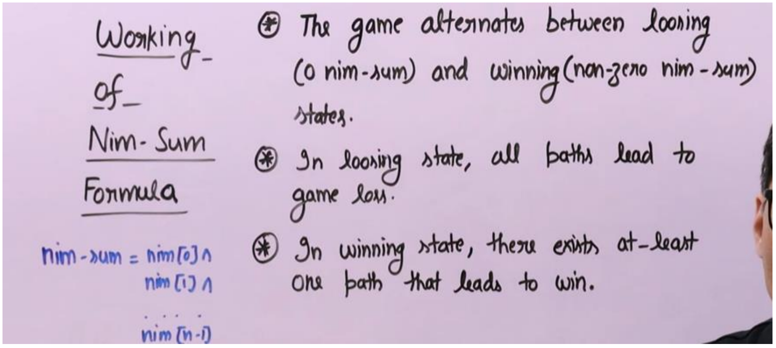
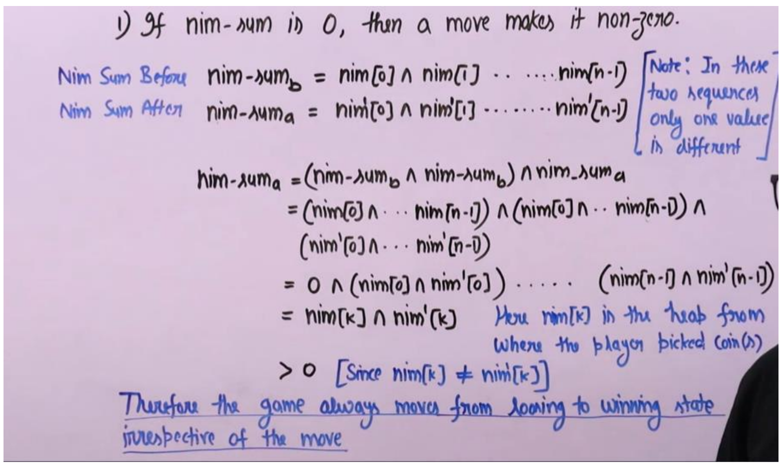
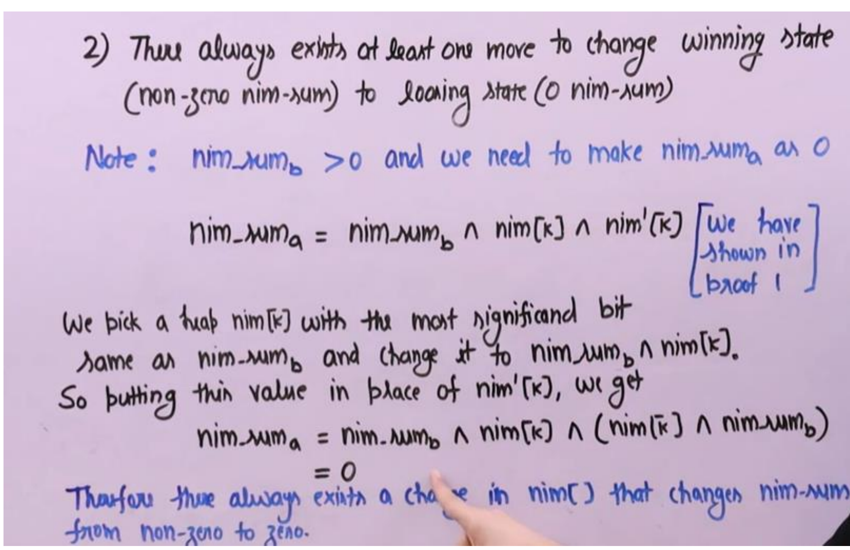
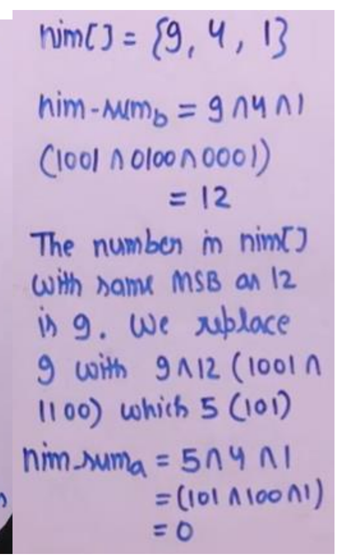
Working of Nim-sum Formula









Working of Sprauge Grundy Theorem

Working of Grundy Number Theorem

Grundy Number for a Composite Game:

$$g(x_1, x_2 \cdots x_n) = g_1(x_1) \wedge g_2(x_2) \cdots\cdots g_n(x_n)$$

Remember: Grundy Number of a Single Impartial Game

$$g(x) = \text{mex}(\{g\text{-values of all possible states after a move}\})$$

For example, in a single pile game

$$g(x) = \text{mex}(\{g(x - \text{pick}[0]), g(x - \text{pick}[1]), \cdots g(x - \text{pick}[\text{size}-1])\})$$

Working of Grundy Number Theorem

Grundy Number for a Composite Game:

$$g(x_1, x_2 \cdots x_n) = g_1(x_1) \wedge g_2(x_2) \cdots\cdots g_n(x_n)$$

Let $b = g_1(x_1) \wedge g_2(x_2) \cdots \wedge g_n(x_n)$

1) For every non-negative integer $a < b$, the value 'a' must exists among combined g-values after a move.

$$g(x_1, x_2 \cdots x_n) = \text{mex}(\{g(x_1', x_2 \cdots x_n), g(x_1'', x_2 \cdots x_n), \cdots$$
$$g(x_1, x_2', \cdots x_n), g(x_1, x_2'' \cdots x_n), \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$g(x_1, x_2 \cdots x_n'), g(x_1, x_2 \cdots x_n'') \cdots\})$$

2) The value b should not exist after a move.

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1) For every non-negative 'a' such that $a < b$, the value 'a' must exist among all possible g-values of the combined game. $b = g_1(x_1) \wedge g_2(x_2) \cdots g_n(x_n)$

Proof: There always exists a $g_k(x_k)$ such that if we replace it with the following $g_k(x'_k)$ we get the value 'a'.

$$g_k(x_k') = g_k(x_k) \wedge a \wedge b$$

The g-value of the combined game after the above move

$$= g_1(x_1) \wedge g_2(x_2) \cdots g_k(x_k') \cdots\cdots g_n(x_n)$$
$$= g_1(x_1) \wedge g_2(x_2) \cdots\cdots g_k(x_k) \wedge a \wedge b \cdots g_n(x_n)$$
$$= b \wedge a \wedge b$$
$$= a$$

100101	b
100001	a
000100	

How to find $g_k(x_k)$?

The leading bit in $a \wedge b$ must be set in b because $b > a$

Therefore there must exist a $g_k(x_k)$ with the same leading set bit. And we can reduce it to $g_k(x_k) \wedge a \wedge b$ because

$$g_k(x_k) \wedge a \wedge b < g_k(x_k)$$

② The value 'b' can not exist after a move

Let a move be $g_k(x_k)$ to $g_k(x_k')$

For the value 'b' to exist after a move

$$g_1(x_1) \wedge \cdots g_k(x_k) \cdots \wedge g_n(x_n) = g_1(x_1) \wedge \cdots g_k(x_k') \cdots \wedge g_n(x_n)$$

Which means

$$g_k(x_k) = g_k(x_k')$$

Which contradicts because g-value of a game cannot be same after a move.