Indian Institute of Technology Patna
Department of Mathematics
MA225: Probability and Statistics
B.Tech. 2nd year

Tutorial Sheet-9

1. You have 1000 dollars to put in an account with interest rate R, compounded annually. That is, if X_n is the value of the account at year n, then

$$X_n = 1000(1+R)^n, \text{ for } n = 0, 1, 2, \dots.$$

The value of R is a random variable that is determined when you put the money in the bank, but it does not not change after that. In particular, assume that $R \sim Uniform(0.04, 0.05)$.

   (a) Find all possible sample functions for the random process $\{X_n,\ n = 0, 1, 2, \dots\}$.
   (b) Find the expected value of your account at year three. That is, find $E[X_3]$.
   (c) Find the mean functions for the random processes.
   (d) Find the correlation functions and covariance functions for the random processes

2. Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \text{ for all } t \in [0, \infty),$$

where A and B are independent normal $N(1, 1)$ random variables.

   (a) Find all possible sample functions for this random process.
   (b) Define the random variable $Y = X(1)$. Find the PDF of Y.
   (c) Let also $Z = X(2)$. Find $E[YZ]$.
   (d) Find the mean functions for the random processes.
   (e) Find the correlation functions $(R_X(\cdot, \cdot))$ and covariance functions $(C_X(\cdot, \cdot))$ for the random processes

3. Consider the random process $\{X_n, n = 0, 1, 2, \dots\}$, in which $X_i's$ are i.i.d. standard normal random variables.

   (a) Write down $f_{X_n}(x)$ for $n = 0, 1, 2, \dots$.
   (b) Write down $f_{X_m X_n}(x_1, x_2)$ for $m \neq n$.

4. Let A, B, and C be independent normal $N(1, 1)$ random variables. Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \text{ for all } t \in [0, \infty).$$

Also, let $\{Y(t), t \in [0, \infty)\}$ be defined as

$$Y(t) = A + Ct, \text{ for all } t \in [0, \infty).$$

Find cross-correlation $(R_{XY}(t_1, t_2))$ and cross-covariance $(C_{XY}(t_1, t_2))$, for $t_1, t_2 \in [0, \infty)$.

5. Consider the random process $\{X(t), t \in \mathbb{R}\}$ defined as

$$X(t) = \cos(t + U),$$

where $U \sim Uniform(0, 2\pi)$. Show that $X(t)$ is a weak-sense stationary or wide-sense stationary (WSS) process.

6. Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of i.i.d. random variables with mean $EY_i = 0$ and $Var(Y_i) = 4$. Define the discrete-time random process $\{X(n), n \in \mathbb{N}\}$ as

$$X(n) = Y_1 + Y_2 + \cdots + Y_n, \text{ for all } n \in \mathbb{N}.$$

Find $\mu_X(n)$ and $R_X(m, n)$, for all $n, m \in \mathbb{N}$.

7. Let $X(t)$ be a continuous-time WSS process with mean $\mu_X = 1$ and $R_X(\tau) = \begin{cases} 3 - |\tau|, & -2 < \tau < 2 \\ 0, & otherwise \end{cases}$

   (a) Find the expected power in $X(t)$.

   (b) Find $E\left[\left(X(1) + X(2) + X(3)\right)^2\right]$.

8. Let $X(t)$ be a Gaussian process with $\mu_X(t) = t$, and $R_X(t_1, t_2) = 1 + 2t_1t_2$, for all $t, t_1, t_2 \in \mathbb{R}$. Find $P(2X(1) + X(2) < 3)$.

9. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.

   (a) Find the probability of no arrivals in (3,5].

   (b) Find the probability that there is exactly one arrival in each of the following intervals: (0,1], (1,2], (2,3], and (3,4].

10. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ. Find the probability that there are two arrivals in (0,2] and three arrivals in (1,4].

11. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson Process with rate λ. Find its covariance function

$$C_N(t_1, t_2) = Cov(N(t_1), N(t_2)), \text{ for } t_1, t_2 \in [0, \infty)$$

12. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ, and X_1 be its first arrival time. Show that given $N(t) = 1$, then X_1 is uniformly distributed in $(0, t]$. That is, show that

$$P(X_1 \le x \mid N(t) = 1) = \frac{x}{t}, \text{ for } 0 \le x \le t.$$

13. Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_=1$ and $\lambda_2 = 2$, respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

    (a) Find the probability that $N(1) = 2$ and $N(2) = 5$.

    (b) Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

14. Suppose that accidents in Delhi roads involving Blueline buses obey a Poisson process with 9 accidents per month of 30 days. In a randomly chosen month of 30 days:

    (a) What is the probability that there are exactly 4 accidents in the first 15 days?

    (b) Given that exactly 4 accidents occurred in the first 15 days, what is the probability that all 4 occurred in the last 7 days out of these 15 days?

15. Suppose that incoming calls in a call center arrive according to a Poisson process with intensity of 30 calls per hour. What is the probability that no call is received in a 3-minute period? What is the probability that more than 5 calls are received in a 5-minute interval?