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On the transcendentality condition for Gaussian Gabor frames

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ABSTRACT. We give a criterion for higher-dimensional Gaussian Gabor frames, which is a reformulation of one of the main results in [4, Thm 1.1] in more explicit terms. We also show that this density criterion for Gaussian Gabor frames is generic in a certain sense.

1. INTRODUCTION

Let us recall the setting of [4]. We take a Gaussian $g_\Omega(t) := \overline{e^{\pi i t^T \Omega t}}$, where $\Omega \in \mathfrak{h}$ is an element of the Siegel upper half space

$$\mathfrak{h} := \{\Omega \in \mathfrak{gl}(n, \mathbb{C}) : \Omega = \Omega^T, \ \mathrm{Im}\,\Omega \text{ is positive definite}\},$$

a lattice $\Gamma \subset \mathbb{R}^{2n}$, and associate to these the Gaussian Gabor system $\{\pi_\lambda g_\Omega\}_{\lambda\in\Lambda}$ as

$$\pi_\lambda g_\Omega(t) := e^{2\pi i \xi^T t} g_\Omega(t - x), \quad \lambda = (\xi, x) \in \Lambda, \quad \xi^T t := \sum_{j=1}^{n} \xi_j t_j.$$

Then $\{\pi_\lambda g_\Omega\}_{\lambda\in\Lambda}$ is called a Gaussian Gabor frame if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} |(f, \pi_\lambda g_\Omega)|^2 \leq B\|f\|^2 \quad \text{for } f \in L^2(\mathbb{R}^n),$$

where$(\cdot,\cdot)$ is the L^2 inner product.

Multivariate Gaussian Gabor frames have recently been investigated by the first and the last author of this note [4, Thm. 1.1], where a density condition for a certain class of Gaussian Gabor frames is formulated for transcendental lattices Λ in $\mathbb{R}^{2n}$. Recall that a lattice Γ in $\mathbb{C}^n$ is said to be transcendental if the complex torus $X := \mathbb{C}^n/\Gamma$ is Campana simple, i.e. the only positive dimensional analytic subvariety of X is X itself.

In [5] density conditions for bivariate Gaussian Gabor frames for product lattices are established by relating Gabor systems as elements of a quasi-shift invariant space of functions. An almost complete description of lattices that have Gaussian Gabor frames is given in [5], except for lattices given by an irrational rotation.

By our reformulation of [4, Thm. 1.1] in purely combinatorial terms for the lattice we are able to close this gap in [5] and we will also demonstrate that this density criterion is a generic property.

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2. A Density Criterion for Gaussian Gabor Frames

We want to give an equivalent condition on lattices $\Lambda \subseteq \mathbb{R}^{2n}$ that give Gaussian Gabor frames (g_Ω, Λ). By Proposition 1.4 in [4], it suffices to look at the $\Omega = iI$ case, i.e.

$$g_\Omega(t) = e^{-\pi|t|^2}.$$

In this case, by the Bargmann transform, we know that $(e^{-\pi|t|^2}, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^n)$ if and only if

$$\Lambda_{\mathbb{C}} := \{\xi + ix \in \mathbb{C}^n : (\xi, x) \in \Lambda\}$$

defines a frame for the classical Bargmann-Fock space. Hence, by [3, Prop. 2.1] we have the following fact:

Proposition 2.1. *$(e^{-\pi|t|^2}, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^n)$ if and only if $\Lambda_{\mathbb{C}}$ is a set of uniqueness for $\mathcal{F}^\infty$, i.e. if for every holomorphic function F on $\mathbb{C}^n$ we have*

$$\sup_{z\in\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2} \leq 1 \ \textit{and} \ F|_{\Lambda_{\mathbb{C}}} = 0,$$

then $F \equiv 0$ on $\mathbb{C}^n$.

On the other hand, one may use Kähler geometry methods to prove the following result.

Proposition 2.2. *If $\Lambda_{\mathbb{C}}$ is not a set of uniqueness for $\mathcal{F}^\infty$ and $\mathbb{C}^n/\Lambda_{\mathbb{C}}$ has no analytic subvariety of dimension $1 \leq d < n$ then $|\Lambda| \geq 1/n!$.*

Proof. If $\Lambda_{\mathbb{C}}$ is not a set of uniqueness for $\mathcal{F}^\infty$, then the following ω-pluri-subharmonic function (PSH)

$$G(z) := \sup{}^* \left\{ \log(|F(z)|^2 e^{-\pi|z|^2}) : F \in \mathcal{O}(\mathbb{C}^n), \sup_{z\in\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2} = 1, \ F|_{\Lambda_{\mathbb{C}}} = 0 \right\}$$

is not identically equal to $-\infty$, where $\mathcal{O}(\mathbb{C}^n)$ denotes the space of holomorphic functions on $\mathbb{C}^n$, $\sup^*$ denotes the upper semicontinuous regularization of the supremum and by ω-PSH we mean that G is an upper semicontinuous function with $dd^c G + \omega \geq 0$, and ω is the standard Kähler form on $\mathbb{C}^n$.

Note that G is $\Lambda_{\mathbb{C}}$-invariant.

To see this, consider a point $z \in \mathbb{C}^n$, a lattice point $\lambda \in \Lambda_{\mathbb{C}}$, and a holomorphic function F which is a suitable candidate for G. Then we have

$$|F(z+\lambda)|^2 e^{-\pi|z+\lambda|^2} = |F(z+\lambda) e^{-\pi z\overline{\lambda} - \frac{\pi|\lambda|^2}{2}}|^2 e^{-\pi|z|^2}.$$

Let us define

$$\tilde{F}(z) := F(z+\lambda) e^{-\pi z\overline{\lambda} - \frac{\pi|\lambda|^2}{2}},$$

we have that $\tilde{F}$ is also a candidate for G. This means that $G(z)$ is no less than $G(z+\lambda)$, implying $\Lambda_{\mathbb{C}}$-invariance.

As such, we now consider G as a function on the torus $\mathbb{C}^n/\Lambda_{\mathbb{C}}$. We clearly have that $dd^c G(z)$ is bounded from below by $-\omega$, meaning we may use the approximation result of Demailly for quasi-psh functions ([2, Proposition 3.7]) to find a sequence of quasi-PSH functions $(G_m)_m$ which converge to G, such that $G_m > G$, have only analytic singularities, and fulfill the estimate:

$$\nu_x(G) - \frac{n}{m} \leq \nu_x(G_m) \leq \nu_x(G),\ x \in \mathbb{C}^n/\Lambda_{\mathbb{C}}$$

for the Lelong numbers of G and G_m. We have the estimate for G_m

$$dd^c G_m \geq -(1+\varepsilon_m)\omega,$$

for some sequence of numbers $(\varepsilon_m)_m$ which decreases to 0.

Assume now that $\mathbb{C}^n/\Lambda_{\mathbb{C}}$ has no analytic subvariety of dimension $1 \leq d < n$. This then implies that the singularities of the G_m must be isolated. Thus we can bound the Seshadri constant of $\mathbb{C}^n/\Lambda_{\mathbb{C}}$ from below by the Lelong numbers of $\frac{G_m}{1+\varepsilon_m}$ at the point corresponding to the origin. Since these are bounded from below by $\frac{1-(n/m)}{1+\varepsilon_m}$, taking the limit as m approaches infinity, we have that the Seshadri constant of $\mathbb{C}^n/\Lambda_{\mathbb{C}}$ is no less than one. Demailly's mass concentration trick also implies that (see [7, Theorem 2.8]) this bound on the Seshadri constant is equivalent to $|\Lambda| \geq 1/n!$. □

To check that $\mathbb{C}^n/\Lambda_{\mathbb{C}}$ has no analytic subvariety of dimension $1 \leq d < n$ we need the following result (the proof follows directly from the argument in page 164-165 in [6, pp.164-165]).

Lemma 2.3. *Let Γ be a lattice in $\mathbb{C}^n$. Assume that the following linear mapping from singular homology to functionals on Dolbeault cohomology*

$$\mathrm{int}_\Gamma : \oplus_{1\leq k<n} H_{2k}(\mathbb{C}^n/\Gamma, \mathbb{Z}) \to \oplus_{p+q=2k,\, 1\leq k<n,\, p>q\geq 0} (H^{p,q}(\mathbb{C}^n/\Gamma, \mathbb{C}))^* \tag{2.1}$$

defined by

$$\mathrm{int}_\Gamma(S)(\alpha) := \int_S \alpha$$

is injective. Then $\mathbb{C}^n/\Gamma$ has no analytic subvariety of dimension $1 \leq d < n$.

Let us give a more elementary description of this condition. The lattice Γ is given by

$$\Gamma = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_{2n},$$

where $e_1, \ldots, e_{2n} \in \mathbb{C}^n$ are vectors which are linearly independent over $\mathbb{R}$. In this case, the homology groups $H_{2k}(\mathbb{C}^n/\Gamma, \mathbb{Z})$ have bases over $\mathbb{Z}$ given by

$$S_i := \pi(\mathbb{R}e_{i_1} + \ldots + \mathbb{R}e_{i_{2k}}),$$

for the multi-index $i = (i_1, \ldots, i_{2k}) \in \mathbb{N}^{2k}$ with $i_j < i_l$ for $j < l$. (For the sake of convenience we will denote the set of such indices by I_k). Observe that the map π is a projection from $\mathbb{C}^n$ onto the complex torus X. As such, we may now write any cycle $S \in H_{2k}(\mathbb{C}^n/\Gamma, \mathbb{Z})$ as

$$S = \sum_{i\in I_k} a_i S_i,$$

for some integers a_i. This fact should also help explain the singular homology groups on the Torus for those who are unfamiliar: $H_{2k}(\mathbb{C}^n/\Gamma, \mathbb{Z})$ can be thought of as the free Abelian group over the closed "curves" of real dimension $2k$ (modulo some equivalence relation which will not be relevant in what follows). Now, for a (p,q)-form α, where $p+q=k$ and $p>q\geq 0$, we have that

$$\mathrm{int}_\Gamma(S)(\alpha)=\sum_{i\in I_k} a_i\int_{S_i}\alpha.$$

Hence the injectivity of $\mathrm{int}_\Gamma(S)$ is equivalent to the following statement:

If for every closed (p,q) form α, where $p+q=k$, $p>q\geq 0$ $\mathrm{int}_\Gamma(S)(\alpha)=0$, the numbers a_i must all be 0. The reader should of course note that this condition must also be checked for every k less than n. Furthermore, by linearity it suffices to check this condition on a a basis of $H^{p,q}(\mathbb{C}^n/\Gamma,\mathbb{C})$.

We now turn to the case where $n=2$. Hence we need only to check the case $k=1$, which also forces $p=2$ and $q=0$. Additionally, the $(2,0)$ forms are spanned by the single form induced by $dz_1\wedge dz_2$. As such, the condition that $\mathrm{int}_\Gamma(S)(\alpha)$ is injective can be expressed as the condition that the numbers

$$\int_{S_i}\pi_*(dz_1\wedge dz_2)$$

are linearly independent over $\mathbb{Z}$. Explicitly computing these integrals gives the following result:

Remark. *In case $n=2$ and Γ generated by $e_j=(\alpha_j,\beta_j)$, $1\leq j\leq 4$, over $\mathbb{Z}$, then* [6, p.165] *tells us that* $\ker\mathrm{int}_\Gamma=0$ *if and only if*

$$\alpha_j\beta_k-\alpha_k\beta_j, 1\leq j<k\leq 4,$$

are linearly independent over $\mathbb{Z}$.

For $n>2$, we have way more conditions. When we examine the case $k>1$, we must not only look at the group $H^{2k,0}(\mathbb{C}^2/\Gamma,\mathbb{C})$, but also the groups $H^{2k-1,1}(\mathbb{C}^2/\Gamma,\mathbb{C})$ and so on. The group $H^{p,q}(\mathbb{C}^2/\Gamma,\mathbb{C})$ has a basis given by the forms

$$\pi_*(dz_P\wedge d\overline{z}_Q),\ \ P\in\mathbb{N}^p, Q\in\mathbb{N}^q,$$

when we take P and Q to be ascending. We can compute the integrals

$$\int_{S_i}\pi_*(dz_P\wedge d\overline{z}_Q)$$

in a manner similar to the one which gives us the statement in the previous remark, and see that these are the numbers

$$C^i_{P,Q}:=\det\begin{pmatrix} e_{i_1,P_1} & \dots & e_{i_1,P_p} & \overline{e}_{i_1,Q_1} & \dots & \overline{e}_{i_1,Q_q}\\ . & & & & & .\\ . & & & & & .\\ . & & & & & .\\ e_{i_{2k},P_1} & \dots & e_{i_{2k},P_p} & \overline{e}_{i_{2k},Q_1} & \dots & \overline{e}_{i_{2k},Q_q}\end{pmatrix}.$$

We may now reformulate our criterion for the nonexistence of subvarieties of the complex torus in the following way.

Lemma 2.4. *We have* $\ker \operatorname{int}_\Gamma = 0$ *if and only if for every* $0 < k < n$ *there exists no set of integers* $\{a_i\}_{i\in I_k} \subseteq \mathbb{Z}$ *not all equal to* 0 *such that for all ascending multi-indices* P, Q *where* $(P, Q) \in \mathbb{N}^{2k}$ *and* P *has more entries than* Q*,*

$$\sum_{i\in I_k} a_i C^i_{P,Q} = 0.$$

This immediately give the following sufficient criterion which may be easier to check than the original condition.

Corollary 2.5. $\ker \operatorname{int}_\Gamma = 0$ *if for every* $0 < k < n$ *there exist multi-indices* P, Q *where* $(P, Q) \in \mathbb{N}^{2k}$ *and* P *has more entries than* Q *are such that the numbers* $\{C^i_{P,Q}\}_{i\in I_k}$ *are linearly independent over* $\mathbb{Z}$*.*

In general, Lemma 2.3 gives the following explicit version of [4, Thm 1.1].

Theorem 2.6. $(e^{-\pi|t|^2}, \Lambda)$ *is a Gabor frame for* $L^2(\mathbb{R}^n)$ *if one of the following assumptions holds:*

(1) $|\Lambda| < \frac{1}{n!}$ *and* $\ker \operatorname{int}_{\Lambda_\mathbb{C}} = 0$*, where*

$$\Lambda_\mathbb{C} := \{\xi + ix \in \mathbb{C}^n : (\xi, x) \in \Lambda\}.$$

(2) $|\Lambda| < \frac{n!}{n^n}$ *and* $\ker \operatorname{int}_{\Lambda^\circ_\mathbb{C}} = 0$*, where*

$$\Lambda^\circ_\mathbb{C} := \{\eta + iy \in \mathbb{C}^n : \xi^T y - x^T\eta \in \mathbb{Z},\ \forall\, (\xi, x) \in \Lambda\}.$$

Remark. *Almost all lattices in* $\mathbb{R}^{2n}$ *satisfy* $\ker \operatorname{int}_{\Lambda_\mathbb{C}} = 0$ *and* $\ker \operatorname{int}_{\Lambda^\circ_\mathbb{C}} = 0$ *(as we will show later on), hence the above theorem gives an effective Gabor frame criterion in terms of the covolume for almost all lattices.*

Corollary 2.7. *In case* $\Lambda = \mathbb{Z}^2 \times A\mathbb{Z}^2$*, where* A *is a real linear mapping defined by*

$$A(x, y) := (ax + by, cx + dy),$$

then we have that

$$\ker \operatorname{int}_{\Lambda_\mathbb{C}} = 0 \Leftrightarrow \ker \operatorname{int}_{\Lambda^\circ_\mathbb{C}} = 0 \Leftrightarrow ad - bc \notin \mathbb{Q},\ a, b, c, d \text{ are } \mathbb{Z}\text{-linearly independent.}$$

Such a lattice then induces a Gabor frame if additionally, $|ad - bc| < \frac{1}{2}$*. Thus the examples in Theorem 1.5 in* [5] *do not satisfy our assumptions and need not generate Gabor frames for* $L^2(\mathbb{R}^2)$*.*

Proof. In this case the complexified lattice $\Lambda_\mathbb{C}$ is generated by the vectors

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}ia\\ic\end{pmatrix}, \begin{pmatrix}ib\\id\end{pmatrix}.$$

We may now apply the two dimensional version of our criterion for a transcendental lattice which was given in an earlier remark to see that Λ is transcendental precisely when the determinants 1, $bc - ad$, ia, ib, ic, and id are linearly independent over $\mathbb{Z}$. This happens precisely when the condition given in the statement of the Lemma is fulfilled. The case of the symplectic dual can be verified in the same way. ∎

We will now prove the following genericity result:

Theorem 2.8. *Transcendental tori are generic with respect to the choice of the lattice.*

We will first clarify what we mean by generic. Consider the fact that the space of lattices is equal to the space $\mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(2n,\mathbb{Z})$. We say that a property is generic for lattices if there exists an $S \subsetneq \mathrm{Gl}(2n,\mathbb{R})$, which is a set of Lebesgue measure zero in $\mathbb{R}^{4n^2}$ such that the property is fulfilled by all lattices in $(\mathrm{Gl}(2n,\mathbb{R}) \setminus S)/\mathrm{Gl}(2n,\mathbb{Z})$.

Proof. Consider any set of integers $\{a_i\}_{i\in I_k}$. Then the equations

$$\sum_{i\in I_k} a_i C^i_{P,Q} = 0$$

induce a closed variety in $\mathbb{R}^{4n^2}$. It is sufficient to prove that a variety of this form has codimension of at least 1 for some given (P,Q), as this will imply that the variety in this case is of Lebesgue measure zero , and the set in $\mathbb{R}^{4n^2}$ which does not induce a transcendental torus must be contained within this union. This is a simple observation to make since all we must show is that there is a point which does not lie in this subvariety.

To see this, observe in the case $2k \leq n$, and choose P to be of length $2k$ and Q an empty multiindex. Then the polynomials $C^i_{P,Q}$ are linearly independent over $\mathbb{R}[e_{1,1},...,e_{2n,n}]$.

We see this by looking at the "first term" of the $C^i_{P,Q}$. By this, we mean the summand in the determinant which is given by multiplying the elements on the main diagonal. So this term will be of the form

$$e_{i_1,P_1} \cdot ... \cdot e_{i_{2k},P_{2k}}.$$

If we now choose a different index $j \in I_k$. Without loss of generality we may assume that the number i_1 does not occur in j. This means that the variable $e_{i_1 P_1}$ also cannot occur in $C^j_{P,Q}$, making $C^j_{P,Q}$ linearly independent from $C^i_{P,Q}$. Thus, nontrivial linear combinations of the $C^i_{P,Q}$ cannot vanish identically.

Now observe the case where $2k > n$. We now take $P = (1,2,...n)$, and Q to be of length $2k-n$. We need to check that the corresponding polynomials $C^i_{P,Q}$ are linearly independent over $\mathbb{R}[e_{1,1},\bar{e}_{1,1},..,e_{2n,n},\bar{e}_{2n,n}]$.

Once again, this follows from that the term given by multiplying the elements of the diagonal, $e_{i_1,1} \cdot ... \cdot e_{i_n,n} \cdot \overline{e}_{i_{n+1},Q_1} \cdot ... \cdot \overline{e}_{i_{2k},Q_{2k-n}}$ is unique with respect to i.

□

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