# Godbillon-Vey type functional for almost contact manifolds

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#### Abstract

Many contact metric manifolds are critical points of curvature functionals restricted to spaces of associated metrics. The Godbillon-Vey functional was never considered in a variational context in Contact Geometry. Recently we extended this functional from foliations to arbitrary plane fields on a 3-dimensional manifold, so, the following question arises: can one use the Godbillon-Vey functional to find optimal almost contact manifolds? In the paper, we introduce a Godbillon-Vey type functional for a 3-dimensional almost contact manifold, present it in Reinhart-Wood form and find its Euler-Lagrange equations for all variations preserving the Reeb vector field. We construct critical (for our functional) 3-dimensional almost contact manifolds having a double-twisted product structure, these solutions belong to the class  $C_5 \oplus C_{12}$  according to Chinea-Gonzalez classification.

**Keywords**: almost contact manifold, Godbillon-Vey functional, double-twisted product, variation, Chinea-Gonzalez classes

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## 1 Introduction

D. Chinea and C. Gonzalez [5] decomposed the space of certain 3-tensors on an almost contact metric manifold  $M^{2n+1}(\varphi,\omega,T,g)$  into irreducible invariant components under the action of the structural group  $U(n) \times 1$  and developed a Gray-Hervella type classification for almost contact metric (a.c.m.) manifolds. They obtained 12 classes of a.c.m. manifolds, which in dimension three are reduced to five classes:  $C_5$  of  $\beta$ -Kenmotsu manifolds,  $C_6$  of  $\alpha$ -Sasakian manifolds,  $C_9$ -manifolds,  $C_{12}$ -manifolds (called generalized Sasakian space forms) and  $|C| = C_5 \cap C_{12}$  of cosymplectic manifolds. Many works on a.c.m. manifolds are devoted to normal structures, that is to the first three Chinea-Gonzalez classes. Some authors investigate  $C_5 \oplus C_{12}$ -manifolds, consisting of integrable, non-normal manifolds because  $\nabla_T T \neq 0$ , see [3, 4, 6], where  $\nabla$  is the Levi-Civita connection. The elements of  $\bigoplus_{1 \leq i \leq 5} C_i \oplus C_{12}$  are a.c.m. manifolds that are locally a double-twisted product  $B \times_{(u,v)} I$ , where B is an almost Hermitian manifold,  $I \subset \mathbb{R}$  is an open interval and u, v are smooth positive functions on  $B \times I$ . The first result in this topic states that any Kenmotsu manifold, i.e.,  $(\nabla_X \varphi)Y = g(\varphi X, Y)T - \omega(Y)\varphi X$ , is locally a warped product  $B \times_u I$ , where  $u \in C^{\infty}(I)$  and B is a Kähler manifold, see [8]. Note that any two-dimensional almost complex manifold is complex, moreover, it is Kähler. Thus, in dimension three, the class  $\bigoplus_{1 \leq i \leq 5} C_i \oplus C_{12}$  reduces to  $C_5 \oplus C_{12}$ , whose elements are locally a double-twisted product  $B \times_{(u,v)} \overline{I}$ , where B is a 2-dimensional Kähler manifold, see [4, 6].

Finding critical metrics of certain functionals can be considered as an approach to searching for the best metric for a given manifold. Many contact metric manifolds are critical points of curvature functionals restricted to spaces of associated metrics; for example, symplectical manifolds are critical for the total scalar curvature and K-contact manifolds are critical for the total Ricci curvature in the T-direction, e.g., [2, Sect. 5].

The Godbillon-Vey functional was introduced for foliations of codimension one in [7], its changes under infinitesimal deformations of the foliation were studied, for example, in [1], but this functional was never considered in a variational context in Contact Geometry. In [11, 12, 13]

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we extended the Godbillon-Vey functional from foliations to arbitrary plane fields on a 3-dimensional manifold and used the calculus of variations to characterize critical metrics in distinguished classes of almost-product manifolds. So, a question arises: can one use a Godbillon-Vey type functional to find optimal 3-dimensional a.c.m. manifolds, e.g., in  $C_5 \oplus C_{12}$  according to Chinea-Gonzalez classification? To answer the question, we define a new Godbillon-Vey type functional, see (3), present it in Reinhart-Wood form and derive its Euler-Lagrange equations for variations preserving the Reeb vector field T (Theorem 1). For clarity of calculations, we divide variations into two types, each of which has a geometric meaning. These allow us to find new solutions to the problem: what a.c.m. manifolds are in some sense optimal? Since a.c.m. manifolds with the geodesic vector field T are critical for our functional, it is interesting to study the case when the curvature k of T-curves is non-zero. We construct such critical 3-dimensional a.c.m. manifolds having a double-twisted product structure, i.e., solutions belonging to  $C_5 \oplus C_{12}$  (Theorem 2).

# 2 The Reinhart-Wood type formula

An almost contact structure  $(\varphi, \omega, T)$  on a smooth odd-dimensional manifold M consists of an endomorphism  $\varphi$  of TM, a 1-form  $\omega \in \Lambda^1(M)$  and a vector field T satisfying

$$\varphi^2(X) = -X + \omega(X)T \quad (X \in TM), \qquad \omega(T) = 1. \tag{1}$$

The plane field ker  $\omega$  is called the contact distribution. By (1), we get  $\omega \circ \varphi = 0$  and  $\varphi T = 0$ .

Let T be a nonzero vector field on a smooth orientable three-dimensional manifold M. Then for any 1-form  $\omega \in \Lambda^1(M)$  such that  $\omega(T)=1$  there exists a unique tensor  $\varphi \in End(TM)$  such that the restriction of  $\varphi$  on ker  $\omega$  specifies a right-hand rotation and  $M^3(\varphi,\omega,T)$  is an almost contact manifold. In [11, 12, 13], the Godbillon-Vey functional  $gv=\int_M \eta \wedge d\eta$ , where  $\eta=\iota_T d\omega=d\omega(T,\cdot)$ , was extended from foliations to arbitrary plane fields on  $M^3$ . Note that  $\eta(T)=0$ .

We use the formula  $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$  for  $\omega \in \Lambda^1(M)$  and  $X,Y \in TM$ .

**Definition 1.** Given an almost contact structure  $(\varphi, \omega, T)$  on  $M^3$ , define a 1-form  $\eta^* \in \Lambda^1(M)$  by

$$\eta^*(X) := (\iota_T d\omega)(\varphi X) = d\omega(T, \varphi X) \quad (X \in TM), \tag{2}$$

i.e.,  $\eta^* = \eta \circ \varphi$ . Using  $\eta^*$ , we introduce (similarly to gv) the following functional:

$$gv^* = \int_M \eta^* \wedge d\eta^*. \tag{3}$$

If an almost contact manifold  $M^3(\varphi,\omega,T)$  admits a Riemannian metric  $g=\langle\cdot,\cdot\rangle$  such that

$$\langle \varphi X, \varphi Y \rangle = \langle X, Y \rangle - \omega(X) \omega(Y),$$
 (4)

then g is called a compatible metric and we get an a.c.m. manifold. Such a structure (4) is induced on any hypersurface of an almost Hermitian manifold. Putting Y = T in (4), we get  $\omega(X) = \langle X, T \rangle$ ; thus, T is g-orthogonal to ker  $\omega$ . An a.c.m. manifold such that

$$g(X, \varphi Y) = d\omega(X, Y)$$

is called a contact metric manifold. Such manifolds have a geodesic vector field T.

Given a unit vector field T on a Riemannian manifold  $(M^3,g)$ , the unit normal N, the binormal  $B=T\times N$  and the torsion  $\tau$  of T-curves are defined on an open subset U of  $M^3$ , where the curvature k of T-curves is nonzero. Further, we assume that U is nowhere dense, so the set  $M\setminus U$  can be neglected during integration over  $M^3$ . The 1-form in the Godbillon-Vey functional is given by  $\eta=kN^{\flat}$  (i.e.,  $\eta(X)=k\langle N,X\rangle$  for  $X\in TM$ ) on U and  $\eta=0$  on  $M\setminus U$ , see [11, 12]. Thus, for the 1-form  $\eta^*$  in (2), using  $d\omega(T,\varphi X)=-\omega([T,\varphi X])=k\langle N,\varphi X\rangle$  and skew-symmetry of  $\varphi$ , we get

$$\eta^* = \begin{cases} -k(\varphi N)^{\flat} & \text{on } U, \\ 0 & \text{on } M \setminus U. \end{cases}$$

If T is a geodesic vector field (i.e., k=0) then  $\eta=\eta^*=0$ , hence  $gv^*=0$ . Recall that the Levi-Civita connection  $\nabla$  of a metric  $g=\langle\cdot,\cdot\rangle$  is given by:

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle. \tag{5}$$

The non-symmetric second fundamental form h of the distribution  $\ker \omega$  is defined by

$$h_{X,Y} = \langle \nabla_X Y, T \rangle = -\langle \nabla_X T, Y \rangle \qquad (X, Y \in \ker \omega).$$
 (6)

If the distribution  $\ker \omega$  in  $TM^3$  is integrable, then the tensor  $2\mathcal{T}_{X,Y} = \langle [X,Y],T\rangle$  vanishes; in this case, the 2-form h is symmetric. The mean curvature of  $\ker \omega$  is defined by  $H = \frac{1}{2}\operatorname{Tr}_g h$ . The distribution  $\ker \omega$  is said to be totally umbilical (or, totally geodesic) if  $\operatorname{Sym}(h) = Hg$  (h = 0, respectively). Here,  $\operatorname{Sym}(h)(X,Y) = \frac{1}{2}(h(X,Y) + h(Y,X))$  is the symmetric second fundamental form of the distribution  $\ker \omega$ . The following Frenet-Serret formulas are true on U:

$$\nabla_T T = kN, \quad \nabla_T N = -kT + \tau B, \quad \nabla_T B = -\tau N.$$
 (7)

Using (7), we obtain  $k = \langle \nabla_T T, N \rangle$  and  $\tau = -\langle \nabla_T B, N \rangle$ . Recall, see [13, Eq. 139],

$$d\eta(T,B) = k(\tau - h_{B,N}), \quad d\eta(T,N) = T(k) - kh_{N,N}.$$
 (8)

With the volume form  $d \operatorname{vol}_g$  in mind, we derive formulas similar to the following one, see [11, 12]:  $gv = -\int_M k^2(\tau - h_{B,N}) d \operatorname{vol}_g$ , obtained for foliations by B.L. Reinhart and J.W. Wood in [10] with opposite sign by convention.

**Proposition 1.** We obtain

$$gv^* = -\int_M k^2(\tau + h_{N,B}) \, d \, \text{vol}_g.$$
 (9)

*Proof.* We have  $\varphi N = B$  and  $\varphi B = -N$ . From Frenet-Serret formulas and  $\eta^* = \eta \circ \varphi$ , we find

$$d\eta^*(T,B) = T(\eta(\varphi B)) - B(\eta(\varphi T)) - \eta(\varphi[T,B]) = -T(\eta(N)) - k \langle \varphi[T,B], N \rangle$$

$$= -T(k) - k \langle \nabla_T B - \nabla_B T, B \rangle = -T(k) - k h_{B,B},$$

$$d\eta^*(T,N) = T(\eta(\varphi N)) - N(\eta(\varphi T)) - \eta(\varphi[T,N]) = T(\eta(B)) - k \langle \varphi[T,N], N \rangle$$

$$= k \langle B, \nabla_T N - \nabla_N T \rangle = k (\tau + h_{N,B}).$$

Thus, using (8),  $\eta^*(T) = \eta^*(N) = 0$  and  $\eta^*(B) = -k$ , we calculate

$$(\eta^* \wedge d\eta^*)(T, N, B) = \eta^*(B) d\eta^*(T, N) = -k^2(\tau + h_{NB}).$$

Applying to the above the volume form  $d \operatorname{vol}_g$  on  $(M^3, g)$ , we get (9).

# 3 The first variation of $gv^*$

Let a smooth orientable 3-dimensional manifold M be equipped with a non-vanishing vector field T. Then for any Riemannian metric g on M such that g(T,T)=1 there exist a unique 1-form  $\omega \in \Lambda^1(M)$  (given by  $\omega=g(T,\cdot)$ ) and a unique (1,1)-tensor  $\varphi \in End(TM)$  such that the restriction of  $\varphi$  on the plane  $\ker \omega$  specifies a right-hand rotation and  $M^3(\varphi,\omega,T,g)$  is an a.c.m. manifold. Let  $\mathcal{C}(M,T)$  denote the set of all such a.c.m. structures on (M,T).

Let  $M^3(\varphi, \omega, T, g)$  be an almost contact manifold, and  $g_t$  ( $|t| < \varepsilon$ ) a family of Riemannian metrics on  $M^3$  such that  $g_0 = g$  and  $g_t(T, T) \equiv 1$ . By the above, there exists a unique family  $(\varphi_t, \omega_t, T, g_t)$  of a.c.m. structures in  $\mathcal{C}(M, T)$  such that  $\varphi_0 = \varphi$  and  $\omega_0 = \omega$ . Denote by dot the t-derivative at t = 0 of any quantity on M. Since  $\dot{T} = 0$  and  $\dot{g}_{T,T} = 0$  are true, the symmetric (0, 2)-tensor  $\dot{g} = (dg_t/dt)|_{t=0}$  has five independent components on a domain  $U \subset M$  (where  $k \neq 0$ ):

$$\dot{g}_{T,N}, \quad \dot{g}_{T,B}, \quad \dot{g}_{N,N}, \quad \dot{g}_{N,B}, \quad \dot{g}_{B,B}$$

Such variations  $g_t$  that generate on U only nonzero components  $\dot{g}_{N,N}, \dot{g}_{N,B}$  and  $\dot{g}_{B,B}$ , are called  $g^{\top}$ -variations. They preserve T and  $\ker \omega$ ; thus produce trivial Euler-Lagrange equations for the functional gv, see [11, 12]. In contrast, the  $g^{\top}$ -variations are essential for the functional  $gv^*$  and they will be considered in Section 3.1.

Variations  $g_t$  of g that generate on U only nonzero components  $\dot{g}_{T,N}$  and  $\dot{g}_{T,B}$ , are called  $g^{\pitchfork}$ -variations, see [11, 12]. The  $g^{\pitchfork}$ -variations will be considered in Section 3.2.

The following general variational formula for the volume form is true, see [13, p. 162]:

$$(\operatorname{d}\operatorname{vol}_g)^{\cdot} = \frac{1}{2} (\operatorname{Tr}_g \dot{g}) \operatorname{d}\operatorname{vol}_g. \tag{10}$$

Applying arbitrary variation  $g_t$  to (5), one can find how the connection changes, e.g., [13, p. 158]:

$$2\langle (\dot{\nabla}_X Y), Z \rangle = (\nabla_X^t \dot{g})(Y, Z) + (\nabla_Y^t \dot{g})(X, Z) - (\nabla_Z^t \dot{g})(X, Y), \quad X, Y, Z \in \mathfrak{X}_M. \tag{11}$$

The main goal of this section is the following

**Theorem 1.** The Euler-Lagrange equations on U of the functional  $gv^*$  with respect to all variations in C(M,T) are the following:

$$\tau + h_{N,B} = 0, \quad T(k) - k h_{B,B} = 0,$$

$$(h_{N,B} + h_{B,N}) (h_{N,N} + h_{B,B}) = 0, \quad h_{N,N}^2 - h_{B,B}^2 - h_{N,N} h_{B,B} - T(h_{N,N}) - \frac{1}{4} k^2 = 0.$$
 (12)

*Proof.* Using the Euler-Lagrange equations (14) for  $g^{\top}$ -variations (in Section 3.1), we simplify the Euler-Lagrange equations (16) for  $g^{\pitchfork}$ -variations (in Section 3.2), and get (12).

### 3.1 The $g^{\top}$ -variations

**Lemma 1.** For  $g^{\top}$ -variations of metric on U, we obtain  $\dot{\eta} = 0$  and

$$\dot{N} = -\frac{1}{2}\dot{g}_{N,N}N - \dot{g}_{N,B}B, \qquad \dot{B} = -\frac{1}{2}\dot{g}_{B,B}B, \qquad \dot{k} = -\frac{k}{2}\dot{g}_{N,N}, 
(\tau + h_{N,B}) = (h_{N,N} - h_{B,B})\dot{g}_{N,B} + \frac{1}{2}(\tau + h_{N,B})\dot{g}_{B,B} - T(\dot{g}_{N,B}).$$
(13)

*Proof.* Using  $\dot{T} = 0$ , we find

$$\langle \dot{N},T\rangle = \langle \dot{B},T\rangle = 0, \quad \langle \dot{N},N\rangle = -\frac{1}{2}\,\dot{g}_{N,N}, \quad \langle \dot{B},B\rangle = -\frac{1}{2}\,\dot{g}_{B,B}, \quad \langle \dot{N},B\rangle + \langle \dot{B},N\rangle = -\dot{g}_{N,B}.$$

Differentiating  $\langle \nabla_T T, N \rangle = k$ ,  $\langle \nabla_T T, B \rangle = 0$  and using (7) and (11), we find

$$\begin{split} \dot{k} &= \frac{k}{2} \, \dot{g}_{N,N} + (\nabla_T \, \dot{g})_{T,N} - \frac{1}{2} \, (\nabla_N \, \dot{g})_{T,T}, \\ \langle \dot{B}, N \rangle &= - \dot{g}_{N,B} - \frac{1}{k} \, (\nabla_T \, \dot{g})_{T,B} + \frac{1}{2 \, k} \, (\nabla_B \, \dot{g})_{T,T}. \end{split}$$

By the above,  $\langle \dot{N}, B \rangle = \frac{1}{k} (\nabla_T \dot{g})_{T,B} - \frac{1}{2k} (\nabla_B \dot{g})_{T,T}$  is true. Thus,

$$\begin{split} \dot{N} &= -\frac{1}{2}\,\dot{g}_{N,N}\,N + \frac{1}{k}\left((\nabla_T\,\dot{g})_{T,B} - \frac{1}{2}\,(\nabla_B\,\dot{g})_{T,T}\right)B,\\ \dot{B} &= \left(-\,\dot{g}_{N,B} - \frac{1}{k}\,(\nabla_T\,\dot{g})_{T,B} + \frac{1}{2\,k}\,(\nabla_B\,\dot{g})_{T,T}\right)N - \frac{1}{2}\,\dot{g}_{B,B}\,B. \end{split}$$

Next, using (7) and  $\dot{g}_{T,X} = 0$ , we find

$$\begin{split} &(\nabla_N \, \dot{g})_{T,T} = (\nabla_B \, \dot{g})_{T,T} = 0, \quad (\nabla_T \, \dot{g})_{T,B} = -k \, \dot{g}_{N,B}, \quad (\nabla_T \, \dot{g})_{T,N} = -k \, \dot{g}_{N,N}, \\ &(\nabla_T \, \dot{g})_{N,N} = T(\dot{g}_{N,N}) - 2 \, \tau \, \dot{g}_{N,B}, \quad (\nabla_N \, \dot{g})_{N,T} = h_{N,N} \, \dot{g}_{N,N} + h_{N,B} \, \dot{g}_{N,B}. \end{split}$$

From the above we find  $\dot{N}$ ,  $\dot{B}$  and  $\dot{k}$  in (13). Using (6) and (7), we calculate

$$\tau + h_{N,B} = -\langle \nabla_T B, N \rangle + \langle \nabla_N B, T \rangle = \langle \nabla_T N, B \rangle - \langle \nabla_N T, B \rangle = \langle [T, N], B \rangle.$$

Since the Lie bracket does not depend on metric, we get

$$(\tau + h_{N,B})^{\cdot} = \langle [T, N], B \rangle^{\cdot} = \dot{g}_{[T,N],B} + \langle [T, \dot{N}], B \rangle + \langle [T, N], \dot{B} \rangle$$
$$= h_{N,N} \, \dot{g}_{N,B} - \langle [T, B], B \rangle \, \dot{g}_{N,B} - T(\dot{g}_{N,B}) + \frac{1}{2} (\tau + h_{N,B}) \, \dot{g}_{B,B},$$

from which the expression of  $(\tau + h_{N,B})$  in (13) follows.

Our main result in Section 3.1 is the following.

**Proposition 2.** An a.c.m. manifold  $M^3(\varphi, \omega, T, g)$  is critical for the functional  $gv^*$  with respect to  $g^{\top}$ -variations if and only if the following Euler-Lagrange equations hold on U:

$$\tau + h_{N,B} = 0, T(k) - k h_{B,B} = 0.$$
 (14)

*Proof.* Using (9) and (10), we find

$$(gv^*)^{\cdot} = -\int_M \left\{ (k^2(\tau + h_{N,B}))^{\cdot} + \frac{k^2}{2} (\tau + h_{N,B}) (\dot{g}_{N,N} + \dot{g}_{B,B}) \right\} d \operatorname{vol}_g.$$

By (13), we obtain

$$(k^{2}(\tau + h_{N,B})) = 2 k \dot{k} (\tau + h_{N,B}) + k^{2}(\tau + h_{N,B})$$

$$= \frac{k^{2}}{2} (\tau + h_{N,B}) \dot{g}_{B,B} - k^{2}(\tau + h_{N,B}) \dot{g}_{N,N} + k^{2}(h_{N,N} - h_{B,B}) \dot{g}_{N,B} - k^{2}T(\dot{g}_{N,B}).$$

Using the equalities  $T(f) = \operatorname{div}(f \cdot T) - f \cdot \operatorname{div} T$  for any function  $f \in C^1(M)$  and  $\operatorname{div} T = -\operatorname{Tr}_q h = -(h_{N,N} + h_{B,B})$ , see [13], we get

$$k^2 T(\dot{g}_{N,B}) = \operatorname{div}(k^2 \dot{g}_{N,B} T) + \left(k^2 (h_{N,N} + h_{B,B}) - 2 k T(k)\right) \dot{g}_{N,B}.$$

Therefore,

$$(gv^*) \dot{} = -\int_M \left\{ -k^2(\tau + h_{N,B}) \, \dot{g}_{N,N} + k^2(h_{N,N} - h_{B,B}) \dot{g}_{N,B} + \frac{k^2}{2} (\tau + h_{N,B}) \dot{g}_{B,B} - \left( k^2(h_{N,N} + h_{B,B}) - 2 k T(k) \right) \dot{g}_{N,B} + \frac{k^2}{2} (\tau + h_{N,B}) (\dot{g}_{N,N} + \dot{g}_{B,B}) \right\} d \operatorname{vol}_g$$

$$= -\int_M \left\{ \frac{k^2}{2} (\tau + h_{N,B}) (\dot{g}_{B,B} - \dot{g}_{N,N}) + 2 (k T(k) - k^2 h_{B,B}) \dot{g}_{N,B} \right\} d \operatorname{vol}_g.$$

Since  $\dot{g}_{N,N},\dot{g}_{N,B},\dot{g}_{B,B}$  are independent functions on  $M^3$ , this completes the proof of (14).  $\Box$ 

Corollary 1. Let  $\{T, N, B\}$  be an orthonormal frame on a Riemannian manifold  $(M^3, g)$  such that the plane field Span(T, N) is tangent to a Riemanian foliation,  $\nabla_T T$  is nonzero and parallel to N and each T-curve has constant curvature. Set  $\omega = T^{\flat}$  and define  $\varphi$  by  $\varphi(T) = 0$ ,  $\varphi(N) = B$  and  $\varphi(B) = -N$ . Then  $(\varphi, \omega, T, g)$  is critical for  $gv^*$  with respect to  $g^{\top}$ -variations.

*Proof.* Since the plane field Span(T, N) is integrable, the first equation of (14) holds. Since the foliation is Riemannian, we have  $\nabla_B B = 0$ . By this, the second equation of (14) holds.

**Example 1.** We present solutions of (14) with  $k \neq 0$  using orthogonal coordinates in  $\mathbb{R}^3$ .

- (i) Consider  $\mathbb{R}^3 \setminus \{\rho = 0\}$  with cylindrical coordinates  $(\rho, \phi, z)$ . Set  $T = \partial_{\phi}$ ,  $N = -\partial_{\rho}$  and  $B = \partial_z$ . Define  $\varphi$  by  $\varphi(T) = 0$ ,  $\varphi(N) = B$  and  $\varphi(B) = -N$ . The T-curves are circles in  $\mathbb{R}^3$ , the (T, N)-surfaces are horizontal planes  $\{z = const\}$ , and the B-curves vertical lines. Since the distribution  $\operatorname{Span}(T, N)$  is integrable, the first Euler-Lagrange equation of (14) is true. Since T(k) = 0 and  $\nabla_B B = 0$ , also the second Euler-Lagrange equation of (14) is true.
- (ii) Consider  $\mathbb{R}^3 \setminus \{\rho = 0 \text{ or } \theta = \pi/2\}$  with spherical coordinates  $(\rho, \theta, \phi)$ . Let T-curves be circles that are the intersections of spheres  $\{\rho = const\} > 0$  with horizontal planes. Then k = 0 on the plane  $\theta = \pi/2$ , and  $k = \infty$  on the axis  $\rho = 0$ . Set  $B = \partial_{\rho}$  and  $N = B \times T$ . Hence, spheres  $\{\rho = const > 0\}$  compose a foliation tangent to  $\mathrm{Span}(T, N)$ . Define  $\varphi$  by  $\varphi(T) = 0$ ,  $\varphi(N) = B$  and  $\varphi(B) = -N$ . Therefore, the Euler-Lagrange equations (14) are true.

## 3.2 The $g^{\uparrow}$ -variations

**Lemma 2.** For  $g^{\uparrow}$ -variations of metric on U, we obtain

$$\dot{k} = T(\dot{g}_{T,N}) - (\tau + h_{N,B})\dot{g}_{T,B} - h_{N,N}\dot{g}_{T,N}, 
\dot{N} = -\frac{1}{2}\dot{g}_{T,N}T + \frac{1}{k}\left((\tau - h_{B,N})\dot{g}_{T,N} - h_{B,B}\dot{g}_{T,B} + T(\dot{g}_{T,B})\right)B, 
\dot{B} = -\frac{1}{2}\dot{g}_{T,B}T - \frac{1}{k}\left((\tau - h_{B,N})\dot{g}_{T,N} - h_{B,B}\dot{g}_{T,B} + T(\dot{g}_{T,B})\right)N, 
(\tau + h_{N,B}) = \frac{1}{k}\left(h_{N,N} + h_{B,B}\right)\left((\tau - h_{B,N})\dot{g}_{T,N} - h_{B,B}\dot{g}_{T,B} + T(\dot{g}_{T,B})\right) 
+ T\left(\frac{1}{k}\left((\tau - h_{B,N})\dot{g}_{T,N} - h_{B,B}\dot{g}_{T,B} + T(\dot{g}_{T,B})\right)\right) - \frac{k}{2}\dot{g}_{T,B}.$$
(15)

*Proof.* Since  $\{T, N_t, B_t\}$  is an orthonormal frame on  $U_t$  for all  $g_t$ , we get

$$\begin{split} &\dot{g}_{T,N} + 2 \left\langle T, \dot{N} \right\rangle = 0, \quad \dot{g}_{T,B} + 2 \left\langle T, \dot{B} \right\rangle = 0, \\ &\langle \dot{N}, N \rangle = 0, \quad \langle \dot{B}, B \rangle = 0, \quad \langle \dot{N}, B \rangle + \langle \dot{B}, N \rangle = 0. \end{split}$$

Differentiating  $\langle \nabla_T T, N \rangle = k$ ,  $\langle \nabla_T T, B \rangle = 0$  and using (7) and (11), we find  $\dot{k}$  in (15) and

$$k \langle \dot{B}, N \rangle = h_{B,B} \, \dot{g}_{T,B} - (\tau - h_{B,N}) \dot{g}_{T,N} - T(\dot{g}_{T,B}).$$

Therefore,  $k \langle \dot{N}, B \rangle = -h_{B,B} \dot{g}_{T,B} + (\tau - h_{B,N}) \dot{g}_{T,N} + T(\dot{g}_{T,B})$  is true. From the above we find  $\dot{N}$ ,  $\dot{B}$  and  $\dot{k}$  in (15). Finally, we get

$$(\tau + h_{N,B}) = \langle [T, N], B \rangle = \dot{g}_{[T,N],B} + \langle [T, \dot{N}], B \rangle + \langle [T, N], \dot{B} \rangle,$$

from which and known  $\dot{N}$ ,  $\dot{B}$  the expression of  $(\tau + h_{N,B})$  in (15) follows.

Our main result in Section 3.2 is the following.

**Proposition 3.** An a.c.m. manifold  $M^3(\varphi, \omega, T, g)$  is critical for the functional  $gv^*$  with respect to  $g^{\uparrow}$ -variations if and only if the following Euler-Lagrange equations hold on U:

$$k (2\tau + h_{N,B} - h_{B,N}) (h_{N,N} + h_{B,B}) - k (\tau + h_{N,B}) h_{N,N} - T(k) (2\tau + h_{N,B} - h_{B,N}) - k T(\tau + h_{N,B}) = 0 k (h_{N,N} + h_{B,B})^{2} - T(k (h_{N,N} + h_{B,B})) - k (h_{N,N} + h_{B,B}) h_{B,B} + T(T(k)) - T(k) h_{N,N} - k (\tau + h_{N,B})^{2} - \frac{1}{4} k^{3} = 0.$$
(16)

*Proof.* For a  $g^{\uparrow}$ -variation, using (9) and (10) with  $\text{Tr}_g \ \dot{g} = 0$ , we find

$$(gv^*)^{\cdot} = -\int_M (k^2(\tau + h_{N,B}))^{\cdot} d \text{vol}_g.$$

By (13), we obtain

$$(k^{2}(\tau + h_{N,B})) = 2 k \dot{k} (\tau + h_{N,B}) + k^{2}(\tau + h_{N,B})$$

$$= 2 k (\tau + h_{N,B}) (T(\dot{g}_{T,N}) - (\tau + h_{N,B}) \dot{g}_{T,B} - h_{N,N} \dot{g}_{T,N})$$

$$+ k (h_{N,N} + h_{B,B}) ((\tau - h_{B,N}) \dot{g}_{T,N} - h_{B,B} \dot{g}_{T,B} + T(\dot{g}_{T,B}))$$

$$+ k^{2} T (\frac{1}{k} ((\tau - h_{B,N}) \dot{g}_{T,N} - h_{B,B} \dot{g}_{T,B} + T(\dot{g}_{T,B}))) - \frac{1}{2} k^{3} \dot{g}_{T,B}.$$
(17)

Using the equalities  $T(\alpha) = \operatorname{div}(\alpha \cdot T) - \alpha \cdot \operatorname{div} T$  for any function  $\alpha \in C^1(M)$  and  $\operatorname{div} T = -\operatorname{Tr}_g h = -(h_{N,N} + h_{B,B})$ , see [13], we get for any functions  $\alpha, \beta, \gamma \in C^1(M)$ :

$$\alpha T(\dot{\beta}T) = \operatorname{div}(\alpha\beta T) + \left[\alpha(h_{N,N} + h_{B,B}) - T(\alpha)\right]\dot{\beta},$$

$$\alpha T(\beta T(\dot{\gamma})) = \operatorname{div}(\left[\alpha\beta T(\dot{\gamma}) + \alpha\beta(h_{N,N} + h_{B,B})\dot{\gamma}\right]T)$$

$$+ \left[\alpha\beta(h_{N,N} + h_{B,B})^2 + T(\beta T(\alpha)) - \beta T(\alpha)(h_{N,N} + h_{B,B}) - T(\alpha\beta(h_{N,N} + h_{B,B}))\right]\dot{\gamma}.$$

Therefore, with some functions  $f_i$  on M we get

$$\begin{split} &2\,k\,(\tau+h_{N,B})\,T(\dot{g}_{T,N}) = \left[2\,k\,(\tau+h_{N,B})\,(h_{N,N}+h_{B,B}) - 2\,T(k\,(\tau+h_{N,B}))\right]\dot{g}_{T,N} + \mathrm{div}\,f_{1},\\ &k\,(h_{N,N}+h_{B,B})\,T(\dot{g}_{T,B}) = \left[k\,(h_{N,N}+h_{B,B})^{2} - T(k\,(h_{N,N}+h_{B,B}))\right]\dot{g}_{T,B} + \mathrm{div}\,f_{2},\\ &k^{2}T\left(\frac{1}{k}\left((\tau-h_{B,N})\dot{g}_{T,N} - h_{B,B}\,\dot{g}_{T,B}\right)\right) = k\,(\tau-h_{B,N})(h_{N,N}+h_{B,B})\,\dot{g}_{T,N}\\ &-k\,h_{B,B}(h_{N,N}+h_{B,B}),\dot{g}_{T,B}) + \mathrm{div}\,f_{3},\\ &k^{2}T\left(\frac{1}{k}\,T(\dot{g}_{T,B})\right) = \left[k\,(h_{N,N}+h_{B,B})^{2} - T(k\,(h_{N,N}+h_{B,B})) - 2\,T(k)\,(h_{N,N}+h_{B,B})\right.\\ &+ 2\,T(T(k))\left]\,\dot{g}_{T,B} + \mathrm{div}\,f_{4}. \end{split}$$

Integrating (17) and using the above and the Divergence Theorem, gives

$$\begin{split} &(gv^*)^{\cdot} = -\int_{M} \Big\{ 2\,k\,(\tau + h_{N,B})\,T(\dot{g}_{T,N}) + k\,(h_{N,N} + h_{B,B})\,T(\dot{g}_{T,B}) \\ &+ k^2 T \Big( \frac{1}{k} \left( (\tau - h_{B,N}) \dot{g}_{T,N} - h_{B,B} \, \dot{g}_{T,B} \right) \Big) + k^2 T \Big( \frac{1}{k} \,T(\dot{g}_{T,B}) \Big) \\ &- 2\,k\,(\tau + h_{N,B})(\tau + h_{N,B}) \dot{g}_{T,B} - 2\,k\,(\tau + h_{N,B}) h_{N,N} \, \dot{g}_{T,N} \\ &+ k\,(h_{N,N} + h_{B,B}) \Big( (\tau - h_{B,N}) \dot{g}_{T,N} - h_{B,B} \, \dot{g}_{T,B} \Big) - \frac{1}{2} \, k^3 \dot{g}_{T,B} \Big\} d\,\mathrm{vol}_g \\ &= -4 \int_{M} \Big\{ \Big[ k\,(2\,\tau + h_{N,B} - h_{B,N})\,(h_{N,N} + h_{B,B}) - k\,(\tau + h_{N,B}) h_{N,N} - T(k\,(\tau + h_{N,B})) \Big] \, \dot{g}_{T,N} \\ &+ \Big[ k\,(h_{N,N} + h_{B,B})^2 - T(k\,(h_{N,N} + h_{B,B})) - k\,h_{B,B}(h_{N,N} + h_{B,B}) \\ &- T(k)\,(h_{N,N} + h_{B,B}) + T(T(k)) - k\,(\tau + h_{N,B})^2 - \frac{1}{4} \,k^3 \Big] \, \dot{g}_{T,B} \Big\} d\,\mathrm{vol}_g. \end{split}$$

Since  $\dot{g}_{T,N}, \dot{g}_{T,B}$  are independent functions on  $M^3$ , this completes the proof of (16).

### 4 Critical almost contact metric manifolds

Using the Euler-Lagrange equations of Theorem 1, we get the following

**Proposition 4.** Any contact metric manifold  $M^3(\varphi, \omega, T, g)$  is critical for the action  $gv^*$  with respect to all variations in C(M, T).

*Proof.* Since T is a geodesic vector field (k=0), see [2], then  $U=\emptyset$  and (12) become trivial.  $\square$ 

We list the defining conditions (formulated in terms of the covariant derivatives  $\nabla \varphi$ ,  $\nabla T$  and  $\nabla \omega$ ) of any a.c.m. manifold  $(M, \varphi, T, \omega, g)$  which falls in  $C_5 \oplus C_{12}$  or in its subclasses, see [4]:

$$C_{5} \oplus C_{12}: \qquad (\nabla_{X}\varphi)Y = \beta\left(\langle \varphi X, Y \rangle T - \omega(Y)\varphi X\right) - \omega(X)((\nabla_{T}\omega)(\varphi Y)T + \omega(Y)\varphi(\nabla_{T}T)),$$

$$C_{5}: \qquad (\nabla_{X}\varphi)Y = \beta\left(\langle \varphi X, Y \rangle T - \omega(Y)\varphi X\right), \quad \beta = const > 0,$$

$$C_{12}: \qquad (\nabla_{X}\varphi)Y = -\omega(X)((\nabla_{T}\omega)(\varphi Y)T + \omega(Y)\varphi(\nabla_{T}T)),$$

$$|C| = C_{5} \cap C_{12}: \qquad \nabla \varphi = 0 \quad \text{(cosymplectic manifolds)}.$$

The vanishing of the tensor h, see (6), that is ker  $\omega$  defines a totally geodesic foliation, means that the considered manifold belongs to  $C_{12}$ . The vanishing of  $\nabla_T T$  means that the considered manifold belongs to  $C_5$ , namely, it is a  $\beta$ -Kenmotsu manifold. Thus, we get the following

**Proposition 5.** Any 3-dimensional a.c.m. manifold of a class  $C_5$  is critical for the action  $gv^*$  with respect to all variations in C(M,T). The set of 3-dimensional a.c.m. manifolds of a class  $C_{12}$  that are critical for the action  $gv^*$  with respect to all variations in C(M,T) coincides with |C|.

Any a.c.m. structure with a geodesic vector field T (e.g., Propositions 4 and 5) can be called "trivial" solution to the equations (12). What are non-trivial solutions of (12)?

**Lemma 3.** If an a.c.m. manifold  $M^3(\varphi, \omega, T, g)$  is critical for the action  $gv^*$  with respect to all variations in C(M,T), then the distribution Span(T,N) on U is integrable. Moreover, if  $\ker \omega$  is either totally umbilical or integrable, then also the distribution Span(T,B) on U is integrable.

Proof. Using the equalities (6) and  $\tau = \langle \nabla_T N, B \rangle$ , we rewrite the Euler-Lagrange equation (14)<sub>1</sub> as  $\langle [T, N], B \rangle = 0$ ; thus, the distribution  $\operatorname{Span}(T, N)$  is integrable. By the conditions and the equalities  $h_{N,B} + h_{B,N} = 0$  (for totally umbilical  $\ker \omega$ ) or  $h_{N,B} - h_{B,N} = 0$  (for integrable  $\ker \omega$ ) and  $\langle [B, T], N \rangle = \tau - h_{B,N}$ , the second claim is true.

**Proposition 6.** Let an a.c.m. manifold  $M^3(\varphi, \omega, T, g)$  with integrable totally umbilical distribution  $\ker \omega$  and the mean curvature  $H \not\equiv 0$  be a critical for the action  $gv^*$  with respect to all variations in  $\mathcal{C}(M,T)$ . Then the distributions Span(T,N) and Span(T,B) on U are also integrable and the Euler-Lagrange equations (12) for all variations in  $\mathcal{C}(M,T)$  are reduced to the following:  $\tau = 0$  and

$$T(k) = k H, \quad T(H) = -H^2 - (1/4) k^2.$$
 (18)

*Proof.* By conditions,  $h_{N,N} = h_{B,B} = H$  and  $h_{N,B} = h_{N,B} = 0$ . Thus, the Euler-Lagrange equations (12) are reduced to  $\tau = 0$  and (18).

The Euler-Lagrange equations (18) on U can be considered along any T-curve (parameterized by s) as the dynamical system of two ODEs for functions k(s) and H(s):

$$(d/ds)k = kH, \quad (d/ds)H = -H^2 - (1/4)k^2.$$
 (19)

**Lemma 4.** The general solution of the system (19) has the following form:

$$s = \int (H^4 + C_1)^{-1/2} dH + C_2, \quad k(s) = \pm 2\sqrt{-\frac{\mathrm{d}}{\mathrm{d}s} H(s) - H^2(s)},$$

or, in Taylor series form with  $k(0) = k_0 \neq 0$  and  $h(0) = H_0$ ,

$$H(x) = H_0 - \left(H_0^2 + \frac{k_0^2}{4}\right)x + H_0^3 x^2 - H_0^2 \left(H_0^2 + \frac{k_0^2}{4}\right)x^3 + O(x^4),$$
  

$$k(x) = k_0 + k_0 H_0 x - \frac{k_0^3}{8} x^2 - \frac{k_0^3 H_0}{8} x^3 + O(x^4).$$
 (20)

*Proof.* From (19) we get the following ODEs:

$$H'' = 2H^3, \quad k'' = -k^3/4.$$
 (21)

Let  $k(0) = k_0 \neq 0$ ,  $H(0) = H_0$  be the values of solutions of (19) at t = 0. Then the values of the first derivatives of solutions of (21) at t = 0 should be  $k'(0) = k_0 H_0$  and  $H'(0) = -H_0^2 - \frac{1}{4} k_0^2$ , see Fig. 4 with  $k_0 = 1$ ,  $H_0 = 0$  and |s| < 3.4, and the series expansion (20) is valid.

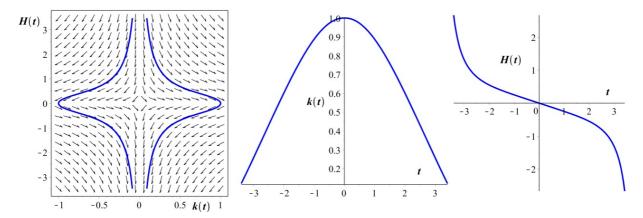


Figure 1: Solution k(s), H(s) of (19) with  $k_0 = 1$  and  $H_0 = 0$  for |s| < 3.4.

**Theorem 2.** There exist 3-dimensional a.c.m. manifolds of a class  $C_5 \oplus C_{12}$  critical for the action  $gv^*$  with respect to all variations in  $\mathcal{C}(M,T)$ . These manifolds have integrable distributions Span(T,N) and Span(T,B) on U,  $\tau=0$  and are presented locally as double twisted products  $B \times_{(u,v)} I$ , where the functions  $H=-2\langle \nabla(\log u), T \rangle$  and  $k=-\langle \nabla(\log v), N \rangle \neq 0$  satisfy (18).

*Proof.* By [4, 6], it is sufficient to build a critical double twisted structure  $B \times_{(u,v)} I$  with metric  $g = (u^2g_B) \oplus (v^2ds^2)$  on a domain  $M = B \times I$  in  $\mathbb{R}^3(x,y,s)$ , where  $B = (-1,1) \times (-1,1) \subset \mathbb{R}^2(x,y)$  and  $u,v:M \to \mathbb{R}$  are positive smooth functions. The second fundamental form h and the mean curvature H of the leaves  $B \times \{s\}$  and the curvature k of the fibers  $\{x\} \times I$  are given by, see [9, 13].

$$h = -\langle \nabla(\log u), T \rangle g^{\top}, \quad H \cdot T = -2 \nabla(\log u), \quad k \cdot N = -\nabla(\log v).$$
 (22)

The leaves  $B \times \{s\}$  are totally umbilical in (M,g). For a critical a.c.m. structure on (M,T), by Proposition 6, we get  $\tau=0$  and the distributions  $\mathrm{Span}(T,N)$  and  $\mathrm{Span}(T,B)$  on U are integrable. Using Lemma 4, we restore functions k and H on  $B \times I \subset \mathbb{R}^3(x,y,s)$  from their arbitrary initial values  $k_0 \neq 0$  and  $H_0$  on B (for s=0). Next, we will show existence of appropriate functions u,v on  $B \times I$ . Let us assume that u=u(s) depends on one variable, thus, by (22), T is parallel to coordinate vector  $\partial_s$  and we can restore u from its arbitrary initial values at s=0 by integration along s-coordinate lines. Then, we assume that v=v(x) depends on one variable, thus, by (22), N is parallel to coordinate vector  $\partial_x$  and we can restore v from its arbitrary initial values at v=v(x)0 by integration along v-coordinate lines.

#### 5 Conclusion

In the article, we applied the calculus of variations approach to finding best a.c.m. structures for a given manifold. We defined a new Godbillon-Vey type functional  $gv^*$  for a 3-dimensional a.c.m. manifold, found its Euler-Lagrange equations for all variations preserving the Reeb vector field and constructed critical 3-dimensional a.c.m. manifolds having a double-twisted product structure, i.e., solutions belonging to the class  $C_5 \oplus C_{12}$  according to Chinea-Gonzalez classification.

In further work, we hope to study the critical a.c.m. manifolds (for  $gv^*$ ) with nonintegrable distribution  $\ker \omega$ . The following tasks also seem interesting: study counterparts  $gv_1^* = \int_M \eta \wedge d\eta^*$  and  $gv_2^* = \int_M \eta^* \wedge d\eta$  of  $gv^*$ ; calculate the second variations of  $gv^*$  and  $gv_i^*$  and find their extrema.

We also intend to study the multidimensional case of  $gv^*$ . For any a.c.m. manifold of dimension  $2n+1 \geq 5$ , one may define one-form  $\eta^* = \eta \circ \varphi = -k (\varphi N)^{\flat}$ , and analogously to the functionals  $gv_s = \int_M \eta \wedge (d\eta)^p \wedge (d\omega)^{n-p}$  for all  $p \geq 1$  in [11, 12], consider the following functionals:

$$gv_s^* = \int_M \eta^* \wedge (d\eta^*)^p \wedge (d\omega)^{n-p}, \quad 1 \le p \le n.$$
 (23)

A question arises: what a.c.m. manifolds, e.g., in  $\bigoplus_{1 \leq i \leq 5} C_i \oplus C_{12}$  due to Chinea-Gonzalez classification, are optimal for functionals (23) with respect to all variations in C(M,T)?

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