Homework 1
ITCS-6114/8114: Algorithms and Data Structures

Due: Thursday, September 9, 2021

Homeworks are due **before midnight** on Thursday, September 9. **Late homeworks will receive no credit.** Homeworks are to be done individually and will be graded on the basis of correctness, clarity, and legibility. Show the steps in your work. Each question is worth **10 points**, for a total of **50 points**.

Be sure to write your name, section, and email address on your homework submission.

Important: Please follow the academic integrity guidelines.
In short, **submit your own work**.

1. For the following, decide whether $T(n) = \Theta(f(n))$, $T(n) = \Omega(f(n))$, $T(n) = O(f(n))$, or none of the above. Justify your answers.

   (a) $T(n) = n^2 + 3n + 4, \qquad f(n) = 6n + \log n.$

   (b) $T(n) = n, \qquad f(n) = (\log n)^{10}.$

   (c) $T(n) = \sum_{i=1}^{n} 3^i, \qquad f(n) = 3^{n-1}.$

   (d) $T(n) = n^{\log c}, \qquad f(n) = c^{\log n}$, where $c > 1$ is a constant.

2. Let `A[]` be an array of n distinct numbers. If index `i<j` and `A[i]` $>$ `A[j]`, then the pair `(i, j)` is called an **inversion** of A. In other words, two elements of the array are considered to form an inversion if they are "out of order".

   (a) List the five inversions of the array `<2, 3, 8, 6, 1>`

   (b) What array with elements from the set $\{1, 2, \ldots, n\}$ has the most inversions? How many does it have? Justify your answer.

   (c) What is the relationship between the running time of insertion sort and the number of inversions in the input array? Justify your answer.

   (d) Give an algorithm that counts the number of inversions in any permutation on n elements in $\Theta(n \log n)$ worst-case time.
   (Hint: Consider which of the sorting algorithms may be modified to solve this problem.)

3. Given a set S of n integers and another integer x, describe an efficient algorithm that determines whether or not there exist two elements in S whose sum is exactly x. Justify the running time of your algorithm.

4. (a) Prove every polynomial $p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0$ with $a_k > 0$ belongs to $\Theta(n^k)$.
   (b) Use a recursion tree to compute an asymptotic solution for the recurrence $J(n) = J(n/2) + J(n/3) + n$.

5. Give asymptotic bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.
   (a) $T(n) = 5T(n/2) + n^2$
   (b) $T(n) = 3T(n/27) + n^{0.25}$
   (c) $T(n) = 16T(n/4) + n^2 + n \log n$
   (d) $T(n) = 3T(n-1) + 1$