Elementary Study Of Fractional Calculus

Under The Guidance Of
Dr. V. S. Kulkarni

SUBMITTED BY
Adasserykudy Abeena Mohammad
Gupta Vandana Vijay
Kanoujiya Ankita Arbind
Siddiqui Hifza Ansar Ahmed
Singh Shreya Shivshyam
Singh Shwetakumari Sunil
Yadav Malathi Murugesan
Yadav Rina Rameshchand

CONTENTS

INTRODUCTION

- The concept of integration and differentiation has been introduced to students of mathematics in elementary calculus. The differential operators $\frac{d}{dx}, \frac{d}{dx^2}, \frac{d}{dx^3}, \ldots\ldots, \frac{d}{dx^n}$.
- It will come as no surprise that the operator $\frac{d^{-1}}{dx^{-1}}$ is nothing but an indefinite integral in disguise.
- Can the order of differentiation be a fraction?
- This question was first raised by L Hospital on 30th September, 1695 in his letter to Gottfried Leibniz asking "What would be the result of $\frac{D^n x}{Dx^m}$ where $n = \frac{1}{2}$ " ?
- To which he replied,

"Although infinite series and geometry are distant relations, infinite series admits only the use of exponents that are positive and negative integers and does not contain fractions, as yet, know the use of fractional exponents. This is an apparent paradox from which one day, useful consequences will be drawn"

- Fractional derivatives have no obvious geometric interpretation along the lines of introduction to derivatives and integral as slopes and area.
- The subject of fractional calculus caught the attention of other great mathematicians too many of whom directly or indirectly contributed to its development.
- In 1819 , Lacroix became the first mathematician to publish a paper that mentioned a fractional derivative.
- Starting with $y = x^m$, where m is a positive integer.
  n^{th} order derivative of t^n $(n \in \mathbb{Z})$

$$\frac{d^n}{dt^n} t^n = n!,$$

  n^{th} order derivative of $t^m, m, n \in \mathbb{Z}$, $m > n$ is

$$\frac{d^n}{dt^n} t^m = \frac{m!}{(m-n)!} t^{m-n}.$$

- Using the Euler's Gamma function (Γ) property,

$$n! = \Gamma(n+1),$$

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

- Gamma Function is defined for positive and negative $\mathbb{R}$ (except for negative integer and zero).
- We let $m, n \in \mathbb{R}$
- We define fractional derivative of order $\alpha \in \mathbb{R}$, $\alpha \geq 0$, of t^β, $\beta \in \mathbb{R}$ as,

$$\frac{d^\alpha}{dt^\alpha} t^\beta = D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \tag{1}$$

Here the condition $\beta > \alpha$ can be relaxed.

$$\frac{d^{0.5}}{dt^{0.5}} t^{0.5} = \Gamma(1+0.5) = \Gamma(1.5),$$

$$\frac{d^{0.5}}{dt^{0.5}} t = \frac{1}{\Gamma(1.5)} t^{0.5},$$

$$\frac{d^{0.5}}{dt^{0.5}} (1) = \frac{1}{\sqrt{\pi t}} \neq 0.$$

- Neils Henrik Abel was the first Mathematician to apply fractional calculus to a "Physical problem" by using the Riemann-Liouville Fractional integral.
- This famous problem known as Abel's Toutochrone Problem, involves a friction less bead sliding along a wire.
- Another mathematician Oliver Heaviside, used fractional calculus in an "Unorthodox" way to solve problems in Partial and Ordinary differential equations.
- Today we call this method as Heavenside's Operational Calculus.Joseph Liouville also used fractional integral to formulate his potential theory problem.

Properties of gamma functions

1. $\Gamma(z+1) = z\Gamma(z)$, $\quad z \in \mathbb{Q}^+$.
2. $\Gamma(n+1) = n!$, $\quad$ where $n = 0, 1, 2, \dots$.
3. $\Gamma(1) = 1$.
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof 1. By definition of gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t} t^z dt,$$

$$= [-t^z e^{-t}]_0^\infty + \int_0^\infty e^{-t} z t^{z-1} dt,$$

$$= \lim_{t\to\infty} (-t^z e^{-t}) - (0e^{-0}) + z \int_0^\infty e^{-t} t^{z-1} dt.$$

Regarding that, $-t^z e^{-t} \to 0$ as $t \to \infty$.

$$\Gamma(z+1) = z \int_0^\infty e^{-t} t^{z-1} dt$$

$$\therefore \Gamma(z+1) = z\Gamma(z), \qquad \text{where } z \in \mathbb{Q}^+.$$

Proof 2. By using property (1),

$$
\begin{aligned}
\Gamma(n+1) &= n\Gamma(n), \\
&= n\Gamma(n-1+1), \\
&= n(n-1)\Gamma(n-1), \\
&= n(n-1)\Gamma(n-2+1), \\
&= n(n-1)(n-2)\Gamma(n-2), \\
&= n(n-1)(n-2)\ldots\ldots 3.2.1.\Gamma(1), \\
&= n(n-1)(n-2)\ldots\ldots 3.2.1, \\
&= n!.
\end{aligned}
$$

$$\therefore \Gamma(n+1) = n!, \qquad \text{where } n = 0, 1, 2, ...,$$

$$
\begin{aligned}
\text{for} \quad n = 0 &\implies \Gamma(0+1) = 0! = 1, \\
\text{for} \quad n = 1 &\implies \Gamma(1+1) = 1! = 1.
\end{aligned}
$$

Proof 3. By definition of gamma function,

$$
\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{z-1} e^{-t} dt, \\
&= \int_0^\infty e^{-t} dt, \\
&= [-e^{-t}]_0^\infty, \\
&= \lim_{t\to\infty} (-e^{-t}) - (-e^{-0}), \\
&= 1, \\
\therefore \Gamma(1) &= 1.
\end{aligned}
$$

Proof 4. By definition of gamma function,

Consider ,

$$\Gamma\left(\frac{1}{2}\right)^2 = \left(\int_0^\infty t^{\frac{-1}{2}} e^{-1} dt\right)^2,$$

$$= \left(2\int_0^\infty e^{-x^2} dx\right)^2,$$

$$= \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2,$$

$$= \left(\int_{-\infty}^\infty e^{-x^2} dx\right)\left(\int_{-\infty}^\infty e^{-y^2} dy\right),$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x+y)} dxdy,$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta,$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta,$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta = \pi,$$

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi,$$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Examples:

Ex 1: Find $D^{\frac{1}{2}}x$.

Sol: Comparing with $D^{\alpha}x^{\beta}$. we have, $\alpha = \frac{1}{2}, \beta = 1$.

By using,

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad \text{where } \alpha, \beta \in \mathbb{R}+,$$

$$D^{\frac{1}{2}}x = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{\frac{1}{2}},$$

$$= \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}},$$

$$= \frac{1}{\Gamma(\frac{1}{2}+1)}x^{\frac{1}{2}},$$

$$= \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}},$$

$$= 2\sqrt{\frac{x}{\pi}}.$$

Ex 2: Find $D^{\frac{3}{2}}x^2$.

Sol: Comparing with $D^{\alpha}x^{\beta}$ we have $\alpha = \frac{3}{2}, \beta = 2$.

By using,

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},$$

where $\alpha, \beta \in \mathbb{R}+$,

$$D^{\frac{3}{2}}x^2 = \frac{\Gamma(2+1)}{\Gamma(2-\frac{3}{2}+1)}x^{2-\frac{1}{2}},$$

$$= \frac{2!}{\Gamma(\frac{1}{2}+1)}x^{\frac{1}{2}},$$

$$= \frac{2}{\Gamma(\frac{1}{2}+1)}x^{\frac{1}{2}},$$

$$= \frac{2}{\frac{1}{2}\sqrt{\pi}}x^{\frac{1}{2}},$$

$$= 4\sqrt{\frac{x}{\pi}}.$$

University of Mumbai

A Mathematical Controversy

- We expect the derivative of a constant value to be zero. Let's check it using definition (1),

$$D^{\alpha}1 = \frac{\Gamma(1)\, x^{-\alpha}}{\Gamma(1-\alpha)} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2}$$

- This definition seems to produce functions for derivatives of value of $\alpha < 1$.
- Consider the following cases using definition (1):

Case 1 : $\alpha = 0, \quad D^{0}1 = 1.$

Case 2 : $\alpha = -1, \quad D^{-1}1 = \frac{x}{\Gamma(2)} = x.$

Case 3 : $\alpha = \frac{1}{2}, \quad D^{\frac{1}{2}}1 = \frac{\Gamma(1)x^{\frac{-1}{2}}}{\Gamma(1-\frac{1}{2})} = \frac{x^{\frac{-1}{2}}}{\sqrt{\pi}} = \frac{1}{\sqrt{x\pi}}.$

- In the first two cases the definition produces the expected result.
- Controversy arises in third case.
- Fractional derivatives when $0 < \alpha < 1$ are the only cases of concern.
- This concerned the mathematicians around the time of Liouville's work.

- This leads to the following proposition,

  Proposition 1: Let $f(x) = x^{\beta}$. Then $D^{\alpha}x^{\beta}$ is a positive constant iff $\alpha = \beta$ provided that, $\alpha < \beta + 1$ and $\beta + 1 > 0$. In this case $D^{\alpha}x^{\beta} = \Gamma(\alpha + 1)$.

  Proof: Let $D^{\alpha}x^{\beta} = C, \quad$ where $C > 0$.

$$\iff \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha} = C, \qquad \forall x > 0,$$

$$\iff C\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)} = x^{\beta-\alpha} > 0, \qquad \forall x > 0,$$

  On taking ln on both sides we get,

$$\iff (\beta-\alpha)\ln x = \ln\left(\frac{C\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\right), \qquad \forall x > 0,$$

$$\iff \beta-\alpha = 0 \quad \text{and} \quad \frac{C\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)} = 1,$$

$$\iff \beta = \alpha \qquad \text{and} \quad C = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)},$$

$$\iff \beta = \alpha \qquad \text{and} \quad C = \Gamma(\alpha+1),$$

$$\implies D^{\alpha}x^{\beta} = \Gamma(\alpha+1).$$

Hence proved.

Examples of functions of the form $f(x) = x^{\frac{m}{n}}$ where $m, n \in \mathbb{Z}$.

Ex 1. Graphical representation of fractional order derivatives of $\sqrt{x}$.

Sol: By applying,

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \qquad \text{where} \quad \alpha, \beta \in \mathbb{R}+.$$

$$D^{\frac{1}{4}}x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{4})}x^{\frac{1}{4}} = 0.9774x^{\frac{1}{4}}.$$

$$D^{\frac{1}{2}} x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} x^0 = \frac{1}{2}\sqrt{\pi}.$$

$$D^{\frac{3}{4}} x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4})} x^{\frac{-1}{4}} = 0.7232 x^{\frac{-1}{4}}.$$

$$D^{1} x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} x^{\frac{-1}{2}} = \frac{1}{2} x^{\frac{-1}{2}}.$$

Below is the graphical representation of these derivatives,

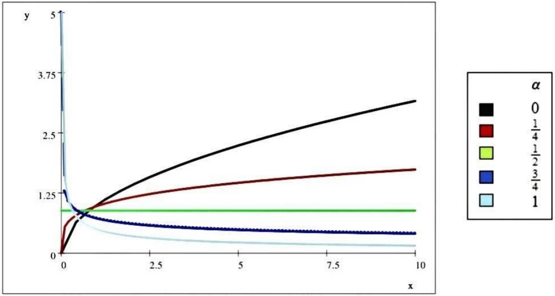


Figure 1: Fractional derivatives of $f(x) = \sqrt{x}$

Remark:

1. As the value of α increases, graph decreases w.r.t fractional derivative of $\sqrt{x}$.
2. Cases of Fractional derivative of $f(x) = \sqrt{x}$

$$D^{\frac{3}{2}} x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(0)} x^{-1}.$$

$$D^{\frac{7}{4}} x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{-1}{4})} x^{\frac{-5}{4}}.$$

   shows the limitations on gamma function.
3. The Gamma function is defined for values than zero.
4. We know $\beta - \alpha + 1 > 0$ and $\beta + 1 > 0$ in,

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad \text{where} \quad \alpha, \beta \in \mathbb{R}+.$$

   From these expressions, we obtain the following restriction on Lacroix's definition of fractional derivative. For

   $D^{\alpha} x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$, where $\alpha, \beta \in \mathbb{R}+$, the following conditions must hold i.e. $\alpha < \beta + 1$ and $\beta > -1$.

Ex 2. Graphical representation of fractional order derivatives of $x^{\frac{-1}{4}}$.

Sol: Here $\beta = \frac{-1}{4} > -1$. Also, here we can take derivatives that satisfy $\alpha < \frac{3}{4}$.

By applying,

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad \text{where} \quad \alpha, \beta \in \mathbb{R}+.$$

$$D^{\frac{1}{8}} x^{\frac{-1}{4}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{8})} x^{\frac{-3}{8}} = 0.8542 x^{\frac{-3}{8}}.$$

$$D^{\frac{1}{4}} x^{\frac{-1}{4}} = \frac{\Gamma(\frac{3}{2})}{\Gamma\frac{1}{2}} x^{\frac{-1}{2}} = 0.6913 x^{\frac{-1}{2}}.$$

$$D^{\frac{3}{8}} x^{\frac{-1}{4}} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{8})} x^{\frac{-5}{8}} = 0.5167 x^{\frac{-5}{8}}.$$

$$D^{\frac{1}{2}} x^{\frac{-1}{4}} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} x^{\frac{-3}{4}} = 0.3380 x^{\frac{-3}{4}}.$$

$$D^{\frac{5}{8}} x^{\frac{-1}{4}} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{8})} x^{\frac{-7}{8}} = 0.1627 x^{\frac{-7}{8}}.$$

Below is the graphical representation of these derivatives,

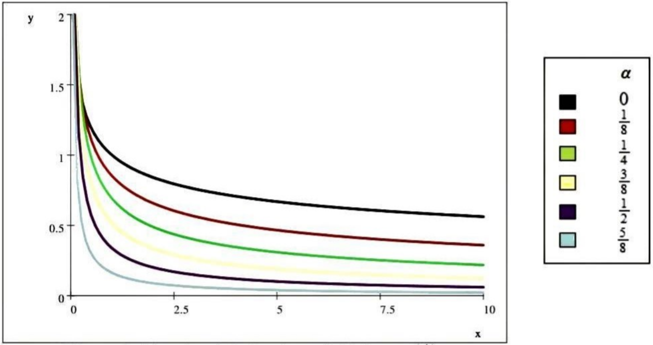


Figure 2: Fractional derivatives of $f(x) = x^{-1/4}$

Remark: As the value of α increases, graph decreases w.r.t fractional derivative of $x^{\frac{-1}{4}}$ and tends to zero.

Fractional Derivative of Exponential Functions

- Joseph Liouville wrote about the theory of fractional calculus in his 1832 memories.
- The derivative of an exponential function, where $n \in \mathbb{Z}_+$ and b is constant, is given by the following formula,

$$\begin{aligned} n &= 1, & D\, e^{bx} &= be^{bx}, \\ n &= 2, & D^2 e^{bx} &= b.be^{bx} = b^2 e^{bx}, \\ n &= 3, & D^3 e^{bx} &= b^2.be^{bx} = b^3 e^{bx}, \end{aligned}$$

and so on ....

$$\text{We get,} \qquad D^n e^{bx} = b^n e^{bx}. \tag{3}$$

- Liouville extended this definition to include derivative of any order α and made the "first major study of fractional calculus." ,

$$D^\alpha e^{bx} = b^\alpha e^{bx}. \tag{4}$$

Justification:

$$\text{Let} \quad f(x) = \sum_{n=0}^{\infty} c_n e^{b_n x}, \qquad \text{where } \operatorname{Re}(b_n) > 0. \tag{5}$$

Taking the derivative yields Liouville's first formula :

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n \, b_n^{\alpha} \, e^{b_n x}. \tag{6}$$

- Here, the value of α can be any value, both real and complex in nature.
- This definition allows for differentiation of functions strictly in the form of (4).
- As an application of Liouville's definition, we can make an extension to the trigonometric functions.

$$\text{Let } f(x) = e^{ibx}, \qquad \text{where } b > 0. \tag{7}$$

Taking the derivative,we get,

$$D^{\alpha}(e^{ibx}) = i^{\alpha} b^{\alpha} e^{ibx},$$

$$D^{\alpha}(e^{ibx}) = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha} b^{\alpha}\big(\cos bx + i\sin bx\big),$$

$$D^{\alpha}(e^{ibx}) = b^{\alpha}\left(cos\frac{\alpha\pi}{2} + isin\frac{\alpha\pi}{2}\right)\big(\cos bx + i\sin bx\big).$$

By De Moivre's Theorem,

$$(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta, \qquad n \in \mathbb{Z},$$

$$D^{\alpha}(\cos bx + i\sin bx) = b^{\alpha}\Big[cos\frac{\alpha\pi}{2}\cos bx - \sin\frac{\alpha\pi}{2}\sin bx + icos\frac{\alpha\pi}{2}\sin bx + i\sin\frac{\alpha\pi}{2}\cos bx\Big],$$

$$D^{\alpha}\cos bx + i\sin bx) = b^{\alpha}\Big[\Big(\cos\frac{\alpha\pi}{2}\cos bx - \sin\frac{\alpha\pi}{2}\sin bx\Big) + i\Big(\cos\frac{\alpha\pi}{2}\sin bx + \sin\frac{\alpha\pi}{2}\cos bx\Big)\Big],$$

$$D^\alpha(\cos bx + i\sin bx) = b^\alpha\left[\cos\left(\frac{\alpha\pi}{2} + bx\right) + i\sin\left(\frac{\alpha\pi}{2} + bx\right)\right],$$

$$D^\alpha(\cos bx) + iD^\alpha(\sin bx) = \left[b^\alpha\cos\left(\frac{\alpha\pi}{2} + bx\right) + ib^\alpha\sin\left(\frac{\alpha\pi}{2} + bx\right)\right].$$

By equating the real and imaginary parts, we obtain:

$$D^\alpha(\cos bx) = b^\alpha\cos\left(\frac{\alpha\pi}{2} + bx\right), \tag{8}$$

$$D^\alpha(\sin bx) = b^\alpha\sin\left(\frac{\alpha\pi}{2} + bx\right). \tag{9}$$

Now, we will consider some examples of exponential , sine and cosine functions and calculate their Fractional derivatives.

Ex 1 :- Graphical representation of fractional order derivatives of e^{bx}.

Soln :- Consider, $f(x) = e^{2x}$,

Applying $D^{\alpha} e^{bx} = b^{\alpha} e^{bx}$ we get ,

$$D^{\frac{1}{4}} e^{2x} = 2^{\frac{1}{4}} e^{2x},$$

$$D^{\frac{1}{2}} e^{2x} = 2^{\frac{1}{2}} e^{2x},$$

$$D^{\frac{3}{4}} e^{2x} = 2^{\frac{3}{4}} e^{2x},$$

$$D^{1} e^{2x} = 2e^{2x}.$$

Below is the graphical representation of these derivatives,

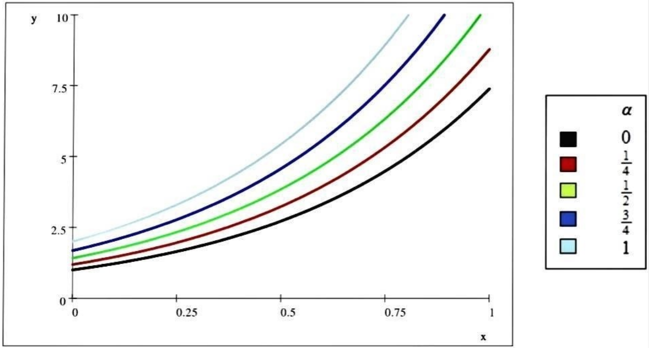


Figure 5: Fractional derivatives of $f(x) = e^{2x}$

Remark: As the value of α increases the graph is increasing that is the higher order derivative of e^{2x} increases.

Ex 2 :- Graphical representation of fractional order derivatives of $\sin bx$

Soln :- Let $f(x) = \sin bx$, where b is a constant.

Consider $f(x) = \sin x$

From equation (2.8),

$$D^{\alpha}(\sin bx) = b^{\alpha} \sin\left(\frac{\alpha\pi}{2} + bx\right)$$

So,

$$D^{\frac{1}{4}} \sin x = \sin\left(x + \frac{\pi}{8}\right).$$

$$D^{\frac{1}{2}} \sin x = \sin\left(x + \frac{\pi}{4}\right).$$

$$D^{\frac{3}{4}} \sin x = \sin\left(x + \frac{3\pi}{8}\right).$$

$$D^{1} \sin x = \sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

Below is the graphical representation of these derivatives,

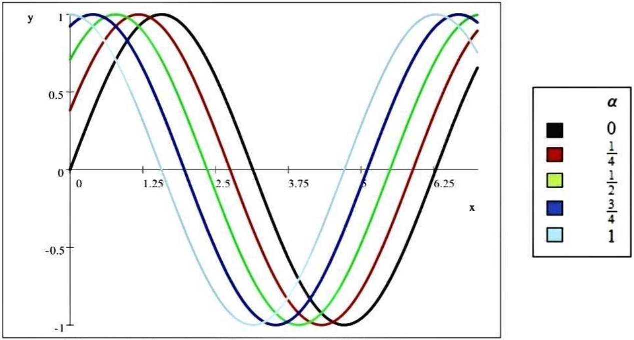


Figure 6: Fractional derivatives of $f(x) = \sin x$

Remark: We can see that sine graph shows a phase shift of $\frac{-\alpha\pi}{2b}$ for the α^{th} derivative.

Ex 3 :- Graphical representation of fractional order derivatives of $\cos bx$.

Soln :- Consider $g(x) = cos(x)$.

From Equation(2.7),

$$D^{\alpha} \cos bx = b^{\alpha} cos\left(bx + \frac{\alpha\pi}{2}\right).$$

So,

$$D^{\frac{1}{4}} \cos x = \cos\left(x + \frac{\pi}{8}\right).$$

$$D^{\frac{1}{2}} \cos x = \cos\left(x + \frac{\pi}{4}\right).$$

$$D^{\frac{3}{4}} \cos x = \cos\left(x + \frac{3\pi}{8}\right).$$

$$D^{1} \cos x = cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

Below is the graphical representation of these derivatives,

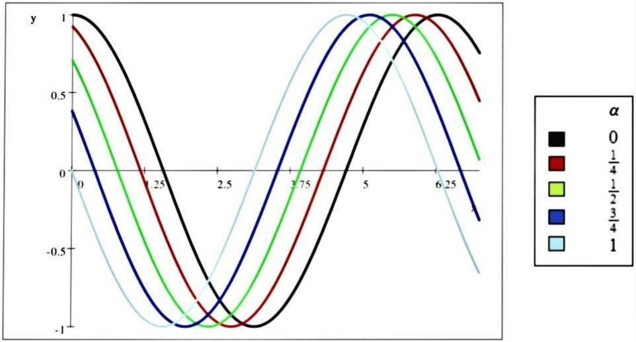


Figure 7: Fractional derivatives of $f(x) = \cos x$

Remark: We can see that cosine graph shows a phase shift of $\frac{\alpha\pi}{2b}$ for the α^{th} derivative.

Finding the Maximum Fractional Derivative Coefficient for $f(x) = x^{\beta}$

Proposition 2:- The fractional derivative of order α for the function $f(x) = x^{\beta}$ is given by $D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(x_0)}x^{(\beta-\alpha)}, \quad \alpha < \beta + 1 \quad \text{and} \quad \beta > -1$. The maximum value of the coefficient $\frac{\Gamma(\beta+1)}{\Gamma(x_0)}$ occurs at $\alpha = \beta - x_0 + 1$, where $x_0 \approx 1.461632$ is the only positive zero of the digamma function.

Proof:- Since, $\quad f(x) = x^{\beta}$.

$$\text{we know, } D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \qquad \text{where} \quad \alpha, \beta \in \mathbb{R}+.$$

$$\text{Let } f(\alpha) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}.$$

By definition of gamma function we have,

$$\Gamma(\beta-\alpha+1) = \int_0^{\infty} e^{-t}t^{\beta-\alpha}dt,$$

$$f^{'}(\alpha) = -\frac{\Gamma(\beta+1)\frac{d}{d\alpha}\Gamma(\beta-\alpha+1)}{[\Gamma(\beta-\alpha+1)]^2},$$

$$f'(\alpha) = 0 \quad \text{iff} \quad \Gamma(\beta-\alpha+1) = 0.$$

Consider $(\beta-\alpha+1) = z$,

$$\implies \frac{d}{d\alpha}[\Gamma(\beta-\alpha+1)] = -\frac{d}{dz}(\Gamma(z)),$$

$$\therefore f'(\alpha) = 0 \iff \frac{d}{d\alpha}[\Gamma(\beta-\alpha+1)] = 0,$$

$$\iff \frac{d}{dz}\Gamma(z) = 0,$$

$$\iff \psi(z) = 0, \quad \text{where digamma function} \quad \psi = \frac{\Gamma^{'}(z)}{\Gamma(z)},$$

$$\iff z = x_0 \approx 1.461632,$$

$$\iff x_0 = \beta - \alpha + 1,$$

$$\iff \alpha = \beta - x_0 + 1.$$

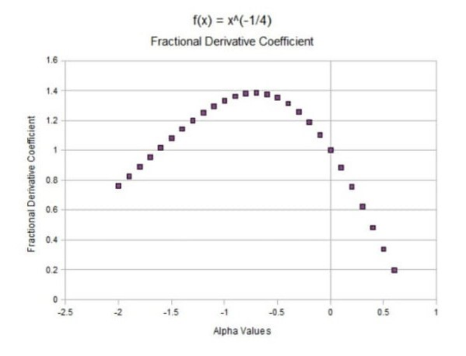
x_0 is the only positive zero of the digamma function. This critical number is indeed a maximum and follows from the fact that the digamma function is an increasing function on the positive real axis.

Digamma function: The digamma function is defined as the logarithmic derivative of the gamma function.

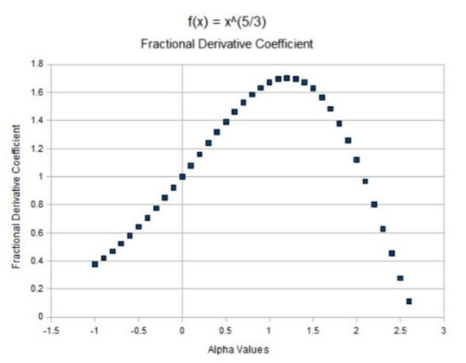
$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

We will verify the above proposition with some example below.

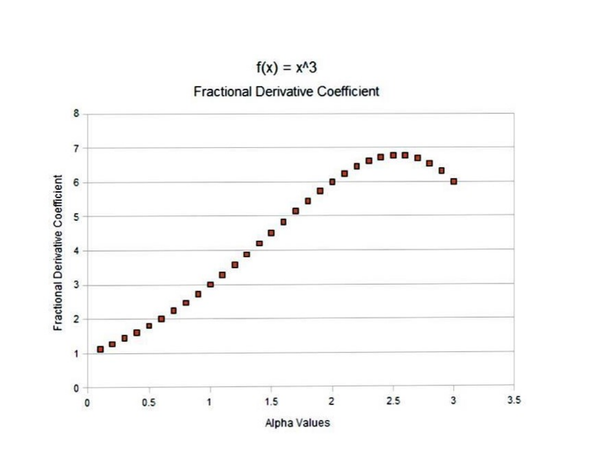
Function(x^{β})	α-value for maximum coefficient: $\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$	numerical value
$x^{\frac{-1}{4}}$	$\alpha = \beta - x_0 + 1$ $= \frac{3}{4} - x_0$ $= \frac{3}{4} - 1.461632$	-0.712



Function(x^β)	α-value for maximum coefficient: $\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$	numerical value
$x^{\frac{5}{3}}$	$\alpha = \beta - x_0 + 1$ $= \frac{8}{3} - x_0$ $= \frac{8}{3} - 1.461632$	1.205



Function(x^{β})	α-value for maximum coefficient: $\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$	numerical value
x^3	$\alpha = \beta - x_0 + 1$ $= 4 - x_0$ $= 4 - 1.461632$	2.538



University of Mumbai

- There are several definition for fractional integral of $f(x)$.
- The Riemann version (when $c < x$) is defined by the integral,

$$ {}_cD_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt, \qquad \text{with} \quad \operatorname{Re}(v) > 0. \tag{10} $$

- The Liouville version (when $c = -\infty$) is defined by the integral,

$$ {}_{-\infty}D_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_{-\infty}^x (x-t)^{v-1} f(t)dt, \qquad \text{with} \quad \operatorname{Re}(v) > 0. \tag{11} $$

- The Riemann-Liouville version (when $c = 0$) is defined by the integral,

$$ {}_0D_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \qquad \text{with} \quad \operatorname{Re}(v) > 0. \tag{12} $$

- The Weyl fractional version is defined by the integral,

$$ {}_xW_\infty^{-v}f(x) = \frac{1}{\Gamma(v)} \int_x^\infty (t-x)^{v-1} f(t)dt, \qquad \text{with} \quad \operatorname{Re}(v) > 0. \tag{13} $$

Note: $f(t)$ must be of the class C, as defined below.
Let $f(t)$ be piecewise continuous on $(0, \infty)$ and integrable on $[0, \infty)$, such functions are said to be of class C.

A Generalized Definition

- We now define the Fractional derivative of any order using the Riemann-Liouville version.
- It works for any function that is continuous and integrable.
  Let n be the smallest positive integer such that, $n > \text{Re}\ (v)$.
  Let $w = n - v$, $0 < \text{Re}\,(w) \leq 1$.
  Using equation (12), define ${}_cD_x^v f(x)$ as,

$$ {}_cD_x^v f(x) = {}_c D_x^n [{}_cD_x^{-w} f(x)], \tag{14} $$

for $x > 0$, provided that the integral ${}_cD_x^v f(x)$ exists.

By denoting the operator ${}_cD_x^{-v}$ in equation (12) by D.

Theorem:- Show that D is a linear operator, i.e. for functions $f, g \in C$ and for scalars α and β the following result holds.

$$D[\alpha f(x) + \beta g(x)] = \alpha Df(x) + \beta Dg(x). \tag{15}$$

Proof: By Riemann-Liouville fractional integral, we have,

$$\begin{aligned}
D[\alpha f(x) + \beta g(x)] &= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} [\alpha f(t) + \beta g(t)] dt, \\
&= \frac{1}{\Gamma(v)} \int_0^x [(x-t)^{v-1} \alpha f(t) + (x-t)^{v-1} \beta g(t)] dt, \\
&= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} \alpha f(t) dt + \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} \beta g(t) dt, \\
&= \frac{\alpha}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt + \frac{\beta}{\Gamma(v)} \int_0^x (x-t)^{v-1} g(t) dt, \\
&= \alpha Df(x) + \beta Dg(x).
\end{aligned}$$

Let us now illustrate the general definition in (14) through an example.

Take $c = 0 \quad \text{and} \quad f(x) = x^{\beta}, \quad \beta > -1.$

Then, ${}_0D_x^{\nu} f(x) = {}_0D_x^{n} \, [{}_0D_x^{-w} \, x^{\beta}].$

Applying the Riemann-Liouville fractional integral in equation (12) we have,

$$ {}_0D_x^{-w} x^{\beta} = \frac{1}{\Gamma(w)} \int_0^x (x-t)^{w-1} t^{\beta} dt. \tag{16} $$

Beta function is defined by the integral as,

$$ B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \mathrm{Re}\,(x) > 0, \ \mathrm{Re}\,(y) > 0. $$

The beta function can also be written as,

$$ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. $$

Change variables, Let $y = \dfrac{x-t}{x}$, $t = x(1-y)$ and $dy = -\dfrac{dt}{x}$.

Consider,

$$\int_0^x (x-t)^{w-1} t^{\beta} dt = \int_0^x \left(\frac{x-t}{x}\right)^{w-1} x^{w-1}\, t^{\beta} dt,$$

$$= \int_1^0 y^{w-1}\, x^{w-1}\, x^{\beta}\, (1-y)^{\beta}\, (-xdy),$$

$$= x^{w+\beta} \int_0^1 y^{w-1}\, (1-y)^{\beta}\, dy,$$

$$\int_0^x (x-t)^{w-1} t^{\beta} dt = x^{w+\beta}\, B(w, \beta+1).$$

Then , by equation (15) we have ,

$${}_0D_x^{-w} x^{\beta} = \frac{1}{\Gamma(w)} \cdot \frac{\Gamma(w)\Gamma(\beta+1)}{\Gamma(w+\beta+1)} x^{(\beta+w)},$$

$${}_0D_x^{-w} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+w+1)} x^{(\beta+w)}, \quad \text{provided} \quad \mathrm{Re}\,(w) > 0,\ x > 0. \qquad (17)$$

$$
\begin{aligned}
\text{So,} \quad {}_0D_x^v\, x^\beta &= {}_0D_x^{n-w}\, x^\beta, \\
&= {}_0D_x^n\, [{}_0D_x^{-w}\, x^\beta], \\
&= {}_0D_x^n \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+w+1)}\, x^{\beta+w} \right], \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+w+1)} \left[\frac{\Gamma(\beta+w+1)}{\Gamma(\beta+w-n+1)}\, x^{\beta+w-n} \right], \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+w-n+1)}\, x^{\beta+w-n}
\end{aligned}
$$

Since, $w = n - v \implies w - n = -v$.

Therefore,

$$
{}_0D_x^v\, x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-v+1)}\, x^{\beta-v}, \quad \beta > -1 \text{ and } x > 0.
$$

Note this is exactly Liouville's definition in (1).

An Elementary Results

1. Functions of the form $f(t) = e^{at}$

Let $f(t) = e^{at}$, where a is a constant.
Then by Riemann-Liouville definition,

$$D^{-v} e^{at} = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} e^{a\xi}\, d\xi, \qquad v > 0.$$

Change variable, Let $x = t - \xi$.
Then,

$$D^{-v} e^{at} = \frac{e^{at}}{\Gamma(v)} \int_0^t x^{v-1} e^{-ax}\, dx.$$

The incomplete gamma function γ^* is defined as,

$$\gamma^*(v, t) = \frac{1}{\Gamma(v)\, t^v} \int_0^t \xi^{(v-1)} e^{-\xi}\, d\xi.$$

By using incomplete gamma function γ^* we get,

$$D^{-v} e^{at} = t^v e^{at} \gamma^*(v, at) = E_t(v, a).$$

This leads to the definition of the fractional integral of $f(t) = e^{at}$ which is given by,

$$E_t\,(v, a) = t^v\, e^{at}\, \gamma^*(v, at),$$
$$E_t\,(v, a) = \frac{e^{at}}{\Gamma(v)} \int_0^t x^{v-1}\, e^{-ax}\, dx.$$

2. Functions of the form $f(t) = \sin at$ and $f(t) = \cos at$

Let $f(t) = \sin at$, where a is a constant then,
By Riemann-Liouville definition,

$$D^{-v}(\sin at) = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,(\sin a\xi)\, d\xi, \qquad v > 0.$$

Changing variable, Let $x = t - \xi$.

$$\therefore D^{-v}(\sin at) = \frac{1}{\Gamma(v)} \int_0^t x^{v-1} \sin a\,(t-x)\, dx, \qquad v > 0. \tag{18}$$

Also can be defined as,

$$S_t(v,a) = \frac{1}{\Gamma(v)} \int_0^t x^{v-1} \sin a\,(t-x)dx,$$

$$\therefore D^{-v}(\sin at) = S_t\,(v,a). \tag{19}$$

Similarly, for $g(t) = \cos at$, where a is a constant.
By Riemann-Liouville definition,

$$D^{-v}(\cos at) = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,(\cos a\xi)\,d\xi, \qquad v > 0.$$

Change variable, Let $x = t - \xi$

$$\therefore D^{-v}(\cos at) = \frac{1}{\Gamma(v)} \int_0^t x^{v-1} \cos a\,(t-x)\,dx, \qquad v > 0. \tag{20}$$

Also can be defined as,

$$C_t(v,a) = \frac{1}{\Gamma(v)} \int_0^t x^{v-1} \cos a\,(t-x)dx,$$

$$\therefore\ D^{-v}(\cos at) = C_t\,(v,a). \tag{21}$$

3. Functions of the form $f(t) = (a - t)^\lambda$

Let $f(t) = (a - t)^\lambda$, where $0 < t < a$.
By Riemann-Liouville definition,

$$D^{-v}(a-t)^\lambda = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}(a-\xi)^\lambda \, d\xi, \qquad Re(v) > 0.$$

Considering,

$$\begin{aligned} \text{Let} \quad x &= \frac{t-\xi}{a-\xi}, \\ \text{then} \quad t-\xi &= x\,(a-\xi), \\ \implies \quad \xi &= \frac{t-ax}{1-x}, \\ a-\xi &= \frac{a-t}{1-x} \qquad \text{and} \qquad t-\xi = \frac{x(a-t)}{1-x}, \\ \text{we also get } d\xi &= \frac{t-a}{(1-x)^2}\, dx. \end{aligned}$$

Upon substitution we obtain the following fractional integral,

$$D^{-v}(a-t)^{\lambda} = \frac{1}{\Gamma(v)} \int_{\frac{t}{a}}^{0} \left[x\left(\frac{a-t}{1-x}\right)\right]^{v-1} \left[\frac{a-t}{1-x}\right]^{\lambda} \left[\frac{t-a}{(1-x)^2}\right] dx.$$

$$D^{-v}(a-t)^{\lambda} = \frac{(a-t)^{\lambda+v}}{\Gamma(v)} \int_{0}^{t/a} x^{v-1}(1-x)^{-v-\lambda-1}\, dx \qquad v > 0.$$

To further simplify we can use incomplete Beta function, i.e.

$$B_{t/a}(v, -\lambda - v) = \int_{0}^{t/a} x^{v-1}(1-x)^{-v-\lambda-1}\, dx.$$

So fractional derivative of $f(t)$ can now be defined as,

$$D^{-v}(a-t)^{\lambda} = \frac{(a-t)^{\lambda+v}}{\Gamma(v)} B_{t/a}(v, -\lambda - v). \tag{22}$$

4. Functions of the form $f(t) = \ln(t)$

Let $f(t) = \ln(t)$.
Then,

$$D^{-\nu} \ln(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \ln(\xi)\, d\xi, \qquad \nu > 0.$$

Change variables, let $\xi = tx$ and $d\xi = t\,dx$,

$$\begin{aligned}
D^{-\nu} \ln(t) &= \frac{1}{\Gamma(\nu)} \int_0^1 (t-tx)^{\nu-1} \ln(tx)\, t\, dx, \\
&= \frac{1}{\Gamma(\nu)} \int_0^1 [t(1-x)]^{\nu-1} \ln(tx)\, t\, dx, \\
&= \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \ln(tx)\, dx, \\
&= \frac{t^\nu \ln(t)}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1}\, dx + \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \ln(x)\, dx, \\
&= \frac{t^\nu \ln(t)}{\nu\,\Gamma(\nu)} + \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \ln(x)\, dx,
\end{aligned}$$

$$D^{-v}\,\ln(t) = \frac{t^{v}\,\ln(t)}{\Gamma(v+1)} + \frac{t^{v}}{\Gamma(v)}\int_0^1 (1-x)^{v-1}\,\ln(x)\,dx. \tag{23}$$

Using the following result from [2], the integral in (3.14) can be rewritten as,

$$\int_0^1 x^{\mu-1}\,(1-x)^{v-1}\,\ln(x)\,dx = B\,(\mu, v)\,[\,\psi\,(\mu) - \psi\,(\mu+v)\,]. \tag{24}$$

where ψ is the digamma function and B is the beta function and Re $\mu > 0$, Re $(v) > 0$.

Let $\mu = 1$ in (23):

$$\int_0^1 (1-x)^{v-1}\,\ln(x)\,dx = B\,(1, v)[\,\psi(1) - \psi\,(1+v)\,],$$

$$\int_0^1 (1-x)^{v-1}\,\ln(x)\,dx = B\,(1, v)[\,-\gamma - \psi\,(1+v)\,].$$

where γ is an Euler's constant defined as,

$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \ln(n)\right) \approx 0.5772157.$

$$\int_0^1 (1-x)^{v-1} \ln(x)\, dx = \frac{-\gamma - \psi(1+v)}{v}.$$

We can rewrite (22) as:

$$D^{-v} \ln(t) = \frac{t^v \ln(t)}{\Gamma(v+1)} + \frac{t^v}{v\,\Gamma(v)}[-\gamma - \psi(1+v)],$$

$$D^{-v} \ln(t) = \frac{t^v}{\Gamma(v+1)}[\ln t - \gamma - \psi(1+v)].$$

The fractional integral of $f(t) = \ln(t)$ is given by,

$$D^{-v} \ln(t) = \frac{t^v}{\Gamma(v+1)}[\ln t - \gamma - \psi(1+v)]. \tag{25}$$

5. Functions of the form $g(t) = tf(t)$

Let $g(t) = tf(t)$.
Then by Riemann-Liouville definition,

$$D^{-v}\,[t\,f(t)] = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,[\,\xi\,f\,(\xi)\,]\,d\xi.$$

Replace $\xi\,f\,(\xi)$ by $[\,t-(t-\xi)\,]\,f\,(\xi)$.

$$\begin{aligned}
D^{-v}[\,t\,f(t)\,] &= \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,[\,t-(t-\xi)\,]\,f(\xi)\,d\xi, \\
&= \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,t\,f(\xi)\,d\xi - \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,(t-\xi)\,f(\xi)\,d\xi, \\
&= \frac{t}{\Gamma(v)} \int_0^t (t-\xi)^{v-1}\,f(\xi)\,d\xi - \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v}\,f(\xi)\,d\xi, \\
&= t\,D^{-v}\,f(t) - v\,D^{-(v+1)}\,f(t).
\end{aligned}$$

$\therefore$ The fractional integral of $g(t) = t\,f(t)$ is given by,

$$D^{-v}[t\,f(t)] = t\,D^{-v}\,f(t) - v\,D^{-(v+1)}\,f(t). \tag{26}$$

The above result can be applied to some functions as,
for $f(t) = t\,e^{at}$,

$$D^{-v}[t\,e^{at}] = t\,E_t\,(v, a) - v\,E_t\,(v+1, a), \tag{27}$$

for $g(t) = t\ \cos at$,

$$D^{-v}[t\ \cos at] = t\,C_t\,(v, a) - v\,C_t\,(v+1, a), \tag{28}$$

for $g(t) = t\ \sin at$,

$$D^{-v}[t\ \sin at] = t\,S_t\,(v, a) - v\,S_t\,(v+1, a). \tag{29}$$

A generalized rule for $D^{-v}\,[\,t^p f(t)\,]$

- We obtained an expression for the functional integral, $D^{-v}\,[\,t\,f(t)\,] = t\,D^{-v}f(t) - v\,D^{-(v+1)}f(t)$, of function of the form $g(t) = t\,f(t)$.
- We can make a generalization of this equation namely compute $D^{-v}[\,t^p\,f(t)\,]$ when p is a non-negative integer.
  We have,

$$D^{-v}[\,t^p\,f(t)\,] = \frac{1}{\Gamma(v)}\int_0^t (t-\xi)^{v-1}\,[\,\xi^p f(\xi)\,]\,d\xi. \tag{30}$$

In addition,

$$\xi^p = [t-(t-\xi)]^p = \sum_{k=0}^{p}\binom{p}{k}(\xi-t)^k\,t^{p-k} = \sum_{k=0}^{p}(-1)^k\binom{p}{k}(t-\xi)^k\,t^{p-k}.$$

Substitute this result into equation (29),

$$D^{-v}[\,t^p f(t)\,] = \frac{1}{\Gamma(v)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}t^{p-k}\int_0^t (t-\xi)^{v+k-1}\,f(\xi)\,d\xi,$$

$$= \frac{1}{\Gamma(v)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} t^{p-k}\, \Gamma(v+k)\, D^{-(v+k)} f(t),$$

$$D^{-v}[\, t^p f(t)\,] = \frac{1}{\Gamma(v)} \sum_{k=0}^{p} (-1)^k \frac{p!}{k!(p-k)!}\, t^{p-k}\, \Gamma(v+k)\, D^{-(v+k)} f(t).$$

Using the formula,

$\binom{-v}{k} = (-1)^k \dfrac{\Gamma(v+k)}{k!\Gamma(v)}$, and (1.1) rewrite the above expression as,

$$D^{-v}[\, t^p f(t)\,] = \sum_{k=0}^{p} \binom{-v}{k} [\, D^k t^p\,]\,[\, D^{-(v+k)} f(t)\,]. \tag{31}$$

Leibniz's Rule for Fractional Integrals in Calculus

Suppose, there are two functions $f(t)$ and $g(t)$, which have the derivative upto n^{th} order. Let us consider now the derivative of the product of these two functions. The first derivative could be written as,

$$D[fg] = f'g + gf'.$$

Now, if we differentiate the above expression again, we get the second derivative,

$$\begin{aligned}
D^2[fg] &= (fg)^{''}, \\
&= [(fg)^{'}]^{'}, \\
&= (f^{'}g)^{'} + (fg^{'})^{'}, \\
&= f^{''}g + f^{'}g^{'} + f^{'}g^{'} + fg^{''}, \\
&= f^{''}g + 2f^{'}g^{'} + fg^{''}.
\end{aligned}$$

Similarly, We can find the third derivative;

$$
\begin{aligned}
D^3[fg] &= (fg)''', \\
&= f''g + 2f'g' + fg'', \\
&= (f''g)' + (2f'g')' + (fg'')', \\
&= f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg''', \\
&= f'''g + 3f''g + 3f'g'' + fg'''.
\end{aligned}
$$

We notice that the coefficient on each term are the numbers in each row of Pascal‘s triangle or the binomial coefficients.
In general, Leibniz rule (or product rule) in calculus is written by,

$$D^n[\,f(t)\,g(t)\,] = \sum_{k=0}^{n} \binom{n}{k} [\,D^k g(t)\,]\,[\,D^{n-k} f(t)\,]. \tag{32}$$

where f and g are n-fold differentiable on an interval.

Proof : We prove this by Principle of Mathematical Induction,
When $n = 1$,

$$D^1[f(t)\, g(t)] = \sum_{k=0}^{1} \binom{1}{k} [D^k g(t)]\, [D^{n-k} f(t)],$$

$$= \binom{1}{0} [g(t)]\, [Df(t)] + \binom{1}{1} [Dg(t)]\, [f(t)],$$

$$= gf' + g' f.$$

which is true for $n = 1$.

Assume, $D^n\, [f(t)\, g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)]\, [D^{n-k} f(t)]$, is true for any positive integer n then.

$$D\left[D^n\left[f(t)g(t)\right]\right] = D\left[\sum_{k=0}^{n}\binom{n}{k}[D^k g(t)][D^{n-k}f(t)]\right],$$

$$= D\left[\binom{n}{0}[g(t)][D^n f(t)] + \binom{n}{1}[Dg(t)][D^{n-1}f(t)] + \dots\right.$$

$$\left. + \binom{n}{n-1}[D^{k-1}g(t)][D^{n-k+1}f(t)] + \binom{n}{n}[D^k g(t)][D^{n-k}f(t)]\right],$$

$$= D\Big[[g(t)][D^n f(t)] + n[Dg(t)][D^{n-1}f(t)] + \dots$$

$$+ n[D^{k-1}g(t)][D^{n-k+1}f(t)] + [D^k g(t)][D^{n-k}f(t)]\Big],$$

$$= g^{'} D^n f + gD^{n+1}f + n[D^2 gD^{n-1}f + DgD^n f] + \dots +$$

$$n[D^k gD^{n-k+1}f + D^{k-1}gD^{n-k+2}f] + [D^{k+1}gDn - kf + D^k gD^{n-k+1}f],$$

$$= g^{'} D^n f + gD^{n+1} f + ng^{''} D^{n-1} f + ng^{'} D^n f + \cdots +$$

$$nD^k gD^{n-k+1} f + nD^{k-1} gD^{n-k+2} f + D^{k+1} gD^{n-k} f + D^k gD^{n-k+1} f,$$

$$= gD^{n+1} f + (n+1)g^{'} D^n f + ng^{''} D^{n-1} f + \cdots +$$

$$nD^{k-1} gD^{n-k+2} f + (n+1)D^k gD^{n-k+1} f + D^{k+1} gD^{n-k} f.$$

$$[D^{n+1}[f(t)g(t)]] = D\left[\sum_{k=0}^{n} \binom{n+1}{k} [D^k g(t)][D^{n-k+1} f(t)]\right].$$

Therefore, equation (31) is true for $n+1$.

Now, equation (31) can be extended to the topic of study by applying it to fractional operators. Using the definition of fractional integral, we have,

$$D^{-v}[f(t)g(t)] = \frac{1}{\gamma(v)} \int_0^t (t-\xi)^{v-1} [f(\xi)g(\xi)]\, d\xi, \quad 0 < t \leq X. \tag{33}$$

Let the function f and g are in C, fg is a member of class C and the fractional integral exist if it satisfies the following condition,

- f must be continuous on $[0,X]$,
- g must be analytic at some point a, $\forall a \in [0, X]$.

$g(\xi)$ can be written using the Taylor series below:

$$g(\xi) = \sum_{k=0}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\xi)^k,$$

$$g(\xi) = g(t) + \sum_{k=1}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\xi)^k. \tag{34}$$

Substitute (33) into (32), We get,

$$D^{-v}[f(t)g(t)] = \frac{1}{\gamma(v)}\int_0^t (t-\xi)^{v-1}\left[f(\xi)\left[g(t) + \sum_{k=1}^{\infty}(-1)^k \frac{D^k g(t)}{k!}(t-\xi)^k\right]\right] d\xi.$$

The order of integration and summation may be interchanged.
Since, f is continuous on $[0, X]$ and $(t-\xi)^k f(\xi)$ is bounded on $[0, t]$.

$$D^{-v}[\,f(t)g(t)\,] = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(v+k)}{k!\Gamma(v)}\,[\,D^k g(t)\,]\,[\,D^{-v-k} f(t)\,],$$

$$D^{-v}[\,f(t)g(t)\,] = \sum_{k=0}^{\infty}\binom{-v}{k}\,[\,D^k g(t)\,]\,[\,D^{-v-k} f(t)\,]. \tag{35}$$

Equation (34) is the Leibniz's Rule for Fractional Integral.

University of Mumbai

- The differintegral is a combined differentiation and integration operator applied to a function f.
- The q - differintegral of f denoted by $D^q f$ is the fractional derivative (if $q > 0$) or fractional integral (if $q < 0$).
- If $q = 0$, then the q^{th} differintegral of a function is the function itself.
  Consider,
  Ordinary derivatives are defined in terms of backward differences as,

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h},$$

$$\frac{d^2 f}{dx^2} = \lim_{h \to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}.$$

We generalize for $n \in \mathbb{N}, \quad and \quad f \in C^n[a,b], \quad a < x < b.$

$$D^n f(x) = \lim_{h \to 0} \frac{\sum_{j=0}^{n} [(-1)^j \binom{n}{j} f(x - jk)]}{h^n}. \tag{36}$$

- Grunwald and Letnikov modified equation (35) and gave meaning to the operator D^q, $q \notin \mathbb{N}$.
- We modify, equation (35) so it will represent n-fold integral for negative integral values of n.

  Consider, a base point a and define for $x > a$, $h_N = \dfrac{x-a}{N}$, $N \in \mathbb{Z}^+$.

  The symbol $\binom{n}{j}$ can be generalized for negative integers as,

$$\binom{-n}{j} \approx \frac{-n(-n-1) - \;............\; - (-n-j+1)}{j!},$$

from (35), for $n = -n$,

$$D_a^{-n} f(x) = \lim_{N \to \infty} h_N^n \left[\sum_{j=0}^{N} (-1)^j \binom{-n}{j} f(x - jh_N) \right]. \tag{37}$$

Note:

- $\binom{n}{j}$ has non-zero values for finitely many values of j.
- $\binom{-n}{j}$ has non-zero values for infinitely many values of j.

$$\therefore D_a^{-1}f(x) = \lim_{N\to\infty} h_N \left[\sum_{j=0}^{N}(-1)^j \binom{-1}{j} f(x - jh_N)\right],$$

$$= \lim_{N\to\infty} h_N \left[\sum_{j=0}^{N} f(x - jh_N)\right],$$

$$= \int_a^x f(z)dz.$$

Similarly,

$$D_a^{-2}f(x) = \lim_{N\to\infty} h_N^2 \left[\sum_{j=0}^{N}(-1)^j \binom{-2}{j} f(x - jh_N)\right],$$

$$= \lim_{N\to\infty} h_N^2 \left[\sum_{j=0}^{N}(j+1) f(x - jh_N)\right],$$

$$= \lim_{N\to\infty} h_N \left[\sum_{j=0}^{N} jh_N f(x - jh_N)\right],$$

$$= \int_a^x (x - z) f(z) dz.$$

Hence , we further similarly get,

$$D_a^{-n} f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!} f(z) dz,$$

$$= \int_a^x dt \int_a^{x_1} dt \,.......... \int_a^{x_{n-1}} f(t) dt.$$

Equation (35) i.e.,

$$D^n f(x) = \lim_{h\to 0} \frac{\sum_{j=0}^{n} \left[(-1)^j \binom{n}{j} f(x - jh\right]}{h^n}.$$

is called the differintegral, as it represents differentiation for $n > 0$ and integration for $n < 0$.

The Fractional Derivative

A General Definition

Let the function f be of class C and let $\mu > 0$.
Let $m = \lceil \mu \rceil$.
The definition of fractional derivative of the function f with order μ as the following,

$$D^{\mu} f(t) = D^{m}[D^{-v} f(t)], \qquad \mu > 0, t > 0. \tag{38}$$

where $v = m - \mu$,
This definition also holds when $\mu = p \in \mathbb{Z}_{+}$,

$$D^{p} f(t) = D^{p+1} \int_{0}^{t} f(\xi) d\xi = D^{p} f(t). \tag{39}$$

A New class of function

Many examples included in next section are of the following form,

$$t^{\lambda}\eta(t), \tag{40}$$

$$t^{\lambda}(\ln t)\eta(t). \tag{41}$$

In this case, $\lambda > -1$ and $\eta(t)$ must be an entire function.
Define a subclass of C as $C^{'}$.
If f belongs to $C^{'}$ it has both a fractional integral and a fractional derivative $C^{'}$ can be defined as a space of functions that are of the form in above equation (39) and equation (40)
The following sections will hut if a fractional integral of a function that belongs to class $C^{'}$ exists,the fractional derivative can be found by replacing v by $-\mu$.
That is,

$$D^{\mu}f(t) = [D^{v}f(t)]_{v=-\mu}. \tag{42}$$

Examples of fractional derivatives

In all of these examples the previous definition (41) is applied.

Ex 1 :- Find the fractional derivative for, $f(t) = e^{at}$.

Soln:- Let $f(t) = e^{at}$, where a is a constant,

$$D^{\mu} e^{at} = D^{m}[D^{-v} e^{at}]. \tag{43}$$

Here $v = m - \mu$... from equation(4.3).

Recall that, $D^{-v}\ e^{at} = E_t(v, a)$.

Also recall the formula,

$D^m E_t(v, a) = E_t(v - m, a) E_t(-\mu, a)$,

$$\text{we have, } D^{\mu} e^{at} = E_t(-\mu, a), \qquad t > 0. \tag{44}$$

Ex 2 :- Find the fractional derivative for, $f(t) = sin(at)$ and $g(t) = cos(at)$.

Soln:- Let $f(t) = \sin(at)$, where a is a constant.

$$D^{\mu} \sin(at) = D^{m}[D^{-v} \sin(at)],$$

Recall that $D^{-v} \sin(at) = S_t(v, a)$,

$$D^{m} S_t(v, a) = S_t(v - m, a) = S_t(-\mu, a). \tag{45}$$

Let $g(t) = \cos(at)$,where a is a constant.

$$D^{\mu} \cos(at) = D^{m}[D^{-v} \cos(at)],$$

Recall that, $D^{-v} \cos(at) = C_t(v, a)$,

$$D^{m} C_t(v, a) = C_t(v - m, a) = C_t(-\mu, a). \tag{46}$$

Ex 3 :- Find the fractional derivative for, $f(t) = (a - t)^{\lambda}$

Soln:- Let $f(t) = (a - t)^{\lambda}$,

We know that,

$$D^{-v}[(a - t^{\lambda})] = \frac{1}{\Gamma(v)} \int_0^t (t - \xi)^{v-1}.(a - \xi)^{\lambda} d\xi, \text{ where } v > 0, 0 < t < a,$$

from (3.13), we know that fractional derivative can be rewritten as,

$$D^{-v}[(a - t)^{\lambda}] = \frac{(a - t)^{\lambda+v}}{\Gamma(v)}.B_{\frac{t}{a}}(v, -\lambda - v)].$$

where the incomplete gamma function and its derivative are given by:

$$B_{\tau}(x, y) = \int_0^{\tau} t^{x-1}(1 - t)^{y-1} dt \text{ and } D_{\tau}B_{\tau}(x, y) = \tau^{x-1}.(1 - \tau)^{y-1}.$$

Let us compute a few derivatives in order to discern a pattern.

$$D[D^{-v}(a-t)^{\lambda}] = -\frac{(\lambda+v)(a-t)^{\lambda+v-1}}{\Gamma(v)} \cdot B_{\frac{t}{a}}(v,-\lambda-v)$$

$$+\frac{(a-t)^{\lambda+v}}{\Gamma(v)} \cdot \frac{1}{a}\left(\frac{t}{a}\right)^{v-1}\left(1-\frac{t}{a}\right)^{-\lambda-v-1},$$

$$D[D^{-v}(a-t)^{\lambda}] = -\frac{(\lambda+v)(a-t)^{\lambda+v-1}}{\Gamma(v)} \cdot B_{\frac{t}{a}}(v,-\lambda-v) + \frac{a^{\lambda+1}t^{v-1}}{\Gamma(v)(a-t)},$$

$$D^{2}[D^{-v}(a-t)^{\lambda}] = \frac{(\lambda+v)(\lambda+v-1)(a-t)^{\lambda+v-2}}{\Gamma(v)} \cdot B_{\frac{t}{a}}(v,-\lambda-v)$$

$$-\frac{(\lambda+v)(a-t)^{\lambda+v-1}}{\Gamma(v)} \cdot a^{\lambda+1}t^{v-1}(a-t)^{-\lambda-v+1}$$

$$+\frac{a^{\lambda+1}}{\Gamma(v)}\left[\frac{(v-1)t^{v-2}}{(a-t)} + \frac{t^{v-1}}{(a-t)^{2}}\right],$$

$$= \frac{(\lambda + v)(\lambda + v - 1)(a-t)^{\lambda+v-2}}{\Gamma(v)} B_{\frac{t}{a}}(v, -\lambda - v)$$

$$+ \frac{a^{\lambda+1}}{\Gamma(v)} \left[\frac{-(\lambda + v - 1)t^{v-1}}{(a-t)^2} + \frac{(v-1)t^{v-2}}{(a-t)} \right],$$

$$D^3[D^{-v}(a-t)^{\lambda}] = -\frac{(\lambda + v)(\lambda + v - 1)(\lambda + v - 2)(a-t)^{\lambda+v-3}}{\Gamma(v)} \cdot B_{\frac{t}{a}}(v, -\lambda - v)$$

$$+ \frac{a^{\lambda+1}}{\Gamma(v)} \left[\frac{(\lambda + v - 1)(\lambda + v - 2)t^{v-1}}{(a-t)^3} - \frac{(\lambda + v - 2)(v-1)t^{v-2}}{(a-t)^2} \right.$$

$$\left. + \frac{(v-1)(v-2)t^{v-3}}{(a-t)} \right].$$

The following proposition establishes the $\mu -$ th $(\mu > 0)$ fractional derivative.

Proposition 3: Let $f(t) = (a-t)^{\lambda}$, $\lambda \in \mathbb{R}$ then, for v>0, the fractional integral is given by,

$$D^{-v}[(a-t)^{\lambda}] = \frac{(a-t)^{\lambda+v}}{\Gamma(v)} B_{\frac{t}{a}}(v, -\lambda-v),$$

and for $\mu > 0$, $\mu = m - v$, where $v > 0$ and $m = \lceil \mu \rceil$,the $\mu -$ th fractional derivative is given by,

$$D^{\mu}(a-t)^{\lambda} = D^{m}[D^{-v}(a-t)^{\lambda}],$$

$$D^{\mu}(a-t)^{\lambda} = \frac{D^{m}(a-t)^{\lambda+v}}{\Gamma(v)} B_{\frac{t}{a}}(v, -\lambda-v)$$

$$+ \frac{a^{\lambda+2}}{\Gamma(v+1)} \sum_{k=1}^{m} \left[(-1)^{m-k} \prod_{j=k}^{m-1} (\lambda + v - j) \right] \frac{D^{k} t^{v}}{(a-t)^{m-k+1}}. \qquad (47)$$

Ex 4:- Find the fractional derivative for, $f(t) = \ln t$

Soln:- Let $f(t) = \ln t$,

from (25), we know that,

$$D^{-v} \ln t = \frac{t^v}{\Gamma(v+1)}[\ln t - \gamma - \Psi(1+v)].$$

Let us look at a few examples.

$$D[D^{-v} \ln t] = \frac{vt^{v-1}}{\Gamma(v+1)}[\ln t - \gamma - \Psi(v+1)] + \frac{t^v}{\Gamma(v+1)t},$$

$$D[D^{-v} \ln t] = \frac{vt^{v-1}}{\Gamma(v+1)}[\ln t - \gamma - \Psi(v+1) + \frac{1}{v}],$$

$$D^2[D^{-v} \ln t] = \frac{v(v-1)t^{v-2}}{\Gamma(v+1)}\left[\ln t - \gamma - \Psi(v+1) + \frac{1}{v}\right] + \frac{vt^{v-1}}{\Gamma(v+1)t},$$

$$D^2[D^{-v} \ln t] = \frac{v(v-1)t^{v-2}}{\Gamma(v+1)}\left[\ln t - \gamma - \Psi(v+1) + \frac{1}{v} + \frac{1}{v-1}\right],$$

$$D^3[D^{-v}\ln t] = \frac{v(v-1)(v-2)t^{v-3}}{\Gamma(v+1)}\left[\ln t - \gamma - \Psi(v+1) + \frac{1}{v} + \frac{1}{v-1}\right] + \frac{v(v-1)t^{v-2}}{\Gamma(v+1)},$$

$$D^3[D^{-v}\ln t] = \frac{v(v-1)(v-2)t^{v-3}}{\Gamma(v+1)}\left[\ln t - \gamma - \Psi(v+1) + \frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-2}\right].$$

The following proposition establishes the μ-th ($\mu > 0$) fractional derivative.

Proposition 4: Let $f(t) = \ln t$. Then, for $v > 0$, the fractional integral is given by,

$$D^{-v}\ln t = \frac{t^v}{\Gamma(v+1)}[\ln t - \gamma - \Psi(1+v)],$$

and for $\mu > 0$, $\mu = m - v$, where $v > 0$ and $m = \lceil \mu \rceil$ the μ-th fractional derivative is given by,

$$D^{\mu}\ln t = D^m[D^{-v}\ln t].$$

Then,

$$D^{\mu}\ln t = \frac{D^m t^v}{\Gamma(v+1)}\left[\ln t - \gamma - \Psi(1+v) + \sum_{k=0}^{m-1}\frac{1}{v-k}\right]. \quad (48)$$

Ex 5 :- Find the fractional derivative for, $f(x) = tf(t)$

Soln :- Using the generalized rule for $D^{\mu}\left[t^{p}f(t)\right]$, consider the case when $p = 1$.

$$D^{\mu}\left[tf(t)\right] = \sum_{r=0}^{1} \binom{\mu}{r} \left[D^{r}t\right]\left[D^{\mu-r}f(t)\right],$$

$$D^{\mu}\left[tf(t)\right] = \binom{\mu}{0}\left[D^{0}t\right]\left[D^{\mu}f(t)\right] + \binom{\mu}{1}\left[Dt\right]\left[D^{\mu-1}f(t)\right],$$

$$D^{\mu}\left[tf(t)\right] = tD^{\mu}f(t) + \mu D^{\mu-1}f(t). \tag{49}$$

Recall that $D^{-v}\left[tf(t)\right] = tD^{-v}f(t) - vD^{-v-1}f(t)$.

This is identical, with the v's replaced by $-\mu$'s.

This general definition can be applied to other functions, we can show that,

$$D^{\mu}\left[te^{at}\right] = tD^{\mu}e^{at} + \mu D^{\mu-1}e^{at}, \tag{50}$$

$$D^{\mu}\left[t\sin at\right] = tD^{\mu}\sin at + \mu D^{\mu-1}\sin at, \tag{51}$$

$$D^{\mu}\left[t\cos at\right] = tD^{\mu}\cos at + \mu D^{\mu-1}\cos at. \tag{52}$$

A Generalized Rule for $D^{\mu}\left[t^p f(t)\right]$

Using Leibniz's Rule for fractional derivatives, we can generalize a formula for $D^{\mu}\left[t^p f(t)\right]$, that is, when f is a polynomial.

$$D^{\mu}\left[t^p f(t)\right] = D^m\left[D^{-(m-\mu)} t^p f(t)\right]. \tag{53}$$

Using Lebniz's rule for the fractional integral and the fact that $v = m - \mu$,

$$D^{-(m-\mu)}\left[t^p f(t)\right] = \sum_{k=0}^{p} \binom{\mu - m}{k} \left[D^k t^p\right]\left[D^{-m+\mu-k} f(t)\right]. \tag{54}$$

Next find the m^{th} or ordinary derivative of (52),

When $n \in \mathbb{N}$, it is obvious that,

$$D^n\left[D^k t^p\right] = D^{n+k} t^p.$$

Use the following result (see Miller and Rose),

$$D^n\left[D^{-m+\mu-k} f(t)\right] = D^{n-m+\mu-k} f(t) \qquad n \in \mathbb{N} \text{ and } f(t) \in C'.$$

Impose this condition upon f and rewrite (52),

$$D^{\mu}\left[t^{p}f(t)\right]=\sum_{k=0}^{p}\binom{\mu-m}{k}D^{m}\left\{\left[D^{k}t^{p}\right]\left[D^{-m+\mu-k}f(t)\right]\right\}.$$

Using Lebniz's rule for fractional integrals we get,

$$D^{\mu}\left[t^{p}f(t)\right]=\sum_{K=0}^{p}\binom{\mu-m}{k}\sum_{j=0}^{m}\binom{m}{j}\left[D^{j+k}t^{p}\right]\left[D^{\mu-k-j}f(t)\right],$$

where $r=j+k$ and $s=k$.

$$D^{\mu}\left[t^{p}f(t)\right]=\sum_{r=0}^{p}\left[\sum_{s=0}^{r}\binom{\mu-m}{s}\binom{m}{r-s}\right]\left[D^{r}t^{p}\right]\left[D^{\mu-r}f(t)\right]. \qquad (55)$$

To evaluate the inner sum, we use algebraic identity,

$$(1+x)^{\mu-m}(1+x)^{m}=(1+x)^{\mu}.$$

Use the Binomial Theorem to expand the term in parentheses.

$$\sum_{s=0}^{r} \binom{\mu - m}{s} \binom{m}{r - s} = \binom{\mu}{r}.$$

This is called **Vandermonde Convolution formula**.
Substitute this expression into (4.20),

$$D^{\mu}\left[t^{p} f(t)\right] = \sum_{r=0}^{p} \binom{\mu}{r} \left[D^{r} t^{p}\right]\left[D^{\mu - r} f(t)\right]. \tag{56}$$

Note that this definition is same as $D^{-v}\left[t^{p} f(t)\right]$, with the v's replaced by $-\mu$'s.
Also, it is important to note that,

$$D^{v}\left[t^{p} f(t)\right] \text{ is valid for f} \in C,$$
$$\text{while } D^{\mu}\left[t^{p} f(t)\right] \text{ is valid for f} \in C'.$$

1. Eigenfunction

The eigenfunctions of the fractional derivatives D^α are defined as the solution of the fractional differential equation,

$$D^\alpha f(t) = \lambda f(t), \tag{57}$$

where λ is the eigenvalue.

The solution of equation (56) can be easily found by means of Laplace transform,

$$L[f(t)] = F(s) = \int_0^\infty f(t)\, exp(-st)dt. \tag{58}$$

The Laplace transform of $D^\alpha f(t)$ is simply $s^\alpha F(s)$.

Applying (57) to (56) the solution in the imaginary space reads,

$$F(s) = \frac{1}{s^{\alpha} - \lambda}. \tag{59}$$

The Laplace transforms of the Mittag-Leffler functions $E_{\alpha,\beta}$(x) is,

$$L[x^{\beta-1}E_{\alpha,\beta}(-\lambda x^{\alpha})] = \frac{s^{\alpha-\beta}}{s^{\alpha} + \lambda}. \tag{60}$$

Using Equation (59), the inverse Laplace transform gives the solution of equation (56).

$$f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}). \tag{61}$$

2. Summation of series

The series,

$$S = \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + ..., \tag{62}$$

can be obtained from the following expansion,

$$e^t = \frac{1}{\Gamma(1)} + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + ..., \tag{63}$$

by applying the operator$I^{\frac{1}{2}}$ to the later. using result $e^t = e^t erf t^{\frac{1}{2}}$, the sum S become,

$$S = I^{\frac{1}{2}} e^t = e^t erf t^{\frac{1}{2}}. \tag{64}$$

It should be noted, that applied the fractional integration for the series summation is advisable when the coefficient of expansion contain Gamma function of fractional argument .This often occurs in the theory of heat and mass transfer.

3. Heat and Mass flux determination

Let us consider the heat of the semi-infinite area with the initial temperature be equal to zero.
The mathematical statement of the problem,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) T = 0, \quad 0 < x < \infty, \tag{65}$$

$$T(0,t) = T_s(t), \tag{66}$$

$$T(\infty,t) = 0, \tag{67}$$

$$T(x,0) = 0. \tag{68}$$

By factorization method, we factorize a differential operator,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) T = 0.$$

According to the well known algebraic formula,

$$a^2 - b^2 = (a - b)(a + b),$$

equation (64) becomes,

$$\left(\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{\partial}{\partial x}\right)\left(\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} + \frac{\partial}{\partial x}\right) T = 0. \tag{69}$$

Let us consider the equation formed by the right multiplier of the differential operator,

$$\left(\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} + \frac{\partial}{\partial x}\right) T = 0. \tag{70}$$

The solution of the latter is the solution of the equation (5.12) as well. It is worth to note that according to Fick's and Fourier's laws, the heat and mass fluxes are equal to the gradient of the concentration and temperature, respectively. So, in order to determine the fluxes we need to know the spatial distribution of the concentration or temperature.

But rewrite the equation (69) in the following form,

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} T = -\frac{\partial}{\partial x} T = q.$$

We arrive at the very important conclusion that the flux at some point $x = x_0$ can be found without knowledge of the spatial temperature distribution as a fractional derivative of $\frac{1}{2}$ order of the temperature with respect to the time.
It should be noted here that the solution of the equation formed by the left multiplier,

$$\left(\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{\partial}{\partial x} \right) T = 0, \tag{71}$$

does not satisfy the condition given by equation (66).
To show this, applying the laplace transform to equation (70) results in the ordinary differential equation of the first order,

$$\frac{d\Im}{dx} = s^{\frac{1}{2}} \Im, \tag{72}$$

where $\Im(x, s) = L[T(x, t)]$ is Laplace transform of T.

By the separation of variable, the solution of equation (71) can be easily found.

$$\Im = const * exp(s^{\frac{1}{2}}x)$$

As x turns to the infinity, $\Im$ turns to the infinity as well. Coming back to equation (66), let us write this equation at $x = 0$:

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}T(0,t) = D^{\frac{1}{2}}T_s(t) = -\frac{\partial}{\partial x}T\bigg|_{x=0} = q_s, \tag{73}$$

where q_s is the heat flux at $x = 0$.

Thus, the heat flux has been found without knowing the spatial temperature distribution. By using the formula for fractional derivative of order α, given as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}d\tau, \tag{74}$$

the heat flux at $x = 0$ can be written in the explicit form,

$$q_s = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t \frac{T_s(\tau)}{(t-\tau)^{1/2}} d\tau,$$
$$q_s = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{T_s(\tau)}{\sqrt{t-\tau}} d\tau. \tag{75}$$

Reference :

[1] Alexander I. Fedrochenko, *Introduction to Fractional Calculus* Institute of Applied Mechanics, National Taiwan University, 2008.

[2] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, pp. 538, 1980.

[3] Lacroix, S.F., *Traite'du calcul Differentiel et du calcul integral*, first edition, Tome Troisieme,seconde edition pp 409-410, (1797-1800).

[4] Leibnitz, G.W., Leibnitzen's *Mathematische schriften, Hildesheim Germany*: Georg Olm, V.2., pp 301-302, 1962.

[5] Miller, R. and Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: John Wiley and Sons, Inc, 1993.

[6] Richard Herrmann, World Scientific., *Fractional Calculus:an introduction for physicists*.

[7] Varsha Daftardar-Gejji *Fractional Calculus and Fractional Differential Equations* Springer International Publishing, 2019.