SUNDAY 20-08-17

01

CO-ORDINATE GEOMETRY

(i) CHANGE OF AXIS

(II) GENERAL EQUATION OF SECOND DEGREE TO REPRESENT ELLIPSE, PARABOLA, and HYPERBOLA

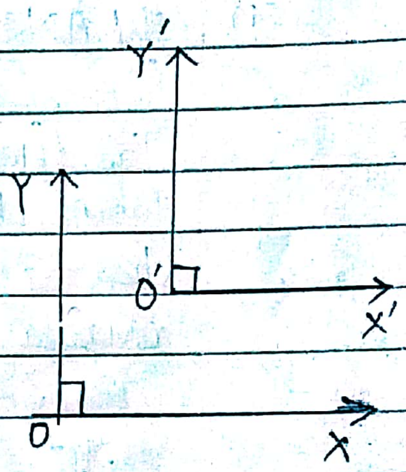
(III) TANGENT AND NORMAL.

CHANGE OF AXIS

The change of axis can be brought about in three ways :-

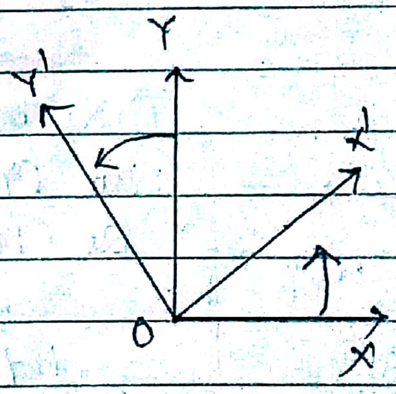
(I) Translation of axis :-

If at a point (h, K) in the x-y plane, new co-ordinate axis x' and y' are chosen so that they are parallel to the original x and y axis respectively, we say that there has been a translation of axis in the plane.



(II) ROTATION OF AXIS :

If the axis in the x-y plane are rotated about the origin through a given angle, we say that there has been a rotation of axis in the plane.



(iii) Translation and Rotation axis:

In this case the two transformations, namely the translation of axis and the rotation of axis are made simultaneously.

A general transformation of rectangular axis can be given by a translation or rotation of axis taken in any order.

We shall deal with them one by one.

(1) TRANSLATION OF AXIS :-

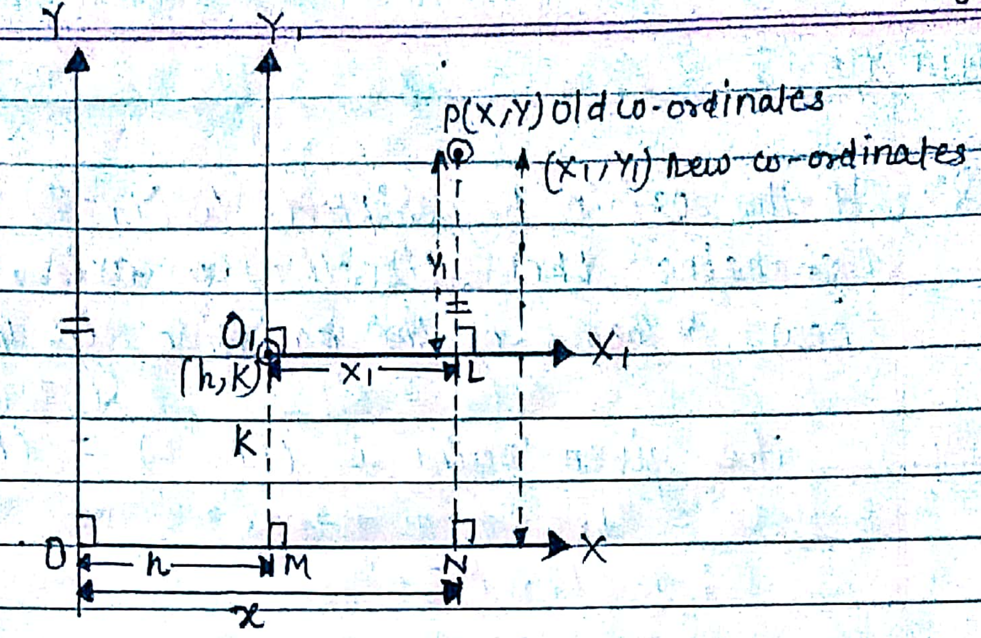
To find the co-ordinates of a point in a plane when the origin is shifted to a new point (h, K), the new axis remaining parallel to the original axis.

Let O be the origin and OX, OY the axis originally. Shift the origin to the point O_1.

Let O_1X_1, O_1Y_1 be the new axis; O_1X_1 being parallel to OX_1, and O_1Y_1 parallel to OY. Let (h, K) be the co-ordinates of O_2 referred to the original axis.

Let P be any point referred to the old axis are (x, y) and that referred to the new axis are (x_1, y_1).

Draw PN parallel to OY, cutting OX in N and O_1X_1 in L.



Then $ON = x$ and $PN = Y$;
$O_1L = x_1$ & $PL = Y_1$.

From O_1 draw O_1M perpendicular to OX.
Then,
$OM = h$ and $O_1M = K$.

Now, $x = ON = OM + MN = OM + O_1L = h + x_1$
$Y = PN = PL + LN = PL + O_1M = y_1 + K$.

Thus we get the formula of transformation, viz,
$$x = x_1 + h$$
$$y = y_1 + K$$

$x_1 = x - h$
$y_1 = y - K$

These express new co-ordinates (x_1, y_1) in terms of old co-ordinates (x, y)

QUESTIONS

Q1. If the origin be shifted to $(1,-2)$ without changing the direction of axis, find the new form of the equation $y^2-4x+4y+8=0$.

Sol The given point is $(1,-2) = (h,K)$ say,

$\therefore h=1, K=-2$

Now $x = x_1+h = x_1+1$
$y = y_1+K = y_1-2$

Substituting the values of x and y in the given equation, we get the transformed equation.

$(y_1-2)^2-4(x_1+1)+4(y_1-2)+8=0$
$\Rightarrow y_1^2-4x_1=0$

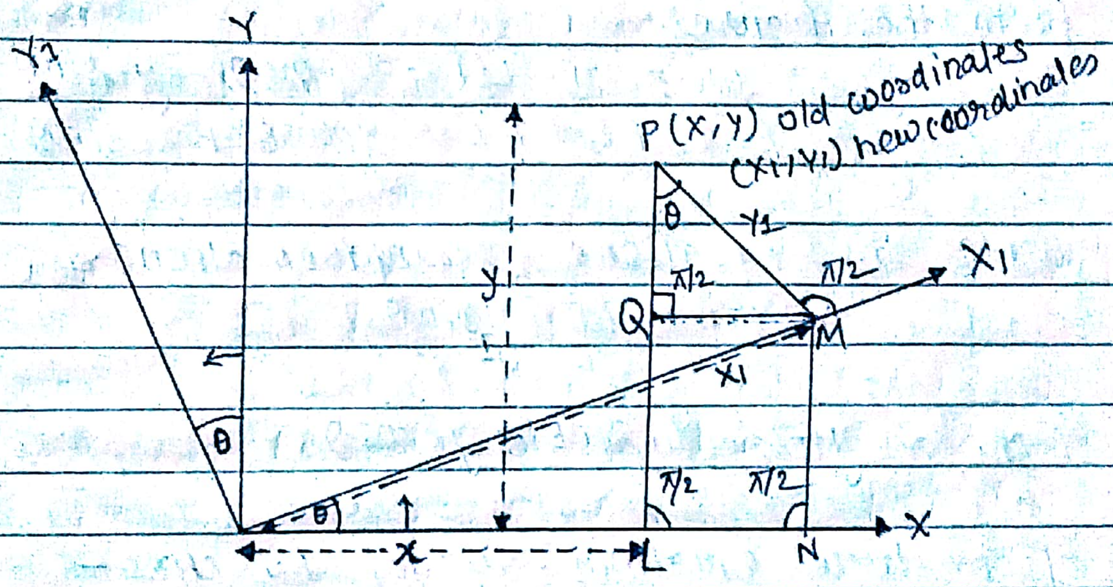
Dropping the suffixes, we get the required equation.

$\boxed{y^2-4x=0}$ Ans

2. ROTATION OF AXIS :-

To find the co-ordinates of a points, when the directions of the axis are changed without changing the position of the origin and the angle between the axis.

Let OX, OY be the old axis and OX_1, OY_1 be the new axis rotated through an angle θ.



Let the co-ordinates of any point P referred to the first and the second system be (x, y) and (x_1, y_1) respectively.

Draw PL and PM perpendecular to OX and and OX_1 respectively.

Draw MN and MQ perpendicular to OX and PL respectively

Then, $\angle MPQ = \theta$, $OL = x$, $PL = y$, $PM = y_1$, $OM = x_1$.

From ΔOMN : $\cos\theta = \dfrac{ON}{OM} = \dfrac{ON}{x_1}$, i.e. $ON = x_1\cos\theta$;

$$\sin\theta = \frac{MN}{OM} = \frac{MN}{x_1}, \text{ i.e, } MN = x_1 \sin\theta.$$

From ΔPQM, $\cos\theta = \frac{PQ}{PM} = \frac{PQ}{y_1}$, i.e., $PQ = y_1 \cos\theta$;

$$\sin\theta = \frac{QM}{PM} = \frac{QM}{y_1}; \text{ i.e } QM = y_1 \sin\theta.$$

But $LN = QM$ (from figure)

$\therefore LN = y_1 \sin\theta$.

From the figure.

$$x = OL = ON - LN = x_1 \cos\theta - y_1 \sin\theta$$

$$y = PL = PQ + QL = y_1 \cos\theta + x_1 \sin\theta$$

Hence, the equations of transformation are

$$x = x_1 \cos\theta - y_1 \sin\theta$$

$$y = x_1 \sin\theta + y_1 \cos\theta.$$

From these equations, we get by cross-multiplication,

$$\frac{x_1}{x\cos\theta + y\sin\theta} = \frac{y_1}{y\cos\theta - x\sin\theta} = \frac{1}{\cos^2\theta + \sin^2\theta}.$$

Thus $\boxed{\begin{aligned} x_1 &= x\cos\theta + y\sin\theta \\ y_1 &= y\cos\theta - x\sin\theta \end{aligned}}$

EX: Find the transformed equation of the straight line $x\cos\alpha + y\sin\alpha = p$. When the axis are rotated through an angle α.

Sol: The equation of transformation are:

$$x = x_1\cos\alpha - y_1\sin\alpha$$
$$y = x_1\sin\alpha + y_1\cos\alpha.$$

Substituting these values of x and y in the given equation we get:

$$(x_1\cos\alpha - y_1\sin\alpha)\cos\alpha + (x_1\sin\alpha + y_1\cos\alpha)\sin\alpha = p$$

$$x_1\cos^2\alpha - y_1\sin\alpha\cos\alpha + x_1\sin^2\alpha + y_1\sin\alpha\cos\alpha = p$$
$$x_1(\cos^2\alpha + \sin^2\alpha) = p$$
$$x_1 = p$$

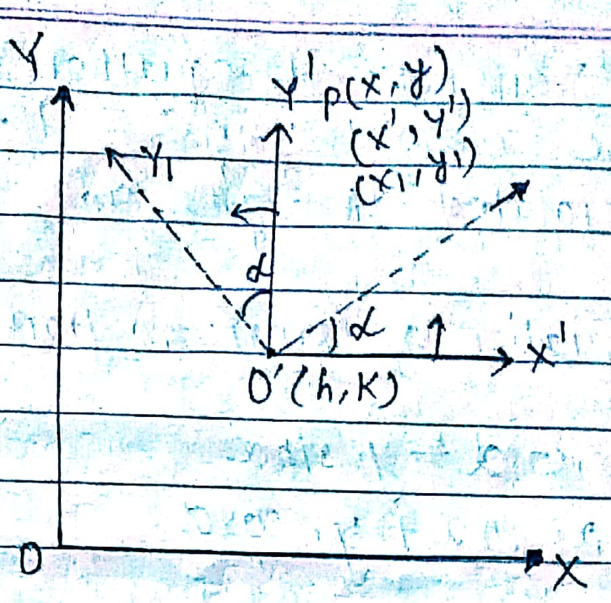
If (x, y) be used as current co-ordinates the new equation is

$$\boxed{x = p}$$

Translation and Rotation of axis :-

To find the co-ordinates of a point in a plane when the origin is shifted to the point (h, K) and also the axis are rotated through an angle α.

Let the origin O be shifted to the point O' whose co-ordinates are (h, K).



From O' draw O'X' and O'Y' parallel to OX and OY respectively. Now rotate the axis O'X' and O'Y' through an angle α so that the new axis are $O'X_1$ and $O'Y_1$

Let P be any point in the plane, whose co-ordinates with respect to OX, OY; O'X', O'Y' and $O'X_1$ $O'Y_1$ are respectively (x, y), (x', y') and (x_1, y_1).

Then, $x = x' + h$ and $y = y' + K$

$$x' = x_1 \cos\alpha - y_1 \sin\alpha \text{ and } y' = x_1 \sin\alpha + y_1 \cos\alpha.$$

Hence
$$x = h + x_1 \cos\alpha - y_1 \sin\alpha$$
$$y = k + x_1 \sin\alpha + y_1 \cos\alpha$$

These are the required relation between the original and the new co-ordinates.

Solved Questions

Q1. Transform to parallel axis to the point $(1, -2)$ for the equation $2x^2 + y^2 - 4x + 4y + 5 = 0$

Sol. Replace x & y with $x+1$ & $y-2$ in above equation

$$2(x+1)^2 + (y-2)^2 - 4(x+1) + 4(y-2) + 5 = 0$$

Now the equation is

$$2(x^2 + 2x + 1) + y^2 - 4y + 4 - 4x - 4 + 4y - 8 + 5 = 0$$

$$2x^2 + \cancel{4x} + 2 + y^2 - \cancel{4y} + \cancel{4} - \cancel{4x} - \cancel{4} + \cancel{4y} - 8 + 5 = 0$$

$$2x^2 + y^2 - 1 = 0$$

$$\boxed{2x^2 + y^2 = 1}$$

Ans.

Q2. Chose the new origin (α, β) direction of axis being a same so that the equation of the curve $5xy + y^2 + 25x - 5y - 65 = 0$ may be convert into $axy + by^2 = 1$. Find a & b?

Sol. $$5xy + y^2 + 25x - 5y - 65 = 0$$

$$5(x+\alpha)(y+\beta) + (y+\beta)^2 + 25(x+\alpha) - 5(y+\beta) - 65 = 0$$

$$5\{xy + x\beta + y\alpha + \alpha\beta\} + \{y^2 + 2\beta y + \beta^2\} + 25x + 25\alpha - 5y - 5\beta - 65 = 0$$

$$5xy + 5x\beta + 5y\alpha + 5\alpha\beta + y^2 + 2y\beta + \beta^2 + 25x + 25\alpha - 5y - 5\beta - 65 = 0$$

$5xy + y^2 + (5\beta + 25)x + (5\alpha + 2\beta - 5)y +$

$y^2 + 5\alpha\beta + \beta^2 + 25\alpha - 5\beta - 65 = 0$ — (1)

Equating Equation coefficent of x & y to 0

Now we, have

$5\beta + 25 = 0$ $\quad$ $5\alpha + 2\beta - 5 = 0$

$\boxed{\beta = -5}$ $\quad$ $5\alpha + 2(-5) - 5 = 0$

$5\alpha = 15$

$\boxed{\alpha = 3}$

putting the value of α & β in eq^n (1)

$5xy + y^2 + 5(-15) + 25 + 75 + 25 - 65 = 0$

$5xy + y^2 \cancel{-75} + \cancel{75} + 50 - 65 = 0$

$5xy + y^2 - 15$

$5xy + y^2 = 15$ — (2)

Now

$axy + y^2 = 1$ — (3)

Dividing equation (2) with 15 and compairing with equation (3)

we get,

$\frac{1}{3}xy + \frac{1}{15}y^2 = 1$ $\quad$ $a = \frac{1}{3}$

$axy + by^2 = 1$ $\quad$ $b = \frac{1}{15}$

(3) Transform the equation $4x^2+3y^2-2xy+3x-7y+5=0$ to parallel axis through the point $(4,-1)$.

Sol we know that the formula of transformation are

$x = x_1+h \qquad y = y_1+K$

$h = 4 \qquad K = -1$

$\therefore\ x = x_1+4 \qquad y = y_1-1.$

Substituting the values of x and y in the given equation. we get:

$$4(x_1+4)^2+3(y_1-1)^2-2(x_1+4)(y_1-1)+3(x_1+4) -7(y_1-1)+5=0$$

$$4(x_1^2+8x_1+16)+3(y_1^2-2y_1+1)-2(x_1y_1-x_1+4y_1-4) +3x_1+12-7y_1+7+5=0$$

$$4x_1^2+3y_1^2-2x_1y_1+37x_1-21y_1+99=0.$$

If (x,y) be used as current co-ordinate the transformed equation is

$$4x^2+3y^2-2xy+37x-21y+99=0$$

(4) Show that the equation $12x^2+2y^2-10xy+11x-5y+2=0$ can be reduced to a homogeneous equation of the second degree by transferring the origin to a properly chosen point.

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Sol. Let the origin be transferred to the point (h, K)

If the point (x, y) becomes (x_1, y_1) referred to new axis then :

$$x = x_1 + h \qquad y = y_1 + K$$

The given eqn becomes :

$$12(x_1+h)^2 + 2(y_1+K)^2 - 10(x_1+h)(y_1+K) + 11(x_1+h) - 5(y_1+K) + 2 = 0$$

$$12x_1^2 + 2y_1^2 - 10x_1y_1 + x_1(24h - 10K + 11) + y_1(4K - 10h - 5) + (12h^2 + 2K^2 - 10hK + 11h - 5K + 2) = 0$$

This equation will be reduced to a homogenous equation of second degree if :

$$24h - 10K + 11 = 0, \quad 4K - 10h - 5 = 0$$
$$12h^2 + 2K^2 - 10hK + 11h - 5K + 2 = 0.$$

$$\therefore \frac{h}{-50+44} = \frac{K}{110-120} = \frac{1}{-96+100}$$

$$h = -\frac{3}{2} \quad \& \quad K = -\frac{5}{2}.$$

NOW

$$12h^2 + 2K^2 - 10hK + 11h - 5K + 2$$
$$= 12 \cdot \frac{9}{4} + 2 \cdot \frac{25}{4} - 10 \cdot \frac{3}{2} \cdot \frac{5}{2} - 11 \cdot \frac{3}{2} + \frac{25}{2} + 2$$
$$= \frac{108 + 50 + 58}{4} - \frac{75}{2} - \frac{33}{2} =$$

$$\frac{216}{4} - \frac{108}{2} = \frac{216-216}{4} = 0.$$

Hence if the origin is transfered to $\left(-\frac{3}{2}, \frac{-5}{2}\right)$ the given equation reduces to $12x^2+2y^2-10xy=0$ Which is a homogenous equation of the degree second.

Sir Q The equation $x^2-y^2=a^2$ is transformed to $xy=K^2$ by change of rectangular axis. Find the inclination of the later axes to the former and the value of K^2?

Sol Let the axes is rotated with an angle θ Replacing the value of x & y with $x_1\cos\theta - y_1\sin\theta$ & $y_1\cos\theta + x_1\sin\theta$ respectively in the given eq^n.

$$(x\cos\theta - y\sin\theta)^2 - (y\cos\theta + x\sin\theta)^2 = a^2$$

$$(x^2\cos^2\theta + y^2\sin^2\theta - 2xy\cos\theta\sin\theta) - (y^2\cos^2\theta + x^2\sin^2\theta + 2xy\cos\theta\sin\theta) = a^2$$

$$\Rightarrow x^2\cos^2\theta - x^2\sin^2\theta + y^2\sin^2\theta - y^2\cos^2\theta - 4xy\cos\theta\sin\theta = a^2$$

$$x^2(\cos^2\theta - \sin^2\theta) + y^2(\sin^2\theta - \cos^2\theta) - 2xy\sin 2\theta = a^2$$

$$x^2\cos 2\theta - y^2\cos 2\theta - 2xy\sin 2\theta = a^2$$

$$(x^2-y^2)\cos 2\theta - 2xy\sin 2\theta = a^2$$

In the new equation

$$\boxed{xy = K^2} \quad \text{—— (2)}$$

There is no term for x^2 & y^2

Hence Equating the coefficient of x^2 & y^2 to zero.

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\therefore \boxed{\theta = \frac{\pi}{4}}$$

putting the value of $\theta = \frac{\pi}{4}$

$$-2xy \sin\frac{\pi}{2} = a^2$$

$$-2xy = a^2$$

$$\boxed{xy = -\frac{a^2}{2}} \quad \text{—— (3)}$$

compairing equation (2) & (3)

$$xy = K^2$$

$$xy = -\frac{a^2}{2}$$

Hence $\boxed{K^2 = -\frac{a^2}{2}}$

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Sr CS2 Find the angle to which the axes must be turned so that the expression $ax^2 + 2hxy + by^2 = 0$ may become an expression in which there is no term involving $x.y$.

Sol

$$ax^2 + 2hxy + by^2 = 0$$

Replacing x & y with $x\cos\theta - y\sin\theta$ & $x\sin\theta + y\cos\theta$ respectively in the given equation.

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2$$

$$\Rightarrow a(x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta) + 2h(x^2\cos\theta\sin\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\cos\theta\sin\theta) + b(x^2\sin^2\theta + y^2\cos^2\theta + 2xy\sin\theta.\cos\theta)$$

Equating the coefficent of xy to zero.

$$\Rightarrow -a2\sin\theta\cos\theta + 2h(\cos 2\theta) + b2\sin\theta\cos\theta = 0$$

$$\Rightarrow -a\sin 2\theta + 2h\cos 2\theta + b\sin 2\theta = 0$$

$$= \sin 2\theta(b-a) + 2h\cos 2\theta$$

$$\sin 2\theta(b-a) = -2h\cos 2\theta$$

$$\tan 2\theta(b-a) = -2h$$

$$\tan 2\theta = \frac{-2h}{(b-a)} = \frac{2h}{(a-b)}$$

$$\Rightarrow 2\theta = \tan^{-1}\frac{2h}{a-b} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}}$$ — Ans

Invariants

To show that if $ax^2+2hxy+by^2=1$ and $a_1x^2+2h_1xy+b_1y^2=1$ represent the same conic if

(i) $a+b = a_1+b_1$

(ii) $(a-b)^2+4h^2 = (a_1-b_1)^2+4h_1^2$

(iii) $(ab-h)^2 = a_1b_1-h_1^2$

Sol (i) Replacing $x = (x\cos\theta - y\sin\theta)$
$y = (x\sin\theta + y\cos\theta)$ respectively x & y.

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2$$

$$= a(x^2\cos^2\theta + y^2\sin^2\theta) - 2xy\cos\theta\sin\theta) + 2h(x^2\sin\theta\cos\theta + xy\cos^2\theta - yx\sin\theta - y^2\sin\theta\cos\theta \; b(x^2\sin^2\theta + y^2\cos^2\theta + 2xy\cos\theta\sin\theta) = 1$$

$$= x^2(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta) + y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta) + xy(-2a\sin\theta\cos\theta + 2h\cos 2\theta + 2b\sin\theta\cos$$

The equation (1) equating at the equation $a_1x^2+2h_1xy+b_1y^2=1$ represent the same conic

Hence the coefficient of the two expression should be same.

$$a\cos^2\theta + h\sin 2\theta + b\sin^2\theta = a_1 \quad \text{—— (2)}$$

$$-a\sin 2\theta + 2h\cos 2\theta + b\sin 2\theta = 2h_1 \quad \text{—— (3)}$$

$$a\sin^2\theta - h\sin 2\theta + b\cos^2\theta = b_1 \quad \text{—— (4)}$$

(1) Adding (2) & (4)

$$\Rightarrow a\cos^2\theta + a\sin^2\theta + h\sin 2\theta - h\sin 2\theta + b\sin^2\theta + b\cos^2\theta$$

$$\Rightarrow a(\sin^2\theta + \cos^2\theta) + b(\sin^2\theta + \cos^2\theta) = a_1 + b_1$$

$$\Rightarrow \boxed{a + b = a_1 + b_1} \text{ proved} \quad \text{—— (A)}$$

subtracting equation (2) − (4)

$$a\cos^2\theta - a\sin^2\theta + h\sin 2\theta + h\sin 2\theta + b\sin^2\theta - b\cos^2\theta = (a_1 - b_1)$$

$$\Rightarrow a(\cos^2\theta - \sin^2\theta) + 2h\sin 2\theta + b(\sin^2\theta - \cos^2\theta) = (a_1 - b_1)$$

squaring both side

$$(a_1 - b_1)^2 = (a\cos 2\theta + 2h\sin 2\theta - b\cos 2\theta)^2$$

$$= a^2\cos^2 2\theta + 4h^2\sin^2 2\theta - b^2\cos^2 2\theta + 4ha\sin 2\theta\cos 2\theta - 4hb\sin\theta\cos\theta - 2ab\cos^2 2\theta \quad \text{—— (5)}$$

o)

o)

(1)

squaring equation ③

$$4h_1^2 = a^2\sin^2 2\theta + 4h^2\cos^2 2\theta + h^2\sin^2 2\theta - 4ha\sin 2\theta\cos 2\theta + 4ha\sin 2\theta\cos 2\theta - 2ab\sin^2 2\theta \quad \text{—— ⑥}$$

Adding eqⁿ ⑤ & ⑥

$$(a_1-b_1)^2 + 4h_1^2 = a^2+4h^2+b^2-2ab$$

$$\boxed{(a_1-b_1)^2 + 4h_1^2 = (a-b)^2+4h^2} \quad \text{—— ⓑ}$$

Now

squaring Equation ①

we get,

$$a^2+b^2+2ab = a_1^2+2a_1b_1+b_1^2$$

subtract eqⁿ ③ − ②

$$\cancel{(a+b)^2} - \cancel{(a-b)^2}$$

$$\cancel{a^2}+\cancel{b^2}+2ab-\cancel{a^2}-\cancel{b^2}+2ab - 4h^2 = \cancel{a_1^2}+\cancel{b_1^2}+2a_1b_1 - \cancel{a_1^2}-\cancel{b_1^2}+2a_1b_1-4h_1^2$$

$$4ab-4h^2 = 4a_1b_1-4h_1^2$$

$$4(ab-h^2) = 4(a_1b_1-h_1^2)$$

$$\boxed{ab-h^2 = a_1b_1-h_1^2} \quad \text{—— ⓒ}$$

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If (x, y) and (x_1, y_1) be the coordinates of same point referred to two sets of rectangular axes with same origin and if $Ux + Vy$ where U & V are independent of (x, y) becomes $(U_1x_1 + V_1y_1)$ show that

$$U^2 + V^2 = U_1^2 + V_1^2$$

Sol.

To Find the condition that the general equation of 2^{nd} degree to represent an ellipse, hyperbola or parabola:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

① condition for ellipse :

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$$

$$A\left(x^2 + \frac{2G}{A}x + \frac{G^2}{A^2} - \frac{G^2}{A^2}\right) + B\left(y^2 + \frac{2F}{B}.y + \frac{F^2}{B^2} - \frac{F^2}{B^2}\right) + C = 0$$

$$\Rightarrow A\left[\left(x + \frac{G}{A}\right)^2 - \frac{G^2}{A^2}\right] + B\left[\left(y + \frac{F}{B}\right)^2 - \frac{F^2}{B^2}\right] + C = 0$$

$$\Rightarrow A\left(x + \frac{G}{A}\right)^2 - \frac{G^2}{A^2} + B\left(y + \frac{F}{B}\right)^2 - \frac{F^2}{B^2} + C = 0$$

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{B^2} - C$$

Now Replacing origin with $\left(-\frac{G}{A}, -\frac{F}{B}\right)$

Putting $x = \left(X - \frac{G}{A}\right)$ $\quad y = \left(Y - \frac{F}{B}\right)$

$\left(-\frac{G}{A}, -\frac{F}{B}\right)$

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Now The equation becomes.

$$A\left(x-\frac{G}{A}+\frac{G}{A}\right)^2+B\left(y-\frac{F}{B}+\frac{F}{B}\right)^2=\frac{G^2}{A}+\frac{F^2}{B}-c$$

$$Ax+By=\frac{G^2}{A}+\frac{F^2}{B}-c=K\text{ (let)}$$

$$Ax^2+By^2=K$$

$$\frac{x^2}{1/A}+\frac{y^2}{1/B}=K$$

$$\frac{x^2}{K/A}+\frac{y^2}{K/B}=1$$

$\because$ K is a positive number then $\frac{K}{A}, \frac{K}{B}$ is a either positive or negative depending on A & B

(i) If $A>0$ and $B>0$

$\Rightarrow AB>0$

The conic section represent an ellipse

(ii) When $A<0$ and $B<0$

$AB>0$

A = Neg

B = Neg

Neg Neg = positive

It Represent imaginary ellipse.

(iii) When $A>0$, and $B<0$ or
i.e. $A<0$ and $B>0$

then $AB<0$ in this case the conic section represent a hyperbola.

Condition that equation $Ax^2+By^2+2Gx+2Fy+C=0$ represent parabola.

Eq^n of Parabola $= y^2 = 4ax$
$x^2 = 4ay$

The equation represent a parabola if either $A=0$ && $B\neq 0$ or, $A\neq 0$ && $B=0$

let $A=0$ && $B\neq 0$

$$By^2+2Gx+2Fy+C=0$$

$$B\left(y^2+\frac{2F}{B}y+\frac{F^2}{B^2}-\frac{F^2}{B^2}\right)+2Gx+C=0$$

$$B\left(y+\frac{F}{B}\right)^2-\frac{F^2}{B}+2Gx+C=0$$

$$\left(y+\frac{F}{B}\right)^2-\frac{F^2}{B^2}+\frac{2Gx}{B}+\frac{C}{B}=0$$

$$\left(y+\frac{F}{B}\right)^2+\frac{2Gx}{B}-\frac{F^2}{B^2}+\frac{C}{B}=0$$

$$\left(y+\frac{F}{B}\right)^2+\frac{2G}{B}\left(x-\frac{F^2}{2GB}+\frac{C}{2G}\right)=0$$

shifting the origin to $\left(\frac{F^2}{2BG}-\frac{C}{2G}, \frac{-F}{B}\right)$

point

$$y^2 + \frac{2G x}{B} = 0$$

$$\left(y - \frac{F}{B} + \frac{F}{B}\right)^2 + \frac{2G}{B}\left(x + \frac{F^2}{2BG} - \frac{c}{2G} - \frac{F^2}{2BG} + \frac{c}{2G}\right) = 0$$

$$\Rightarrow \quad y^2 + \frac{2G}{B} x = 0$$

The eq^n $Ax^2 + By^2 + 2Gx + 2Fy + c = 0$ — ①

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ — ②

represent the same conic

V.V.I Condition that the second equation will represent a (parabola) ellipse

According to invariant

(I) $ab - h^2 = AB$

Now if $AB > 0$ it represent a ellipse

i.e. $ab - h^2 > 0$ it " " "

(II) When $AB < 0$ it represent a hyperbola

$ab - h^2 < 0$ it " " "

Now,

(III) When $AB = 0$ it will represent a parabola

$\Rightarrow$ $ab - h^2 = 0$ it will represent a parabola.

Q1. Find the type of conic which is represented by the following equation ?

(i) $16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$

Sol. From equation :-

$h = 12$

$a = 16$

$b = 9$

Now According to formula

$$ab - h^2 = 16 \times 9 - (12)^2$$
$$= 144 - 144$$
$$= 0.$$

$$ab - h^2 = 0$$
$$AB = 0$$

Hence the given equation represent an parabola.

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DIFFERENTIAL CALCULAS

(i) Successive differentiation.

(ii) Expansion of function.

(iii) Partial differentiation

(iv) Tangent and normal.

01-11-17 01-NOVEMBER 2017

Page

① $\frac{d \sin x}{dx} = +\cos x$

② $\frac{d \cos x}{dx} = -\sin x$

③ $\frac{d \tan x}{dx} = \sec^2 x$

④ $\frac{d \cot x}{dx} = -\operatorname{cosec}^2 x$

⑤ $\frac{d \sec x}{dx} = +\sec x \cdot \tan x$

⑥ $\frac{d \operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cdot \cot x$

⑦ $\frac{d \sin^{-1} x}{dx} = \cancel{\sqrt{1+x^2}} \quad \frac{1}{\sqrt{1-x^2}}$

⑧ $\frac{d \cos^{-1} x}{dx} = \cancel{\frac{1}{\sqrt{x^2-1}}} \quad \frac{-1}{\sqrt{1-x^2}}$

⑨ $\frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}$

⑩ $\frac{d \cot^{-1} x}{dx} = \cancel{\frac{1}{x^2-1}} \quad \frac{-1}{1+x^2}$

⑪ $\frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{\cancel{1+}x^2-1}}$

⑫ $\frac{d \operatorname{cosec}^{-1} x}{dx} = \cancel{\frac{-1}{x\sqrt{1-x^2}}} \quad \frac{-1}{x\sqrt{x^2-1}}$

Questions

① $y = A\sin mx + B\cos mx$ P.T $y_2 = -m^2 y$

Sol Given that

$$y = A\sin mx + B\cos mx$$

Differentiate w.r.t 'x' we get,

$$\frac{dy}{dx} = y_1 = mA\cos mx - mB\sin mx \quad \text{—} ①$$

Again Differentiate ① w.r.t. 'x' we get,

$$y_2 = -m^2 A\sin mx - m^2 \cos mx$$
$$= -m^2 (A\sin mx + B\cos mx)$$
$$\boxed{y_2 = -m^2 y}$$

Proved

② $y = Ae^{mx} + Be^{-mx}$ P.T $y_2 = m^2 y$

Sol Given that,

$$y = Ae^{mx} + Be^{-mx}$$

Differentiate y w.r.t 'x' weget.

$$y_1 = Ame^{mx} - Bme^{-mx}$$
$$y_2 = Am^2 e^{mx} + Bm^2 e^{-mx}$$
$$= m^2 (Ae^{mx} + Be^{-mx})$$
$$\boxed{y_2 = m^2 y}$$ Proved

Date
Page

(3) If $y = ax^{n+1} + bx^{-n}$ prove that $x^2y_2 = n(n+1)\cdot y$

Sol. Given that

$$y = ax^{n+1} + bx^{-n}$$

Diff y w.r.t 'x' we get,

$$y_1 = (n+1)ax^{n} - nbx^{-(n+1)}$$

$$y_2 = n(n+1)ax^{n} + n(n+1)bx^{-(n+2)}$$

$$y_2 = n(n+1)\left[ax^{n} + bx^{-(n+2)}\right]$$

$$= n(n+1)\left[\frac{ax^{n+1}}{x^2} + \frac{bx^{-n}}{x^2}\right]$$

$$y_2 = n(n+1)\left[\frac{ax^{n+1} + bx^{-n}}{x^2}\right]$$

$$x^2y_2 = n(n+1)\cdot(ax^{n+1} + bx^{-n})$$

$$\boxed{x^2y_2 = n(n+1)\cdot y}$$ Proved.

(4) $y = \log\sin x$ P.T $y_3 = \frac{2\cos x}{\sin^3 x}$

Sol. ~~Giv~~.

Given that,

$y = \log \sin x$

Differentiate y w.r.t 'x' we get,

$y_1 = \frac{1}{\sin x} \cos x = \cot x$

$y_2 = -\operatorname{cosec}^2 x$

$y_3 = -2\operatorname{cosec} x(-\operatorname{cosec} x \cdot \cot x)$

$= 2\operatorname{cosec}^2 x \cdot \cot x$

$= 2 \frac{1}{\sin^2 x} \frac{\cos x}{\sin x} = \frac{2\cos x}{\sin^3 x}$

$\boxed{y_3 = \frac{2\cos x}{\sin^3 x}}$ Proved.

(5) $y = a\sin(\log x)$ P.T $x^2 y_2 + x y_1 + y = 0$

Sol Given that

$y = a\sin(\log x)$

differentiate y w.r.t 'x' we get,

$y_1 = \frac{a\cos \log x}{x}$

$\boxed{x y_1 = a\cos \log x}$ ——— (1)

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Differentiate ① w.r.t 'x' we get,

$$y_1 + xy_2 = \frac{-a\sin x \log x}{x}$$

$$xy_1 + x^2y_2 = -a\sin x \log x$$

NOW $x^2y_2 + xy_1 + y$

$= -a\sin x \log x + a\sin \log x = 0$

$= 0$

$$\boxed{x^2y_2 + xy_1 + y = 0}$$ proved

⑥ $x = \cos(\log y)$

Sol $\cos^{-1}x = \log y$

Differentiate both side w.r.t 'x' we get,

$$\frac{-1}{\sqrt{1-x^2}} = \frac{1}{y}y_1$$

$$y_1/y = \frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{-y = y_1\sqrt{1-x^2}} \quad \text{—— ①}$$

Differentiate w.r.t 'x' we get,

$$\frac{1}{\cancel{2}\sqrt{1-x^2}}\, y_1 \cdot \cancel{2}x + y_2\sqrt{1-x^2} = -y_1$$

Multiplying both side by $\sqrt{1-x^2}$ we get,

$$-xy_1 + y_2(1-x^2) = -y_1\sqrt{1-x^2}$$

$$(1-x^2)y_2 - xy_1 + y_1\sqrt{1-x^2} = 0$$

$$\boxed{(1-x^2)y_2 - xy_1 - y = 0}$$

proved

7. $y = a\cos(\log x)$ P.T $x^2y_2 + xy_1 + y = 0$

Sol

Given that,

$$y = a\cos\log x$$

Differentiate y w.r.t 'x' we get,

$$y_1 = \frac{-a\sin\log x}{x}$$

$$xy_1 = -a\sin\log x \quad \text{—— (1)}$$

Again differentiate (1) w.r.t 'x' we get,

$$y_1 + xy_2 = \frac{-a\cos\log x}{x}$$

$$xy_1 + x^2y_2 = -a\cos\log x$$

$$\boxed{x^2y_2 + xy_1 + y = 0}$$ Proved

8 $x = \sin(\log y)$ P.T $(1-x^2)y_2 - xy_1 = y$

Sol Given that,

$\sin^{-1}x = \log y$.

Differentiate both side w.r.t 'x' we get

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}y_1$$

$$y = y_1\sqrt{1-x^2} \quad —① $$

Differentiate ① w.r.t 'x' we get,

$$\frac{1}{\cancel{2}\sqrt{1-x^2}}\,-\cancel{2}xy_1 + y_2\sqrt{1-x^2} = y$$

$$-xy_1 + y_2(1-x^2) - y = 0$$

$$(1-x^2)y_2 - xy_1 - y = 0$$

$$\boxed{(1-x^2)y_2 - xy_1 = y}$$ Proved

Date
Page

(9) $y = (\tan^{-1} x)^2$ P.T $y_2(1+x^2)^2 + 2xy_1(1+x^2) = 2$

Sol Given that,

$$y = (\tan^{-1} x)^2$$

Differentiate y w.r.t 'x' we get,

$$y_1 = 2\tan^{-1} x \frac{1}{1+x^2}$$

$$(1+x^2)y_1 = 2\tan^{-1} x \quad \text{——} \quad (1)$$

Differentiate (1) w.r.t 'x' we get,

$$2x \cdot y_1 + y_2(1+x^2) = 2\frac{1}{1+x^2}$$

$$2xy_1(1+x^2) + y_2(1+x^2)^2 = 2$$

$$\boxed{y_2(1+x^2)^2 + 2xy_1(1+x^2) = 2}$$

Proved

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LEBNITZ'S THEOREM

If u & v are the function of x possessing derivative of the n^{th} order.

$$(UV)_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \cdots + U V_n$$

Proof

When $n=1$

$$(UV)_1 = \frac{d(UV)}{d} = \frac{du}{dx} \cdot V + u \cdot \frac{dv}{dx} = u_1 v + u v_1$$

$$(UV)_2 = \frac{d}{dx}(u_1 V + U V_1) = \frac{d(U_1 V)}{dx} + \frac{d}{dx}(U V_1)$$

$$= u_2 V + u_1 V_1 + U_1 V_1 + U V_2$$

$$= U_2 V + 2 u_1 V_1 + U V_2$$

Hence the expression is satisfied when $n=1$ & $n=2$ let the expression is true for $n=m$. where m is positive integer.

$$(UV)_m = U_m V + {}^mC_1 U_{m-1} V_1 + {}^mC_2 U_{m-2} V_2 + \cdots U V_m$$

$$(UV_{m+1}) = U_{m+1} V + U_m V_1 + {}^mC_1 (U_m V_1 + U_{m-1} V_2)$$
$$+ {}^mC_2 (U_{m-1} V_2 + U_{m-2} V_3)$$
$$+ \cdots + U_1 V_m + U V_{m+1}$$

$$= U_{m+1} V + U_m V_1 (1 + {}^mC_1) + U_{m-1} V_2 ({}^mC_1 + {}^mC_2)$$
$$+ U_{m-2} V_3 ({}^mC_2 + {}^mC_3) + \cdots + U V_{m+1}$$

$= U_{m+1}V + {}^{m+1}c_1 \cdot U_m V_1 + {}^{m+1}c_2\, U_{m-1} V_2 + {}^{m+1}c_3$

$U_{m-2} V_3 + \cdots + U V_{m+1}$

Hence the given expression is true for $n=1$, $n=2$ & $n = m+1$

we can recursively proved that the expression true for any positive integer (n)

Hence prove the lebnitz theorem -

Questions V.V.I

① $y = \log x$ ③ $y = \frac{1}{a+bx}$

② $y = \sin x$ ④ $y = e^{ax}$

Question V.V.I

Find derivative of $n+1$

$y = e^{ax} \sin bx$

$y = e^{ax} \sin bx$

Diff y w.r.t 'x' we get,

$y_1 = e^{ax} \cos bx + a e^{ax} \sin bx$

$y_1 = e^{ax}(b\cos bx + a\sin bx)$

let $b = r\sin\theta$

$a = r\cos\theta$

$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\frac{b}{a}$

$y_1 = e^{ax}(r\sin\theta\cos bx + r\cos\theta\sin bx)$

$= re^{ax}(\sin bx + \theta)$

$y_2 = re^{ax}\sin(bx+\theta)$

$y_2 = r[e^{ax}\cos(bx+\theta) \cdot b + ae^{ax}\sin(bx+\theta)]$

$y_2 = re^{ax}[r\sin\theta\cos(bx+\theta) + r\cos\theta\sin(bx+\theta)]$

$y_2 = r^2e^{ax}\sin x(bx+2\theta)$

$$\boxed{y_n = \cancel{x}\, r^n e^{ax}\sin(bx+n\theta)}$$

Q1 Find the n^{th} differential coefficient of $\frac{1}{x+a}$

Sol let $y = \frac{1}{x+a} = (x+a)^{-1}$

$y_1 = (-1)(x+a)^{-2}$

$y_2 = (-1)\{(-2)(x+a)^{-3}\}$

$= (-1)(-2)(x+a)^{-3}$

$y_3 = (-1)(-2)\{(-3)(x+a)^{-4}\}$

$= (-1)(-2)(-3)(x+a)^{-4}$

$y_n = (-1)(-2)(-3) \ldots$ to n factors $\times (x+a)^{-(n+1)}$

$= \frac{(-1)^n \cdot 1.2.3 \ldots n}{(x+a)^{n+1}} = \frac{(-1)^n n!}{(x+a)^{n+1}}$

(2) Find the nth differential coefficient of $\log(1+x)$

Sol

let $y = \log(1+x)$

Then $y_1 = \frac{1}{1+x} = (1+x)^{-1}$

$y_2 = (-1)(1+x)^{-2}$

$y_3 = (-1)(-2)(1+x)^{-3}$

$\ldots\ldots\ldots\ldots$

Similarly $y_n = (-1)(-2)(-3) \ldots$ to $(n-1)$ factors $\times (1+x)^{-n}$

$= \frac{(-1)^{n-1} \cdot 1.2.3 \ldots (n-1)}{(1+x)^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Date
Page

(3) find the nth differential coefficient of $\sin 2x \cos 3x$.

Sol

let $y = \sin 2x \cos 3x$

which is $= \frac{1}{2} \cdot 2\sin 2x \cos 3x$

$$= \frac{1}{2}\{\sin(2x+3x) + \sin(2x-3x)\}$$

$$= \frac{1}{2}(\sin 5x - \sin x)$$

$y_n = \frac{1}{2}\{$nth d.c of $\sin 5x -$ nth d.c $\sin x\}$

$$= \frac{1}{2}\left\{5^n \sin\left(5x + \frac{n\pi}{2}\right) - \sin\left(x + \frac{n\pi}{2}\right)\right\}$$

Ans.

07/11/2017

STATE AND PROOF MACLAURIN'S THEROM

Under certain circumstances is the function $f(x)$ can be expand in power of x then.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \cdots \text{ to } \infty$$

proof. let us assume that the possibility of expanding $f(x)$ in a conservant series of positive integral power of f so that.

$$f(x) = A_0 + A_1 x + A_2\frac{x^2}{\lfloor 2} + A_3\frac{x^3}{\lfloor 3} + \cdots \text{ to } \infty \quad \text{—— (1)}$$

Where $A_0, A_1, A_2, A_3, \ldots\ldots$ are constant to be determined net containing x.

Differentiate (1) w.r.t 'x'

$$f'(x) = 0 + A_1 + A_2\frac{2x}{\lfloor 2} + A_3\frac{3x^2}{\lfloor 3} + \cdots$$

$$= 0 + A_1 + A_2 \cdot x + A_3\frac{x^2}{2} + \cdots \infty \quad \text{—— (2)}$$

Differentiate (2) w.r.t 'x'

$$f''(x) = A_2 + A_3\frac{2x}{2} + \cdots \text{ to } \infty$$

$$f''(x) = A_2 + A_3 x + \cdots \text{ to } \infty \quad \text{—— (3)}$$

put $x=0$ in equation ① ② ③ we get.

$f(0) = A_0, f'(0) = A_1, f''(0) = A_2$ - - - - -
- - - so own

Substituting the value of these constant in equation ① then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^2}{\lfloor 3} f'''(0) + \cdots \text{to } \infty$$

Hence prove maclurin's theorem.

② Under certain circumstances if a function $f(x+h)$ can be expanded in a series of power of h then $f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots$

Proof By assumption let $f(x+h)$ can be written as.

$$f(x+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \cdots \text{to } \infty \quad \text{— ①}$$

Differentiating successively expression

$$f'(x+h) = A_1 + 2A_2 h + 3A_3 h^2 + \cdots 4A_4 h^3 + \cdots \infty \quad \text{— ②}$$

$$f''(x+h) = 2A_2 + 3 \cdot 2 A_3 h + 4 \cdot 3 A_4 h^2 + \cdots \infty \quad \text{— ③}$$

$$f'''(x+h) = 3 \cdot 2 \cdot A_3 + 4 \cdot 3 \cdot 2 \cdot A_4 + \cdots \infty \quad \text{— ④}$$

putting $h = 0$ in the succesievly equation

$f(x) = A_0$

$f'(x) = A_1$

$f''(x) = 2A_2 \qquad A_2 = \frac{1}{\lfloor 2} f''(x)$

$f'''(x) = \lfloor 3 A_3$

$A_3 = \frac{1}{\lfloor 3} f'''(x)$

putting the value of A_0, A_1, A_2, A_3 etc in the equation — ① we get,

$f(x+h) = f(x) + f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots$ to ∞

Hence prove maclaurin's theorem.

Q1. Expand e^x using maclaurin's theorem ?

sol $f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots$ to ∞ —— ①

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

Now $x = 0$

$f(0) = 1$; $f'(0) = 1$ $f''(0) = 1$ $f'''(0) = 1$

Put these value in equation (1) we get,

$$\boxed{f(x) = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \text{ to } \infty}$$

Hence this is expansion of e^x using maclaurin's theorem.

Q2. Expand $\sin x$, $\cos x$, $\tan x$, $\tan^{-1}x$ by using maclaurin's theorem

Sol (1) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots$ to ∞

$f(x) = \sin x$.

$f^{I}(x) = \cos x$.

$f^{II}(x) = -\sin x$.

$f^{III}(x) = -\cos x$.

$f^{IV}(x) = \sin x$.

Now $x = 0$

$f(0) = 0$; $f'(0) = 1$; $f''(0) = 0$; $f'''(0) = -1$

$f^{IV}(0) = 0$

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By Maclaurin's theorem.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f^{iv}(0) + \cdots$$

$$f(x) = 0 + x + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \cdots \infty$$

$$f(x) = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots \text{ to } \infty$$

$$\boxed{\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots \text{ to } \infty}$$

proved

(2) $f(x) = \cos x$

Sol $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = +\cos x$

$f^{v}(x) = \sin x$

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Now $x = 0$

$f(0) = 1$; $f'(x) = 0$; $f''(x) = -1$; $f'''(x) = 0$

∞

$f^{IV}(0) = +1$ $f^{V}(0) = 0$

By Maclaurin's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f^{IV}(0) + \frac{x^5}{\lfloor 5}f^{V}(0) + \text{-----} \text{ to } \infty$$

$$f(x) = 1 + x(0) + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3}f(0) + \frac{x^4}{\lfloor 4}(+1) + \frac{x^5}{\lfloor 5}(0) + \text{--} \text{ to } \infty.$$

$$f(x) = 1 + 0 + \frac{x^2}{\lfloor 2}(-1) + 0 + \frac{x^4}{\lfloor 4}(+1) + 0 + \text{--} \text{ to } \infty$$

$$\boxed{\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \text{---} \text{ to } \infty}$$

proved

③ $f(x) = \tan x$.

Sol $f(x) = \tan x$

$$f'(x) = \frac{d}{dx}(\tan x) = \sec^2 x = \frac{1}{\cos^2 x}$$

$$f''(x) = \frac{d^2(\tan x)}{dx^2} = \frac{d}{dx}(\sec^2 x) = 2\tan x \sec^2 x$$

$$f'''(x) = \frac{d^3}{dx^3}\tan x = \frac{d^2}{dx}(\sec^2 x)$$

$$= \frac{d}{dx}(2\tan x \sec^2 x)$$

$$= -2(\cos(2x)-2)\sec^4 x$$

$$= -2(\cos(2x)-2)\frac{1}{\cos^4 x}$$

Now, $x = 0$

$$f(0) = \tan(0) = 0$$

$$f'(0) = \frac{1}{\cos^2(x)} = \frac{1}{\cos^2(0)} = \frac{1}{1} = 1$$

$$f''(0) = 2\tan(0)\sec^2(0) = 0$$

$$f'''(0) = -2(\cos(0)-2)\frac{1}{\cos^4(0)}$$

$$= -2(1-2) = 2$$

By Maclaurin's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f^{iv}(0) + \cdots$$

$$= 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(2) + \frac{x^4}{\lfloor 4} \cdots \text{to}$$

$$\boxed{\tan x = x + \frac{x^3}{3} + \cdots \text{ to } \infty}$$

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$f(x) = \tan^{-1}x.$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

* Partial Differentiation

It is a function $u = f(x,y)$ be a function of x & y then partial differention of u is defined as $\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$

If x is assumed as variable y will be constant similarly partial differention of u with respect y is defined as

$$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Ex:- $u = x^3 + 2x^2y + 3xy^2 - y^3$

find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = 3x^2 + 4xy + 3y^2$$

$$\frac{\partial u}{\partial y} = 2x^2 + 6xy - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 4y \quad \text{——— (1)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4x + 6y \quad \text{——— (2)}$$

$$\frac{\partial^2 u}{\partial y^2} = 6x - 6y \quad \text{——— (3)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 4x + 6y \quad \text{——— (3)}$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

Important Questions

* Euler's Theorem.

$f(x,y) = A_0 x^3 + A_1 x^2 y + A_2 x y^2 + A_3 y^3 \rightarrow 3^{rd}$ degree

$f(x,y) = A_0 x^4 + A_1 x^3 y + A_2 x y^3 + A_3 y^4 \rightarrow 4^{th}$ degree homogeneous function.

n^{th} Degree Homogeneous function.

$f(x,y)_n = A_0 x^n + A_1 x^{n-1} y + A_2 x y^{(n-2)} + \ldots \rightarrow$

$\ldots \quad A_n y^n$

$= x^n \left(A_0 + A_1 \frac{y}{x} + A_2 \left(\frac{y}{x}\right) + \ldots + A_n \left(\frac{y}{x}\right)^n\right.$

$f(x,y) = x^n f\left(\frac{y}{x}\right)$

For homogenous function $f(x,y)$ Euler's theorem said that;

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

n is the degree of the function.

Prove $\quad f(x,y) = x^n f\left(\frac{y}{x}\right)$ —— 1

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof

$$\frac{\partial f}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

let $\frac{y}{x} = z$

$$\frac{\partial f(z)}{\partial z} \times \frac{\partial z}{\partial x} = \frac{\partial f(z)}{\partial x}$$

$$\boxed{x\frac{\partial f}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right)} \quad \text{—— } 2$$

$$\frac{\partial f}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\boxed{y\frac{\partial f}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right)} \quad \text{—— } 3$$

Adding 2 + 3

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

from 1 $\quad \boxed{f = x^n f\left(\frac{y}{x}\right)}$

proved

Questions

Q1. $u = \frac{xy}{x+y}$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Sol degree of $\frac{xy}{x+y} = \frac{xy}{x(1+\frac{y}{x})} = 1$

According to Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nf(x)$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \cdot u = u}$$

L.H.S = R.H.S proved

Q2. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Sol.

$$\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}} = u$$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

~~[illegible]~~ degree of the expression

$$\frac{x+y}{\sqrt{x}+\sqrt{y}} - \frac{1}{2}$$

let $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = v$

$$v = \cos u$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y}$$

$$= -\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

Acc to Euler's theorem,

$$-\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}v$$

$$-\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0}$$ proved.

Q3. If $V = \tan^{-1}\frac{x^3+y^3}{x-y}$ P.T $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \sin 2V$

Sol Degree of the expression $\frac{x^3+y^3}{x-y} = \frac{x^{\cancel{3}\,2}\left(1+\frac{y^3}{x^3}\right)}{\cancel{x}\left(1-\frac{y}{x}\right)}$

Degree $(n) = 2$

let $\tan v = \frac{x^3+y^3}{x-y} = u$ $\qquad \boxed{u = \tan V}$

$$x\frac{\partial u}{\partial x} = x\sec^2 v\frac{\partial u}{\partial x}$$

$$y\frac{\partial u}{\partial y} = y\sec^2 v\frac{\partial u}{\partial y}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\sec^2 v\frac{\partial u}{\partial x} + y\sec^2 v\frac{\partial u}{\partial y}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sec^2 v\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

By Euler's theorem,

$$x\sec^2 v\frac{\partial v}{\partial x}+y\sec^2 v\frac{\partial v}{\partial y} = 2u = 2\tan v$$

$$\sec^2 v\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right) = 2\tan v$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = \frac{2\tan v}{\sec^2 v} = \frac{2\sin v\cos^{\cancel{2}} v}{\cancel{\cos v}}$$

$$= 2\sin v.\cos v$$

$$= \sin 2v$$

$$\boxed{x\frac{dv}{\partial x}+y\frac{\partial v}{\partial y} = \sin 2v}$$

Proved