Indian Institute of Technology Patna
Department of Mathematics
MA225: Probability and Statistics
B.Tech. 2nd year

Tutorial Sheet-8

1. Let X and Y have joint PDF defined as $f_{X,Y}(x,y) = k \frac{1+x+y}{(1+x)^4(1+y)^4}, x > 0, y > 0$ (and 0 otherwise). Find the value of k. (9/2)

2. Let $f_{X,Y}(x,y) = k\left(x^2 + y^2\right), \quad 0 \le x \le 1, 0 \le y \le 2$ and 0 elsewhere. Find k so that $f_{X,Y}(x,y)$ is a PDF. Find marginal densities of X and Y and joint CDF of (X,Y).

3. Find expected value of $(X + Y)^2$ where $f_{X,Y}(x,y) = (1/8)(x+y), \quad 0 \le x \le 2, 0 \le y \le 2$ and 0 elsewhere. (3/2)

4. Let the joint PDF of a bivariate random variable (X,Y) be $f_{X,Y}(x,y) = 3/4, \quad -1 \le x \le 1, x^2 \le y < 1$ and 0 elsewhere. Find the marginal densities and corresponding means and variances.

5. Let a continuous random vector (X_1, X_2) with the joint probability density function $f(x_1, x_2) = 6x_1, \ 0 < x_1 < x_2 < 1; \ = 0, \ otherwise$. Find the covariance and correlation between X_1 and X_2. (1/40, 0.179)

6. A continuous random vector (X_1, X_2) has density function $f(x_1, x_2) = 1, \ -x_2 < x_1 < x_2, \ 0 < x_2 < 1; \ = 0, \ otherwise$. Show that X_1 and X_2 are uncorrelated.

7. If X and Y are random variables and a, b, c, d are any numbers provided only that $a \neq 0, \ c \neq 0$, then
$$Corr(aX + b, cY + d) = \frac{ac}{|ac|} Corr(X, Y).$$

8. The variables X and Y are connected by the equation $aX + bY + c = 0$. Show that the correlation between them is -1 if the signs of a and b are alike and $+1$ if they are different.

9. Consider the PDF $f_{X,Y}(x,y) = 3x^2 e^{-x} y(1-y), 0 < x, 0 < y < 1$ (and 0 otherwise). Calculate the marginal densities $f_X(x), f_Y(y)$ and guess the corresponding distribution. Calculate kurtosis and skewness of X and Y.

10. Let X and Y have joint PDF defined as $f_{X,Y}(x,y) = \frac{y}{2}(1+x), \quad 0 < x < 2, 0 < y < 1$ (and 0 otherwise). Find $P(X + Y > 1)$. (43/48)

11. Let (X,Y) be a bivariate random variable where X is the input and Y is the output of a channel. Let $P(X = 0) = 0.5, P(Y = 1 \mid X = 0) = 0.1, P(Y = 0 \mid X = 1) = 0.2$. Then find the joint PMF of (X,Y), marginal PMF of X and Y, check for independency between X and Y.

12. If the joint density of the X and Y is given by $f_{X,Y}(x,y) = 0.5, \ x > 0, y > 0, x + y < 2$ (and 0 otherwise). Evaluate $P(X \le 1, Y \le 1), \ P(X + Y < 1), \ P(X > 2Y)$. (0.5, 0.25, 1/3)

13. Determine c so that $f_{X,Y}(x,y)$ defined as: $f_{X,Y}(x,y) = cxy^2 \quad 0 < x < y < 1$ (and 0 otherwise), becomes a PDF. Also, find the conditional density $f_{X/y}$ and $f_{Y/X}$ for both rv X and Y, respectively. ($c = 10$)

14. If the RVs X and Y have the joint PDF $f_{X,Y}(x,y) = e^{-x-y}, x > 0, y > 0$, compute the following probabilities: (i) $P(X \le x)$; (ii) $P(Y \le y)$; (iii) $P(X < Y)$; (iv) $P(X + Y \le 3)$.

15. Consider the function $f_{X,Y}(x,y)$ defined by: $f_{X,Y}(x,y) = 8xy, 0 < x \le y < 1$. (i) Verify that $f_{X,Y}(x,y)$ is, indeed, a PDF. (ii) Determine the marginal and conditional PDFs. (iii) Calculate the quantities: $EX, EX^2, \text{Var}(X), EY, EY^2, \text{Var}(Y), E(XY), E(X \mid Y = y), E(Y \mid X = x), \text{Var}(X \mid Y = y), \text{Var}(Y \mid X = x), \text{Cov}(X, Y)$. Are X and Y independent.

16. Let RVs X and Y represent the number of orders for a large turbine in July and August. The joint distributio for X and Y is defined as

$Y \downarrow X \rightarrow$	0	1	2
0	0.05	0.05	0.10
1	0.10	0.25	0.05
2	0.10	0.15	0.05
3	0.05	0.05	0.00

Find the marginal distribution of X and Y. Also, find the conditional distributions $p_{Y|0}$, $p_{Y|1}$, $p_{Y|2}$ and the conditional expected values of each.

17. Let RVs X and Y represent the number of customers waiting for service in two lines in a bank.

$Y \downarrow X \rightarrow$	0	1	2	3
0	0.05	0.21	0	0
1	0.2	0.26	0.08	0
2	0	0.06	0.07	0.02
3	0	0	0.03	0.02

Their joint PMF is given in tabular form. (i) Find $F_{X,Y}(2,1), P(2 \le X \le 3, 0 \le Y \le 2)$. (ii) Derive the marginal and conditional PMFs involved. (iii) Find $E(X \mid Y = 0)$ and $E(Y \mid X = 2)$.

18. Suppose the RV Y is distributed as $P(\lambda)$ and that the conditional PDF of a RV X, given $Y = y$, is $B(y, p)$. Then show that: (i) The marginal PMF of X is $P(\lambda p)$. (ii) The conditional PMF $p_{Y|X}(y \mid x)$ is Poisson with parameter $\lambda(1 - p)$ over the set: $x, x + 1, \ldots$.

19. (i) For a fixed $y > 0$, consider the function $P_{X,Y}(x, y) = \frac{e^{-y}y^x}{x!}, x = 0, 1, \ldots,$ and show that it is the conditional PDF of a RV, given that another RV $Y = y$. (ii) Now, suppose that the marginal PDF of Y is $\text{Exp}(1)$. Determine the joint PDF of the RVs X and Y. (iii) Show that the marginal PMF of X is given by: $p_X(x) = \left(\frac{1}{2}\right)^{x+1}, x = 0, 1, \ldots$.

20. Consider the PDF $f(x, y) = e^{-(x+y)}$ $x > 0, y > 0$. Calculate $P(X > 1), P(X < Y \mid X < 2Y), P(1 < X + Y < 2)$.

21. Consider the joint PDF of (X, Y) as: $f(x, y) = 6x^2y, 0 < x < 1, 0 < y < 1$ and 0 otherwise. Find $P(X + Y < 1), P(X > Y), P(0 < X < 3/4, \ 1/3 < Y < 2), P(X < 1 \mid Y < 2)$.