$ $\quad P(U|E) = \dfrac{60}{100}$

$P(U \cap E) = \dfrac{60}{100} \cdot P(E)$

$= \dfrac{60}{100} \cdot \dfrac{15}{100}$

$P(E|U) = \dfrac{P(U \cap E)}{P(U)}$

$= \dfrac{\frac{60}{100} \cdot \frac{15}{100}}{\frac{20}{100}} = \dfrac{45}{100} = 0.45$

14)

B = Better than market avg

W = worse " " "

A = same as " "

E = "Good buy" rated by Analyst

$P(B) = \frac{25}{100} = \frac{1}{4}$ $P(W) = \frac{25}{100} = \frac{1}{4}$ $P(A) = \frac{50}{100} = \frac{1}{2}$

$P(E|B) = \frac{40}{100} = \frac{4}{10}$ $P(E|W) = \frac{10}{100} = \frac{1}{10}$ $P(E|A) = \frac{20}{100} = \frac{2}{10}$

$$\therefore P(E) = P(B) \cdot P(E|B) + P(W) \cdot P(E|W) + P(A) \cdot P(E|A)$$

$$= \frac{1}{4} \cdot \frac{4}{10} + \frac{1}{4} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{2}{10}$$

$$= \frac{4+1+4}{40}$$

$$= \frac{9}{40}$$

$$P(B|E) = \frac{P(B) \cdot P(E|B)}{P(E)}$$

$$= \frac{\frac{1}{4} \cdot \frac{4}{10}}{\frac{9}{40}}$$

$$= \frac{4}{9}$$

15) E_1 = The sum is 5

E_2 = " " " 7

E = ~~The~~ ~~first~~ 5 occurs first

$P(E_1) = \frac{4}{36}$

$P(E_2) = \frac{6}{36}$

$P(E_1 \cup E_2) = \frac{10}{36}$

$P(E_1^c \cap E_2^c) = \frac{26}{36}$

Let D be the event that ~~was~~ ~~been~~ neither 5 nor 7 occurs

$\therefore D = E_1^c \cap E_2^c$

$P(E) = P(E_1) + P(D) \cdot P(E_1) + P(D)\, P(D)\, P(E_1) + \cdots$

$= \frac{4}{36} + \left(\frac{26}{36}\right) \cdot \frac{4}{36} + \left(\frac{26}{36}\right)^2 \cdot \frac{4}{36} + \cdots$

$= \left(\frac{4}{36}\right) \left[1 + \frac{26}{36} + \left(\frac{26}{36}\right)^2 + \cdots \right]$

$= \left(\frac{4}{36}\right) \cdot \left[\frac{1}{1 - \left(\frac{26}{36}\right)}\right]$

$= \frac{4}{10}$

$= \frac{2}{5}$

16) # ways to select two socks from n socks $= \binom{n}{2}$

" " " two ~~both~~ red socks from 3 red socks $= \binom{3}{2}$

$\therefore \dfrac{\binom{3}{2}}{\binom{n}{2}} = \frac{1}{2} \Rightarrow n^2 - n - 12 = 0 \Rightarrow (n-4)(n+3) = 0$

$\Rightarrow n = 4.$

17) Since E and F are mutually exclusive, we have

$$P(E \cup F) = P(E) + P(F)$$

Now, let D be the event that event E occurs before event F.

$$\therefore P(D) = P(E) + P(E^c \cap F^c) \cdot P(E) + \{P(E^c \cap F^c)\}^2 \cdot P(E) + \{P(E^c \cap F^c)\}^3 \cdot P(E) + \cdots$$

$$= P(E) \cdot \left[1 + P(E^c \cap F^c) + \{P(E^c \cap F^c)\}^2 + \{P(E^c \cap F^c)\}^3 + \cdots \right]$$

$$= P(E) \left[\frac{1}{1 - P(E^c \cap F^c)} \right]$$

$$= \frac{P(E)}{P(E \cup F)}$$

$$= \frac{P(E)}{P(E) + P(F)}$$

18) B_1 = Box 1 is selected

B_2 = Box 2 " "

R = The ball is red

$$P(R \mid B_1) = \frac{999}{1000} \qquad P(R \mid B_2) = \frac{1}{1000}$$

$$P(B_1) = P(B_2) = \frac{1}{2}$$

$$P(R) = P(R \mid B_1) \cdot P(B_1) + P(R \mid B_2) \cdot P(B_2) = \frac{1}{2}.$$

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)}$$

$$= 0.999$$

19) $B_1 =$ Box 1 is selected

$B_2 =$ Box 2 is selected

$D =$ The bulb is defective

$E =$ ~~sobe~~ selecting two bulbs from a box is defective

$P(B_1) = P(B_2) = \frac{1}{2}$

$P(\cancel{B}D|B_1) = \frac{10}{100}$

$P(D|B_2) = \frac{5}{100}$

$P(E|B_1) = \frac{10}{100} \cdot \frac{10}{100}$

$= \frac{100}{(100)^2} = \frac{1}{100}$

$P(E|B_2) = \frac{5}{100} \cdot \frac{5}{100}$

$= \frac{1}{400}$

(i) $P(E) = P(B_1) \cdot P(E|B_1) + P(B_2) \cdot P(E|B_2)$

$= \frac{1}{2} \cdot \left[\frac{1}{100} + \frac{1}{400}\right]$

$= \frac{5}{800}$

(ii) $P(B_1|E) = \frac{P(B_1) \cdot P(E|B_1)}{P(E)}$

$= \frac{\frac{1}{2} \cdot \frac{1}{100}}{\frac{5}{800}}$

$= \frac{4}{5}$

20) C_1 = Fair coin

C_2 = Biased coin

H = ~~pr~~ head appears both time

$P(C_1) = P(C_2) = \frac{1}{2}$

$P(H|C_1) = \frac{1}{4}$

$P(H) = P(C_1).P(H/C_1) + P(C_2).P(H/C_2)$

$= \frac{1}{2}.[\frac{1}{4} + 1]$

$= \frac{5}{8}$

$P(H|C_2) = 1$

$P(C_1|H) = \dfrac{P(C_1).P(H|C_1)}{P(H)}$

$= \dfrac{\frac{1}{2}.\frac{1}{4}}{\frac{5}{8}}$

$= \frac{1}{5}$