Practical 1:

Topic: Limits and continuity

Q1 $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$= \lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{a+2x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$= \lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \left(\frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right)$

$= \lim_{x \to a} \frac{(a-x)}{3(a-x)} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{3a+x} + 2\sqrt{x})}$

$= \frac{1}{3} \lim_{x \to a} \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}}$

$= \frac{1}{3} \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$= \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}}$

$= \frac{1}{3} \frac{2\sqrt{a}}{\sqrt{3a} \cdot \sqrt{a}}$

$= \frac{2}{3\sqrt{3}}$

2] $\lim\limits_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$= \lim\limits_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$= \lim\limits_{y \to 0} \frac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y} + \sqrt{a})}$

$= \lim\limits_{y \to 0} \frac{y}{y\sqrt{a+y}(\sqrt{a+y} + \sqrt{a})}$

$= \lim\limits_{y \to 0} \frac{1}{\sqrt{a+y}(\sqrt{a+y} + \sqrt{a})}$

$= \frac{1}{\sqrt{a}+(\sqrt{a}+\sqrt{a})}$

$= \frac{1}{\sqrt{2} \cdot 2\sqrt{a}}$

$= \frac{1}{2a}$

3] $\lim\limits_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \frac{\pi}{6}$

as $x \to \frac{\pi}{6}$, $h \to 0$

$= \lim\limits_{h \to 0} \frac{\cos(h+\frac{\pi}{6}) - \sqrt{3}\sin(h+\frac{\pi}{6})}{\pi - 6(h+\frac{\pi}{6})}$

$= \lim\limits_{h \to 0} \frac{\cos h \cos\frac{\pi}{6} - \sin h \sin\frac{\pi}{6} - \sqrt{3}\left(\sin h \cos\frac{\pi}{6} + \cos h \sin\frac{\pi}{6}\right)}{\pi - 6\left(\frac{6h+\pi}{6}\right)}$

$= \lim\limits_{h \to 0} \frac{\cos h \frac{\sqrt{3}}{2} - \sin h \frac{1}{\sqrt{2}} - \sqrt{3}\left(\sin h \frac{\sqrt{3}}{2} + \cos h \frac{1}{2}\right)}{\pi - 6h + \pi}$

$= \lim\limits_{h \to 0} \frac{\cos \frac{\sqrt{3}}{2} h - \sin \frac{h}{2} - \sin \frac{3h}{2} - \cos \frac{\sqrt{3}}{2} h}{-6h}$

$= \lim\limits_{h \to 0} \frac{-\sin \frac{4h}{2}}{-6h}$

$= \lim\limits_{h \to 0} \frac{\sin 4h}{3+2h}$

$= \frac{1}{3} \lim\limits_{h \to 0} \frac{\sin h}{h}$

$= \frac{1}{3}$

4] $\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

By rationalising both;

$$= \lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2+5-x^2+3}{x^2+3-x^2-1}\,\frac{(\sqrt{x^2+3}+\sqrt{x^2+1})}{(\sqrt{x^2+5}+\sqrt{x^2-3})}\right]$$

$$= \lim_{x\to\infty} \frac{8\,(\sqrt{x^2+3}+\sqrt{x^2+1})}{2\,(\sqrt{x^2+5}+\sqrt{x^2-3})}$$

$$= 4\lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+\sqrt{\left(x^2+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get,

$= 4$

Q5 $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0<x\le\frac{\pi}{2}$
$= \frac{\cos x}{\pi-2x}$, for $\frac{\pi}{2}<x<\pi$ } at $x=\frac{\pi}{2}$

Solution : $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

$\therefore f(\pi/2) = 0$

f at $x=\pi/2$ defined

- RHS = $\lim_{x\to\pi/2^+} \frac{\cos x}{\pi-2x}$

Put $x-\pi/2 = h$ where $x\to\pi/2, h\to 0$

$$= \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi-2\left(h+\frac{\pi}{2}\right)}$$

$$= \lim_{h\to 0} \frac{\cos(h+\pi/2)}{\pi-2\left(\frac{2h+\pi}{2}\right)}$$

$$= \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$$

$$= \lim_{h\to 0} \frac{\cos h\cdot\cos\frac{\pi}{2} - \sin h\cdot\sin\frac{\pi}{2}}{-2h}$$

$$= \lim_{h\to 0} \frac{-\sin h}{-2h}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{2}$$

$$LHS = \lim_{x \to \pi/2^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$$

$$= \lim_{x \to \pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$$

$$= \lim_{x \to \pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2} \cdot \sin x}$$

$$= \lim_{x \to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$$

$\because$ LHS $\neq$ RHS

$\therefore$ f is not continuous at $x = \pi/2$

3] $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & 0 < x < 3 \\ x+3 & 3 \leq x < 6 \\ \dfrac{x^2-9}{x+3} & 6 \leq x < 9 \end{cases}$ at $x = 3$ & $x = 6$

Solution: $f(3) = \dfrac{x^2-9}{x-3}$

$= 0$

f at $x=3$ is defined

$$RHS = \lim_{x \to 3^+} x+3$$

$$= 6$$

$$LHS = \lim_{x \to 3^-} \frac{x^2-9}{x-3}$$

$$= \lim_{x \to 3^-} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3^-} (x+3)$$

$$= 6$$

$\therefore$ RHS = LHS

$\therefore$ f is continuous at $x = 3$

for $x = 6$

$$f(6) = \frac{x^2-9}{x+3}$$

$$= \frac{36-9}{9}$$

$$= \frac{27}{9}$$

$$= 3$$

$$RHS = \lim_{x \to 6^+} \frac{x^2-9}{x+3}$$

$$= \lim_{x \to 6^+} \frac{(x-3)(x+3)}{(x+3)}$$

$$= \lim_{x \to 6^+} (x-3)$$

$$= 3$$

$$LHS = \lim_{x \to 6^-} x+3$$

$$= 9$$

$\because$ LHS $\neq$ RHS

$\therefore$ f is not continuous at $x = 6$

8] $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x<0 \\ = k & x=0 \end{cases}$ at $x=0$

solution : f is continuous at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \dfrac{1-\cos 4x}{x^2} = k$

$\lim_{x\to 0} \dfrac{2\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \dfrac{\sin^2 2x}{x^2} = k$

$2 \lim_{x\to 0^+} \left(\dfrac{\sin 2x}{x}\right)^2 = k$

$2(2)^2 = k$

$\therefore k = 8$

9] $f(x) = \begin{cases} (\sec^2 x)^{\cot^2 x} & x \neq 0 \\ = k & x=0 \end{cases}$ at $x=0$

solution = using $\tan^2 x - \sec^2 x - 1$

$\therefore \sec^2 x = 1+\tan^2 x$

& $\cot^2 x = \dfrac{1}{\tan^2 x}$

$= \lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$

$= \lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$

we know that

$\lim_{x\to 0} (1+px)^{1/px} = e$

$\therefore k = e$

$f(x) = \begin{cases} \dfrac{\sqrt{3} - \tan x}{\pi - 3x} & x \neq \pi/3 \\ = k & x = \pi/3 \end{cases}$ at $x=\pi/3$

solution: Put $x - \pi/3 = h$

where $x \to \pi/3$, $h \to 0$

$f(\pi/3+h) = \dfrac{\sqrt{3} - \tan(\pi/3+h)}{\pi - 3\left(\frac{\pi}{3}+h\right)}$

$\lim_{h\to 0} \dfrac{\sqrt{3} - \tan(\pi/3+h)}{\pi - 3(\pi/3+h)}$

$\lim_{h\to 0} \dfrac{\sqrt{3} - \dfrac{\tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\pi - \pi - 3h}$

$\lim_{h\to 0} \dfrac{\dfrac{(\sqrt{3} - \sqrt{3}\times\sqrt{3}\cdot\tan h) - (\sqrt{3}+\tan h)}{1-\tan \pi/3 \cdot \tan h}}{-3h}$

$\lim_{h\to 0} \dfrac{-4\tan h}{-3h(1-\sqrt{3}\tan h)}$

$\dfrac{4}{3} \lim_{h\to 0} \dfrac{\tan h}{h} \lim_{h\to 0} \dfrac{1}{(1-\sqrt{3}\tan h)}$

$\dfrac{4}{3} \cdot \dfrac{1}{1-\sqrt{3}(0)}$

$= \dfrac{4}{3}$

Q.1] $f(x) = \dfrac{1-\cos 3x}{x\tan x}$ } at $= 0$
$\quad = 9$

$f(x) = \dfrac{1-\cos 3x}{x\tan x}$

$\lim_{x\to 0} \dfrac{2\sin^2 3x/2x}{x\tan x}$

$\lim_{x\to 0} \dfrac{\dfrac{2\sin^2 3x/2}{x^2} \times x^2}{\dfrac{x\tan x}{x^2} \times x^2}$

$= 2\lim_{x\to 0} \dfrac{\left(\frac{3}{2}\right)^2}{1}$

$= 2 \times \frac{9}{4}$

$= \frac{9}{2}$

$\lim_{x\to 0} f(x) = \frac{9}{2}$, $9 = f(0)$

$\therefore f$ is not continuous at $x=0$

Redefine function

$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x\tan x}, & x \neq 0 \\ = \frac{9}{2}, & x = 0 \end{cases}$

Now $\lim_{x\to 0} f(x) = f(0)$

2] $f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin x^\circ}{x^2}, & x \neq 0 \\ = \pi/6, & x = 0 \end{cases}$ at $x=0$

$\lim_{x\to 0} \dfrac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$

$\lim_{x\to 0} \dfrac{e^{3x}-1}{x} \cdot \lim_{x\to 0} \dfrac{\sin(\pi x/180)}{x}$

$\lim_{x\to 0} 3 \cdot \dfrac{e^{3x}-1}{3x} \quad \lim_{x\to 0} \dfrac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$3\lim_{x\to 0} \dfrac{e^{3x}-1}{3x} \quad \lim_{x\to 0} \dfrac{\sin(\pi x/180)}{x}$

$3\log e \; \frac{\pi}{180}$

$\frac{\pi}{60}$

$f(0)$

f is continuous at $x=0$

Q8] $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x = 0$

is continuous at $x = 0$

solution: $\therefore$ given f is continuous at $x \to 0$

$$\lim_{x \to 0} f(x) = f(0)$$

$$= \lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}$$

$$= f(0)$$

$$\lim_{x \to 0} \frac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$$

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{x} + \lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$\log e + \lim_{x \to 0} \left(\frac{2 \sin^2 x/2}{x} \right)$$

Multiply with 2 on both,

$$= 1 + 2 \times \frac{1}{4}$$

$$= \frac{3}{2}$$

$$= f(0)$$

Q9] $f(x) = \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ $x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$\lim_{x \to \pi/2} \frac{2 - 1 + \sin x}{\cos^2 x (\sqrt{2} + \sqrt{1} + \sin x)}$$

$$\lim_{x \to \pi/2} \frac{1 + \sin x}{1 - \sin^2 x (\sqrt{2} + \sqrt{1 + \sin x})}$$

$$\lim_{x \to \pi/2} \frac{1}{(1 - \sin x)(\sqrt{2} + \sqrt{1 + \sin x})}$$

$$= \frac{1}{2(\sqrt{2} + \sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}$$

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{4\sqrt{2}}$$

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Practical 2:

Topic: Derivative

Q1] Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable

a] $\cot x$

Solution: $\therefore f(x) = \cot x$

$$\therefore Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

$$= \lim_{x \to a} \frac{\tan x - \tan a}{(x-a)\tan x \tan a}$$

$\therefore$ put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$\therefore Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a) \cdot \tan(a+h) \tan a}$$

$$\therefore Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \cdot \tan(a+h) \cdot \tan a}$$

$$\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$= \lim_{h \to 0} \frac{\tan(a - a - h) \cdot (1 + \tan a \tan(a+h))}{h \cdot \tan(a+h) \cdot \tan a}$$

$$= \lim_{h \to 0} \frac{-\tan h}{h} \times \frac{1 + \tan a \tan(a+h)}{\tan(a+h) \tan a}$$

$$= -1 \times \frac{1 + \tan^2 a}{\tan^2 a}$$

$$= -\frac{\sec^2 a}{\tan^2 a}$$

$$= -\frac{1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$\therefore Df(a) = -\operatorname{cosec}^2 a$

$\therefore f$ is differentiable $\forall a \in \mathbb{R}$

b] $\operatorname{cosec} x$

Solution: $\therefore f(x) = \operatorname{cosec} x$

$$\therefore Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec}(x) - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sin x - \sin a}{(x-a)\sin a \sin x}$$

$\therefore$ put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$\therefore Df(h) = \lim_{h\to 0} \dfrac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$

$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$= \lim_{h\to 0} \dfrac{2\cos\left(\frac{a+a+h}{2}\right)\cdot \sin\left(\frac{a-a-h}{2}\right)}{h\cdot \sin a \cdot \sin(a+h)}$

$= \lim_{h\to 0} \dfrac{-\sin\frac{h}{2}}{h/2}\cdot\frac{1}{2} \times \dfrac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \cdot \sin(a+h)}$

$= -\frac{1}{2} \times \dfrac{2\cos\left(\frac{2a+h}{2}\right)}{\sin(a+h)(\sin a)}$

$= -\dfrac{\cos a}{\sin^2 a}$

$= -\cot a \cdot \operatorname{cosec} a$

$\therefore Df(a) = -\cot a \cdot \operatorname{cosec} a$

$\therefore f$ is differentiable $\forall a \in R$

$\sec x$

Solution: $\therefore f(x) = \sec x$

$\therefore Df(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$

$= \lim_{x\to a} \dfrac{\sec x - \sec a}{x-a}$

$= \lim_{x\to a} \dfrac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$

$= \lim_{x\to a} \dfrac{\cos a - \cos x}{(x-a)\cos a \cos x}$

$\therefore$ put $x - a = h$

$x = a+h$

as $x \to a,\ h \to 0$

$\therefore Df(h) = \lim_{h\to 0} \dfrac{\cos(a) - \cos(a+h)}{h \cdot \cos a \cdot \cos(a+h)}$

$\left[\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$

$= \lim_{h\to 0} \dfrac{-2\sin\left(\frac{a+a+h}{2}\right)\cdot \sin\left(\frac{a-a-h}{2}\right)}{h \cdot \cos a \cos(a+h)}$

$= \lim_{h\to a} \dfrac{-2\sin\left(\frac{2a+h}{2}\right)\sin -\frac{h}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}}$

$\cdot -\frac{1}{2}$

$= -\frac{1}{2} \times \frac{-2\cos\sin\left(\frac{2a+0}{2}\right)}{\cos a \cdot \cos(a+0)}$

$= -\frac{1}{2} \times \frac{-2\sin a}{\cos^2 a}$

$= \frac{\sin a}{\cos a \cdot \cos a}$

$= \tan a \cdot \sec a$

$\therefore Df(a) = \tan a \sec a$

$\therefore f$ is differentiable $\forall a \in R$.

Q2] If $f(x) = 4x+1$, $x \le 2$
$= x^2+5$, $x > 0$, at $x = 2$
then find function is differentiable or not.

Solution:

LHD: $Df(2^-) = \lim_{x\to 2^-} \frac{f(x) - f(2)}{x-2}$

$= \lim_{x\to 2^-} \frac{4x+1-\cancel{x^2+5}\ 4(2)+1}{x-2}$

$= \lim_{x\to 2^-} \frac{4x+1-9}{x-2}$

$= \lim_{x\to 2^-} \frac{4x-8}{x-2}$

$= \lim_{x\to 2^-} \frac{4(x-2)}{(x-2)}$

$= 4$

RHD: $Df(2^+) = \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$

$= \lim_{x\to 2^+} \frac{x^2-4}{x-2}$

$= \lim_{x\to 2^+} (x+2)$

$= 2+2$

$= 4$

$\because$ LHD = RHD

$\therefore f$ is differentiable at $x = 2$

Q3] If $f(x) = 4x+7$, $x < 3$
$= x^2+3x+1$, $x \ge 3$ at $x = 3$
then find f is differentiable or not?

Solution:

RHD: $Df(3^+) = \lim_{x\to 3^+} \frac{f(x) - f(3)}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+9+1)}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x+1-19}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+3x-18}{x-3}$

$= \lim_{x\to 3^+} \frac{x^2+6x-3x-18}{x-3}$

$$= \lim_{x\to 3^+} \frac{(x+6)(x-3)}{x-3}$$

$$= 3+6$$

$$= 9$$

$$LHD = Df(2^-) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4(x-3)}{(x-3)}$$

$$= 4$$

$\because$ LHD $\neq$ RHD

$\therefore f$ is not differentiable at $x=3$

Q.4] If $f(x) = 8x-5,\ x \leq 2$
$= 3x^2-4x+7,\ x>2$ at $x=2$
then find f is differentiable or not

Solution:

$$RHD: Df(2^+) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-4x+7-11}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-4x-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2-6x+2x-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(3x+2)}{x-2}$$

$$= \lim_{x\to 2^+} (3x+2)$$

$$= 3\times 2+2$$

$$= 8$$

$$LHD = Df(2^-) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x-5-11}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x-16}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8(x-2)}{x-2}$$

$$= 8$$

$\because$ LHD = RHD

$\therefore f$ is differentiable at $x=3$

Practical 3:

Title: Application of derivatives.

Q.1] Find the intervals in which function f is increasing or decreasing

a] $f(x) = x^3 - 5x - 11$.

Solution: f is increasing if & only if $f'(x) > 0$

$\because f(x) = x^3 - 5x - 11$

$\therefore f'(x) = 3x^2 - 5$

$\therefore 3x^2 - 5 > 0$

$\therefore x > \pm\sqrt{\frac{5}{3}}$

$+ \quad - \quad +$

$-\infty \quad -\sqrt{\frac{5}{3}} \quad \sqrt{\frac{5}{3}} \quad \infty$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

Now f is decreasing if & only if $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore x < \pm\sqrt{\frac{5}{3}}$

$+ \quad - \quad +$

$-\infty \quad -\sqrt{\frac{5}{3}} \quad \sqrt{\frac{5}{3}} \quad \infty$

$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

b] $f(x) = x^2 - 4x$

Solution: f is increasing if & only if $f'(x) > 0$

$\because f(x) = x^2 - 4x$

$\therefore f'(x) = 2x - 4$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$\therefore x > 2$

$- \quad +$

$-\infty \quad 2 \quad \infty$

$\therefore x \in (2, \infty)$

Now f is decreasing if & only if $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$\therefore x < 2$

$- \quad +$

$-\infty \quad 2 \quad \infty$

$\therefore x \in (-\infty, 2)$

$\boxed{c}$ $f(x) = 2x^3 + x^2 - 20x + 4$

Solution: f is increasing if & only if

$f'(x) > 0$

$\because f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 6x^2 + 12x - 10x - 20 > 0$

$\therefore 6x(x+2) - 10(x+2) > 0$

$\therefore (x+2)(6x-10) > 0$

$\therefore x \geqslant -2, \frac{5}{3}$

$+ \quad - \quad +$

$-\infty \quad -2 \quad \frac{5}{3} \quad \infty$

$\therefore x \in (-\infty, -2) \cup (\frac{5}{3}, \infty)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$\therefore (x+2)(6x-10) < 0$

$\therefore x \lessdot -2, 5/3$

$+ \quad - \quad +$

$-\infty \quad -2 \quad \frac{5}{3} \quad \infty$

$x \in (-2, \frac{5}{3})$

$\boxed{d}$ $f(x) = x^3 - 27x + 5$

Solution: f is increasing if & only if

$f'(x) > 0$

$\therefore f(x) = x^3 - 27x + 5$

$\therefore f'(x) = 3x^2 - 27$

$\therefore 3x^2 - 27 > 0$

$\therefore 3(x^2 - 9) > 0$

$\therefore x^2 - 9 > 0$

$\therefore x \geqslant 3, -3$

$+ \quad - \quad +$

$-\infty \quad -3 \quad 3 \quad \infty$

$\therefore x \in (-\infty, -3) \cup (3, \infty)$

Now f is decreasing if & only if

$f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore 3(x^2 - 9) < 0$

$\therefore x^2 - 9 < 0$

$\therefore x \lessdot 3, -3$

$+ \quad - \quad +$

$-\infty \quad -3 \quad 3 \quad \infty$

$\therefore x \in (-3, 3)$

e] $f(x) = 69 - 24x - 9x^2 + 2x^3$

solution: f is increasing if & only if $f'(x) > 0$

$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore -(6x^2 + 18x + 24) > 0$

$\therefore \cancel{-6(x^2 + 3x + 4) > 0}$

$\therefore \cancel{-6(x^2 + 4x - x + 4) > 0}$

$\therefore \cancel{-((x+4)(x-1))}$

$\therefore -24 - 18x + 6x^2 > 0$

$\therefore -6(-4 - 3x + x^2) > 0$

$\therefore x^2 - 3x - 4 > 0$

$\therefore x^2 - 4x + x - 4 > 0$

$\therefore (x-4)(x+1) > 0$

$\therefore x > 4, -1$

$-\infty$ + -1 − 4 + ∞

$x \in (-\infty, -1) \cup (4, \infty)$

Now f is decreasing if & only if $f'(x) < 0$

$\therefore -24 - 18x + 6x^2 < 0$

$\therefore 6(-4 - 3x + x^2) < 0$

$\therefore (x-4)(x+1) < 0$

$\therefore x < 4, -1$

$-\infty$ + -1 − 4 + ∞

$x \in (-1, 4)$

Q2] Find the intervals in which function is concave upwards & concave downwards

a] $y = 3x^2 - 2x^3$

solution: $y = f(x)$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

$\because f$ is concave upward if & only if $f''(x) > 0$

$\therefore 6 - 12x > 0$

$\therefore 6(1 - 12x) > 0$

$\therefore (1 - 12x) > 0$

$\therefore -(2x - 1) > 0$

$-\infty$ + $\frac{1}{2}$ − ∞

$x \in (-\infty, \frac{1}{2})$

$\because f$ is concave downward if & only if $f''(x) < 0$

$\therefore 6(1 - 12x) < 0$

$\therefore -(2x - 1) < 0$

$-\infty$ + $\frac{1}{2}$ − ∞

$x \in (\frac{1}{2}, \infty)$

6] $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

solution: $\therefore\ y = y(x)$

$\therefore\ y(x) = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\therefore\ y'(x) = 4x^3 - 18x^2 + 24x + 5$

$\therefore\ y''(x) = 12x^2 - 36x + 24$

$\therefore$ y is concave upward if & only if

$y''(x) > 0$

$\therefore\ 12x^2 - 36x + 24 > 0$

$\therefore\ 12(x^2 - 3x + 2) > 0$

$\therefore\ x^2 - 3x + 2 > 0$

$\therefore\ (x-2)(x-1) > 0$

$\therefore\ x > 2, 1$

$+ \quad - \quad +$

$-\infty \quad 1 \quad 2 \quad \infty$

$x \in (-\infty, 1) \cup (2, \infty)$

$\therefore$ y is concave downward if & only if

$y''(x) < 0$

$\therefore\ 12x^2 - 36x + 24 < 0$

$\therefore\ 12(x^2 - 3x + 2) < 0$

$\therefore\ x^2 - 3x + 2 < 0$

$\therefore\ (x-2)(x-1) < 0$

$\therefore\ x < 2, 1$

$+ \quad - \quad +$

$-\infty \quad 1 \quad 2 \quad \infty$

$x \in (1, 2)$

] $y = x^3 - 27x + 5$

solution: $\therefore\ y = y(x)$

$\therefore\ y(x) = x^3 - 27x + 5$

$\therefore\ y'(x) = 3x^2 - 27$

$\therefore\ y''(x) = 6x$

$\therefore$ y is concave upward if & only if

$y''(x) > 0$

$\therefore\ 6x > 0$

$\therefore\ x > 0$

$\therefore\ x > 0$

$- \quad +$

$-\infty \quad 0 \quad \infty$

$\therefore\ x \in (0, \infty)$

$\therefore$ y is concave downward if & only if

$y''(x) < 0$

$\therefore\ 6x < 0$

$\therefore\ x < 0$

$\therefore\ x < 0$

$- \quad +$

$-\infty \quad 0 \quad \infty$

$\therefore\ x \in (-\infty, 0)$

d] $y = 69 - 24x - 9x^2 + 2x^3$

Solution: $\therefore y = f(x)$

$\therefore f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore f''(x) = -18 + 12x$

$\therefore f$ is concave upward if & only if

$f''(x) > 0$

$\therefore -18 + 12x > 0$

$\therefore 6(2x - 3) > 0$

$\therefore 2x - 3 > 0$

$\therefore x > 3/2$

$-\infty \quad - \quad \frac{3}{2} \quad + \quad \infty$

$\therefore x \in \left(+\frac{3}{2}, \infty\right)$

$\therefore f$ is concave downwards if & only if

$f''(x) < 0$

$\therefore -18 + 12x < 0$

$\therefore 6(2x - 3) < 0$

$\therefore 2x - 3 < 0$

$\therefore x < 3/2$

$-\infty \quad - \quad \frac{3}{2} \quad + \quad \infty$

$\therefore x \in \left(-\infty, \frac{3}{2}\right)$

e] $y = 2x^3 + x^2 - 20x + 4$

Solution: $\therefore y = f(x)$

$\therefore f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore f''(x) = 12x + 2$

$\therefore f$ is concave upwards if & only if

$f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 2(6x + 1) > 0$

$\therefore 6x + 1 > 0$

$\therefore x > -1/6$

$-\infty \quad - \quad -\frac{1}{6} \quad + \quad \infty$

$\therefore x \in \left(-\frac{1}{6}, \infty\right)$

$\therefore f$ is concave downward if & only if

$f''(x) < 0$

$\therefore 12x + 2 < 0$

$\therefore 2(6x + 1) < 0$

$\therefore 6x + 1 < 0$

$\therefore x < -1/6$

$-\infty \quad - \quad -1/6 \quad + \quad \infty$

$\therefore x \in \left(-\infty, -\frac{1}{6}\right)$

Practical 4: Application of derivatives / Newton method

Q.1] Find maximum & minimum values of following functions

a] $f(x) = x^2 + \frac{16}{x^2}$

solution: $\therefore f(x) = x^2 + 16(x^{-2})$
$\therefore f'(x) = 2x + 16(-2x^{-3})$
$\therefore f'(x) = 2x - \frac{32}{x^3}$

Now consider $f'(x) = 0$
$\therefore 2x - \frac{32}{x^3} = 0$
$\therefore 2x = \frac{32}{x^3}$ ... (1)
$\therefore 2x^4 = 32$
$\therefore x^4 = 16$
$\therefore x^2 = 4$
$\therefore x = \pm 2$
$\therefore f''(x) = 2 + 96/x^4$
$\therefore f''(2) > 0$
$\therefore f$ has minima at $x = 2$
$\therefore f''(-2) > 0$
$\therefore f$ has minima at $x = -2$
$\therefore f(-2) = (-2)^2 + 16/(-2)^2 = 8$
$\therefore f(+2) = (2)^2 + 16/(2)^2 = 8$

b) $f(x) = 3 - 5x^3 + 3x^5$

solution: $\therefore f(x) = 3 - 5x^3 + 3x^5$
$\therefore f'(x) = -15x^2 + 15x^4$

Now consider $f'(x) = 0$
$\therefore -15x^2 + 15x^4 = 0$
$\therefore -15x^2(1 - x^2) = 0$
$\therefore 1 - x^2 = 0$
$\therefore x^2 = 1$
$\therefore x = 0, +1, -1$
$\therefore f''(x) = -30x + 60x^3$
$\therefore f''(-1) = -30 < 0$
$\therefore f$ is maxima at $x = -1$
$\therefore f''(0) = 0$
$\therefore f''(1) = 30 > 0$
$\therefore f$ is minima at $x = 1$
$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5 = 1$
$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5 = 5$

c) $f(x) = x^3 - 3x^2 + 1$ in $\left[-\frac{1}{2}, 4\right]$

solution: $\therefore f(x) = x^3 - 3x^2 + 1$
$\therefore f'(x) = 3x^2 - 6x$

Now consider $f'(x) = 0$
$\therefore 3x^2 - 6x = 0$
$\therefore 3x(x - 1) = 0$
$\therefore x - 1 = 0$
$\therefore x = 1, 0$

$\therefore$ we calculate the values of f at the ends points $-\frac{1}{2}, 4$ as well as critical points $x = 0, 1$

at $x = -\frac{1}{2}$, $\therefore f(-\frac{1}{2}) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 1$

$= \frac{1}{8}$

at $x = 0$, $\therefore f(0) = (0)^3 - 3(0)^2 + 1$
$= 1$

at $x = 1$, $\therefore f(1) = (1)^3 - 3(1)^2 + 1$
$= -1$

at $x = 4$, $\therefore f(4) = (4)^3 - 3(4)^2 + 1$
$= 17$

$\therefore$ Absolute maximum values are $\frac{1}{8}, 1, 17$ at $x = -\frac{1}{2}, 1, 4$. and absolute minimum value is -1 at $x = 1$

d] $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Solution: $\therefore f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Now consider $f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$\therefore x^2 - x - 2 = 0$

$\therefore x^2 - 2x + x - 2 = 0$

$\therefore (x-2)(x+1) = 0$

$\therefore x = 2, -1$

$\therefore$ we calculate the values of f at the end points $-2, 3$ as well as critical points $x = -1, 2$

at $x = -2$, $\therefore f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1$
$= -3$

at $x = -1$, $\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$
$= 8$

at $x = 2$, $\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$
$= -19$

at $x = 4$, $\therefore f(4) = 2(4)^3 - 3(4)^2 - 12(4) + 1$
$= 33$

$\therefore$ Absolute maximum values are $8, 33$ at $x = -1, 4$ and absolute minimum values are $-3, -19$ at $x = -2, 2$

Q.2] Find the root of following equation by Newton's Method (Take 2 iteration only) correct upto 4 decimal

a] $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

Solution: $\therefore f(x) = x^3 - 3x^2 - 55x + 9.5$

$\therefore f'(x) = 3x^2 - 6x - 55$

$\therefore x_0 = 0$

$\therefore$ I[st] iteration:

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ [By Newton's Method]

$$\therefore x_1 = 0 - \frac{0^3 - 3(0)^2 - 55(0) + 9.5}{3(0)^2 - 6(0) - 55}$$

$$\therefore x_1 = 0 - \left(\frac{9.5}{-55}\right) = 0.1727$$

II nd iteration:

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{(0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5}{3(0.1727)^2 - 6(0.1727) - 55}$$

$$= 0.1712$$

III rd iteration:

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.1712 - \frac{(0.1712)^3 - 3(0.1712)^2 - 55(0.17..) + 9.5}{3(0.1712)^2 - 6(0.1712) - 55}$$

$$= 0.1712$$

IV th iteration:

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.1712 - \frac{(0.1712)^3 - 3(0.1712)^2 - 55(0.17..) + 9.5}{3(0.1712)^2 - 6(0.1712) - 55}$$

$$= 0.1712$$

Now we have 2 approximations that agree to 4 decimal places. So we will assume that solution is approximately

$$x_4 = 0.1712$$

$f(x) = x^3 - 4x - 9$ in $[2,3]$

Solution: $\therefore f(x) = x^3 - 4x - 9$ in $[2,3]$

To get the value at x_0 we take midpoint at the interval $[2,3]$

$\therefore x = 2.5$

$\therefore f(x) = x^3 - 4x - 9$

$\therefore f'(x) = 3x^2 - 4$

$\therefore$ Ist iteration:

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{[By Newton's method]}$$

$$\therefore x_1 = 2.5 - \frac{(2.5)^3 - 4(2.5) - 9}{3(2.5)^2 - 4}$$

$$= 2.7288$$

$\therefore$ IInd iteration:

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7288 - \frac{(2.7288)^3 - 4(2.7288) - 9}{3(2.7288)^2 - 4}$$

$$= \cancel{2.6502}\ 2.7067$$

$\therefore$ IIIrd iteration:

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7067 - \frac{(2.7067)^3 - 4(2.7067) - 9}{3(2.7067)^2 - 4}$$

$$= 2.7065$$

$\therefore$ IVth iteration:

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7065 - \frac{(2.7065)^3 - 4(2.7065) - \ldots}{3(2.7065)^2 - 4}$$

$$= 2.7065$$

Now we have got 2 approximations that agree to 4 decimal places. so we will assume that solution is approximately $x_4 = 2.7065$

5] $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1,2]$

Solution: $\therefore f(x) = x^3 - 1.8x^2 - 10x + 17$

To get the value at x_0 we take the midpoint at the interval $[1,2]$

$\therefore \quad x_0 = 1.5$

$\therefore \quad f'(x) = 3x^2 - 3.6x - 10$

I[st] iteration:

$$\therefore x_1 = x_0 - \frac{f(x)}{f'(x)}$$

$$= 1.5 - \frac{(1.5)^3 - 1.8(1.5)^2 - 10(1.5) + 17}{3(1.5)^2 - 3.6(1.5) - 10)}$$

$$= 1.6632$$

II[nd] iteration:

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.6632 - \frac{(1.6632)^3 - 1.8(1.66\ldots - 10(1.6632) + 17}{3(1.6632)^2 - 3.6(1.66\ldots - 10}$$

$$= 1.6618$$

III[rd] iteration:

$$\therefore x_3 = x_2 - \frac{f(x)}{f'(x)}$$

$$= 1.6618 - \frac{(1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17}{3(1.6618)^2 - 3.6(1.6618) - 10}$$

$$= 1.6618$$

IV[th] iteration:

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.6618 - \frac{(1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17}{3(1.6618)^2 - 3.6(1.6618) - 10}$$

$$= 1.6618$$

$\therefore$ Now we have got 3 approximations that agree to 4 decimal places so we can stop. we will assume that the solution is approximately $x_4 = 1.6618$

Practical 5

Title: Integration

Q.1] Solve the following integration:

a] $\int \frac{dx}{\sqrt{x^2+2x-3}}$

Solution: First rewrite the integrand

$$\frac{1}{\sqrt{x^2+2x-3}} = \frac{1}{\sqrt{(x+1)^2-4}}$$

put $x+1 = u$

$$= \int \frac{dx}{\sqrt{(x+1)^2-4}}$$

$$= \int \frac{du}{\sqrt{u^2-4}}$$

$$= \log |u + \sqrt{u^2-4}| + c$$

$$= \log |x+1+\sqrt{(x+1)^2-4}| + c$$

$$= \log |x+1+\sqrt{x^2+2x-3}| + c$$

$$\therefore \int \frac{dx}{\sqrt{x^2+2x-3}} = \log |x+1+\sqrt{x^2+2x-3}|$$

$\int (4e^{3x}+1)\, dx$

Solution: $I = \int (4e^{3x}+1)\, dx$

$$= \int 4e^{3x}\, dx + \int 1\, dx$$

put $4e^{3x} \cancel{+1} = t$

on differentiation

$$12e^{3x}\, dx = dt$$

$$e^{3x}\, dx = \frac{dt}{12}$$

$$\therefore \quad I = \int 4 \frac{dt}{12} + x$$

$$= \frac{1}{3} \int dt + x$$

$$= \frac{1}{3} [t] + x + c$$

$$\therefore \quad I = \frac{4e^{3x}+1}{3} + x + c$$

b] $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

Solution: $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$$= 2\int x^2\, dx - 3\int \sin x\, dx + 5\int \sqrt{x}\, dx$$

$$= 2 \cdot \frac{x^3}{3} - 3(-\cos x) + \frac{5x^{3/2}}{3/2} + c$$

$$= \frac{2x^3}{3} + 3\cos x + \frac{5 \times 2x^{3/2}}{3} + c$$

$$= \frac{2x^3}{3} + 3\cos x + \frac{10x^{3/2}}{3} + c$$

$$\therefore I = \frac{2x^3}{3} + 3\cos x + \frac{10x^{3/2}}{3} + c$$

4) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

solution: $I = \int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$$\therefore I = \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right) dx$$

$$\therefore I = \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$$

$$\therefore I = \int (x^{3-1/2} + 3x^{1-1/2} + 4x^{-1/2})\,dx$$

$$\therefore I = \int (x^{5/2} + 3x^{1/2} + 4x^{-1/2})\,dx$$

$$\therefore I = \int x^{5/2}\,dx + \int 3x^{1/2}\,dx + \int 4x^{-1/2}\,dx$$

$$\therefore I = \int x^{5/2}\,dx + 3\int x^{1/2}\,dx + 4\int x^{-1/2}\,dx$$

$$\therefore I = \frac{x^{7/2}}{7/2} + \frac{3x^{3/2}}{3/2} + 4 \times 2 \times x^{1/2} + c$$

$$\therefore I = \frac{2x^{7/2}}{7} + \frac{2\cancel{6}x^{3/2}}{3} + 8\sqrt{x} + c$$

$$\therefore \int \frac{x^3+3x+4}{\sqrt{x}}\,dx = \frac{2x^{7/2}}{7} + 2x^{3/2} + 8\sqrt{x} + c$$

5) $\int t^7 \sin(2t^4)\,dt$

solution: $I = \int t^7 \sin(2t^4)\,dt$

put $u = 2t^4$

$du = 2 \times 4t^3$

$$\therefore I = \int t^7 \sin(2t^4) \frac{1}{2 \times 4t^3}\,dt$$

$$= \int t^4 \sin(2t^4) \times \frac{1}{8}\,dt$$

$$= \frac{t^4 \sin(2t^4)}{8}\,dt$$

substitute t^4 with $u/2$

$$\therefore I = \int \frac{u/2 \sin(u)}{8}\,du$$

$$= \int \frac{u \sin(u)}{16}\,du$$

$$= \frac{1}{16} \int u \sin(u)\,du$$

Now let's put $u = u$

$dv = \sin(u)\,du$

$du = 1$

$v = -\cos(u)$

$$\therefore I = -\int -\cos(u)\,du + \frac{1}{16}(u \times (-\cos(u)))$$

$$= \frac{1}{16} \times u(-\cos u) + \int \cos(u)\,du$$

$$= \frac{1}{16} \times (u \times -\cos u + \sin(u))$$

$$\therefore \int t^7 \sin(2t^4)\,dt = \frac{1}{16}(2t^4(-\cos(2t^4)) + \sin(2t^4)$$

$\therefore \int t^7 \sin(2t^4)\,dt = \frac{-t^4\cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + c$

6] $\int \sqrt{x}(x^2-1)\,dx$

Solution : $I = \int \sqrt{x}(x^2-1)\,dx$

$= \int \sqrt{x}\,x^2 - \sqrt{x}\,dx$

$= \int x^{1/2} x^2 - x^{1/2}\,dx$

$= \int x^{5/2} - x^{1/2}\,dx$

$= \int x^{5/2}\,dx - \int x^{1/2}\,dx$

$= \frac{x^{7/2}}{7/2} - \frac{x^{3/2}}{3/2} + c$

$\int \sqrt{x}(x^2-1)\,dx = \frac{2x^{7/2}}{7} - \frac{2x^{3/2}}{3} + c$

$\int \frac{\cos x}{\sqrt[3]{\sin(x)^2}}\,dx$

Solution : $\therefore I = \int \frac{\cos x}{\sin^{3/2}}\,dx$

put $t = \sin x$

$dt = \cos x$

$\therefore I = \int \frac{\cos x}{\sin(x)^{3/2}} \times \frac{1}{\cos x}\,dt$

$= \int \frac{1}{\sin^{3/2}}\,dt$

$= \int \frac{1}{t^{3/2}}\,dt$

$= \frac{-1}{(\frac{3}{2}-1)\,t^{2/3-1}}$

$= \frac{-1}{-\frac{1}{3}t^{-1/3}}$

$= \frac{t^{1/3}}{1/3}$

$= 3t^{1/3}$

$= 3\sqrt[3]{t}$

$\therefore \int \frac{\cos x}{\sqrt[3]{\sin(x)^2}}\,dx = 3\sqrt[3]{t}$

Return substitution $t = \sin(x)$

$\therefore \int \frac{\cos x}{\sqrt[3]{\sin(x)^2}} = 3\sqrt[3]{\sin(x)} + c$

$\int e^{\cos^2 x} \sin 2x\,dx$

Solution : $I = \int e^{\cos^2 x} \sin 2x\,dx$

Now put $\cos^2 x = t$

$2\cos x(-\sin x\,dx) = dt$

$-\sin 2x\,dx = dt$

$I = \int e^t\,dt$

$I = e^t + C$

$\int \cancel{\cos}\, e^{\cos^2 x} \sin 2x\,dx = e^{\cos^2 x} + c$

Now put $\cos x = t$

$-\sin x \, dx = dt$

$\therefore \quad I = \int e^{t^2}(-dt)$

$= -\int e^{t^2} dt$

$\therefore \quad I = \mp e^{t^2} + C$

$\therefore \int e^{\cos^2 x} \sin x \, dx = \mp e^{\cos^2 x} + c$

10] $\int \frac{x^2-2x}{x^3-3x^2+1} \, dx$

put $x^3-3x^2+1 = t$

$I = \int \frac{x^2-2x}{x^3-3x^2+1} \times \frac{1}{3x^2-6x} \, dt$

$= \int \frac{x^2-2x}{x^3-3x^2+1} \times \frac{1}{3(x^2-2x)} \, dt$

$= \int \frac{1}{x^3-3x^2+1} \times \frac{1}{3} \, dt$

$= \int \frac{1}{3(x^3-3x^2+1)} \, dt$

$= \int \frac{1}{3t} \, dt$

$= \frac{1}{3} \int \frac{1}{t} \, dt$

$= \frac{1}{3} \log|t| + C$

$= \frac{1}{3} \log|x^3-3x^2+1| + C$

$\int \frac{x^2-2x}{x^3-3x^2+1} = \frac{1}{3} \log|x^3-3x^2+1| + c$

8] $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

solution: $I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$I = \int x^{-3} \sin(x^{-2}) \, dx$

Now put $x^{-2} = t$

$\therefore -2x^{-3} dx = dt$

$\therefore \quad x^{-3} dx = -\frac{dt}{2}$

$\therefore \quad I = -\int \frac{\sin(t) \, dt}{2}$

$\therefore \quad I = -\frac{1}{2} \int \sin(t) \, dt$

$\therefore \quad I = -\frac{1}{2}(-\cos(t)) + C$

$\therefore \quad I = \frac{\cos(t)}{2} + C$

$\therefore \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx = \frac{\cos(x^{-2})}{2} + c$

Practical 6:

Title: Application of Integration & Numerical Integration

Q.1] Find the length of the following curve:

a] $x = t\sin t$, $y = 1-\cos t$ $t \in [0, 2\pi]$

Solution: $L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

$\because x = t\sin t$

$\therefore \frac{dx}{dt} = \cancel{\sin t + t\cos t}$ $1-\cos t$

$\because y = 1-\cos t$

$\therefore \frac{dy}{dt} = \sin t$

$\therefore L = \int_0^{2\pi} \sqrt{(\sin t)^2 + (\cancel{\sin t + t\cos t})^2}\, dt$

$= \int_0^{2\pi} \sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{1-2\cos t+1}\, dt$

$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt$

$= \int_0^{2\pi} 2\sin t/2\, dt$

$= \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$

$= (-4\cos\pi) - (-4\cos 0)$

$= 4+4$

$= 8$

$\therefore \boxed{L = 8}$

b] $y = \sqrt{4-x^2}$ $x \in [-2, 2]$

Solution: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$\because y = \sqrt{4-x^2}$

$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$

$\therefore L = \int_{-2}^{2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$

$$= \int_{-2}^{2} 2\sqrt{\frac{1}{4-x^2}}\,dx$$

$$= \int_{-2}^{2} 2\cdot\sqrt{(4-x^2)^{-1}}\,dx$$

$$= \int_{-2}^{2} 2\left((4-x^2)^{-1}\right)^{1/2}\,dx$$

$$= \int_{-2}^{2} 2\cdot(4-x^2)^{-1/2}\,dx$$

$$= 2\left[\frac{(4-x^2)^{1/2}}{1/2}\right]_{-2}^{2}$$

$$= 4\left[(4-4)^{1/2}-(4-4)^{1/2}\right]$$

$$= 0$$

$y = x^{3/2}$ in $[0,4]$

Solution: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\because y = x^{3/2}$

$\therefore \frac{dy}{dx} = \frac{3}{2}x^{1/2}$

$$\therefore L = \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \frac{1}{2}\int_0^4 (4+9x)^{1/2}\,dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{3/2\cdot 9}\right]_0^4$$

$$= \frac{1}{2}\cdot\frac{2}{27}\left[(4+9x)^{3/2}\right]_0^4$$

$$= \frac{1}{27}\left[(4+36)^{3/2}-(4)^{3/2}\right]$$

$$= \frac{1}{27}\left[(40)^{3/2}-(4)^{3/2}\right]$$

$$= \frac{1}{27}\left[36^{3/2}\right]$$

$$= 9.0734$$

$\therefore L = 9.0734$

Q) $x = 3\sin t,\ y = 3\cos t \quad t\in[0,2\pi]$

Solution: $L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,dt$

$\because x = 3\sin t$

$\therefore \frac{dx}{dt} = 3\cos t$

$\because y = 3\cos t$

$\therefore \frac{dy}{dt} = -3\sin t$

$$\therefore L = \int_0^{2\pi} \sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t+9\cos^2 t}\,dt$$

$$= \int_0^{2\pi} 3\sqrt{\sin^2 t+\cos^2 t}\,dt$$

$$= 3\int_0^{2\pi} dt$$
$$= 3[t]_0^{2\pi}$$
$$= 6\pi$$
$$\therefore \boxed{L = 6\pi}$$

e] $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1,2]$

Solution: $L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$\because x = \frac{1}{6}y^3 + \frac{1}{2y}$

$\therefore \frac{dx}{dy} = \frac{3y^2}{6} - \frac{1}{2y^2}$

$$\therefore L = \int_1^2 \sqrt{1+\left(\frac{3y^2}{6} - \frac{1}{2y^2}\right)^2}\, dy$$
$$= \int_1^2 \sqrt{1+\left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\, dy$$
$$= \int_1^2 \sqrt{1+\frac{1}{4}\left(y^4 - 2 + \frac{1}{y^4}\right)}\, dy$$
$$= \int_1^2 \sqrt{\left(\frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}\right)+1}\, dy$$
$$= \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}\, dy$$
$$= \int_1^2 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy$$

$$= \left[\frac{y^3}{6} - \frac{1}{2y}\right]_1^2$$
$$= \frac{2^3}{6} - \frac{1}{6} - \frac{1}{4} + \frac{1}{2}$$
$$= \frac{8}{6} - \frac{1}{6} - \frac{1}{4} + \frac{2}{4}$$
$$= \frac{7}{6} - \frac{3}{4}$$
$$= \frac{17}{12}$$
$$\therefore \boxed{L = \frac{17}{12}}$$

f) $y = \sqrt{4-x^2}$ on $x \in [-2,2]$

Solution: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$\because y = \sqrt{4-x^2}$

$\therefore \frac{dy}{dx} = -\frac{x}{\sqrt{4-x^2}}$

$$\therefore L = \int_a^b \sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\, dx$$
$$= \int_a^b \sqrt{1+\frac{x^2}{4-x^2}}\, dx$$

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$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\, dx$$

$$= 2\int_{-2}^{2} \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}\, dx$$

$$= 2\int_{-2}^{2} \frac{1}{2\sqrt{1-(x/2)^2}}\, dx$$

$\Rightarrow$ Now put $\frac{x}{2} = \sin(u)$

$$\frac{1}{2}\, dx = \cos(u)\, du$$

$$dx = 2\cos(u)\, du$$

$$= 2\int_{-2}^{2} \frac{1}{2\sqrt{1-\sin^2 x}} \cdot 2\cos(u)\, du$$

$$= 2\int_{-2}^{2} \frac{1}{\sqrt{\cos^2 x}} \cdot \cos(u)\, du$$

$$= 2\int_{-2}^{2} u\, du$$

$$= 2u$$

$$= 2\cdot\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 2\cdot\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$\therefore \boxed{L = 2\pi}$$

Q3 using simpson's rule solve the following

a] $\int_0^2 e^{x^2}\, dx$ with $n=4$

solution : For $n=4$, we have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

$\therefore$ we compute the values of $y_0, y_1, y_2, \cdots, y_4$

x	0	0.5	1	1.5	2
$y = e^{x^2}$	1	1.2840	2.7183	9.4877	54.5982

$$\therefore \int_0^2 e^{x^2} \approx \frac{0.5}{3}\left(1 + \cancel{4 \cdot 1.2840} + \cancel{2 \cdot 2.7183} + \cancel{4 \cdot 9.4877} + 54.5982\right)$$

5.1360 5.4366 37.9508

$$\approx \frac{0.5}{3}(69.0882)$$

$$\approx 17.353625 45$$

b] $\int_0^4 x^2\, dx$ with $n=4$

solution : For $n=4$, we have $\Delta x = \frac{4-0}{4} = 1$

$\therefore$ we compute the values of $y_0, y_1, \cdots, y_4$

x	0	1	2	3	4	5
$y = x^2$	0	1	4	9	16	25

$$\therefore \int_0^4 x^2\, dx \approx \frac{1}{3}\left[0 + 4(1) + 2(4) + 4(9) + 2(16) + 25\right]$$

$$\approx \frac{64}{3}$$

$$\approx 21.333$$

c] $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n=6$

Solution: For $n=6$, we have $\Delta x = \frac{\pi/3 - 0}{6}$

we compute the values of $y_0, y_1, y_2, \ldots, y_6$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$	$6\pi/18$
$y=\sqrt{\sin x}$	0	0.4167	0.584	0.707	0.801	0.875	0.930

$\therefore \int_0^{\pi/3} \sqrt{\sin x}\, dx \approx \frac{\pi/18}{3}[0 + 4(0.4167) + 2(0.584) + 4(0.707) + 2(0.801) + 4(0.875) + 0.930]$

≈ 0.681

AU 08/01/2024

Practical 7:

Title: Differential Equations

Q.1] Solve the following differential equation:

a] $x\frac{dy}{dx} + y = e^x$

Solution: $x\frac{dy}{dx} + y = e^x$ ... ①

Now dividing eq ① by x

$\therefore \frac{dy}{dx} + \frac{y}{x} = \frac{e^x}{x}$

Now comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

$\therefore P(x) = 1/x$

$\therefore Q(x) = e^x/x$

$\therefore I.F = e^{\int P dx}$

$= e^{\int 1/x\, dx}$

$= e^{\log x}$

$= x$

Solution is

$y(I.F) = \int Q(I.F)\, dx + c$

$\therefore y(x) = \int \frac{e^x}{x} \cdot x\, dx + c$

$\therefore yx = \int e^x\, dx + c$

$\therefore \boxed{yx = e^x + c}$

b) $e^{x}\frac{dy}{dx}+2e^{x}y=1$

solution: $e^{x}\frac{dy}{dx}+2e^{x}y=1 \quad \cdots (1)$

Now dividing eq (1) by e^{x}

$\therefore \frac{dy}{dx}+2y=\frac{1}{e^{x}}$

Now comparing with $\frac{dy}{dx}+P(x)\frac{y}{dx}=Q$

$\therefore P(x)=2$

$\therefore Q(x)=1/e^{x}$

$\therefore I.F=e^{\int P dx}$

$=e^{\int 2 dx}$

$=e^{2x}$

$\therefore$ solution is

$y(I.F)=\int Q(I.F)\,dx+c$

$\therefore ye^{2x}=\int \frac{1}{e^{x}}\cdot e^{2x}dx+C$

$\therefore ye^{2x}=\int e^{2x-x}dx+C$

$\therefore ye^{2x}=\int e^{x}\,dx+C$

$\therefore \boxed{ye^{2x}=e^{x}+C}$

c) $x\frac{dy}{dx}=\frac{\cos x}{x}-2y$

solution: $x\frac{dy}{dx}=\frac{\cos x}{x}-2y \quad \cdots (1)$

Now dividing eq (1) by x

$\therefore \frac{dy}{dx}=\frac{\cos x}{x^{2}}-\frac{2y}{x}$

$\therefore \frac{dy}{dx}+\frac{2y}{x}=\frac{\cos x}{x^{2}}$

Now comparing with $\frac{dy}{dx}+P(x)\frac{y}{dx}=Q(x)$

$\therefore P(x)=2/x$

$\therefore Q(x)=\cos x/x^{2}$

$\therefore I.F=e^{\int P dx}$

$=e^{\int 2/x\, dx}$

$=e^{\log x^{2}}$

$\therefore I.F=x^{2}$

$\therefore$ solution is

$y(I.F)=\int Q(I.F)\,dx+c$

$\therefore y(x^{2})=\int \frac{\cos x}{x^{2}}\cdot x^{2}\,dx+c$

$\therefore x^{2}y=\int \cos x+c$

$\therefore \boxed{x^{2}y=\sin x+c}$

d) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

solution: $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2} \quad \cdots (1)$

Now Dividing eq (1) by x

$\therefore \frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3} \quad \cdots (2)$

Now comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

$\therefore P(x) = 3/x$

$\therefore Q(x) = \sin x / x^3$

$\therefore I.F = e^{\int P dx}$

$= e^{\int 3/x \, dx}$

$= e^{\log x^3}$

$\therefore I.F = x^3$

$\therefore$ solution is

$y(I.F) = \int Q(I.F)\,dx + C$

$\therefore y(x^3) = \int \frac{\sin x}{x^3} \cdot x^3 \,dx + C$

$\therefore x^3 y = \int \sin x + C$

$\therefore \boxed{x^3 y = -\cos x + C}$

$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

solution: $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x \quad \ldots (1)$

Now diving eq (1) by e^{2x}

$\therefore \frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

Now comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

$\therefore P(x) = 2$

$\therefore Q(x) = 2x/e^{2x}$

$\therefore I.F = e^{\int 2dx}$

$= e^{2x}$

$\therefore I.F = e^{2x}$

$\therefore$ solution is

$y(I.F) = \int Q(I.F)\,dx + c$

$\therefore y(e^{2x}) = \int \frac{2x}{e^{2x}} \cdot e^{2x}\,dx + c$

$\therefore ye^{2x} = \int 2x\,dx + c$

$\therefore \boxed{ye^{2x} = x^2 + c}$

f] $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

solution: $\sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$\therefore \frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y \, dy}{\tan y}$

$\therefore \int \frac{\sec^2 x \, dx}{\tan x} = \int -\frac{\sec^2 y \, dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\therefore \log|\tan x - \tan y| = c$

$\therefore \boxed{\tan x - \tan y = e^c}$

g] $\frac{dy}{dx} = \sin^2(x - y + 1)$

solution: Put $x - y + 1 = v$

Differentiating on both sides

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore 1 - \frac{dv}{dx} = \frac{dy}{dx}$

$\therefore 1 - \frac{dv}{dx} = \sin^2 v$

$\therefore \frac{dv}{dx} = 1 - \sin^2 v$

$\therefore \frac{dv}{dx} = \cos^2 v$

$\therefore \frac{dv}{\cos^2 v} = dx$

$\therefore \int \sec^2 v \, dv = \int dx$

$\therefore \tan^2 v = x + c$

$\therefore \boxed{\tan^2(x + y - 1) = x + c}$

h] $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y - 6}$

solution: Put $2x + 3y = v$

$\therefore 2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore dy/dx = 1/3(dv/dx - 2)$

$\therefore \frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$

$\therefore \frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\therefore \frac{dv}{dx} = \frac{v - 1 + 2v + 4}{v+2}$

$\therefore \frac{dv}{dx} = \frac{3v+3}{v+2}$

$\therefore \frac{v+2}{3v+3} \, dv = dx$

$\therefore \int \frac{v+2}{3v+3} \, dv = \int dx$

$\therefore \frac{1}{3} v + \frac{1}{3} \log(3v+3) = x + C$

$\therefore \frac{1}{3}(2x+3y) + \frac{1}{3} \log(3(2x+3y)+3) = x + C$

$\therefore \boxed{\frac{1}{3}(2x+3y) + \frac{1}{3} \log(6x+9y+3) = x+C}$

Practical 8 :

Title : Euler's Method

Q.1] Using Euler's method find the following

a] $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2$, $h = 0.5$ find $y(2)$

Solution : $\therefore x_0 = 0$, $y_0 = 2$, $h = 0.5$

$f(x, y) = y + e^x - 2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5000
1	0.5	2.5	2.1487	~~2.7487~~ 3.5744
2	1	3.5744	4.2927	5.7208
3	1.5	5.7208	8.2025	9.8221
4	2	9.8221		

$\therefore \boxed{y(2) = 9.8221}$

b] $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, find $y(1)$

$h = 0.2$

solution : $x_0 = 0$, $y_0 = 0$, $f(x_n, y_n) = 1 + y^2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2000
1	0.2	0.2000	1.0400	0.4080
2	0.4	0.4080	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8530	1.2942
5	1	1.2942		

$\therefore \boxed{y(1) = 1.2942}$

c] $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$ find $y(1)$

solution: $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_n, y_n) = \sqrt{x/y}$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2106
3	0.6	1.2106	0.7040	1.3514
4	0.8	1.3514	0.7694	1.5053
5	1	1.5053		

$\therefore \boxed{y(1) = 1.5053}$

d] $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$ find $y(2)$ for $h = 0.5$ & $h = 0.25$

solution: For $h = 0.5$:

$x_0 = 1$, $y_0 = 2$, $h = 0.5$, $f(x_n, y_n) = 3x^2 + 1$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4.0000	4.0000
1	1.5	4	7.7500	7.8750
2	2	7.8750		

$\therefore \boxed{y(2) = 7.8750}$

For $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4.0000
1	1.25	4	5.6875	5.4219
2	1.50	5.4219	7.7500	7.3594
3	1.75	7.3594	10.1875	9.9063
4	2	9.9063		

$\therefore \boxed{y(2) = 9.9063}$

e) $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$, find $y(1.2)$ with $h = 0.2$

Solution: $x_0 = 1$, $y_0 = 1$, $f(x_n, y_n) = \sqrt{xy} + 2$, $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.60000
1	1.2	1.60000		

$\therefore \boxed{y(1.2) = 1.60000}$

Practical 9

Title: Limits & Partial order Derivatives

Q1) Evaluate the following limits

i) $\lim_{(x,y)\to(-4,-1)} \frac{x^3 - 3x + y^2 - 1}{xy + 5}$

$= \lim_{(x,y)\to(-4,-1)} \frac{x^3 - 3x + y^2 - 1}{xy + 5}$

Apply limit

$= \frac{(-4)^3 + 3\times4 + (-1)^2 - 1}{(-4)(-1) + 5}$

$= \frac{-64 + 12 + 1 - 1}{9}$

$= \frac{-52}{9}$

ii) $\lim_{(x,y)\to(2,0)} \frac{[y+1](x^2 + y^2 - 4x)}{x + 3y}$

$= \lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$

Apply limit

$= \frac{(0+1)((2)^2 + (0)^2 - 4(2))}{2 + 3(0)}$

$= \frac{1(4-8)}{2}$

$= -2$

3] $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2-z^2}{x^3-x^3y^2}$

Apply limit

$= \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)}$

$= \frac{0}{0}$

$= 0$

$\therefore$ limit does not exist

Q.2] Find f_x, f_y for each of the following of

1] $f[x,y] = xye^{x^2} + y^2$

$\therefore f_x = y(1e^{x^2}+y^2) + xy(e^{x^2}+y^2 \cdot 2x)$
$= y \cdot e^{x^2}+y^2 + 2xy\, e^{x^2}+y^2$

$\therefore f_y = x(1e^{x^2}+y^2) + xy(e^{x^2}+y^2 \cdot 2y)$
$= x \cdot e^{x^2}+y^2 + 2xy\, e^{x^2}+y^2$

2] $f(x,y) = e^x \cos y$

$\therefore f_x = \cos y\, e^x$
$\therefore f_y = e^x \cdot \sin y$

3] $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = y^2 3x^2 - 3xy^2 x + 6 + 0$

$\therefore f_x = 3x^2y^2 - 6xy$

$\therefore f_y = x^3 2y - 3x^2 + 3y^2$
$= 2x^3y - 3x^2 + 3y^2$

Q.3] using definition find values of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+y^2}$

$\therefore f_x[a,b] = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

$\therefore f_y(a,b) = \lim_{h\to 0} \frac{f(a,h+b) - f(a,b)}{h}$

According to given $(a,b) = (0,0)$

$f_x = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{2h-0}{h}$

$= 2$

$f_y = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$

$= \frac{0-0}{h}$

$= 0$

$\therefore f_x = 2,\ f_y = 0$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h \to 0}$

Q 4] Find all second order partial derivatives of f. Also verify whether $f_{xy} = f_{yx}$

I] $f(x,y) = \frac{y^2 - xy}{x^2}$

$\therefore f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad f_{yy} = \frac{\partial^2 f}{\partial^2 y}$

Applying $\frac{u}{v}$ rule;

$f_x = \frac{x^2 [0-y] - [y^2 - xy] 2x}{x^4}$

$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^4}$

$\therefore f_x = \frac{x^2y - 2xy^2}{x^4}$

$f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$

$= \frac{2x^5y - 2x^4y^2 - (4x^5y - 8x^4y^2)}{x^8}$

$\therefore f_{xx} = \frac{-2x^5y + 6x^4y^2}{x^8}$

$\therefore f_y = \frac{1}{x^2}[2y - x]$

$\therefore f_{yy} = \frac{1}{x^2} \cdot 2 = \frac{2}{x^2}$

$\therefore f_{xy} = \frac{2y - x}{x^2}$

$= \frac{x^2(-1) - (2y-x)(2x)}{x^4}$

$= \frac{-x^2 - 4xy + 2x^2}{x^4}$

$= \frac{x^2 - 4xy}{x^4}$

$= \frac{x - 4y}{x^3}$

$\therefore f_{yx} = \frac{x^2y - 2xy}{x^4}$

$= \frac{x - 4xy}{x^3}$

$= f_{xy}$

$\therefore$ Hence proved

2] $f(x,y) = x^3 + 3x^2y^2 - \log[x^2+1]$

$f_x = \frac{\partial}{\partial x^2} x^3 + 3x^2y^2 - \log[x^2+1]$

$f_x = 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$

$f_y = \frac{\partial}{\partial y^2} x^3 + 3x^2y^2 - \log[x^2+1]$

$= 0 + 6xyx^2 - 0$

$f_y = 6x^2y$

$$f_{xx} = \frac{\partial}{\partial x} f_x$$
$$= \frac{\partial}{\partial x}\left[3x^2+6xy^2-\frac{2x}{x^2+1}\right]$$
$$f_{xx} = 6x^2+6y^2-\frac{4x-2x^2+2}{(x^2+1)^2}$$

$$f_{yy} = \frac{\partial}{\partial y} f_y$$
$$= \frac{\partial}{\partial y} 6x^2y$$
$$f_{yy} = 6x^2$$

$$f_{xy} = \frac{\partial}{\partial y} f_x$$
$$= \frac{\partial}{\partial y}\left[3x^2+6xy^2-\frac{2x}{x^2+1}\right]$$

$$f_{xy} = 12xy$$

$$f_{yx} = \frac{\partial}{\partial x} f_y$$
$$f_{yx} = 12xy$$

$\therefore f_{xy} = f_{yx}$

$\therefore$ Hence proved

8) $f(x,y) = \sin(xy)+e^{x+y}$

$f(x,y) = \sin(xy)+e^x e^y$

$$f_x = \frac{\partial}{\partial x}(\sin(xy)+e^x e^y)$$
$$f_x = y\cos(xy)+e^x e^y$$
$$f_y = \frac{\partial}{\partial y}(\sin(xy)+e^x e^y)$$
$$= x\cos(xy)+e^x e^y$$
$$f_{xx} = \frac{\partial}{\partial x} f_x$$
$$= \frac{\partial}{\partial x} y\cos(xy)+e^x e^y$$
$$f_{xx} = y^2\sin(xy)+e^{x+y}$$
$$f_{yy} = \frac{\partial}{\partial y} f_y$$
$$= \frac{\partial}{\partial y}(x\cos(xy)+e^x e^y)$$
$$f_{yy} = -x^2\sin^2(xy)+e^x e^y$$
$$f_{xy} = \frac{\partial}{\partial y} y\cdot\cos(x+y)+e^x e^y$$
$$f_{xy} = -xy^2\sin(xy)+\cos(xy)+e^x e^y$$
$$f_{yx} = \frac{\partial}{\partial x} f_y$$
$$f_{yx} = -xy^2\sin(xy)+\cos(xy)+e^x e^y$$

$\therefore f_{xy} = f_{yx}$ Hence proved

Q5] Find the linearization of $f(x,y)$ at given point

1] $f(x,y) = 1-x+\sin xy$ at $\pi/2$

$f(\pi/2,0) = 1-\pi/2+0+\sin\pi/2$

$f(\pi/2,0) = \frac{2-\pi}{2}$

$f_x = -1+y\cos x$

$f_y = \sin x$

$f_x(\pi/2,0) = -1$

$f_y(\pi/2,0) = 1$

$L(x,y) = f(\pi/2,0) + f_x(\pi/2,0)(x-\pi/2) + f_y(\pi/2,0)(y-0)$

$= \frac{2\pi}{2} + (-1)(x-\frac{\pi}{2}) + 1(y)$

$= 1-\pi/2 - x + \pi/2 + y$

$L(x,y) = 1-x+y$

2] $f(x,y) = \log x + \log y$

$f(1,1) = \log 1 + \log 1$

$= 0$

$f(x) = 1/x$ $f_y = (1/y)$

$f(1,1) = 1$ $f_y(1,1) = 1$

$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$L(x,y) = x+y-2$

3] $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{1+1} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$

$f_y = \frac{1}{2\sqrt{x^2+y^2}} \times 2y$

$f(1,1) = \frac{1}{\sqrt{2}}$ $f_y = \frac{1}{\sqrt{2}}$

$\therefore L(1,1) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}$

$= \frac{2+x+y-2}{\sqrt{2}}$

$L(x,y) = \frac{x+y}{\sqrt{2}}$

Practical 10:

Title: Directional derivative, gradient vector & maxima, minima, tangent & normal vectors.

Q.1] Find the directional derivative of the following at given points & in the direction given vector

a] $f(x,y) = x+2y-3 \quad a=(1,-1) \quad u = 3i-j$

Solution: $|\bar{u}| = \sqrt{(3)^2+(-1)^2} = \sqrt{10}$

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = 1+2(-1)-3 = -4$

$\therefore f(a+hu) = f\left(1+\frac{3h}{\sqrt{10}}\right), \left(-1-\frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1+\frac{3h}{\sqrt{10}}\right)+2\left(-1-\frac{h}{\sqrt{10}}\right)-3$

$= 1+\frac{3}{\sqrt{10}} - 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$

$= 1+\frac{3h}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

$= \lim_{h\to 0} \frac{-4+h/\sqrt{10}+4}{h}$

$= \frac{1}{\sqrt{10}}$

b] $f(x) = y^2-4x+1 \quad a=(3,4) \quad u = i+5j$

solution: Here $u = i+5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2+(5)^2} = \sqrt{26}$

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$f(a) = f(3,4) = (4)^2-4(3)+1$
$= 5$

$f(a+hu) = f(3,4)+h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5}{\sqrt{26}}\right)$

$f_{(x,y)}(a+hu) = \left(4+\frac{h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right)+1$

$= 16+\frac{25h^2}{26}+\frac{40h}{\sqrt{26}}-12-\frac{4h}{\sqrt{26}}+1$

$= \frac{25h^2}{\sqrt{26}} + \frac{36h}{\sqrt{26}} + 5$

$D_u f(x) = \lim_{h\to 0} \frac{\frac{25h^2}{26}+\frac{36h}{\sqrt{26}}+5-5}{h}$

$= \frac{h\left(\frac{25h}{26}+\frac{36}{\sqrt{26}}\right)}{h}$

$= 36/\sqrt{26}$

3) $2x+3y \quad a=(1,2),\ u=(3\hat{i}+4\hat{j})$

solution: since $u=3\hat{i}+4\hat{j}$ is not a unit vector

$|u| = \sqrt{(3)^2+(4)^2} = \sqrt{25} = 5$

unit vector along u is $\frac{u}{|u|} = \frac{1}{5}(3,4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(a) = f(1,2) = 2(1)+3(2) = 2+6 = 8$

$f(a+hu) = f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= f\left(1+\frac{3h}{5}, 2+\frac{4h}{5}\right)$

$f(a+hu) = 2\left(1+\frac{3h}{5}\right) + 3\left(2+\frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$

$= \dfrac{\frac{18h}{5}+8-8}{h}$

$= \frac{18}{5}$

Find gradient vector for the following function at given point

a) $f(x,y) = x^y + y^x \rightarrow at\ (1,1)$

$f(x) = x^y \log x \quad yx^{y-1} + y^x \log y$

$f(y) = x^y \log x + xy^{x-1}$

$\nabla f(x,y) = (f_x, f_y)$

$= (yx^{y-1} + y^x \log y,\ x^y \log x + xy^{x-1})$

$f(1,1) = (1+0, 1+0)$

$= (1,1)$

b) $f(x,y) = (\tan^{-1}x)\, y^2 \quad at\ (1,-1)$

$f_x = \frac{1}{1+x^2}\, y^2$

$f_y = 2y \tan^{-1}x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, 2y\tan^{-1}x\right)$

$f(1,-1) = \left(\frac{1}{2}, -\frac{\pi}{2}\right)$

3] $f(x,y,z) = xyz - e^{x+y+z}$ $a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z})$

$= (-1 \cdot 0 - e^{1+(-1)+0}, (1)(0) - e^{(1)+(-1)+0}, (1)(-1) - e^{(1)+(-1)+0})$

$= (-1, -1, -2)$

Q3] Find the equation of tangent & normal to each of the following curves at given points

I] $x^2\cos y + e^{xy} = 2$ at $(1,0)$

solution: $f_x = \cos y\, 2x + e^{xy}\, y$

$f_y = x^2(-\sin y) + e^{xy}\cdot x$

$(x_0,y_0) = (1,0)$

$\therefore x_0 = 1,\ y_0 = 0$

equation of tangent

$f_x(x-x_0) + f_y(y-y_0) = 0$

$f_x(x_0,y_0) = \cos 0\, 2(1) + e^0\, 0$ 82

$= 1(2) + 0$

$= 2$

$f_y(x_0,y_0) = (1)^2 - (\sin 0) + e^0 \cdot 1$

$= 0 + 1 \cdot 1$

$= 1$

$\therefore 2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$\boxed{2x + y - 2 = 0}$

equation of normal

$ax + by + c = 0$

$bx + ay + d = 0$

$(1)(1) + 2(y) + d = 0$

$1 + 2y + d = 0$

$1 + 2(0) + d = 0$

$\therefore d = -1$

$\therefore \boxed{x + 2y - 1 = 0}$

2] $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 2y + 3$

$(x_0,y_0) = (2,-2)$

$x_0 = 2$

$y_0 = -2$

$f_x(x_0,y_0) = 2(2)-2 = 2$

$f_y(x_0,y_0) = 2(-2)+3 = -1$

Equation of tangent

$f_x(x-x_0) + f_y(y-y_0) = 0$

$2(x-2) + (-1)(y+2) = 0$

$2x-2-y-2 = 0$

$\boxed{2x-y-4=0}$

Equation of normal:

$ax+by+c=0$

$bx+ay+d=0$

$(-1)(x)+2(y)+d=0$

$-x+2y+d=0$

$-2+2(-2)+d=0$

$-6+d=0$

$d=6$

$\therefore \boxed{-x+2y+6=0}$

Q.4] Find the equation of tangent & normal line to each of the following surface

I] $x^2-2yz+3y+xz=7$ at $(2,1,0)$

Solution: $f_x = 2x+z$

$f_y = 2z+3$

$f_z = -2y+x$

$(x_0,y_0,z_0) = (2,1,0)$

$\therefore x_0 = 2,\ y_0 = 1,\ z_0 = 0$

$f_x(x_0,y_0,z_0) = 2(2)+0 = 4$

$f_y(x_0,y_0,z_0) = 2(0)+3 = 3$

$f_z(x_0,y_0,z_0) = -2(1)+2 = 0$

Equation of tangent

$f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0) = 0$

$4(x-2)+3(y-1)+0(z-0) = 0$

$4x-8+3y-3 = 0$

$4x+3y-11 = 0$

Equation of normal

$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$

$\therefore \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$

II] $3xyz-x-y+z=-4$ at $(1,-1,2)$

Solution: $f_x = 3yz-1$

$f_y = 3xz-1$

$f_z = 3xy+1$

$(x_0,y_0,z_0) = (1,-1,2)$

$f_x(x_0,y_0,z_0) = -7$

$f_y(x_0,y_0,z_0) = 5$

$f_z(x_0,y_0,z_0) = -2$

Equation of tangent

$-7(x-1)+5(y+1)-2(z-2)=0$

$-7x+7+5y+5-2z+4=0$

$-7x+5y-2z+16=0$

Equation of normal

$$\frac{x-x_0}{f_x}=\frac{y-y_0}{f_y}=\frac{z-z_0}{f_z}$$

$$\frac{x-1}{-7}=\frac{y+1}{5}=\frac{z-2}{-2}$$

Q.5] Find the local maxima & minima for the following function

1] $f(x,y)=3x^2+y^2-3xy+6x-4y$

Solution: $f_x=6x-3y+6$

$f_y=2y-3x-4$

$\therefore f_x=0$

$6x-3y+6=0$

$3(2x-y+2)=0$

$2x-y+2=0$

$2x-y=-2 \quad -①$

$\therefore f_y=0$

$2y-3x-4=0$

$2y-3x=4 \quad -②$

multiply eqn ① with 2

$4x-2y=-4$

$2y-3x=4$

$x=0$

substitute value of x in eq^n ①

$2(0)-y=-2$

$y=2$

$\therefore$ critical points are $(0,2)$

$\therefore r=f_{xx}=6$

$\therefore t=f_{yy}=2$

$\therefore s=f_{xy}=-3$

Here $r>0$

$=rt-s^2$

$=6(2)-(-3)^2$

$=3>0$

$\therefore f$ has maxima at $(0,2)$

$= 3x^2 + y^2 - 3xy + 6x - 4y$
$= 3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(y)$
$= -4$

2] $f(x,y) = 2x^4 + 3x^2y - y^2$
solution: $f_x = 8x^3 + 6xy$
$f_y = 3x^2 - 2y$

$f_x = 0$ $\therefore 8x^3 + 6xy = 0$
$\therefore 2x(4x^2 + 3y) = 0$
$\therefore 4x^2 + 3y = 0$ — ①

$f_y = 0$ $3x^2 - 2y = 0$ — ②
Multiply eq ① by ③ & ② by 4
$12x^2 + 9y = 0$
$-12x^2 - 8y = 0$
$\therefore y = 0$

substitute value of y in eq ①
$4x^2 + 3(0) = 0$
$4x^2 = 0$
$x = 0$
critical point is (0,0)
$\therefore r = f_{xx} = 0$
$\therefore t = -2$
$\therefore s = 0$

$\therefore r = 0$
$\therefore rt - s^2 = 0(-2) - (0)^2$
$= 0(-2) - 0$
$= 0$

$\therefore rt - s^2 = 0$ & $r = 0$
$\therefore$ Nothing to say; they are saddle points

3] $f(x,y) = x^2 - y^2 + 2x + 8y - 70$
$f_x = 2x + 2$
$f_y = -2y + 8$
$f_x = 0$ $\therefore x = -1$
$f_y = 0$ $\therefore y = 4$
$\therefore$ critical point is (-1,4)

$\therefore r = f_{xx} = 2$
$t = f_{yy} = -2$
$s = f_{xy} = 0$

$\therefore r > 0$
$\therefore rt - s^2 = 2(-2) - (0)^2$
$= -4 < 0$
$f(x,y)$ at (-1,4)
$= (-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$
$= 1 - 16 - 2 + 32 - 70$
$= 17 + 30 - 70$
$= 37 - 70 = 33$