Computer Vision-CS385

Geometric Transformation

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- Images can be oriented after applying a specific transformation to the models.
- Known as Affine Transformation or Geometric Transformation in Computer Vision.

Objective

- Our objective is to understand Geometric Transformation.
- Two types: 2D and 3D transformation.

2D TRANSFORMATIONS & MATRICES

- Transformation in 2D is basically matrix transformation.
- With transformation, we can move a line, change shape, etc.

$$[B] = [T]\,[A]$$

  - $[A]$ = co-ordinate of points on which we apply transformation
  - $[B]$ = co-ordinate of transformed points
  - $[T]$ = geometric transformation matrix/operator

Inference

- If $[A]$ and $[T]$ are known, transformed points are obtained by calculation of $[B]$.

GENERAL TRANSFORMATION OF 2D POINTS

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \qquad \begin{aligned} x' &= ax + cy \\ y' &= bx + dy \end{aligned}$$

- $\begin{bmatrix} T \end{bmatrix}$ is the transformation matrix with four scalar parameters.
- (x, y) are the points that are to be transformed.
- (x', y') are the transformed co-ordinates of (x, y).
- So, we are pre-multiplying operator $\begin{bmatrix} T \end{bmatrix}$ with $\begin{bmatrix} A \end{bmatrix}$.
- We can also do post-multiplication i.e. $\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} T \end{bmatrix}$.
- Solution has to be intact i.e. $x' = ax + cy; y' = bx + dy$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a & c \\ b & d \end{bmatrix}; \qquad \begin{aligned} x' &= ax + cy \\ y' &= bx + dy \end{aligned}$$

SOLID BODY TRANSFORMATION

- Transformation equation is valid for all set of points and lines of the object being transformed.
- A solid transformation preserves distances between every pair of points.

SPECIAL CASES OF 2D MATRIX

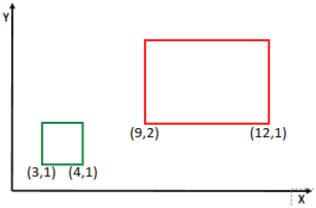
- When $a = d = 1, b = c = 0$, so $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
- $\begin{bmatrix} T \end{bmatrix}$ = Identity matrix and $x' = x; y' = y$.
- $\begin{bmatrix} T \end{bmatrix}$ = Identity, transformation do not change the structure of the solid body.

SCALING

SPECIAL CASES OF 2D MATRIX

- $a = d \neq 0, b = c = 0$, so
  $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = ax; y' = dy.$$
- x is now scaled by a factor 'a' and y by a factor 'd'.

- $a, d > 1$, ENLARGEMENT
  $0 < a, d < 1$, COMPRESSION
- If a = d , UNIFORM.
- If $a \neq d$, NON-UNIFORM.



EXAMPLE

- $a = 3, d = 2$, Non-uniform scaling $a \neq d$, Expansion $a, d > 0$

REFLECTION

SPECIAL CASES OF 2D MATRIX

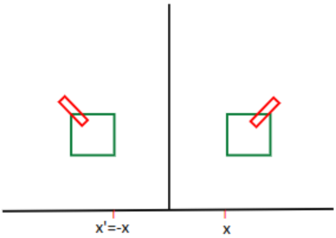
- a **and/or** $d < 0, b = c = 0$, reflection along an axis.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; x' = -x$$

Reflection around Y-axis.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; y' = -y$$

Reflection around X-axis.



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; \text{Special Case}$$

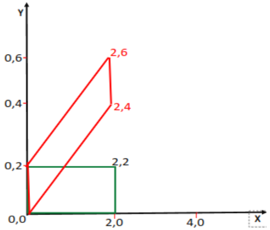
Reflection around a plane (3D case).

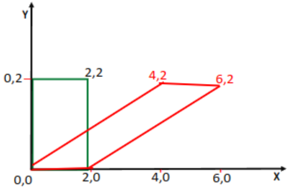
SHEAR

SPECIAL CASES OF 2D MATRIX

- $a = d = 1$. Let $c = 0, b = 2$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

- $x' = x; y' = 2x + y$
  So, y' depends linearly on x. This effect is called Shear.
- If $c = 2, b = 0$, shear will be proportional to X-axis.



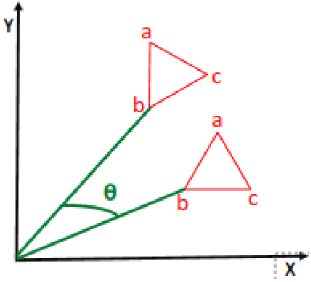


ROTATION

SPECIAL CASES OF 2D MATRIX

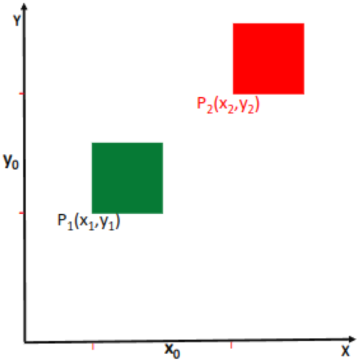
- $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rotation always around origin.
- Counter-clockwise direction is positive.



TRANSLATION

- Translate from point $P_1(x_1, y_1)$ to another point $P_2(x_2, y_2)$.
- Translation by x_0 in X and y_0 in Y means $x_2 = x_0 + x_1, y_2 = y_0 + y_1$.
- Realized through matrix operation:
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



- Translations are used in Scaling and Rotation, when objects/lines are not centered around origin.

ANOMALY

- Five basic transformations: Scaling, Reflection, Rotation, Sheer, Translation.
- Scaling, Reflection, Rotation, Sheer are realized by $[B] = [T]\,[A]$.
- Translation cannot be realized by $[B] = [T]\,[A]$.

SOLUTION

- Move from 2x2 transformation matrix to 3x3 transformation matrix to realize all 2D transforms.
- Concept of Homogeneous co-ordinate system.

HOMOGENEOUS COORDINATES

- A 2D point (x, y) is represented using a triplet (x, y, w) and the 2D transformation matrix $\left[T\right]$ will be a 3×3 matrix.

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & c & x_0 \\ b & d & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

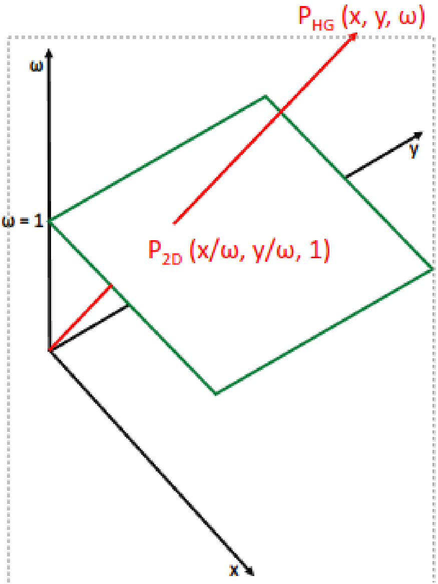
- (x, y, w) are co-ordinates before transformation and after transformation we get (x', y', w').

$$\begin{cases} x' = ax + cy + wx_0 \\ y' = bx + dy + wy_0 \\ w' = w \end{cases}$$

- Remember, we are not in 3D space. We are still talking about 2D transformations.
- What is the role of w, also called homogeneous term?

HG CONTD...

- (x, y, w) is the homogeneous representation of a point.
- Divide the first 2 elements by w i.e., $\{\frac{x}{w}, \frac{y}{w}\}$ gives the cartesian co-ordinates for the homogeneous points.
- w=1 represents the cartesian plane in the HG system.



PROPERTY

- 2 Homogeneous co-ordinates (x_1, y_1, w_1) and (x_2, y_2, w_2) may represent same point iff they are multiples of one another. Ex: $(1, 2, 3)$ & $(3, 6, 9)$.
- Hence, there is no unique homogeneous representation of a point.
- All triplets of the form $\{tx, ty, tw\}$ form a line in the x, y, w space.
- Cartesian co-ordinates are just the plane $w = 1$ in this space.
- $w = 0$? Points at infinity. $(x, y, 0)$ is the "Ideal Point".

TRANSLATION

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad w = 1$$

- Homogeneous Co-ordinate concept is created to capture translation as matrix multiplication.

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \frac{1}{s} \end{bmatrix} \xrightarrow[\text{Transform}]{\text{Cartesian}} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} sx \\ sy \\ 1 \end{bmatrix}$$

- Uniform Scaling can be captured using a single parameter.

REVISITING 2D TRANSFORMATION

$$T = \begin{bmatrix} a & c & p \\ b & d & q \\ m & n & s \end{bmatrix}$$

- Parameters involved in scaling, rotation, reflection, & shear: a, b, c, d.
  - If $B = TA$, translation parameters: p, q.
  - If $B = AT$, translation parameters: m, n.
  - s: Special case for uniform scaling.
- If $B = TA$, what is the role of m, n? Perspective Transform.

COMPOSITE TRANSFORMATIONS

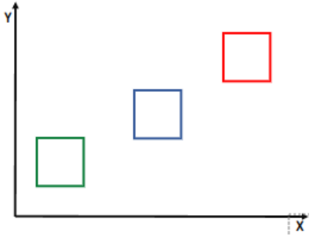
- If we want to apply a series of transformations T_1, T_2, T_3 to a point 'p', we can do it in 2 ways.
  1. $p' = T_1 * p \rightarrow p'' = T_2 * p' \rightarrow p''' = T_3 * p''$
  2. Calculate $T = T_3 * T_2 * T_1$, and then $p''' = T * p$
- Method 2 saves large no. of computational time.

Note

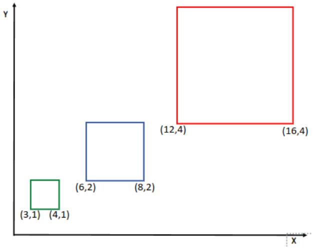
$T_3 * T_2 * T_1$. As per convention, since T_1 is applied first it has to be the rightmost transformation, and then T_2 and so on.

SOME EXAMPLES

- Translate by tx_1, ty_1 and then by tx_2, ty_2



- Scale by a_1, b_1 and then by a_2, b_2



SOME EXAMPLES

- Translate by tx_1, ty_1 and then by tx_2, ty_2

$$\begin{bmatrix} 1 & 0 & tx_1 + tx_2 \\ 0 & 1 & ty_1 + ty_2 \\ 0 & 0 & 1 \end{bmatrix}$$

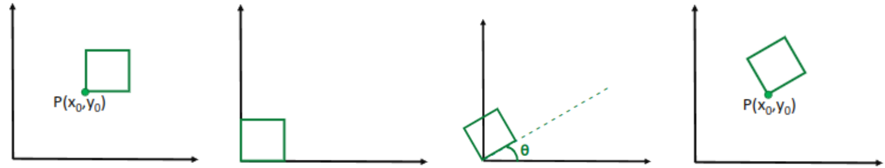
- Scale by a_1, b_1 and then by a_2, b_2

$$\begin{bmatrix} a_1 * a_2 & 0 & 0 \\ 0 & b_1 * b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotate by θ_1 and then by θ_2
  - Replace θ by $(\theta_1 + \theta_2)$
  - Calculate T_1 for θ_1 and T_2 for θ_2 and multiply them

$$\begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY POINT P



House at $P(x_0, y_0)$

Translate P to origin: T_1

Rotate by θ: T_2

Translate back to P: T_3

What will be the transformation matrix? T

$$T = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_3} * \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_2} * \underbrace{\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_1}$$

SCALING ABOUT AN ARBITRARY POINT P(x_0, y_0)

- T_1: Translate P to origin
  T_2: Scale
  T_3: Translate P back

- $$T = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_3} * \underbrace{\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_2} * \underbrace{\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_1}$$

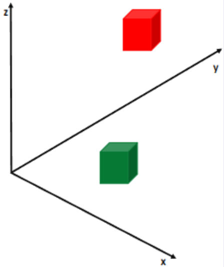
- $T = T_3(x_0, y_0) * T_2(S_x, S_y) * T_1(-x_0, -y_0)$

TRANSFORMATIONS IN 3D

- Homogeneous representation of a 3D point (x,y,z) is (xw,yw,zw,w), where w is the homogeneous term.

TRANSLATION

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

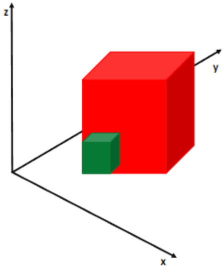


TRANSFORMATIONS IN 3D CONTD...

SCALING

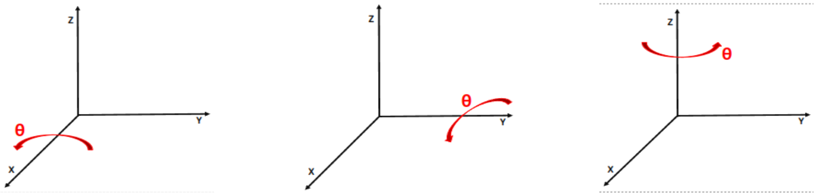
$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

can be changed for uniform scaling



ROTATION

- In 2D, we were rotating around origin. In 3D, rotation will be around an axis.



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REFLECTION

Reflection around XY plane, Z -ve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection around XZ plane, Y -ve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection around YZ plane, X -ve

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Around X-axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Around Y-axis

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Around Z-axis

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Around origin? All diagonal elements, except 'w', = -1.

SUMMARY 3D TRANSFORMATION

$$T = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & s \end{bmatrix}$$

- 9 parameters involved in scaling, rotation, reflection, & shear: $a, b, c, e, f, g, i, j, k$.
    - If $B = TA$, translation parameters: d, h, l.
    - If $B = AT$, translation parameters: m, n, o.
    - s: Special case for uniform scaling.
- If $B = TA$, what is the role of m, n, o? Perspective Projection.

PROJECTION

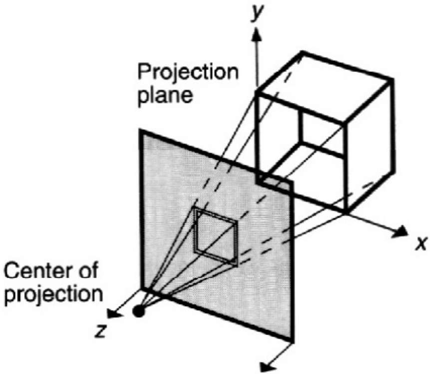
- A scene is where the camera is.
- Projection is required to define the position and attributes of the camera.
- Map the scene into a 2D image, since display screen is always 2D.
- Defined by straight projection rays (projectors) emanating from the object, passing through the projection plane and meeting on the center of projection (COP).

Two types

- Parallel Projection
- Perspective Projection

PERSPECTIVE PROJECTION

- Distance between the COP and projection plane is finite.
- Projectors are not parallel.

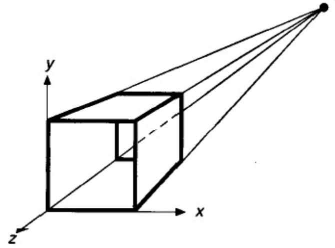


Perspective foreshortening

- Closer is the object to the COP, larger is its projection.
- Further the object from the COP, smaller is its projection.

VANISHING POINTS - VP

- Two parallel railway tracks appear to meet at a point on the horizon.
- This point is called the VP.



- Projection plane is XY plane.
- All lines parallel to z axis appear to come from a point.
- All lines parallel to x and y axis remain parallel.

- Realized by the parameters m, n, o in the transformation matrix.
- Derivation not within the purview of geometric transformation.

Thank you