# Problem Set 2

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1. Blood testing. Suppose that a large number, n, of blood samples are to be screened for a relatively rare disease. If each sample is assayed individually, n tests will be required. On the other hand, it is possible, assuming that the disease is rare, that some savings can be achieved through some pooling procedure. The purpose of this exercise is examine some common pooling procedures.

   Consider the following scheme for group testing. The original lot of samples is divided into two groups and each of the subgroups is tested as a whole. If either subgroup tests positive, it is divided in two and the procedure is repeated. If any of the groups thus obtained tests positive, test every member of that group. (We will assume that the test method is sensitive enough; a group tests positive if and only if at least one person is positive in that group.)

   - Find the expected number of tests performed.
   - Compare it to the number of tests performed with no groupings. For which value of $p$ is this group testing scheme inferior to testing every individual?
   - Consider now the following scheme. The n samples are first grouped into $m$ groups of $k$ samples each, or $n = mk$. Each group is then tested; if a group tests positively, each individual in the group is tested. Find the expected number of tests performed.

2. Two aces. The following problem is called the two aces problem. This problem dating back to 1936 has been attributed to the English mathematician J. H. C. Whitehead. This problem was also submitted to Marilyn vos Savant by the master of mathematical puzzles Martin Gardner, who remarks that it is one of his favorites. A bridge hand (13 cards) has been dealt. (a) If the hand has one ace, what is the probability that the hand has another ace? (b) Given the hand has the ace of hearts, what is the probability that the hand has another ace? Are these probabilities equal?

3. Let $X$ and $Y$ be independent normal random variables each having parameters $\mu$ and $\sigma^2$. Show that $X + Y$ is independent of $X - Y$ .

4. Generating discrete random variables. In this exercise we wish to sample from a discrete distribution but we only have available a routine that samples uniform random variables.

   - Show that the following method for generating discrete random variables works. Suppose for concreteness that $X$ takes on values 0, 1, 2, ... with probabilities $p_0$, $p_1$, $p_2$, ..., Let $U$ be a uniform random variable. If $U < p_0$, return $X = 0$. If not replace $U$ by $Up_0$, and if the new $U$ is less than $p_1$, return $X = 1$. If not, decrement $U$ by $p_1$, compare to $p_2$, etc.
   - Write a computer program to implement this strategy. Test your program by sampling from the Poisson distribution with parameter $\lambda = 5$. Generate 1000 samples and graph your results