28/07/2018

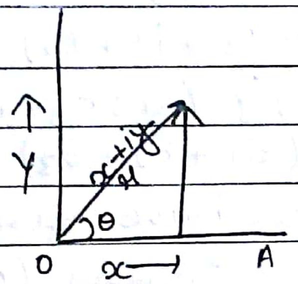
Maths Syllabous

1. Abstract algebra
2. Trigonometry
3. Differential eqn
4. Probability theory.

Trigonometry

Sin (A+B)

(Complex Number



$OA = r\cos\theta = x$

$AB = r\sin\theta = y$

$x+iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

De-Moivre's Thm :-

Statement :— for any value of n (integer, fraction, real and imaginary)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof :— when n is positive integer

we have, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \cdot \sin\theta_2)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

Again, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

Similarly Proceeding, we have for n factors

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) \cdots (\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)$

In this result put $\theta_1 = \theta_2 = \dots = \theta_n = \theta$.

Then we get $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Case-II :— when n is negative integer.

Let $n = -m$, where m is a positive integer.

Now, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}, \text{ by case I}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta + i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta$$

Case III. When n is a fraction.

Let $n = r/s$ where s is a +ve integer and r is any integer.

From Case I or II, we have

$$(\cos\theta + i\sin\theta)^r = \cos r\theta + i\sin r\theta$$

$$= \cos\frac{r}{s}.s\theta + i\sin\frac{r}{s}.s\theta.$$

$$= \left[\cos\frac{r}{s}\theta + i\sin\frac{r}{s}\theta\right]^s.$$

Taking s^{th} root of both the sides,

$(\cos\theta + i\sin\theta)^{r/s}$ has one of its values

$$= \cos\frac{r}{s}\theta + i\sin\frac{r}{s}\theta.$$

Now putting $\frac{r}{s} = n$,

$(\cos\theta + i\sin\theta)^n$ has one of its values =

$$\cos n\theta + i\sin n\theta.$$

This completes the proof of the theorem.

Q) Prove that $\dfrac{(\cos 3\theta + i\sin 3\theta)^5 (\cos 2\theta - i\sin 2\theta)^3}{(\cos 4\theta + i\sin 4\theta)^9 (\cos 5\theta + i\sin 5\theta)^9} = 1.$

Soln:- $L.H.S = \dfrac{(\cos 3\theta + i\sin 3\theta)^5 (\cos 2\theta + i\sin 2\theta)^3}{(\cos 4\theta + i\sin 4\theta)^9 (\cos 5\theta + i\sin 5\theta)^{11}}$

$= \dfrac{[\cos(5.3\theta) + i\sin(5.3\theta)][\cos(3.2\theta) - i\sin(3.2\theta)]}{[\cos(-9.4\theta) + i\sin(-9.4\theta)][\cos(9.5\theta) + i\sin(9.5\theta)]}$

$= \dfrac{(\cos 15\theta + i\sin 15\theta)(\cos 6\theta - i\sin 6\theta)}{(\cos 36\theta - i\sin 36\theta)(\cos 45\theta + i\sin\theta\, 45\theta)}$

$= \dfrac{\cos 9\theta + i\sin 9\theta}{\cos 9\theta + i\sin 9\theta} = 1 = R.H.S$ proved

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Q) $(\cos\theta + i\sin\theta)^{-1}$

$= \dfrac{1}{\cos\theta + i\sin\theta} \times \dfrac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \dfrac{\cos\theta - i\sin\theta}{1} = \cos(-\theta) + i\sin(-\theta)$

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8) If $\left[1+i\frac{x}{a}\right]\left[1+i\frac{x}{b}\right]\left[1+i\frac{x}{c}\right]\ldots = A+iB$, prove

that $\left[1+\frac{x^2}{a^2}\right]\left[1+\frac{x^2}{b^2}\right]\left[1+\frac{x^2}{c^2}\right]\ldots = A^2+B^2$

and $\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\ldots = \tan^{-1}\frac{B}{A}$.

Soln:— Let $1 = r_1\cos\theta_1$, $\frac{x}{a} = r_1\sin\theta_1$;

$\therefore\ 1^2+\frac{x^2}{a^2} = r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta_1$

$\Rightarrow 1+\frac{x^2}{a^2} = r_1^2(\cos^2\theta_1+\sin^2\theta_1) = r_1^2$.

also, $\frac{r_1\sin\theta_1}{r_1\cos\theta_1} = \frac{x}{a}$

$\Rightarrow\ \tan\theta_1 = \frac{x}{a}$

$\theta_1 = \tan^{-1}\frac{x}{a}$.

assuming likewise for other factors,

we have $\left[1+i\frac{x}{a}\right]\left[1+i\frac{x}{b}\right]\left[1+i\frac{x}{c}\right]\ldots = A+iB$

$\rightarrow (r_1\cos\theta_1+ir_1\sin\theta_1)(r_2\cos\theta_2+ir_2\sin\theta_2)$

$(r_3\cos\theta_2+ir_3\cos\theta_3)\ldots = A+iB$

$\Rightarrow r_1r_2r_3\ldots[(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$

$(\cos\theta_3+i\sin\theta_3)\ldots] = A+iB$

$\Rightarrow r_1r_2r_3\ldots[\cos(\theta_1+\theta_2+\theta_3+\ldots)$

$+ i\sin(\theta_1+\theta_2+\theta_3+\ldots)] = A+iB$,

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Equating real and imaginary part seperately from both sides,

we get $r_1 r_2 r_3 \ldots \cos(\theta_1+\theta_2+\theta_3+\ldots) = A$ —(1)

and $r_1 r_2 r_3 \ldots \sin(\theta_1+\theta_2+\theta_3+\ldots) = B$ —(2)

squaring and adding (1) and (2), we get.

$$A^2+B^2 = r_1^2 r_2^2 + r_3^2 \ldots \left[\cos^2(\theta_1+\theta_2+\theta_3\ldots) + \sin^2(\theta_1+\theta_2+\theta_3+\ldots)\right]$$

$$A^2+B^2 = r_1^2 + r_2^2 + r_3^2 \ldots$$

$$A^2+B^2 = \left[1+\frac{x^2}{a^2}\right]\left[1+\frac{x^2}{b^2}\right]\left[1+\frac{x^2}{c^2}\right] \ldots$$

Hence the first part is proved.

again, dividing (2) by (1), we get.

$$\frac{r_1 r_2 r_3 \ldots \sin(\theta_1+\theta_2+\theta_3+\ldots)}{r_1 r_2 r_3 \ldots \cos(\theta_1+\theta_2+\theta_3+\ldots)} = \frac{B}{A}$$

$$\tan(\theta_1+\theta_2+\theta_3+\ldots) = \frac{B}{A}.$$

$$\theta_1+\theta_2+\theta_3 \ldots = \tan^{-1}\frac{B}{A}$$

$$\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{b} + \tan^{-1}\frac{x}{c} = \tan^{-1}\frac{B}{A}.$$

This prove the second part.

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Q) Prove that.

(1) $$\left[\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n = (\sin\theta+i\cos\theta)^n.$$

$$= \cos\left(\tfrac{1}{2}n\pi-n\theta\right)+i\sin\left(\tfrac{1}{2}n\pi-n\theta\right).$$

Soln :- Here $1+\sin\theta+i\cos\theta$

$= (\sin^2\theta+\cos^2\theta)(\sin\theta+i\cos\theta)$

$= (\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta)+(\sin\theta+i\cos\theta)$

$= (\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta+1)$

$\therefore$ expression $= \left\{\dfrac{(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta+1)}{1+\sin\theta-i\cos\theta}\right\}^n$

$= (\sin\theta+i\cos\theta)^n$ [1st part]

$= \{\cos(\tfrac{1}{2}\pi-\theta)+i\sin(\tfrac{1}{2}\pi-\theta)\}^n$

$= \cos n(\tfrac{1}{2}\pi-\theta)+i\sin n(\tfrac{1}{4}\pi-\theta)$

$= \cos(\tfrac{1}{2}n\pi-n\theta)+i\sin(\tfrac{1}{2}n\pi-n\theta)$. [2nd part]

~~Second part :-~~

Second method :-

Let $1+\sin\theta = r\cos\phi$, $\cos\theta = r\sin\phi$.

Then, $r^2 = (1+\sin\theta)^2+\cos^2\theta = 2(1+\sin\theta)$,

$$\tan\phi = \frac{\cos\theta}{1+\sin\theta} = \frac{\cos^2\tfrac{1}{2}\theta-\sin^2\tfrac{1}{2}\theta}{(\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta)^2}$$

$$= \frac{\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta}{\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta}.$$

$$= \frac{1-\tan\tfrac{1}{2}\theta}{1+\tan\tfrac{1}{2}\theta} = \tan\left[\frac{\pi}{4}-\frac{\theta}{2}\right]; \quad \therefore \phi = \frac{\pi}{4}-\frac{\theta}{2}.$$

$\therefore$ L.H.S $= \left[\dfrac{r\cos\phi + ir\sin\phi}{r\cos\phi - ir\sin\phi}\right]^n = \left[\dfrac{\cos\phi + i\sin\phi}{\cos\phi - i\sin\phi}\right]^n$

$= (\cos\phi + i\sin\phi)^n \cdot (\cos\phi - i\sin\phi)^{-n}$

$= (\cos n\phi + i\sin n\phi) \cdot (\cos n\phi + i\sin n\phi)$

$= (\cos n\phi + i\sin n\phi)^2$

$= (\cos 2\phi + i\sin 2\phi)^n$.

$= \left[\cos\left(\pi/2 - \theta\right) + i\sin\left(\pi/2 - \theta\right)\right]^n$

$= \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$.

4. Prove that.

$$(a+ib)^{m/n} + (a-ib)^{m/n} = 2(a^2+b^2)^{\frac{m}{2n}} \cos\left[\frac{m}{n}\tan^{-1}\frac{b}{a}\right]$$

Soln:- on putting $a = r\cos\theta$, $b = r\sin\theta$ we get,

$r^2 = a^2 + b^2$ and $\tan\theta = b/a$.

L.H.S $= \{r(\cos\theta + i\sin\theta)\}^{m/n} + \{r(\cos\theta - i\sin\theta)\}^{m/n}$

$= r^{m/n}\left[\left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right) + \left[\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta\right]\right]$

$= \left(\sqrt{a^2+b^2}\right)^{m/n} 2\cos\frac{m}{n}\theta$

$= 2(a^2+b^2)^{\frac{m}{2n}} \cdot \cos\left[m/n \tan^{-1} b/a\right)$

= R.H.S Proved.

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5. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that.

$x_1 . x_2 . x_3 \dots$ to $\infty = -1$.

→ Putting $r = 1, 2, 3, \dots$ successively in given relation, $x_1 . x_2 . x_3 \dots$ to ∞

$= \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]\left[\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right]\left[\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right] \dots$ to ∞

$= \cos\left[\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \text{to } \infty\right] + i\sin\left[\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \text{to } \infty\right]$

$= \cos\frac{\frac{1}{2}\pi}{1-\frac{1}{2}} + i\sin\frac{\frac{1}{2}\pi}{1-\frac{1}{2}}$ (summing up the infinity G.P., series, C.R. being <1)

$= \cos\pi + i\sin\pi = -1$

proved.

Q) Find the equation whose roots are the n^{th} powers of the roots $x^2 - 2x\cos\theta + 1 = 0$.

→ Soln:- from the given equations,

$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \frac{2\cos\theta \pm 2i\sin\theta}{2}$

$x = \cos\theta \pm i\sin\theta$.

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If α, β are the roots of the given equation, then

$\alpha = \cos\theta + i\sin\theta,$

$\beta = \cos\theta - i\sin\theta.$

Now, $\alpha^n + \beta^n = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n$

$= (\cos n\theta + i\sin n\theta) + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$\alpha^n \beta^n = (\alpha\beta)^n = \{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)\}^n$

$= (\cos^2\theta + \sin^2\theta)^n = 1$

$\therefore$ required equation is $x^2 - (\alpha^n + \beta^n)x + \alpha^n\beta^n = 0.$

$= x^2 - 2x\cos n\theta + 1 = 0.$

proved.

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Q) $x^7 + 1 = 0.$

Soln: Here $x^7 = -1,$

or $x = (-1)^{1/7}.$

$\therefore x = (\cos\pi + i\sin\pi)^{1/7}$

$= \{\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\}^{1/7}$

$= \cos\frac{1}{7}(2k+1)\pi + i\sin\frac{1}{7}(2k+1)\pi.$

Putting $k = 0, 1, 2, 3, 4, 5, 6$ successively we get the following seven roots:

$\cos\frac{1}{7}\pi + i\sin\frac{1}{7}\pi, \quad \cos\frac{3}{7}\pi + i\sin\frac{3}{7}\pi,$

$\cos\frac{5}{7}\pi + i\sin\frac{5}{7}\pi, \ldots, \cos\frac{3}{7}(-\pi) + i\sin\frac{3}{7}(-\pi).$

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Q) $x^7 + 1 = 0$

$\Rightarrow x^7 = -1$

$x^7 = \cos\pi + i\sin\pi$

$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$

where $k = 0, 1, 2, 3, 4, 5, 6$.

$\theta = 2n\pi + \theta$

where $n = 0, 1, 2$;

$x = \{\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\}^{n = \frac{1}{7}}$

Using De-moivers thm :—

$x = \cos\frac{(2k+1)}{7}\pi + i\sin\frac{(2k+1)\pi}{7}$

$k = 0$.

$x = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

when $k = 1$

$x = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$

when $k = 2$

$x = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

when $k = 3$

$x = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7} = x = \cos\pi + i\sin\pi$

when $k = 4$

$x = \cos\frac{9}{7}\pi + i\sin\frac{9}{7}\pi$.

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when $K=5$

$x = \frac{11}{7}\pi + i\sin\frac{11}{7}\pi$

when $K=6$

$\cos\frac{13}{7}\pi + i\sin\frac{13\pi}{7}$

when $K=7$

$x = \cos\frac{15\pi}{7} + i\sin\frac{15}{7}\pi$

$\cos\left[2\pi+\frac{\pi}{7}\right] + i\sin\left[2\pi+\frac{\pi}{7}\right]$

Q) Solve:—

$x^7+x^4+x^3+1=0.$

Soln:— Here $x^7+x^4+x^3+1$

$= x^4(x^3+1) + (x^3+1)$

$= (x^3+1)(x^4+1)$

$\therefore$ $(x^3+1)(x^4+1)=0$, which is the same as the given eqn;

$\therefore$ either $x^3+1=0$, or $x^4+1=0$

i.e., $x = (-1)^{1/3} = (\cos\pi + i\sin\pi)^{1/3}$

$= \{\cos(2K\pi+\pi) + i\sin(2K\pi+\pi)\}^{1/3}$

$= \{\cos\frac{1}{3}(2K+1)\pi + i\sin\frac{1}{3}(2K+1)\pi,$

where $K = 0, 1, 2.$

similarly, $x = \{\cos(2K\pi+\pi) + i\sin(2K\pi+\pi)\}^{1/4}$

$= \cos\frac{1}{4}(2K+1)\pi + i\sin\frac{1}{4}(2K+1)\pi,$

where $K=0, 1, 2, 3.$

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$\therefore$ the required roots are $\cos\frac{1}{3}(2K+1)\pi + i\sin\frac{1}{3}(2K+1)\pi$, where $K = 0, 1, 2$.

and $\cos\frac{1}{4}(2K+1)\pi + i\sin\frac{1}{4}(2K+1)\pi$,

where $K = 0, 1, 2, 3$.

Q) $x^4+x^3+x^2+x+1=0$ Solve by De-Moivre's thm.

$\rightarrow$ Multiplying with $(x-1)$ we get.

$(x-1)(x^4+x^3+x^2+x+1)=0$;

or $x^5-1=0$ — (1)

$x^5=1$

or $x^5 = \cos 0 + i\sin 0$

$= \cos 2n\pi + i\sin 2n\pi$;

$\therefore x = (\cos 2n\pi + i\sin 2n\pi)^{1/5}$

$= \frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}$, where $n = 0, 1, 2, 3, 4$

Q) $x^5-1=0$. Prove that the sum of the nth power of the roots, n being an integer not divisible by 5, is zero.

Soln:— Here $x^5=1$;

$\therefore x^5 = \cos 0 + i\sin 0$

$= \cos 2n\pi + i\sin 2n\pi$;

$\therefore x = (\cos 2n\pi + i\sin 2n\pi)^{1/5}$

$= \cos\frac{2}{5}n\pi + i\sin\frac{2}{5}n\pi$,

where $n = 0, 1, 2, 3, 4$.

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$\therefore$ the roots are 1,

$\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$, $\cos\frac{4}{5}\pi + i\sin\frac{4}{5}\pi$,

$\cos\frac{6}{5}\pi + i\sin\frac{6}{5}\pi$, $\cos\frac{6}{5}\pi + i\sin\frac{8}{5}\pi$.

Putting $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi = \alpha$;

$\cos\frac{4}{5}\pi + i\sin\frac{4}{5}\pi = \left(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi\right)^2$

$= \alpha^2$ etc.

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

also. $\alpha^5 = \left(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi\right)^5$

$= \cos 2\pi + i\sin 2\pi = 1$.

Sum of the nth power of the roots

$= 1 + \alpha^n + \alpha^{2n} + \alpha^{3n} + \alpha^{4n}$

$= \frac{1-(\alpha^n)^5}{1-\alpha^n} = \frac{1-(\alpha^5)^n}{1-\alpha^n} = \frac{1-1}{1-\alpha^n}$

$= \frac{0}{\text{non-zero quantity}}$ $= 0.$

04-08-18

Application of De-Moivere's thm in expansion.

* To expand $\cos x$ and $\sin x$

Soln:— $(\cos\theta + i\sin\theta)^n = \cos^n\theta + {}^nC_1 \cos^{n-1}\theta\,(i\sin\theta)$
$+ {}^nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3 \cos^{n-3}\theta\,(i\sin\theta)^3$
$+ {}^nC_4 \cos^{n-4}\theta\,(i\sin\theta)^4 + \dots + (i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta = \cos^n\theta + n\cos^{n-1}\theta\,(i\sin\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(-\sin^2\theta)$

$+ \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\,(-i\sin^3\theta) + \frac{n(n-1)(n-2)(n-3)}{4!}$

$(\cos^{n-4}\theta\,\sin^4\theta) + \dots$

equating equal part.

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2}\cos^{n-2}\theta\,\sin^2\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\,\sin^4\theta - 1$$

Let $n\theta = x \longrightarrow n = \frac{x}{\theta}$

for x to be constant when n is very large, θ is very small.

$$\cos x = \cos^n\theta - \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)}{2!}\cos^{n-2}\theta\sin^2\theta +$$

$$\frac{\frac{x}{\theta}\left(\frac{x}{\theta}-1\right)\left(\frac{x}{\theta}-2\right)\left(\frac{x}{\theta}-3\right)}{4!}\cos^{n-4}\theta\,\sin^4\theta$$

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$= \cos^n\theta - \frac{x(x-\theta)}{2!}\cos^{n-1}\theta\left[\frac{\sin\theta}{\theta}\right]^2 + \frac{x(x-\theta)(x-2\theta)(x-3\theta)}{4!}$

$\cos^{x-4}\theta\left[\frac{\sin\theta}{\theta}\right]^4$

when θ is very small

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

Equating imaginary part and proceeding in the same way.

$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots$

Q) Use De-Moivre's theorem to show that

$\cos 5\theta = \cos\theta\,[16\cos^4\theta - 20\cos^2\theta + 5]$.

→ soln:- we have $\cos 5\theta + i\sin 5\theta$

$= (\cos\theta + i\sin\theta)^5$

$\cos^5\theta + {}^5C_1\cos^4\theta\,(i\sin\theta) + {}^5C_2\cos^3\theta\,(i\sin\theta)^2$

$+ {}^5C_2\cos^2\theta\,(i\sin\theta)^3 + {}^5C_4\cos\theta\,(i\sin\theta)^4$

$+ {}^5C_5\,(i\sin\theta)^5$.

equating real parts from both sides, we get.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta$

$(1 - 2\cos^2\theta + \cos^4\theta)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$= \cos\theta\,[16\cos^4\theta - 20\cos^2\theta + 5]$

Q) Express $\cos^6\theta$ in terms of cosines of multiples of θ.

→ Soln: Let $x = \cos\theta + i\sin\theta$,

then $\frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - \sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta$.

$\Rightarrow (2\cos\theta)^6 = \left[x + 1/x\right]^6 = x^5 + {}^6C_1\, x^5 \cdot \frac{1}{x}$

$+ {}^6C_2\, x^4 \cdot \frac{1}{x^2} + {}^6C_3\, x^3 \cdot \frac{1}{x^3} + {}^6C_4\, x^2 \cdot \frac{1}{x^4}$

$+ {}^6C_5\, x \cdot \frac{1}{x^5} + {}^6C_6\, \frac{1}{x^6}$.

But ${}^6C_1 = 6$; ${}^6C_2 = \frac{6\times5}{2\times1} = 15$; ${}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20$

${}^6C_4 = {}^6C_2 = 15$; ${}^6C_5 = {}^6C_1 = 6$; ${}^6C_6 = 1$.

$\therefore (2\cos\theta)^6 = \left[x^6 + \frac{1}{x^6}\right] + 6\left[x^4 + \frac{1}{x^4}\right] + 15\left[x^2 + \frac{1}{x^2}\right]$

$+ 20$.

$\because x = \cos\theta + i\sin\theta$

and $\frac{1}{x} = \cos\theta - i\sin\theta$,

$\therefore x^n = \cos n\theta + i\sin n\theta$ and

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$.

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$\therefore\ x^6 + \frac{1}{x^6} = 2\cos 6\theta$;

$x^4 + \frac{1}{x^4} = 2\cos 4\theta$;

$x^2 + \frac{1}{x^2} = 2\cos 2\theta$;

Hence $2^6 \cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

This is required result.

Q) To expand $\cos^n\theta$ in terms of cosines of multiple angles, n being a positive integer.

B.S

Let $z = \cos\theta + i\sin\theta$

$\therefore \frac{1}{z} = \cos\theta - i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$;

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$ [from de Moivre's thm]

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$, for all integral n.

We have, $(2\cos\theta)^n = \left[z + \frac{1}{z}\right]^n$

$= z^n + {}^nC_1 z^{n-1}\frac{1}{z} + {}^nC_2 z^{n-2}\frac{1}{z^2} + \dots +$

${}^nC_2 z^2 \cdot \frac{1}{z^{n-2}} + {}^nC_{n-1} z \frac{1}{z^{n-1}} + {}^nC_n \frac{1}{z^n} \dots (1)$

Now, grouping in pairs the terms equidistant from the begining and the end and noting that ${}^nC_r = {}^nC_{n-r}$,

$$2^n \cos^n\theta = \left[z^n + \frac{1}{z^n}\right] + {}^nC_1 \left[z^{n-2} + \frac{1}{z^{n-2}}\right] + {}^nC_2 \left[z^{n-4} + \frac{1}{z^{n-4}}\right]$$

$+ \dots$

$$= 2\cos n\theta + n \cdot 2 \cos(n-2)\theta + \frac{n(n-1)}{2!} 2\cos(n-4)\theta + \dots$$

$$\therefore 2^{n-1} \cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!} \cos(n-4) + \dots \quad \dots (2)$$

7) $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta$

$\left[z + \frac{1}{z}\right]^6 = (2\cos\theta)^6 = 2^6 \cos^6\theta$

$z^6 \cos^6\theta = z^6 + {}^6C_1 z^5 \cdot \frac{1}{z} + {}^6C_2 z^4 \cdot \frac{1}{z^2}$.

Q) Show that $\frac{\pi^2}{2\cdot4} - \frac{\pi^4}{2\cdot4\cdot6\cdot8} + \frac{\pi^6}{2\cdot4\cdot6\cdot8\cdot10} - \dots$ to $\infty = 1$

Soln:- L.H.S.

$$= \frac{1}{1\cdot2} \cdot \frac{\pi^2}{2^2} - \frac{1}{1\cdot2\cdot3\cdot4} \cdot \frac{\pi^4}{2^4} + \frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6} \cdot \frac{\pi^6}{2^6} - \dots \text{ to } \infty$$

$$\rightarrow 1 - \left[1 - \frac{1}{\lfloor 2} \left[\frac{\pi}{2}\right]^2 + \frac{1}{\lfloor 4} \left(\frac{\pi}{2}\right)^4 - \frac{1}{\lfloor 6} \left[\frac{\pi}{2}\right]^6 + \dots \text{ to } \infty\right]$$

Teacher's Signature.........

$1 - \cos\frac{1}{2}\pi = 1 =$ R.H.S

09-08-18

Complex Arguments

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} +$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!}$

$e^{ix} = 1 + \frac{ix}{1!} + \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \dots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right]$

$= \cos x + i\sin x$ — (1)

similarly,

$e^{-ix} = \cos x - i\sin x$ — (2)

adding (1) and (2)

$e^{ix} + e^{-ix} = 2\cos x$

$\cos x = \frac{1}{2}\left[e^{ix} + e^{-ix}\right]$ — (3)

subtracting (2) from (1)

$e^{ix} - e^{-ix} = 2i\sin x$

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$$\sin x = \frac{1}{2i}[e^{ix} - e^{-ix}] \quad — (4)$$

eqn (3) and (4) represent the defination of sinx and cosx resp.
eqn ~~(2)~~

8) Using the defination of sinx and cosx prove that $\sin^2 x + \cos^2 x = 1$

$\rightarrow$ $\sin^2 x + \cos^2 x$

$$= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left[\frac{e^{ix} + e^{-ix}}{2}\right]^2$$

$$= -\frac{1}{4}(e^{2ix} - 2 + e^{-2ix}) + \frac{1}{4}(e^{2ix} + 2 + e^{-2ix}) = 1.$$

9) $\cos 2x = 2\cos^2 x - 1$

We have,

$$2\cos^2 x - 1 = 2\left[\frac{e^{ix} + e^{-ix}}{2}\right]^2 - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2e^{ix}e^{-ix}}{2} - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2 - 2}{2}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2}$$

$$= \cos 2x$$

Q) $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$

$$\rightarrow \frac{e^{ix} - e^{-ix}}{2!} \cdot \frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} - e^{-iy}}{2i}$$

$$\rightarrow \frac{1}{4i}\left(2e^{ix}e^{iy} - 2e^{-ix}e^{-iy}\right)$$

$$= \frac{1}{2i}\left(e^{i(x+y)} - e^{-i(x+y)}\right)$$

$$= \sin(x+y).$$

Replacing y by $-y$, we get

$$\sin(x-y) = \sin x \cdot \cos(-y) + \cos x \sin(-y)$$

$$= \sin x \cos y - \cos x \sin y.$$

Similarly (iii) can also be established.

formulae for $\sin 2x$, $\cos 2x$, $\sin 3x$, ... can also be easily established.

8) Logarithm of complex Number.

$$e^z = w$$

where z and w are complex quantities; z is called the logarithm of w.

$$z = x + iy$$

$$\log e^z = \log w$$

$$z = \log w$$

$$e^{ix} = w$$

$$\log e^{ix} = \log w$$

$$ix = \log w \rightarrow \text{Principal Value}$$

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$e^{2\pi n + ix} = \omega$

$ix + 2\pi x = \log \omega$

8) $\log(\alpha + i\beta)$,

$\alpha = r\cos\theta$

$\beta = r\sin\theta$, then

$r^2 = \alpha^2 + \beta^2$ $\tan\theta = \beta/\alpha$

$r = +\sqrt{\alpha^2+\beta^2}$

$\theta = \tan^{-1}\beta/\alpha$.

$\log(\alpha+i\beta) = \log(r\cos\theta + ir\sin\theta)$

$= \log\{r(\cos\theta + i\sin\theta)\}$

$= \log(r.e^{i\theta})$

$= \log r + \log i\theta$

$= \log\sqrt{r^2+\beta^2} + i\tan^{-1}\frac{\beta}{\alpha}$.

$\therefore \log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\left[\beta/\alpha\right]\right\}$

principal value put $n=0$.

$$\boxed{\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}}$$

P.T

Q) $\tan\left[i \log \frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2-b^2}$

Putting $a = r\cos\theta$, $b = r\sin\theta$, we get.
$r^2 = a^2+b^2$ and $\tan\theta = b/a$.

$\therefore$ L.H.S $= \tan\left[i \log \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right]$

$= \tan\left\{ i \log \frac{e^{-i\theta}}{e^{i\theta}} \right\}$

$= \tan\{ i \log e^{-2i\theta} \} = \tan\{ i(-2i\theta)\}$

$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2b/a}{1-b^2/a^2}$

$= \frac{2ab}{a^2-b^2}$ = R.H.S proved.

11/08/18

Q) Express $(\alpha+i\beta)^{x+iy}$ in the form of $A+iB$.

Soln:$\longrightarrow$ $(\alpha+i\beta)^{x+iy} = e^{\log(\alpha+i\beta)^{x+iy}}$

$= e^{(x+iy)\log(\alpha+i\beta)}$

$(x+iy)\log(\alpha+i\beta) = (x+iy)\left[\frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}\right]$

$= x\frac{1}{2}\log(\alpha^2+\beta^2) + ix\tan^{-1}\frac{\beta}{\alpha} + iy\frac{1}{2}\log(\alpha^2+\beta^2)$
$- y\tan^{-1}\frac{\beta}{\alpha}$

Teacher's Signature............

$= x \frac{1}{2} \log(\alpha^2+\beta^2) - y\tan^{-1}\frac{\beta}{\alpha} + i\left[x\tan^{-1}\frac{\beta}{\alpha} + y\frac{1}{2}\log(\alpha^2+\beta^2)\right]$

$= p+iq$

$= e^{(x+i\beta)\log(\alpha+i\beta)} = e^{p+iq} = e^{p}\cdot e^{iq}$

$= e^{p}[\cos q + i\sin q]$

$= e^{p}\cos q + ie^{p}\sin q$

$\boxed{P = A + iB}$

Where $A = e^{p}\cos q$

$B = e^{p}\sin q.$

Q) Find the general value of i^{i}.

$\rightarrow i^{i} = e^{\log i^{i}}$

$= e^{i\log i}$

$\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$

Principal

$\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left[\tan^{-1}\frac{\beta}{\alpha} + 2\pi n\right)$

$i\log i = i\left\{i\left[\frac{\pi}{2}+2\pi n\right]\right\} = -\left[2\pi n + \frac{\pi}{2}\right]$

$= \frac{-4\pi n + \pi}{2}$

$= -\left[\frac{4n+1}{2}\right]\pi$

Teacher's Signature........

$$= e^{-\frac{1}{2}(4n+1)\pi}.$$

8) $(i^i)^i = \cos\theta - i\sin\theta$ where $\theta = [4n+1]\frac{\pi}{2}$.

Q) If $i^{x+iy} = x+iy$ Prove that
$$x^2+y^2 = e^{-(4n+1)\pi y}.$$

Soln:— $\because i^{x+iy} = x+iy,$

$\therefore e^{(x+iy)\log i} = x+yi.$

But $\operatorname{Log} i = 2n\pi i + \log i$
$$= 2n\pi i + \log(\cos\tfrac{1}{2}\pi + i\sin\tfrac{1}{2}\pi)$$
$$= 2n\pi i + \log e^{i\pi i}$$
$$= 2n\pi i + \frac{1}{2}\pi i = \frac{1}{2}(4n+1)\pi i$$

$\therefore x+iy = e^{(x+iy)\log i} = e^{(x+iy)\frac{1}{2}(4n+1)\pi i}$
$$= e^{-\frac{1}{2}(4n+1)y\pi} \cdot e^{i\frac{1}{2}(4n+1)\pi x}$$
$$= e^{-\frac{1}{2}(4n+1)y\pi}\{\cos\tfrac{1}{2}(4n+1)\pi x + i\sin\tfrac{1}{2}(4n+1)\pi x\}.$$

Seperating real and imaginary part.
$$x = e^{-\frac{1}{2}(4n+1)y\pi}\cos\tfrac{1}{2}(4n+1)\pi x;$$
$$y = e^{-\frac{1}{2}(4n+1)y\pi}\sin\tfrac{1}{2}[4n+1]\pi x.$$

$$\therefore x^2+y^2 = \{e^{-\frac{1}{2}(4n+1)yx}\}^2 \cdot \{\cos^2\tfrac{1}{2}[4n+1]\pi x + \sin^2\tfrac{1}{2}[4n+1]\pi x\}$$
$$= e^{-(4n+1)\pi y}.$$

16/08/18

Hyperbolic function

$$\sin hx = \frac{e^x - e^{-x}}{2}$$

$$\cos hx = \frac{e^x + e^{-x}}{2}$$

Relation between hyperbolic function and Trigonometrical function.

$$\cos hx = \frac{e^x + e^{-x}}{2} = \frac{e^{-i.ix} + e^{i.ix}}{2} = \frac{e^{i.ix} + e^{-i.ix}}{2}$$

$$= \cos ix$$

$$\sin hx = \frac{e^x - e^{-x}}{2} = \frac{e^{-i.ix} - e^{i.ix}}{2} = \frac{-(e^{i.ix} - e^{-i.ix})}{2}$$

$$= i \sin x$$

$$\tan hx = i \tan ix$$

Relation between hyperbolic inverse and Inverse Circular function.

$$\left.\begin{aligned} \cos hx &= \cos ix \\ \sin hx &= -i \sin ix \\ \tan hx &= -i \tan ix \end{aligned}\right\}$$

(1) Cosine

$\cos hx = y$

$\cos h^{-1} y = x$

$\cos hx = \cos ix$

$ix = \cos h^{-1} x$

$\cosh^2 x - \sinh^2 x = 1$

$$\cosh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\left[\frac{e^x - e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{1}{4}[4] = 1$$

Q) If $\sin(A+iB) = x+iy$ Prove that

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

$\sin(A+iB) = \sin A \cdot \cos iB + \cos A \cdot \sin iB$

$= \sin A \cdot \cosh B + \cos A \cdot i \sinh B.$

$= \sin A \cdot \cosh B + i \sinh B \cos A.$

Relation between Inverse circular function and hyperbolic function.

(i) $\cosh y = \cos iy = x$

$\cosh y = x \Rightarrow \cosh^{-1}x = y$ — (1)

$\cos iy = x \Rightarrow \cos^{-1}x = iy$ — (2)

from (1) and (2)

$\cos^{-1}x = i\cosh^{-1}x$

(ii) $\sinh^{-1}x = -i\sin^{-1} ix$

(iii) $\tanh^{-1}x = -i\tan^{-1} ix$

(ii) $\sinh y = -i\sin iy = x$ (Let)

$\sinh y = x$

$\sinh^{-1}x = y$ — (1)

$-i\sin iy = x$

$\sin iy = ix$

$\sin^{-1} ix = iy$ — (2)

from (1) and (2)

$\sin^{-1} ix = iy$

$\sin^{-1} ix = i\sinh^{-1}x$

$i\sin^{-1} ix = \sinh^{-1}x$.

→ Express $\tan^{-1}(x+iy)$ in the form of $A+iB$.

$A+iB = \tan^{-1}(x+iy)$

$\tan(A+iB) = x+iy$ — (1)

$\tan(A-iB) = x-iy$ — (2)

Adding eqⁿ (1) and (2)

$$\tan 2A = \tan\{(A+iB) + (A-iB)\}$$

$$= \frac{\tan(A+iB) + \tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-x^2-y^2}$$

$$A+iB = \tan^{-1}(x+iy)$$
$$A-iB = \tan^{-1}(x-iy)$$
$$2A = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$
$$2A = \tan^{-1}\frac{2x}{1-x^2-y^2}$$

$$A = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$$

$$\tan 2iB = \tan\{A+iB-(A-iB)\}$$
$$= \frac{\tan(A+iB) - \tan(A-iB)}{1+\tan(A+iB)\cdot\tan(A-iB)}$$

$$\tan 2iB = \frac{2iy}{1+(x+iy)(x-iy)}$$

$$\tan 2iB = \frac{2iy}{1+x^2+y^2}$$

$$\tan hx = -i \tan ix$$

$$i \tan hx = \tan ix$$

$$i \tan h 2B = \frac{2y}{1+x^2+y^2}$$

$$2B = \tan h^{-1} \frac{2y}{1+x^2+y^2}$$

$$B = \frac{1}{2} \tan h^{-1} \frac{2y}{1+x^2+y^2}$$

form of $A + iB$ $\left(\frac{1}{2} \tan^{-1} \frac{2x}{1-x^2-y^2} + 2i \frac{1}{2} \tan h^{-1} \frac{2y}{1+x^2+y^2} \right)$

Q) Separate $\cos^{-1}(x+iy)$ into real and imaginary part.

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Gregory's Series

Q) State and Prove gregory's series or to express θ in ascending power's of $\tan\theta$ or,

PT, $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{3}\tan^5\theta + \dots$ to ∞.

proof:- $\log(\cos\theta + i\sin\theta) = \log\{\cos\theta(1+i\tan\theta)\}$

$\rightarrow \log e^{i\theta} = \log\cos\theta + \log(1+i\tan\theta).$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots \infty$

$\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} \dots \infty$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$

$i\theta = \log + 1 - i\tan\theta - \frac{(i\tan\theta)^2}{2} + \frac{(i\tan\theta)^3}{3}$

$- \frac{(i\tan\theta)^4}{4} + \frac{(i\tan\theta)^5}{5}$

$= \log\cos\theta + \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \text{to } \infty = 0$

$= \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + \dots \infty = \theta$

— (1)

$-\log\cos\theta = \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \dots + \infty$

Q)

$$\log \sec\theta = \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \dots \infty \quad -(2)$$

Let $\tan\theta = x$

$\theta = \tan^{-1} x$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \text{ to } \infty$$

11) 8) Show that

$$\frac{\pi}{4} = \frac{2}{3} + \frac{1}{7} - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right)$$

$$\text{R.H.S :— } 2\left[\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} \dots\right) + \left[\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{7^5} + \frac{1}{5}\cdot\frac{1}{7^5}\right]$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} \dots$$

$$= \tan^{-1}\frac{2 - 1/3}{1 - 1/9} + \tan^{-1} 1/7$$

$$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} \quad \left[\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{(a+b)}{a - ab}\right]$$

put in formula then

$$\tan^{-1}\frac{25}{25} = \tan^{-1} 1 = \frac{\pi}{4} \text{ proved.}$$

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Show that

Q1 $\frac{\pi}{4} = \left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left[\frac{1}{2^5}+\frac{1}{3^5}\right]$

Q3 $\tan^{-1}\sec\theta = \frac{\pi}{4} + \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2}.$

Q2 $2\left[x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right] = \frac{2x}{1-x^2} - \frac{1}{3}\left[\frac{2x}{1-x^2}\right]^3 + \frac{1}{5}\left[\frac{2x}{1-x^2}\right]^5$

~~30~~-08-~~18~~

Q) Prove that under proper restriction?

$\tan^{-1}(\sec\theta) = \frac{\pi}{4} + \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \ldots$ to ∞.

→ from Gregory's Series, if

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Differential Eqⁿ

integration / differentiation

$y = mx + m^2$ { algebric Equation }

$\frac{dy}{dx} = m$

$\left[y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 \right]$ (differential eqⁿ)

Degree - 2
Order - 1

$x\frac{dy}{dx} + y = 0$

$x\,dy + y\,dx = 0$

$x\,dy = -y\,dx$

$\frac{dy}{y} = \frac{-dx}{x}$

1) Separation of Variables -

$\log y = -\log x + \log K$

$\log xy = \log K$

$xy = K$

8) $(1+y^2)\,ax + (1+x^2)\,dy = 0$

Q) $\frac{dy}{dx} = e^{x+y} + x^2 \cdot e^y$

Q) $y\,dx - x\,dy = xy\,dx$

Q) $(x+y)^2 \frac{dy}{dx} = a^2$

soln:- $(1+y^2)\,ax + (1+x^2)\,dy = 0$

$(1+y^2)\,dx = -(1+x^2)\,dy$

$\frac{dx}{(1+x^2)} = -\frac{dy}{(1+y^2}$

On intigerating both side

$\tan^{-1}x = -\tan^{-1}y + K$

$\tan^{-1}x + \tan^{-1}y = K$

$\tan^{-1}\frac{x+y}{1-xy} = K$

7-marks

2) Multiplying the equation with integrating factor.

$$\left[\frac{dy}{dx} + Py = Q\right]$$

P & Q function of x.

Multiplying the eqn with $e^{\int P dx}$

$$I.F = e^{\int P \cdot dx}$$

Q) $\frac{dy}{dx} + \frac{y}{x} = x^3$ | $P = 1/x$, $Q = x^3$

Soln:- $I.F = e^{\int 1/x \, dx} = e^{\log x} = x$

$$x\frac{dy}{dx} + y = x^4$$

$$= \frac{d}{dx}(x \cdot y) = x^4$$

$$d(x \cdot y) = x^4 dx$$

$$\int d(x \cdot y) = \int x^4 dx$$

$$xy = \frac{x^5}{5} + K$$

Q2) $\frac{dy}{dx} + y \cot x = 2\cos x$ | $P = \cot x$, $Q = 2\cos x$

Soln $I.F = e^{\int \cot x \, dx} = e^{\log \sin x}$

$= \sin x$

$\frac{dy}{dx} \sin x + y \cot x \sin x = 2\cos x \cdot \sin x$

$\sin x \frac{dy}{dx} + y . \cos x = \sin 2x$

$\frac{d}{dx}(y \sin x) = \sin 2x$

$d(y \sin x) = \sin 2x \, dx$

On Integrating both side ;

$\int dy \sin x = \int \sin 2x \, dx$

$y \sin x = \frac{\cos 2x}{2} + K$

1st september 2018

* Differential eqn of first order but not first degree.

$\left(\frac{dy}{dx}\right)^2 + 7\left[\frac{dy}{dx}\right] + 12 = 0$

(i) Factorising

(ii) Non- Factorising

(iii) clairavt's method.

(1) Let $\frac{dy}{dx} = P$ Factorising Method

$P^2 - 7P + 12 = 0$

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$P^2 - 3P - 4P + 12 = 0$

$P(P-3) - 4(P-3) = 0$

$P-3=0$ or $P-4=0$

$P=4$ or $P=4$

$\frac{dy}{dx} = 3$ or $\frac{dy}{dx} = 4$

$\frac{dy}{dx} = 3dx$ or $dy = 4dx$

$\boxed{y = 3x + K}$ $\boxed{y = 4x + K_2}$

$y - 3x - K_1 = 0$ $y - 4x - K_2 = 0$

Soln :—

H/W

Q) $P^2 - P(e^x - e^{-x}) + 1 = 0$

Q) $P^2 - 2xP - 3x^2 = 0$

Q) $P(P+x) = y(x+y)$ I.F

Q4) $P(P+x) = y(x+y)$

$P^2 + Px = xy + y^2$

$P^2 - y^2 + Px - xy = 0$

$(P-y)(P+y) + x(P-y) = 0$

$(P-y)(P+y+x) = 0$

$P=y$ or $P+y+x=0$

$P+y+x=0$

$\frac{dy}{dx} + y + x = 0$

$\frac{dy}{dx} + y = -x$

$I.F = e^{\int dx} = e^x$

multiplying the eq^n with e^x.

$$e^x \frac{dy}{dx} + e^x y = -x e^x$$

$$\frac{d}{dx}(e^x y) = -x e^x$$

$$e^x y = -\int x e^x dx$$

$$\int uv \, dx = 4$$

$$u = x \quad , \quad v = e^x$$

$$e^x y = x \int e^x dx - \left[\frac{dx}{dy}\left(\int e^x dx\right)\right] dx$$

$$e^x y + x e^x - e^x + K = 0$$

$$\boxed{e^x(x + y - 1) + K = 0}$$

5 $$\left[\frac{dy}{dx}\right]^2 + 2y \cot x \frac{dy}{dx} = y^2$$

6 $$xyp^2 - (x^2 - y^2) - xy = 0$$

7 $$p^2 - 9p + 18 = 0$$

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6/09/2018

Non factorising technique :-

$y = f(P, x)$

$\frac{dy}{dx} = \frac{d}{dx}[f(P, x)]$ $\left[\because \frac{dy}{dx} = P\right]$

$P = \frac{d}{dx}[f(P, x)]$

eqn solvable for y

$x = f(P, y)$ (∴ eqn solvable for x)

$\frac{dx}{dy} = \frac{d}{dy}[f(P, y)]$

$\frac{1}{P} = \frac{d}{dy}[f(P, y)]$

$y + Px = x^4 p^2$

→ $y + Px = x^4 p^2$ — (1)

$\frac{dy}{dx} + P + x\frac{dP}{dx} = 4x^3 P^2 + x^4 2P \frac{dP}{dx}$

$P + P + x\frac{dP}{dx} = 2Px^3\left(2P + x\frac{dP}{dx}\right)$

$$2P + x\frac{dP}{dx} = 2Px^3\left(2P + x\frac{dP}{dx}\right)$$

$$\rightarrow \left[2P + x\frac{dP}{dx}\right](2x^3P - 1) = 0$$

$$\rightarrow 2P + x\frac{dP}{dx} = 0$$

$$\rightarrow 2x^3P - 1 = 0$$

$$\rightarrow 2P = -x\frac{dP}{dx}$$

$$\frac{dx}{x} = -\frac{dP}{2P}$$

Integrating both side, we get.

$$\int \frac{dx}{x} = -\int \frac{dP}{2P}$$

$$\log|x| = -\frac{1}{2}\log|P| + \log|c|$$

$$\log|x| = \frac{-1}{2}\log|P| + \log|c|$$

$$\log|x| = -\log\sqrt{P} + \log|c|$$

$$\log x\sqrt{P} = \log c$$

$$c = x\sqrt{P}$$

Squaring we get.

$$c^2 = x^2 P$$

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$$P = \frac{c^2}{x^2} \quad — (2)$$

Put these value in eqn (1)

$$y + \frac{c^2}{x^2} \times x = x^4 \frac{c^4}{x^4}$$

$$\boxed{y + \frac{c^2}{x} = c^4}$$

$$x^2 \left(\frac{dy}{dx}\right)^2 + y(2x-y)\frac{dy}{dx} + y^2 = 0 \quad — (i)$$

$$x^2 p^2 + 2yp - y^2 p + y^2 = 0$$

$$(xp + y)^2 - y^2 p = 0$$

$$(xp + y)^2 = y^2 p$$

$$xp + y = y\sqrt{p}$$

$$xp + y(1 - \sqrt{p}) = 0$$

$$y(1 - \sqrt{p}) = -xp$$

$$y = \frac{-xp}{1 - \sqrt{p}}$$

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$$x = \frac{y\sqrt{p} - 1}{p}$$

$$x = y\left[\frac{1}{\sqrt{p}} - \frac{1}{p}\right]$$

$$\frac{dx}{dy} = y\left[\frac{-1}{2p^{3/2}} + \frac{1}{p^2}\right] + \left[\frac{1}{\sqrt{p}} - \frac{1}{p}\right]$$

$$\frac{1}{p} = y\left[\frac{-1}{2p^{3/2}} + \frac{1}{p^2}\right] + \left[\frac{1}{\sqrt{p}} - \frac{1}{p}\right]$$

$$\frac{2}{p} - \frac{1}{\sqrt{p}} = y\left(\frac{1}{p^2} - \frac{1}{2p^{3/2}}\right)$$

$$\frac{2\sqrt{p} - p}{p\sqrt{p}} = \frac{y}{2p}\left(\frac{2}{p} - \frac{1}{\sqrt{p}}\right)$$

$$\frac{2.\sqrt{p} - p}{p\sqrt{p}} = \frac{y}{2p}\left[\frac{2\sqrt{p} - p}{p\sqrt{p}}\right]$$

$$1 = \frac{y}{2p}$$

$$y = 2p$$

$$p = y/2 \quad - \text{(ii)}$$

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Putting these in eqn (i) we get

$$\frac{x^2y^2}{4} + 2xy \times \frac{y}{2} - y^2 \times \frac{y}{2} + y^2 = 0$$

$$\rightarrow \frac{x^2y^2}{4} + xy^2 - \frac{y^3}{2} + y^2 = 0.$$

$$\rightarrow \boxed{y^2\left(\frac{x^2}{4} + x - \frac{y}{2} + 1\right) = 0}$$

08/09/18

clairauts equations

$y = px + f(p)$ — (1), where $p = \frac{dy}{dx}$

$$y = px + \frac{1}{p}$$

differentiating eqn (1)

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$\frac{dp}{dx}(x+f'(p)) = 0$

$\frac{dp}{dx} = 0 \Rightarrow p = c$ —(2)

or $x + f'(p) = 0$ —(3)

Eliminating p from eqn (1) using eqn (2) to get general solution of this equation

$y = cx + f(c)$ is the general solution

(1) General Solution or complete primitive.

→ It contain arbitrary constant.

It is $y = cx + f(c)$.

Eliminating p from eqn (1) using eqn (3).

The new eqn form is called singular solution.

and singular solution doesn't contain arbitrary constant.

(Q) To find general and singular solution of the given eqn $y = px + \frac{a}{p}$

$y = px + \frac{a}{p}$

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(i) General solution, Put $p = c$

$$y = cx + \frac{a}{c} \quad \text{— (1)}$$

(ii) singular solution

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \left[\frac{a}{p^2}\right]\frac{dp}{dx}$$

$$p = p + \frac{dp}{dx}\left[x - \frac{a}{p^2}\right]$$

$$\frac{dp}{dx}\left[x - \frac{a}{p^2}\right] = 0$$

$$\frac{dp}{dx} = 0 \Rightarrow p = c \quad \text{— (2)}$$

$$x - \frac{a}{p^2} = 0$$

$$x = \frac{a}{p^2} \Rightarrow p = \sqrt{\frac{a}{x}} \quad \text{— (3)}$$

$f(p)$

3) $y = px + \sqrt{a^2p^2 + b^2}$ find the singular solution of the eqn.

$$x + f'(p) = 0$$

$$x = -f'(p)$$

$$x = -\frac{2a^2 p}{2\sqrt{a^2p^2 + b^2}} \quad \text{— (2)}$$

Differentiating eqn (1)

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{2a^2p}{2a^2p^2+b^2}\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2+b^2}}\right) = 0$$

$$\rightarrow \sqrt{a^2p^2+b^2} = \frac{a^2p}{x} \quad \text{— (3)}$$

using eqn (1), (2) and (3) we get eqn (4)

$$y = p\left[x - \frac{a^2}{x}\right] \quad \text{— (4)}$$

find the value of p using eqn (2)
Put the value of p in eqn (4).

(8) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

11/09/18

Reducing Into clairauth's Equation

$y = 2Px + y^2 p^3$

$\rightarrow$ Let $y^2 = Y$ and $x = X$

$2y\,dy = dY \Rightarrow dx = dX$

$$2y\frac{dy}{dx} = \frac{dY}{dX}$$

$$2y\frac{dy}{dx} = \frac{1}{2y}\frac{dY}{dX} \Rightarrow P$$

$$y = 2\cdot\frac{1}{2y}\, p\cdot x + y^2 \frac{1}{8y^3} p^3$$

$$y = \frac{P}{y}x + \frac{P^3}{8y}$$

$$= Px + \frac{P^3}{8y}$$

$$= Px + \frac{P^3}{8}$$

$$\left[Y = PX + \frac{P^3}{8} \right] \quad \text{--- (1)} \quad \text{clairaute form}$$

$$Y = PX + \frac{P^3}{8} \quad \text{--- (1)}$$

$$\frac{dY}{dX} = P + X\frac{dP}{dx} + \frac{3P^2}{8} \times \frac{dP}{dx}$$

$$x\frac{dp}{dx} + \frac{3p^2}{8} \cdot \frac{dp}{dx} = 0$$

$$\frac{dp}{dx}\left[x + \frac{3p^2}{8}\right] = 0$$

$$p \cdot c = \quad (2)$$

$$x + \frac{3p^2}{8} = 0 \quad - (3)$$

Eqn (1) and (2) is the general soln and eqn (1) and (3) is singular soln.

$$y = cx + \frac{c^3}{8} \quad \Big\} \text{ from } (1)$$

From eqn (3)

$$x = \frac{3p^2}{8}$$

$$x^2 = \left[\frac{3p^2}{5}\right]^2$$

$$x^2 = \frac{9p^4}{64}$$

$$\frac{64}{9}x^2 = p^4$$

$$yp = p^2 x + \frac{p^4}{8}$$

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$$= -\frac{8x}{3} \cdot x + \frac{\cancel{64}^{8}}{9} x^2 \cdot \frac{1}{\cancel{8}}$$

$$= (yp)^2 = \left[\frac{-8x^2}{3} + \frac{8x^2}{9}\right]^2$$

8) $x^2\left[y - x\frac{dy}{dx}\right] = y\frac{dy}{dx}$ P.g no - 111.

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