Tutorial Sheet 1.

1)

(i) $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$

Since A and B are the only events in S.

we have, ~~$A \cap f$~~ $S = A \cup B$

If $A \cap B = \phi$, then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B)$$
$$= \frac{1}{3} + \frac{1}{4}$$
$$= \frac{7}{12}$$

but $P(A \cup B) = P(S) = 1$.

So, $A \cap B \neq \phi$.

(ii) Given that $A \cap B = \phi$

~~Also~~, Then, $P(A \cap B) = 0$

$\Rightarrow P((A \cap B)^c) = 1$

$\Rightarrow P(A^c \cup B^c) = 1$

$\Rightarrow P(A^c) + P(B^c) - P(A^c \cap B^c) = 1$

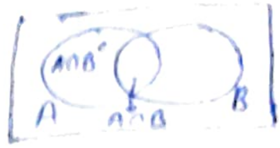
$\Rightarrow P(B^c) = 1 - P(A^c) + P(A^c \cap B^c)$

$\Rightarrow P(B^c) = P(A) + P(A^c \cap B^c)$

$\therefore P(B^c) \geq P(A)$.

2)

i) $A = (A \cap B^c) \cup (A \cap B)$



Then $P(A) = P(A \cap B^c) + P(A \cap B)$ [$\because$ $(A \cap B^c)$ and $(A \cap B)$ are disjoint]

$\Rightarrow P(A) = P(A \cap B^c) + P(B)$ [$\because$ $B \subset A \Rightarrow A \cap B = B$]

$\Rightarrow P(A \cap B^c) = P(A) - P(B)$

ii) From (i) we have

$P(A) - P(B) = P(A \cap B^c) \geq 0$

$\Rightarrow P(A) \geq P(B)$

3)

a) $(A^c \cup B)^c \cup (A^c \cup B^c)^c$

$= ((A \cap B)^c)^c \cup ((A^c)^c \cap B^c)$

$= (A \cap B) \cup (A \cap B^c)$

$= A$

b) $(A \cup B) \cap (A \cap B)^c$

$= (A \cup B) \cap (A^c \cup B^c)$

$= \{(A \cup B) \cap A^c\} \cup ((A \cup B) \cap B^c)$

$= (A^c \cap B) \cup (A \cap B^c)$

$= (A \cap B^c) \cup (B \cap A^c)$

4)
$A = \{x : 2 \le x \le 5\}$ $\quad B = \{x : 3 \le x \le 6\}$

$A \cup B = \{x : 2 \le x \le 6\}$

$A \cap B = \{x : 3 \le x \le 5\}$

$(A \cup B) \cap (A \cap B)^c = (A \cap B^c) \cup (B \cap A^c)$

$= \{x : 2 \le x < 3\} \cup \{x : 5 < x \le 6\}$

$= \{x : x \in [2,3) \cup (5,6]\}$

5)
a) $P(A) = P(B) = P(A \cap B)$

$P((A \cap B^c) \cup (B \cap A^c))$

$= P(((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c))$

$= P((A \cup B) \cap (A^c \cup B^c))$

$= P(A \cup B) + P(A^c \cup B^c) - P((A \cup B) \cup (A^c \cup B^c))$

$= P(A \cup B) + P(A^c \cup B^c) - P(S)$

$\cancel{= P(A) + P(B) - P(A \cap B}$

$= P(A \cup B) + P(A^c) + P(B^c) - P(A^c \cap B^c) - P(S)$

$= P(A \cup B) + P(A^c) + P(B^c) - 1 + P(A \cup B) - P(S)$

$= P(A \cup B) + P(A^c) + P(B^c) - 2$

$= 2P(A \cup B) + P(A^c) + P(B^c) - 2$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A)$

$\therefore P((A \cap B^c) \cup (B \cap A^c)) = 2P(A) + P(A^c) + P(B^c) - 2$

$= 2P(A) + 1 - P(A) + 1 - P(B) - 2$

$= \cancel{P(A)}\ 0$

b) $P(A) = P(B) = 1$

$\Rightarrow$ ~~P(A)~~

Now, $A \subseteq A \cup B$

~~P(A) ≤ P(B)~~

$\therefore P(A) \leq P(A \cup B)$

$\Rightarrow \quad 1 \leq P(A \cup B) \leq 1$

$\therefore \quad P(A \cup B) = 1.$

Now, $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 1 = 1 + 1 - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = 1.$

6) ~~Given~~ $B \subset C$

$\Rightarrow B \cap A \subset C \cap A$

$\Rightarrow P(B \cap A) \leq P(C \cap A)$

$$\Rightarrow \frac{P(B \cap A)}{P(A)} \leq \frac{P(C \cap A)}{P(A)}$$

$\Rightarrow P(B|A) \leq P(C|A)$

7) $P(A \cup B) = \frac{3}{4}$

$P(A \cap B) = \frac{1}{4}$

$P(A^c) = \frac{2}{3}$

$\Rightarrow 1 - P(A) = \frac{2}{3}$

$\Rightarrow P(A) = \frac{1}{3}$

$\therefore P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$= \frac{3}{4} - \frac{1}{3} + \frac{1}{4}$

$= \frac{2}{3}$

$P(A \cap B^c) = P(A) - P(A \cap B)$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{1}{12}$

8) $ax^2 + bx + c = 0$ — (1)

The discriminant of the quadratic equation (1) is

$D = b^2 - 4ac$

Now, eqn (1) has real roots if $D \geq 0$

and complex roots if $D < 0$

a) Since the coefficients of eqn (1) ~~can~~ is be determined by throwing an ordinary die, then a, b and c must be from the set $\{1, 2, 3, 4, 5, 6\}$

Let A be the event that $D \geq 0$

and B " " " " $D < 0$

Now, for $D \geq 0$,

$b^2 \geq 4ac$

If $b = 1$, then $1 \geq 4ac$ So, ~~no such pair~~ there is no such outcome

If $b = 2$,

$\Rightarrow 4 \geq 4ac$

$\Rightarrow ac \leq 1$

Then the coefficients for which $D \geq 0$ are $\{(1,1,1)\}$

If $b = 3$,

$\Rightarrow 9 \geq 4ac$

Then the coefficients for which $D \geq 0$ are $\{(1,3,1), (2,3,1),$
$(1,3,2)\}$

If $b = 4$,

$\Rightarrow 4 \geq ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,4,1), (2,4,1),$
$(3,4,1), (4,4,1),$
$(2,4,2), (1,4,4),$
$(1,4,2), (1,4,3)\}$

If $b = 5$,

$\Rightarrow 25 \geq 4ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,5,1), (2,5,1),$
$(3,5,1), (4,5,1),$
$(5,5,1), (6,5,1),$
$(2,5,2), (1,5,2),$
$(1,5,3), (1,5,4),$
$(1,5,5), (1,5,6),$
$(2,5,3), (3,5,2)\}$

If $b=6$,

$+ 9 \geq ac$

$\therefore$ The coefficients for which $D \geq 0$ are $\{(1,6,1),(2,6,1),(3,6,1),(4,6,1),(5,6,1),(6,6,1),(2,6,2),(1,6,2),(1,6,3),(1,6,4),(1,6,5),(1,6,6),(2,6,3),(2,6,4),(3,6,3),(3,6,2),(4,6,2)\}$

$\therefore P(A) = \frac{1+3+8+14+17}{6^3} = \frac{43}{216}$

(b) $P(B) = 1 - P(A)$

$= 1 - \frac{43}{216}$

$= \frac{173}{216}$

9) We use induction on n.

Base case for $n=2$ $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$

$\Rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

$= \sum_{i=1}^{2} P(A_i) - (2-1)$

Induction hyp. Assume that the statement is true for n.

Induction step.

$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \cap A_{n+1}) \geq P(A_1 \cap A_2 \cap \cdots \cap A_n) + P(A_{n+1}) - 1$

$\geq \sum_{i=1}^{n} P(A_i) - (n-1) - 1 + P(A_{n+1})$

$= \sum_{i=1}^{n+1} P(A_i) - \{(n+1) - 1\}$

$\therefore$ The statement is true for $n+1$.

$\therefore$ The statement is true.

10) Let A be the event that no students share the same birthday. Then A^c is our required event

Now, $P(A) = \dfrac{\binom{365}{n}}{(365)^n}$ [∴ $|S| = 365^n$ and no students haves same birthday so A can occur in = $\binom{365}{n}$ ways]

∴ $P(A^c) = 1 - \dfrac{\binom{365}{n}}{(365)^n}$

11) Let M be the event that a person is male

F " " " " " " Female

and S " " " " " " Smoker

Given that, $P(M) = \frac{40}{100} = \frac{2}{5}$

and $P(F) = \frac{60}{100} = \frac{3}{5}$

and $P(S|M) = \frac{50}{100} = \frac{1}{2}$

and $P(S|F) = \frac{30}{100} = \frac{3}{10}$

Now, $P(S) = P(S \cap M) + P(S \cap F)$

$= P(M) \cdot P(S|M) + P(F) \cdot P(S|F)$

$= \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{3}{10}$

$= \frac{19}{50}$

$$P(M|S) = \frac{P(S\cap M)}{P(S)}$$

$$= P(S|M) \cdot \frac{P(M)}{P(S)}$$

$$= \frac{1}{2} \cdot \frac{\cancel{2}}{5} \cdot \frac{\cancel{50}\,10}{19}$$

$$= \frac{10}{19}$$

12>

(i) Let A be the event that both coins show head.

B " " " " the first coin shows a head.

$\therefore A = \{(H,H)\}$

$B = \{(H,H), (H,T)\}$

$\therefore$ ~~[illegible]~~ $A\cap B = \{(H,H)\}$

$$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{\frac{1}{2^2}}{\frac{2}{2^2}} = \frac{1}{2}$$

(ii) Let C be the event that at least one of them is head.

Then $C = \{(H,T), (H,H), (T,H)\}$

$\therefore C\cap A = \{(H,H)\}$

$$\therefore P(A|C) = \frac{P(C\cap A)}{P(C)} = \frac{\frac{1}{4}}{3/4} = \frac{1}{3}$$

13) Let a die is trown n number of times.

Let A be the event that no ~~6~~ six occurs.

~~Since no six occurs~~ ~~then the number of ways event~~

A ~~occur~~

Now, event A occurs in 5^n number of ways.

$$\therefore \quad P(A) = \frac{5^n}{6^n}$$

For, $P(A) < \frac{1}{2}$

we have, $\left(\frac{5}{6}\right)^n < \frac{1}{2}$

For, $n = 1$, $\quad \frac{5}{6} > \frac{1}{2}$

for $n = 2$, $\quad \frac{25}{36} > \frac{1}{2}$

for $n = 3$, $\quad \frac{125}{216} > \frac{1}{2}$

for $n = 4$, $\quad \frac{5^4}{6^4} < \frac{1}{2}$

So, the die has to be thrown 4 times, ~~such~~.

14) Let A be the event that 6 doesnot occur in 4 throws of a die. Our required event is A^c.

The total number of outcomes for throwing a die 4 times is $= 6^4$

The total number of ways event A occurs is $= 5^4$.

$\therefore P(A) = \frac{5^4}{6^4}$

$\therefore P(A^c) = 1 - \frac{5^4}{6^4} = 0.517$

Now let B be the event that no double six occurs in 24 throws with two die.

~~Now,~~ The total number of ~~ca~~ outcome for throwing two die 24 times is $= (6^2)^{24}$

the number of ways event B occurs is $= (6^2-1)^{24} = (35)^{24}$.

[The total number of outcomes for throwing two die is $6^2=36$ and double six occurs in only one of them. So no double six occurs in $(6^2-1) = 35$ ways]

$\therefore P(B) = \frac{(35)^{24}}{(36)^{24}}$

$\therefore P(B^c) = 1 - \frac{(35)^{24}}{(36)^{24}}$

$= 0.491$

So, $P(A^c) > P(B^c)$

$\therefore$ The chance of seeing 6 at least once in 4 throws of a die is higher than seeing a double six at least once in 24 throws with two die.

$\therefore$ The former one is suitable ~~for~~ ~~to~~ to bet.

15) Let E_1 be the event that A solves the problem

E_2 " " " " B " " "

E_3 " " " " C " " "

$\therefore P(E_1) = \frac{1}{2}$ $P(E_2) = \frac{3}{4}$ and $P(E_3) = \frac{1}{4}$.

Now, it is given that the events E_1, E_2 and E_3 are independent

Then, $P(E_1 \cup E_2 \cup E_3) = 1 - P(E_1^c \cap E_2^c \cap E_3^c)$

$$= 1 - P(E_1^c) \cdot P(E_2^c) \cdot P(E_3^c)$$

$$= 1 - \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{3}{4}\right) \left(1 - \frac{1}{4}\right)$$

$$= 1 - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4}$$

$$= 1 - \frac{3}{32}$$

$$= \frac{29}{32}$$

If E_1 & E_2 are independent events then
$P(E_1^c \cap E_2^c) = 1 - P(E_1 \cup E_2)$
$= 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2)$
$= (1 - P(E_1)) \cdot (1 - P(E_2))$
$= P(E_1^c) \cdot P(E_2^c)$

16) Let A_1 and A_2 be the two boxes where A_1 contains 1 black and 1 white marble and A_2 contains 2 black and 1 white marble.

Let B be the event that the selected marble is black

and W " " " " " " " white.

Now, $P(A_1) = P(A_2) = \frac{1}{2}$.

Now, $P(B|A_1) = \frac{1}{2}$

and $P(B|A_2) = \frac{2}{3}$

$$\therefore P(B) = P(A_1).P(B|A_1) + P(A_2).P(B|A_2)$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{3}$$

$$= \frac{7}{12}$$

17) i) Let E be the event that none of the men selects their own hat.

We use principle of inclusion-exclusion.

Let A_i be the event that i-th man selects his own hat.

then $|A_i| = (N-1)!$ $\forall$ $1 \le i \le N$

$|A_i \cap A_j| = (N-2)!$ for $i \neq j$

$|A_1 \cap A_2 \cap \dots \cap A_N| = 1.$

$$\therefore |E| = N! - \binom{N}{1}.(N-1)! + \binom{N}{2}.(N-2)! - \dots + (-1)^N\, 1$$

$$= N!\left(\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^N \frac{1}{N!}\right)$$

$$\therefore P(E) = \frac{N!\left(\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^N \frac{1}{N!}\right)}{N!}$$

$$= \left(\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^N \frac{1}{N!}\right)$$

$$= \sum_{i=2}^{N} (-1)^i \frac{1}{i!}$$

ii) Let E_2 be the event that exactly k of the men select their own hats.

$$\therefore |E_2| = \binom{N}{K} \cdot \left[(N-K)! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{(N-K)} \cdot \frac{1}{(N-K)!} \right) \right]$$

$$\therefore P(E_2) = \frac{\binom{N}{K} \cdot (N-K)! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{N-K} \cdot \frac{1}{(N-K)!} \right)}{N!}$$

$$= \frac{1}{K!} \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{N-K} \frac{1}{(N-K)!} \right)$$

$$= \frac{1}{K!} \sum_{i=2}^{N-K} (-1)^{i} \cdot \frac{1}{i!}$$

(iii)

For $N=3$,

$$P(E) = \sum_{i=2}^{3} (-1)^{i} \cdot \frac{1}{i!}$$

$$= \frac{1}{2!} - \frac{1}{3!}$$

$$= \frac{1}{3}$$

For $N=4$,

$$P(E) = \sum_{i=2}^{4} (-1)^{i} \frac{1}{i!}$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

$$= \frac{3}{8}$$

iv) We can ~~a~~ rewrite the expression $\sum_{i=2}^{N} (-1)^i \frac{1}{i!}$ as $\sum_{i=0}^{N} (-1)^i \frac{1}{i!}$

$$\therefore P(E) = \sum_{i=0}^{N} (-1)^i \frac{1}{i!}$$

As $N \to \infty$, $\sum_{i=0}^{N} (-1)^i \frac{1}{i!} \to e^{-1}$

$$\therefore P(E) \to e^{-1} \text{ as } N \to \infty$$

18) Let W_K be the event that encountering a white ball by the K.th draw. and B_i be the event of drawing i black balls followed by a white ball.

$$\therefore W_K = B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_{K-1}$$

The events B_i, B_j for $i \neq j$ are mutually exclusive.

Now, $P(B_0) = \frac{m}{m+n}$

$$P(B_1) = \frac{n}{(m+n)} \cdot \frac{m}{(m+n-1)}$$

$$P(B_2) = \frac{n}{m+n} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{m}{(m+n-2)}$$

$\vdots$

$$P(B_{K-1}) = \frac{n}{(m+n)} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{(n-2)}{(m+n-2)} \cdots \frac{(n-K+2)}{(m+n-K+2)} \cdot \frac{m}{(m+n-K+1)}$$

$$\therefore P(W_K) = P(B_0) + P(B_1) + \cdots + P(B_{K-1})$$

$$= \frac{m}{m+n} + \frac{n}{(m+n)} \cdot \frac{m}{(m+n-1)} + \cdots + \frac{n(n-1)\cdots(n-K+2) \cdot m}{(m+n)(m+n-1)\cdots(m+n-K+1)}$$

$$= \frac{n! \cdot m}{(m+n)!} \sum_{i=0}^{K-1} \frac{(m+n-(i+1))!}{(n-i)!}$$

19) Let ~~A~~ us consider that player A starts the game.

Let W_i be the event that a white ball is drawn by A at i-th draw

Then $P(A \text{ wins}) = P(W_1) + P(W_2) + P(W_3) + \cdots$

Now, $P(W_1) = \dfrac{m}{m+n}$

For W_2, A draws a black ball, then B draws a black ball and then A draw a white ball.

$$\therefore \; P(W_2) = \frac{n}{m+n} \cdot \frac{(n-1)}{(m+n-1)} \cdot \frac{m}{(m+n-2)}$$

$$\therefore \; P(W_k) = \frac{n}{(m+n)} \cdot \frac{(n-1)}{(m+n-1)} \cdots \frac{n-2k+3}{m+n-2k+3} \cdot \frac{m}{m+n-2k+2}$$

The process will terminate after all the balls are drawn from the box.

$$\therefore \; P(A \text{ wins}) = \frac{m}{m+n} + \frac{n(n-1).m}{(m+n)(m+n-1)(m+n-2)} + \frac{n(n-1)(n-2).(n-3).m}{(m+n)(m+n-1)(m+n-2)(m+n-3)} + \cdots$$

20)

i) If m is the largest number drawn then remaining $k-1$ balls are drawn from the set of $\{1, 2, \ldots, m-1\}$ balls.

$\therefore$ $P(m \text{ is the largest number}) = \dfrac{\binom{m-1}{k-1}}{\binom{n}{k}}$

ii) Since the largest number drawn is ~~from~~ less or equal to m, the largest number must be from the set $\{1, 2, \ldots, m\}$.

Therefore, the event ~~that~~ of ~~largest~~ ~~n~~ selecting k balls such that largest number drawn is ~~same~~ ~~same~~ less or equal to m is same as selecting k balls from the set $\{1, 2, \ldots, m\}$.

$\therefore$ $P(\text{largest number drawn is} \leq m) = \dfrac{\binom{m}{k}}{\binom{n}{k}}$

21) Let A be the event that ~~all the k balls~~ none of the k balls ~~come~~ ~~out~~ is white.

$\therefore$ The required event is A^c.

Now, $P(A) = \dfrac{\binom{n}{k}}{\binom{m+n}{k}}$

$\therefore$ $P(A^c) = 1 - \dfrac{\binom{n}{k}}{\binom{m+n}{k}}$

22) Let W be the event that the ball drawn from A is white.

Now, consider the following events.

W_0 = No white ball is transferred from B to A

W_1 = one " " " B to A

W_2 = two " balls are " B to A.

i)

$\therefore P(W) = P(W|W_0).P(W_0) + P(W|W_1)P(W_1) + P(W|W_2).P(W_2)$

Now, $P(W_0) = \frac{\binom{8}{2}}{\binom{12}{2}}$, $P(W_1) = \frac{\binom{4}{1}.\binom{8}{1}}{\binom{12}{2}}$, $P(W_2) = \frac{\binom{4}{2}}{\binom{12}{2}}$

and $P(W|W_0) = \frac{\binom{6}{1}}{\binom{13}{1}}$ $P(W|W_1) = \frac{\binom{7}{1}}{\binom{13}{1}}$ $P(W|W_2) = \frac{\binom{8}{1}}{\binom{13}{1}}$

$\therefore P(W) = \frac{6}{13} \cdot \frac{8\times7}{12\times11} + \frac{7}{13} \cdot \frac{4\times8\times2}{12\times11} + \frac{8}{13} \cdot \frac{4\times3}{12\times11}$

$= \frac{20}{39}$

ii) Let E be the event that at least one white ball was transferred to A.

$\therefore E = W_1 \cup W_2$

$\therefore P(E|W) = P(W_1|W) + P(W_2|W)$ [$\because W_1$ and W_2 are mutually exclusive]

$= \frac{P(W|W_1).P(W_1) + P(W|W_2).P(W_2)}{P(W)}$

$= \frac{\frac{7}{13} \cdot \frac{4\times8\times2}{12\times11} + \frac{8}{13} \cdot \frac{4\times3}{12\times11}}{\frac{20}{39}} = \frac{34}{35}$