Parameter estimation

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Exponential Random Variable

An exponential random variable has PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where λ is (+ve)

① Non (-)ve

② $\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx$

$$= \left[-e^{-\lambda x}\right]_0^{\infty} = -\left(\frac{1}{e^{\infty}} - \frac{1}{e^0}\right)$$

$$= 1$$

* An exponential random variable is used to model the amount of time until an incident of interest takes place, for example:–

① Breaking down of some equipment

② Burning out of light bulb

③ An accident occuring

If X is exponential random variable

$$P(X \geq a) = \int_a^{\infty} f_X(x)dx = \int_a^{\infty} \lambda e^{-\lambda x} dx$$

$$P(X \geq a) = e^{-\lambda a} \quad \text{—— } ★$$

$\Gamma 2 = 1$

The mean and variance of X:–

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \frac{\Gamma 2}{\lambda^2} = \frac{\Gamma 2}{\lambda} = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \lambda \int_{-\infty}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda\left[\frac{\Gamma 3}{\lambda^3}\right] = \frac{2}{\lambda^2}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Q: The time until a small meteorite first lands anywhere in Sahara desert is modeled as an exponential random variable with mean of 10 days. The time is currently mid-night. What is the probability that a meteorite first lands some time between 6 AM to 6 PM of the first day?

⇒ Let X be the time elapsed until the

$$\frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$$

$$P(1/4 \le X \le 3/4) = \int_{1/4}^{3/4} \lambda e^{-\lambda x} dx = \frac{1}{10} \int_{1/4}^{3/4} e^{-1/10 x} dx$$

$$= \frac{1}{10} \frac{\left[e^{-x/10}\right]_{1/4}^{3/4}}{-1/10} = -\left(e^{-3/40} - e^{-1/40}\right)$$

$$= e^{-1/40} - e^{-3/40}$$

$$= \cancel{0.075}$$

$$= 0.04756$$

$$\simeq 0.0476$$

$$P(X \ge 1/4) - P(X > 3/4)$$

$$= e^{-\lambda a} - e^{-\lambda b}$$

$$= e^{-1/40} - e^{-3/40}$$

Cumulative Distributive Function (CDF)

The CDF of a random variable X is denoted by F_X & provides the probability $P(X \le x)$. For every x, we have

$$F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} p_X(k) & \text{If } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{If } X \text{ is continuous} \end{cases}$$

A

ex:- X is discrete & $X = 1, 2, \ldots 10$

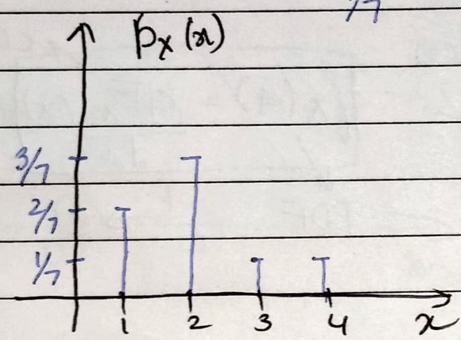
$$F_X(5) = P(X \leq 5)$$
$$= p_X(1) + p_X(2) + p_X(3) + p_X(4) + p_X(5)$$

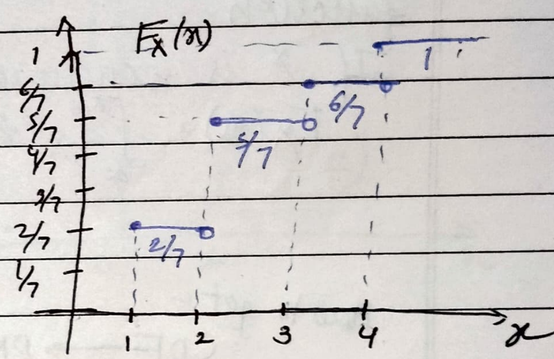
$$= \sum_{k \leq 5} p_X(k)$$

Let X be a discrete random variable

Is it a PMF? $P_X(x) = \begin{cases} 2/7 & x=1 \\ 3/7 & x=2 \\ 1/7 & x=3 \\ 1/7 & x=4 \end{cases}$

Assume $X = 1, 2, 3, 4$

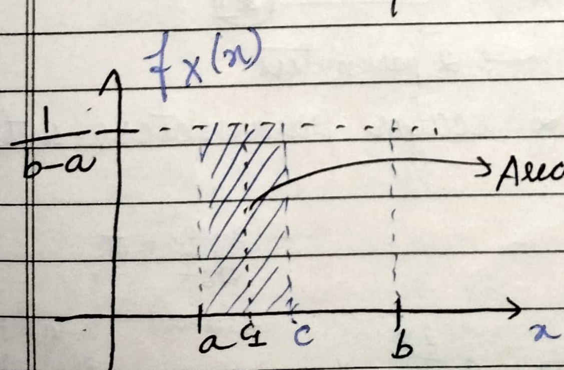


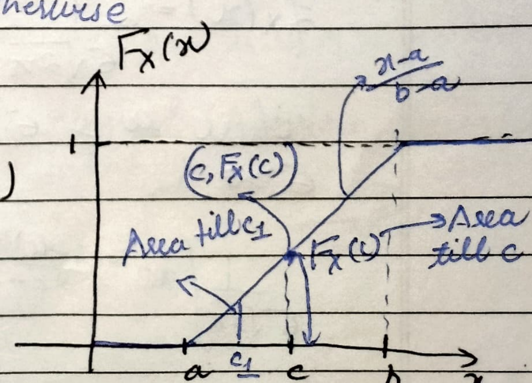


Discrete → CDF is step function

Let X be a continuous (uniform) random variable with PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$





$E[Y] = E[X]+4$ $N(x+4)$ $x+4$

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$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b-a}\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt = \frac{x-a}{b-a}$$

Properties of CDF :-

* Now If $x \le y$ $F_X(x) \le F_Y(y)$

$F_X(x)$ tends to 0 as $x \to -\infty$ and tends to 1 as $x \to \infty$.

If X is discrete, then $F_X(x)$ is piecewise constant function.

If X is continuous then $F_X(x)$ is a continuous function.

If X is continuous then $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ & $\boxed{f_X(x) = \frac{dF_X(x)}{dx}}$ → CDF, PDF, ↳ (*)

How to get :-
CDF ⟶ PMF

Normal Random Variable :-

A continuous random variable X is said to be normal (or Gaussian) if it has a PDF of the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty \quad \text{——— (*)}$$

↳ 2 parameters

where μ & σ are 2 scalar parameters with $\sigma > 0$

$$\frac{1}{\sigma\sqrt{2\pi}}\int x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \qquad -\frac{(x-\mu)^2}{2\sigma^2} = t$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int (t+\mu)\, e^{-\frac{t^2}{2\sigma^2}}\,dx \qquad (x-\mu)^2 = t$$

Exponential is senseless
↳ $-ve$ X

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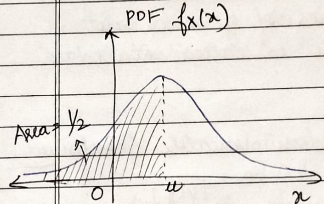
$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} t e^{-\frac{t^2}{2\sigma^2}}\,dt + \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \mu e^{-t^2/2\sigma^2}\,dt$$

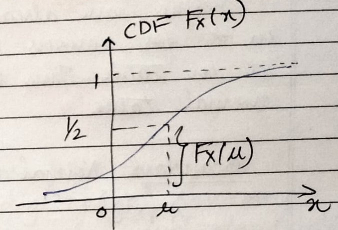
$\Rightarrow$

The mean & variance

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu$$

$$Var[X] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma^2$$





the PDF is symmetric around mean μ

Standard Normal random variable

A normal random variable with $\mu = 0$ & $\sigma^2 = 1$ is said to be standard normal random variable (denoted by Y). It's PDF will be $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Area is 99.9% under $[-3,3]$

Find CDF of standard normal random variable

$$\Phi(y) = P(Y \le y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-t^2/2}\,dt \quad \text{——— (*)}$$

Phi

Example:- We know that this integral can be evaluated for all y

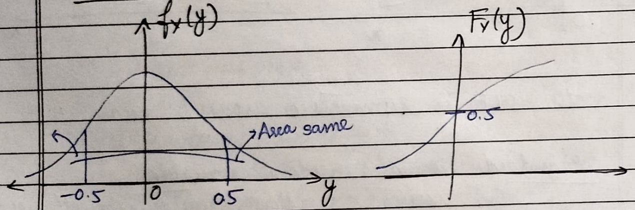
Let Y be a standard normal random variable taking values in $[1, 100]$

$$F_Y(5) = P(Y \le 5) = \int_{-\infty}^{5} f_Y(y)\,dy$$

$$F_Y(8) = P(Y \le 8) = \int_{-\infty}^{8} f_Y(y)\,dy$$

People have already ~~created~~ calculated the integrals in * for various values of y and put it in table. This table is called standard normal table.

Standard Normal random variable (continued)



$$\Phi(0.5) = P(Y \le 0.5) = P(Y \ge 0.5)$$
$$= 1 - P(Y \le 0.5)$$
$$= 1 - \Phi\, 0.5$$

In general:-
$$\Phi(-y) = 1 - \Phi(y)$$

Let X be a normal random with mean μ and variance σ^2. Define another random variable

$$Y = \frac{X-\mu}{\sigma}$$

Y is a linear function of X, hence it is also a Normal random variable.

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\left(E[X]-\mu\right) = \frac{1}{\sigma}(\mu-\mu)$$

$E[X] = \mu$

$$= 0$$

$$V[Y] = Var\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2}\left(Var[X]\right) = \left(\frac{\sigma^2}{\sigma^2}\right) = 1$$

$Var(aX+b) = a^2 Var(X)$

Thus Y is a standard normal variable.

$\Rightarrow$ Prob

Example:- The annual snowfall at a particular location is modeled using normal random variable with mean 60 inches and standard derivation of $\sigma = 20$. What is the probability that this year's snowfall will be atleast 80 inches?

Soln:- Let X be the snowfall. It is given by X is normal random variable with $\mu = 60$ & $\sigma = 20$. Define

$$Y = \frac{X-60}{20} \qquad \left(Y = \frac{X-\mu}{\sigma}\right)$$

We know that Y is a standard

$$P(X \ge 80) = P\left(\frac{X-60}{20} \ge \frac{80-60}{20}\right)$$
$$= P(Y \ge 1) = 1 - P(Y \le 1)$$
From Table
$$= 1 - 0.8413$$
$$= 0.1587$$

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1. $Y = X^2$ $\quad f_X(x) = \frac{1}{2} e^{-|x|} \quad -\infty < x < \infty$

$$= \begin{cases} \frac{1}{2} e^{-x} & x > 0 \\ \frac{1}{2} e^{x} & x < 0 \end{cases}$$

$F_Y(y) = P(Y \le y)$
$= P(X^2 \le y)$
$= P(-\sqrt{y} \le X \le \sqrt{y})$
$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = 2\int_0^{\sqrt{y}} \cancel{f} \frac{1}{\cancel{2}} e\, f_X(x)\,dx$
$= 2\int_0^{\sqrt{y}} \frac{1}{2} e^{-|x|}\,dx = \cancel{2} - \left[e^{-|x|}\right]_0^{\sqrt{y}}$
$= -(e^{-\sqrt{y}} - 1) = 1 - e^{-\sqrt{y}}$

2. $Y = \frac{2}{X} + 3$ $\quad f_X(x) = \begin{cases} x^2(2x + 3/2) & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$\cancel{F}$ $\quad Var(Y) = Var\left(\frac{2}{X} + 3\right) = Var\left(\frac{2}{X}\right)$ $\quad Var(X+Y) = Var(X) + Var(Y) - Cov(X,Y)$
$= 4V\left(\frac{1}{X}\right) = 4\left(E\left(\frac{1}{X^2}\right) - \left(E\left(\frac{1}{X}\right)\right)^2\right)$

$E\left(\frac{1}{X}\right) = \frac{\cancel{1}}{\cancel{X}} \int_0^1 \frac{1}{x} x^2 \left(2x + \frac{3}{2}\right) = \int_0^1 2x^2 + \frac{3x}{2}\,dx$
$= \frac{2}{3}(x^3)_0^1 + \frac{3}{2\times 2}(x^2)_0^1$
$= \frac{2}{3}(1) + \frac{3}{4}(1) = \frac{8+9}{12} = \frac{17}{12}$

$E\left(\frac{1}{X^2}\right) =$

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5. (a) $P(X \le a) = P(X > a)$
$\int_{-\infty}^{a} 3x^2\,dx = 1 - P(X \le a) = 1 - \int_{-\infty}^{a} 3x^2\,dx$
$\Rightarrow \cancel{6}\int_{-\infty}^{a} x^2\,dx = 1 \quad \Rightarrow 6\int_0^a x^2\,dx = 1$
$\Rightarrow 2[x^3]_0^a = 1 \quad \Rightarrow 2[a^3] = 1 \quad \Rightarrow a^3 = 1/2$
$\Rightarrow a = (1/2)^{1/3}$

(b) $P(X \ge b) = 0.05$
$\Rightarrow 1 - P(X < b) = 5/100$
$\Rightarrow 1 - b^3 = 5/100 \quad \Rightarrow b^3 = \frac{95}{100} \quad \Rightarrow b = \left(\frac{95}{100}\right)^{1/3}$

6. (A) $P(0 \le x \le 2)$

$f_{XY}(x,y)$
Joint PDF

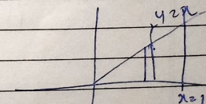
Marginal Density Function of X
$f_X(x) = \int_{Y=} f_{XY}\,dy$

Marginal Density Function of Y
$f_Y(y) = \int_X f_{XY}\,dx$

Conditional Density Function :-
$f_{Y|X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$
Y given X

X given Y
$f_{X/Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)}$

6. (a) $f(x,y) = 2 \quad 0 < x < 1 \quad 0 < y < x$
(a) $f_X(x) = \int_0^x 2\,dx = 2x$



(c) $f_Y(y) = \cancel{\int_0^1 2\,dy = 2}$ $\quad \int_{x=y}^{x=1} 2\,dx = 2(1-y)$

(b) $E(X \cdot Y) = E(X)\,E(Y)$ X & Y are independent random variables

$f_{Y/X}(x,y) = \dfrac{f_{XY}(x,y)}{f_X(x)} =$ $\quad f_{X,Y}(x,y) = f_X(x) f_Y(y)$

& $\neq 2x \times 2(1-y)$

$= \dfrac{2}{2x} = \dfrac{1}{x}$

$f_{X/Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$

7. $f(x) = 1 \quad 0 < x < 1$

$Y = \dfrac{X}{1+X}$

Let X be a continuous random variable with PDF $f_X(x)$ and we have given transformation $Y = V(x)$

i) If y is continuously differentiable ($\frac{dy}{dx} \to$ continuous)

ii) If y is either non-decreasing or non-increasing for all values of x for which $f_X(x) \neq 0$.

$$\boxed{g_Y(y) = f_X(x(y)) \left|\frac{dx}{dy}\right|}$$ is PDF of Y

→ X in terms of Y

$Y = \dfrac{X}{1+X}$ $\Rightarrow Y(1+X) = X$

$\Rightarrow YX + Y = X$

$\Rightarrow \cancel{X(Y=)}\; Y = X(1-Y)$

$\Rightarrow X = \dfrac{Y}{1-Y}$

$f_X\left(\dfrac{y}{1-y}\right)$ $\quad f_X(x), x$

$2\left(\dfrac{y}{1-y}\right)^2$

X is random variable

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$Y = \dfrac{X}{1+X}$ $\quad Y' = \dfrac{(1+X) - X}{(1+X)^2} = \dfrac{1}{(1+X)^2} \geq 0$ increasing

$g_Y(y) = f_X(x(y)) \left|\dfrac{dx}{dy}\right|$ $\quad X = \dfrac{Y}{1-Y}$

$= f_X\left(\dfrac{Y}{1-Y}\right) \dfrac{1}{(1-Y)^2}$ $\quad \dfrac{dx}{dy} = \dfrac{(1-Y) - Y(-1)}{(1-Y)^2} = \dfrac{1}{(1-Y)^2}$

$= \dfrac{1}{(1-Y)^2}$

Independent

8:- 20% chance of receiving an offer from each company $\quad p = \frac{20}{100} = \frac{1}{5}$

P(atleast one offer) $= 1 - P(\text{no offer})$

$= 1 - {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^4$

$= 1 - \left(\frac{4}{5}\right)^4 =$

No:-

9:- $\mu = 60$ inches $\quad \sigma = 1$ inches $\to P(|x-\mu| < k\sigma) > 1 - \frac{1}{k^2}$

Chebyshev Inequality

$P(4'10'' \leq X \leq 5'2'')$

$P(58 \leq X \leq 62) \geq A$

$\Rightarrow P(58-60 \leq X-60 \leq 62-60) \geq A$

$\Rightarrow P(-2 \leq X-60 \leq 2) \geq A$

$\Rightarrow P(|x-60| < 2) \geq 1 - \frac{1}{2^2}$

$k\sigma = 2$

$\boxed{k = 2}$

$1 - \frac{1}{4}$

$\frac{3}{4}$

~~Atleast 75%~~

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Joint PDFs of multiple random variables:-

Let X and Y be continuous random variables associated with same experiment. Then X & Y are jointly continuous if there is a non(-)ve function $f_{X,Y}(x,y)$ such that

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

Function $f_{X,Y}$ satisfies following property:-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

This $f_{X,Y}(x,y)$ is joint PDF of X & Y.

Joint CDFs:-

Let X and Y be 2 random variables associated with same experiment, their joint

We have

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = f_{X,Y}(x,y)$$

The marginal PDFs of X & Y can be obtained from the joint PDF as follows:-

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

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Expectation:-

$$E[ax+by+cz] = aE[X] + bE[y] + cE[z]$$

Conditioning one random variable

Ex:- The speed of a typical vehicle that drives past a police radar is modeled as an exponential random variable X with mean 50 miles/hr. The police radar's measurement Y of the vehicles' speed has an error which is modeled as a normal random variable with zero mean & standard deviation equal to 1/10th of the vehicle's speed. What is the joint PDF of X & Y.

$\Rightarrow$ $\frac{1}{\lambda} = 50 \Rightarrow \lambda = \frac{1}{50}$ $\quad \mu = 0 \quad \sigma = \frac{x}{10}$ $\quad \sigma = \frac{\mu}{?}$

variance $\sigma^2 = \frac{x^2}{100}$ $\quad$ Given that $X = x$

$$f_{Y|X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi}(x/10)} e^{\left(\frac{-(y-x)^2}{2(x/10)^2}\right)}$$

Independence:- 2 continuous random variables X and Y are independent if their joint PDF is the product of the marginal PDFs.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \text{for all } x,y$$

Note that,

$$f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

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If random variable are independent

$f_{X/Y}(x/y) = f_X(x)$ ②

for all y with $f_Y(y) > 0$ and all x

In general, if we have 3 random variables x, y and z

then $\iint f_{XY}(x/y) = \int f_X(x) \int f_X(y)$

ex:- The waiting time in hours between successive speeders spotted by a radar is continuous random variable with CDF

$$P(X<x) = F(x) = \begin{cases} 1-e^{-8x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the probability of waiting less than 12 min between successive speeders.

$$P(X = x) = \frac{dF(x)}{dx} = 8e^{-8x}$$

$$P\left(X < \frac{12}{60}\right) = \int_{-\infty}^{1/5} f = F(0.2) = \int_{0}^{0.2} 8e^{-8x}\,dx = 1-e^{-1.6}$$

$$= \frac{8}{-8}\left(e^{-8x}\right)_0^{0.2} = -1\left(e^{-1.6} - 1\right)$$

$$= 1 - e^{-1.6}$$

Q:- (1)
$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(X) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x x\,dx = \frac{x^2}{2} \qquad 0 \leq x \leq 1$$

$$F(X) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 x\,dx + \int_1^x (2-x)\,dx = \left[\frac{x^2}{2}\right]_0^1 \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left(2x - \frac{x^2}{2} - \left(2 - \frac{1}{2}\right)\right)$$

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$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2} = 2x - \frac{x^2}{2} - 1$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx + \int_2^{\infty} f(x)\,dx$$

$$= \cancel{2x - \frac{x^2}{2}} \quad \frac{1}{2} + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{2} + \left(4 - 2 - \left(2 - \frac{1}{2}\right)\right) = \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)$$

$$= 1 \qquad x \geq 2$$

$$F(X) = \begin{cases} x^2/2 & 0 \leq x < 1 \quad F(X<1) \\ 2x - x^2/2 - 1 & F(X \leq x) \quad 1 \leq X \leq 2 \\ 1 & x \geq 2 \end{cases}$$

6 − 5 − 4

6 − 5 + 6

Q:-
$$F(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \leq x < 1 \\ 3/5 & 1 \leq x < 2 \\ 4/5 & 2 \leq x < 3 \\ 9/10 & 3 \leq x < 3.5 \\ 1 & x \geq 3.5 \end{cases}$$

$3/5 - 1/2$

$$P(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1/10 & x = 1 \\ 1/5 & x = 2 \\ 1/10 & x = 3 \\ 1/10 & x = 3.5 \end{cases}$$

Q:- The Department of Energy puts projects out on bid & generally estimates what a reasonable bid should be. We call the estimate 'b'. The Department of energy has determined that the density function of winning bid is

$$f(y) = \begin{cases} 5/8b & 2b/5 < y < 2b \\ 0 & \text{otherwise} \end{cases}$$

Find $F(y)$? What is the probability that winning bid is less than estimate 'b'?

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$$F(y) = \int_{2b/5}^{x} \frac{5}{8b}\,dt = \frac{5[t]_{2b/5}^{x}}{8b} = \frac{5}{8b}\left(x - \frac{2b}{5}\right)$$

$$= \begin{cases} \frac{5x}{8b} - \frac{1}{4} & \frac{2b}{5} \le y < 2b \\ 1 & y \ge 2b \end{cases}$$

$$P(Y \le b) = \frac{5x}{8b} - \frac{1}{4} = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}$$

$$\int_{2b/5}^{b} \frac{5}{8b} \quad \frac{5}{8b}\left[b - \frac{2b}{5}\right] = \frac{5}{8b}\left(\frac{3b}{5}\right) = 3/8$$

Q:- The weekly demand for a certain drink, in thousands of litres, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$ where X has the following PDF:-

$$f_X(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of the weekly demand for the drink?

$$E(g(X)) = E[X^2 + X - 2]$$

$$= E[X^2] + E[X] - 2$$

$$= \int_1^2 x^2\, 2(x-1) + \int_1^2 x\, 2(x-1) - 2$$

$$= 2\left(\frac{x^4}{4} - \frac{x^3}{3}\right)_1^2 + 2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^2 - 2$$

$$= 2\left(4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}\right) + 2\left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right) - 2$$

$$= 8 - \frac{16}{3} - \frac{1}{2} + \frac{2}{3} + \frac{16}{3} - 4 - \frac{2}{3} + 1 - 2$$

$$= 4 - \frac{3}{2} = \frac{5}{2} = 2.5$$

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$X = 1, 2, 3$

$$P(Y \le y) \quad P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$$

Derived Distribution :- Calculating the PDF of $Y = g(X)$ where X is a continuous random variable:-

ex:- Let X be uniform on $[0, 1]$ & let $Y = \sqrt{X}$. Find PDF of Y.

Sol^n:- PDF of X

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of Y will be $[0, 1]$

First we will find CDF then PDF

$$F_Y(y) = P(Y \le y)$$

$$= P(\sqrt{X} \le y)$$

$$= P(X \le y^2)$$

$$= \int_0^{y^2} f_X(x)\,dx = \int_0^{y^2} 1\,dx = y^2$$

$$F_Y(y) = y^2$$

PDF of Y :- $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d(y^2)}{dy} = 2y$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le \\ 0 & \text{otherw} \end{cases}$$

$$\int_{-\sqrt{y}}^{\sqrt{y}} 1\,dx = \int_0^{\sqrt{y}} 1\,dx = \sqrt{y}$$

PDF of Y :- $\frac{d\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$

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Covariance & Correlation :- We will study a quantitative measure of the strength and direction of the relationship b/w 2 random variables.

Let X and Y be 2 random variables. The covariance of X and Y denoted by Cov(X,Y) is defined as:-

$$Cov(X,Y) = E[(X-E(X))(Y-E(Y))]$$

A simpler formula is:-

$$Cov(X,Y) = E[XY] - E[X]E[Y] \quad \text{—— (3)}$$

$$E[XY - YE[X] - XE[Y] + E[X]E[Y]]$$

$$E[XY] - E[Y]E[X] - \cancel{E[X]E[Y]} + \cancel{E[X]E[Y]}$$

$$= E[XY] - E[X]E[Y]$$

* If X & Y are independent then Cov(X,Y) = 0, we say that X & Y are uncorrelated

Reverse not true (Think example)

The Correlation coefficient $\rho(X,Y)$ of 2 random variables X & Y that have non-zero variances is defined as:-

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \quad \text{—— (A)}$$

↳ rho

$$-1 \le \rho \le 1$$

So

$$\rho(X,Y) = \frac{\int xy f_{XY}(x,y) - \int x f_X(x) \int y f_Y(y)}{\sqrt{}}$$

$\int_{-\infty}^{\infty} x^2 \sin x$ $\qquad \int_{-\infty}^{\infty} e^{-x} \sin x$

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$$\frac{E[XY] - E[X]E[Y]}{\sqrt{E[X-E[X]]^2 \, E[Y-E[Y]]^2}}$$

Moment Generating Function:-

Let X be a random variable, then n-th-order moment of X is defined as $E[X^n]$.

The moment generating function (MGF) associated with random variable X is defined as:-

$$M_X(s) = E[e^{sX}] \quad \text{—— (1)}$$

where s is a scalar parameter

$X \to e^X \to e^{sX}$

For a discrete random variable X

$$M(s) = \sum_x e^{sx} p_X(x)$$

For a continuous random variable X

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

→ (2)

ex:- MGF of Poisson Random Variable

$$M(s) = \cancel{\int_{-\infty}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!}dx} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{sx}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[1 + e^{s}\lambda + \frac{e^{2s}\lambda^2}{2!} + \frac{e^{3s}\lambda^3}{3!} + \dots\right]$$

$1 + e^s\lambda + \frac{(e^s\lambda)^2}{2!}$ $\quad = e^{\lambda} e^{s}$ $\quad (e^s\lambda)^3$

$e^x = 1 + x + \dots$ $\quad = e^{-\lambda} \sum_{k=0}^{\infty} \frac{e^{sk}\lambda^k}{k!}$ $\quad e^{e^s\lambda}$

$$= e^{-\lambda} e^{e^s\lambda} = e^{\lambda(e^s - 1)}$$

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MGF of exponential random variable

$$M(s) = \int_{-\infty}^{\infty} e^{sx} \lambda e^{-\lambda x} = \left[\frac{\lambda e^{(s-\lambda)x}}{(s-\lambda)}\right]_{0}^{\infty}$$

$$= \lambda\left[0 - \frac{1}{s-\lambda}\right] = \frac{-\lambda}{s-\lambda} \qquad s<\lambda$$

$$M(s) = \frac{\lambda}{\lambda - s}$$

From MGF to moments:-

Let X be a continuous random variable then MGF of X is defined as

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

Differentiate both side wrt s

$$\frac{d}{ds}M(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \qquad \text{when } \int \text{ \& } \frac{d}{dx} \text{ can be interchanged}$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds} e^{sx} f_X(x)\,dx$$

$$\frac{d}{ds}(M(s)) = \int_{-\infty}^{\infty} x e^{sx} f_X(x)\,dx$$

Consider the special case when $s=0$ we get

$$\left.\frac{d}{ds}M(s)\right|_{s=0} = \int_{-\infty}^{\infty} x f_X(x)\,dx = E[X]$$

In general, we have:-

$$\left.\frac{d^n M(s)}{ds^n}\right|_{s=0} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = E[X^n]$$

Random variable $X \xrightarrow{①}$ MGF of X

$\downarrow ②$

~~nth moment~~

$E[X^n] \longleftarrow$ Differentiate and put $s=0$

$E[X]$ 0.5 $\quad P(x\geq 1) = 0.7$

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example:- Let X be a random variable

$$p_X(x) = \begin{cases} 1/2 & x=2 \\ 1/6 & x=3 \\ 1/3 & x=5 \end{cases}$$

$$E[X] = 1 + \frac{1}{2} + \frac{5}{3} = \frac{6+3+10}{6} = \frac{19}{6}$$

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{2s}}{2} + \frac{e^{3s}}{6} + \frac{e^{5s}}{3}$$

$$\left.\frac{d}{ds}M(s)\right|_{s=0} = e^{s} + \frac{e^{3s}}{2} + \frac{5e^{5s}}{3}$$

$$= 1 + \frac{1}{2} + \frac{5}{3} = \frac{19}{6}$$

$$\frac{d}{ds}\left(\frac{\lambda}{\lambda - s}\right) = \pm\lambda\left.\left(\frac{1}{\lambda - s}\right)^2\right|_{s=0} = \frac{1}{\lambda}$$

$$\boxed{M_X(0) = 1} \qquad \int_{-\infty}^{\infty} f_X(x) = 1$$

Markov Inequality:- If a random variable X can take only non-negative values, then

$$P(X\geq a) \leq \frac{E[X]}{a} \quad \text{for all } a>0$$

$\hookrightarrow$ Not for Normal random variable as it is $-ve$

Chebyshev Inequality:-

If X is a random variable with mean μ & variance σ^2, then

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \quad \forall\, c>0$$

Ex:- Let X be uniform random variable in the interval $[0,4]$

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$f_X(x)$

$E[X] = 2 = \mu \quad Var = 4/3 = 6$

$P(|X-2| \geq 1) \leq \frac{16}{9}$

$P((x-2) \leq -1 \text{ and } (x-2) \geq 1)$

$P((X-2) \geq 1)$

$P(X \geq 3) = \int_3^4 \frac{1}{4} dx = \frac{1}{4} = 0.25$

$P(X \leq 1) = \int_0^1 \frac{1}{4} dx = 1/4 = 0.25$

$P(X \geq 3)$ and $P(X \leq 1) = 0.5$

$C = 2$

$P(|X-2| \geq 2) = P((X-2) \leq -2 \text{ and } (X-2) > 2)$
$= P(X \leq 0 \text{ and } X > 4)$
$= 0$

The Weak law of large numbers :-

If $X_1, X_2, \ldots$ be independent identically distributed (iid) random variables with mean μ. For every $\varepsilon > 0$, we have

$$P\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right) \longrightarrow 0 \text{ as } n \to \infty$$

are independent R.V.

$10^{-3}, 10^{-2}, 10^{-1}$

$P(|y-\mu| \geq 10^{-3}) = P(y-\mu < -10^{-3} \text{ or } y-\mu > 10^{-3})$
$= P(y < -10^{-3} + \mu \text{ or } y > 10^{-3} + \mu)$

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Explaination :- Let $X_1, X_2, \ldots$ have mean μ and variance σ^2. Define a new random variable

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \quad \text{— (2)}$$

$E[M_n] = \frac{1}{n}[E[X_1] + E[X_2] + \cdots + E[X_n]]$
$= \frac{1}{n}[\mu + \mu + \cdots + \mu] = \frac{1}{n}(n\mu) = \mu$

$Var(M_n) = \frac{1}{n^2}[Var(X_1) + Var(X_2) + \cdots + Var(X_n)]$
$= \frac{1}{n^2}(\sigma^2 + \sigma^2 + \cdots + \sigma^2) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$P(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \quad n \to \infty$
$\to 0$

The Central Limit Theorem (CLT) $\quad Cov(X, Y)$

Derived Distributions :-
Voters → IID

Q:- We load
Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common mean μ & variance σ^2. Define

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Then CDF of Z_n converges to standard normal CDF.

$$E[Z_n] = \frac{1}{\sigma\sqrt{n}}\left[E[X_1+\cdots+X_n]-n\mu\right]$$

Q:- We load on a plane 100 packages whose weights are independent random variables that are uniformly distributed b/w 5 & 50 kg. What is the probability that the total wt. exceed 3000kg?

$P(S_{100} > 3000) = ?$

S_{100} = sum of the weight of 100 packages

⇒ $p = \frac{1}{50-5} = \frac{1}{45}$ $\quad E = \frac{a+b}{2} = \frac{50+5}{2} = 27.5$

$E[X_i] = \mu = 27.5$

$Var(X_i) = \frac{(50-5)^2}{12} = \frac{45\times 45}{12} = 168.75 = \sigma^2$

$Z = \frac{X_1+X_2+\cdots+X_{100} - 100\times 27.5}{\sqrt{168.75}\sqrt{100}} = \frac{X}{\sigma}$

$Z = \frac{S_{100} - 100\times 27.5}{\sqrt{16875}}$ $\quad S_{100} = \sqrt{16875}\,Z + 2750$

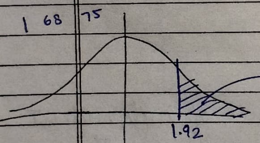
$P(S_{100} > 3000) = P(\sqrt{16875}\,Z + 2750 > 3000)$

$= P\left(Z > \frac{250}{\sqrt{16875}}\right)$ $\quad$ 168

$= P(Z > 1.92)$ $\quad a > b$

$= 0.5 - 0.4726$ $\quad$ 0.4726

$= 0.0274$



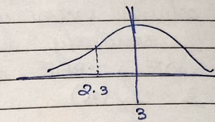
Q:- A certain type of storage battery lasts on average, 3 yrs with a standard deviation of 0.5 yrs. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 yrs.

$\sigma = 1/2$ $\quad \mu = 3$

$Z = 3$

$P(\cancel{X} < 2/3) = P(X < 2.3)$

$= 0.5 - 0.$



$Z = \frac{X-\mu}{\sigma} = \frac{X-3}{0.5}$

$P(Z<2.3) = P\left(\frac{X-3}{1/2} < 2.3\right) = P($ $\quad X-$

$P(Z<-1.4)$

$= 1 - P(Z<1.4)$

$P(X<2.3) = 0.0808$

Multidimensional Random Variable

→ To develop a vector / matrix notation that will allow us to represent large sequence of random variables with a compact notation.

Joint, Marginal & conditional PMFs, CDFs & PDFs

Defn:- For a set of N random variables $X_1, X_2, \ldots X_N$, the joint PMF, CDF & PDF are given as

$$P_{X_1,X_2,\ldots X_N}(x_{k_1}, x_{k_2}, \ldots x_{k_N}) = P(X_1 = x_{k_1}, X_2 = x_{k_2}, \ldots X_N = x_{k_N})$$

$$F_{X_1,X_2,\ldots X_N}(x_1, x_2, \ldots x_N) = P(X_1 \le x_1, X_2 \le x_2, \ldots X_N \le x_n)$$

$$f_{X_1,X_2,\ldots X_N}(x_1, x_2, \ldots x_N) = \frac{\partial^N}{\partial x_1 \partial x_2 \ldots \partial x_N} F_{X_1,X_2,\ldots X_N}(x_1, x_2, \ldots x_N)$$

Above notation seems cumbersome for large N.

2 → 1 बार Integrate
10 → 9 बार Integrate

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Introduce:- Let $X = [x_1, x_2, \dots X_N]^T$ be a column vector with N random variables & similarly $x = [x_1, x_2, \dots x_N]$. Then PMF, CDF & PDF are expressed as $P_X(x)$, $F_X(x)$ & $f_X(x)$

$$P_{X_1, X_2, \dots X_M}(x_{k_1}, x_{k_2} \dots x_{k_M}) = \sum_{k_{m+1}} \sum_{k_{m+2}} \dots \sum_{k_N} P_{X_1, X_2, \dots X_N}(x_{k_1}, x_{k_2}, \dots x_{k_N})$$

$$F_{X_1, X_2 \dots X_M}(x_1, x_2, \dots x_m) = F_{X_1, X_2, \dots X_N}(x_1, x_2, \dots x_M, \infty, \infty)$$

$$f_{X_1, X_2, \dots X_M}(x_1, x_2, \dots x_M) = \int\int_{-\infty}^{\infty} \dots \int f_{X_1, X_2, \dots X_N}(x_1, x_2, \dots x_N)\, dx_{M+1}\, dx_{M+2} \dots dx_N$$

Conditional PMF & PDF:-

$$P_{X_1 X_2, \dots X_M | X_{M+1} X_{M+2}, \dots X_N}(x_{k_1}, x_{k_2} \dots x_{k_M} | x_{k_{m+1}}, x_{k_{m+2}} \dots x_{k_N})$$
$$= \frac{P(X_1 = x_{k_1}, X_2 = x_{k_2}, \dots x_N = x_{k_N})}{P(X_{M+1} = x_{k_{m+1}}, X_{M+2} = x_{k_{m+2}}, \dots)}$$

Expectation Corelation / Covariance Matrix:-

Def^n:- For a random vector $X = [x_1, x_2, \dots x_N]^T$ the corelation matrix is defined as $R_{XX} = E[XX^T]$ i.e. the (i,j)th element of $N \times N$ matrix R_{XX} is $E[X_i X_j]$. Similarly, the covariance matrix C_{XX} is defined as $C_{XX} = E[(X-\underset{\text{vector}}{\mu})(X-\mu)^T]$ i.e. the $(i,j)^{th}$ element of C_{XX} is $Cov(X_i, X_j)$

Proof:-

$\mu = E[X_1]E[X_2]$

S Dhamraja
V G Kulkarni
Stochastic Process
↳ Random

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Theorem:- The matrices R_{XX} & C_{XX} are symmetric & positive definite.

A matrix $Z_{n \times n}$ +ve definite $\Leftrightarrow x^T Z x > 0 \ \forall x \in \mathbb{R}^n \setminus \{0\}$
↳ which is non-zero

Proof:-

Gaussian Random Variable in multiple dimensions (why only Gaussian not others?)

Stochastic Process:- → Collection of random variables

Consider a system that evolves randomly in time. For example:-

- Stock market index
- The inventory in warehouse
- Queue of customers at a service station
- Water level in a reservoir
- State of the machines

These systems may be observed at discrete time points $n = 0, 1, \dots$

That is:- every second, every minute, every hour, every day, every week

X_n = the state of system at time n

or X_n = NSE index at the end of nth day

= # of unsold car on a dealer's lot at the starting of nth day

These systems may be observed continuously in time $X(t)$ =

$X(t)$ = no. of failed machines in a machine stop at the time 't'

= the amt. of money in a bank account at time 't'

2 → 1 बार Integrate
10 → 9 बार Integrate

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$$F_{X_1, X_2 \ldots X_M}(x_1, x_2, \ldots x_m) = F_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots x_M, \infty, \infty)$$

$$f_{X_1, X_2, \ldots X_M}(x_1, x_2, \ldots x_M) = \int_{-\infty}^{\infty}\int \ldots \int f_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots x_N)\, dx_{M+1}\, dx_{M+2} \ldots dx_N$$

Conditional PMF & PDF:–

$$P_{X_1 X_2, \ldots X_M | X_{M+1} X_{M+2}, \ldots X_N}(x_{k_1}, x_{k_2} \ldots x_{k_M} | x_{k_{m+1}}, x_{k_{m+2}} \ldots x_{k_N}) = \frac{P(X_1 = x_{k_1}, X_2 = x_{k_2}, \ldots x_N = x_{k_N})}{P(X_{m+1} = x_{k_{m+1}}, X_{M+2} = x_{k_{m+2}}, \ldots)}$$

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Proof:

$\mu = E[X_1]E[X_2]$

S. Dhawan
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Stochastic Process
↳ Random

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↳ which is non-zero

Proof:–

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(why only Gaussian not others?)

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These systems may be observed continuously in time $X(t)$ =
$X(t)$ = no. of failed machines in a machine shop at the time 't'
= the amt. of money in a bank account at time 't'

$X_0 = 100$, $X_{10} = 97$, $X_{20} = 101$ → State space

Topic: Like Natural Number

Date: Like Real Number

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We say that $\{X_n, n \geq 0\}$ is a discrete-time stochastic process describing the system.

We say $\{X(t), t \geq 0\}$ is a continuous time stochastic process.

* A stochastic process is a collection of random variables $\{X(t), t \in T\}$ indexed by the parameter t taking values in the parameter space T (which is usually subset of $\mathbb{R}$)

* The random variables takes values in the set S, called the state-space of the stochastic process.

Following 4 combinations are possible:-

1. Discrete-state, discrete-time → Cricket score

a) The no. of individuals in a country at the end of year t → stochastic process. $\{X_t; t \in T\}$ with $T = \{0, 1, 2 \dots\}$ & $S = \{0, 1, 2, \dots\}$

b) A motor insurance company reviews the status of its customers yearly. Based on this, 3 discounts 5%, 10%, 15% are given depends on the accident record of the driver.

X_t = percentage of discount at the end of year t. Then $T = \{0, 1, 2, \dots\}$ & $S = \{5, 10, 15\}$

2. Discrete-state continuous Time stochastic Process

No claim Bonus (NCB) intra trading

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a) The no. of incoming calls X_t in an interval $[0, t]$. Then $T = \{t : 0 \leq t < \infty\}$ & $S = \{0, 1, 2, \dots\}$

b) The no. of cars X_t parked at a commercial center in the time interval $[0, t]$. Again $T = \{t : 0 \leq t < \infty\}$, $S = \{0, 1, 2, \dots\}$

3. Continuous state, Discrete-time

a) The share price of an asset at the close of trading on day t with $T = \{0, 1, 2, \dots\}$ & $S = \{x : 0 \leq x < \infty\}$

4) Continuous state, Continuous-time

a) The value of NSE index at time t where $T = \{t : 0 \leq t < \infty\}$ & $S = \{x : 0 \leq x < \infty\}$

Let $\{X(t), t \in T\}$ be a stochastic process with state space S. Let $x : T \to S$ be a function. We may think that $\{x(t), t \in T\}$ as a possible evolution (trajectory) of $\{X(t), t \in T\}$. The functions x are called the sample paths of the stochastic process. infinitely many

Remark:- In general, the set of all possible sample paths, called the sample space of the stochastic process is uncountable.

For finite state → infinite paths

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Independent Increments :- If for all $t_0, t_1, t_2 \ldots t_n$ s.t. $t_0 < t_1 < t_2 \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$

are independent (or equivalently $X_{t+\tau} - X_{\tau}$ is independent of X_s for $s < \tau$), then the process $\{X_t; t \in T\}$ is said to be a process with independent increments.

Stationary Increments :- A stochastic process $\{X_t; t \in T\}$ is said to have stationary increments if $X_{t_2+\tau} - X_{t_1+\tau}$ has the same

distribution as $X_{t_2} - X_{t_1}$ for all choices of t_1, t_2 & $\tau > 0$.

Stationary Process :- If for arbitrary $t_1, t_2 \ldots, t_n$ st $t_1 < t_2 < \ldots < t_n$, the joint distributions of the vector random variables $(X_{t_1}, X_{t_2}, \ldots X_{t_n})$ & $(X_{t_1+h}, X_{t_2+h}, \ldots X_{t_n+h})$ are the same

for all $h > 0$. Then the stochastic process is said to be stationary stochastic process of order n (or simply a stationary process). The stochastic process $\{X_t; t \in T\}$ is said to be a strong stationary stochastic process or strictly stationary process if the above property is satisfied for all n.

Markov Process

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Consider a system that is modelled by a discrete time stochastic process $\{X_n, n \geq 0\}$ with a countable state-space S, say $S = \{0, 1, 2, \ldots\}$. Consider a fixed value of n that we shall call "the present time" or just the "present". Then X_n is called the present (state) of the system, $\{X_0, X_1, X_2, \ldots X_{n-1}\}$ is called the ~~pr~~ past of the system and $\{X_{n+1}, X_{n+2}, \ldots\}$ is called the future of the system. If $X_n = i$ & $X_{n+1} = j$. It is said that the system has jumped (or made a transition) from state i to state j from time n to $n+1$.

Markov Property :- If the present state of the system is known, the future of the system is ~~t~~ independent of the past system.

Or in other words :- the present state of the system contains all the relevant information needed to predict the future of the system in a probabilistic sense.

Tossing a coin regularly

$X(t) \to$ no. of heads till time 't'

Def :- Let $\{X_t; t \geq 0\}$ be a stochastic process defined over a probability space $(\Omega, \mathcal{F}, P)$ & with state space $(\mathbb{R}, \mathcal{B})$. We say that $\{X_t; t \geq 0\}$ is a Markov process if for any $0 \leq t_1 < t_2 \ldots < t_n$ and for any $B \in \mathcal{B}$.

$$P(X_{t_n} \in B / X_{t_1}, \ldots X_{t_{n-1}}) = P(X_{t_n} \in B / X_{t_{n-1}})$$

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Remark :- Any stochastic process which has independent increments is a Markov process.

(i) X_n : no. of heads in first n tosses

ex :- (ii) Let $Y_0, Y_1, \ldots, Y_n$ be non (-)ve independent & identically distributed random variables.

The sequence $\{X_n ; n \geq 0\}$ with

$X_0 = Y_0$

$X_n = X_0 + Y_1 + \ldots + Y_n$

is a Markov Process

$$P(X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= \frac{P(X_{n+1} = j, X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)}{P(X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)}$$

$$= \frac{P(X_0 = i_0)\, P(Y_1 = \overset{X_1 - X_0}{i_1 - i_0}) \ldots P(Y_n = i - i_{n-1})\, P(Y_{n+1} = \overset{X_{n+1} - X_n}{j - i})}{P(X_0 = i_0)\, P(Y_1 = i_1 - i_0) \ldots P(Y_n = i - i_{n-1})}$$

$$= P(Y_{n+1} = j - i)$$

$$= P(X_{n+1} = j \mid X_n = i) \qquad P\left(\frac{X_{n+1} \cap X_n}{X_n}\right) = \frac{P(Y_{n+1}) P(X_n)}{P(Y_n)}$$

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$$\frac{P(X_{n+1} = j, X_n = i)}{P(X_n = i)} \qquad Y_{n+1} = X_{n+1} - X_n$$

$$Y_{n+1} = j - i$$

$$\hookrightarrow \frac{P(Y_{n+1} = j - i)\, P(X_n = i)}{P(X_n = i)}$$

$X_n \rightarrow$ always non-decreasing Why?

A discrete state Markov process is known as markov chain

Based on T:

- Discrete Time Markov Chain (DTMC)
- Continuous Time Markov Chain (CTMC)

1st order Markov Property

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Discrete Time Markov Chain (DTMC) :-

A sequence of random variables $\{X_n\}_{n \in N}$ with discrete state space is called a DTMC if it satisfies the condition :-

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) \quad \text{--- (1)}$$

for all $n \in N$ & for all $i_0, i_1, \ldots, i_{n-1}, i, j \in S$

with $P(X_0 = i_0, \ldots, X_n = i_n) > 0$

In other words, eqn (1) implies that if we know the present state "$X_n = i$", the knowledge of past history "$X_{n-1}, X_{n-2}, \ldots, X_0$" has no influence on the probabilistic structure of the future state X_{n+1}

Example :- X_n = "no. of heads obtained in the first n tosses"

Do it :- If $\{X_n ; n \geq 0\}$ is a Markov chain, then for all $i_0, i_1, \ldots, i_n \in S$ we have

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_0)\, P(X_1 = i_1 / X_0 = i_0) \ldots P(X_n = i_n \mid X_{n-1} = i_{n-1})$$

Defn :- (Transition Probability) :-

Let $\{X_n ; n \geq 0\}$ be a markov chain. The probabilities $p_{ij} = P(X_{n+1} = j \mid X_n = i)$ are called transition probabilities.

Defn :- A markov chain $\{X_n ; n \geq 0\}$ is called homogeneous or a Markov chain with stationary probabilities if the transition probabilities do not depend on n.

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Remark:— Because of the

Def^n – (Transition Probability Matrix)

The matrix

$$P = (p_{ij}) = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & \cdots & \\ p_{20} & & & \\ \vdots & & & \end{pmatrix}$$

is called the transition probability matrix or stochastic matrix. Note that

$$\left.\begin{array}{ll} p_{ij} \geq 0 & \forall \; i,j \in S \\ \sum_j p_{ij} = 1 & \forall \; i \in S \end{array}\right\} — (*)$$

Explanation of (*) p_{ij} is a conditional probability, so $p_{ij} \geq 0$. Now

$$\sum_{j \in S} p_{ij} = \sum_{j \in S} P(X_{n+1} = j \mid X_n = i) \quad (i \text{ to } j) \; (n \to n+1)$$

$j \in S$ → State of (100, 101)

$$= P(X_{n+1} \in S \mid X_n = i)$$

→ State space ($\Omega \to H, T$ $P(\text{outcome} \in \Omega)$)

$$= 1 \quad (\text{since } X_{n+1} \in S \text{ with probability } 1)$$

Example:— On any given day, Kuldip is cheerful (C), normal (N) or depressed (D). If he is cheerful today, then he will be C, N or D tomorrow with probabilities 0.5, 0.4 & 0.1 respectively. If he is feeling so-so today, then he will be C, N or D tomorrow with probabilities 0.3, 0.4, 0.3 respec. If he is gloomy today, then he will be C, N or D with probabilities 0.2, 0.3, 0.5

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respectively.

Let X_n denote the mood of Kuldip on the nth day. Then $\{X_n, n \geq 0\}$ is a 3 state discrete time Markov chain (state 0 = C, state 1 = N, state 2 = D) with transition probability matrix

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

Ques:— Is a DTMC completely characterised by its transition probability matrix? No

Def^n:— Let $a_i = P(X_0 = i)$, $i \in S$ & $a = [a_i]_{i \in S}$ be a row vector representing the PMF of X_0 we say that a is the initial distribution of the DTMC.

Next Page continuation

How to characterise a stochastic process?

To provide a mathematically precise method to describe a stochastic process unambiguously.

Stochastic process – collection of random variables.

To start with – "How one describes a single random variable"

A single random variable is completely described by its CDF $F(x) = P(X \leq x)$ $-\infty < x < \infty$

A multivariate random variable $(X_1, X_2, \ldots X_n)$ is completely described by its joint CDF

$$F(x_1, x_2, \ldots x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_n \leq x_n)$$

for all $-\infty < x_i < \infty$ $i = 1, 2, \ldots n$

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Theorem:- A DTMC $\{X_n, n \geq 0\}$ is completely described by its initial distribution a & the transition probability matrix P.

Proof:- Do it yourself

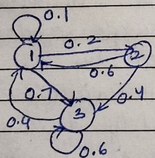
Hint:- Show that the finite dimensional joint PMF $P(X_0 = i_0, X_1 = i_1, \ldots\ldots X_n = i_n)$ can be computed in terms of a & P.

Thus if the parameter set T is finite, the stochastic process $\{X_n;\ n \geq 0\}$ is a multivariate random variable & hence is completely described

We have already discussed about transition probability matrix. The transition probability matrix of a DTMC can be represented graphically by its transition diagram which is a directed arc from node i to node j if $p_{ij} > 0$ & to itself if $p_{ii} > 0$

Q:- State space $\{1, 2, 3\}$

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0 & 0.6 \end{bmatrix}$$



Example:- Coming back to characterisation, we have shown that a DTMC $\{X_n, n \geq 0\}$ is completely characterised by initial distribution a & transition probability matrix P i.e. we can find

$$P(X_0 = i_0, X_1 = i_1, \ldots\ldots X_n = i_n)$$
$$= a_{i_0} p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-2} i_{n-1}} p_{i_{n-1} i_n} \quad \text{where}$$

$$a = P(X_0 = i_0) \quad \& \quad p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Note that

$$P = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix} \quad \text{where } S = \{\cancel{1, 2, \ldots, n}\}\ \{i_0, i_1, i_2, \ldots\ldots i_n\}$$

$$a = [\underbrace{a_{i_0}}_{P(X_0 = i_0)}, a_{i_1}, \ldots \underbrace{a_{i_n}}_{P(X_n = i_n)}]$$

Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{1, 2, 3, 4\}$ &

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0 & 0.5 & 0.0 \\ 0.6 & 0.2 & 0.1 & 0.\cancel{6}1 \end{bmatrix}$$

& $a = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$

$a_1 = P(X_0 = 1) = 0.25 \qquad a_2 = P(X_0 = 2) = 0.25$

$a_3 = P(X_0 = 3) = 0.25 \qquad a_4 = P(X_0 = 4) = 0.25$

Further:- $i_0 = 1,\ i_1 = 2,\ i_2 = 3,\ i_3 = 4$. For

$$P(X_0 = 1, X_1 = 3, X_2 = 1, X_3 = 4)$$
$$= P(X_0 = i_0, X_1 = i_2, X_2 = i_0, X_3 = i_3)$$
$$= a_{i_0} p_{i_0 i_2} p_{i_2 i_0} p_{i_0 i_3}$$

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$$= a_1 p_{13} p_{31} p_{14} = 0.25 \times 0.3 \times 0.5 \times 0.4$$
$$= 0.015$$