CS 307-Optimization Algorithms and Techniques

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Syllabus

- Part I--Introduction
  - Introduction to Optimization
  - Math Foundations
- Part II-- Linear optimization
  - Linear Optimization
- Part III-- Non-linear optimization

Books

- An Introduction to Optimization: Foundations and Fundamental Algorithms, Niclas Andr ´easson, Anton Evgrafov, and Michael Patriksson, 2nd and 3rd ed.
- Convex optimization, Stephen Boyd and Lieven Vandenberghe, 1st ed., 2004

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Introduction to Optimization

- **What is it?**
  - Is a discipline that provides scientific methods for the purpose of solving real life problems that helps us in determining the best utilization of limited resources.
- Suppose we want to maximize the profit or minimize the cost then maximization of the profit or minimization of cost is the optimization of profit/cost.
- In a related field called **operation research (O.R.)**, we obtain the optimal solution for decision making problems with the help of optimization techniques.

A bit of History of OR

- OR came into existence during world war II in Britain.
- A group of scientists were called under a roof to solve strategic and tactical military operations and the field evolved.
- OR in India started with the establishment of regional research laboratory at Hyderabad around 1949.
- At the same time, one more unit was established at defence research laboratory.
- In 1955, operation research society of India was formed.

Applications

- In Agriculture--
- In Defence Operations--
- In Finance--
- In Marketing--
- In Insurance--

Important Definitions

- **Solution-** A set of values of the decision variables which satisfy the constraints of the given LPP is said to be a solution of that LPP.
- **Feasible Solution-** A solution in which values of decision variables satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as feasible solution.
- **Infeasible Solution**- A solution in which values of decision variables do not satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as infeasible solution.
- **Basic solution-** Suppose there are m equations representing constraints (limited available resources) containing m + n variables in an allocation problem. The solution obtained by setting any n variables equal to zero and solving for the remaining m variables and the remaining n variables are non – basic variables.

  The maximum number of possible basic solutions is given by the formula

$$C_n^{m+n}$$

Contd...

- **Basic Feasible Solution**--A basic solution for which all the basic variables are non – negative is called the basic feasible solution.
  - **Degenerate Solution**-A basic feasible solution is known as degenerate if value of at least one basic variable is zero.
  - **Non-Degenerate Solution**- A basic feasible solution is known as non- degenerate if value of all basic variables are non-zero and positive.
- **Optimum Basic Feasible Solution**- A basic feasible solution which optimizes i.e. maximise or minimise the objective function value of the given LPP is called optimum basic feasible solution.
- **Unbounded Solution**- A solution in which value of the objective function of the given LPP increase/decrease indefinitely is called an unbounded solution

Ex.

- Example. Determine all the basic feasible solutions of the equations:

$$2\mathrm{x}_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6\mathrm{x}_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Ex.

- Example. Determine all the basic feasible solutions of the equations:

$$2\mathrm{x}_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6\mathrm{x}_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

- Sol: max possible sol. 6. BFS- (0,1/2,0,0)

Modeling of Optimization Problems

- Ex. **(a staff planning problem)** Consider a hospital ward which operates 24 hours a day. At different times of day, the staff requirement differs. Table below shows the demand for reserve wardens during six work shifts. Each member of staff works in 8 hour shifts. The goal is to fulfill the demand with the least total number of reserve wardens.

Shift	1	2	3	4	5	6
Hours	0–4	4–8	8–12	12–16	16–20	20–24
Demand	8	10	12	10	8	6

Understanding the Modeling

- To optimize = to do something as well as is possible
- **To do something**:
  - We identify activities which we can control and influence **(variables)**. The remaining quantities are **constants** in the problem.

Understanding the Modeling

- **As well as:**
  - How good a vector of activity levels is measured by a real-valued function of the variable values.
  - This quantity is to be given a highest or lowest value, that is, we minimize or maximize, depending on our goal; this defines **the objective function**

Understanding the Modeling

- **Is possible:**
  - Normally, the activity levels cannot be arbitrarily large, since an activity often is associated with the utilization of resources (time, money, raw materials, labour, etcetera) that are limited.
  - These restrictions on activities form **constraints** on the possible choices of the variable values.

Three Components

- **Variables:**

  – x_j := number of reserve wardens whose first shift is j = 1, 2, . . . , 6.

- **Objective function:** Wish to minimize the total number of reserve wardens, that is, the objective function, which we call f, is to

$$\text{mimimize } f(x) := x_1 + x_2 + \ldots + x_6 = \sum_{j=1}^{6} x_j$$

- **Constraints:** Two types

  – **Demand:** Can be written as an inequality constraints like

$$x_6 + x_1 \geq 8$$

  – **Logical:** Sign $x_j \geq 0$, x_j Integer, j = 1, 2, . . . , 6.

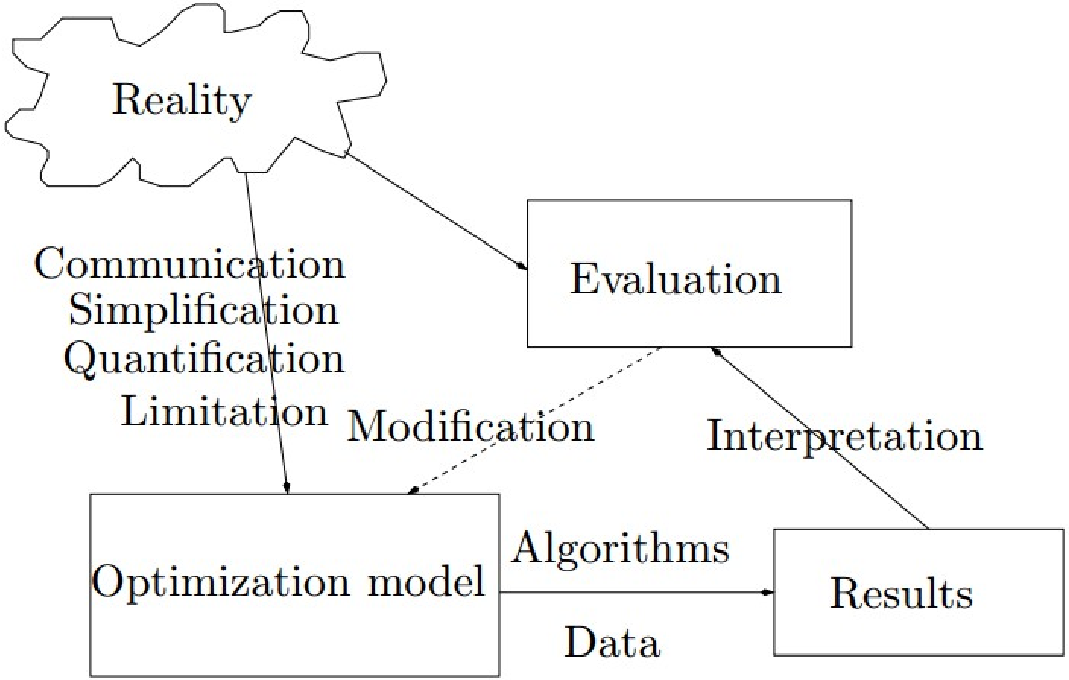
Summarizing the Optimization Problem

$$
\begin{aligned}
&\underset{x}{\text{minimize}} f(\boldsymbol{x}) := \textstyle\sum_{j=1}^{6} x_j, \\
&\text{subject to } x_1 + x_6 \geq 8, \qquad \text{(last shift: 1)} \\
&x_1 + x_2 \geq 10, \qquad \text{(last shift: 2)} \\
&x_2 + x_3 \geq 12, \qquad \text{(last shift: 3)} \\
&x_3 + x_4 \geq 10, \qquad \text{(last shift: 4)} \\
&x_4 + x_5 \geq 8, \qquad \text{(last shift: 5)} \\
&x_5 + x_6 \geq 6, \qquad \text{(last shift: 6)} \\
&x_j \geq 0, \quad j = 1, \ldots, 6, \\
&x_j \text{ integer}, \quad j = 1, \ldots, 6.
\end{aligned}
$$

Success Story and Real-life Examples

- (applications of staffing optimization problems):
  - It has been reported that a 1990 staffing problem application for the Montreal municipality bus company, employing 3,000 bus drivers and 1,000 metro drivers and ticket salespersons and guards, saved some 4 million Canadian dollars per year.
  - In an application from 1986, scientists collaborating with United Airlines considered their crew scheduling problem and helped save $6M.

Flow Chart of Modeling Process



Classification of Optimization Models

$$\boldsymbol{x} \in \mathbb{R}^n: \text{ vector of decision variables } x_j, \quad j = 1, 2, \ldots, n;$$

$$f: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}: \text{ objective function;}$$

$$X \subseteq \mathbb{R}^n: \text{ ground set defined logically/physically;}$$

$$g_i: \mathbb{R}^n \to \mathbb{R}: \text{ constraint function defining restriction on } \boldsymbol{x}:$$

$$g_i(\boldsymbol{x}) \geq b_i, \quad i \in \mathcal{I}; \qquad \text{(inequality constraints)}$$

$$g_i(\boldsymbol{x}) = d_i, \quad i \in \mathcal{E}. \qquad \text{(equality constraints)}$$

We let $\mathrm{b}_i \in \mathbb{R} (i \in \mathcal{I})$ and $d_i \in \mathbb{R} (i \in \mathcal{E})$

Contd...

- The optimization problem is

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}), \\ \text{subject to} & g_i(\boldsymbol{x}) \geq b_i, \quad i \in \mathcal{I} \\ & g_i(\boldsymbol{x}) = d_i, \\ & \boldsymbol{x} \in X \end{array}$$

The problem type depends on the nature of the functions f and g_i, and the set X.

Contd...

- **Linear Programming (LP):** objective function $f(x)$ is linear and constrains $g_i(x)$ are affine.
- **Nonlinear programming (NLP):** Some function(s) f, g_i $(i \in I \cup E)$ are nonlinear
- **Continuous optimization:** f, g_i $(i \in I \cup E)$ are continuous on an open set containing X; X is closed and convex.
- **Integer programming (IP):** $X \subseteq \{0, 1\}^n$ (binary) or $X \subseteq Z^n$ (integer).

Contd...

- **Unconstrained optimization:** $I \cup E = \emptyset$; $X = \mathbb{R}^n$
- **Constrained optimization:** $I \cup E \neq \emptyset$ and/or $X \subset \mathbb{R}^n$
- **Differentiable optimization** f, gi ($i \in I \cup E$) are at least once continuously differentiable on an open set containing X (that is, "in C^1 on X," which means that ∇f and ∇g_i ($i \in I \cup E$) exist there and the gradients are continuous); further, X is closed and convex.

Contd...

- **Non-differentiable optimization:** At least one of f, g_i ($i \in I \cup E$) is non-differentiable.
- **(CP) Convex programming:** f is convex; g_i ($i \in I$) are concave; g_i ($i \in E$) are affine; and X is closed and convex.
- **Non-convex programming:** The complement of the above

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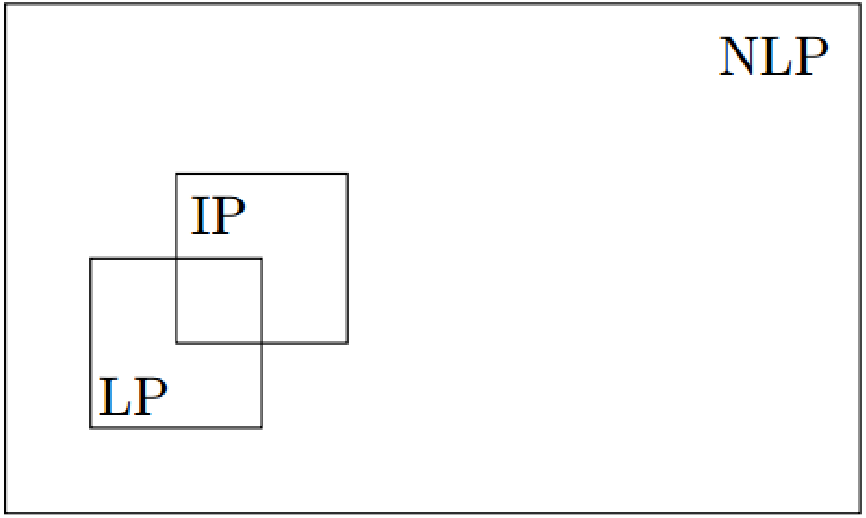

Relation between LP, NLP, and IP

Some Extra Stuff

- **Infinite-dimensional problems:** (that is, problems formulated in functional spaces rather than vector spaces);
- **Implicit functions f and/or gi (i ∈ I ∪E):** then, no explicit formula can then be written down; this is typical in engineering applications, where the value of, say, f(x) can be the result of a simulation;
- **Multiple-objective optimization:** "minimize {f1(x), f2(x), . . . , fp(x)}";
- **Optimization under uncertainty, or, stochastic programming:** (that
- is, where some of f, gi (i ∈ I∪E) are only known probabilistically).