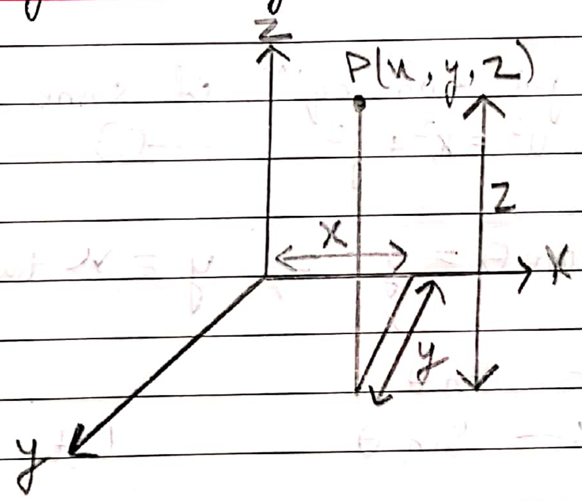
Date 27/02/2019

CO-ORDINATE GEOMETRY – 3D

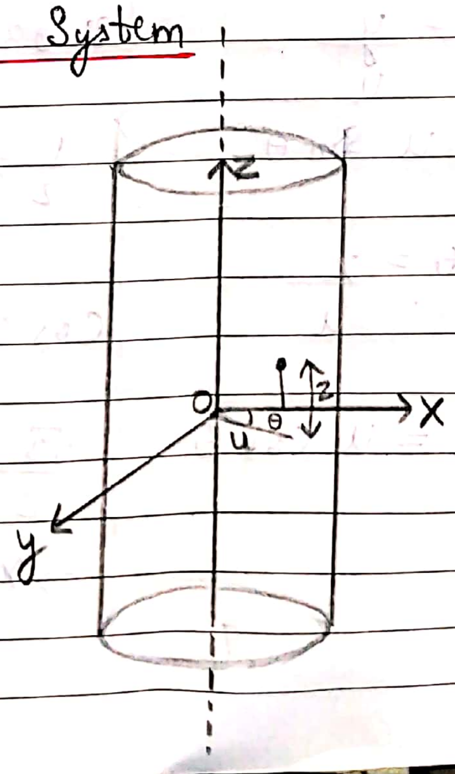
⓪ Transformation of co-ordinate system.

(i) Cartesian System (x, y, z)
(ii) Cylindrical System
(iii) Spherical or plane System.

(i) Cartesian System (x, y, z)



(ii) Cylindrical System



(u, θ, z)

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Transformation of Cartesian System to Cylindrical System :—

$NR = y$

$OR = x$

$u = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \tan^{-1}\frac{y}{x}$

Here, we get two eqⁿ of same variable.

$u^2 = x^2+y^2$ —— (I)

$\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$ —— (II)

~~To find :— Sin θ~~

To find :— $\sin\theta$

$\sin\theta = \frac{y}{u}$

$\Rightarrow y = u\sin\theta$

$\Rightarrow \cos\theta = \frac{x}{u}$

$\Rightarrow x = u\cos\theta$

Let $u = 3,\ \theta = 30°,\ z = 2$

$\sin 30° = \frac{y}{u}$

$\frac{1}{2} = \frac{y}{3} \Rightarrow y = \frac{3}{2}$

$\cos 30° = \frac{x}{u}$

$\frac{\sqrt{3}}{2} = \frac{x}{3}$

$\Rightarrow x = \frac{3\sqrt{3}}{2}$

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Q1. Transform the following co-ordinate into cylindrical (3, 4, 5).

⟶ Given that,

$x = 3,\ y = 4,\ z = 5$

So, $u = \sqrt{x^2+y^2} = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Now, $\theta = \tan^{-1}\frac{y}{x}$

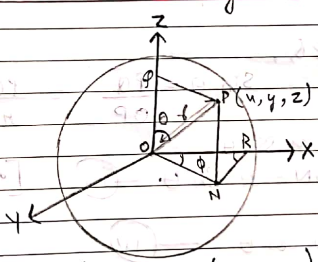
$= \tan^{-1}\frac{4}{3}$

So, the cylindrical system co-ordinate

$= (u, \theta, z)$

$= \left(5, \tan^{-1}\frac{4}{3}, 5\right)$

(iii) Spherical or Polar System :—



r = Distance of point $P(x, y, z)$ from O

θ = Angle made by OP with Z axis

ϕ = Angle made by the plane PON with x-axis

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Transformation of Spherical or Polar system to cartesian system.

Draw $NR \perp OX$ &

$PQ \perp OZ$

we know,

$OR = x$

$RN = y$

$OQ = z$

Now, spherical or polar system to cartesian system conversion.

From, ΔORN

$$\cos\phi = \frac{OR}{ON} = \frac{x}{ON}$$

$$x = ON \cos\phi \quad \text{—— (I)}$$

$$\sin\phi = \frac{NR}{ON}$$

$$\Rightarrow y = ON \sin\phi \quad \text{—— (II)}$$

From ΔOPQ,

$$\sin\theta = \frac{PQ}{OP} = \frac{PQ}{r}$$

$$PQ = r\sin\theta$$

$$\Rightarrow ON = r\sin\theta \quad \text{—— (III)} \quad [\because PQ = ON]$$

$$OQ = r\cos\theta \quad \text{—— (IV)}$$

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from eq (I) & (III)

$$x = r\sin\theta \cdot \cos\phi = ON\cos\phi$$

from eq (II) & (III) $\quad (\because \sin\phi = \frac{NR}{ON})$

$$y = r\sin\theta \sin\phi = ON\sin\phi \quad \text{—— (II)}$$

from eq (IV)

$$z = r\cos\theta$$

Transformation from cartesian system to polar system :

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos\theta = \frac{z}{r} \Rightarrow \theta = \cos^{-1}\frac{z}{y}$$

$$\tan\phi = \frac{y}{x} \Rightarrow \phi = \tan^{-1}\frac{y}{x}$$

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Ques.1. Find the distance between the pointer $\left(2\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{6}\right)$, $\left(2\sqrt{3}, -1, -4\right)$ which
polar
Cartesian

$\longrightarrow$ $x = ON \cos\phi$

$y = ON \sin\phi$

$ON^2 = x^2 + y^2$

$r^2 = x^2 + y^2 + z^2$

$\phi = \frac{\pi}{6}$

$\tan\phi = \frac{y}{x}$

$\cos\theta = \frac{z}{r}$

$\frac{1}{\sqrt{2}} = \frac{z}{r}$

$z = \frac{1}{\sqrt{2}} \times 2\sqrt{3}$

$z = \sqrt{6}$

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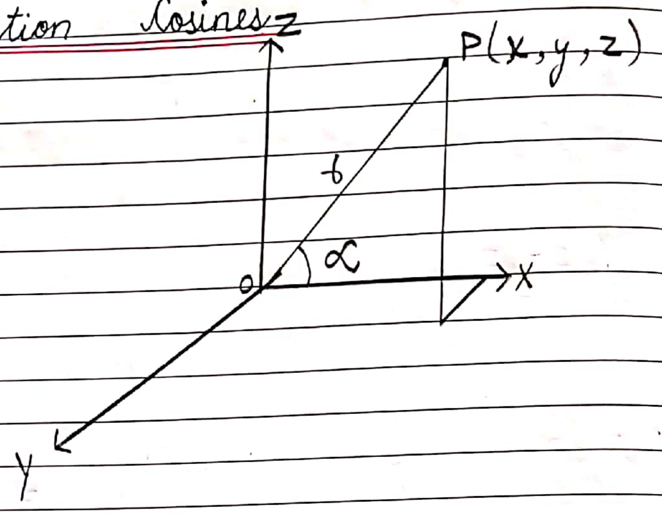
Direction Cosines



α, β, γ is the angle made by line OP with OX, OY, OZ respectively.

Let $OP = r$

$$\cos\alpha = \frac{x}{OP} = \frac{x}{r}$$

$$x = r\cos\alpha$$

Similarly, $y = r\cos\beta$

$z = r\cos\gamma$

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ is called the direction cosine of the line of and it is represented as l, m, n.

Hence,

$$x = lr, \quad y = mr, \quad z = nr$$

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Ques Prove that, $l^2 + m^2 + n^2 = 1$

Sol $OP^2 = (x-0)^2 + (y-0)^2 + (z-0)^2$

$r^2 = x^2 + y^2 + z^2$

$r^2 = (lr)^2 + (mr)^2 + (nr)^2$

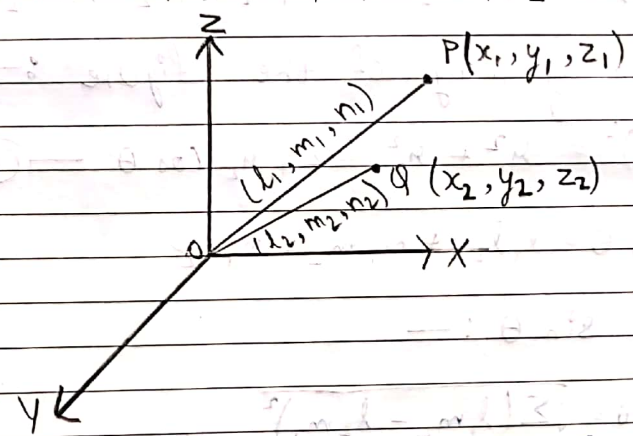
$r^2 = r^2(l^2 + m^2 + n^2)$

$l^2 + m^2 + n^2 = 1$

Ques Angle between two lines. If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosine of two lines, then find the angle between the two straight line or prove that :—

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

Sol



Let the two points are P, Q

Let co-ordinates of point $P = (x_1, y_1, z_1)$

$Q = (x_2, y_2 + z_2)$

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So, According to question

$x_1 = l_1 r_1$, $y_1 = m_1 r_1$, $z_1 = n_1 r_1$

$x_2 = l_2 r_2$, $y_2 = m_2 r_2$, $z_2 = n_2 r_2$

$$(PQ)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$
$$= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$$

$$= r_1^2 (l_1^2 + m_1^2 + n_1^2) + r_2^2 (l_2^2 + m_2^2 + n_2^2)$$
$$- 2 r_1 r_2 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$(PQ)^2 = r_1^2 + r_2^2 - 2 r_1 r_2 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

from geometry of the figure :—

$$(PQ)^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta \quad \text{—— (ii)}$$

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

for $\sin\theta$:—

$$\sin\theta = \sqrt{\Sigma (l_1 m_2 - l_2 m_1)^2}$$
$$= \sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$$

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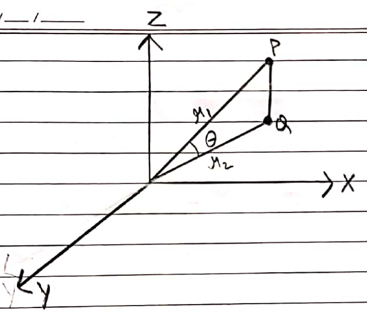

Que. If $l_1 m_1 n_1$ & $l_2 m_2 n_2$ are the direction cosine of two straight line. prove that :—

$$\sin\theta = \sqrt{\Sigma (l_1 m_2 - l_2 m_1)^2}$$

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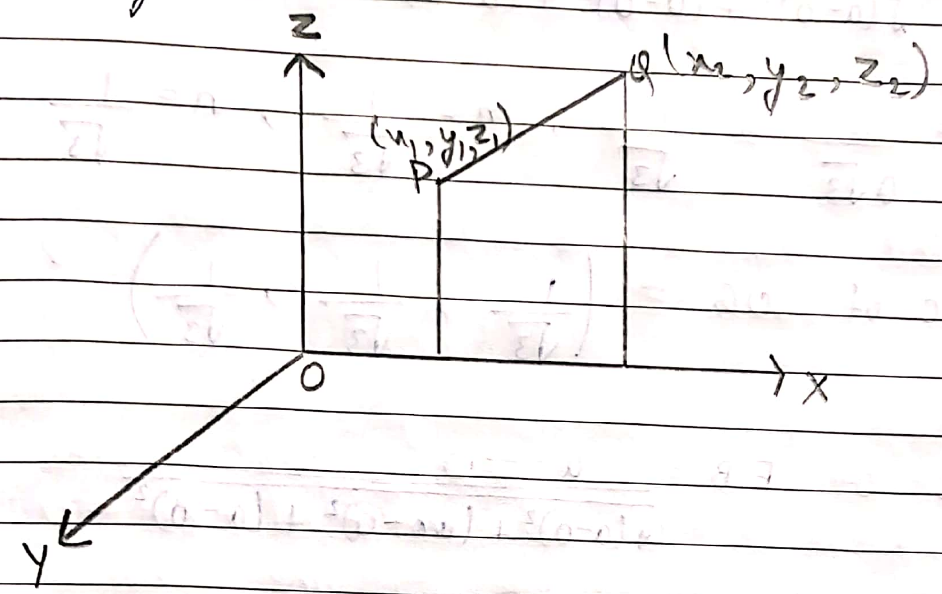
- Direction Cosine of a line when two points are given.



Direction cosines of PQ are

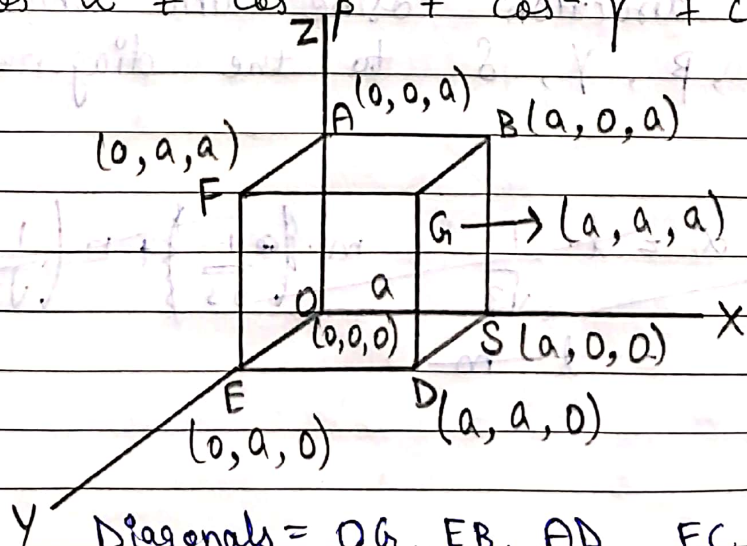
$$\frac{x_1 - x_2}{r}, \quad \frac{y_1 - y_2}{r}, \quad \frac{z_1 - z_2}{r}$$

where,

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Ques. A line makes an angle $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

Ans



The length of ages are = a

Diagonals = OG, EB, AD, FC

Direction cosines of OG

$$\frac{a-0}{\sqrt{(a-0)^2+(a-0)^2+(a-0)^2}}$$

$$l = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}, \quad m = \frac{1}{\sqrt{3}}, \quad n = \frac{1}{\sqrt{3}}$$

$$\text{D.C of } OG = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\text{D.C of } EB = \frac{a-0}{\sqrt{(a-0)^2+(a-0)^2+(a-0)^2}}$$

$$l = \frac{1}{\sqrt{3}}, \quad m = -\frac{1}{\sqrt{3}}, \quad n = \frac{1}{\sqrt{3}}$$

$$\text{D.C of } AD = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$\text{D.C of } FC = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Let the direction cosine line which makes angle $\alpha, \beta, \gamma, \delta$ to the diagonal is (l, m, n)

$$\cancel{OG = \cos\alpha = l.\frac{1}{\sqrt{3}} + m.\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right)}$$

$$\cancel{= l - m}$$

$$O.G = \cos\alpha = l.\frac{1}{\sqrt{3}} + m.\frac{1}{\sqrt{3}} + n.\frac{1}{\sqrt{3}}$$

$$= \frac{(l+m+n)}{\sqrt{3}}$$

$$E.B = \cos\beta = l.\frac{1}{\sqrt{3}} + m.\left(\frac{-1}{\sqrt{3}}\right) + n.\frac{1}{\sqrt{3}}$$

$$= \frac{(l-m+n)}{\sqrt{3}}$$

$$A.D = \cos\gamma = l.\frac{1}{\sqrt{3}} + m.\frac{1}{\sqrt{3}} + n.\left(\frac{-1}{\sqrt{3}}\right)$$

$$= \frac{(l+m-n)}{\sqrt{3}}$$

$$F.C = \cos\delta = l.\left(\frac{-1}{\sqrt{3}}\right) + m.\frac{1}{\sqrt{3}} + n.\frac{1}{\sqrt{3}}$$

$$= \frac{(-l+m+n)}{\sqrt{3}}$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$\Rightarrow \left(\frac{l+m+n}{\sqrt{3}}\right)^2 + \left(\frac{l-m+n}{\sqrt{3}}\right)^2 + \left(\frac{l+m-n}{\sqrt{3}}\right)^2 + \left(\frac{-l+m+n}{\sqrt{3}}\right)^2$$

$$\Rightarrow \frac{l^2+m^2+n^2}{3} + \frac{l^2-m^2+n^2}{3} + \frac{l^2+m^2-n^2}{3} + \frac{-l^2+m^2+n^2}{3}$$

$$\Rightarrow \frac{\cancel{l^2}+\cancel{m^2}+\cancel{n^2}+l^2-\cancel{m^2}+n^2+l^2+m^2-\cancel{n^2}-\cancel{l^2}+m^2+n^2}{3}$$

$$\Rightarrow \frac{l^2+n^2+l^2+m^2+m^2+n^2}{3}$$

Ques1. Find the angles between two diagonal of a cube

Ques2. Show that the straight lines whose direction cosines are given by the equation $2l+2m-n=0$ & $mn+nl+lm=0$ are perpendicular to one another.

Ans2.

$2l+2m-n=0$ —— (I)

$mn+nl+lm=0$ —— (II)

Eliminating n for equation (II)

$n = 2(l+m)$

$m\,2(l+m) + 2(l+m)l + lm = 0$

$2m^2 + 2l^2 + 5l = 0$

Dividing equation with ml

$$\frac{2m}{l} + 2\frac{l}{m} + 5 = 0$$

let $\frac{m}{l} = t$

$$2t + \frac{2}{t} + 5 = 0$$

$2t^2 + 2 + 5t = 0$

Product of root

let $t_1, t_2 = \frac{2}{2} = 1$ [where t_1 & t_2 are roots]

$$\Rightarrow \frac{m_1}{l_1}\cdot\frac{m_2}{l_2} = 1$$

$\Rightarrow m_1 m_2 = l_1 \cdot l_2$ —— (III)

Similarly eliminating m from equation (II)

$2m = n-2l$

$\Rightarrow m \cdot \frac{1}{2}(n-2l)$

$\Rightarrow \frac{1}{2}(n-2l)n + nl + l\cdot\frac{1}{2}(n-2l) = 0$

$\Rightarrow n(n-2l) + 2nl + l(n-2l) = 0$

$\Rightarrow n^2 - \cancel{2ln} + \cancel{2nl} + nl - 2l^2 = 0$

$\Rightarrow n^2 + nl - 2l^2 = 0$

Dividing with nl

$$\frac{n}{l} + 1 - \frac{2l}{n} = 0$$

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let $\frac{n}{l} = 2$

Product

$$\frac{n_1}{l_1} \cdot \frac{n_2}{l_2} = \frac{-2}{1} = -2$$

$$n_1 n_2 = -2 l_1 l_2 \quad \text{—— (IV)}$$

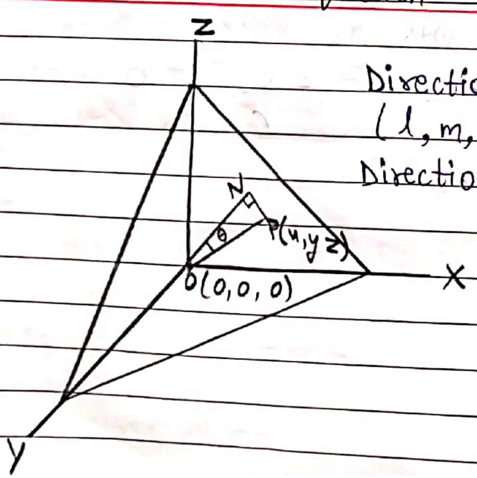
$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= l_1 l_2 + l_1 l_2 - 2 l_1 l_2 = 0$$

$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}$$

◉ Equation of length :—

PLANE

(i) Normal form of the equation of a plane



Direction cosine of ON (l, m, n) ON = OP

Direction cosine of OP

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$$\Rightarrow \frac{x_1 - 0}{OP}, \frac{y_1 - 0}{OP}, \frac{z_1 - 0}{OP}$$

$$\cos\theta = l \cdot \frac{x_1}{OP} + m \cdot \frac{y_1}{OP} + n \cdot \frac{z_1}{OP}$$

$$OP \cos\theta = l x_1 + m y_1 + n z_1$$
$$P = l x_1 + m y_1 + n z_1$$

$$\Rightarrow l x_1 + m y_1 + n z_1 = P$$

Hence the locus of the point is

$$\Rightarrow lx + my + nz = P$$

Ques 1. Find the equation of the plane $P(2, 4, 2)$ when $\alpha = 60°$, $\beta = 45°$ & $\gamma = 60°$ & $P = 3$

Ques 2. Find the equation of the plane passes through the point and perpendicular to OP.

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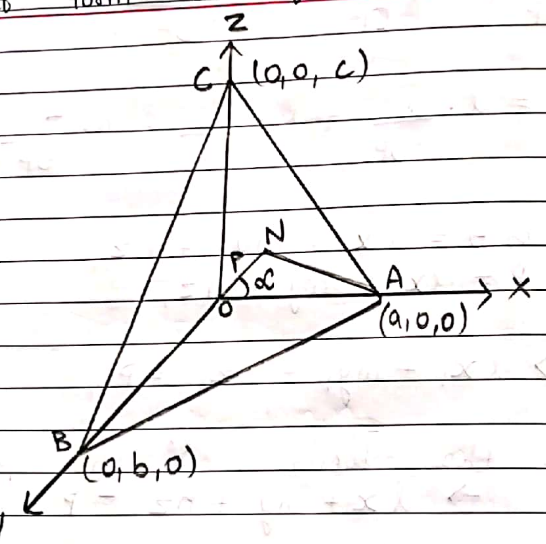
(iii) Intercept form of equation of plane



The plane ABC cut intercept a, b, c from x, y, z axis

Let ON be perpendicular to the plane.
The direction cosine of ON(P) is l, m, n.

In Δ ONA,

$$\cos\alpha = \frac{ON}{OA} = \frac{P}{a}$$

$$\Rightarrow l = \frac{P}{a}$$

Similarly,

$$m = \frac{P}{b} \quad \& \quad n = \frac{P}{c}$$

From the eq^n of normal form of plane

$$lx + my + nz = P$$

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$$\Rightarrow \frac{P}{a}x + \frac{P}{b}y + \frac{P}{c}z = P$$

$$\Rightarrow P\left[\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right] = P$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{\cancel{P}}{\cancel{P}} \; 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Ist degree equation ~~repen~~ representing a plane is (general eq^n)

$$\boxed{ax + by + cz + d = 0}$$

$$lx + my + nz = P \quad \text{— (I)}$$
$$ax + by + cz = -d \quad \text{— (II)}$$

From (I) & (II)

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{P}{-d}$$

$$\Rightarrow \sqrt{\frac{l^2 + m^2 + n^2}{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

l, m, n is the d.c of the perpendicular to the plane as well as plane itself.
(a, b, c are called direction cosines)

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Ques 1. Find intercept made by co-ordinate axis by the plane $x + 2y - 2z = 9$. Find the length of the normal form the original to the plane and also the direction cosine of the normal.

ANGLE BETWEEN TWO PLANE

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

where (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosine of the normal of the two planes.

~~When the direction ratio given~~

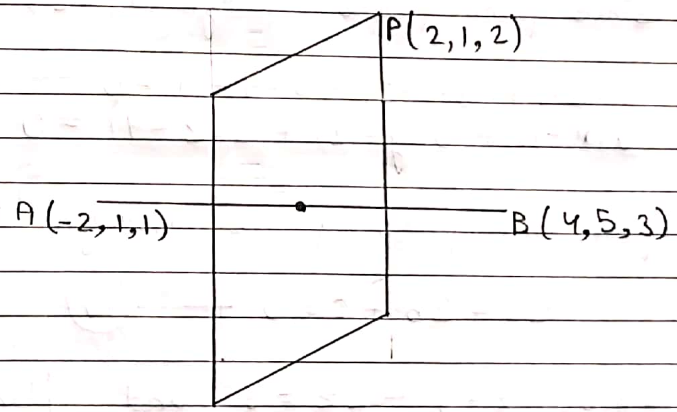
* When the direction ratio is given, then

$$\boxed{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left(\sqrt{a_1^2 + b_1^2 + c_1^2}\right)\left(\sqrt{a_2^2 + b_2^2 + c_2^2}\right)}}$$

* When both the planes are perpendicular to each other, then

$$\boxed{\cos\theta = a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$$

Ques 1. In the following plane. Find the equation of the plane that passes through the point $P(2,1,2)$ and perpendicular to the line AB. Such that : $A(-2,1,1)$ & $B(4,5,3)$



$ax + by + cz + d = 0$

$2a + b + 2c + d = 0$

$a(x-2) + b(y-1) + c(z-2) = 0$

$AB = (6, 4, 2)$

$\Rightarrow 6(x-2) + 4(y-1) + 2(z-2) = 0$

$\Rightarrow 6x + 4y + 2z = 0$

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Ques. 1. $(1,-2,4)$, $(3,-4,5)$ are per perpendicular to $x+y-2z=6$

Solution $\therefore$ $ax+by+cz+d=0$ —— (I)

Since plane passes through $(1,-2,4)$

$a\cdot 1+b\cdot(-2)+c\cdot 4+d=0$ —— (II)

$a(x-1)+b(y+2)+c(z-4)=0$ —— (III)

Again the plane passes through $(3,-4,5)$

$2a-2b+c=0$ —— (IV)

Since, $x+y-2z=6$ and plane 1

$a+b-2c=0$ —— (V)

Solving eq^n (IV) & (V)

$2a-2b+c=0$
$a+b-2c=0$

$$\frac{a}{4-1}=\frac{b}{1+4}=\frac{c}{2+2}=k$$

$a=3k$, $b=5k$, $c=4k$

$\Rightarrow 3(x-1)+5(y+2)+4(z-4)=0$ Ans

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Q 2. Find the equation of the plane through $(1,4,3)$ and perpendicular to the line of intersection of the planes $3x+4y+7z+4=0$ and $x-y+2z+3=0$

Solution: Let the general equation of the plane.

$ax+by+cz+d=0$ —— (1)

The plane passes through $(1,4,3)$

$a\cdot 1+b\cdot 4+c\cdot 3+d=0$ —— (2)

$a(x-1)+b(y-4)+c(z-3)=0$ —— (3)

a,b,c be the direction ratio of line $3x+4y+7z+4=0$ and $x-y+2z+3=0$

Then, $3a+4b+7c=0$, $a-b+2c=0$

$$\frac{a}{15}=\frac{b}{1}=\frac{c}{-7}$$

So, direction ratio of the line of intersection of planes are $15, 1, -7$

Equation of plane is

$15(x-1)+1(y-4)-7(z-3)=0$

$15x-15+y-4-7z+21=0$

$15x+y-7z+2=0$ Ans

~~Q Find the equation~~

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Ques 3. Find the equation of the plane passing through the intersection of planes $2x + 5y - 5z = 6$ and $2x + 7y - 8z = 7$ and the points $(-1, 4, 3)$

Solution

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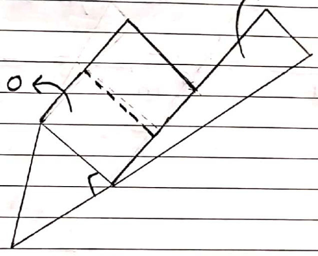
Ques 4. The plane $x - 2y + 3z - 4 = 0$ is rotated to ~~[illegible]~~ right angle above the line of the intersection with the plane $2x + 3y - 4z - 5 = 0$. Find the equation of the plane in its new position.

Solution



The equation shows the new position of plane

$\Rightarrow x - 2y + 3z - 4 + \lambda(2x + 3y - 4z - 5) = 0$ ——— (I)

$\Rightarrow (1 + 2\lambda)x + (3\lambda + 2)y + (3 - 4\lambda)z - 4 - 5\lambda = 0$ ——— (II)

$\Rightarrow 1(1 + 2\lambda) + (-2)(3\lambda - 2) + 3(3 - 4\lambda) = 0$

$\Rightarrow 1 + 2\lambda - 2(3\lambda - 2) + 3(3 - 4\lambda) = 0$

$\Rightarrow 1 + 2\lambda - 6\lambda + 4 + 9 - 12\lambda = 0$

$\Rightarrow -16\lambda + 14 = 0$

$\Rightarrow 16\lambda = 14$

$\Rightarrow \lambda = \frac{14}{16}$

$\Rightarrow \lambda = \frac{7}{8}$

Putting in eq^n (I)

$22x + 5y - 4z = 67$

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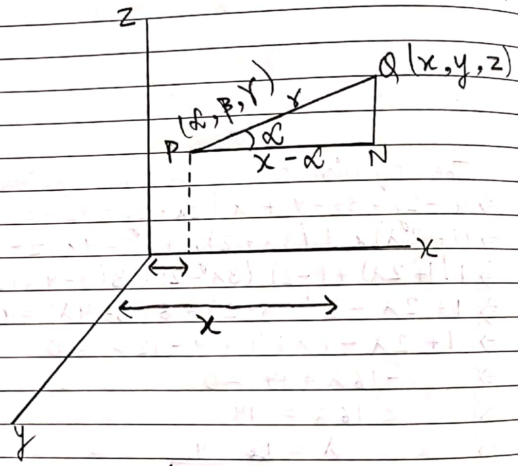
Equation of Straight line

- Symmetric form

→ If a straight line passes through the point α, β, γ with direction cosine l, m, n then the equation is,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$



$$\cos\alpha = \frac{x-\alpha}{r} = l \qquad \frac{x-\alpha}{l} = r$$

$$\frac{y-\beta}{r} = m \qquad \frac{y-\beta}{m} = r$$

$$\frac{z-\gamma}{r} = n \qquad \frac{z-\gamma}{n} = r$$

If x_1, y_1, z_1 and x_2, y_2, z_2 are two given points then the eq^n of the straight line will be,

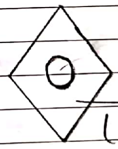
$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2} = r$$

Ques Find the co-ordinates of the point of the intersection of line,

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4} = r \quad \text{with plane}$$

$$2x - 2y - z = 7$$

Solution



$\rightarrow 2x - 2y - z = 7$ (Equation of the plane)

$x = 2r+1$, $y = -3r+2$, $z = 4r-3$ — ①

Putting in equation of the plane

$$2(2r+1) - 2(-3r+2) - (4r-3) = 7$$
$$4r + 2 + 6r + 4 - 4r + 3 - 7 = 0$$
$$+6r = 6$$
$$r = 1$$

Putting in eq^n ①

$x = 3$, $y = -1$, $z = 1$

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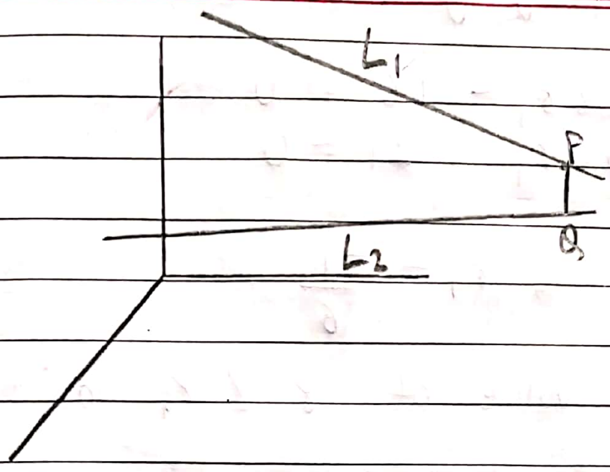
~~Shortest~~ Shortest Distance Between Two Straight Line



$PQ \perp L_1$

$PQ \perp L_2$

Then PQ is shortest distance.

Q.1. Find shortest distance between

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \&$$

$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$ and obtain its eq^n.

$\rightarrow$ $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$$

The co-ordinate of P is given by,

$x_1 = 2r+1 \quad , \quad y_1 = 3r+2 \quad , \quad z_1 = 4r+3$

$x_2 = 3r_1+2 \quad , \quad y_2 = 4r_1+4 \quad , \quad z_2 = 5r_1+5$

Direction ratio of PQ

$(3r_1 - 2r + 1 \quad , \quad 4r_1 - 3r + 2 \quad , \quad 5r_1 - 4r + 2)$

The straight line $PQ \perp L_1$

$2(3r_1 - 2r + 1) + 3(4r_1 - 3r + 2) + 4(5r_1 - 4r + 2) = 0$ (I)

Again $PQ \perp L_2$

$3(3r_1 - 2r + 1) + 4(4r_1 - 3r + 2) + 5(5r_1 - 4r + 2) = 0$ (II)

After simplifying the eq^n 1 & 11

$29r - 38r_1 - 16 = 0$
$38r - 50r_1 - 21 = 0$

$r = \frac{1}{3}, \quad r_1 = \frac{-1}{6}$

Putting the value of r & r_1 in the given eq^n

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \frac{1}{3}$

$\frac{x-1}{2} = \frac{1}{3}$

$3(x-1) = 2$
$3x - 3 = 2$
$3x = 2+3$
$3x = 5$
$x = \frac{5}{3}$

$\frac{y-2}{3} = \frac{1}{3}$

$3(y-2) = 3$
$3y - 6 = 3$
$3y = 3+6$
$3y = 9$
$y = \frac{\cancel{9}3}{\cancel{3}} = 3$

$\frac{z-3}{4} = \frac{1}{3}$

$3(z-3) = 4$
$3z - 9 = 4$
$3z = 4+9$
$3z = 13$
$z = \frac{13}{3}$

$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} = \frac{-1}{6}$

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$\frac{x-2}{3} = \frac{-1}{6}$

$6(x-2) = -3$
$6x - 12 = -3$
$6x = -3+12$
$6x = 9$
$x = \frac{\cancel{9}^3}{\cancel{6}_2} = \frac{3}{2}$

$\frac{y-4}{4} = \frac{-1}{6}$

$6(y-4) = -4$
$6y - 24 = -4$
$6y = -4+24$
$6y = 20$
$y = \frac{\cancel{20}^{10}}{\cancel{6}_3} = \frac{10}{3}$

$\frac{z-5}{5} = \frac{-1}{6}$

$6(z-5) = -5$
$6z - 30 = -5$
$6z = -5+30$
$6z = 25$
$z = \frac{25}{6}$

$P\left(\overset{x_1}{\frac{5}{3}}, \overset{y_1}{3}, \overset{z_1}{\frac{13}{3}}\right), \quad Q\left(\overset{x_2}{\frac{3}{2}}, \overset{y_2}{\frac{10}{3}}, \overset{z_2}{\frac{25}{6}}\right)$

$PQ = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

$=$

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Q. Shortest Distance given by

$$\delta = \begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sin\theta$$

$$\rightarrow \begin{vmatrix} \alpha & \beta & \gamma \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{vmatrix} \quad \begin{vmatrix} l & m & n \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$\sin\theta = \sqrt{1+4+1} = \sqrt{6}$$

$$\delta = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \div \sqrt{6}$$

$$\delta = [1(15-16) - 2(10-12) + 2(8-9)] \div \sqrt{6}$$

$$\delta = \boxed{-1+4-2} \div \sqrt{6}$$

$$= [-3+4] \div \sqrt{6}$$

$$= \frac{1}{\sqrt{6}}$$ Ans

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VECTOR ANALYSIS

$\vec{a} \cdot \vec{b} = ab\cos\theta$

$\vec{a} \times \vec{b} = ab\sin\theta\ \hat{n}$

Product of three vectors

(i) scalar product

(ii) vector product

* **Scalar product of three vectors**

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

Q. Prove that scalar product of three vectors represent the volume of a parallelopiped.

→ Area of OACD $= ab\sin\theta$

Height of parallelopiped $= OM = c\cos\alpha$

Volume $= ab\sin\theta\ c\cos\alpha$

$= abc\sin\theta\cos\alpha$

Now,

$\vec{c} \cdot (\vec{a} \times \vec{b})$

$= \vec{c} \cdot ab\sin\theta\,\hat{n}$ ($\hat{n}$ is unit vector ⊥ to OACB)

$= \vec{c} \cdot \hat{n}\ ab\sin\theta$

$= |\vec{c}| \cdot |\hat{n}| \cdot \cos\alpha \cdot ab\sin\theta$

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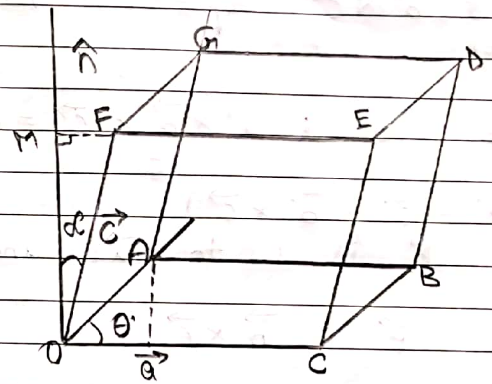
$= c\,ab\sin\theta\cos\alpha$

$= abc\sin\theta\cos\alpha$



Q1. $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

Q2. $\vec{a} = 4\hat{i} + 5\vec{j} + \vec{k}$

$\vec{b} = -(\vec{j} + \vec{k})$

$\vec{c} = 3\vec{i} + 9\vec{j} + 4\vec{k}$

$\vec{d} = 4(-\vec{i} + \vec{j} + \vec{k})$

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* Vector product of three vectors

$\vec{a} \times (\vec{b} \times \vec{c})$ represent vector product of three vectors.

Q Prove that : $\vec{a} \times \vec{b} \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

→ $\vec{P} = \vec{a} \times (\vec{b} \times \vec{c})$

Then $\vec{P}$ is $\perp$ to $\vec{a}$ and $\vec{b} \times \vec{c}$

$$\vec{P} \perp \vec{b} \times \vec{c}$$

It implies $\vec{P}$ lies in the plane of $\vec{b}$ & $\vec{c}$.

$\vec{P} = x\vec{b} + y\vec{c}$ —— (1)

$\vec{P}\cdot\vec{a} = (x\vec{b} + y\vec{c})\cdot\vec{a}$

$\Rightarrow (x\vec{b} + y\vec{c})\cdot\vec{a} = 0$

$\Rightarrow x\vec{b}\cdot\vec{a} + y\vec{c}\cdot\vec{a} = 0$

$\Rightarrow x(\vec{b}\cdot\vec{a}) = -y(\vec{c}\cdot\vec{a})$

$\Rightarrow \dfrac{x}{\vec{c}\cdot\vec{a}} = \dfrac{-y}{\vec{b}\cdot\vec{a}} = \lambda$

$\Rightarrow \dfrac{x}{\vec{a}\cdot\vec{c}} = \dfrac{-y}{\vec{a}\cdot\vec{b}} = \lambda$

$\Rightarrow x = \lambda(\vec{a}\cdot\vec{c})$

$\Rightarrow y = -\lambda(\vec{a}\cdot\vec{b})$

Putting the value of x & y in eq (1)

$\vec{P} = \lambda(\vec{a}\cdot\vec{c})\vec{b} - \lambda(\vec{a}\cdot\vec{b})\cdot\vec{c}$

Putting $\lambda = 1$

$\vec{P} = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

$= \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

* Product of four vectors

• Scalar product of four vectors

→ It is represented as $(\vec{a} \times \vec{b})\cdot(\vec{c} \times \vec{d})$

$$(\vec{a} \times \vec{b})\cdot(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$(\vec{a} \times \vec{b})\cdot(\vec{c} \times \vec{d}) = \vec{a} \times \vec{b}\cdot\vec{m}$ when $\vec{m} = \vec{c} \times \vec{d}$

$\Rightarrow \vec{a} \times \vec{b}\cdot\vec{m} = \vec{a}\cdot(\vec{b} \times \vec{m})$ [dot and cross are interchanged]

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$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{m}) = \vec{a}(\vec{b} \times (\vec{c} \times \vec{d}))$

$\Rightarrow \vec{a}[(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}]$

$\Rightarrow (\vec{b} \cdot \vec{d}) \cdot (\vec{a} \cdot \vec{c}) - (\vec{b} \cdot \vec{c}) \cdot (\vec{a} \cdot \vec{d})$

$\Rightarrow \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}$

Q Prove that : $(\vec{a} \times \vec{b})^2 = a^2 \cdot b^2 - (\vec{a} \cdot \vec{b})^2$

$\rightarrow$ Vector product of four vectors.

It is represented as $\vec{a} \times \vec{b} \times (\vec{c} \times \vec{d})$

Geometrical interpretation

$\vec{P} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$\vec{P} \perp (\vec{a} \times \vec{b})$ and $\vec{P} \perp (\vec{c} \times \vec{d})$

$\vec{P}$ lies in the plane containing $\vec{a}$ & $\vec{b}$. Hence, P represent a vector parallel to the line of intersection of two planes. ~~One~~ One of each is parallel to the plane containing $\vec{a}$ & $\vec{b}$ and other containing $\vec{c}$ & $\vec{d}$.

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$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \cdot \vec{b} \cdot \vec{d}]\vec{c} - [\vec{a} \cdot \vec{b} \cdot \vec{c}]\vec{d}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{m}$

$\Rightarrow \vec{m} \times (\vec{c} \times \vec{d})$

$\Rightarrow (\vec{m} \cdot \vec{d}) \cdot \vec{c} - (\vec{m} \cdot \vec{c}) \cdot \vec{d}$

$\Rightarrow [\vec{a} \times \vec{b} \cdot \vec{d}]\vec{c} - [\vec{a} \times \vec{b} \cdot \vec{c}]\vec{d}$

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Work Done and Moment

(1) Work Done $W = \vec{F} \cdot \vec{d}$

(2) Moment $M = \vec{d} \times \vec{F}$

Q1. $\vec{F_1} = 4\vec{i} + \vec{j} - 3\vec{k}$
$\vec{F_2} = 3\vec{i} + \vec{j} - \vec{k}$

$\vec{OA} = \vec{i} + 2\vec{j} + 3\vec{k}$ to the
$\vec{OB} = 5\vec{i} + 4\vec{j} + \vec{k}$

Find total work done

→ Net force $\vec{F} = \vec{F_1} + \vec{F_2}$
$= 7\vec{i} + 2\vec{j} - 4\vec{k}$

Displacement $= \vec{OB} - \vec{OA}$
$\vec{d} = 4\vec{i} + 2\vec{j} - 2\vec{k}$

$W = \vec{F} \cdot \vec{d}$
$= 7\vec{i} \cdot 4\vec{i} + 2\vec{j} \cdot 2\vec{j} + 4\vec{k} \cdot 2\vec{k}$

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Q2. $\vec{F} = 5\vec{i} + \vec{k}$

$\vec{OP} = \vec{i} - \vec{j} + 2\vec{k}$
$\vec{OA} = 3\vec{i} + 2\vec{j} + \vec{k}$

→ $\vec{M} = \vec{PA} \times \vec{F}$
where, $\vec{PA} = \vec{OA} - \vec{OP}$

⇒ $\vec{OP} + \vec{PA} = \vec{OA}$
$\vec{PA} = 2\vec{i} + 3\vec{j} - \vec{k}$

$\vec{F} = 5\vec{i} + \vec{k}$

$$\vec{M} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 5 & 0 & 1 \end{vmatrix}$$

$= 3\vec{i} - 7\vec{j} - 15\vec{k}$

$|\vec{M}| = 9 - 49 - 225$
$= \sqrt{283}$

(i) Moment of force about a point (vector)
(ii) Moment of force about a line (l) (scalar)
= Dot product
$= \vec{M} \cdot \vec{l}$ where $\vec{l} = a\vec{i} + b\vec{j} + c\vec{k}$

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- Moment of force along x-axis
  $= (3\vec{i} - 7\vec{j} - 15\vec{k}) \cdot \vec{i}$

- Moment of force along y-axis
  $= (3\vec{i} - 7\vec{j} - 15\vec{k}) \cdot \vec{j}$

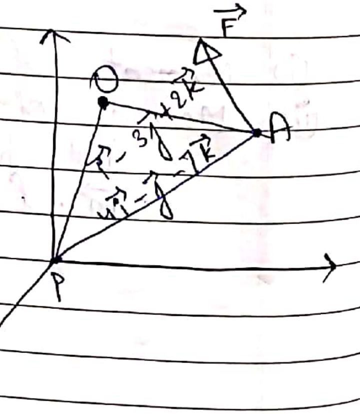
- Moment of force along z axis
  $= (3\vec{i} - 7\vec{j} - 15\vec{k}) \cdot \vec{k}$

Que. A force of magnitude 6 acts along the vector $9\vec{i} + 6\vec{j} - 2\vec{k}$ and passes through the point A whose vector is $4\vec{i} - \vec{j} - 7\vec{k}$. Find the moment of the force about the point O whose vector is $\vec{i} - 3\vec{j} + 2\vec{k}$ also find the moment of the force about the line O and parallel to co-ordinate axis.

→ Find the unit vector of

$$\left[\frac{9\vec{i} + 6\vec{j} - 2\vec{k}}{\sqrt{9^2 + 6^2 + 2^2}}\right]$$

$$\vec{F} = \boxed{\frac{9\vec{i} + 6\vec{j} - 2\vec{k}}{11}}$$



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Theorem : The necessary and sufficient condition for the vector $a(t)$ to have constant magnitude is $a \cdot \frac{da}{dt} = 0$

Proof : Let a be a vector function of the scalar variable t.
Let $|a| = a =$ constant
Then $a \cdot a = a^2 =$ constant

$\therefore \frac{d(a \cdot a)}{dt} = 0$ or $a \cdot \frac{da}{dt} + \frac{da}{dt} \cdot a = 0$

or, $2a \cdot \frac{da}{dt} = 0$ or $a \cdot \frac{da}{dt} = 0$

Therefore, the condition is necessary.

Condition is sufficient if $a \cdot \frac{da}{dt} = 0$, then

$a \cdot \frac{da}{dt} + \frac{da}{dt} \cdot a = 0$

or, $\frac{d}{dt}(a \cdot a) = 0$

or $a \cdot a =$ constant

or $a^2 =$ constant

or $|a| =$ constant.

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Gradient, Divergence & Curl

Partial differential of $f = f(x, y, z)$

$$\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$$

Del operator →

$$\vec{\nabla} = \vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}$$

Gradient → If f is a scalar point function and if is defined, and differentiable at every point in space then gradient is

$$\vec{\nabla} f = \vec{i}\frac{df}{dx} + \vec{j}\frac{df}{dy} + \vec{k}\frac{df}{dz}$$

Divergence → If f is a vector point function then divergence is,

$$\vec{\nabla}\cdot\vec{f} = \frac{df}{dx} + \frac{df}{dy} + \frac{df}{dz}$$

Curl →

$$\vec{\nabla}\times\vec{f} = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\times\vec{f}$$

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Solenoid Vector →

$$\text{if } \vec{\nabla}\cdot\vec{f} = 0$$

$$m\vec{a}\cdot\vec{b} = \vec{a}\cdot m\vec{b}$$
$$m\vec{a}\times\vec{b} = \vec{a}\times m\vec{b}$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$
$$\vec{a}\cdot\vec{b}\times\vec{c} = \vec{a}\times\vec{b}\cdot\vec{c}$$

Q1. $\vec{\nabla}\cdot(\phi\vec{A}) = (\vec{\nabla}\phi)\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$

Proof: $\left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\cdot(\phi\vec{A})$

$$\Sigma\vec{i}\frac{d}{dx}\cdot(\phi\vec{A})$$

$$\Sigma\vec{i}\frac{d}{dx}\cdot(\phi\vec{A})$$

$$\Rightarrow \Sigma\vec{i}\left[\phi\frac{d\vec{A}}{dx} + \frac{d\phi}{dx}\vec{A}\right]$$

$$\Rightarrow \Sigma\vec{i}\cdot\phi\frac{d\vec{A}}{dx} + \Sigma\vec{i}\frac{d\phi}{dx}\cdot\vec{A}$$

$$\Rightarrow \phi\,\Sigma\vec{i}\cdot\frac{d\vec{A}}{dx} + \vec{A}\cdot\Sigma\vec{i}\cdot\frac{d\phi}{dx}$$

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$\Rightarrow \phi \, \Sigma \vec{i} \frac{d}{dx} \cdot \vec{A} + \vec{A} \cdot \Sigma \vec{i} \frac{d}{dx}(\phi)$

$\Rightarrow \phi(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla}\phi)$

$\Rightarrow (\nabla \phi) \cdot \vec{A} + \phi(\vec{\nabla} \cdot \vec{A})$

Q2. $\vec{\nabla} \times (\phi \vec{A}) = \vec{\nabla}\phi \times \vec{A} + \phi(\vec{\nabla} \times \vec{A})$

Q3. $\vec{\nabla}(\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A}(\vec{\nabla} \times \vec{B})$

Q4. $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$

curl gradient $\phi = 0$

Q5. $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$

divergence ~~of~~ curl $A = 0$

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INTEGRATION

- Integration of rational function
- Integration of irrational function
- Definite Integrals
- Area of length of curve
- Volume and surface area

$\int \frac{1}{x} \cdot dx = \log x$ | $\int e^x \cdot dx = e^x$ | $\int \frac{1}{x-1} \cdot dx = \log|x-1|$

☆ Integration of irrational function using partial fraction.

Q1. $\int \frac{x-2}{(x-1)(x-5)} \cdot dx$

$\longrightarrow \frac{x-2}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$

$(x-2) = A(x-5) + B(x-1)$

put $x = 5$

$5-2 = B(5-1)$

$3 = 4B$

$B = \frac{3}{4}$

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Put $x = 1$

$(1-2) = A(1-5)$

$4A = 1$

$A = \frac{1}{4}$

Q2. $\displaystyle\int \frac{dx}{x^3-1} = \int \frac{dx}{(x-1)(x^2+x+1)}$

$\rightarrow \displaystyle\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$\displaystyle\frac{1}{(x-1)(x^2+x+1)} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}$

~~1 = A(x²+x+1) + B(x+c)(x−1)~~

$1 = A(x^2+x+1) + (Bx+C)(x-1)$

$1 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$

$1 = (A+B)x^2 + (A-B+C)x + A - C$

$A + B = 0$ | $A - B + C = 0$ | $A - C = 1$

$A = -B$ | $-B - B + C = 0$ | $-B - 2B = 1$

$A = -\left(-\frac{1}{3}\right)$ | $-2B + C = 0$ | $-3B = 1$

$A = \frac{1}{3}$ | $C = 2B$ | $B = -\frac{1}{3}$

$C = 2 \times -\frac{1}{3}$

$C = -\frac{2}{3}$

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$\displaystyle\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x-1}\,dx$

$\displaystyle = \frac{1}{3}\log(x-1) - \frac{1}{3}\int \frac{x-2}{x^2+x-1}\,dx$

Let $\displaystyle I = \int \frac{x-2}{x^2+x-1}\,dx$

$\displaystyle = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2} - 2}{x^2+x+1}\cdot dx$

$\displaystyle = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\cdot dx - \frac{4}{3}\int \frac{1}{x^2+x+1}\cdot dx$

$\displaystyle = \frac{1}{2}\log(x^2+x+1) - \frac{4}{3}\int \frac{1}{x^2+x+1}\cdot dx$

$\displaystyle\int \frac{1}{x^2+x+1} = \int \frac{1}{x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\cdot dx$

$\displaystyle = \int \frac{1}{(x+\frac{1}{2})^2 - \frac{3}{4}}\cdot dx$

$\displaystyle = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2}$

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$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}$$

$$I_1 = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}$$

$$\int \frac{x-2}{x^2+x+1}\, dx = \frac{1}{2} \log(x^2+x+1) + \frac{4}{3}\left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{2x}{\sqrt{}}\right.$$

Q1. $\int \frac{dx}{\sin x + \sin 2x}$

Q2. $\int \frac{x^2+1}{x(x^2-1)} \cdot dx$

Q3. $\int \frac{dx}{\sin x\,(3+2\cos x)}$

Q4. $\int \frac{dx}{1+3e^x + \cancel{2e^x}\, 2e^{2x}}$

Q5. $\int \frac{x^2-1}{x^4+1} \cdot dx$