① $\quad X_n = 1000\,(1+R)^n \quad ; \quad n = 0, 1, 2 \cdots\cdots$

$\quad R \sim Uni\,(0.04, 0.05).$

ⓐ Here, the randomness in X_n comes from the R.V. R. As soon as you know R, you know the entire sequence X_n. for $n = 0, 1, \cdots$. If $R = r$., then

$$X_n = 1000\,(1+r)^n \quad ; \quad \text{for all } n \in \{1, 2 \cdots\}.$$

Thus, the sample functions are of the form

$$f(n) = 1000\,(1+r)^n \quad ; \quad n = 0, 1, 2 \cdots\cdots$$

where $\quad r \in [0.04, 0.05].$

For any $r \in [0.04, 0.05]$, you obtain a sample function for the random process X_n.

ⓑ. Expected value of account at year 3. OR $E[X_3]$.

The expe random variable X_3 is given by

$$X_3 = 1000\,[1+R]^3.$$

Let $Y = (1+R) \sim Uni\,(1.04, 1.05]$. So,

$$f_Y(y) = \begin{cases} 100 & ; \quad 1.04 < y < 1.05 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Now, we need to calculate ;

$$E[X_3] = E[1000(1+R)^3] = 1000 \, E[Y^3].$$

$$= 1000 \int_{1.04}^{1.05} 100 \, y^3 \, dy = \frac{100\,000}{4} [y^4]_{1.04}^{1.05}$$

$$E(X_3) = \frac{100000}{4}[1.05^4 - 1.04^4] \approx \underline{\underline{1,141.2}}.$$

ⓒ For $\{X_n ; n = 0,1,2 --- \}$, the mean function is —

$$\mu_x(n) = E[X_n] = \cdot E[1000(1+R)^n].$$

$$= 1000 \, E[Y^n]. \qquad ; \quad \therefore \quad Y \sim U[1.04, 1.05)$$

$$= 1000 \int_{1.04}^{1.05} y^n \cdot 100 \, dy = \frac{100000}{n+1}[y^{n+1}]_{1.04}^{1.05}$$

$$\mu_x(n) = \frac{10^5}{n+1} [1.05^{n+1} - 1.04^{n+1}] \quad ; \quad n = 0,1,2 ---$$

ⓓ $R_x(m,n) = E[X_m \cdot X_n] = E[1000(1+R)^m \cdot 1000(1+R)^n].$

$$= 10^6 E(Y^m Y^n) = 10^6 \, E[Y^{m+n}]$$

$$= 10^6 \int_{1.04}^{1.05} 100 \, y^{m+n} \, dy = \frac{10^8}{m+n+1} [y^{m+n+1}]_{1.04}^{1.05}$$

$$R_x(m,n) = \frac{10^8}{m+n+1} [1.05^{m+n+1} - 1.04^{m+n+1}].$$
$$; \quad m,n \in \{0,1,2 --- \}$$

$$C_x(m,n) = R_x(m,n) - E[X_m] \cdot E[X_n]$$

$$= R_x(m,n) - E\left[1000(1+R)^m\right] E\left[1000(1+R)^n\right]$$

$$= R_x(m,n) \quad 1000 \, E[Y^m] \, 1000 \, E(Y^n).$$

$$= R_x(m,n) - 10^6 \, E[Y^m] \, E(Y^n).$$

$$= R_x(m,n) - 10^6 \int_{1.04}^{1.05} 100 \, y^m \, dy \cdot \int_{1.04}^{1.05} 100 \, y^n \, dy$$

$$= R_x(m,n) - \frac{10^{10}}{(m+1)(n+1)} \left[y^{m+1}\right]_{1.04}^{1.05} \left[y^n\right]_{1.04}^{1.05}$$

$$C_x(m,n) = R_x(m,n) - \frac{10^{10}}{(m+1)(n+1)} \left[1.05^{m+1} - 1.04^{m+1}\right] \cdot \left[1.05^{n+1} - 1.04^{n+1}\right].$$

② $\{X(t) \, ; \, t \in [0,\infty)\}.$

$$X(t) = A + B(t) \, ; \, \forall \, t \in [0,\infty).$$

where $A, B \overset{inde}{\sim} N(1,1)$. R.Vs.

ⓐ Here, we note that the randomness in $X(t)$ comes from the two R.Vs. A and B. The random variable A can take any real value $a \in \mathbb{R}$. The R.V. B can also take any real value $b \in \mathbb{R}$. As soon as we know the values of A and B, the entire process of $X(t)$ is known.

In particular, if $A = a$ & $B = b$, then

$$X(t) = a + bt \quad ; \quad t \in [0, \infty).$$

Thus, the sample functions are of the form

$$f(t) = a + b(t) \quad ; \quad t \geqslant 0. \quad ; \quad a, b \in \mathbb{R}.$$

(b). R.V. $Y = X(1)$.

$$\therefore \quad Y = X(1) = A + B \cdot 1 = A + B.$$

Also, $X \sim N(1,1)$, $\quad Y \sim N(1,1)$ and $X, Y \to$ indep.

$$E(Y) = E(A + B) = E(A) + E(B) = 2.$$

$$V(Y) = V(A + B) = V(A) + V(B) = 2$$

$$\therefore \quad Y = X(1) \sim N(2, 2).$$

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sqrt{2}} \, e^{-\frac{1}{2}\left(\frac{y-2}{\sqrt{2}}\right)^2} \quad ;$$

(c) $\quad Z = X(2). \quad , \quad Y = X(1) = A + B)$

$$Z = X(2) \quad ; \quad Z = A + 2B$$

$$E(YZ) = E[(A + B)(A + 2B)] = E[A^2 + 3AB + 2B^2]$$

$$= E[A^2] + 3E(AB) + 2E(B^2).$$

$$= [V(A) + E(A)^2] + 3E(A) \cdot E(B) + 2[V(B) + E(B)^2]$$

$$= 2 + 3 + 2 \cdot 2 \implies \boxed{E(YZ) = 9}$$

(d) For $\{X(t) ; t \in [0, \infty)\}$

$$\mu_x(t) = E[X(t)] = E[A + B(t)]$$

$$= E(A) + t E(B).$$

$$\mu_x(t) = 1 + t \quad ; \quad t \in [0, \infty).$$

(e). $R_x(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$.

$$= E[(A + Bt_1)(A + Bt_2)]$$

$$= E[A^2 + AB(t_1 + t_2) + B^2 t_1 t_2]$$

$$= E[A^2] + E[AB](t_1 + t_2) + E[B^2] t_1 t_2$$

$$= 2 + E(A) E(B) \cdot (t_1 + t_2) + 2 t_1 t_2.$$

$$R_x(t_1, t_2) = 2 + t_1 + t_2 + 2 t_1 t_2 \quad ; \quad t_1, t_2 \in [0, \infty)$$

$$C_x(t_1, t_2) = R_x(t_1, t_2) - E[X(t_1)] E[X(t_2)]$$

$$= R_x(t_1, t_2) - E[A + Bt_1] E[A + Bt_2]$$

$$= R_x(t_1, t_2) - (1 + t_1)(1 + t_2) \quad ; \quad t_1, t_2 \in [0, \infty).$$

③. Random Process : $\{X_n \; ; \; n = 0, 1, 2 \cdots \}$

where $\quad X_i \sim N(0,1) \quad \forall \; i.$

ⓐ $\quad \because \quad X_n \sim N(0,1).$

$$f_{X_n}(x) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2}} \; ; \qquad x \in \mathbb{R}.$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \because X_i \text{ are iid.}$

ⓑ $\quad f_{X_m, X_n}(x_1, x_2) = f_{X_m}(x_1) \cdot f_{X_n}(x_2) \; ;$

$$= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x_2^2}{2}}$$

$$f_{X_m, X_n}(x_1, x_2) = \frac{1}{2\pi} \, e^{-\left\{ \frac{x_1^2 + x_2^2}{2} \right\}} \; ; \quad x_1, x_2 \in \mathbb{R}$$

④. $A, B, C \stackrel{iid}{\sim} N(1,1).$

$\qquad X(t) = A + Bt \; ; \quad t \in [0, \infty)$

$\qquad Y(t) = A + Ct \; ; \quad t \in [0, \infty)$

$\qquad \mu_X(t) = E[X(t)] = E[A + Bt] = E(A) + t\,E(B)$

$\qquad \qquad \mu_X(t) = 1 + t \; ; \quad t \in [0, \infty)$

and $\quad \mu_Y(t) = E[Y(t)] = 1 + t \; ; \quad t \in [0, \infty).$

$$R_{xy}(t_1,t_2) = E[X(t_1) \, Y(t_2)]$$

$$= E[(A + Bt_1)(A + Ct_2)]$$

$$= E[A^2 + ACt_2 + ABt_1 + BCt_1t_2]$$

$$= E[A^2] + t_2 E[AC] + t_1 E[AB] + t_1t_2 E[BC]$$

$$= E[A^2] + t_2 E(A)E(C) + t_1 E(A)E(B) + t_1t_2 E(B)E(C).$$

$$R_{xy}(t_1,t_2) = 2 + t_2 + t_1 + t_1t_2 \quad ; \quad t_1, t_2 \in [0, \infty).$$

$$C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - \mu_x(t_1) \cdot \mu_y(t_2).$$

$$= R_{xy}(t_1,t_2) - (1+t_1)(1+t_2).$$

$$= 2 + t_1 + t_2 + t_1t_2 - 1 - t_2 - t_1 - t_1t_2$$

$$C_{xy}(t_1,t_2) = 1 \quad ,$$

⑤ Given random process $\{X(t) \; ; \; t \in \mathbb{R}\}$.

$$X(t) = \cos(t + U). \quad ; \quad U \sim \text{Uni}(0, 2\pi).$$

For $X(t)$, to be WSS (Weak-Sense Stationary) process.
We need to show:

(i), $\mu_x(t) = \mu_x$; for all $t \in \mathbb{R}$ and

(ii). $R_x(t_1,t_2) = R_x(t_1 - t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

$$\mu_x(t) = E[X(t)] = E[\cos(t+U)].$$

$$= \int_0^{2\pi} \cos(t+u) \cdot \frac{1}{2\pi} \, du = \frac{1}{2\pi} [\sin(t+u)]_0^{2\pi}$$

$$= \frac{1}{2\pi} (\sin(2\pi+t) - \sin(t)).$$

$$\mu_x(t) = \frac{1}{2\pi} (\sin t - \sin t) = 0. \qquad \forall \ t \in \mathbb{R}.$$

Also, $R_x(t_1, t_2) = E[X(t_1) X(t_2)].$

$$= E[\cos(t_1+U) \cdot \cos(t_2+U)]$$

$$\because \cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)].$$

$$= E\left[\frac{1}{2} \cos(t_1+t_2+2U) + \frac{1}{2} \cos(t_1-t_2) \right]$$

$$= E\left[\frac{1}{2} \cos(t_1+t_2+2U) \right] + E\left[\frac{1}{2} \cos(t_1-t_2) \right]$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{\cos(t_1+t_2+2U)}{2\pi} \, du + \frac{1}{2} \cos(t_1-t_2)$$

$$= \frac{1}{4\pi} \left[\frac{\sin(t_1+t_2+2U)}{2} \right]_0^{2\pi} + \frac{1}{2} \cos(t_1-t_2).$$

$$= \frac{1}{8\pi} \left[\sin(4\pi+t_1+t_2) - \sin(t_1+t_2) \right] + \frac{1}{2} \cos(t_1-t_2)$$

$$= \frac{1}{8\pi} \left(\sin(t_1+t_2) - \sin(t_1+t_2) \right) + \frac{1}{2} \cos(t_1-t_2)$$

$$= \frac{1}{2} \cos(t_1-t_2) \ ; \ t_1, t_2 \in \mathbb{R}.$$

Thus, $X(t)$ is WSS. process.

$$R_X(t_1 - t_2) = E[X(t_1 - t_2) X(0)] = E[\cos(t_1 - t_2 + u) \cos(u)]$$

$$= E\left[\tfrac{1}{2}\cos(t_1 - t_2 + 2u) + \tfrac{1}{2}\cos^1(t_1 - t_2)\right]$$

⑥ . $\mu_{X(n)} = E[X(n)] = E[Y_1 + Y_2 + \cdots + Y_n]$ $; n \in \mathbb{N}.$

$$= E(Y_1) + E(Y_2) + \cdots + E(Y_n) \quad = \underline{\underline{0}}$$

Let $m \leq n$., then

$$R_X(m, n) = E[X(m) X(n)]$$

$$= E[X(m)(X(m) + Y_{m+1} + Y_{m+2} + \cdots + Y_n)]$$

$$= E[X(m)] \, E[X(m) + Y_{m+1} + \cdots + Y_n]$$

$$= E[X(m)^2 + X(m)(Y_{m+1} + \cdots + Y_n)]$$

$$= E(X(m)^2) + E[X(m)(Y_{m+1} + \cdots + Y_n)]$$

$$= E(X(m)^2) + \underline{E(X(m)) E(Y_{m+1} + \cdots + Y_n)}$$

$$= V(X(m)) + E(X(m)) + \overset{\downarrow}{0}$$

$$= V(Y_1) + \cdots + V(Y_m)$$

$$R_X(m, n) = 4m.$$

Similarly, for $m \geq n$, then

$$R_X(m, n) = E[X(m) X(n)] = 4n.$$

We conclude;

$$R_X(m, n) = 4 \cdot \min(m, n).$$

⑦ Given $\mu_x = 1.$, $R_x(\tau) = \begin{cases} 3 - |\tau| & ; -2 \le \tau \le 2 \\ 1. & ; \text{ otherwise.} \end{cases}$ ⑩

ⓐ. The expected power in $X(t)$ at time t is $E[X(t)^2]$ which is given by

$$R_x(0) = E[X(0)^2]$$

$$R_x(0) = 3 - |0| = 3.$$

ⓑ. $\cancel{E[X(t)]}$ $E\left[(X(1) + X(2) + X(3))^2\right]$

$$= E\left[X(1)^2 + X(2)^2 + X(3)^2 + 2X(1)X(2) \right.$$
$$\left. + 2X(2)X(3) + 2X(3)X(1)\right]$$

$$= E(X(1)^2) + E(X(2)^2) + E(X(3)^2)$$
$$+ 2E(X(1),X(2)) + 2E(X(2),X(3)) + 2E(X(3),X(1))$$

$$= R_x(0) + R_x(0) + R_x(0) + 2R_x(1-2)$$
$$+ 2R_x(1-3) + 2R_x(2-3)$$

$$= 3 + 3 + 3 + 2R_x(-1) + 2R_x(-2) + 2R_x(-1)$$

$$= 9 + 2 \times 2 + 2 \times 1 + 2 \times 2$$

$$= 9 + 4 + 2 + 4$$

$$= 19 \quad \underline{Ans}.$$

⑧. $\mu_x(t) = t$, $R_x(t_1, t_2) = 1 + 2t_1 t_2$; $t, t_1, t_2 \in R$

Let $Y = 2X(1) + X(2)$. Then Y is a normal random

variable. We have

$$E[Y] = 2 E[X(1)] + E[X(2)].$$
$$= 2 \times 1 + 2 = \underline{\underline{4}}.$$

$$Var(Y) = 4 V(X(1)) + V(X(2)) + 4 cov(X(1), X(2)).$$

Now; $V(X(1)) = E(X(1)^2) - E(X(1))^2$

$$= R_x(1,1) - \left(\mu_x(1)\right)^2$$

$$= 1 + 2 \cdot 1 \cdot 1 - 1^2 = 3 - 1$$

$$Var(X(1)) = \underline{\underline{2}}$$

$$Var(X(2)) = E(X(2)^2) - E(X(2))^2$$

$$= R_x(2,2) - \left(\mu_x(2)\right)^2$$

$$= 1 + 2 \cdot 2 \cdot 2 - 2^2 = 1 + 8 - 4$$

$$Var(X(2)) = \underline{\underline{5}}$$

and $Cov(X(1), X(2)) = E(X(1), X(2)) - E(X(1)) E(X(2))$

$$= R_x(1,2) - \mu_x(1) \mu_x(2)$$

$$= 1 + 2 \cdot 1 \cdot 2 - 1 \cdot 2 = 5 - 2$$

$$Cov(X(1), X(2)) = \underline{\underline{3}}$$

∴ $Var(Y) = 4 \times 2 + 5 + 4 \times 3 = \underline{\underline{25}}$

So, $Y \sim N(4, 25)$.

Now, $P(Y < 3) = P\left(\dfrac{Y-4}{25} < \dfrac{3-4}{25}\right)$

$\qquad\qquad = P(Z < -1/5) = P(Z < -0.20)$

$\qquad P(Y < 3) = \underline{\underline{0.42}}$: (Approx.)

⑨ Given $\{N(t) ; t \in [0, \infty)\}$ be Poisson Process with $\lambda = 0.5$

ⓐ No arrival in $[3, 5]$.

Let Y be the number of arrivals in $(3, 5]$, then

$Y \sim$ Poisson (0.5×2) $\qquad\qquad$ $[\because \; Y \sim \text{Poi } (\lambda t).]$

ie, $Y \sim$ Poisson (1).

$P[Y = 0] = \dfrac{e^{-1} \, (1)^0}{0!} = e^{-1} = \underline{\underline{0.37}}$

ⓑ Let Y_1, Y_2, Y_3 & Y_4 be the number of arrivals in the intervals $(0, 1]$, $(1, 2]$, $(2, 3]$ and $(3, 4]$.

$Y_i \sim$ Poisson (0.5) ; $i = 1, 2, 3, 4$, and independent.

since Poi

$P(Y_1 = 1, Y_2 = 1, Y_3 = 1, Y_4 = 1) = P(Y_1 = 1) \cdot P(Y_2 = 1) \cdot P(Y_3 = 1) \cdot P(Y_4 = 1)$

$\qquad\qquad\qquad\qquad = \left[0.5 \, e^{-0.5}\right]^4$

$\qquad\qquad\qquad\qquad \approx \underline{\underline{8.5 \times 10^{-3}}}$

(10) Since the two intervals $(0,2]$ and $(1,4]$ are not disjoint. Thus we can't multiply the probabilities. for each interval to obtain the desired probability.

$$(0,2] \cap (1,4] = (1,2].$$

Let , X, Y & Z be the number of arrivals in $(0,1]$, $(1,2]$ and $(2,4]$. resp. Then X, Y and Z are indeper

and $X \sim P(\lambda \cdot 1)$, $Y \sim Poi(\lambda \cdot 1)$

$$Z \sim P(\lambda \cdot 2).$$

Let A be the event that there are two arrivals in $(0,2]$ and three arrivals in $(1,4]$. So.

$$P(A \cap B) = P(A|B) \cdot P(B).$$

$$P(A) = P(X+Y=2 \text{ and } Y+Z=3)$$

$$= \sum_{k=0}^{\infty} P(X+Y=2 \text{ and } Y+Z=3 \,|\, Y=k) \cdot P(Y=k).$$

$$= P(X=2, Z=3 | Y=0) \cdot P(Y=0) + P(X=1, Z=2 | Y=1) \cdot P(Y=1)$$

$$+ P(X=0, Z=1 | Y=2) \cdot P(Y=2).$$

$$= P(X=2, Z=3) \cdot P(Y=0) + P(X=1, Z=2) \cdot P(Y=1)$$

$$+ P(X=0, Z=1) P(Y=2) -$$

$$= P(X=2) \cdot P(Z=3) \cdot P(Y=0) + P(X=1) \cdot P(Z=2) \cdot P(Y=1))$$

$$+ P(X=0) \cdot P(Z=1) \cdot P(Y=2).$$

$$=$$

⑪ Let's assume $t_1 \geq t_2 \geq 0$. Then by independent increment ⑭ property of Poi. Process, these two R.V. $N(t_1) - N(t_2)$ & $N(t_2)$ are independent. we can write.

$$C_N(t_1, t_2) = \text{Cov}(N(t_1), N(t_2))$$
$$= \text{Cov}(N(t_1) - N(t_2) + N(t_2), N(t_2)).$$
$$= \text{Cov}(N(t_1) - N(t_2), N(t_2)) + \text{Cov}(N(t_2), N(t_2)).$$
$$= 0 + \text{Cov}(N(t_2), N(t_2)).$$
$$= \text{Var}(N(t_2))$$

$$\text{Cov}(t_1, t_2) = \lambda t_2 \quad ; \text{ since } N(t_2) \sim \text{Poi}(\lambda t_2).$$

similarly, if $t_2 \geq t_1 \geq 0$, we conclude
$$C_N(t_1, t_2) = \lambda t_1$$

Therefore,
$$C_N(t_1, t_2) = \lambda \min(t_1, t_2) \quad \text{for} \quad t_1, t_2 \in [0, \infty).$$

⑫ For $0 \leq x \leq t$, we can write
$$P(X_1 \leq x \mid N(t) = 1) = \frac{P(X_1 \leq x, N(t) = 1)}{P(N(t) = 1)}$$

we know that
$$P(N(t) = 1) = \lambda t \, e^{-\lambda t} \quad ; \quad N(t) \sim \text{Exp}(\lambda t).$$

and $P(X_1 \leq x, N(t) = 1) = P[\text{one-arrival in } (0, x] \text{ and no arrival in } (x, t])$
$$= [\lambda x \, e^{-\lambda x}][e^{-\lambda(t-x)}]$$
$$= \lambda x \, e^{-\lambda t}$$

Thus, $P(x, \leq x | N(t) = 1) = \frac{x}{t}$; for $0 \leq x \leq t$.

(13) $N_1(t)$ with $\lambda_1 = 1$, $N_2(t)$ with $\lambda = 2$

$N(t) = N_1(t) + N_2(t)$ is Poisson process with $\lambda = 1 + 2 = 3$.

(a) $P[N(1) = 2, N(2) = 5] = P[$two arrival in $(0,1]$ and three arrival in $(1,2]$ $]$

$$= \left(\frac{e^{-3} 3^2}{2!} \right) \left(\frac{e^{-3} 3^3}{3!} \right) \approx 0.05$$

(b) $P[N_1(1) = 1 | N(1) = 2] = \dfrac{P[N_1(1) = 1, N(1) = 2)}{P(N(1) = 2)}$

$$= \dfrac{P(N_1(1) = 1, N_2(1) = 1)}{P(N(1) = 2)}$$

$$= \dfrac{P(N_1(1) = 1) \cdot P(N_2(1) = 1)}{P(N(1) = 2)}$$

$$= (e^{-1})(2e^{-2}) \Big/ \left[\frac{e^{-3} 3^2}{2!} \right].$$

$$= \underline{\underline{4/9}} \quad \text{Ane}$$