Inertial and gravity waves

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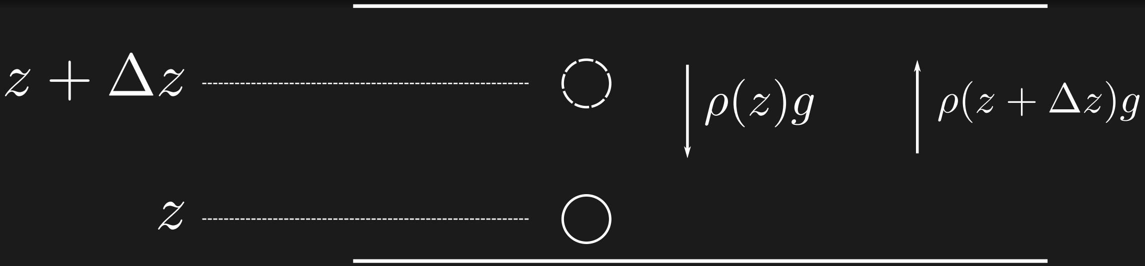
Waves vs Modes

- Waves are travelling, don't care about boundaries till they reflect. They are 'local' in nature.
- Modes are aware of boundaries and are 'global' in nature.

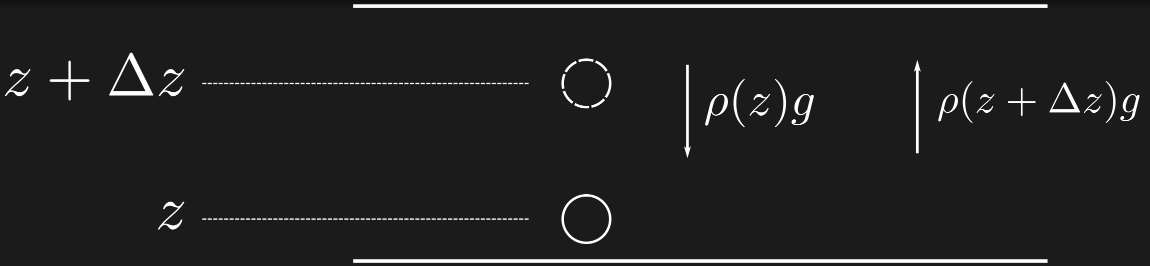
Types of waves in planetary fluid dynamics

Type of wave	Restoring force(s)
Acoustic (p-modes)	$-\nabla p$
Inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}$
Surface/Internal gravity (f-/g-modes)	$\rho' \boldsymbol{g}$
Inertia-gravity or gravito-inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g}$
Alfvén	$\boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetoacoustic	$-\nabla p + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magneto-Coriolis (MC)	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetic, Archimedean, Coriolis (MAC)	$-2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$

Internal gravity oscillations



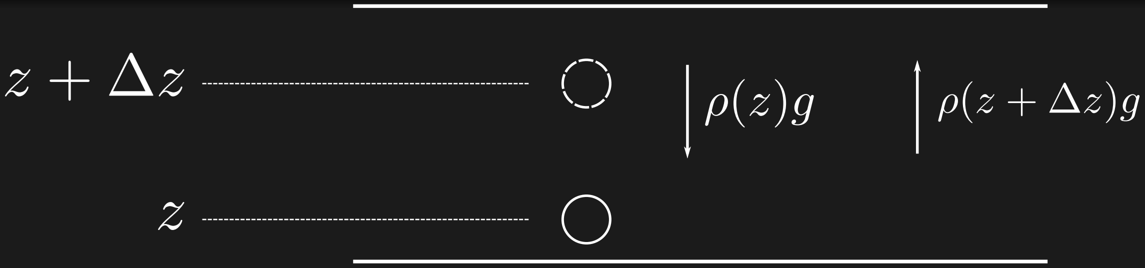
Internal gravity oscillations



Equation of motion of blob:

$$
\begin{aligned}
\rho_0 \frac{d^2}{dt^2} \Delta z &= \rho_0(z + \Delta z) g - \rho_0(z) g \\
&= \left(\rho_0(z) + \frac{d\rho_0}{dz} \Delta z \right) g - \rho_0(z) g
\end{aligned}
\tag{1}
$$

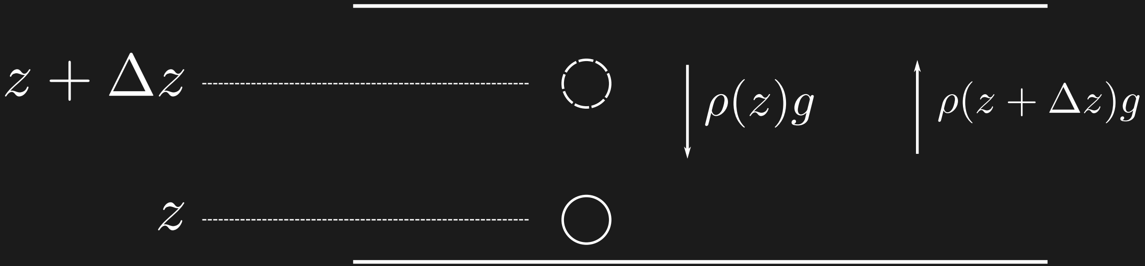
Internal gravity oscillations



$$\frac{d^2}{dt^2}\Delta z + N^2 \Delta z = 0 \tag{2}$$

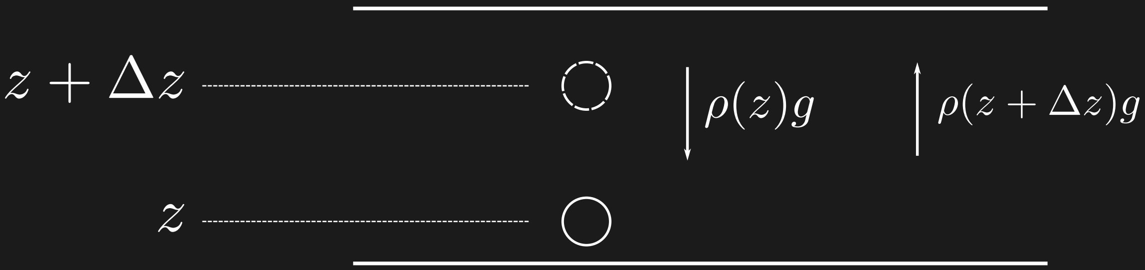
$$N = \sqrt{-\frac{g}{\rho_0}\frac{d\rho_0}{dz}} \tag{3}$$

Internal gravity oscillations



What happens when $N^2 > 0$ and vice versa?

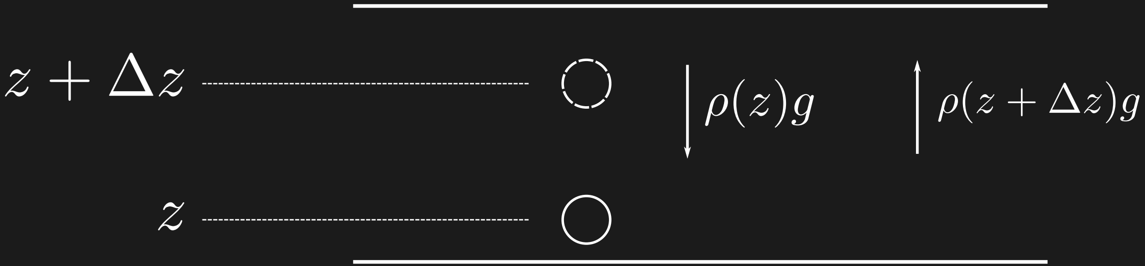
Internal gravity oscillations



What happens when $N^2 > 0$ and vice versa?

- $N^2 > 0$: Internal gravity waves
- $N^2 < 0$: Convective instability

Internal gravity oscillations



What happens when $N^2 > 0$ and vice versa?

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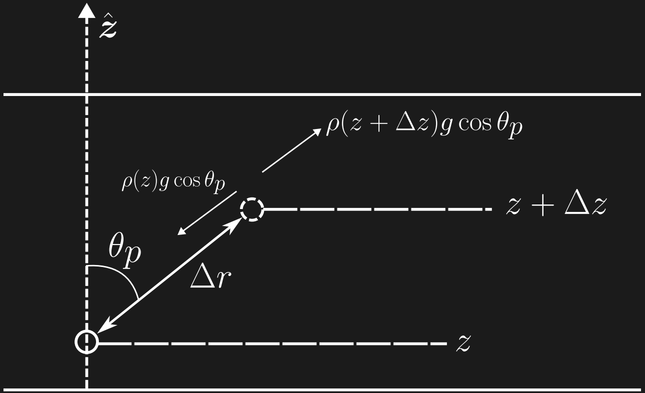
N is called the Brunt–Väisälä frequency or buoyancy frequency.

At an angle?


$\hat{z}$
$\rho(z+\Delta z)g\cos\theta_p$
$\rho(z)g\cos\theta_p$
$z+\Delta z$
θ_p
Δr
z

At an angle?

$$
\begin{aligned}
\rho_0 \frac{d^2}{dt^2}\Delta r &= (\rho_0(z+\Delta z)g - \rho_0(z)g)\cos\theta_p \\
&= \left[\left(\rho_0(z) + \frac{d\rho_0}{dz}\Delta z\right) g - \rho_0(z) g\right]\cos\theta_p \\
&= \left(\frac{d\rho_0}{dz}\Delta r \cos\theta_p\right)\cos\theta_p
\end{aligned}
\tag{4}
$$

Frequency of oscillation, $\omega = N\cos\theta_p$

When pressure variations are important

- In previous example, we assumed the parcel did not change its density when moving up.
- This is no longer true if there are significant pressure variations across height.
- Requires a simple change, replace density by potential density ρ_θ

$$N = \sqrt{-\frac{g}{\rho_\theta}\frac{d\rho_\theta}{dz}} = \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}} \tag{5}$$

When pressure variations are important

$$\theta = T\left(\frac{P_o}{P}\right)^{(R/C_p)}$$

Thus,

$$N = \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}} = \sqrt{g\left[\frac{1}{T}\frac{dT}{dz} - \frac{R}{C_p}\frac{1}{P}\frac{dP}{dz}\right]} \tag{6}$$

Schwarzschild criterion

For convection to occur:

$$
\begin{aligned}
& N^2 < 0 \\
\Rightarrow & g\left[\frac{1}{T}\frac{dT}{dz} - \frac{R}{C_p}\frac{1}{P}\frac{dP}{dz}\right] < 0 \\
\Rightarrow & \frac{P}{T}\frac{dT}{dP} < \frac{R}{C_p} \\
\Rightarrow & \frac{d\ln T}{d\ln P} < \left(\frac{d\ln T}{d\ln P}\right)_{ad}
\end{aligned}
\tag{7}
$$

Internal gravity waves

- Wave involves a coherent oscillation of a domain of parcels/particles
- Need a continuum analysis

Linearized equations of Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p - \frac{\rho'}{\rho_0}\hat{\boldsymbol{z}} \tag{9}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\rho_0}{dz} = 0 \tag{10}$$

Useful vector relations

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{11}$$

$$\begin{aligned}
\nabla \times \nabla \times (f\hat{\boldsymbol{z}}) &= (\dots)\hat{\boldsymbol{x}} + (\dots)\hat{\boldsymbol{y}} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f\hat{\boldsymbol{z}} \qquad (12)\\
&= (\dots)\hat{\boldsymbol{x}} + (\dots)\hat{\boldsymbol{y}} - \nabla_H^2 f\hat{\boldsymbol{z}}
\end{aligned}$$

Linearized equations of Boussinesq approximation

Taking double curl of equation (9) and only considering the z-component,

$$-\frac{\partial}{\partial t}\nabla^2 w = \frac{g}{\rho_0}\nabla_H^2 \rho' \tag{13}$$

Differentiating this wrt time and using equation (10), we get

$$\frac{\partial^2}{\partial t^2}\nabla^2 w = \frac{g}{\rho_0}\frac{dT_0}{dz}\nabla_H^2 w = -N^2\nabla_H^2 w \tag{14}$$

Dispersion relation

Assuming, $w = \hat{w}e^{i(k_x x + k_y y + k_z z - \omega t)}$, equation (14) becomes,

$$-\omega^2 k^2 = -N^2 k_h^2 \Rightarrow \omega^2 = N^2 \frac{k_h^2}{k^2} \tag{15}$$

where, $k^2 = k_x^2 + k_y^2 + k_z^2$ and $k_h = k_x^2 + k_y^2$. Thus,

$$\omega = \pm N \cos\theta \tag{16}$$

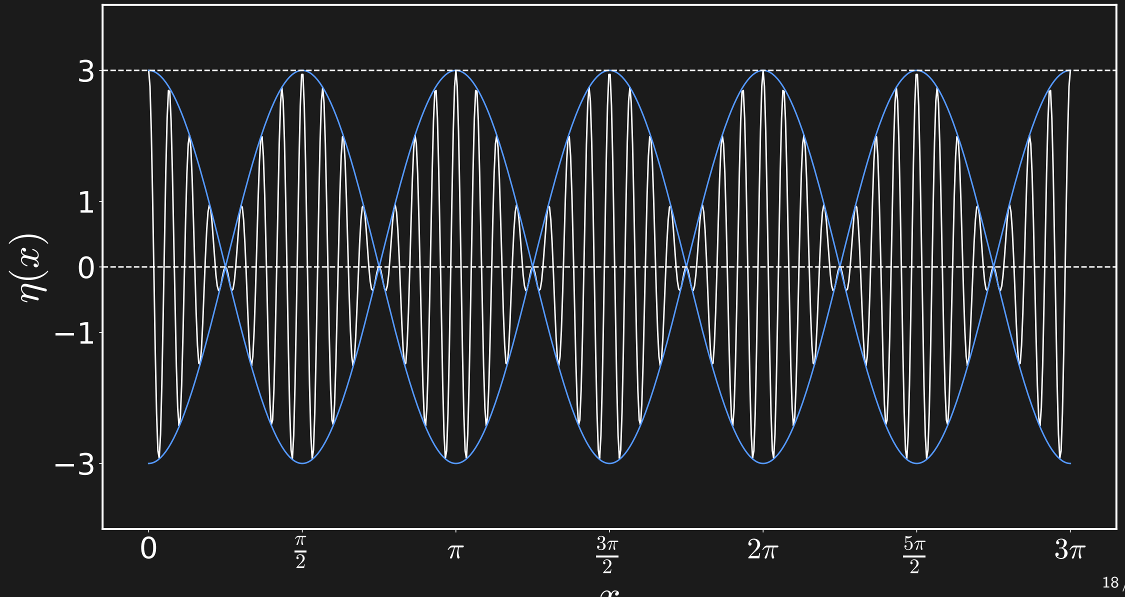
where, θ is the angle to the *horizontal*.

Particle motion

$$\nabla \cdot \boldsymbol{u} = 0 \Rightarrow \boldsymbol{k} \cdot \boldsymbol{u} = 0$$

Thus, particles move perpendicular to the wavevector. This can also be seen in the definitions of θ_p and θ .

Phase and Group velocities



Phase and Group velocities

$$c_{\text{phase}} = \frac{\omega}{k}\frac{\boldsymbol{k}}{k} = N\frac{k_h}{k^3}\boldsymbol{k} \tag{17}$$

$$\begin{aligned} c_{\text{group}} &= \nabla_{\boldsymbol{k}}\omega(\boldsymbol{k}) = \frac{d\omega}{dk_x}\hat{\boldsymbol{x}} + \frac{d\omega}{dk_y}\hat{\boldsymbol{y}} + \frac{d\omega}{dk_z}\hat{\boldsymbol{z}} \\ &= \frac{N}{k^3}\left[\frac{k_z^2}{k_h}(k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}) - k_z k_h\hat{\boldsymbol{z}}\right] \end{aligned} \tag{18}$$

$$c_{\text{phase}} \cdot c_{\text{group}} = \frac{N}{k^3}(k_z^2 k_h^2 - k_z^2 k_h^2) = 0$$

Movie

Movie
Movie 2

Inertial Oscillations

Inertial Oscillations

A fluid in solid body rotation is stratified in angular momentum.

Inertial Oscillations

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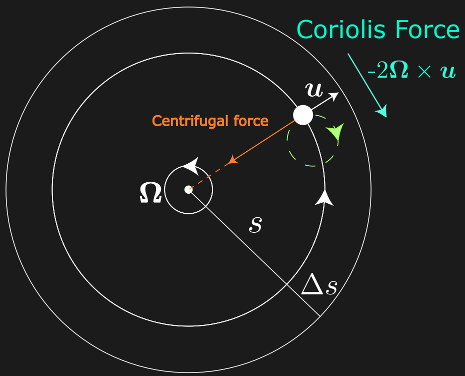

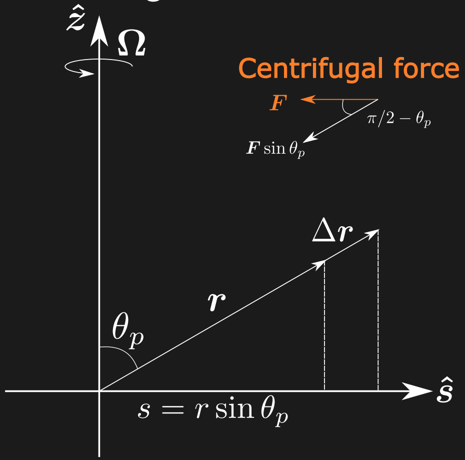

Inertial Oscillations

$$\ddot{\Delta r} + 4\Omega^2 \sin^2\theta_p \Delta r = 0$$

Oscillation frequency,

$$\omega = \pm 2\Omega \sin\theta_p$$

Inertial waves

Taylor-Proudman theorem:

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Inertial waves

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In the non-steady state

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} \tag{20}$$

Inertial waves

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Taking curl:

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{u}) = 2\Omega\frac{\partial \boldsymbol{u}}{\partial z} \tag{21}$$

Inertial waves

Taking curl again,

$$-\frac{\partial}{\partial t}\nabla^2\boldsymbol{u} = 2\Omega\frac{\partial}{\partial z}\nabla \times \boldsymbol{u}$$

Differentiating wrt times and using equation (21), we get

$$\frac{\partial^2}{\partial t^2}\nabla^2\boldsymbol{u} + 4\Omega^2\frac{\partial^2\boldsymbol{u}}{\partial z^2} = 0 \tag{22}$$

Inertial waves

Consider an ansatz, $\boldsymbol{u}, p \propto e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$. Substituting in the continuity equation,

$$\boldsymbol{u}\cdot\boldsymbol{k} = 0 \tag{23}$$

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Substituting in (22), we obtain

$$\omega^2 k^2 - 4\Omega^2 k_z^2 = 0 \Rightarrow \omega = \pm 2\Omega \frac{k_z}{k} = \pm 2\Omega\cos\theta \tag{24}$$

What do these physically mean?

Inertial waves

Consider an ansatz, $\boldsymbol{u}, p \propto e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$. Substituting in the continuity equation,

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- The velocity perturbations are perpendicular to the wave propagation direction $\Rightarrow$ transverse waves (like on a string).

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What do these physically mean?

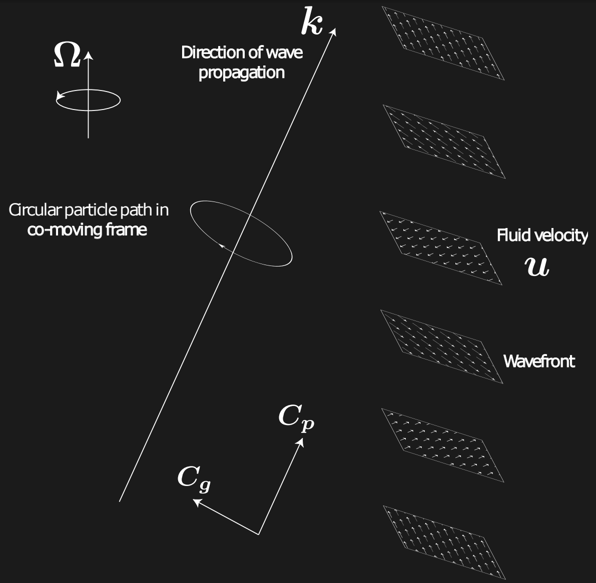
- The velocity perturbations are perpendicular to the wave propagation direction $\Rightarrow$ transverse waves (like on a string).
- The frequency of the waves lie between $\pm 2\Omega$.
- Frequency depends on the angle θ from rotation axis.

Inertial waves

$$c_{\text{phase}} = \frac{\omega(\boldsymbol{k})}{|\boldsymbol{k}|}\hat{\boldsymbol{k}} = \pm\frac{2\Omega\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{k}}}{|\boldsymbol{k}|}\hat{\boldsymbol{k}}$$

$$c_{\text{group}} = \nabla_{\boldsymbol{k}}\omega(\boldsymbol{k}) = \pm 2\Omega\frac{\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}}{|\boldsymbol{k}|}$$

Inertial waves



Inertial waves

Movie

Inertia-gravity waves

What happens when both forces are present?

Inertia-gravity waves

$$\nabla \cdot \boldsymbol{u} = 0 \tag{25}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p - \frac{\rho'}{\rho_0} g\hat{\boldsymbol{z}} - 2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{u} \tag{26}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\rho_0}{dz} = 0 \tag{27}$$

Inertia-gravity waves

Take curl of (26),

$$\frac{\partial}{\partial t}\nabla \times \boldsymbol{u} = -\nabla \times \frac{\rho'}{\rho_0} g\hat{\boldsymbol{z}} + 2\Omega\frac{\partial \boldsymbol{u}}{\partial z} \tag{28}$$

Taking the z-component gives,

$$\frac{\partial \omega_z}{\partial t} = 2\Omega\frac{\partial w}{\partial z} \tag{29}$$

Taking the curl of (28) and taking the z-component gives

$$-\frac{\partial}{\partial t}\nabla^2 w = \frac{g}{\rho_0}\nabla_H^2 \rho' + 2\Omega\frac{\partial \omega_z}{\partial z} \tag{30}$$

Inertia-gravity waves

Differentiating wrt time and using equations (27) and (29), we get

$$
\begin{aligned}
-\frac{\partial^2}{\partial t^2}\nabla^2 w &= -\frac{g}{\rho_0}\frac{d\rho_0}{dz}\nabla_H^2 w + 4\Omega^2\frac{\partial^2 w}{\partial z^2} \\
\Rightarrow -\frac{\partial^2}{\partial t^2}\nabla^2 w &= N^2\nabla_H^2 w + 4\Omega^2\frac{\partial^2 w}{\partial z^2}
\end{aligned} \tag{31}
$$

Inertia-gravity wave

Using the wave solution, $w = \hat{w}e^{i(k_x x + k_y y + k_z z - \omega t)}$, we get:

$$\omega^2 k^2 = N^2 k_h^2 + 4\Omega^2 k_z^2 \tag{32}$$

$$\omega^2 = N^2 \left(\frac{k_h}{k}\right)^2 + 4\Omega^2 \left(\frac{k_z}{k}\right)^2 \tag{33}$$

Inertia-gravity wave

When gravity and rotation are misaligned,

$$\omega^2 = N^2 \left(\frac{k_h}{k}\right)^2 + f^2 \left(\frac{k_z}{k}\right)^2 = N^2 \cos^2\theta + f^2 \sin^2\theta \tag{34}$$

where, $\theta =$ angle wrt horizontal.
This limits the range of ω to $f \leq \omega \leq N$.

Movie

Movie

Particle paths

Note that the continuity equation still gives $\boldsymbol{k} \cdot \boldsymbol{u} = 0$. Particles go in elliptical paths perpendicular to the wave propagation direction.

Phase and group velocities

$$c_{\text{phase}} = \frac{\omega}{k}\frac{\boldsymbol{k}}{k} = \frac{\omega}{k^2}\boldsymbol{k} \tag{35}$$

$$\begin{aligned} c_{\text{group}} &= \nabla_{\boldsymbol{k}}\omega(\boldsymbol{k}) = \frac{d\omega}{dk_x}\hat{\boldsymbol{x}} + \frac{d\omega}{dk_y}\hat{\boldsymbol{y}} + \frac{d\omega}{dk_z}\hat{\boldsymbol{z}} \\ &= \frac{N^2 - f^2}{\omega k^2}\left[k_z^2(k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}) - k_z k_h^2\hat{\boldsymbol{z}}\right] \end{aligned} \tag{36}$$

$$c_{\text{phase}} \cdot c_{\text{group}} = \frac{N^2 - f^2}{k^4}\left[k_z^2 k_h^2 - k_z^2 k_h^2\right] = 0 \; .$$