$$3P\dot{R} + R\dot{P} = -3R_uTk_cHP.$$

Now, $P = P_A + \rho_L g z$

$$\frac{dz}{dt} = \beta R$$

$$\dot{P} = \rho_L g \dot{z}$$
$$= \rho_L g \beta R$$

Substituting it into main equation:-

$$3P\dot{R} + R(\rho_L g \beta R) = -3R_uTk_cHP$$

$$\Rightarrow 3P\dot{R} + \rho_L g \beta R^2 = -3R_uTk_cHP.$$

Solve for $\dot{R} \Rightarrow$

$$3P\dot{R} = -3R_uTk_cHP - \rho_L g \beta R^2$$

$$\frac{dR}{dt} = -R_uTk_cH - \frac{\rho_L g \beta}{3P} R^2$$

Eliminating time:-

$$\frac{dR}{dP} = \frac{\dot{R}}{\dot{P}} = \frac{-R_uTk_cH - \frac{\rho_L g \beta R^2}{3P}}{\rho_L g \beta R}$$

Given Problem:—

For Physical Vapor Deposition:—

Assumptions:—

- 1D growth :— Crystal grows only in the $+x$ direction from $x = 0$ to $x = L(t)$.
- Instantaneous crystallization :— every molecule arriving at the interface freezes with 100% efficiency.
- Isothermal interface :— Interface temperature $T_{int}(t)$ is prescribed.
- Constant properties
- Negligible vapor-side resistance : mass flux $J_m(t)$ is taken as known (or clamped to an empirical growth rate).

Governing Equations:—

- Heat conduction of the solid:—

$$\rho_s c_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L(t)$$

- Stefan (energy) condition at $x = L(t)$

$$\rho_s \Delta H \frac{dL}{dt} = -k_s \frac{\partial T}{\partial x}\bigg|_{x = L(t)}$$

Since, $u_0 = R_0^2$ rearranging:-

$$R^2 P^{2/3} = R_0^2 P^{2/3} - \frac{6A}{5}\left[P^{5/3} - P_0^{5/3}\right]$$

Disappearance condition $R \to 0$:-

Set $R = 0$ at $P = P_{end}$,

$$0 = R_0^2 P_0^{2/3} - \frac{6A}{5}\left[P_{end}^{5/3} - P_0^{5/3}\right]$$

$$\Rightarrow P_{end}^{5/3} = P_0^{5/3} + \frac{5 R_0^2 P_0^{2/3}}{6A}$$

$$\Rightarrow P_{end} = \left(P_0^{5/3} + \frac{5 R_0^2 P_0^{2/3}}{6A}\right)^{3/5}$$

$$z_{end} = \frac{P_{end} - P_A}{\rho_L g}$$