Probability Theory

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Probability is the meause of how likely an event will occur.

Example:- We may model winning outcomes of on election for a party as if they are random when they are result of many components such as age, qualification, religion, gender etc. of the votters

Probability in this case would be the long run proportion of time winning outcome in repeated independent trial

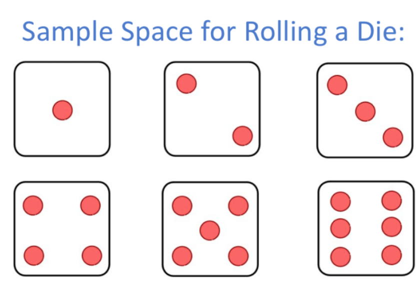
Probability Terminology:- Let us consider an example of rolling a die.

Sample space:- Sample space of an experiment consists of the set of all possible outcomes.

In the case of rolling a die, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Sample Space for Rolling a Die:



6 outcomes

The integers 1 - - - 6 represent the number of dots on the six faces of the die.

Sample points:- The six possible outcomes are the sample points of the experiment.

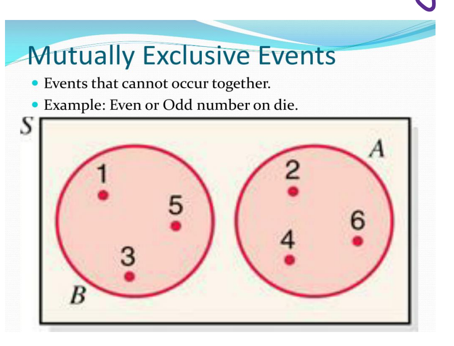
Event:- An event is a subset of 'S' and may consider any number of sample points.
For example, an event A is defined as

$$A = \{2, 4\}$$

The compliment of an event A, denoted by $\bar{A}$, consists of all the sample points in 'S' that are not in A. Thus,

$$\bar{A} = \{1, 3, 5, 6\}$$

Mutually Exclusive events:- Two events are said to be mutually exclusive if they have no sample points in common.



Union of events:- The union (sum) of two events is an event that consists all the sample points in the two events. For the events A and B above

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$

The above implies $A \cup \bar{A} = S$ (sample space)

Intersection of events:- Intersection two events is an event that consists of the points that are common to the two events.

If $A = \{2, 4\}$, $B = \{1, 2, 3\}$

$$A \cap B = \{2\}$$

For mutually exclusive events (see fig)

$$A \cap B = \phi \text{ (Null event)}$$

Thus, $A \cap \bar{A} = \phi$

The concept of Union and Intersection can be extented to more than two events in a straightforward manner.

Probability rules:- Associated with each event A in S has probability of occurrence P(A), which is defined as

$$\boxed{P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}}$$

① The probability of an event 'A' satisfies the condition

$$\boxed{P(A) \geq 0}$$

② The probability of the sample space 'S'

$$\boxed{P(S) = 1}$$

① and ② imply that $\boxed{0 \leq P(A) \leq 1}$

③ Let A_i for $i = 1, 2, \ldots$ are mutually exclusive events in 'S', that is

$$A_i \cap A_j = \phi, \quad i \neq j = 1, 2, \ldots$$

Then, the probability of union of these mutually exclusive events satisfies the condition

$$\boxed{P\left(\bigcup_i A_i\right) = \sum_i P(A_i)}$$

Example:- Let in the experiment of rolling a die each outcome is equiprobable (die is fair)

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Let us consider an event $A = \{2, 4\}$. Since the event A consists of two mutually exclusive subevents or outcomes

$$P(A) = P(2) + P(4) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Consider another event $B = \{1, 3, 6\}$

$$P(B) = P(1) + P(3) + P(4) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ because A and B are mutually exclusive.

$$P(S) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

Conditional probability:- It is often required to find the probability of an event B under the condition that an event A has occurred.

This prob is called the conditional probability of B given A and is denoted as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad P(A) \neq 0$$

Similarly

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0$$

Multiplication rule:- If A and B are events in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$, then

$$\boxed{P(A \cap B) = P(A)\, P(B|A) = P(B)\, P(A|B)}$$

Independent Events:- If events A and B are such that $P(A \cap B) = P(A)\,P(B)$, they are called independent events.

Assuming $P(A) \neq 0$, $P(B) \neq 0$, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

Similarly $P(B|A) = P(B)$

This implies that the prob of A does not depend on the occurrence or non-occurrence B. This justifies the term INDEPENDENT.

The concept can be extented for more than two events.

Example:- Consider an experiment of tossing a fair die. Let two events $A = \{1, 3, 6\}$, $B = \{1, 2, 3\}$
Find the prob $P(B|A)$

$$P(B|A) = \frac{P(A \cap B)}{P(A)},$$

The event $A \cap B = \{1, 3\} \Rightarrow P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

$$P(A) = 3/6 = 1/2$$

Thus,

$$\boxed{P(B|A) = \frac{1/3}{1/2} = 2/3}$$