Unit → 1

Partial Differential Equations

- A differential equation containing partial derivatives of a function of two or more independent variables is called a partial differential equ^{n}s

For e.g. $$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

or $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

→ **Notations in Partial Differential equation**

Let z is a func^n of two independent variables x & y then,

- $\boxed{p = \frac{\partial z}{\partial x}}$ • $\boxed{q = \frac{\partial z}{\partial y}}$

- $\boxed{r = \frac{\partial^2 z}{\partial x^2}} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$

- $\boxed{s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$

- $\boxed{t = \frac{\partial^2 z}{\partial y^2}} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$

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→ **Order of PDE**

Order of PDE is the highest ordered derivative present in the equation

→ **Degree of PDE**

Degree of PDE is the greatest exponent (power) of the highest ordered derivative present in the equation.

★ **Formation of PDE**

→ By the elimination of arbitary constants
→ By the elimination of arbitary functions

Q- Form partial differential equations from the following equations by eliminating the arbitary constants

(i) $z = ax + by + ab$

Sol^n ⇒

$$p = \frac{\partial z}{\partial x} = a \quad , \quad q = \frac{\partial z}{\partial y} = b$$

$$\therefore \boxed{z = px + qy + pq} \text{ Ans}$$

(ii) $z = ax + a^2y^2 + b$

Sol^n

$$p = \frac{\partial z}{\partial x} = a \qquad q = \frac{\partial z}{\partial y} = a^2(2y) = a^2 \cdot 2y$$

$$\Longrightarrow \boxed{q = p^2 \cdot 2y} \text{ Ans}$$

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(iii) $z = (x+a)(y+b)$

Sol^n ⇒

$$p = \frac{\partial z}{\partial x} = (y+b) \qquad q = \frac{\partial z}{\partial y} = (x+a)$$

$$\Rightarrow \boxed{z = qp}$$

(iv) $az + b = a^2x + y$

Sol^n ⇒

$$a \cdot \frac{\partial z}{\partial x} + 0 = a^2 + 0 \qquad \Big| \qquad a\frac{\partial z}{\partial y} + 0 = 0 + 1$$

$$\Rightarrow \boxed{p = \frac{\partial z}{\partial x} = a} \qquad \Big| \qquad \boxed{q = \frac{\partial z}{\partial y} = \frac{1}{a}}$$

$$\Rightarrow \boxed{pq = 1} \text{ Ans}$$

Q- Form the PDE by eliminating the arbitary function from the following:

(i) $z = f(x^2 - y^2)$

Sol^n ⇒

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x)$$

$$\Rightarrow p = f'(x^2 - y^2)(2x) \quad \text{—} (1)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \Rightarrow q = f'(x^2 - y^2)(-2y) \quad \text{—} (2)$$

from (1) & (2)

$$\frac{p}{q} = \frac{x}{-y} \Rightarrow -py = qx \Rightarrow \boxed{qx + py = 0} \text{ Ans}$$

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(ii) $z = x + y + f(xy)$

Sol^n

$$p = \frac{\partial z}{\partial x} = 1 + 0 + f'(xy) \cdot y$$

$$q = \frac{\partial z}{\partial x} = 0 + 1 + f'(xy) \cdot x$$

$$\Rightarrow p - 1 = f'(xy)\, y \quad \text{—} (1)$$

$$q - 1 = f'(xy)\, x \quad \text{—} (2)$$

from (1) & (2)

$$\frac{p-1}{q-1} = \frac{y}{x} \Rightarrow px - x = qy - y$$

$$\Rightarrow \boxed{px - qy = x - y} \text{ Ans}$$

(iii) $z = \phi(x) \cdot \phi(y)$

$$p = \frac{\partial z}{\partial x} = \phi'(x) \cdot \phi(y) \quad \text{—} (1)$$

$$q = \frac{\partial z}{\partial y} = \phi(x) \cdot \phi'(y) \quad \text{—} (2)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \phi'(x) \cdot \phi'(y)$$

On (1) × (2) $\quad pq = \phi'(x) \cdot \phi(y) \cdot \phi(x) \cdot \phi'(y)$

$$= (\phi(x) \cdot \phi(y)) \cdot (\phi'(x) \cdot \phi'(y)) = zs$$

$$\Rightarrow \boxed{pq = zs} \text{ Ans}$$

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★ Linear Partial Differential Equation of First Order

If p and q occur in the first degree only and are not multiplied together, then it is called linear partial differential equation of the first order.

★ Lagrange's Equation

$$\boxed{Pp + Qq = R}$$

where P, Q, R $\Rightarrow$ func^n of x, y, z

→ To solve the equation $Pp + Qq = R$

(i) Form the Lagrange's Auxilliary equ^{n}s or subsidiary equations.

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

(ii) Solve the auxiliary equ^n to get two independent solutions

$\boxed{u = a}$ and $\boxed{v = b}$ (a, b are constants)

(iii) Then $\phi(u, v)$ is the general solution of the equation $[Pp + Qq = R]$

Q- Solve PDE:

(i) $yzp - xzq = xy$

Sol^n ⇒

Here; $P = yz$, $Q = -xz$ & $R = xy$

∴ Auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \underset{\text{I}}{\frac{dx}{yz}} = \underset{\text{II}}{\frac{dy}{-xz}} = \underset{\text{III}}{\frac{dz}{xy}}$$

Taking I & II $\quad \frac{dx}{y\cancel{z}} = \frac{dy}{-x\cancel{z}} \Rightarrow -x\,dx = y\,dy$

$$\Rightarrow \boxed{x\,dx + y\,dy = 0}$$

On integrating $\frac{x^2}{2} + \frac{y^2}{2} = \frac{C_1}{2}$

$$\Rightarrow \boxed{x^2 + y^2 = C_1}$$

Taking II & III $\quad \frac{dy}{-\cancel{x}z} = \frac{dz}{\cancel{x}y} \Rightarrow y\,dy = -z\,dz$

$$\Rightarrow y\,dy + z\,dz = 0$$

On integrating $\frac{y^2}{2} + \frac{z^2}{2} = \frac{C_2}{2}$

$$\Rightarrow \boxed{y^2 + z^2 = C_2}$$

∴ $\phi(x^2+y^2, y^2+z^2) = 0$ is the complete sol^n of the eqn^n

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$\frac{y^2 z}{x} p + xzq = y^2 \Rightarrow y^2 z\, p + x^2 z\, q = xy^2$

Here, $P = y^2 z$, $Q = x^2 z$, $R = y^2 x$

$\therefore$ Auxiliary equ^n $\quad \underset{\text{I}}{\frac{dx}{y^2 z}} = \underset{\text{II}}{\frac{dy}{x^2 z}} = \underset{\text{III}}{\frac{dz}{xy^2}}$

eq^n I & II $\quad \frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}} \Rightarrow x^2 \cdot dx = y^2 dy$

$\Rightarrow x^2 \cdot dx - y^2 dy = 0$

On integrating $\quad \frac{x^3}{3} - \frac{y^3}{3} = \frac{C_1}{3}$

$\Rightarrow \boxed{x^3 - y^3 = C_1}$

eq^n I & III $\quad \frac{dx}{\cancel{y^2} z} = \frac{dz}{x \cancel{y^2}} \Rightarrow x \cdot dx = z \cdot dz$

$\Rightarrow x\,dx - z\,dz = 0$

On integrating $\quad \frac{x^2}{2} - \frac{z^2}{2} = \frac{C_2}{2}$

$\Rightarrow \boxed{x^2 - z^2 = C_2}$

$\phi(x^3 - y^3, x^2 - z^2) = 0$ is the sol^n of the given equ^n

) $y^2 p - xy\, q = x(z - 2y)$

$\Rightarrow$ Here $P = y^2$, $Q = -xy$, $R = x(z - 2y)$

$\therefore$ Auxiliary equ^n $\quad \underset{\text{I}}{\frac{dx}{y^2}} = \underset{\text{II}}{\frac{dy}{-xy}} = \underset{\text{III}}{\frac{dz}{x(z-2y)}}$

eq^n I & II $\quad \frac{dx}{\cancel{y^2}} = \frac{dy}{-x\cancel{y}} \Rightarrow -x\,dx = y\,dy$

$\Rightarrow x\,dx + y\,dy = 0$

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On integrating $\frac{x^2}{2} + \frac{y^2}{2} = \frac{C_1}{2}$

$$\Rightarrow \boxed{x^2 + y^2 = C_1}$$

Taking II & III $\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$

$$\Rightarrow z \cdot dy - 2y \cdot dy = -y\,dz$$

$$\Rightarrow 2y \cdot dy = z \cdot dy + y\,dz$$

On integrating $\Rightarrow y^2 = zy + yz + C_2$

$$\Rightarrow \boxed{y^2 - 2yz = C_2}$$

$\therefore \phi(x^2+y^2,\ y^2-2yz)$ is the solⁿ of the eqnⁿ.

Q- Solve : $x^2 p + y^2 q = (x+y)z$

Solⁿ Here, $P = x^2$; $Q = y^2$; $R = (x+y)z$

$\therefore$ Auxiliary Eqnⁿ $\underset{\text{I}}{\frac{dx}{x^2}} = \underset{\text{II}}{\frac{dy}{y^2}} = \underset{\text{III}}{\frac{dz}{(x+y)z}}$

Taking I & II $\frac{dx}{x^2} = \frac{dy}{y^2}$ On integrating $-\frac{1}{x} = -\frac{1}{y} + C_1$

$$\Rightarrow \boxed{-\frac{1}{x} + \frac{1}{y} = C_1}$$

Formula to create IV fraction $\left[\dfrac{m_1 dx + m_2 dy + m_3 dz}{m_1 P + m_2 Q + m_3 R}\right]$

$$\underset{\text{I}}{\frac{dx}{x^2}} = \underset{\text{II}}{\frac{dy}{y^2}} = \underset{\text{III}}{\frac{dz}{(x+y)z}} = \underset{\text{IV}}{\frac{dx-dy}{x^2-y^2}}$$

Taking III & IV

$$\frac{dz}{\cancel{(x+y)}z} = \frac{dx-dy}{\cancel{(x+y)}(x-y)}$$

$$\Rightarrow \frac{dz}{z} = \frac{dx-dy}{(x-y)}$$

on integrating $\log z = \log(x-y) + \log C_2$

$$\Rightarrow \log\left(\frac{z}{x-y}\right) = \log C_2$$

$$\Rightarrow \boxed{\frac{z}{x-y} = C_2}$$

$\therefore \phi\left(-\frac{1}{x}+\frac{1}{y}, \frac{z}{x-y}\right) = 0$ is the sol^n of the given equ^n

Q - Solve: $(mz-ny)p + (nx-lz)q = ly-mx$

Sol^n Here, $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

$\therefore$ Auxiliary Equ^n $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$ — (1)

Taking l, m, n as multiplier, equ^n (1) becomes

$$\underset{\text{I}}{\frac{dx}{mz-ny}} = \underset{\text{II}}{\frac{dy}{nx-lz}} = \underset{\text{III}}{\frac{dz}{ly-mx}} = \underset{\text{IV}}{\frac{l\,dx+m\,dy+n\,dz}{0}}$$

Taking I & II $\dfrac{dx}{mz-ny} = \dfrac{l\,dx+m\,dy+n\,dz}{0}$

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$$l\,dx + m\,dy + n\,dz = 0$$

$$\Rightarrow \boxed{lx + my + nz = C_1}$$

Again, Taking x, y, z as multipliers, equ^n (1) becomes

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_2}{2}$$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = C_2}$$

$\therefore$ $\boxed{\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0}$ is the sol^n of the equ^n

Q- Solve: $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

Sol^n: Here, $P = x^2(y-z)$ $\quad Q = y^2(z-x)$ $\quad R = z^2(x-y)$

$\therefore$ Auxiliary Equ^n $\quad \dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$ — (1)

Taking $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ as multipliers, equ^n (1) becomes

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

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$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}\cdot dy + \frac{1}{z^2}dz = 0$$

On integrating

$$\Rightarrow \boxed{\frac{-1}{x} - \frac{1}{y} - \frac{1}{z} = C_1}$$

Now, Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, equ^n ① becomes

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x}\cdot dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}dz = 0$$

On integrating

$$\Rightarrow \log x + \log y + \log z = \log C_2$$

$$\Rightarrow \log xyz = \log C_2$$

$$\Rightarrow \boxed{xyz = C_2}$$

$\therefore \boxed{\phi\left(\frac{-1}{x} - \frac{1}{y} - \frac{1}{z}, xyz\right) = 0}$ is the ~~[illegible]~~ sol^n of the equ^n

Q- Solve the PDE: $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Sol^n ⇒ Here, $P = x(y^2+z)$, $Q = -y(x^2+z)$, $R = z(x^2-y^2)$

$\therefore$ Auxiliary equ^n $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$ — ①

On Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers equ^n ① becomes

Note → Some of the common multipliers

$1, 1, 1 \qquad -1, -1, -1 \qquad x, y, z \qquad x^2, y^2, z^2$

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \qquad \frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}\cdot dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}\cdot dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\Rightarrow \log x + \log y + \log z = \log c_1$$

$$\Rightarrow \boxed{xyz = c_1}$$

Now, again taking $x, y, -1$ as multipliers eqn (1) becomes

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\cdot dx + y\,dy - dz}{0}$$

$$\Rightarrow x\cdot dx + y\,dy - dz = 0$$

On integrating

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c_2$$

$$\Rightarrow \boxed{x^2+y^2-2z = c_2}$$

$\therefore \boxed{\phi(xyz, x^2+y^2-2z) = 0}$ is the complete solution of the eqn

Q. Solve: $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$

Sol^n:

Here, $P = x^2-y^2-yz$, $Q = x^2-y^2-zx$, $R = z(x-y)$

$\therefore$ Auxiliary Eqn^n $\dfrac{dx}{x^2-y^2-yz} = \dfrac{dy}{x^2-y^2-zx} = \dfrac{dz}{z(x-y)}$ — (1)

On Taking 1, −1, −1 as multipliers eqn^n (1) becomes

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)} = \frac{dx-dy-dz}{0}$$

$$\Rightarrow dx-dy-dz=0$$

On integrating

$$\Rightarrow \boxed{x-y-z=c_1}$$

Now, taking $x, -y, 0$ as multipliers eqn^n (1) becomes

$$\underset{\text{I}}{\frac{dx}{x^2-y^2-yz}} = \underset{\text{II}}{\frac{dy}{x^2-y^2-zx}} = \underset{\text{III}}{\frac{dz}{z(x-y)}} = \underset{\text{IV}}{\frac{x\cdot dx-y\,dy}{(x-y)(x^2-y^2)}}$$

Taking III & IV $\dfrac{dz}{z\cancel{(x-y)}} = \dfrac{x\cdot dx-y\,dy}{\cancel{(x-y)}(x^2-y^2)}$

$$\Rightarrow \frac{dz}{z} = \frac{x\cdot dx-y\,dy}{x^2-y^2}$$

$$\log z = \frac{1}{2}\log(x^2-y^2) + \log c_2$$

$$\Rightarrow \boxed{\frac{z}{\sqrt{x^2-y^2}} = c_2}$$

$\therefore \boxed{\phi\left(x-y-z, \dfrac{z}{\sqrt{x^2-y^2}}\right)=0}$ is the complete sol^n of the given eqn

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★ Charpit's Method (Sol^n for non-linear PDE)

→ **Charpit's Auxiliary Equation**

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Solve

Q- $2zx - px^2 - 2qxy + pq = 0$

Sol^n

Let $f = 2zx - px^2 - 2qxy + pq = 0$

$\therefore$ $\boxed{\frac{\partial f}{\partial x} = 2z - 2px - 2qy}$ $\boxed{\frac{\partial f}{\partial y} = -2qx}$

$\boxed{\frac{\partial f}{\partial z} = 2x}$ $\boxed{\frac{\partial f}{\partial p} = -x^2 + q}$ $\boxed{\frac{\partial f}{\partial q} = -2xy + p}$

Charpit's Auxiliary Eqn

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{2z - \cancel{2px} - 2qy + \cancel{2px}} = \frac{dq}{-\cancel{2qx} + \cancel{2qx}} = \frac{dz}{px^2 - pq + 2qxy - pq}$$

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$$= \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$$

$$\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{px^2-2pq+2qxy} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$$

Now, $\frac{dq}{0} = \frac{dF}{0}$

$$\Rightarrow \boxed{dq = 0}$$

on integrating

$$\Rightarrow \boxed{q = C_1}$$

Let $\boxed{q = a}$

Now, putting $q = a$ in the given differential equ^n

$$2zx - px^2 - 2axy + pa = 0$$

$$\Rightarrow p(-x^2+a) = -2axy - 2xz$$

$$\Rightarrow p = \frac{2axy - 2xz}{(-x^2+a)}$$

$$\Rightarrow \boxed{p = \frac{2x(z-ay)}{(x^2-a)}}$$

Now, $dz = \overset{\to p}{\frac{\partial z}{\partial x}} \cdot dx + \overset{\to q}{\frac{\partial z}{\partial y}} dy$

$$\Rightarrow dz = \left\{\frac{2x(z-ay)}{x^2-a}\right\} dx + a \cdot dy$$

$$\Rightarrow dz - a\,dy = (z-ay)\left\{\frac{2x}{x^2-a}\right\} dx$$

$$\Rightarrow \frac{dz - x\,dy}{(z - xy)} = \frac{2x \cdot dx}{x^2 - x}$$

on integrating we get

$$\Rightarrow \log(z - xy) = \log(x^2 - x) + \log c$$

$$\Rightarrow \log(z - xy) = \log\{(x^2 - x) \cdot c\}$$

$$\Rightarrow z - xy = (x^2 - x)c$$

$$\Rightarrow \boxed{z = (x^2 - x) \cdot c + xy} \text{ Ans}$$

Q. Solve by using the Charpit's Method

$$p^2 + qy = z$$

Sol^n $\Rightarrow p^2 + qy - z = 0$

Let $f = p^2 + qy - z = 0$

$$\therefore \boxed{\frac{\partial f}{\partial x} = 0} \quad \boxed{\frac{\partial f}{\partial y} = q} \quad \boxed{\frac{\partial f}{\partial z} = -1} \quad \boxed{\frac{\partial f}{\partial p} = 2p} \quad \boxed{\frac{\partial f}{\partial q} = y}$$

$\therefore$ Charpit's Auxiliary eqn^n

$$\frac{dp}{0 + p(-1)} = \frac{dq}{q + q(-1)} = \frac{dz}{-p(2p) - q(y)} = \frac{dx}{-2p} = \frac{dy}{-y} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-2p^2 - qy} = \frac{dx}{-2p} = \frac{dy}{-y} = \frac{dF}{0}$$

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$$\Rightarrow \frac{dq}{0} = \frac{dF}{0}$$

$\Rightarrow dq = 0$ On integrating, we get $\boxed{q = c_1}$

Let $\boxed{q = a}$

Now, on putting $q = a$ in the given eqn, we get

$$p^2 + ay = z \Rightarrow \boxed{p = \sqrt{z - ay}}$$

Now, $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\Rightarrow dz = \sqrt{z - ay}\, dx + ay$$

$$\Rightarrow dz - ay = \sqrt{z - ay}\, dx$$

$$\Rightarrow \frac{dz - ay}{\sqrt{z - ay}} = dx$$

On integrating, we get

$$\Rightarrow 2\sqrt{z - ay} = x + c$$

$$\Rightarrow \sqrt{z - ay} = \frac{x + c}{2}$$

$$\Rightarrow \boxed{z = \left(\frac{x + c}{2}\right)^2 + ay}$$ Ans

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Q- Solve: $(p^2 + q^2)y = qz$

Sol^n $\Rightarrow$ $(p^2 + q^2)y - qz = 0$

Let $f = (p^2 + q^2)y - qz = 0$

$\therefore \boxed{\frac{\partial f}{\partial x} = 0}$ $\boxed{\frac{\partial f}{\partial y} = p^2+q^2}$ $\boxed{\frac{\partial f}{\partial z} = -q}$ $\boxed{\frac{\partial f}{\partial p} = 2py}$

$\boxed{\frac{\partial f}{\partial q} = 2qy - z}$

$\therefore$ Charpit's Auxiliary Equation

$$\frac{dp}{0+p(-q)} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2y-2q^2y+zp} = \frac{-dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow p^2dp = -pq\,dq$$

$$\Rightarrow p\,dp = -q\,dq \Rightarrow \boxed{p\,dp + q\,dq = 0}$$

on integrating, we get $\boxed{p^2+q^2 = a^2}$

$\cancel{\boxed{q = \sqrt{a^2-p^2}}}$ $\cancel{\boxed{p = \sqrt{a^2-q^2}}}$

putting in the given equ^n.

$\cancel{a^2-q^2+qy=z}$ $a^2y = qz$

$$\Rightarrow \boxed{q = \frac{a^2y}{z}}$$

$$p^2+q^2 = a^2 \Rightarrow p = \sqrt{a^2-q^2}$$

$$\Rightarrow p = \sqrt{a^2 - \frac{a^4y^2}{z^2}} \Rightarrow \boxed{p = \frac{a}{z}\sqrt{z^2-a^2y^2}}$$

Now,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow z\,dz = \sqrt{z^2-a^2y^2}\,\{a\,dx\} + a^2y\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

on integrating, we get

$$\sqrt{z^2-a^2y^2} = ax + C$$

$$\Rightarrow z^2 - a^2y^2 = (ax+c)^2$$

$$\Rightarrow \boxed{z^2 = (ax+c)^2 + a^2y^2}$$

Cauchy's Method of Characteristics

For partial differential equation,

$$\boxed{a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x,y) + Ku}\ ;\ u(0,y) = h(y)$$

Auxiliary Equ^n

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y)+Ku}$$

- Use Cauchy's Method of characteristics to solve the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y\ ;\ u(x,0)=0$$

Sol^n ⇒ Auxiliary Equ^n $\quad \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{f(x,y)+ku}$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$$

$$\frac{dx}{1} = \frac{dy}{1} \Rightarrow dx - dy = 0$$

on integrating, we get

$$\boxed{x - y = C_1} \Rightarrow x = y + C_1$$

Now $\dfrac{dy}{1} = \dfrac{du}{x+y} \Rightarrow du = (x+y)\,dy$

$$\Rightarrow du = (y + C_1 + y)\,dy$$

$$\Rightarrow du = (C_1 + 2y)\,dy$$

on integrating we get

$$\boxed{u(x,y) = C_1 y + y^2 + C_2}$$

Let $C_2 = g(C_1) \Rightarrow \boxed{C_2 = g(x-y)}$

$$\boxed{u(x,y) = (x-y)y + y^2 + g(x-y)}$$

for find $g(x-y)$ use given cond^n i.e. $u(x,0)=0$

$$0 = (x-0)0 + 0 + g(x)$$

$$\Rightarrow \boxed{g(x) = 0} \Rightarrow \boxed{g(x-y) = 0}$$

$$\therefore u(x,y) = (x-y)y + y^2 \Rightarrow \boxed{u(x,y) = xy}$$ Ans

$u_x \to \frac{\partial u}{\partial x}$ $\quad u_y = \frac{\partial u}{\partial y}$

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Q- Use the method of characteristics to solve:

$$u_x - u_y = 0 \quad ; \quad u(x,0) = x$$

Sol^n

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Auxiliary Equ^n $\quad \frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$

$$\Rightarrow -dx = dy \Rightarrow dx + dy = 0$$

on integrating, we get

$$\boxed{x + y = C_1}$$

Now $\quad \frac{dy}{-1} = \frac{du}{0} \Rightarrow du = 0$

on integrating, we get

$$\boxed{u = C_2}$$

$$\Rightarrow \boxed{u(x,y) = C_2}$$

Let $C_2 = g(C_1) \Rightarrow \boxed{C_2 = g(x+y)}$

$$\boxed{u(x,y) = g(x+y)}$$

$$u(x,0) = g(x) \Rightarrow \boxed{g(x) = x}$$

$$\Rightarrow \boxed{g(x+y) = x+y}$$

$$\Rightarrow \boxed{u(x,y) = x+y}$$

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Q- Use the method of characteristics of solve the first order PDE:

$$\boxed{u_x - yu = 0} \quad ; \quad u(0,y) = 1$$

Sol^n $\Rightarrow$ $u_x = uy \Rightarrow \dfrac{du}{dx} = uy$

$\therefore$ Auxiliary Eqn^n $\dfrac{dx}{1} = \dfrac{dy}{0} = \dfrac{du}{yu}$

$$\frac{dx}{1} = \frac{dy}{0} \Rightarrow dy = 0$$

$$\Rightarrow \boxed{y = C_1}$$

$$\frac{du}{uy} = \frac{dx}{1} \Rightarrow \frac{du}{u} = y\,dx$$

$$\Rightarrow \frac{du}{u} = C_1 \cdot dx$$

$$\Rightarrow \boxed{\log u(x,y) = C_1 x + C_2}$$

Let $C_2 = g(C_1)$

$\Rightarrow C_2 = g(y)$

$$\Rightarrow \boxed{\log(u(x,y)) = yx + g(y)}$$

using cond^n

$$\Rightarrow \log(u(0,y)) = 0 + g(y)$$

$$\Rightarrow \boxed{g(y) = 0}$$

$$\boxed{\log u(x,y) = yx}$$

$$\Rightarrow \boxed{u = e^{xy}} \text{ Ans}$$

Homogeneous Linear PDE with constant coefficients

Consider an equ^n:

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

$$\Rightarrow \left[a_0 D^n + a_1 D^{n-1} (D')^1 + a_2 D^{n-2} (D')^2 + \cdots + a_n (D')^n\right] z = F(x,y)$$

where $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

or $\boxed{\phi(D, D') z = F(x,y)}$

whose

Complete solution	=	Complementary Function (CF)	+	Particular Integral (PI)

Complementary Function (C.F.) Calculation

Step 1) Write $\phi(D, D') z = F(x,y)$ as $\phi(D, D') z = 0$

Step 2) Write Auxiliary equ^n by putting $D = m$, $D' = 1$
i.e. $\phi(m, 1) = 0$

Step 3) Obtain the roots of A.E. such $m_1, m_2, m_3 \cdots$

Step 4) Write C.F. as per roots

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Case I For all distinct roots $(m_1, m_2, m_3 - - -)$

$$C.F. = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + - - -$$

Case II For two equal roots $(m_1, m_1, m_2, m_3 - - - - -)$

$$C.F. = f_1(y+m_1x) + xf_2(y+m_1x) + f_3(y+m_2x) + f_4(y+m_3x) - - -$$

Case III For three equal roots $(m_1, m_1, m_1 - - - - -)$

$$C.F. = f_1(y+m_1x) + xf_2(y+m_1x) + x^2f_3(y+m_1x) + - - - -$$

Q. Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

Sol^n: Given, $\{D^2 - DD' - 6(D')^2\}z = 0$

For C.F.

A.E. $\boxed{m^2 - m - 6 = 0}$

$m = -2, 3$

$\therefore \boxed{C.F. = f_1(y+3x) + f_2(y-2x)}$ Ans

$\therefore z = f_1(y+3x) + f_2(y-2x)$ Ans

Q. Solve: $4r - 12s + 9t = 0$

$$\Rightarrow \left[4D^2 - 12DD' + 9(D')^2\right] z = 0$$

for C.F. A.E. $\Rightarrow 4m^2 - 12m + 9 = 0 \Rightarrow 4m^2 - 6m - 6m + 9 = 0$

$$\boxed{m = \frac{3}{2}, \frac{3}{2}}$$

$\therefore$ C.F. $= f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$

P.I. $= 0$

$$\Rightarrow z = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$$

Q- $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

Soln $\Rightarrow$ Given, $D^4 - (D')^4 = 0$

for C.F.

A.E. $\Rightarrow m^4 - 1 = 0 \Rightarrow (m^2)^2 - (1)^2 = 0$

$$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow \boxed{m = \pm 1, \pm i}$$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

P.I. $= 0$

Hence, $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$ Ans

Q- Solve: $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

Soln $\Rightarrow$ for C.F.

A.E. $\Rightarrow m^3 - 6m^2 + 12m - 8 = 0$

$$\Rightarrow m = 2, 2, 2$$

$\therefore$ C.F. $= f_1(y + 2x) + x f_2(y + 2x) + x^2 f_3(y + 2x)$

P.I. $= 0$

Hence, $z = f_1(y+2x) + x f_2(y+2x) + x^2 f_3(y+2x)$ Ans

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→ Rules to find P.I. (Particular Integral)

Case I When $\phi(x,y)$ is function of $ax+by$

Q. Solve: $\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$

Sol^n

$$\{D^3 - 3D^2D' + 4(D')^3\}u = e^{x+2y}$$

for C.F. $[D^3 - 3D^2D' + 4(D')^3]u = 0$

A.E. $\Rightarrow m^3 - 3m^2 + 4 = 0$

$\Rightarrow m = -1, 2, 2$

Hence, $\boxed{C.F. = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)}$

for P.I.

$$P.I. = \frac{1}{D^3 - 3D^2 \cdot D' + 4(D')^3} e^{x+2y}$$

$D \to a$ & $D' \to b$

Here $a = 1$ & $b = 2$

$$\therefore P.I. = \frac{1}{(1)^2 - 3(1)^2 2 + 4(2)^3} \iiint e^u \, du\, du\, du$$

where $u = x+2y$

$$= \frac{1}{1-6+32} \iiint e^u \, du\, du\, du$$

$$\Rightarrow P.I. = \frac{1}{27} e^{x+2y}$$

$\therefore$ Complete solution = C.F. + P.I.

$$u = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

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Q. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$

Sol^n Given, $[D^2 + 3DD' + 2(D')^2]z = x+y$

for C.F. A.E. $\Rightarrow m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$\therefore \boxed{C.F. = f_1(y-x) + f_2(y-2x)}$

for P.I. $P.I. = \frac{1}{D^2 + 3DD' + 2(D')^2} \cdot x+y$

$D \Rightarrow 1$ & $D' \Rightarrow 1$

$= \frac{1}{(1)^2 + 3(1)(1) + 2(1)^2} \iint u \, du \, du$ where $u = x+y$

$= \frac{1}{1+3+2} \cdot \frac{1}{2} \int u^2 \cdot du = \frac{1}{6} \times \frac{1}{2} \times \frac{1}{3} \cdot u^3$

$\Rightarrow \boxed{P.I. = \frac{1}{36}(x+y)^3}$

Hence, Complete Solution = C.F. + P.I.

$\Rightarrow \boxed{z = f_1(y-x) + f_2(y-2x) + \frac{1}{36}(x+y)^3}$

Q. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

Sol^n Given, $[D^2 - 2DD' + (D')^2]z = \sin(2x+3y)$

for C.F. A.E. $\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow \boxed{m = 1, 1}$

$\therefore \boxed{C.F. = f_1(y+x) + x f_2(y+x)}$

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for P.I. $P.I. = \frac{1}{D^2 - 2DD' + (D')^2} \cdot \sin(2x+3y)$

$D \Rightarrow 2$ & $D' \Rightarrow 3$

$P.I. = \frac{1}{(2)^2 - 2(2)(3) + (3)^2} \iint \sin u \, du \, du$ where $u = 2x+3y$

$= \frac{1}{4-12+9} \left[-\int \cos u \cdot du \right]$

$= -\sin u = -\sin(2x+3y)$

Hence, Complete Sol^n = C.F. + P.I.

$\Rightarrow z = f_1(y+x) + x f_2(y+x) - \sin(2x+3y)$

Q. Solve: $r + s - 2t = \sqrt{2x+y}$

Sol^n: Given,

$[D^2 + DD' - 2(D')^2] z = \sqrt{2x+y}$

for C.F. A.E. $\Rightarrow m^2 + m - 2 = 0 \Rightarrow \boxed{m = 1, -2}$

$\therefore \boxed{C.F. = f_1(y+x) + f_2(y-2x)}$

for P.I.

$P.I. = \frac{1}{D^2 + DD' - 2(D')^2} \cdot \sqrt{2x+y}$

putting $D = 2$ & $D' = 1$

$P.I. = \frac{1}{(2)^2 + (2)(1) - 2(1)^2} \iint \sqrt{u} \, du \, du$ where $u = 2x+y$

$= \frac{1}{4+2-2} \left[\frac{2}{3} \int u^{\frac{3}{2}} \cdot du \right]$

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$$= \frac{1}{\cancel{4}}\left[\frac{\cancel{4}}{15}u^{5/2}\right] = \frac{+1}{15}(2x+y)^{5/2}$$

Hence, Complete solution = C.F. + P.I.

$$\Rightarrow z = f_1(y+x) + f_2(y-2x) + \frac{1}{15}(2x+y)^{5/2}$$

Q- Solve: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Sol^n:

Given, $[D^2 - 2DD']z = \sin x \cos 2y$

for C.F. A.E. $\Rightarrow m^2 - 2m = 0 \Rightarrow \boxed{m = 0, 2}$

$\therefore$ C.F. $= f_1(y) + f_2(y+2x)$

for P.I. $P.I. = \frac{1}{D^2 - 2DD'} \sin x \cos 2y$

$$= \frac{1}{D^2 - 2DD'} \cdot \frac{1}{2}\left\{\sin(x+2y) + \sin(x-2y)\right\}$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D^2 - 2DD'}\left\{\sin(x+2y)\right\}}_{P_1} + \underbrace{\frac{1}{D^2 - 2DD'}\left\{\sin(x-2y)\right\}}_{P_2}\right]$$

~~$D = 1$ & $D' = 2$~~ ~~$D = 1$ & $D' = -2$~~

$$\cancel{= \frac{1}{2}\left[\frac{1}{(1)^2 - 2(1)(2)}\right.}$$

$$P_1 = \frac{1}{D^2 - 2DD'} \sin(x+2y)$$

$D = 1$ & $D' = 2$

$$P_1 = \frac{1}{(1)^2 - 2(1)(2)} \iint \sin u \cdot du \cdot du \quad \text{where } u = x+2y$$

$$= \frac{1}{1-4}\left[-\int \cos u \cdot du\right]$$

$$= \frac{1}{3}\sin u = \frac{1}{3}\sin(x+2y)$$

$$P_2 = \frac{1}{D^2 - 2DD'}\sin(x-2y)$$

$D = 1$ & $D' = -2$

$$= \frac{1}{(1)^2 - 2(1)(-2)}\iint \sin u \, du \, du \quad \text{where } u = x-2y$$

$$= \frac{1}{5}\left[-\int \cos u \, du\right] = -\frac{1}{5}\sin u = -\frac{1}{5}\sin x-2y$$

$$\Rightarrow P.I. = \frac{1}{2}\left[\frac{1}{3}\sin(x+2y) - \frac{1}{5}\sin(x-2y)\right]$$

Hence, Complete Solution = C.F. + P.I.

$$\Rightarrow z = f_1(y) + f_2(y+2x) + \frac{1}{2}\left[\frac{1}{3}\sin(x+2y) - \frac{1}{5}\sin(x-2y)\right]$$

Q. Solve $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x+y)$

Sol: Given,

$$\left[D^3 - 4D^2D' + 4D(D')^2\right] z = 4\sin(2x+y)$$

for C.F. A.E. $\Rightarrow m^3 - 4m^2 + 4m = 0$

$$\Rightarrow \boxed{m = 0, 2, 2}$$

$$\therefore \boxed{C.F. = f_1(y) + f_2(y+2x) + x f_3(y+2x)}$$

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for P.I. $\quad P.I. = \dfrac{1}{D^3 - 4D^2D' + 4D(D')^2} \cdot 4\sin(2x+y)$

putting $D = 2$ & $D' = 1$

$$\cancel{= 4\left[\frac{1}{(2)^3 - 4(2)^2(1) + 4(2)(1)^2} \int \sin u \, du\right] \text{ where } u = 2x+y}$$

$$\cancel{= 4\left[\frac{1}{8 - 16 + 8}\right]} \Rightarrow \text{case of failure}$$

$$\therefore \quad P.I. = 4\left[\frac{x}{3D^2 - 8DD' + 4(D')^2} \sin(2x+y)\right]$$

putting $D = 2$ & $D' = 1$.

$\Rightarrow$ case of failure

$$\therefore \quad P.I. = 4\left[\frac{x^2}{6D - 8D'} \sin(2x+y)\right]$$

putting $D = 2$ & $D' = 1$

$$= 4x^2\left[\frac{1}{6(2) - 8(1)} \int \sin u \cdot du\right] \text{ where } u = 2x+y$$

$$= \cancel{4}x^2\left[\frac{-\cos u}{\cancel{4}}\right] = -x^2\cos(2x+y)$$

Hence, Complete sol^n = C.F. + P.I.

$$\Rightarrow z = f_1(y) + f_2(y+2x) + x f_3(y+2x) - x^2\cos(2x+y)$$

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Q. Solve : $(D^2+5DD'+6(D')^2)z = \dfrac{1}{y-2x}$

Sol^n

for C.F. A.E. $m^2+5m+6=0$

$\Rightarrow \boxed{m=-2,-3}$

Hence, $\boxed{C.F. = f_1(y-2x) + f_2(y-3x)}$

for P.I.

$$P.I. = \frac{1}{D^2+5DD'+6(D')^2}(y-2x)^{-1}$$

putting $D=-2$ & $D'=1$

$\Rightarrow$ case of failure

$$\therefore P.I = \frac{x}{2D+5D'}(y-2x)^{-1}$$

putting $D=-2$ & $D'=1$

$$P.I. = x\left[\frac{1}{2(-2)+5(1)}\int u^{-1}\,du\right] \quad \text{where } u=y-2x$$

$$= x\left[\frac{1}{1}\log u\right] = x\log(y-2x)$$

Hence, Complete sol^n = C.F. + P.I.

$$z = f_1(y-2x) + f_2(y-3x) + x\log(y-2x)$$

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Case II When $\phi(x,y)$ is of the form $x^m y^n$

Q- Solve: $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

Sol^n

Given, $D^3 - (D')^3 = x^3 y^3$

for C.F.

A.E. $\Rightarrow m^3 - 1 = 0 \Rightarrow m = 1, \cancel{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$

$-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ C.F. $= f_1(y+x) + f_2(y+\omega_1 x) + f_3(y+\omega_2 x)$

where $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ & $\omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

for P.I. $\quad$ P.I. $= \frac{1}{D^3 - (D')^3} x^3 y^3$

$$= \frac{1}{D^3\left[1 - \left(\frac{D'}{D}\right)^3\right]} x^3 y^3 = \frac{1}{D^3}\left[1 - \left(\frac{D'}{D}\right)^3\right]^{-1} x^3 y^3$$

$$= \frac{1}{D^3}\left[1 + \left(\frac{D'}{D}\right)^3\right] x^3 y^3 = \frac{1}{D^3}\left[x^3 y^3 + \frac{1}{D^3} 6x^3\right]$$

$$= \frac{1}{D^3} x^3 y^3 + \frac{1}{D^6} 6x^3$$

$$= y^3\left[\frac{x^6}{4 \times 5 \times 6}\right] + 6\left[\frac{x^9}{4 \times 5 \times 6 \times 7 \times 8 \times 9}\right]$$

$$= \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

Hence, complete solution is

$$z = f_1(y+x) + f_2(y+\omega_1 x) + f_3(y+\omega_2 x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

Q Solve: $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

Sol^n

for C.F. A.E. $m^2 - 6m + 9 = 0$

$$\Rightarrow \boxed{m = 3, 3}$$

$\therefore$ C.F. $= f_1(y+3x) + x f_2(y+3x)$

for P.I. $\Rightarrow$ $P.I. = \dfrac{1}{D^2 - 6DD' + 9(D')^2}(12x^2 + 36xy)$

$$= \underbrace{\frac{1}{D^2 - 6DD' + 9(D')^2} 12x^2}_{P_1} + \underbrace{\frac{1}{D^2 - 6DD' + 9(D')^2} 36xy}_{P_2}$$

$$P_1 = \frac{1}{D^2 - 6DD' + 9(D')^2} 12x^2 = \frac{1}{D^2\left[1 - 6\frac{D'}{D} + 9\left(\frac{D'}{D}\right)^2\right]} 12x^2$$

$$= \frac{12}{D^2}\left[1 - 6\frac{D'}{D} + 9\left(\frac{D'}{D}\right)^2\right]^{-1} x^2$$

$$P_1 = \frac{1}{D^2 - 6DD' + 9(D')^2} 12x^2$$

putting $D = 1$ & $D' = 0$

$$= 12 \cdot \frac{1}{(1)^2 - 6(1)(0) + 9(0)} \iint u^2 \, du \quad \text{where } u = x$$

$$= 12\left[\frac{u^4}{3\times 4}\right] = u^4 = x^4$$

$$= \frac{1}{D^2 - 6DD' + 9(D')^2}\, 36xy = \frac{1}{(D-3D')^2}\, 36xy$$

$$= \frac{36}{D^2\left[1-\frac{3D'}{D}\right]^2}\, xy = \frac{36}{D^2}\left[1-\frac{3D'}{D}\right]^{-2} xy$$

$$\cancel{= \frac{36}{D^2}\left[1+6\frac{D'}{D}\right] xy = \frac{36}{D^2}} = \frac{36}{D^2}\left[1+6\frac{D'}{D}\right] xy$$

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}x\right]$$

$$= \frac{36}{D^2}xy + \frac{216}{D^3}x$$

$$= 36y\left[\frac{x^3}{2\times 3}\right] + 216\left[\frac{x^4}{2\times 3\times 4}\right]$$

$$= 6x^3y + 9x^4$$

$$P.I. = x^4 + 6x^3y + 9x^4 = 10x^4 + 6x^3y$$

ce, complete sol^n = C.F. + P.I.

$$z = f_1(y+3x) + x f_2(y+3x) + 6x^3y + 10x^4$$

★ General Method to find the P.I.

Q- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Sol:

Given, $[D^2 + DD' - 6(D')^2]z = y\cos x$

for C.F. A.E. $m^2 + m - 6 = 0 \Rightarrow (m-2)(m+3) = 0$

$\Rightarrow \boxed{m = 2, -3}$

$\therefore$ C.F. $= f_1(y+2x) + f_2(y-3x)$

for P.I. $P.I. = \frac{1}{(D-2D')(D+3D')} y\cos x$

Let $y = C_1 + 3x$

$\Rightarrow P.I. = \frac{1}{D-2D'} \int ((C_1+3x)\cos x \cdot dx$

$= \frac{1}{D-2D'} \left[(C_1+3x)\sin x - (3)(-\cos x) \right]$

$= \frac{1}{D-2D'} \left[y\sin x + 3\cos x \right]$

Let $y = C_2 - 2x$

$\Rightarrow P.I. = \int \left[(C_2-2x)\sin x + 3\cos x \right] dx$

$= \left[(C_2-2x)(-\cos x) - (-2)(-\sin x) \right] + 3\left[\sin x\right]$

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$$= -y\cos x - 2\sin x + 3\sin x$$

$$\Rightarrow \boxed{P.I. = -y\cos x + \sin x}$$

Hence, Complete sol^n = C.F. + P.I.

$$z = f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x$$

Q- Solve the linear PDE $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$

Sol^n ⇒ Given, $[D^2 + DD' - 2(D')^2]z = (y-1)e^x$

for C.F. $[D^2 + DD' - 2(D')^2]z = 0$

A.E. ⇒ $m^2 + m - 2 = 0 \Rightarrow m^2 + 2m - m - 2 = 0$

$\Rightarrow m(m+2) - 1(m+2) = 0$

$\Rightarrow (m+2)(m-1) = 0 \Rightarrow \boxed{m = -2, 1}$

$\therefore \boxed{C.F. = f_1(y-2x) + f_2(y+x)}$

for P.I. $P.I. = \frac{1}{(D+2D')(D-D')}(y-1)e^x$

Let $y = c + 2x$

$$\Rightarrow P.I. = \frac{1}{D-D'}\int (c+2x-1)e^x\,dx$$

$$= \frac{1}{D-D'}\left[(c+2x-1)e^x - (2)e^x\right]$$

$$= \frac{1}{D-D'}\left[(y-1)e^x - 2e^x\right]$$

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Let $y = c - x$

$$\Rightarrow P.I. = \int \left((c-x-1)e^{x} - 2e^{x}\right) dx$$

$$= \left[(c-x-1)e^{x} - (-1)e^{x}\right] - 2e^{x}$$

$$P.I. = (y-1)e^{x} + e^{x} - 2e^{x}$$

$$\Rightarrow \boxed{P.I. = (y-1)e^{x} - e^{x}}$$

$$\Rightarrow \boxed{P.I. = (y-2)e^{x}}$$

Hence, Complete sol^n = C.F. + P.I.

$$\Rightarrow \boxed{z = f_1(y-2x) + f_2(y+x) + (y-2)e^{x}}$$

Q. Solve: $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$

Sol^n ⇒ for C.F.

$$(D^2 + 2DD' + D'^2)z = 0$$

A.E. $m^2 + 2m + 1 = 0 \Rightarrow m^2 + m + m + 1 = 0$

$$\Rightarrow m(m+1) + 1(m+1) = 0$$

$$\Rightarrow (m+1)(m+1) = 0$$

$$\Rightarrow m = -1, -1$$

$$\therefore C.F. = f_1(y-x) + x f_2(y-x)$$

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for P.I. $\quad P.I. = \dfrac{1}{(D^2+2DD'+D'^2)}(2\cos y - x\sin y)$

$$= \underbrace{\frac{1}{D^2+2DD'+D'^2}\,2\cos y}_{P_1} - \underbrace{\frac{1}{D^2+2DD'+D'^2}\,x\sin y}_{P_2}$$

$$P_1 = \frac{1}{D^2+2DD'+D'^2}\,2\cos y = 2\cdot\frac{1}{D^2+2DD'+D'^2}\cos(0+y)$$

putting $D=0$ & $D'=1$

$$\Rightarrow P_1 = 2\cdot\frac{1}{(0)^2+2(0)(1)+(1)^2}\iint \cos u\cdot du\,du$$

where $u=y$

$$= 2\left[\int \sin u\cdot du\right] = -2\cos u$$

$$\Rightarrow \boxed{P_1 = -2\cos y}$$

$$P_2 = \frac{1}{(D+D')(D+D')}\,x\sin y$$

Let $y = c+x$

$$\Rightarrow P_2 = \frac{1}{(D+D')}\int x\sin(c+x)\cdot dx$$

$$= \frac{1}{D+D'}\left[x(-\cos(c+x)) - 1(-\sin(c+x))\right]$$

$$= \frac{1}{D+D'}\left[-x\cos y + \sin y\right]$$

Let. $y = c+x$

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$$\Rightarrow P_2 = \int -x\cos(c+x) + \sin(c+x) \cdot dx$$

$$= \left[-x\sin(c+x) - (-1)(-\cos(c+x))\right] - \cos(c+x)$$

$$= -x\sin y - \cos y - \cos y$$

$$\Rightarrow P_2 = -x\sin y - 2\cos y$$

$$P.I. = P_1 - P_2 = -\cancel{2\cos y} + x\sin y + \cancel{2\cos y}$$

$$\Rightarrow \boxed{P.I. = x\sin y}$$

Hence, complete sol^n = C.F. + P.I.

$$\boxed{z = f_1(y-x) + x f_2(y-x) + x\sin y}$$ Ans

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Non-Homogeneous Linear PDE

When all the derivatives are not of same order, then the linear PDE is called Non-Homogeneous PDE

Method to Find C.F.

For the solution of non-homogeneous PDE, we resolve $\phi(D, D')$ into linear factor of the form $(D - mD' - a)$

Hence, for $(D - m_1D' - a_1)(D - m_2D' - a_2)(D - m_3D' - a_3)z = 0$

$$C.F. = e^{a_1x} f_1(y + m_1x) + e^{a_2x} f_2(y + m_2x) + e^{a_3x} f_3(y + m_3x) + \cdots + e^{a_nx} f_n(y + m_nx)$$

In case of repeated roots

e.g. $\Rightarrow$ $(D - mD' - a)^2 z = 0$

$$CF = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$$

Solve: $(D + D' - 1)(D + 2D' - 2)z = 0$

$\Rightarrow$ $C.F. = e^{x} f_1(y - x) + e^{2x} f_2(y - 2x)$

Complete Solution = C.F. + P.I.

Here P.I. = 0

$\therefore z = e^{x} f_1(y - x) + e^{2x} f_2(y - 2x)$ Ans

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Q- Solve: $DD'(D+2D'+1)z=0$

Sol^n ⇒

$C.F. = f_1(y) + f_2(x) + e^{-x} f_3(y-2x)$

Complete solution = C.F. + P.I. Here P.I. = 0

$\Rightarrow z = f_1(y) + f_2(x) + e^{-x} f_3(y-2x)$

Q- Solve: $(D+4D'+5)^2 z = 0$

Sol^n ⇒ $C.F. = e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$

Complete Solution = C.F. + P.I. Here P.I. = 0

$z = e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$ Ans

Q- Solve: $(D^2 - DD' - 2D)z = 0$

Sol^n Given, $D(D-D'-2)z = 0$

$C.F. = f_1(y) + e^{2x} f_2(y+x)$

Complete Sol^n = C.F. + P.I.

$z = f_1(y) + e^{2x} f_2(y+x)$ Ans

Q- Solve: $(D+1)(D+D'-1)z = 0$

Sol^n ⇒

$C.F. = e^{-x} f_1(y) + e^{x} f_2(y-x)$

Complete Sol^n = C.F. + P.I.

$z = e^{-x} f_1(y) + e^{x} f_2(y-x)$ Ans

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P.I. Calculation

e I When $F(x,y) = e^{ax+by}$

Solve: $s + ap + bq + abz = e^{mx+ny}$

Sol^n

Given, $\dfrac{\partial^2 z}{\partial x \partial y} + a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} + abz = e^{mx+ny}$

$$\Rightarrow \left[D D' + Da + D'b + ab\right] z = e^{mx+ny}$$

$$\Rightarrow \left[D(D'+a) + b(D'+a)\right] z = e^{mx+ny}$$

$$\Rightarrow (D+b)(D'+a) z = e^{mx+ny}$$

er C.F. $(D+b)(D'+a) = 0$

$$\therefore C.F. = e^{-bx} f_1(y) + e^{-ax} f_2(x)$$

er P.I.

$$P.I. = \frac{1}{(D+b)(D+a')} e^{mx+ny}$$

putting $D = m$ & $D' = n$

$$\Rightarrow P.I. = \frac{1}{(m+b)(n+a)} \iint e^{u} \, du \, du \quad \text{where } u = mx+ny$$

$$= \frac{1}{(m+b)(n+a)} e^{u} = \frac{1}{(m+b)(n+a)} e^{mx+ny}$$

$$\Rightarrow P.I. = \frac{1}{(m+b)(n+a)} e^{mx+ny}$$

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Complete Sol^n = C.F. + P.I.

$$\Rightarrow z = e^{-bx} f_1(y) + e^{-ax} f_2(x) + \frac{e^{mx+ny}}{(m+b)(n+a)} \text{ Ans}$$

Case II When $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+y) \text{ or } \cos(ax+y)$$

Here $\boxed{D^2 = -a^2}$, $\boxed{DD' = -ab}$, $\boxed{D'^2 = -b^2}$ replacement done

Q Solve: $(D-D'-1)(D-D'-2)z = \sin(2x+3y)$

Sol^n for C.F. $(D-D'-1)(D-D'-2)z = 0$

$$\Rightarrow C.F. = e^{x} f_1(y+x) + e^{2x} f_2(y+x)$$

for P.I. $P.I. = \dfrac{1}{(D-D'-1)(D-D'-2)} \sin(2x+3y)$

$$D^2 - DD' - 2D - DD' + D'^2 + 2D' - D + D' + 2$$

$$\Rightarrow D^2 - 2DD' - 3D + 3D' + 2 + D'^2$$

$$\Rightarrow P.I. = \frac{1}{D^2 - 2DD' - 3D + 3D' + 2 + D'^2} \sin(2x+3y)$$

putting $D^2 = -4$, $DD' = -6$ & $D'^2 = -9$

$$\Rightarrow P.I. = \frac{1}{(-4) - 2(-6) - 3D + 3D' + 2 + (-9)} \sin(2x+3y)$$

$$\Rightarrow P.I. = \frac{1}{(-3D+3D'+1)} \sin(2x+3y)$$

$$= \frac{(-3D+3D'-1)}{(-3D+3D'+1)(-3D+3D'-1)} \sin(2x+3y)$$

$$= \frac{(-3D+3D'-1)}{(-3D+3D')^2-(1)^2} \sin(2x+3y)$$

$$= \frac{(-3D+3D'-1)}{9D^2+9D'^2-18DD'-1} \sin(2x+3y)$$

putting $D^2=-4$, $D'^2=-9$ & $DD'=-6$

$$= \frac{(-3D+3D'-1)}{9(-4)+9(-9)-18(-6)-1} \sin(2x+3y)$$

$$= \frac{(-3D+3D'-1)}{-10} \sin(2x+3y)$$

$$= \frac{1}{10}\left[3D\sin(2x+3y) - 3D'\sin(2x+3y) + \sin(2x+3y)\right]$$

$$\Rightarrow P.I. = \frac{1}{10}\left[6\cos(2x+3y) - 9\cos(2x+3y) + \sin(2x+3y)\right]$$

$$\Rightarrow P.I. = -\frac{3}{10}\cos(2x+3y) + \frac{1}{10}\sin(2x+3y)$$

Hence, Complete sol^n = C.F. + P.I.

$$z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{10}\left[\sin(2x+3y) - 3\cos(2x+3y)\right]$$

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$(D - mD' - a)$

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Solve: $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

$\Rightarrow \{(D^2-1) - D'(D-1)\}z = \cos(x+2y) + e^y$

$\Rightarrow \{(D-1)(D+1) - D'(D-1)\}z = \cos(x+2y) + e^y$

$\Rightarrow (D-1)(D-D'+1)z = \cos(x+2y) + e^y$ — (1)

for C.F. $(D-1)(D-D'+1)z = 0$

$\therefore C.F. = e^x f_1(y) + e^{-x} f_2(y+x)$

for P.I.

$$P.I. = \underbrace{\frac{1}{D^2 - DD' + D' - 1}\cos(x+2y)}_{P_1} + \underbrace{\frac{1}{D^2 - DD' + D' - 1} e^y}_{P_2}$$

$$P_1 = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y)$$

putting $D^2 = -1$, $DD' = -2$

$$P_1 = \frac{1}{(-1) - (-2) + D' - 1}\cos(x+2y)$$

$$= \frac{1}{D'}\cos(x+2y) \Rightarrow \boxed{P_1 = \frac{\sin(x+2y)}{2}}$$

$$P_2 = \frac{1}{D^2 - DD' + D' - 1} e^y$$

putting $D = 0$ & $D' = 1$

$$= \frac{1}{(0)^2 - (0)(1) + 1 - 1} e^y$$

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$x \quad x^2 \quad x^3 \quad x^n y^n$

$\Rightarrow$ case of failure

$$\therefore P_2 = \frac{x}{2D - D'} e^y$$

putting $D = 0$ & $D' = 1$

$$\Rightarrow P_2 = \frac{x}{2(0) - (1)} \int e^u \cdot du \quad \text{where } u = y$$

$$P_2 = -x e^u \Rightarrow \boxed{P_2 = -x e^y}$$

$$P.I. = P_1 + P_2$$

$$\Rightarrow \boxed{P.I. = \frac{\sin(x+2y)}{2} - x e^y}$$

Hence, Complete sol^n = C.F. + P.I.

$$\boxed{z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{\sin(x+2y)}{2} - x e^y}$$ Ans

III When $F(x,y) = x^m y^n$

$$P.I. = \frac{1}{\phi(D, D')} x^m y^n = [\phi(D, D')]^{-1} x^m y^n$$

Solve: $(D - D' - 1)(D - D' - 2) z = e^{3x - y} + x$

$\Rightarrow$ for C.F. $(D - D' - 1)(D - D' - 2) z = 0$

$$\therefore C.F. = e^x f_1(y+x) + e^{2x} f_2(y+x)$$

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$$P.I. = \underbrace{\frac{1}{(D-D'-1)(D-D'-2)} e^{3x-y}}_{P_1} + \underbrace{\frac{1}{(D-D'-1)(D-D'-2)} x}_{P_2}$$

$$P_1 = \frac{1}{(D-D'-1)(D-D'-2)} e^{3x-y}$$

putting $D = 3$ & $D' = -1$

$$P_1 = \frac{1}{\{(3)-(-1)-1\}\{(3)-(-1)-2\}} \iint e^{u} \cdot du\, du$$

where $u = 3x-y$

$$= \frac{1}{(3)(2)} e^{u} \Rightarrow \boxed{P_1 = \frac{1}{6} e^{3x-y}}$$

$$P_2 = \frac{1}{(D-D'-1)(D-D'-2)} x$$

$$= \frac{1}{-1(1-(D-D'))} \cdot \frac{1}{-2\left(1-\left(\frac{D-D'}{2}\right)\right)} x$$

$$= \frac{1}{2}\left[\frac{1}{[1-(D-D')]\left[1-\left(\frac{D-D'}{2}\right)\right]}\right] x$$

$$= \frac{1}{2}\left\{[1-(D-D')]^{-1}\left[1-\left(\frac{D-D'}{2}\right)\right]^{-1}\right\} x$$

$$= \frac{1}{2}\left[\{1+(D-D')\}\left\{1+\frac{D-D'}{2}\right\}\right] x$$

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$$= \frac{1}{2}\left[1 + \frac{D}{2} - \frac{D'}{2} + D - D' + \frac{D^2 - DD' - D'D + D'^2}{2}\right]x$$

neglecting higher order

$$= \frac{1}{2}\left[1 + \frac{D}{2} - \frac{D'}{2} + D - D'\right]x$$

$$= \frac{1}{2}\left[x + \frac{1}{2} - 0 + 1 - 0\right]$$

$$\Rightarrow \boxed{P_2 = \frac{1}{2}\left(x + \frac{3}{2}\right)}$$

$$P.I. = P_1 + P_2$$

$$\Rightarrow \boxed{P.I. = \frac{1}{6}e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right)}$$

Hence, complete solution = C.F. + P.I.

$$\Rightarrow \boxed{z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6}e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right)}$$

Q- Solve $D(D+D'-1)(D+3D'-2)z = x^2 - 4xy + 2y^2$

Soln

for C.F. $D(D+D'-1)(D+3D'-2)z = 0$

$$\therefore C.F. = f_1(y) + e^{x} f_2(y-x) + e^{2x} f_3(y-3x)$$

for P.I. $P.I. = \dfrac{1}{D(D+D'-1)(D+3D'-2)}(x^2 - 4xy + 2y^2)$

$$= \frac{1}{D} \cdot \frac{1}{-1[1-(D+D')]} \cdot \frac{1}{-2\left[1-\left(\frac{D+3D'}{2}\right)\right]}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{\left[1-(D+D')\right]^{-1}\left[1-\left(\frac{D+3D'}{2}\right)\right]^{-1}\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{\left[1+(D+D')+(D+D')^2\right]\left[1+\left(\frac{D+3D'}{2}\right)+\left(\frac{D+3D'}{2}\right)^2\right]\right\}$$

$$(x^2-4xy+2y^2)$$

neglecting higher order terms

$$= \frac{1}{2D}\left[1+\frac{D}{2}+\frac{3D'}{2}+\left(\frac{D^2+9D'^2+6DD'}{4}\right)+(D+D')\right.$$

$$\left.+\left(\frac{D^2+4DD'+3D'^2}{2}\right)+(D^2+D'^2+2DD')\right]$$

$$(x^2-4xy+2y^2)$$

$$\cancel{= \frac{1}{2D}\left[(x^2-4xy+2y)+\frac{1}{2}(2x-4y)+\frac{3}{2}(-4x+4y)\right.}$$

$$\cancel{+\frac{1}{4}(2+9(4)+6(-4))+(2x-4y-4x+4y)}$$

$$\cancel{\left.+\frac{1}{2}(2+4(-4)+3(4)\right.}$$

$$= \frac{1}{2D}\left[4+2D+6D'+D^2+9D'^2+6DD'+4D+4D'+2D^2+8DD'\right.$$

$$\left.\frac{+8D'^2+4D^2+4D'^2+8DD'}{4}\right]$$

$$(x^2-4xy+2y^2$$

$$= \frac{1}{8D}\left[4+7D^2+19D'^2+22DD'+6D+10D'\right](x^2-4xy+2y^2)$$

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$$= \frac{1}{8D}\left[4(x^2-4xy+2y^2)+7(2)+19(4)+22(-4)+6(2x-4y)+10(4x+4y)\right]$$

$$= \frac{1}{8D}\left[4(x^2-4xy+2y^2)+12x-24y-40x+40y+2\right]$$

$$= \frac{1}{8D}\left[4(x^2-4xy+2y^2)-28x+16y+2\right]$$

$$= \frac{1}{8}\left[4\left(\frac{x^3}{3}-\frac{4x^2y}{2}+2xy^2\right)-\frac{28x^2}{2}+16xy+2x\right]$$

$$= \frac{1}{2}\left[\frac{4}{4}\left(\frac{x^3}{3}-\frac{4x^2y}{2}+2xy^2\right)-\frac{28x^2}{4\times 2}+\frac{16xy}{4}+\frac{2x}{4}\right]$$

$$\Rightarrow P.I. = \frac{1}{2}\left[\frac{x^3}{3}-2x^2y+2xy^2-\frac{7}{2}x^2+4xy+\frac{x}{2}\right]$$

Hence, Complete Sol^n = C.F. + P.I.

$$\Rightarrow z = f_1(y)+e^{x}f_2(y-x)+e^{2x}f_3(y-3x)$$
$$+\frac{1}{2}\left[\frac{x^3}{3}-2x^2y+2xy^2-\frac{7}{2}x^2+4xy+\frac{x}{2}\right] \text{ Ans}$$

Case V or $D = D + a$, $D' = D' + b$

Case IV When $F(x,y) = e^{ax+by} \cdot V$ where V is a function of x and y.

$$P.I. = \frac{1}{\phi(D, D')} e^{ax+by} V = e^{ax+by} \frac{1}{\phi(D+a, D'+b')} V$$

V can be either (i) e^{ax+by}

(ii) $\sin(ax+by)$ or $\cos(ax+by)$

(iii) $x^m y^n$ (iv) constant (say 1).

Q. Solve: $(D-3D'-2)^3 z = 6e^{2x}\sin(3x+y)$

Sol^n:

for C.F. $(D-3D'-2)^3 z = 0$

$\therefore$ C.F. $= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x)$

for P.I. $P.I. = \frac{1}{(D-3D'-2)^3} 6e^{2x}\sin(3x+y)$

on putting $D = D+2$ & $D' = D'+0$

$$P.I. = 6e^{2x} \frac{1}{\{(D+2)-3(D'+0)-2\}^3} \sin(3x+y)$$

$$= 6e^{2x} \frac{1}{(\underbrace{D-3D'}_{\text{Homogeneous}})^3} \sin(3x+y)$$

on putting $D = 3$ & $D' = 1 \Rightarrow$ case of failure

$$\therefore P.I. = 6e^{2x} \frac{x}{3[D-3D']^2} \sin(3x+y)$$

again on putting $D = 3$ & $D' = 1 \Rightarrow$ case of failure

$$\therefore P.I. = 6e^{2x} \frac{x^2}{6[D-3D']} \sin(3x+y)$$

again on putting $D = 3$ & $D' = 1 \Rightarrow$ case of failure

$$\therefore P.I. = \cancel{6}e^{2x} \cdot \frac{x^3}{\cancel{6}} \sin(3x+y)$$

$$\Rightarrow \boxed{P.I. = x^3 e^{2x} \sin(3x+y)}$$

Hence, Complete Sol^n $\Rightarrow$ C.F. + P.I.

$$z = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x) + x^3 e^{2x} \sin(3x+y)$$

Q Find the Particular Integral of $(D^2 - D')z = x e^{ax + a^2 y}$

Sol^n $\Rightarrow$

$$P.I. = \frac{1}{(D^2 - D')} x e^{ax + a^2 y}$$

On putting $D = D + a$ & $D' = D' + a^2$

$$\Rightarrow P.I. = \frac{e^{ax + a^2 y}}{[(D+a)^2 - (D' + a^2)]} x$$

$$= e^{ax + a^2 y} \frac{1}{[D^2 + \cancel{a^2} + 2aD - D' - \cancel{a^2}]} x$$

$$= e^{ax+a^2y} \frac{1}{[D^2+2aD-D']} x$$

$$= e^{ax+a^2y} \frac{1}{2aD\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]} x$$

$$= \frac{e^{ax+a^2y}}{2a} \frac{1}{D}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1} x$$

$$= \frac{e^{ax+a^2y}}{2a} \frac{1}{D}\left[1-\frac{D^2}{2aD}+\frac{D'}{2aD}\right] x$$

$$= \frac{e^{ax+a^2y}}{2a} \frac{1}{D}\left[1-\frac{D}{2a}+\frac{D'}{2aD}\right] x$$

$$= \frac{e^{ax+a^2y}}{2a} \frac{1}{D}\left[x-\frac{1}{2a}+0\right]$$

$$= \frac{e^{ax+a^2y}}{2a}\left[\frac{x^2}{2}-\frac{x}{2a}\right]$$

$$\boxed{P.I. = \frac{e^{ax+a^2y}}{4a^2}\left[ax^2-x\right]}$$ Ans

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★ Equation Reducible to PDE with constant coefficients

Q- Solve: $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$

Soln ⇒

Note → for reducing

put $\boxed{x = e^X}$ & $\boxed{y = e^Y}$ such that $\boxed{X = \log x}$ & $\boxed{Y = \log y}$

put

$\boxed{x\frac{\partial z}{\partial x} = Dz}$, $\boxed{y\frac{\partial z}{\partial y} = D'z}$

$\boxed{x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z}$ $\boxed{y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z}$

$\boxed{x^3\frac{\partial^3 z}{\partial x^3} = D(D-1)(D-2)z}$ and $\boxed{xy\frac{\partial^2 z}{\partial x \partial y} = DD'z}$

--- & so on

Given, $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$

$$\Rightarrow \left[D(D-1) - 4DD' + 4D'(D'-1) + 6D'\right]z = (e^X)^3(e^Y)^4$$

$$\Rightarrow \left[D^2 - D - 4DD' + 4(D')^2 - 4D' + 6D'\right]z = e^{3X}\cdot e^{4Y}$$

$$\Rightarrow \left[D^2 - 4DD' - D + 2D' + 4(D')^2\right]z = e^{3X+4Y}$$

$$\Rightarrow \left[D^2 + 4D'^2 - 4DD' - D + 2D'\right]z = e^{3X+4Y}$$

$$\Rightarrow \left[(D-2D')^2 - 1(D-2D')\right]z = e^{3X+4Y}$$

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$$\Rightarrow (D-2D')(D-2D'-1)z = e^{3X+4Y}$$

for C.F. $(D-2D')(D-2D'-1)z = 0$

$\therefore \cancel{C.F. = f_1(y+2x) + e^x f_2(y+2x)}$

$$\therefore C.F. = f_1(Y+2X) + e^X f_2(Y+2X)$$

$$= f_1(\log y + 2\log x) + x f_2(\log y + 2\log x)$$

$$= f_1(\log yx^2) + x f_2(\log yx^2)$$

$$\Rightarrow \boxed{C.F. = g_1(yx^2) + x g_2(yx^2)}$$

for P.I.

$$P.I. = \frac{1}{(D-2D')(D-2D'-1)} e^{3X+4Y}$$

$$= \frac{1}{D^2 - D - 4DD' + 4D'^2 + 2D'} e^{3X+4Y}$$

on putting $D = 3$ & $D' = 4$

$$\Rightarrow P.I. = \frac{1}{(3)^2 - (3) - 4(3)(4) + 4(4)^2 + 2(4)} \int e^u \, du$$

where $u = 3X+4Y$

$$= \frac{1}{30} e^u = \frac{1}{30} e^{3X+4Y}$$

$$\Rightarrow \boxed{P.I. = \frac{1}{30} x^3 y^4}$$

Complete Solution = C.F. + P.I.

$$\boxed{z = g_1(yx^2) + x g_2(yx^2) + \frac{1}{30} x^3 y^4}$$ Ans

Q- Solve: $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^m y^n$

Sol^n Let $x = e^X$ & $y = e^Y$ such that $X = \log x$ & $Y = \log y$

Let $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$ & $DD' = \frac{\partial^2}{\partial X \partial Y}$

then

$$[D(D-1) + 2DD' + D'(D'-1)]z = e^{mX} \cdot e^{nY}$$

$$\Rightarrow [D^2 - D + 2DD' + D'^2 - D']z = e^{mX+nY}$$

$$\Rightarrow [(D+D')^2 - 1(D+D')]z = e^{mX+nY}$$

$$\Rightarrow (D+D')(D+D'-1)z = e^{mX+nY}$$

for C.F. $(D+D')(D+D'-1)z = 0$

$$\therefore C.F. = f_1(Y-X) + e^X f_2(Y-X)$$

$$= f_1(\log y - \log x) + x f_2(\log y - \log x)$$

$$= f_1\left(\log\left(\frac{y}{x}\right)\right) + x f_2\left(\log\left(\frac{y}{x}\right)\right)$$

$$C.F. = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right)$$

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for P.I. $P.I. = \dfrac{1}{D^2 - D + 2DD' + D'^2 - D'} e^{mX+nY}$

on putting $D = m$ & $D' = n$

$$= \frac{1}{(m)^2 - m + 2(m)(n) + (n)^2 - n} \iint e^{u} du\, du \quad \text{where } u = mX + nY$$

$$= \frac{1}{m^2 - m + 2mn + n^2 - n} e^{u}$$

$$= \frac{1}{(m+n)^2 - 1(m+n)} e^{mX+nY}$$

$$\Rightarrow \boxed{P.I. = \frac{1}{(m+n)(m+n-1)} x^m y^n}$$

Complete Sol^n = C.F. + P.I.

$$\boxed{z = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right) + \frac{1}{(m+n)(m+n-1)} x^m y^n}$$

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Unit → 2

★ Classification of Linear PDE of Second Order

Second order PDE in two independent variables x & y

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0}$$

Based on the calculation of $B^2 - 4AC$

→ if $B^2 - 4AC < 0 \Rightarrow$ elliptic

→ if $B^2 - 4AC = 0 \Rightarrow$ parabolic

→ if $B^2 - 4AC > 0 \Rightarrow$ Hyperbolic

Q- Classify $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

Sol^n Here $A = 1$, $B = 1$ & $C = 1$

$$B^2 - 4AC = (1)^2 - 4(1)(1) = -3 < 0 \Rightarrow \text{elliptic}$$

Q- Classify $t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$

Sol^n On comparing with the standard equ^n, we get
$A = x$, $B = 2$ & $C = t$

$$B^2 - 4AC = 4 - 4xt$$

equ^n is elliptic if $4 - 4xt < 0$ i.e. $tx > 1$

equ^n is parabolic if $4 - 4xt = 0$ i.e. $tx = 1$

equ^n is hyperbolic if $4 - 4xt > 0$ i.e. $tx < 1$

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Q- Show that the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

Sol^n

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$A = c^2$, $B = 0$, $C = -1$

$B^2 - 4AC = (0)^2 - 4(c^2)(-1) = 4c^2$

$\because c > 0 \Rightarrow 4c^2 > 0$

Hence, the equ^n is hyperbolic.

Q- Show that the equ^n $z_{xx} + 2x z_{xy} + (1-y^2) z_{yy} = 0$ is elliptic for values of x & y in the region $x^2+y^2 < 1$ parabolic on the boundary and hyperbolic outside this region

Sol^n

$$z_{xx} + 2x z_{xy} + (1-y^2) z_{yy} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + 2x \frac{\partial^2 z}{\partial x \partial y} + (1-y^2) \frac{\partial^2 z}{\partial y^2} = 0$$

Here, $A = 1$, $B = 2x$ & $C = (1-y^2)$

$B^2 - 4AC = (2x)^2 - 4(1)(1-y^2) = 4x^2 - 4 + 4y^2$

If elliptic, $B^2 - 4AC < 0$

$\Rightarrow 4x^2 + 4y^2 - 4 < 0$

$\Rightarrow 4(x^2+y^2) < 4$

$\Rightarrow \boxed{x^2+y^2 < 1}$

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if parabolic, $B^2 - 4AC = 0$

$\Rightarrow 4x^2 + 4y^2 - 4 = 0 \Rightarrow 4(x^2+y^2) = 4$

$\Rightarrow x^2 + y^2 = 1$

if hyperbolic, $B^2 - 4AC > 0 \Rightarrow 4x^2 + 4y^2 - 4 > 1$

$\Rightarrow 4(x^2+y^2) > 4 \Rightarrow x^2 + y^2 > 1$

★ Method of Separation of Variables

Q- Use the method of Separation of variables to solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ given that $u(x,0) = 6e^{-3x}$

Boundary cond^n

Sol^n ⇒

Let $\boxed{u = XT}$ be the sol^n.

where X is func^n of x only and T is the func^n of t only

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial (XT)}{\partial x} = T\frac{dX}{dx}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

Now, $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \Rightarrow \boxed{T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT}$

on dividing above equ^n with XT

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1 = -p^2 \text{ (Let)}$$

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$$\frac{1}{X}\frac{dX}{dx} = -p^2 \Rightarrow \int \frac{dX}{X} = \int -p^2 dx$$

$$\Rightarrow \log X = -p^2 x + \log C_1$$

$$\Rightarrow \log X - \log C_1 = -p^2 x$$

$$\Rightarrow \log\left(\frac{X}{C_1}\right) = -p^2 x \Rightarrow \frac{X}{C_1} = e^{-p^2 x}$$

$$\Rightarrow \boxed{X = C_1 e^{-p^2 x}}$$

$$\frac{2}{T}\frac{dT}{dt} + 1 = -p^2 \Rightarrow \frac{2}{T}\frac{dT}{dt} = -p^2 - 1$$

$$\Rightarrow \int \frac{dT}{T} = \int \left(\frac{-p^2-1}{2}\right) dt$$

$$\Rightarrow \log T = \left(\frac{-p^2-1}{2}\right)t + \log C_2$$

$$\Rightarrow \log T - \log C_2 = \left(\frac{-p^2-1}{2}\right)t$$

$$\Rightarrow \log\left(\frac{T}{C_2}\right) = \left(\frac{-p^2-1}{2}\right)t$$

$$\Rightarrow \frac{T}{C_2} = e^{\left(\frac{-p^2-1}{2}\right)t}$$

$$\Rightarrow \boxed{T = C_2 e^{\left(\frac{-p^2-1}{2}\right)}}$$

$$u = XT = C_1 C_2 e^{-p^2 x - \left(\frac{p^2+1}{2}\right)t}$$

$$u(x,0) = C_1 C_2 e^{-px^2} \Rightarrow C_1 C_2 e^{-px^2} = 6e^{-3x}$$

$$\Rightarrow \boxed{C_1 C_2 = 6} \quad \boxed{p^2 = 3} \quad \therefore \boxed{u = 6e^{-3x-2t}}$$ Ans

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Q- Use the method of separation of variables to solve the $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ given that $v=0$ when $t \to \infty$
$v=0$ at $x=0$ & $x=l$

Sol^n:-

Let $\boxed{v = XT}$ be the solution
where X is the func^n of x only & T is the func^n of t only

$$\therefore \frac{\partial^2 v}{\partial x^2} = T\frac{d^2X}{dx^2} \quad \& \quad \frac{\partial v}{\partial t} = X\frac{dT}{dt}$$

Now, $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \Rightarrow T\frac{d^2X}{dx^2} = X\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt} = (-p^2) \text{ (Let)}$$

Case I

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \Rightarrow \frac{d^2X}{dx^2} = -p^2X \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

$$\Rightarrow (D^2+p^2)X = 0$$

A.E. $\Rightarrow m^2+p^2=0 \Rightarrow m = \pm ip$

$$C.F. = C_1\cos px + C_2\sin px$$

$P.I. = 0$

$$\therefore \boxed{X = C_1\cos px + C_2\sin px}$$

Case II

$$\frac{1}{T}\frac{dT}{dt} = -p^2 \Rightarrow \int\frac{dT}{T} = \int -p^2\,dt$$

$$\Rightarrow \log T = -p^2t \log C_3 \Rightarrow \log T - \log C_3 = -p^2t$$

$$\Rightarrow \log\left(\frac{T}{C_3}\right) = -p^2t \Rightarrow \frac{T}{C_3} = e^{-p^2t} \Rightarrow \boxed{T = C_3e^{-p^2t}}$$

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$\cancel{V(x,t)}$ $V = XT = (C_1 \cos px + C_2 \sin px) C_3 e^{-p^2 t}$

$$\boxed{V(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-p^2 t}}$$

$V(0,t) = C_1 C_3 e^{-p^2 t} \Rightarrow 0 = C_1 C_3 e^{-p^2 t}$

$\because C_3 \neq 0$ & $e^{-p^2 t} \neq 0$

$\therefore \boxed{C_1 = 0}$

$$\boxed{V(x,t) = C_2 C_3 e^{-p^2 t} \sin px}$$

$V(l,t) = C_2 C_3 \sin pl \, e^{-p^2 t}$

$0 = C_2 C_3 \sin pl \, e^{-p^2 t}$

$\Rightarrow \sin pl = 0$

$\sin pl = \sin n\pi$

$\Rightarrow \boxed{p = \frac{n\pi}{l}}$

$\cancel{V(x,t)}$ $V(x,t) = C_2 C_3 \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{l^2}\right)t}$

$\Rightarrow V(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{l^2}\right)t}$

or the general sol^n is

$$\boxed{V = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n^2\pi^2}{l^2}\right)t} \sin\left(\frac{n\pi x}{l}\right)}$$

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★ One-Dimensional Wave Equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

Thus, $y = 0$ when $x = 0$ and $x = l$ (Boundary conditions)

Also, if the string is made to vibrate by pulling it into a curve $\boxed{y = f(x)}$ and then releasing

Thus, $y = f(x)$ when $t = 0$

(velocity) $\frac{dy}{dt} = 0$ when $t = 0$ Initial conditions

Q- Solve: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Sol^n

Let $y(x,t) = XT$ be the sol^n

$\therefore$ $\boxed{\frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2}}$ and $\boxed{\frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}}$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \Rightarrow \boxed{X \frac{d^2 T}{dt^2} = c^2 \cdot T \frac{d^2 X}{dx^2}}$$

On dividing with XT on both sides

$$\frac{1}{T} \frac{d^2 T}{dt^2} = c^2 \frac{1}{X} \frac{d^2 X}{dx^2}$$

$$\Rightarrow \boxed{\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2}}$$

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Case I

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = p^2 \text{ (Let)}$$

- $$\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \Rightarrow \frac{d^2X}{dx^2} - p^2X = 0$$

$$\therefore \text{A.E.} \Rightarrow m^2 - p^2 = 0 \Rightarrow \boxed{m = \pm p}$$

$$\therefore \text{C.F.} = C_1 e^{px} + C_2 e^{-px}$$

P.I. = 0

$$\Rightarrow \boxed{X = C_1 e^{px} + C_2 e^{-px}}$$

- $$\frac{1}{c^2T}\frac{d^2T}{dt^2} = p^2 \Rightarrow \frac{d^2T}{dt^2} - c^2p^2 \cdot T = 0$$

$$m^2 - c^2p^2 = 0 \Rightarrow \boxed{m = \pm cp}$$

$$\therefore \boxed{T = C_3 e^{cpt} + C_4 e^{-cpt}}$$

Hence, the solution of the equ^n becomes

$$\Rightarrow y = XT$$

$$\Rightarrow \boxed{y = (C_1 e^{px} + C_2 e^{-px})(C_3 e^{cpt} + C_4 e^{-cpt})}$$

Case II

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -p^2 \text{ (Let)}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

$$m^2 + p^2 = 0 \Rightarrow \boxed{m = \pm ip}$$

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$$\therefore \boxed{X = C_1 \cos px + C_2 \sin px}$$

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = -p^2 \Rightarrow \frac{d^2T}{dt^2} + c^2p^2 T = 0$$

$$m^2 + c^2p^2 = 0 \Rightarrow \boxed{m = \pm icp}$$

$$\therefore \boxed{T = C_3 \cos cpt + C_4 \sin cpt}$$

Hence, the solution of equ^n becomes

$$\Rightarrow y = XT$$

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

Case III $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0$ (Let)

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$

$$A.E. \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$$

$$\therefore \boxed{X = C_1 + x C_2}$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = 0 \Rightarrow \frac{d^2T}{dt^2} = 0$$

$$A.E. \Rightarrow m^2 = 0 \Rightarrow m_1 = 0, 0$$

$$\therefore \boxed{T = C_3 + t C_4}$$

Hence, the solution of the equ^n becomes

$$\Rightarrow y = XT$$

$$\Rightarrow \boxed{y = (C_1 + x C_2)(C_3 + t C_4)}$$

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$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

Applying boundary cond^{n}s,

$y(0,t)=0$

$$0 = C_1(C_3 \cos cpt + C_4 \sin cpt)$$

$\because C_3 \cos cpt + C_4 \sin cpt \neq 0$

$$\therefore \boxed{C_1 = 0}$$

$$y(x,t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$$

Applying $y=0$ at $x=l$

$$0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$$

$\because C_2 \neq 0$

$$\Rightarrow \sin pl = 0 = \sin n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

~~Sol^n becomes~~

Hence, the solution of wave equation is:

$$y(x,t) = C_2 \sin\left(\frac{n\pi x}{l}\right)\left(C_3 \cos \frac{n\pi ct}{l} + C_4 \sin \frac{n\pi ct}{l}\right)$$

~~Or general sol^n $\Rightarrow$~~

$$\cancel{y(x,t) = \sum_{n=1}^{\infty} (a_n \cos}$$

$$y(x,t) = \underbrace{C_2 C_3}_{a_n} \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l} + \underbrace{C_2 C_4}_{b_n} \sin\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l}$$

or general solution $\Rightarrow$

$$\boxed{y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}}$$

$$y_t = \sum_{n=1}^{\infty}\left[a_n\left(-\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + b_n\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right]\sin\frac{n\pi x}{l}$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty}\left[-a_n \sin\frac{n\pi ct}{l} + b_n \cos\frac{n\pi ct}{l}\right]\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$\frac{dy}{dx} = 0$ when $t = 0$

$$\Rightarrow 0 = \sum_{n=1}^{\infty}\left[b_n\right]\frac{n\pi c}{l}\sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \boxed{b_n = 0}$$

Hence sol^n after applying initial conditions are

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\cdot\sin\frac{n\pi x}{l}}$$

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

or $\boxed{f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}}$

which represent Fourier series. Thus,

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx}$$

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Q- A string is stretched and fastened to two points l apart. motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$

from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at a time 't' is given by

$$y(x,t) = A \sin \frac{n\pi x}{l} \cos \frac{\pi c t}{l}$$

Sol^n $\Rightarrow$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The suitable solution is

$$y = XT = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

or general sol^n

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{l} + b_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l}$$

Here, solution after applying initial cond^{n}s are

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi c t}{l} \cdot \sin \frac{n\pi x}{l}} \quad \text{— (1)}$$

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at $y(x,0) = A \sin \frac{\pi x}{l}$

Now, equ^n (1)

$$y(x,0) = \sum_{n=1}^{\infty} a_n \cdot \sin \frac{n\pi x}{l}$$

$$\Rightarrow A \sin \frac{\pi x}{l} = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \cdots$$

$$\Rightarrow a_1 = A, \quad a_2 = a_3 = a_4 \cdots = 0$$

Now

$$\boxed{y(x,t) = A \cos \frac{\pi c t}{l} \cdot \sin \frac{\pi x}{l}}$$

A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position. Find the displacement $y(x,t)$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

The suitable solution is

$$y = XT = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

or general sol^n

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{l} + b_n \sin \frac{n\pi c t}{l}\right) \sin \frac{n\pi x}{l}$$

Hence, solution after applying initial cond^{n}s are:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l} \quad \text{—} (1)$$

at $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

Now, equ^n (1)

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$ $\Rightarrow \cancel{y_0\left[\frac{3\sin\theta - \sin 3\theta}{4}\right]}$

$$\Rightarrow y_0\left\{\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}\right\} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow y_0\left[\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}\right] = a_1 \sin\frac{\pi x}{l} + a_2 \sin\left(\frac{2\pi x}{l}\right) + a_3 \sin\frac{3\pi x}{l} + a_4 \sin\frac{4\pi x}{l} + \cdots$$

$\therefore a_1 = \frac{3y_0}{4}$, $a_2 = 0$, $a_3 = -\frac{y_0}{4}$, $a_4 = a_5 = \cdots = 0$

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Hence, the solution of the displacement

$$y(x,t) = \frac{3y_0}{4} \sin\frac{\pi x}{l} \cdot \cos\frac{\pi c t}{l} - \frac{y_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi c t}{l}$$

Q- A tightly stretched flexible string has it ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x) = \mu x(l-x)$, μ is a constant and then released. Find the displacement $y(x,t)$ of any point x of the string at any time $t>0$.

Sol^n ⇒

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The suitable solution of equ^n above is

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

or general sol^n

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi c t}{l} + b_n \sin\left(\frac{n\pi c t}{l}\right)\right) \sin\frac{n\pi x}{l}$$

Hence, the solution after applying initial conditions are

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c t}{l} \cdot \sin\frac{n\pi x}{l}$$

$$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$\mu x(l-x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

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where, $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx$

$$= \frac{2}{l}\int_0^l \mu x(l-x)\sin\frac{n\pi x}{l}\cdot dx$$

$$= \cancel{\frac{2}{l}\left[(\mu x(l-x))\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (\mu l - 2\mu x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + (-2\mu)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l}$$

$$= \frac{2\mu}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\cdot dx$$

$$= \frac{2\mu}{l}\left[\cancel{(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)}^{\to 0} - \cancel{(l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)}^{\to 0} + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2\mu}{l}\left[0 + 0 - 2\frac{\{\cos n\pi - \cos 0\}}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$a_n = \frac{4\mu l^2}{n^3\pi^3}\left[1-\cos n\pi\right]$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty}\frac{4\mu l^2}{n^3\pi^3}\left(1-(-1)^n\right)\cos\frac{n\pi ct}{l}\cdot\sin\frac{n\pi x}{l}$$

Ans

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Q- A string is stretched between two fixed points $(0,0)$ and $(l,0)$ and released at rest from the initial deflection given by.

$$f(x) = \begin{cases} \left(\frac{2k}{l}\right)x & 0 < x < \frac{l}{2} \\ \left(\frac{2k}{l}\right)(l-x) & \frac{l}{2} < x < l \end{cases}$$

Find the deflection of the ~~shape~~ string at any time.

Sol^n

The suitable sol^n for the equ^n

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

is

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

or general sol^n

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

Hence, the solution after applying initial cond^{n}s are:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \cdot \sin \frac{n\pi x}{l}}$$

$$\Rightarrow \quad y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

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where, $a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\cdot dx$

$$= \frac{2}{l}\left[\int_0^{l/2}\left(\frac{2k}{l}\right)x\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx + \int_{l/2}^{l}\left(\frac{2k}{l}\right)(l-x)\cdot\sin\frac{n\pi x}{l}\cdot dx\right]$$

$$= \frac{4k}{l^2}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left\{\left[\frac{l}{2}\left(\cancel{\frac{-\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}}\right) + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right] + \left[\frac{l}{2}\left(\cancel{\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}}\right) + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]\right\}$$

$$= \frac{4k}{\cancel{l^2}} \times \frac{\cancel{l^2}}{n^2\pi^2}\left\{\frac{\sin n\pi}{2} + \frac{\sin n\pi}{2}\right\}$$

$$\Rightarrow \boxed{a_n = \frac{8k}{n^2\pi^2}\cdot\sin\frac{n\pi}{2}}$$

$$\therefore \boxed{y(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right)\cdot\left(\cos\frac{n\pi ct}{l}\right)\left(\sin\frac{n\pi x}{l}\right)}$$ Ans

Q- Find the deflection $u(x,t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$\sin\pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x \quad 0 \le x \le 1.$

The suitable solution is

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

or general solⁿ

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Hence, the solⁿ after applying initial condⁿ are:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \cdot \sin \frac{n\pi x}{l}}$$

~~as it is of unit length~~

$$\therefore y(x,t) = \sum_{n=1}^{\infty} a_n (\cos n\pi ct)(\sin n\pi x)$$

at $t=0$, $y(x,0) = \sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x$

$$y(x,0) = \sum_{n=1}^{\infty} a_n (\sin n\pi x)$$

$$\Rightarrow \sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x = a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x + a_4 \sin 4\pi x + a_5 \sin 5\pi x + \cdots$$

$$\therefore a_1 = 1, \quad a_3 = \frac{1}{3}, \quad a_5 = \frac{1}{5}, \quad a_2 = a_4 = \cdots = 0$$

$$\therefore \boxed{y(x,t) = (\cos \pi ct) \sin \pi x + \frac{1}{3} \cos(3\pi ct) \sin 3\pi x + \frac{1}{5} \cos 5\pi ct \sin 5\pi x}$$ Ans

★ One Dimensional Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

→ Solution of Heat Equation

$$u(x,t) = XT$$

- $\frac{\partial u}{\partial t} = \frac{d}{dt}(XT) = X\frac{dT}{dt}$

- $\frac{\partial^2 x}{\partial x^2} = \frac{d^2}{dx^2}(XT) = T\frac{d^2X}{dx^2}$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2}$$

On dividing by XT on both sides, we get

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{c^2}{X}\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{c^2}\cdot\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Case I When $\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = p^2$ (Let)

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$$\frac{1}{c^2 T}\frac{dT}{dt} = p^2 \Rightarrow \frac{dT}{dt} = p^2 c^2 T$$

$$\Rightarrow \int \frac{1}{T}\, dT = \int p^2 c^2 dt$$

$$\Rightarrow \log T = p^2 c^2 t + \log C_1$$

$$\Rightarrow \log\left(\frac{T}{C_1}\right) = p^2 c^2 t$$

$$\Rightarrow \boxed{T = e^{p^2 c^2 t} \cdot C_1}$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} = p^2 \Rightarrow \frac{d^2 X}{dx^2} - p^2 X = 0$$

A.E. $m^2 - p^2 = 0 \Rightarrow m^2 = p^2$

$$\Rightarrow \boxed{m = \pm p}$$

C.F. $= C_2 e^{px} + C_3 e^{-px}$

P.I. $= 0$

$$\Rightarrow \boxed{X = C_2 e^{px} + C_3 e^{-px}}$$

$$\boxed{u = XT = C_1 e^{p^2 c^2 t}\left(C_2 e^{px} + C_3 e^{-px}\right)}$$

Case II $\frac{1}{c^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2 X}{dx^2} = -p^2$

$$\frac{1}{c^2 T}\frac{dT}{dt} = -p^2 \Rightarrow \frac{1}{T}\frac{dT}{dt} = -p^2 c^2 \Rightarrow \int \frac{1}{T}\, dT = -p^2 c^2 \int dt$$

$$\Rightarrow \log T = -p^2 c^2 t + \log C_1$$

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$$\Rightarrow \log\left(\frac{T}{C_1}\right) = -p^2c^2t$$

$$\Rightarrow \boxed{T = e^{-p^2c^2t} \cdot C_1}$$

- $$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

$$\Rightarrow \underline{\underline{A.E}}\ m^2 + p^2 = 0 \Rightarrow m = \pm ip$$

$$\therefore C.F. = C_2\cos px + C_3 \sin px$$

$$\underline{\underline{P.I.}} = 0$$

$$\therefore \boxed{X = C_2\cos px + C_3 \sin px}$$

$$\boxed{u = XT = C_1 e^{-p^2c^2t}(C_2\cos px + C_3 \sin px)}$$

<u>Case II</u> $$\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = 0$$

- $$\frac{1}{X}\frac{d^2X}{dx^2} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$

$$m^2 = 0 \Rightarrow m = 0, 0$$

$$\therefore \boxed{X = C_1 + xC_2}$$

- $$\frac{1}{c^2T}\frac{dT}{dt} \Rightarrow dT = 0 \Rightarrow \boxed{T = C_3}$$

$$\boxed{u = XT = (C_1 + xC_2)C_3}$$

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Q- A rod of length l with insulated sides is initially at a uniform temperature u_0. It's ends are suddenly cooled at $0°C$ and are kept at that temperature. Find the temperature func^n $u(x,t)$.

Sol^n ⇒

The solution of the equ^n is:

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2p^2t}$$

As per question, boundary cond^{n}s are:

$$u(0,t) = u(l,t) = 0 \quad \text{—— (3)}$$

and $u(x,0) = u_0$ (initial cond^n) —— (4)

Now, applying boundary cond^{n}s

$$u(0,t) = C_1 \cdot C_3 e^{-c^2p^2t}$$

$$\Rightarrow 0 = C_1 \cdot C_3 e^{-c^2p^2t}$$

$\because C_3 \neq 0$ and $e^{-c^2p^2t} \neq 0$

$$\therefore \boxed{C_1 = 0}$$

Now, sol^n becomes

$$\boxed{u(x,t) = C_2 \cdot C_3 \sin px\, e^{-c^2p^2t}}$$

Now applying another boundary cond^n

$$u(l,t) = C_2 C_3 \sin pl\, e^{-c^2p^2t}$$

$\because C_2 \neq 0$, $C_3 \neq 0$, $e^{-c^2p^2t} \neq 0$

$\therefore \sin pl = 0$

$\Rightarrow \sin pl = \sin n\pi$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Now, sol^n becomes

$$\boxed{u(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t}}$$

On applying initial cond^{n}s, we get

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l} \int_0^l u_0 \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx$

$$= \frac{2u_0}{l} \int_0^l \sin\left(\frac{n\pi x}{l}\right) \cdot dx = \frac{2u_0}{l} \left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right]_0^l$$

$$= \frac{2u_0}{\cancel{l}} \times \frac{\cancel{l}}{n\pi} \left[1 - (-1)^n \right]$$

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$$b_n = \begin{cases} 0 & , \text{when } n \text{ is even} \\ \dfrac{4u_0}{n\pi} & , \text{when } n \text{ is odd} \end{cases}$$

Hence, the temperature func^n

$$\boxed{u(x,t) = \frac{4u_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2}{l^2} t}}$$

Q- Find the temperature in a bar of length 2 where ends are kept at zero and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Sol^n The solution of equ^n is:

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2 p^2 t} \quad \text{--- (1)}$$

As per the question, ~~initial~~ boundary cond^{n}s are

$$u(0,t) = u(2,t) = 0$$

and $u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

applying $x=0$, $u(0,t) = C_1 C_3 e^{-c^2 p^2 t}$

$$\Rightarrow 0 = C_1 C_3 e^{-c^2 p^2 t} \Rightarrow C_1 = 0$$

put $C_1 = 0$ in equ^n (1), we get

$$\boxed{u(x,t) = C_2 C_3 \sin px\, e^{-c^2 p^2 t}} \quad \text{--- (2)}$$

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applying $x=2$, $\Rightarrow u(2,t) = C_2 C_3 \sin p2\, e^{-c^2p^2t}$

or $0 = C_2 C_3 \sin 2p\, e^{-c^2p^2t}$

or $\sin 2p = 0 \Rightarrow \sin 2p = \sin n\pi$

$$\Rightarrow \boxed{p = \frac{n\pi}{2}}$$

Now, general sol^n

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\, e^{-\frac{c^2\pi^2n^2}{4}t}}$$

Now, $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$

$$\sin\frac{\pi x}{2} + 3\sin\left(\frac{5\pi x}{2}\right) = b_1 \sin\frac{\pi x}{2} + b_2 \sin\left(\frac{2\pi x}{2}\right)$$

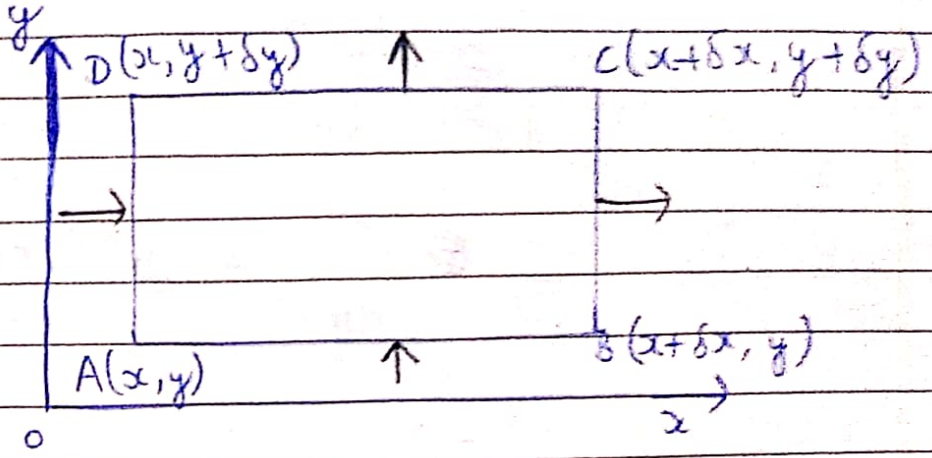
$$b_3 \sin\left(\frac{3\pi x}{2}\right) + b_4 \sin\left(\frac{4\pi x}{2}\right) + b_5 \sin\left(\frac{5\pi x}{2}\right) + \cdots$$

$\therefore b_1 = 1$, $b_5 = 3$, $b_2 = b_3 = b_4 = b_6 = \cdots = 0$

Hence, sol^n is

$$\boxed{u(x,t) = \sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2c^2}{4}t} + 3\sin\frac{5\pi x}{2}\, e^{-\frac{25c^2\pi^2}{4}t}}$$

Two Dimensional Heat Equation



Then, the flow is called two dimensional and defined as:

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

where $c^2 = \frac{K}{s\rho}$

In steady state, u is independent of t, i.e. $\frac{\partial u}{\partial t} = 0$

Therefore, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

This is known as Laplace's eqn in two dimensions.

Solution of Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x,y) = XT$

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{d^2X}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2}$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow Y\frac{d^2X}{dx^2} = -X\frac{d^2Y}{dy^2}$$

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On dividing both sides with XY, we get

$$\boxed{\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}}$$

<u>Case I</u> $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} = p^2$ (Let)

- $\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \Rightarrow \frac{d^2X}{dx^2} - p^2X = 0$

$$\Rightarrow m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$$\therefore \boxed{X = C_1 e^{px} + C_2 e^{-px}}$$

- $\frac{-1}{Y}\frac{d^2Y}{dy^2} = p^2 \Rightarrow \frac{d^2Y}{dy^2} + p^2Y = 0$

$$\Rightarrow m^2 + p^2 = 0 \Rightarrow m = \pm ip$$

$$\therefore \boxed{Y = C_3 \cos py + C_4 \sin py}$$

$$u(x,t) = XY = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py)$$

<u>Case II</u> $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} = 0$ (Let)

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0 \rightarrow \frac{d^2X}{dx^2} = 0$$

$$\Rightarrow m^2 = 0 \Rightarrow m = 0, 0$$

$$\therefore \boxed{X = C_1 + xC_2}$$

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- $\frac{-1}{Y}\frac{d^2Y}{dy^2}=0 \Rightarrow \frac{d^2Y}{dy^2}=0$

$\Rightarrow m^2=0 \Rightarrow m=0,0$

$\therefore \boxed{Y=C_3+yC_4}$

$$\boxed{u(x,y)=XY=(C_1+xC_2)(C_3+yC_4)}$$

Case III

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{-1}{Y}\frac{d^2Y}{dy^2}=-p^2 \text{ (Let)}$$

- $\frac{1}{X}\frac{d^2X}{dx^2}=-p^2 \Rightarrow \frac{d^2X}{dx^2}+p^2X^2=0$

A.E. $m^2+p^2=0 \Rightarrow m=\pm ip$

$\therefore \boxed{X=C_1\cos px+C_2\sin px}$

- $\frac{-1}{Y}\frac{d^2Y}{dy^2}=-p^2 \Rightarrow \frac{d^2Y}{dy^2}-p^2Y=0$

A.E. $m^2-p^2=0 \Rightarrow m^2=p^2$

$\Rightarrow \boxed{m=\pm p}$

$\therefore \boxed{Y=C_3e^{py}+C_4e^{-py}}$

$$\boxed{u(x,t)=XY=(C_1\cos px+C_2\sin px)(C_3e^{py}+C_4e^{-py})}$$

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Hence, the solution of 2D heat equation (Laplace Equation) is

$$\boxed{u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})}$$

Q- Use Seperation of variable method to solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

subject to boundary conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin\frac{n\pi x}{l}$

Sol^n

The solution of 2D-heat equation (Laplace equ^n) is

$$\boxed{u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})} \quad \text{--- (1)}$$

applying $u(0,y) = 0$ in equ^n (1)

$$0 = (C_1)(C_3 e^{py} + C_4 e^{-py}) \Rightarrow \boxed{C_1 = 0}$$

Hence, equ^n (1) reduces to

$$\boxed{u(x,y) = (C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})} \quad \text{--- (2)}$$

applying $u(x,0) = 0$ in equ^n (2), we get

$$\Rightarrow 0 = (C_2 \sin px)(C_3 + C_4)$$

$$\Rightarrow C_3 + C_4 = 0 \Rightarrow \boxed{C_3 = -C_4}$$

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Hence, equ^n (2) reduces to

$$u(x,y) = (c_2 \sin px)\, c_3 \left(e^{py} - e^{-py}\right) \quad \text{--- (3)}$$

applying $u(l,y)=0$ in equ^n (3), we get

$$0 = (c_2 \sin pl)\, c_3 \left(e^{py} - e^{-py}\right)$$

$$\Rightarrow \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Hence, equ^n (3) reduces to

$$u(x,y) = c_2 c_3 \sin\frac{n\pi x}{l} \left(e^{\frac{n\pi}{l}y} - e^{-\frac{n\pi y}{l}}\right) \quad \text{--- (4)}$$

applying $u(x,a) = \sin\frac{n\pi x}{l}$ in equ^n (4), we get

$$\sin\frac{n\pi x}{l} = c_2 c_3 \left(\sin\frac{n\pi x}{l}\right) \left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]$$

$$\therefore c_2 c_3 \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right) = 1$$

$$\Rightarrow \boxed{c_2 c_3 = \frac{1}{\left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right)}}$$

$$\Rightarrow c_2 c_3 = \frac{1}{2\sinh\left(\frac{n\pi a}{l}\right)}$$

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Now, eqn ⑨ becomes,

$$u(x,y) = \frac{\sin\left(\frac{n\pi x}{l}\right) \; 2\sinh\left(\frac{n\pi y}{l}\right)}{2\sinh\left(\frac{n\pi a}{l}\right)}$$

$$\Rightarrow \boxed{u(x,y) = \frac{\sin\left(\frac{n\pi x}{l}\right)\cdot\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)}}$$

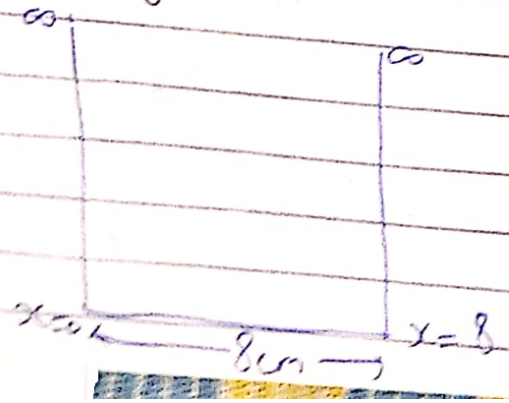
Q. A rectangular plate with insulated surfaces is 8cm wide and so long compared to it's width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one ~~[illegible]~~ short edge $y=0$ is given by

$$u(x,0) = 100\sin\frac{\pi x}{8} \qquad 0<x<8$$

While the two edges $x=0$ and $x=8$ as well as the other short edge are kept at $0°C$ show that the steady state temperature at any point of the plate is given by

$$u(x,y) = 100\, e^{-\pi y/8} \sin\frac{\pi x}{8}$$

Sol:



Sol^n $u(0,y) = u(8,y) = 0$ and $\lim_{y\to\infty} u(x,y) = 0$

The solution of 2D-heat equation (Laplace Equation) is

$$\boxed{u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})} \quad -(1)$$

<u>Boundary cond^{n}s</u> $u(0,y) = u(8,y) = 0$

$$\lim_{y\to\infty} u(x,y) = 0$$

and $u(x,0) = 100 \sin \frac{\pi x}{8}$

Applying $u(0,y) = 0$ in eqn (1), we get

$$0 = (C_1)(C_3 e^{py} + C_4 e^{-py}) \Rightarrow \boxed{C_1 = 0}$$

Hence eqn (1) reduces to

$$u(x,y) = (C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(2)$$

Applying $u(8,y) = 0$ in eqn (2), we get

$$0 = (C_2 \sin 8p)(C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow \sin p8 = 0 \Rightarrow \sin 8p = \sin n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{8}}$$

Hence, equ^n (2) becomes

$$u(x,y) = \left(C_2 \sin\frac{n\pi x}{8}\right)\left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right) \quad \text{—(3)}$$

Applying $\lim_{y\to\infty} u(x,y) = 0$ in equ^n (3), we get

$$0 = C_2 \sin\left(\frac{n\pi x}{8}\right)\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{8}}\right]$$

$$\Rightarrow \boxed{C_3 = 0}$$

Hence, equ^n (3) becomes

$$u(x,y) = C_2 C_4 \sin\frac{n\pi x}{8}\, e^{-\frac{n\pi y}{8}} \quad \text{—(4)}$$

or $$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}\, e^{-\frac{n\pi y}{8}}} \quad \text{—(5)}$$

On applying $u(x,0) = 100 \sin\frac{\pi x}{8}$ in equ^n (5)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}$$

$$100 \sin\frac{\pi x}{8} = b_1 \sin\frac{\pi x}{8} + b_2 \sin\frac{\pi x}{8} + b_3 \sin\frac{3\pi x}{8} + \cdots$$

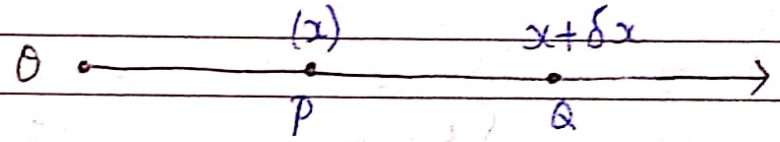
$$\therefore b_1 = 100, \quad b_2 = b_3 = b_4 = \cdots = 0$$

$\therefore$

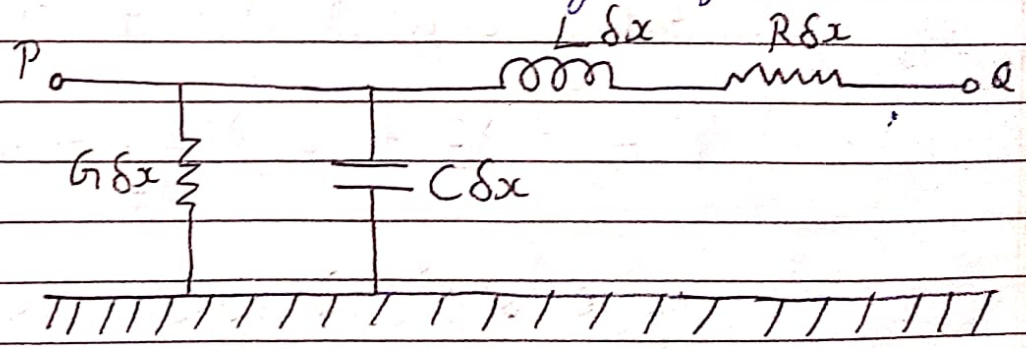
$$\boxed{u(x,y) = 100 \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}}$$ Ans

★ Transmission Line Equation

- Consider the flow of electricity in an insulated cable.
- Let V be the potential and I be the current at time 't' at a point P of the cable at a distance x from a given point O (generator).
- Let $V+\delta V$, $I+\delta I$ be the potential and current at time 't' at a point Q of the cable at a distance $(x+\delta x)$.



- Let R, L, C, and G be the resistance, inductance, capacitance and leakage to ground (conductance) per unit length of the cable.



$\Rightarrow$ Hence, $R\delta x$, $L\delta x$, $C\delta x$ and $G\delta x$ are the line parameters for δx length.

Now, potential drop across the segment PQ

$$\Rightarrow -\delta V = (R\delta x)I + L\delta x \frac{\partial I}{\partial t}$$

$$\Rightarrow -\frac{\delta V}{\delta x} = RI + L\frac{\partial I}{\partial t}$$

as $\delta x \to 0$ $$\boxed{-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t}} \quad (1)$$

Now, current

$$-\delta I = (G\delta x)V + (C\delta x)\frac{\partial V}{\partial t}$$

$$\Rightarrow -\frac{\delta I}{\delta x} = GV + C\frac{\partial V}{\partial t}$$

as $\delta x \to 0$ $$\boxed{-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t}} \quad (2)$$

~~Now~~ On differentiating equ^n (1) w.r.t. 'x' we get

$$-\frac{\partial^2 V}{\partial x^2} = R\frac{\partial I}{\partial x} + L\frac{\partial^2 I}{\partial x \partial t} \quad (3)$$

~~putting equ^n (2) in equ^n (3), we get~~

On differentiating equ^n (2) w.r.t. 't' we get

$$-\frac{\partial^2 I}{\partial x \partial t} = G\frac{\partial V}{\partial t} + C\frac{\partial^2 V}{\partial t^2} \quad \text{—(4)}$$

Now, on putting equ^n (2) & equ^n (4) in equ^n (3), we get

$$-\frac{\partial^2 V}{\partial x^2} = R\left[-GV - C\frac{\partial V}{\partial t}\right] + L\left[-G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = RGV + RC\frac{\partial V}{\partial t} + LG\frac{\partial V}{\partial t} + LC\frac{\partial^2 V}{\partial t^2}$$

$$\rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC+LG)\frac{\partial V}{\partial t} + RGV} \quad \text{—(5)}$$

Transmission line equation for voltage

for current

on differentiating equ^n (2) w.r.t. 'x', we get

$$-\frac{\partial^2 I}{\partial x^2} = G\frac{\partial V}{\partial x} + C\frac{\partial^2 V}{\partial x \partial t} \quad \text{—(6)}$$

on differentiating equ^n (1) w.r.t. 't' we get

$$-\frac{\partial^2 V}{\partial x \partial t} = R\frac{\partial I}{\partial t} + L\frac{\partial^2 I}{\partial t^2} \quad \text{—(7)}$$

Now, on putting equ^n (1) & (7) in equ^n (6) we get

$$-\frac{\partial^2 I}{\partial x^2} = G\left[-RI - L\frac{\partial I}{\partial t}\right] + C\left[-R\frac{\partial I}{\partial t} - L\frac{\partial^2 I}{\partial t^2}\right]$$

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$$\rightarrow \frac{\partial^2 I}{\partial x^2} = GRI + GL\frac{\partial I}{\partial t} + RC\frac{\partial I}{\partial t} + LC\frac{\partial^2 I}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC+LG)\frac{\partial I}{\partial t} + RGI} \quad (8)$$

Transmission Line Current Eqn.

Telephone Equations

- $$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC+LG)\frac{\partial V}{\partial t} + RGV$$

- $$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC+LG)\frac{\partial I}{\partial t} + RGI$$

* If $L=G=0$,

$$\boxed{\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}} \text{ and } \boxed{\frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}}$$

These are called Telegraph equations. (Similar to 1-D heat flow.)

* If $R=G=0$

$$\boxed{\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}} \text{ and } \boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}}$$

These are called Radio Equations (Similar to wave equation)

If $L = C = 0$

$$\boxed{\frac{\partial^2 V}{\partial x^2} = RGV} \quad \text{and} \quad \boxed{\frac{\partial^2 I}{\partial x^2} = RGI}$$

These are called equation for submarine cable

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV$$

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGI$$

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★ Statistics

It is the science which deals with methods of collecting, classifying, presenting, comparing and interpreting numerical data collected to give answer on any sphere of enquiry.

★ Measures of Central Tendency

- Consider a figure which is used to represent a whole series, should neither have the lowest value nor the highest in the series.

- It contains a value somewhere b/w these two limits, possibly is the centre, where most of the items of the series cluster.

- Such figures are called Measure of Central Tendency (or Averages)

1) Arithmetic Average (or Mean)

2) Median

3) Mode

★ Arithmetic Mean

Arithmetic Mean $\bar{x} = \frac{x_1 + x_2 + x_3 + - - - + x_n}{n}$

$\Rightarrow \boxed{\bar{x} = \frac{\Sigma x}{n}}$ where n is the total number of items / observations.

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* With frequency distribution

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + f_3 + \cdots + f_n} = \frac{\Sigma fx}{\Sigma f}$$

or $\boxed{\bar{x} = \frac{\Sigma fx}{N}}$

* Case of continuous series having equal class intervals of width 'h'

Let $\boxed{u = \frac{x-a}{h}}$ or $\boxed{x = a + hu}$

$$\boxed{\bar{x} = a + h\frac{\Sigma fu}{N}}$$

Q- Find the mean from the following data

Marks	No. of students
Below 10	5
Below 20	9
Below 30	17
Below 40	29
Below 50	45
Below 60	60
Below 70	70
Below 80	78
Below 90	83
Below 100	85

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Sol^n ⇒

Marks	No. of Student (f)	x	$u = \frac{x-a}{h}$	fu
0 - 10	5	5	-5	-25
10 - 20	4	15	-4	-16
20 - 30	8	25	-3	-24
30 - 40	12	35	-2	-24
40 - 50	16	45	-1	-16
50 - 60	15	55	0	0
60 - 70	10	65	1	10
70 - 80	8	75	2	16
80 - 90	5	85	3	15
90 - 100	2	95	4	8
	$\Sigma f = 85$			$\Sigma fu = -56$

Let $a = 55$, $h = 10$

$$\bar{x} = a + h \times \frac{\Sigma fu}{\Sigma f} = 55 + 10 \times \left(\frac{-56}{85}\right)$$

$$\Rightarrow \bar{x} = 55 - 6.5882 = 48.4118 \text{ Marks}$$

★ Median

- It is the central item of series when it is arranged in ascending or descending order of magnitude.

- It divides the series into two halves: one half all items are less than median whereas in other half all items are having higher value than median.

- When n is odd, $\left(\frac{n+1}{2}\right)^{th}$ value gives the median

When n is even, A.M. of $\left(\frac{n}{2}\right)^{th}$ and ~~$\left(\frac{n+1}{2}\right)^{th}$~~ $\left(\frac{n}{2}+1\right)^{th}$ values gives the median

Q- Consider the marks obtained by a batch of 4 students in a certain class in Maths & Physics

Roll No.	1	2	3	4
Marks in Maths	53	54	52	32
Marks in Physics	58	55	25	32

In which subject is the level of knowledge of students is higher?

Soln ⇒ Directly calculate ni karne hy jayenge, pahale ascending order mein arrange krenge

Marks in Maths 32 52 53 54

$n = 4 \Rightarrow$ A.M. 2nd & 3rd $\Rightarrow \frac{52+53}{2} = 52.5$

Marks in Physics 25 32 55 58

$n = 4 \Rightarrow$ A.M. of 2nd & 3rd $\Rightarrow \frac{32+55}{2} = 43.5$

The level of knowledge of students is higher in Maths as compared to Physics

Obtain the Median

x	f	cumulative frequency C.F.
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
	$\Sigma f = 120$	

$\frac{N+1}{2} = \frac{120+1}{2} = 60.5$ ← Isse just bdi C.F. ke corresponding values lete hai

$\Rightarrow$ $\boxed{\text{Median} = 5}$ Ans

Mode

Mode is the most commonly or frequently occuring value in a series.

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Moments — Statistical Tools used in statistical investigations

→ Moment about Mean (central moments)
→ Moment about arbitrary number (raw moments)
→ Moment about origin

★ **Moments about Mean**

Case I If $x_1, x_2, x_3 ---- x_n$ are the values of variables then r^{th} moment μ_r about mean $\bar{x}$ is

$$\boxed{\mu_r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^r}{n}}$$

where $r = 0, 1, 2 ---- 3$

Q- Find the first four moments for the series

$x: 3, 6, 8, 10, 18$

Sol^n ⇒

$$\bar{x} = \frac{3+6+8+10+18}{5} = \frac{45}{5} = 9$$

x	$x-\bar{x}$	$(x-\bar{x})^2$	$(x-\bar{x})^3$	$(x-\bar{x})^4$
3	-6	36	-216	1296
6	-3	9	-27	81
8	-1	1	-1	1
10	1	1	1	1
18	9	81	729	6561
	$\Sigma x-\bar{x} = 0$	$\Sigma(x-\bar{x})^2 = 128$	$\Sigma(x-\bar{x})^2 = 486$	$\Sigma(x-\bar{x})^4 = 7940$

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$\mu_1 = 0$

$$\mu_2 = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{128}{5} = 25.6$$

$$\mu_3 = \frac{\Sigma(x-\bar{x})^3}{n} = \frac{486}{5} = 97.2$$

$$\mu_4 = \frac{\Sigma(x-\bar{x})^4}{n} = \frac{7940}{5} = 1588$$

<u>Case II</u> If $x_1, x_2, x_3 - - - - x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2 - - - - f_n$, then r^{th} moment μ_r about mean $\bar{x}$ is

$$\boxed{\mu_r = \sum_{i=1}^{n} \frac{f_i(x_i-\bar{x})^r}{N}} \quad \text{where } N = \sum_{i=1}^{n} f_i$$

and $r = 0, 1, 2, 3 - - - -$

<u>for $r = 0$</u> $\boxed{\mu_0 = 1}$

<u>for $r = 1$</u> $\mu_1 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x}) = 0$

<u>for $r = 2$</u> $\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^2 = \left(\text{Standard Deviation}\right)^2 = \text{Variance}$

<u>for $r = 3$</u> $\mu_3 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^3$

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for $r=4$

$$\mu_4 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^4$$

Q- Calculate $\mu_1, \mu_2, \mu_3, \mu_4$ for the following distribution

Marks	5-15	15-25	25-35	35-45	45-55	55-65
No. of students	10	20	25	20	15	10

Soln

Marks	No. of Students (f)	Mid value (x)	fx	$f(x-\bar{x})$	$f(x-\bar{x})^2$	$f(x-\bar{x})^3$	$f(x-\bar{x})^4$
5-15	10	10	100	-240	5760	-138240	3317760
15-25	20	20	400	-280	3920	-54880	768320
25-35	25	30	750	-100	400	-1600	6400
35-45	20	40	800	120	720	4320	25920
45-55	15	50	750	240	3840	61440	983040
55-65	10	60	600	260	6760	175760	4569760
	$\Sigma f = 100$		$\Sigma fx = 3400$	$\Sigma f(x-\bar{x}) = 0$	$\Sigma f(x-\bar{x})^2 = 21400$	$\Sigma f(x-\bar{x})^3 = 46800$	$\Sigma f(x-\bar{x})^4 = 9671200$

$x-\bar{x}$
-24
-14
-4
16
26

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{3400}{100} = 34$$

$$\mu_1 = 0$$

$$\mu_2 = \frac{\Sigma f(x-\bar{x})^2}{N} = \frac{21400}{100} = 214 \text{ Ans}$$

$$\mu_3 = \frac{\Sigma f(x-\bar{x})^3}{N} = \frac{46800}{100} = 468 \text{ Ans}$$

$$\mu_4 = \frac{\Sigma f(x-\bar{x})^4}{N} = \frac{9671200}{100} = 96712 \text{ Ans}$$

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★ Moment about an arbitary number (Raw Moments)

If $x_1, x_2, x_3 - - - x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2, f_3 - - - - f_n$ respectively, then r^{th} moment (μ'_r) about the number $x = A$ is

$$\boxed{\mu'_r = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - A)^r}$$

where $r = 0, 1, 2 - - -$ and $N = \sum_{i=1}^{n} f_i$

for $r=0$ $\boxed{\mu'_0 = 1}$

for $r=1$ $\boxed{\mu'_1 = \bar{x} - A}$

for $r=2$

$$\boxed{\mu'_2 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - A)^2}$$

for $r=3$

$$\boxed{\mu'_3 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - A)^3}$$

for $r=4$

$$\boxed{\mu'_4 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - A)^4}$$

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★ Moments about Origin

If $x_1, x_2, x_3 - - - - x_n$ be the values of a variable x with corresponding frequencies $f_1, f_2, - - - f_n$ respectively then r^{th} moment about the origin V_r is

$$\boxed{V_r = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^r}$$

for $r = 0, 1, 2 - -$ and $N = \sum_{i=1}^{n} f_i$

for $r=0$ $\boxed{V_0 = 1}$

for $r=1$ $\boxed{V_1 = \bar{x}}$

for $r=2$ $\boxed{V_2 = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2}$

for $r=3$ $\boxed{V_3 = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^3}$

★ Relations

Relationship b/w μ_r & μ'_r

- $\boxed{\mu_1 = 0}$
- $\boxed{\mu_2 = \mu'_2 - (\mu'_1)^2}$
- $\mu_3 = \mu'_3 (\mu'_1)^0 - 3\mu'_2 (\mu'_1)^1 + 2\mu'_1 (\mu'_1)^2$

$\Rightarrow \boxed{\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu'^3_1}$

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- $\mu_4 = \mu'_4 - 4\mu'_3(\mu'_1)^1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$

$\Rightarrow \boxed{\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1}$

Relationship b/w V_r and μ_r

- $\boxed{V_1 = \bar{x}}$
- $\boxed{V_2 = \mu_2 + \bar{x}^2}$
- $\boxed{V_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3}$
- $\boxed{V_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4}$

Q- The first three moments of a distribution about the value '2' of the variable are 1, 16 and -70. Show that mean is 3, variance is 15 & μ_3 is -86.

Sol^n ⇒

Given, $A = 2$, $\mu'_1 = 1$, $\mu'_2 = 16$, $\mu'_3 = -70$

$\mu'_1 = \bar{x} - A \Rightarrow 1 = \bar{x} - 2 \Rightarrow \boxed{\bar{x} = 3} \Rightarrow$ Mean

$\mu_2 = \mu'_2 - (\mu'_1)^2 =$ variance

$\Rightarrow \mu_2 = 16 - (1)^2 = 15$

$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$

$= -70 - 3 \times 16 \times 1 + 2(1)^3 = -86$

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Q- Calculate the variance and third central moment from the frequency data

x_i	0	1	2	3	4	5	6	7	8
f_i	1	9	26	59	72	52	29	7	1

Sol^n

x	f	$u = \frac{x-a}{h}$	fu	fu^2	fu^3	fu^4
0	1	-4	-4	16	-64	256
1	9	-3	-27	81	-243	729
2	26	-2	-52	104	-208	416
3	59	-1	-59	59	-59	59
4	72	0	0	0	0	0
5	52	1	52	52	52	52
6	29	2	58	116	232	464
7	7	3	21	63	189	567
8	1	4	4	16	64	256
	256		$\Sigma fu = -7$	$\Sigma fu^2 = 507$	$\Sigma fu^3 = -37$	$\Sigma fu^4 = 2799$

Let $a = 4$, $h = 1$

$$\mu_1' = \left(\frac{\Sigma fu}{\Sigma f}\right) h = \frac{-7}{256} \times 1 = -0.02734$$

$$\mu_2' = \left(\frac{\Sigma fu^2}{N}\right) h^2 = \frac{507}{256} \times (1)^2 = 1.9804$$

$$\mu_3' = \left(\frac{\Sigma fu^3}{N}\right) h^3 = \frac{-37}{256} \times (1)^3 = -0.1445$$

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Thus, $\boxed{\mu_1 = 0}$

$\mu_2 = \text{Variance} = \mu_2' - \mu_1'^2 = 1.9804 - (-0.02734)^2$

$= 1.97$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1'^2 + 2\mu_1'^3 = -0.1445 - 3 \times 1.9804 \times (-0.2734)^2 + 2(-0.02734)^3$

$\Rightarrow \mu_3 = -0.14898$

Q- The first four moments of a distribution about the value '4' of the variable are $-1.5, 17, -30$ and 108

Find the moments about mean, about origin. Also find the moments about the point $x = 2$

Sol^n $\Rightarrow$ Given $A = 4$, $\mu_1' = -1.5$, $\mu_2' = 17$, $\mu_3' = -30$, $\mu_4' = 108$

$\rightarrow$ Moment about mean

$\boxed{\mu_1 = 0}$

$\mu_2 = \mu_2' - (\mu_1')^2 = 17 - (-1.5)^2 = 14.75$ Ans

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= -30 - 3 \times 17 \times (-1.5) + 2(-1.5)^3 = 39.75$ Ans

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 108 - 4(-30)(-1.5) + 6 \times 17 \times (-1.5)^2 - 3(-1.5)^4 = 142.31$ Ans

$\mu_1 = 0$, $\mu_2 = 14.75$, $\mu_3 = 39.75$, $\mu_4 = 142.31$

→ Moment about origin

$\mu'_1 = \bar{x} - a \Rightarrow -1.5 = \bar{x} - 4 \Rightarrow \bar{x} = 4 - 1.5$

$\Rightarrow \bar{x} = 2.5$

$$\boxed{\nu_1 = \bar{x} = 2.5}$$

$\nu_2 = \mu_2 + (\bar{x})^2 = 14.75 + (2.5)^2 = 21$ Ans

$\nu_3 = \mu_3 + 3\mu_2\bar{x} + (\bar{x})^3 = 39.75 + 3 \times 14.75 \times 2.5 + (2.5)^3$

$= 166$ Ans

$\nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + (\bar{x})^4$

$= 142.31 + 4 \times 39.75 \times 2.5 + 6 \times 14.75 \times (2.5)^2 + (2.5)^4$

$= 1132$ Ans

→ Moment about the pt. '2'

$\mu'_1 = \bar{x} - A = 2.5 - 2 = 0.5$

$\mu'_2 = \mu_2 + (\mu'_1)^2 = 14.75 + (0.5)^2 = 15$

$\mu'_3 = \mu_3 + 3\mu'_2\mu'_1 - 2(\mu'_1)^3 = 39.75 + 3 \times 15 \times 0.5 - 2(0.5)^3$

$= 85$

$\mu'_4 = \mu_4 + 4\mu'_3\mu'_1 - 6\mu'_2(\mu'_1)^2 + 3(\mu'_1)^4$

$= 142.31 + 4 \times 85 \times 0.5 - 6 \times 15 \times (0.5)^2 + 3(0.5)^4$

$= 289.9975$

★ Moment Generating Function

* In case of continuous variable

$$\boxed{M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \cdot dx}$$

* In case of discrete variable

$$\boxed{M_x(t) = \sum_{x=0}^{n} e^{tx} \cdot f(x)}$$

Q Obtain the moment generating function of the random variable x having probability distribution

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Sol^n

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^1 e^{tx} \cdot x \cdot dx + \int_1^2 e^{tx} (2-x) \cdot dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right) \right]_0^1 + \left[(2-x)\left(\frac{e^{tx}}{t}\right) - (-1)\left(\frac{e^{tx}}{t^2}\right) \right]_1^2$$

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$$= \left[\left(\cancel{\frac{e^t}{t}} - 0\right) - \left(\frac{e^t}{t^2} - \frac{1}{t^2}\right)\right] + \left[\left(0 - \cancel{\frac{e^t}{t}}\right) + \left(\frac{e^{2t}}{t^2} - \frac{e^t}{t^2}\right)\right]$$

$$= \frac{-e^t + 1 + e^{2t} - e^t}{t^2} = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$M_x(t) = \frac{(e^t - 1)^2}{t^2} \quad \text{Ans}$$

$\Rightarrow$ Mean, $V_1 = \left[\frac{d}{dt}(M_x(t))\right]_{t=0}$

Similarly, $V_2 = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0}$

and variance $\mu_2 = V_2 - \bar{x}^2$

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Q- Find the moment generating function of the discrete binomial distribution given by

$$P(x) = {}^nC_x \, p^x \, q^{n-x} \quad (\text{where } q = 1-p)$$

Also find the first and second moments about the mean.

Soln

$$M_x(t) = \sum e^{tx} \cdot f(x)$$

$$\Rightarrow M_x(t) = \sum e^{tx}\left[{}^nC_x \, p^x \, q^{n-x}\right]$$

$$= \sum {}^nC_x \, (pe^t)^x \, q^{n-x}$$

$$= {}^nC_0 (pe^t)^0 q^{n-0} + {}^nC_1 (pe^t)^1 q^{n-1} + \cdots + {}^nC_n (pe^t)^n q^0$$

$$\boxed{M_x(t) = (pe^t + q)^n}$$ (By using Binomial Theorem)

Now,

$$V_1 = \left[\frac{d}{dt}\left(M_x(t)\right)\right]_{t=0} = \left[\frac{d}{dt}\left\{(pe^t+q)^n\right\}\right]_{t=0}$$

$$= \left[n(pe^t+q)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= \left[n\,\cancel{(p+q)}^{1\,n-1} \cdot p\right] \Rightarrow \boxed{V_1 = np} = \bar{x} \text{ (Mean}$$

$$\cancel{V_2 = \left[\frac{d^2}{dt^2}\left(M_x(t)\right)\right]_{t=0} = \left[n(n-1)(pe^t+q)^{n-2}(pe^t)^2\right]_{t=0}}$$

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$$\nu_2 = \left[\frac{d^2}{dt^2}\left(M_x(t)\right)\right]_{t=0}$$

$$= np\left[e^t(n-1)(q+pe^t)^{n-2}\cdot pe^t + e^t(q+pe^t)^{n-1}\right]_{t=0}$$

$$= \cancel{np\left[(n-1)(p+q)^{n-2}p + (p+q)^{n-1}\right]}$$

$$= \cancel{np\left[np^2+1\right]}$$

$$= np\left[(n-1)p\,(q+p)^{n-2} + (q+p)^{n-1}\right]$$

$$= np\left[(n-1)p+1\right]$$

$$= np\left[np-p+1\right]$$

$$= np\left[np+q\right]$$

$$\Rightarrow \boxed{\nu_2 = n^2p^2 + npq}$$

→ <u>Moment about Mean</u>

$$\boxed{\mu_1 = 0}$$

$$\mu_2 = \nu_2 - \bar{x}^2 = \cancel{n^2p^2} + npq - \cancel{n^2p^2}$$

$$\Rightarrow \boxed{\mu_2 = npq}$$

Hence, $\boxed{\text{Mean} = np}$

S.D. = $\sqrt{\mu_2} = \sqrt{npq}$ Ans

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Q- Find the moment generating function of the discrete Poission distribution given by $P(x) = e^{-\lambda} \frac{\lambda^x}{\lfloor x}$

Also find the first and second moments about the mean.

Sol^n ⇒

$$M_x(t) = \sum e^{tx} f(x)$$

$$= \sum e^{tx} \left(e^{-\lambda} \frac{\lambda^x}{\lfloor x} \right) = e^{-\lambda} \sum \frac{(e^t \lambda)^x}{\lfloor x}$$

$$= e^{-\lambda} \left[1 + \frac{e^t \lambda}{\lfloor 1} + \frac{(e^t \lambda)^2}{\lfloor 2} + - - - - - - \infty \right]$$

$$= e^{-\lambda} \left[e^{(e^t \lambda)} \right] = e^{(-\lambda + e^t \lambda)}$$

$$\Rightarrow \boxed{M_x(t) = e^{\lambda(e^t - 1)}}$$

Now,

$$\nu_1 = \left[\frac{d}{dt} (M_x(t)) \right]_{t=0} = \left[e^{\lambda(e^t - 1)} \cdot \lambda e^t \right]_{t=0} = \lambda$$

$$\Rightarrow \boxed{\nu_1 = \bar{x} = \lambda}$$

$$\nu_2 = \left[\frac{d^2}{dt^2} (M_x(t)) \right]_{t=0} = \lambda \left[e^t \cdot e^{\lambda(e^t - 1)} + e^t \cdot e^{\lambda(e^t - 1)} \cdot \lambda e^t \right]_{t=0}$$

$$\Rightarrow \boxed{\nu_2 = \lambda(\lambda + 1)}$$

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→ Moment about Mean

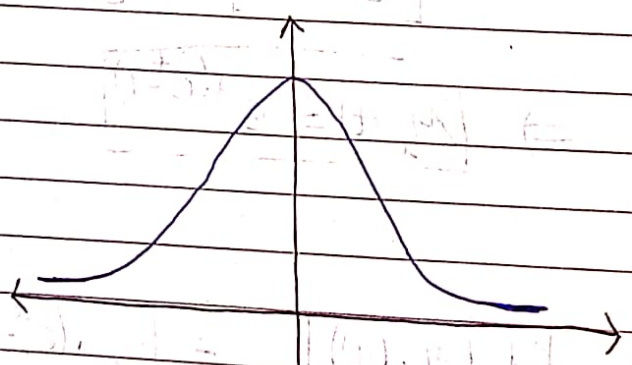
$$\boxed{\mu_1 = 0}$$

$$\mu_2 = V_2 - \bar{x}^2 = \lambda(\lambda+1) - \lambda^2 = \lambda^2 + \lambda - \lambda^2$$

$$\Rightarrow \boxed{\mu_2 = \lambda}$$

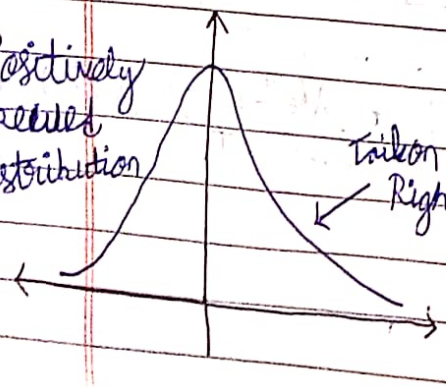
Skewness

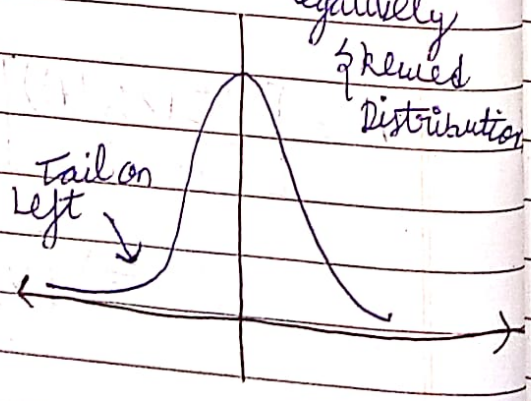
* For a symmetrical distribution, the frequencies are symmetrically distributed about the mean.



Symmetrical Distribution

Mean
Median
Mode





Skewness means lack of symmetry in a frequency distribution.

Objective of measuring skewness is to estimate the extent to which a distribution is distorted from a perfectly symmetrical distribution

Karl's Pearson Method to measure Skewness Coefficient

Karl's Pearson's coefficient of skewness

$$\boxed{SK_p = \frac{A.M. - Mode}{S.D.}}$$

$$\boxed{e = 3Median - 2A.M.}$$

Karl's Pearson's coefficient of skewness

$$\boxed{SK_p = \frac{3(A.M. - Median)}{S.D.}}$$

Karl Pearson's coefficient of skewness of a distribution is 0.32. It's standard deviation is 6.5 and mean is 29.6
Find the mode of the distribution

$SK_p = 0.32$, $S.D. = 6.5$ & $A.M. = 29.6$

$$SK_p = \frac{A.M. - Mode}{S.D.} \Rightarrow 0.32 = \frac{29.6 - Mode}{6.5}$$

$$\Rightarrow Mode = 29.6 - 6.5 \times 0.32$$
$$= 27.52 \text{ Ans}$$

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→ Method of Moment for Measuring Skewness

Moment coefficient of skewness $\sqrt{\beta_1} = \dfrac{\mu_3}{\sqrt{\mu_2^3}}$

Q- The first three central moments of a distribution are 0, 15, -31. Find the moment coefficient of skewness

Sol^n ⇒

$\mu_1 = 0, \quad \mu_2 = 15, \quad \mu_3 = -31$

$$\sqrt{\beta_1} = \frac{\mu_3}{\sqrt{\mu_2^3}} = \frac{-31}{\sqrt{(15)^3}} = -0.53 \text{ Ans}$$

Q- Calculate the moment coefficient of skewness for the following distribution

Classes	2.5-7.5	7.5-12.5	12.5-17.5	17.5-22.5	22.5-27.5	27.5-32.5
Frequency	8	15	20	32	23	17

32.5-37.5 5

Classes	Frequency	x	$u = \frac{x-A}{h}$	fu	fu^2	fu^3
2.5-7.5	8	5	-3	-24	72	-216
7.5-12.5	15	10	-2	-30	60	-120
12.5-17.5	20	15	-1	-20	20	-20
17.5-22.5	32	20	0	0	0	0
22.5-27.5	23	25	1	23	23	23
27.5-32.5	17	30	2	34	68	136
32.5-37.5	5	35	3	15	45	135

Let $a = 20$, $h = 5$

$$\mu_1' = \frac{\Sigma f u}{\Sigma f} \times h = \frac{-2}{120} \times 5 = -0.083$$

$$\mu_2' = \frac{\Sigma f u^2}{\Sigma f} \times h^2 = \frac{288}{120} \times (5)^2 = 60$$

$$\mu_3' = \frac{\Sigma f u^3}{\Sigma f} \times h^3 = \frac{-62}{120} \times (5)^3 = -64.5$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 60 - (-0.083)^2 = 59.993$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= -64.5 - 3 \times 60 \times (-0.083) + 2(-0.083)^3$$

$$\Rightarrow \boxed{\mu_3 = -49.561}$$

Moment coefficient of skewness $= \dfrac{\mu_3}{\sqrt{\mu_2^3}} = \dfrac{-49.561}{\sqrt{(59.993)^3}}$

$$= -0.106 \text{ Ans}$$

Kurtosis

- Kurtosis refers to the bulginess (projection) of the curve of frequency distribution.

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- It is denoted by β_2, and calculated as

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$$

where
$\mu_2 \rightarrow$ Second moment
$\mu_4 \rightarrow$ fourth moment
} about mean

If $\beta_2 > 3 \Rightarrow$ more sharply peaked than the normal curve
The distribution is called <u>Laptakurtic</u>

If $\beta_2 = 3 \Rightarrow$ Neither flat nor sharply peaked
The distribution is called <u>Mesokurtic</u>.

If $\beta_3 < 3 \Rightarrow$ More flatter than the normal curve
The distribution is called <u>Platykurtic</u>

Q The first four moments about mean of a frequency distribution are 0, 100, −7 and 35000. Discuss the Kurtosis of the distribution

<u>Soln</u>

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{35000}{(100)^2} = 3.5$$

$$\beta_2 > 3 \Rightarrow \text{Laptakurtic}$$

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Q- The first four moments of a distribution about $x=4$ are 1, 4, 10 and 45. Obtain the various characteristics of the distribution on the basis of the given information. Comment upon the nature of the distribution

Sol^n

$\mu'_1 = 1$; $\mu'_2 = 4$; $\mu'_3 = 10$; $\mu'_4 = 45$

- $\boxed{\mu_1 = 0}$
- $\mu_2 = \mu'_2 - (\mu'_1)^2 = 4 - (1)^2 = 3$
- $\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$
  $= 10 - 3\times4\times1 + 2(1)^3 = 0 \Rightarrow \boxed{\mu_3 = 0}$
- $\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$
  $= 45 - 4\times10\times1 + 6\times4\times(1)^2 - 3(1)^4$
  $\mu_4 = 26$

Skewness $= \dfrac{\mu_3}{\sqrt{\mu_2^3}} = 0 \Rightarrow$ Symmetric

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{26}{(3)^2} = 2.89 < 3$

Hence, the distribution is platykurtic

Q- The following table represents the height of a batch of 100 students.

Height (in cm)	59	61	63	65	67	69	71	73	75
No. of Students	0	2	6	20	40	20	8	2	2

Sol^n ⇒

Height (in cm)	No. of Students	$u = \frac{x-a}{h}$	fu	fu^2	fu^3	fu^4
59	0	-4	0	0	0	0
61	2	-3	-6	18	-54	162
63	6	-2	-12	24	-48	96
65	20	-1	-20	20	-20	20
67	40	0	0	0	0	0
69	20	1	20	20	20	20
71	8	-2	16	32	64	128
73	2	3	6	18	54	162
75	2	4	8	32	128	512
	$\Sigma f = 100$		12	164	144	1100

Let $a = 67$, $h = 2$

$$\mu'_1 = \frac{\Sigma fu}{\Sigma f} \times h = \frac{12}{100} \times 2 = 0.24$$

$$\mu'_2 = \frac{\Sigma fu^2}{\Sigma f} \times h^2 = \frac{164}{100} \times (2)^2 = 6.56$$

$$\mu'_3 = \frac{\Sigma fu^3}{\Sigma f} \times h^3 = \frac{144}{100} \times (2)^3 = 11.52$$

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$$\mu_4' = \frac{\sum f u^4}{\sum f} \times h^4 = \frac{1100}{100} \times (2)^4 = 176$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 6.56 - (0.24)^2 = 6.5024$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 176 - 4 \times 11.52 \times 0.24 + 6 \times 6.56 \times (0.24)^2 - 3(0.24)^4$$

$$\Rightarrow \mu_4 = 167.197$$

Hence, $\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{167.197}{(6.5024)^2}$

$$\Rightarrow \beta_2 = 3.95 > 3$$

Hence, distribution is Leptokurtic

Curve Fitting

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- Curve Fitting means an exact relationship between two variables by algebraic equations.

→ **Fit a Straight line**

- $\boxed{y = a + bx}$

Normal Equations

$$\Sigma y = na + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

- $\boxed{y = ax + bx^2}$

Normal Equations

$$\Sigma xy = a\Sigma x^2 + b\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^3 + b\Sigma x^4$$

Q- By the method of least squares, find the straight line that best fits the following data:

x	1	2	3	4	5
y	14	27	40	55	68

Sol^n ⇒

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x = 15$	$\Sigma y = 204$	$\Sigma x^2 = 55$	$\Sigma xy = 748$

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$$y = a + bx$$

<u>Normal Equations</u>

$$\Sigma y = na + b\Sigma x \quad \text{— (1)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{— (2)}$$

Here, $n = 5$

$$204 = 5a + 15b \quad \text{— (3)}$$

$$748 = 15a + 55b \quad \text{— (4)}$$

$$\boxed{a = 0} \qquad \boxed{b = 13.6}$$

$$\Rightarrow \boxed{y = 13.6x}$$

→ <u>Fit an exponential curve</u>

$$\boxed{y = ae^{bx}}$$

$$\log y = \log(ae^{bx})$$

$$\Rightarrow \log y = \log a + bx \cdot \log e$$

$$\Rightarrow \boxed{Y = A + BX}$$

where, $Y = \log y$, $A = \log a$, $B = b \log e$, $X = x$

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

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$$y = ae^{bx}$$

Taking log both sides, we get

$$\log y = \log a + bx \log e$$

or $\boxed{Y = A + BX}$

where $\boxed{Y = \log y}$ $\boxed{A = \log a}$, $\boxed{B = b\log e}$ $\boxed{X = x}$

Q- Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least square.

x	1	5	7	9	12
y	10	15	12	15	21

Sol^n:

$$y = ae^{bx}$$

$$\Rightarrow \log y = \log a + bx \log e$$

$$\Rightarrow Y = A + BX$$

$\boxed{Y = \log y}$, $\boxed{A = \log a}$ $\boxed{B = b\log e}$, $\boxed{X = x}$

Normal eqn^s

$$\Sigma Y = nA + B\Sigma X \quad \text{—(1)}$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \text{—(2)}$$

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x	y	$X = x$	$Y = \log y$	XY	X^2
1	10	1	1.0000	1.0000	1
5	15	5	1.1761	5.8805	25
7	12	7	1.0792	7.5537	49
9	15	9	1.1761	10.5849	81
12	21	12	1.3222	15.8664	144
		34	5.7536	40.8855	300

$$5.7536 = 5A + 34B \quad \text{—} (3)$$
$$40.8855 = 34A + 300B \quad \text{—} (4)$$

on solving (3) & (4), we get

$$\boxed{A = 0.9766} \qquad \boxed{B = 0.0255}$$

$$A = \log a \Rightarrow 0.9766 = \log a$$
$$\Rightarrow \boxed{a = 9.4754}$$

$$B = b \log e \Rightarrow b = \frac{0.0255}{\log e} \Rightarrow \boxed{b = 0.0587}$$

$$\Rightarrow \boxed{y = 9.4754\, e^{0.059x}}$$ Ans

L-8 Fit curve $\boxed{y = ax^b}$

Taking log on both sides, $\log y = \log a + b \log$

or $\boxed{Y = A + BX}$

where $\boxed{Y = \log y}$ $\boxed{A = \log a}$ $\boxed{B = b}$ $\boxed{X = \log x}$

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Hence, the normal equ^{n}s

$$\boxed{\begin{aligned}\Sigma Y &= nA + B\Sigma X\\ \Sigma XY &= A\Sigma X + B\Sigma X^2\end{aligned}}$$

- Fit the curve $\boxed{y = ab^x}$

Taking log on both sides, we get

$$\log y = \log a + x \log b$$

or $\boxed{Y = A + BX}$

where, $\boxed{Y = \log y}$ $\boxed{A = \log a}$ $\boxed{X = x}$ $\boxed{B = \log b}$

Hence, the normal equ^{n}s are

$$\boxed{\begin{aligned}\Sigma Y &= nA + B\Sigma X\\ \Sigma XY &= A\Sigma X + B\Sigma X^2\end{aligned}}$$

Q- Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

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Sol^n:

$$y = ab^x$$

$$\Rightarrow \log y = \log a + x \log b$$

$$\Rightarrow \boxed{Y = A + BX}$$

where $\boxed{Y = \log y}$ $\boxed{A = \log a}$ $\boxed{X = x}$ $\boxed{B = \log b}$

Normal Eqn^{n}s

$$\Sigma Y = nA + B\Sigma X \quad \text{--- (1)}$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \text{--- (2)}$$

x	y	$Y = \log y$	$X = x$	XY	X^2
2	8.3	0.9191	2	1.8382	4
3	15.4	1.1875	3	3.5625	9
4	33.1	1.5198	4	6.0792	16
5	65.2	1.8142	5	9.0712	25
6	127.4	2.1052	6	12.6312	36
		$\Sigma Y = 7.5458$	$\Sigma X = 20$	$\Sigma XY = 33.1823$	$\Sigma X^2 = 90$

Here $n = 5$,

$$7.5458 = 5A + 20B$$

$$33.1823 = 20A + 90B$$

$$\Rightarrow \boxed{A = 0.30952} \quad \boxed{B = 0.29991}$$

or $\boxed{A = 0.31}$ $\boxed{B = 0.3}$

$A = \log a$ $\quad B = \log b$

$\Rightarrow a = \text{antilog} A$ $\quad \Rightarrow b = \text{antilog} B$

$\Rightarrow \boxed{a = 2.0417}$ $\quad \Rightarrow \boxed{b = 1.9952}$

$$y = 2.0417(1.9952)^x \text{ Ans}$$

Q- The pressure of the gas corresponding to various values V is measured by the following data:

$V(cm^3)$	50	60	70	90	100
$P(kg\,cm^{-2})$	64.7	51.3	40.5	25.9	78

Fit the data to the eqn $\boxed{PV^{\gamma}=C}$

Sol^n:

$$PV^{\gamma}=C \Rightarrow P=\frac{C}{V^{\gamma}} \Rightarrow P=CV^{-\gamma}$$

Taking log on both sides, we get

$$\log P = \log C - \gamma \log V$$

$$\Rightarrow \boxed{Y = A + BX}$$

where $\boxed{Y=\log P}$ $\boxed{A=\log C}$ $\boxed{B=-\gamma}$ $\boxed{X=\log V}$

Normal eqn's $\Rightarrow$

$$\Sigma Y = nA + B\Sigma X \quad -(1)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad -(2)$$

$V(cm^3)$	$P(kg\,cm^{-2})$	$Y=\log P$	$X=\log V$	XY	X^2
50	64.7	1.8109	1.6990	3.0767	2.8865
60	51.3	1.7101	1.7782	3.0408	3.1618
70	40.5	1.6075	1.8451	2.9659	3.4044
90	25.9	1.4133	1.9542	2.7619	3.8191
100	78	1.8921	2	3.7842	4
		$\Sigma Y = 8.4339$	$\Sigma X = 9.2265$	$\Sigma XY = 15.6236$	$\Sigma X^2 = 17.2718$

Here $n=5$

$$\Rightarrow \quad 8.4339 = 5A + 9.2765B$$
$$15.6296 = 9.2765A + 17.2718B$$

$$\Rightarrow \quad \boxed{A = 2.2276} \quad \boxed{B = -0.2915}$$

$$\gamma = -B \Rightarrow \boxed{\gamma = 0.2915}$$

$$A = \log C \Rightarrow C = \text{antilog}(A) \Rightarrow C = \text{antilog}(2.2276)$$
$$\Rightarrow \boxed{C = 168.888}$$

Hence, the required relation is $\boxed{PV^{0.2915} = 168.888}$ Ans

- <u>Polynomial Fit - Second Degree Parabola</u>

$$\boxed{y = a + bx + cx^2}$$

The normal equ^ns

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad \text{— (1)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{— (2)}$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{— (3)}$$

Q- Fit a second degree polynomial to the following data

x	0	1	2
y	1	6	17

Sol^n ⇒

$$y = a + bx + cx^2$$

The normal equ^{n}s are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad (2)$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad (3)$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	6	1	1	1	6	6
2	17	4	8	16	34	68
$\Sigma x = 3$	$\Sigma y = 24$	$\Sigma x^2 = 5$	$\Sigma x^3 = 9$	$\Sigma x^4 = 17$	$\Sigma xy = 40$	$\Sigma x^2y = 74$

$$24 = 3a + 3b + 5c \quad (4)$$

$$40 = 3a + 5b + 9c \quad (5)$$

$$74 = 5a + 9b + 17c \quad (6)$$

$\boxed{a = 1}$ $\boxed{b = 2}$ $\boxed{c = 3}$

Hence, the required curve is

$$\boxed{y = 1 + 2x + 3x^2}$$ Ans

Other Curve Types

- $y = ax + bx^2$

Normal equ^n

$$\Sigma xy = a\Sigma x^2 + b\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^3 + b\Sigma x^4$$

$y = ax^2 + \frac{b}{x}$ or $y = ax^2 + bx^{-1}$

rmal equ^n

$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x$

$\Sigma x^{-1} y = a\Sigma x + b\Sigma x^{-2}$

$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x$

$\frac{y}{x} = a\Sigma x + b\Sigma \frac{1}{x^2}$

$y = a + \frac{b}{x} + \frac{c}{x^2}$

Normal equ^{n}s

$\Sigma y = na + b\Sigma \frac{1}{x} + c\Sigma \frac{1}{x^2}$

$\Sigma \frac{y}{x} = a\Sigma \frac{1}{x} + b\Sigma \frac{1}{x^2} + c\Sigma \frac{1}{x^3}$

$\Sigma \frac{y}{x^2} = a\Sigma \frac{1}{x^2} + b\Sigma \frac{1}{x^3} + c\Sigma \frac{1}{x^4}$

$y = ax + \frac{b}{x}$

Normal equ^{n}s

$\Sigma xy = a\Sigma x^2 + nb$

$\Sigma \frac{y}{x} = na + b\Sigma \frac{1}{x^2}$

$y = \frac{C_0}{x} + C_1\sqrt{x}$

Normal equ^{n}s

$\Sigma \frac{y}{x} = C_0\Sigma \frac{1}{x^2} + C_1\Sigma \frac{1}{\sqrt{x}}$

$\Sigma \sqrt{x}\, y = C_0\Sigma \frac{1}{\sqrt{x}} + C_1\Sigma x$

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Unit → 3 (continued)

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Correlation

In a bivariate distribution, if the change in one variable affects a change in other variable, the variables are said to be correlated.

If the two variable deviate in same direc^n, correlation is said to be direct or positive.

If the two variable deviate in opposite direction i.e. if the increase in one variable results in a corresponding decrease in the other, correlation is said to be inverse or negative.

Karl Pearson's Coefficient of Correlation

Correlation coefficient b/w two variables x and y is given as:

$$\boxed{r_{xy} = \frac{\Sigma(x_i - \bar{x}) \cdot \Sigma(y_i - \bar{y})}{\sqrt{\Sigma(x_i - \bar{x})^2 \cdot \Sigma(y_i - \bar{y})^2}}}$$

or $$\boxed{r_{xy} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x \sigma_y}}$$

where, $\sigma_x = \text{S.D. of } x = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$

$\sigma_y = \text{S.D. of } y = \sqrt{\frac{\Sigma(y_i - \bar{y})^2}{n}}$

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Another formula

$$r(x,y) = \frac{n\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2} \cdot \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

Note ⇒ Correlation coefficient is independent of change of origin and scale.

If $u = \frac{x-a}{h}$, $v = \frac{y-b}{h}$

then,

$$r(u,v) = \frac{n\Sigma uv - \Sigma u \cdot \Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2} \cdot \sqrt{n\Sigma v^2 - (\Sigma v)^2}}$$

Q- Find the coefficient of correlation b/w the values of x and y.

x	1	3	5	7	8	10
y	8	12	15	17	18	20

Sol⇒

$$r(x,y) = \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

x	y	x^2	y^2	xy
1	8	1	64	8
3	12	9	144	36
5	15	25	225	75
7	17	49	289	119
8	18	64	324	144
10	20	100	400	200
$\Sigma x = 34$	$\Sigma y = 90$	$\Sigma x^2 = 248$	$\Sigma y^2 = 1446$	$\Sigma xy = 582$

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Here, $n = 6$

$$r(x,y) = \frac{6(582) - (34)(90)}{\sqrt{6(248) - (34)^2} \cdot \sqrt{6(1446) - (90)^2}}$$

$$\Rightarrow \boxed{r(x,y) = 0.9879}$$

<u>Rank Correlation</u> It is defined as:

$$\boxed{r = 1 - \left[\frac{6\Sigma D_i^2}{n(n^2-1)}\right]}$$

$n \rightarrow$ no. of observation

$D_i \rightarrow (x_i - y_i)$

and $\Sigma D_i \rightarrow 0$ (serves as a check on calculations)

Rank is given

Compute the rank correlation coefficient for the following

Person	A	B	C	D	E	F	G	H	I	J
Rank in Maths	9	10	6	5	7	2	4	8	1	3
Rank in Physics	1	2	3	4	5	6	7	8	9	10

$$r = 1 - \left[\frac{6\Sigma D_i^2}{n(n^2-1)}\right]$$

Person	R_1	R_2	$D = R_1 - R_2$	D^2
A	9	1	8	64
B	10	2	8	64
C	6	3	3	9
D	5	4	1	1
E	7	5	2	4
F	2	6	-4	16
G	4	7	-3	9
H	8	8	0	0
I	1	9	-8	64
J	3	10	-7	49
			0	280

Here $n = 10$

$$r = 1 - \left[\frac{6 \times 280}{10((10)^2 - 1)}\right]$$

$$\Rightarrow r = 1 - 1.6969$$

$$\Rightarrow \boxed{r = -0.6969}$$

Sol

X
10
15
12
17
13
16
24
14
22

Note

Case II Rank is not given.

Q- The marks secured by recruits in the selection test (X) and in the proficiency test (Y) are given below:

Serial No.	1	2	3	4	5	6	7	8	9
X	10	15	12	17	13	16	24	14	22
Y	30	42	45	46	33	34	40	35	39

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Calculate the rank correlation coefficient

R_1 for X	Y	R_2 for Y	$D = R_1 - R_2$	D^2
9	30	9	0	0
5	42	3	2	4
8	45	2	6	36
3	46	1	2	4
7	33	8	-1	1
4	34	7	-3	9
1	40	4	-3	9
6	35	6	0	0
2	39	5	-3	9
			0	$\Sigma D^2 = 72$

→ • Agr rank given nahi hoti hai to rank assign krni pdti.
• Maximum value ko Rank 1 assign krdo aur descending order mein rank assign krte jao
Series (given) ko ascending / descending order mein nahi diya hai, jaha pr jo value hai uske according rank assign krni hai.

$$r = 1 - \left[\frac{6 \Sigma D_i^2}{n(n^2-1)}\right]$$

where $n = 9$

$$\Rightarrow r = 1 - \left[\frac{6 \times 72}{9(9^2-1)}\right]$$

$$\Rightarrow \boxed{r = 0.4} \text{ Ans}$$

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Case III Tied Rank

When rank are repeated Then a factor is to be added. Hence,

$$r = 1 - 6\left\{\frac{\Sigma D^2 + \frac{1}{12} m_1(m_1^2-1) + \frac{1}{1}}{n(n^2-1)}\right.$$

Q- Obtain the Rank Correlation coefficie

X : 68 64 75 50 64 80 75
Y : 62 58 68 45 81 60 68

Sol^n ⇒

X	R_1 for X	Y	R_2 for Y	$D = R_1 - R_2$
68	4	62	5	-1
64	6	58	7	-1
75	2.5	68	3.5	-1
50	9	45	10	-1
64	6	81	1	5
80	1	60	6	-5
75	2.5	68	3.5	-1
40	10	48	9	1
55	8	50	8	0
64	6	70	2	4
				0

Note → Jb bhi rank repeat hoti hai to, Ae le kr us value ko rank assign krna

Jaise upar X mein 75 do baar repeat ho rhi repeat hoti hata to rank 2, 3 hoti isliye mean leng $\frac{2+3}{2} = 2.5$ aur isko assign ko

djustment

$\frac{1}{2} m_2(m_2^2-1) + \cdots$

t for the data

40	55	64
48	50	70

D^2
1
1
1
1
25
25
1
1
0
16
$\Sigma D^2 = 72$

rithmetic mean
hata hai

a hai ~~to~~ ogr
inka Arithmetic

Next value ko rank 4 assign kroge because 2, 3 use kr li hai rank

No. 64 $\Rightarrow \frac{5+6+7}{3} = 6$ (rank)

Jitni values repeat ho rhi hai $\left\{ \begin{array}{l} m_1 = 2 \\ m_2 = 3 \\ m_3 = 2 \end{array} \right\}$ Jitni baar values repeat hongi

Here the first value is 75 repeated two times so $m_1 = 2$
then second value is 64 repeated three times so $m_2 = 3$
and third repeated value is 68 two times so $m_3 = 2$

Hence,
$$r = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12} m_1(m_1^2-1) + \frac{1}{12} m_2(m_2^2-1) + \frac{1}{2} m_3(m_3^2-1)\right\}}{n(n^2-1)}$$

$$= 1 - \frac{6\left\{72 + \frac{1}{12}(2)(2^2-1) + \frac{1}{12}(3)(3^2-1) + \frac{1}{2}(2)(2^2-1)\right\}}{(10)(10^2-1)}$$

$$= 1 - \frac{6 \times 75}{990} \Rightarrow r = 1 - 0.4545$$

$$\Rightarrow \boxed{r = 0.5454} \text{ Ans}$$

★ Regression

* The term "Regression" stands for some sort of functional relationship b/w two or more related variables.

* The basic difference from curve fitting is that in regression any of the variables may be considered as independent or dependent while in curve fitting one variable cannot be dependent.

* Regression is the ~~[illegible]~~ estimation or prediction of unknown values of one variable from known values of another variable.

→ Linear Regression

- If the relationship between variables is straight line or all the points lies on a straight line, the regression is called linear and the straight line is called as Line of Regression (i)

- Line of Regression of y on x

$$\boxed{y - \bar{y} = b_{yx}(x - \bar{x})}$$

$$\boxed{b_{yx} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}}$$ Regression Coefficient of y on x

Line of Regression of x on y

$$\boxed{x - \bar{x} = b_{xy}(y - \bar{y})}$$

$$\boxed{b_{xy} = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma y^2 - (\Sigma y)^2}}$$ regression coefficient of x on y

Regression coefficient in terms of correlation

$$\boxed{b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}}$$

where r = coefficient of correlation
σ_y = S.D. of y-series
σ_x = standard deviation of x-series

$$\boxed{b_{xy} = r\frac{\sigma_x}{\sigma_y}}$$

Properties of Regression Coefficients

1) Correlation coefficient is the geometric mean of regression coefficients.

$$\sqrt{b_{yx} \cdot b_{xy}} = r$$

or $$\sqrt{r \cdot \frac{\sigma_y}{\sigma_x}} \cdot \sqrt{r \cdot \frac{\sigma_x}{\sigma_y}} = r$$ proved.

2) $b_{yx} \cdot b_{xy} = r^2 \leq 1$.

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(iii) $\frac{b_{yx} + b_{xy}}{2} > r$

Arithmetic Mean of regression coefficients is greater than the correlation coefficient

Angle b/w two lines of Regression

If θ is the acute angle b/w the two regression lines, then

$$\boxed{\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x \cdot \sigma_y}{\sigma_x^2 + \sigma_y^2}}$$

Line of regressions of y on x and x on y are:

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}), \quad x - \bar{x} = \frac{r\sigma_x}{\sigma_y}(y - \bar{y})$$

$y = m_1 x$ $\qquad$ $y = m_2 x$

$$\boxed{m_1 = r\frac{\sigma_y}{\sigma_x}} \qquad \boxed{m_2 = \frac{\sigma_y}{r\sigma_x}}$$

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{\sigma_y}{r\sigma_x} - \frac{\sigma_y r}{\sigma_x}}{1 + \frac{\sigma_y}{r\sigma_x} \cdot \frac{\sigma_y r}{\sigma_x}}$$

$$\Rightarrow \tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

- If the regression coefficients are 0.8 and 0.2 what would be the value of coefficient of correlation?

Sol^n

$$r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{0.8 \times 0.2} \Rightarrow \boxed{r = 0.4}$$ Ans

Q- The following data regarding the heights (y) and weights (x) of 100 college students are given:

$\Sigma x = 15000$, $\Sigma x^2 = 2272500$, $\Sigma y = 6800$

$\Sigma y^2 = 46302$ and $\Sigma xy = 1022250$

Find the equ^n of regression line of height on weight

$$y - \bar{y} = b_{yx}(x - \bar{x}) \qquad \boxed{n = 100}$$

$$b_{yx} = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{100 \times 1022250 - 15000 \times 6800}{100 \times 2272500 - (15000)^2}$$

$$\Rightarrow \boxed{b_{yx} = 0.1}$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15000}{100} \Rightarrow \boxed{\bar{x} = 150} \qquad \bar{y} = \frac{\Sigma y}{n} = \frac{6800}{100} \Rightarrow \boxed{\bar{y} = 68}$$

Required equ^n $y - 68 = 0.1(x - 150)$

$$\Rightarrow y - 68 = 0.1x - 15 \Rightarrow \boxed{y = 0.1x + 53}$$ Ans

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Q- The following results were obtained from records of age (x) and systolic blood pressure (Y) of a group of 10 man.

	x	y
Mean	53	142
Variance	130	

and $\Sigma(x-\bar{x})(y-\bar{y}) = 1220$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.

Sol^n

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

Here, for regression line of y on x we need.

$$b_{yx} = r\cdot\frac{\sigma_y}{\sigma_x} = \left[\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}\right]\cdot\frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x^2}$$

$$\Rightarrow b_{yx} = \frac{1220}{10\times(130)}$$

$$\Rightarrow \boxed{b_{yx} = 0.938}$$

$$\boxed{y-142 = 0.938(x-53)} \Rightarrow y-142 = 0.938x-49.714$$

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$$\boxed{y = 0.938x + 92.286}$$

putting $x = 45$, $y = 0.938 \times 45 + 92.286$

$$\Rightarrow \boxed{y = 134.496}$$

Q- The following table gives age(x) in years of cars and annual maintenance (y) in hundred rupees.

x :	1	3	5	7	9
y :	15	18	21	23	22

Estimate the maintenance cost for a 4 years old car after finding the regression equation.

Sol^n y on x $\quad y - \bar{y} = b_{yx}(x - \bar{x})$

$\bar{x} = \frac{1+3+5+7+9}{5} \Rightarrow \boxed{\bar{x} = 5}$ $\quad \bar{y} = \frac{15+18+21+23+22}{5} \Rightarrow \boxed{\bar{y} = 19.8}$

Here $\boxed{n = 5}$

x	y	x^2	xy
1	15	1	15
3	18	9	54
5	21	25	105
7	23	49	161
9	22	81	198
$\Sigma x = 25$	$\Sigma y = 99$	$\Sigma x^2 = 165$	$\Sigma xy = 533$

$$b_{yx} = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{5 \times 533 - 25 \times 99}{5 \times 165 - (25)^2}$$

$$\Rightarrow \boxed{b_{yx} = 0.95}$$

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$$y - 19.8 = 0.95(x-5)$$

$$\Rightarrow y - 19.8 = 0.95x - 4.75$$

$$\Rightarrow \boxed{y = 0.95x + 15.05}$$

<u>On putting</u> $x = 4$

$$y = 0.95 \times 4 + 15.05$$

$$\Rightarrow \boxed{y = 18.85}$$ maintenance cost

Q- In a partially destroyed laboratory record of an analysis of a correlation data, the following results are legible:

Variance of $x = 9$

Regression equ^{n}s : $8x - 10y + 66 = 0$ $\quad 40x - 18y = 214$

What were (a) mean values of x & y?
(b) standard deviation of y?
(c) coefficient of correlation b/w x & y?

Sol^n

Variance of x, $\sigma_x^2 = 9$

$$\Rightarrow \sigma_x = 3$$

$8x - 10y + 66 = 0$

$\Rightarrow 10y = 8x + 66$

$\Rightarrow \boxed{y = \frac{8}{10}x + \frac{66}{10}} \Rightarrow \boxed{b_{yx} = \frac{8}{10}}$

$40x - 18y = 214$

$40x = 18y + 214$

$\boxed{x = \frac{18}{40}x + \frac{214}{40}}$ $\quad \boxed{b_{xy} = \frac{18}{40}}$

Note → The regression line passes from mean pt.s

$$8\bar{x} - 10\bar{y} = -66$$
$$40\bar{x} - 18\bar{y} = 214$$

On solving $\boxed{\bar{x} = 13}$ $\boxed{\bar{y} = 17}$

$\boxed{\bar{x} = 13}$ $\boxed{\bar{y} = 17}$

$\cancel{b_{yx} = r}$ $r = \sqrt{b_{yx} \cdot b_{xy}} \Rightarrow r = \sqrt{\frac{8}{10} \times \frac{18}{40}}$

$\Rightarrow \boxed{r = 0.6}$

$\boxed{b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}} \Rightarrow \frac{8}{10} = 0.6 \cdot \frac{\sigma_y}{3}$

$\Rightarrow \sigma_y = \frac{8 \times 3}{10 \times 0.6}$

$\Rightarrow \boxed{\sigma_y = 4}$ Ans

Non-Linear Regression

Regression will be called non-linear if their exists a relationship other than a straight line b/w the variables under consideration.

~~Multiple Linear~~

e.g. $y = a + bx + cx^2$

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→ Multiple Linear Regression

- In this, the dependent variable is a func^n of two or more linear or non-linear independent variables.

- Regression equ^n of variable 'y' as a func^n of x and z is

$$\boxed{y = a + bx + cz}$$

Q- Find the multiple linear regression of X_1 on X_2 and X_3 from the data relating to three variables

X_1	4	6	7	9	13	15
X_2	15	12	8	6	4	3
X_3	30	24	20	14	10	4

Sol^n → $X_1 = a + bX_2 + cX_3$

Normal equations

$$\Sigma X_1 = na + \Sigma X_2 + c\Sigma X_3$$

$$\Sigma X_1X_2 = a\Sigma X_2 + b\Sigma X_2^2 + c\Sigma X_2X_3$$

$$\Sigma X_1X_3 = a\Sigma X_3 + b\Sigma X_2X_3 + c\Sigma X_3^2$$

Here $\boxed{n = 6}$

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X_1	X_2	X_3	X_1X_2	X_2X_3	X_3X_1	X_2^2	X_3^2
4	15	30	60	450	120	225	900
6	12	24	72	288	144	144	576
7	8	20	56	160	140	64	400
9	6	14	54	84	126	36	196
13	4	10	52	40	130	16	100
15	3	4	45	12	60	9	16
$X_1 = 54$	$\Sigma X_2 = 48$	$\Sigma X_3 = 102$	$\Sigma X_1X_2 = 339$	$\Sigma X_2X_3 = 1034$	$\Sigma X_3X_1 = 720$	$\Sigma X_2^2 = 494$	$\Sigma X_3^2 = 2188$

$$54 = 6a + 48b + 102c \quad \text{—} (1)$$
$$339 = 48a + 494b + 1034c \quad \text{—} (2)$$
$$720 = 102a + 1034b + 2188c \quad \text{—} (3)$$

by solving.

$\boxed{a = 16.47}$ $\boxed{b = 0.3899}$ $\boxed{c = -0.623}$

Hence,

$$\boxed{X_1 = 16.47 + 0.389X_2 - 0.623X_3}$$ Ans

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★ Trial and Event

* If an experiment is repeated under the same conditions and results in any one of the several possible outcomes.
  Then the experiment is called a Trial.

* The possible outcomes of trial are called events or cases

For example

→ Tossing of a coin is a trial and heading up of head or tail is an event.

→ Throwing a die a trial and turning up 1 or 2 or 3 or 4 or 5 or 6 is an event.

★ Sample Space

- A set of all possible outcomes of an experiment is called Sample Space of that experiment.
- It is denoted by S

★ Exhaustive Events

The total number of all possible outcomes in any trial is known as Exhaustive Events.

e.g. • In tossing a coin, there are two exhaustive

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cases i.e. Head and tail.

→ In throwing a dice, there are 6 exhaustive cases i.e. (1-6)

→ In throwing two dices, the exhaustive cases are $6 \times 6 = 6^2$

★ **Favourable Event**

It is the total number of possible outcomes in which the specified event happens

★ **Mutually Exclusive Events**

Events are said to be mutually exclusive if the happening of any one of them rules out the happening of all others.

★ **Equally Likely Events**

* Events are said to be equally likely if there is no preference given to any event.

For example i) Drawing card from a well shuffled pack, 52 different cases are equally likely.

★ **Independent Events**

If happening of event 'A' has nothing to do with happening of event 'B', then A and B are independent events.

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★ Dependent Events

If outcome of one event (A) is affected by other event (B), then they are called dependent events.

★ Probability

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases, and 'm' of them are favourable to the happening of an event 'E', then the probability of E is

$$p = P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

$$q = P(\bar{E}) = \frac{\text{Unfavourable number of cases}}{\text{exhaustive number of cases}} = \frac{n-m}{n}$$

$q = 1 - p$ $\quad$ $P(E) + P(\bar{E}) = 1$

$\Rightarrow p + q = 1$

Q- A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.

Soln

Exhaustive cases $\Rightarrow {}^{18}C_2$

favourable cases (white ball) $\Rightarrow {}^{7}C_2$

$$p = \frac{\text{favour}}{\text{exhaus.}} = \frac{{}^{7}C_2}{{}^{18}C_2} \Rightarrow p = \frac{7}{51} \quad \text{Ans}$$

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Q- A perfect cube die is thrown. Find the probability that

(i) an even number comes up (ii) a perfect square comes up

Sol^n ⇒ Exhaustive cases $= {}^6C_1 = 6$

(i) favourable cases (even no.) $= {}^3C_1$

$\therefore p = \frac{\text{favour.}}{\text{exhau.}} = \frac{{}^3C_1}{6} \Rightarrow p = \frac{3}{6} = \frac{1}{2}$ Ans

(ii) favourable cases (a perfect square – 1, 4) $= {}^2C_1 = 2$

$p = \frac{\text{favour.}}{\text{Exhaus.}} = \frac{2}{6} \Rightarrow p = \frac{1}{3}$ Ans

L-2A

★ Properties of Probability

1) The probability of any event is always less than or equal to 1 and non-negative i.e.

$$\boxed{0 \leq P(E) \leq 1}$$

2) The probability of certain event (i.e. the event contains all the outcomes) is unity i.e. $P(E) = 1$

3) If E is any event, then the probability of not happening of E is

$$\boxed{P(\bar{E}) = 1 - P(E)}$$

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★ Addition Theorem of Probability (Theorem - Total Probability)

For mutually exclusive events $\Rightarrow P(A+B) = P(A) + P(B)$

Q- An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same color.

Soln

Exhaustive cases $\Rightarrow {}^{20}C_2$

Probability of 2 black balls, $P(A) = \dfrac{{}^{10}C_2}{{}^{20}C_2} = \dfrac{9}{38}$

Probability of 2 white balls, $P(B) = \dfrac{{}^{10}C_2}{{}^{20}C_2} = \dfrac{9}{38}$

Total Probability, $P = P(A) + P(B) = \dfrac{9}{38} + \dfrac{9}{38}$

$\Rightarrow P = \dfrac{18}{38} \Rightarrow \boxed{P = \dfrac{9}{19}}$ Ans

★ Multiplication Law of Probability (Compound Probability)

- The probability of simultaneous occurance of two events is equal to the probability of one of the events multiplied by the conditional probability of the other.

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- Consider two events A and B, then

$$P(A \cap B) = P(A) \times P\left(\frac{B}{A}\right)$$

where $P\left(\frac{B}{A}\right)$ represents the conditional probability of occurence of B when the event A has already happened.

→ If A and B are independent events then

$$P\left(\frac{B}{A}\right) = P(B)$$

or $P(A \cap B) = P(A) \times P(B)$

or $\boxed{P(AB) = P(A) \times P(B)}$

Q- A bag contains four white and two black balls and a second bag contains three of each color. A bag is selected at random and a ball is then drawn at random from the bag chosen. What is the probability that the ball drawn is white?

Ans-

4W 2B	3W 3B
Bag 1	Bag 2

Probability of selecting Bag I $\Rightarrow P(B_1) = \frac{1}{2}$

Probability of selecting ~~Bag~~ Bag II $\Rightarrow P(B_2) = \frac{1}{2}$

Probability of white ball from Bag I $= \frac{{}^4C_1}{{}^6C_1} = \frac{2}{3}$

Probability of white ball from Bag II $= \frac{{}^3C_1}{{}^6C_1} = \frac{1}{2}$

Then, probability of drawing a whiteball from first bag is

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

Similarly, the probability of drawing a white ball from second bag is $= \frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$

Hence, Total Probability $= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ Ans

L-2B

Q- A problem in mechanics is given to three students A, B, C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that the problem will be solved?

Soln

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

The probability that problem will not be solved by anyone

= (Probability of A not solving the problem)

× (Probability of B not solving the problem)

× (Probability of C not solving the problem)

$$= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) = \frac{1}{4}$$

Probability that the problem will be solved

$$= 1-\frac{1}{4} = \frac{3}{4} \text{ Ans}$$

Q- A can hit a target 4 times in 5 shots, B 3 times in 4 shots, C twice in 3 shots. They fire a valley. What is the probability that at least two shots hits?

Sol^n

Probability of A's hitting the target $= \frac{4}{5}$

Probability of B's hitting the target $= \frac{3}{4}$

Probability of C's hitting the target $= \frac{2}{3}$

Sc I A, B, C all hit the target

$$\Rightarrow \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{2}{5}$$

Sc II A & B hit the target and C misses it

$$\Rightarrow \frac{4}{5} \times \frac{3}{4} \times \left(1-\frac{2}{3}\right) = \frac{1}{5}$$

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Case III A and C hit the target, & B misses it

$$\Rightarrow \frac{4}{5} \times \left(1-\frac{3}{4}\right) \times \frac{2}{3} = \frac{2}{15}$$

Case IV B and C hit the target and A misses it

$$= \left(1-\frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$$

Thus, the probability that atleast two shots hits

$$= \frac{2}{5} + \frac{1}{5} + \frac{2}{15} + \frac{1}{10} = \frac{12+6+4+3}{30} = \frac{5}{6} \text{ Ans}$$

Q- An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter.
What is the probability that it is a white ball

Soln

- Let the probability of drawing two balls from first urn and both are white

$$P(A) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{15}{26}$$

- Let the probability of drawing two balls from first urn and both are black

$$\Rightarrow P(B) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{1}{26}$$

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Let probability of drawing two balls from first urn and one is black, one is white.

$$P(C) = \frac{{}^{10}C_1 \times {}^{3}C_1}{{}^{13}C_2} = \frac{5}{13}$$

Now, when two balls are put in the second urn, the possible cases in second urn are:

(i) 5 white & 5 black balls
(ii) 3 white & 7 black balls
(iii) 4 white & 6 black balls

Let the probability of drawing a white ball from second urn is

$$P(W) \Rightarrow \text{i)}\ P\left(\frac{W}{A}\right) = \frac{{}^{5}C_1}{{}^{10}C_1} = \frac{1}{2}$$

$$\text{ii)}\ P\left(\frac{W}{B}\right) = \frac{{}^{3}C_1}{{}^{10}C_1} = \frac{3}{10}$$

$$\text{iii)}\ P\left(\frac{W}{C}\right) = \frac{{}^{4}C_1}{{}^{10}C_1} = \frac{2}{5}$$

Hence, required probability

$$= P(A) \cdot P\left(\frac{W}{A}\right) + P(B) \cdot P\left(\frac{W}{B}\right) + P(C) \cdot P\left(\frac{W}{C}\right)$$

$$= \frac{15}{26} \times \frac{1}{2} + \frac{1}{26} \times \frac{3}{10} + \frac{5}{13} \times \frac{2}{5} = \frac{59}{130}$$

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★ Baye's Theorem

If $E_1, E_2, E_3 ---- E_n$ are mutually exclusive events with $P(E_i) \neq 0$ of a random experiment then for any arbitary event A of the sample space with $P(A) > 0$, we have

$$\boxed{P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{A}{E_i}\right)}}$$

Q- A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

Sol^n

2W 3R	4W 5R
Bag X	Bag Y

Let $E_1 \rightarrow$ ball is drawn from bag X

$E_2 \rightarrow$ ball is drawn from bag Y

$A \rightarrow$ ball is red

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$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\cdot P\left(\frac{A}{E_2}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$.

$P\left(\frac{A}{E_1}\right) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5}$ | $P\left(\frac{A}{E_2}\right) = \frac{{}^5C_1}{{}^9C_1} = \frac{5}{9}$

$$\therefore P\left(\frac{E_2}{A}\right) = \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5} + \frac{1}{2}\times\frac{5}{9}}$$

$$\Rightarrow P\left(\frac{E_2}{A}\right) = \frac{25}{52} \text{ Ans}$$

Q- In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?

Sol^n

Let $E_1 \rightarrow$ selected bolt is manufactured by machine A

$E_2 \rightarrow$ selected bolt is manufactured by machine B

$E_3 \rightarrow$ selected bolt is manufactured by machine C

$A \rightarrow$ defective bolt

As per question

$P(E_1) = 0.25$, $P(E_2) = 0.35$, $P(E_3) = 0.4$

Let A denote that the selected bolt is defective

$P\left(\frac{A}{E_1}\right) = 0.05$, $P\left(\frac{A}{E_2}\right) = 0.04$, $P\left(\frac{A}{E_3}\right) = 0.02$

Required Probability

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \cdot P\left(\frac{A}{E_2}\right)}{P(E_1) P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + P(E_3) P\left(\frac{A}{E_3}\right)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$

$$\Rightarrow P\left(\frac{E_2}{A}\right) = \frac{28}{69} = 0.41 \text{ Ans}$$

Q- The contents of urns I, II and III are as follows:

→ 1 white, 2 black and 3 red balls
→ 2 white, 1 black and 1 red balls
→ 4 white, 5 black and 3 red balls.

One urn is choosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III?

Soln ⇒ Let E_1 → urn I is choosen E_3 → urn III is choosen
E_2 → urn II is choosen

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1W 2B 3R	2W 1B 1R	4W 5B 3R
Urn I	Urn II	Urn III

Let A → two ball selected one is white and one is red

Here $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

$$P\left(\frac{A}{E_2}\right) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$$

$$P\left(\frac{A}{E_3}\right) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + P(E_3) \cdot P\left(\frac{A}{E_3}\right)}$$

$$\Rightarrow P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}}$$

$$\Rightarrow P\left(\frac{E_1}{A}\right) = \frac{33}{118}$$

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Similarly, $P\left(\frac{E_2}{A}\right) = \dfrac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}}$

$$\Rightarrow P\left(\frac{E_2}{A}\right) = \frac{55}{118}$$

$P\left(\frac{E_3}{A}\right) = \dfrac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}}$

$$\Rightarrow P\left(\frac{E_3}{A}\right) = \frac{30}{118} = \frac{15}{59}$$

L-4

★ Discrete Random Variables

The discrete random variable has countable number of distinct values.

For example, $X = \{1, 4, 5, 7\}$ is discrete random variable

★ Probability Distribution & Probability mass function

Probability distribution

X	x_1	x_2	x_3	----------	x_n
$P(X=x)$	$P(X=x_1) = P(x_1)$	$P(X=x_2) = P(x_2)$	$P(X=x_3) = P(x_3)$	----	$P(X=x_n) = P(x_n)$

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$\boxed{P(X=x) = P(x)}$ ⇒ Probability mass func^n

Q- In an experiment, three coins are tossed simultaneously. If the number of heads is the random variable, find the probability func^n for this random variable.

Sol^n

No. of heads ← X	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

★ Mean and Variance of Discrete Random Variable

X : x_1 x_2 x_3 - - - - x_n
P(X): p_1 p_2 p_3 - - - - p_n

then, ~~Mean~~ mean $\boxed{\mu = \Sigma p_i x_i}$ or also called expected value E(X).

and variance $\sigma^2 = \Sigma p_i (x_i - \mu)^2$

and standard deviation, $\sigma = \sqrt{\text{variance}}$

Q- Find the standard deviation for the following discrete distribution:

x :	8	12	16	20	24
p(x) :	$\frac{1}{8}$	1/6	3/8	1/4	1/12

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x	$p(x)$	$x_i p_i$	$(x_i-\mu)^2$	$p_i(x_i-\mu)^2$
8	$\frac{1}{8}$	1	64	8
12	$\frac{1}{6}$	2	16	2.67
16	$\frac{3}{8}$	6	0	0
20	$\frac{1}{4}$	5	16	4
24	$\frac{1}{12}$	2	64	5.33
		$\mu = 16$		$\Sigma = 20$

Variance, $\sigma^2 = \Sigma p_i (x_i - \mu)^2 = 20$

Standard Deviation, $\sigma = \sqrt{\text{Variance}} = \sqrt{20}$
$= 4.4720$ Ans

Q- An urn contains 4 white and 3 red balls. Three balls are drawn, with replacement, from this urn. Find μ, σ^2 and σ for the number of red balls drawn.

Soln Let X be number of red balls

X	0	1	2	3
$p(X)$	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

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0 red balls $\Rightarrow \dfrac{{}^4C_3}{{}^7C_3} = \dfrac{4}{35}$

1 red ball $\Rightarrow \dfrac{{}^4C_2 \times {}^3C_1}{{}^7C_3} = \dfrac{18}{35}$

2 red balls $\Rightarrow \dfrac{{}^4C_1 \times {}^3C_2}{{}^7C_3} = \dfrac{12}{35}$

3 red balls $\Rightarrow \dfrac{{}^3C_3}{{}^7C_3} = \dfrac{1}{35}$

X	P(x)	$x_i p_i$	$(x_i-\mu)^2$	$p_i \cdot (x_i-\mu)^2$
0	$\frac{4}{35}$	0	$\frac{81}{49}$	$\frac{324}{1715}$
1	$\frac{18}{35}$	$\frac{18}{35}$	$\frac{4}{49}$	$\frac{72}{1715}$
2	$\frac{12}{35}$	$\frac{24}{35}$	$\frac{25}{49}$	$\frac{60}{343}$
3	$\frac{1}{35}$	$\frac{3}{35}$	$\frac{144}{49}$	$\frac{144}{1715}$
		$\mu = \frac{9}{7}$		$\Sigma = \frac{24}{49}$

Variance, $\sigma^2 = \Sigma p_i (x_i-\mu)^2 = \dfrac{24}{49}$

S.D., $\sigma = \sqrt{\text{variance}} = \cancel{\sqrt{49} = 7}$

$\sqrt{\dfrac{24}{49}} = 0.7$

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★ Continuous Random Variable

If random variable X takes ~~place~~ on any value in a whole observation interval, X is called a continuous random variable

* The probability distribution of continuous random variable is called Probability Density func^n (PDF) or simply density func^n.

* PDE is represented by $f(x)$ and having the following properties:

(i) $f(x) \geq 0$ for any value of x

(ii) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

(iii) $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x) \cdot dx$

Q- If $f(x)$ has probability density Cx^2, $0 < x < 1$. Determine C and find the probability that $\frac{1}{3} < x < \frac{1}{2}$ i.e.

$P\left(\frac{1}{3} < x < \frac{1}{2}\right)$

Sol^n

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$\Rightarrow \int_{0}^{1} Cx^2 \cdot dx = 1$$

$$\Rightarrow C\left[\frac{x^3}{3}\right]_0^1 = 1 \Rightarrow C \times \frac{1}{3} = 1 \Rightarrow \boxed{C = 3}$$

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$$P(x_1 \leq X < x_2) = \int_{x_1}^{x_2} f(x) \cdot dx$$

$$\Rightarrow P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} 3x^2 \cdot dx$$

$$= \left[x^3\right]_{1/3}^{1/2} = \frac{19}{216} \text{ Ans}$$

★ Mean and Variance of Continuous Random Variable

(i) Mean value about origin $V_1 = \int_a^b x \cdot f(x) \cdot dx \Leftarrow E(X)$

Expected value

where $f(x)$ is PDF of a random variable X where X ranges from a to b

(ii) $V_2 = \int_a^b x^2 \cdot f(x) \cdot dx \Leftarrow E(X^2)$

(iii) Variance $\mu_2 = V_2 - V_1^2 = E(X^2) - (E(X))^2$

(iv) Standard Deviation $= \sqrt{\text{Variance}} = \sqrt{E(X^2) - (E(X))^2}$

Q- A random variable X has the uniform distribution given by.

$$f(x) = \begin{cases} \frac{1}{2\pi} & 0 \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

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Determine $E(x)$, $E(x^2)$ and σ_x.

Sol^n

$$E(x) = \int_a^b x \cdot f(x) \cdot dx = \int_0^{2\pi} x \cdot \frac{1}{2\pi} \cdot dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{\cancel{2}\pi} \times \frac{\cancel{4}\pi^2}{\cancel{2}}$$

$$\Rightarrow \boxed{E(x) = \pi}$$

$$E(x^2) = \int_a^b x^2 f(x) \cdot dx = \int_0^{2\pi} x^2 \cdot \frac{1}{2\pi} \cdot dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\cancel{2}\pi} \times \frac{\cancel{8}^{4}\pi^3}{3}$$

$$\Rightarrow \boxed{E(x^2) = \frac{4\pi^2}{3}} \text{ Ans}$$

Standard Deviation, $\sigma_x = \sqrt{E(x^2) - (E(x))^2} = \sqrt{\frac{4\pi^2}{3} - (\pi)^2}$

$$\Rightarrow \sigma_x = \frac{\pi}{\sqrt{3}} \text{ Ans}$$

Q- A random variable X is distributed at random b/w the values 0 and 1 so that it's probability density function is: $f(x) = kx^2(1-x^3)$, where k is a constant. Find the value of k. Using this value of k, find it's mean and variance.

Sol^n

$$\int_{-\infty}^{\infty} f(x)\cdot dx = 1$$

$$\Rightarrow \int_{0}^{1} k x^2(1-x^3)\cdot dx = 1$$

$$\Rightarrow k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{3} - \frac{1}{6}\right] = 1 \Rightarrow k \times \frac{1}{6} = 1 \Rightarrow \boxed{k=6}$$

$$\therefore \quad f(x) = 6x^2(1-x^3)$$

Mean $\Rightarrow$ $V_1 = \int_a^b x\cdot f(x)\cdot dx = \int_0^1 x\cdot 6x^2(1-x^3)\,dx$

$$= \cancel{6\left[x^3\right.} = 6\int_0^1 (x^3 - x^6)\cdot dx = 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1$$

$$\Rightarrow V_1 = 6\left[\frac{1}{4} - \frac{1}{7}\right] \Rightarrow \boxed{V_1 = \frac{9}{14}}$$

$$V_2 = \int_a^b x^2\cdot f(x)\cdot dx = \int_0^1 x^2\cdot 6x^2(1-x^3)\cdot dx = 6\int_0^1 (x^4 - x^7)\cdot dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 \Rightarrow V_2 = 6\left[\frac{1}{5} - \frac{1}{8}\right] \Rightarrow V_2 = \frac{9}{20}$$

Variance, $\sigma^2 = E(x^2) - (E(x))^2 = \frac{9}{20} - \left(\frac{9}{14}\right)^2 = \frac{9}{245}$ Ans

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★ Binomial Probability Distribution

- discovered by Swiss mathematician Jacob James Bernoulli in the year 1700

- It is concerned with trials of a repetitive nature in which only outcome is
  → occurence / non-occurence
  → success / failure
  → acceptance / rejection

* Let there be 'n' independent trials in an experiment.
Let p be the probability of a success and q be the probability of a failure in a single trial.

* Let a random variable X denote the number of success in 'n' trials.

* The probability of 'r' success in 'n' trials is

$$\boxed{P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}}$$

Recurrence or Recursion Formula

$$\boxed{P(r+1) = \frac{n-r}{r+1} \frac{p}{q} P(r)}$$

Using above formula, we can find P(1), P(2) ---- P(3) --- if P(0) is known

Mean of Binomial Distribution

$$P(r) = {}^nC_r\, q^{n-r} p^r$$

So, mean $\mu = \sum_{r=0}^{n} r \cdot P(r) = \sum_{r=0}^{n} r \cdot {}^nC_r\, q^{n-r} \cdot p^r$

$$= 1\, {}^nC_1\, q^{n-1} p^1 + 2 \cdot {}^nC_2\, q^{n-2} p^2 + \cdots$$
$$\cdots + n\, {}^nC_n\, q^{n-n} p^n$$

~~by using Binomial expansion~~

$$= np\,[q+p]^{n-1}$$

$(\because p+q=1)$

$$\Rightarrow \boxed{\mu = np}$$

Variance of Binomial Distribution

Variance, $\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2$

$$= \left\{\sum_{r=0}^{n} [r + (r-1)r]\, P(r)\right\} - \mu^2$$

$$= \left\{\sum_{r=0}^{n} r P(r)\right\} + \left\{\sum_{r=2}^{n} (r-1) r P(r)\right\} - \mu^2$$

or $\sigma^2 = \mu + \left\{\sum_{r=2}^{n} r(r-1)\, {}^nC_r\, q^{n-r} \cdot p^r\right\} - \mu^2$

$$= \mu + \left[\frac{2n(n-1)}{2\times1} q^{n-2} p^2 + \frac{3\times2\, n(n-1)(n-2)}{3\times2\times1} q^{n-3} p^3 + \cdots\right.$$

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$$= \mu + n(n-1)p^2\left[q^{n-2} + (n-2)q^{n-3}p + \cdots + p^{n-2}\right] - \mu^2$$

$$= \mu + n(n-1)p^2\left[q+p\right]^{n-2} - \mu^2$$

$$= \mu + n(n-1)p^2 - \mu^2 \qquad (\because q+p=1)$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np - np^2 = np(1-p) = npq$$

$$\Rightarrow \boxed{\sigma^2 = npq}$$

- <u>Moments of Binomial Distribution</u>

→ <u>Moment about mean of Binomial Distribution</u>

$\boxed{\mu_1 = 0}$ $\boxed{\mu_2 = npq}$ $\boxed{\mu_3 = npq(q-p)}$

$$\boxed{\mu_4 = 3n^2p^2q^2 + npq(1-6qp)}$$

→ <u>Coefficient of Skewness</u>

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(npq)^2(q-p)^2}{(npq)^3} = \frac{(q-p)^2}{npq}$$

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Thus $\boxed{\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}}}$ gives the measure of skewness

→ Coefficient of Kurtosis

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}}$$

(Ex 7)

Q- Comment on the following:
For a binomial distribution mean is 6 and variance is 2.

Sol^n

$np = 6 \quad \& \quad npq = 2$

$\Rightarrow 6q = 2 \Rightarrow q = \frac{2}{6} = \frac{1}{3} = 1.5$

Here, q is greater than 1, which is not possible as $p+q=1$

∴ The statement is false.

Q- A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of success.

Sol^n

The success is getting 1 or 6 on a roll

∴ Let success $p = \frac{2}{6} \Rightarrow \boxed{p = \frac{1}{3}}$

$q = 1 - \frac{1}{3} \Rightarrow \boxed{q = \frac{2}{3}}$

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No. of trials, $n = 3$

$$\text{Mean} = np \Rightarrow \mu = 3 \times \frac{1}{3} \Rightarrow \boxed{\mu = 1}$$

$$\text{Variance} = npq \Rightarrow \sigma^2 = 3 \times \frac{1}{3} \times \frac{2}{3} \Rightarrow \boxed{\sigma^2 = \frac{2}{3}}$$

Q- If 10% of the bolts produced by a machine are defective determine the probability that out of 10 bolts choosen at random

(i) 1 (ii) None (iii) atmost 2 bolts will be defective

Sol^n

Let p be the probability of defective bolts

$$\therefore \boxed{p = 0.1} \qquad \boxed{q = 0.9}$$

Here, $n = 10$

$$P(r) = {}^nC_r \, q^{n-r} \, p^r$$

(i) $r = 1 \Rightarrow P(1) = {}^{10}C_1 (0.1)^1 (0.9)^{10-1}$

$= 0.3874$ Ans

(ii) $r = 0 \Rightarrow P(0) = {}^{10}C_0 (0.1)^0 (0.9)^{10-0}$

$= 0.3486$ Ans ~~$\Rightarrow 0.3874$ Ans~~

(iii) at most two bolts defective, $P(x \leq 2) = P(0) + P(1) + P(2)$

$P(2) = {}^{10}C_2 (0.1)^2 (0.9)^{10-2} = 0.1937$

$P(x \leq 2) = 0.3486 + 0.3874 + 0.1937 = 0.9297$ Ans

Q- During a war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?

Soln

Let p be the probability of arriving a ship safely.

Given, $\boxed{q = \frac{1}{9}}$ $\therefore$ $\boxed{p = \frac{8}{9}}$

Here, $n = 6$ & $r = 3$

$P(r) = {}^{n}C_r \, p^r q^{n-r}$

$\Rightarrow P(3) = {}^{6}C_3 \left(\frac{8}{9}\right)^3 \left(\frac{1}{9}\right)^{6-3}$

Probability that 3 ships would arrive safely. $\Rightarrow$ $\boxed{P(3) = 0.0193}$ Ans

Q- If the probability of hitting a target is 10% and 10 shots are fired independently. What is the probability that the target will be hit at least one?

Soln

Let p be the probability of hitting the target.

$\therefore$ $\boxed{p = 0.1}$ & $\boxed{q = 0.9}$

Here $\boxed{n = 10}$

Taking into this, $P(0) = {}^{10}C_0 (0.1)^0 (0.9)^{10-0}$

$\Rightarrow P(0) = 0.3486$

Probability that the target will be hit atleast once, $P(x \geq 1) = 1 - P(0) = 1 - 0.3486 = 0.6514$ Ans

6- Six dice are thrown 729 times. How many times do you expect atleast three dice to show a five or six?

Ans-

$\boxed{n = 6}$ $\boxed{N = 729}$

Let p be the probability that the die shows a five or six

$\therefore p = \frac{2}{6} \Rightarrow \boxed{p = \frac{1}{3}}$ & $\boxed{q = \frac{2}{3}}$

$P(x \geq 3) = P(3) + P(4) + P(5) + P(6)$

$= {}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{6-3} + {}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{6-4} + {}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{6-5} + {}^6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^{6-6}$

$\Rightarrow P(x \geq 3) = 0.2195 + 0.0823 + 0.0165 + 0.0014$

$\Rightarrow P(x \geq 3) = 0.3197$

Experiment is repeated 729 times

for 729 times $\Rightarrow 729 \times 0.3197 = 233$ Ans

L-8 ★ Poisson Distribution

* Poisson Distribution is a limiting case of Binomial distribution
* Discovered by S.D. Poisson in 1837
* If n and p are known we can find the probability using binomial distribution
* But as $n \to \infty$ & $p \to 0$ such that np always finite
  Let $np \to \lambda$ we get Poisson distribution

From Binomial Distribution $P(X=r) = {}^nC_r\, q^{n-r} p^r$

$$= \left\{ \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} \right\} (1-p)^{n-r} \cdot p^r$$

$$= \left\{ \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} \right\} \left(1-\frac{\lambda}{n}\right)^{n-r} \left(\frac{\lambda}{n}\right)^r \quad \left(\begin{matrix} np \to \lambda \\ p \to \frac{\lambda}{n} \end{matrix}\right)$$

$$= \frac{\lambda^r}{\lfloor r} \left\{ \frac{n(n-1)(n-2)\cdots(n-r+1)}{n^r} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r} \right\}$$

$$= \frac{\lambda^r}{\lfloor r} \left\{ \cancel{\left(\frac{n}{n}\right)} \cancel{\left(\frac{n-1}{n}\right)} \cancel{\left(\frac{n-2}{n}\right)} \cdots \cancel{\left(\frac{n-r+1}{n}\right)} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\cancel{\left(1-\frac{\lambda}{n}\right)^r}} \right\}$$

$$= \frac{\lambda^r}{\lfloor r} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^r}{\lfloor r} \lim_{n\to\infty} \left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}$$

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Since, $\boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}$ the Naperian Base

Hence, $\boxed{P(X = r) = \frac{\lambda^r e^{-\lambda}}{\lfloor r}}$ $\left(r = 0, 1, 2, 3 \ldots \;\&\; \lambda = np\right)$

Recurrence or Recursion Formula $\boxed{P(r+1) = \frac{\lambda}{r+1} \cdot P(r)}$

- <u>Mean of Poisson Distribution</u>

$$\mu = \sum_{r=0}^{\infty} r \cdot P(r) = \sum_{r=0}^{\infty} r \cdot \frac{\lambda^r}{\lfloor r} \cdot e^{-\lambda}$$

$$= e^{-\lambda} \cdot \sum_{r=1}^{\infty} \frac{\lambda^r}{\lfloor r-1}$$

$$= e^{-\lambda} \left\{ \frac{\lambda}{\lfloor 0} + \frac{\lambda^2}{\lfloor 1} + \frac{\lambda^3}{\lfloor 2} + \frac{\lambda^4}{\lfloor 3} + \cdots \infty \right]$$

$$= \lambda e^{-\lambda} \left\{ 1 + \lambda + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^3}{\lfloor 3} + \cdots \infty \right\}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} \Rightarrow \boxed{\mu = \lambda}$$

- <u>Variance of Poisson Distribution</u>

Variance, $\sigma^2 = \left\{ \sum_{r=0}^{\infty} r^2 \cdot P(r) \right\} - \mu^2$

$$= \sum_{r=0}^{\infty} r^2 \frac{\lambda^r e^{-\lambda}}{\lfloor r} - \mu^2$$

$$= \left[e^{-\lambda} \sum_{r=1}^{\infty} \frac{r^2 \lambda^r}{\lfloor r} \right] - \mu^2$$

$$= e^{-\lambda} \left[\frac{\lambda}{\lfloor 1} + \frac{(2)^2 \lambda^2}{\lfloor 2} + \frac{(3)^2 \lambda^3}{\lfloor 3} + \cdots \infty \right] - \mu^2$$

$$= \lambda e^{-\lambda} \left[1 + 2\lambda + \frac{3\lambda^2}{\lfloor 2} + \frac{4\lambda^3}{\lfloor 3} + \cdots \infty \right] - \mu^2$$

$$= \lambda e^{-\lambda} \left[\left(1 + \lambda + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^3}{\lfloor 3} + \cdots \infty \right) + \left(\lambda + \frac{2\lambda^2}{\lfloor 2} + \frac{3\lambda^3}{\lfloor 3} + \cdots \infty \right) \right] - \mu^2$$

$$= \lambda e^{-\lambda} \left[e^{\lambda} + \lambda e^{-\lambda} \right] - \mu^2$$

$$= \lambda e^{-\lambda} \left[e^{\lambda} + \lambda e^{-\lambda} \right] - \lambda^2$$

$$= \lambda + \cancel{\lambda^2} - \cancel{\lambda^2} \implies \boxed{\sigma^2 = \lambda}$$

- <u>Moments of Poisson Distribution</u>

→ Moments about mean

$\boxed{\mu_1 = 0}$ $\boxed{\mu_2 = \lambda}$ $\boxed{\mu_3 = \lambda}$ $\boxed{\mu_4 = 3\lambda^2 + \lambda}$

→ coefficient of skewness $\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \frac{1}{\sqrt{\lambda}}$

→ coefficient of kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$$

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Q- If the variance of the Poisson Distribution is 2. Find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relation of the Poisson distribution. Also, find $P(r \geq 4)$

Sol^n

Given, $\sigma^2 = 2 \Rightarrow \boxed{\lambda = 2}$ & $\cancel{\mu = 2}$

$$P(r) = \frac{\lambda^r e^{-\lambda}}{\lfloor r}$$

$$\therefore P(0) = \frac{(2)^0 e^{-2}}{\lfloor 0} = 0.1353$$

from recurrence relation $\boxed{P(r+1) = \frac{\lambda}{r+1} P(r)}$

- $r = 0 \Rightarrow P(1) = \lambda \cdot P(0) = 2 \times 0.1353 = 0.2706$
- $r = 1 \Rightarrow P(2) = \frac{\lambda}{2} P(1) = \frac{2}{2} \times 0.2706 = 0.2706$
- $r = 2 \Rightarrow P(3) = \frac{\lambda}{3} \cdot P(2) = \frac{2}{3} \times 0.2706 = 0.1804$
- $r = 3 \Rightarrow P(4) = \frac{\lambda}{4} \cdot P(3) = \frac{2}{4} \times 0.1804 = 0.0902$

Now, $P(r \geq 4) = 1 - [P(0) + P(1) + P(2) + P(3)]$

$= 1 - [0.1353 + 0.2706 + 0.2706 + 0.1804]$

$\boxed{P(r \geq 4) = 0.1491}$ Ans

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Q- Using Poisson Distribution, find the probability that the ace of Spades will be drawn from a pack of well shuffled cards, at least once in 104 consecutive trials.

Soln: Let p be the probability of getting ace of Spades

$\therefore \boxed{p = \frac{1}{52}}$ Here $n = 104$

$\lambda = np = 104 \times \frac{1}{52} \Rightarrow \boxed{\lambda = 2}$

Probability of getting ace of spade atleast once

$\Rightarrow P(r \geq 1) = 1 - P(0) = 1 - \frac{(2)^0 e^{-2}}{\lfloor 0}$

$\Rightarrow P(r \geq 1) = 0.8647$ Ans

***

Q- Fit a Poisson Distribution to the following data and calculate theoritical frequencies:

Deaths:	0	1	2	3	4	
frequencies:	122	60	15	2	1	200

Soln:

$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{60 + 30 + 6 + 4}{200} = 0.5$

Hence, $\boxed{\lambda = 0.5}$

$P(r) = N \cdot \frac{\lambda^r e^{-\lambda}}{\lfloor r} = 200 \cdot \frac{(0.5)^r e^{-0.5}}{\lfloor r}$

r	$N \cdot P(r)$	Theoretical Frequency
0	$200 \times 0.6065 = 121.3$	121
1	$200 \times 0.3033 = 60.66$	61
2	$200 \times 0.0758 = 15.16$	15
3	$200 \times 0.0126 = 2.52$	3
4	$200 \times 0.0016 = 0.32$	0

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Q- A manufacturer knows that the condensors he makes contain on an average 1% of defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 4 or more faulty condensors.

Sol^n Let p be the probability of getting faulty condensor

$\therefore \boxed{p = 0.01}$ $\quad n = 100$

$\therefore \lambda = np = 0.01 \times 100$

$\Rightarrow \boxed{\lambda = 1}$

$\therefore P(r) = \dfrac{\lambda^r e^{-\lambda}}{\lfloor r} = \dfrac{e^{-1}}{\lfloor r}$

Probability that 4 or more faulty condensor will be there

$\Rightarrow P(r \geq 4) = 1 - [P(0) + P(1) + P(2) + P(3)]$ *

$= \cancel{1 - e^{-1}[\ ]} = 1 - e^{-1}\left[\dfrac{1}{\lfloor 0} + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3}\right]$

$\Rightarrow P(r \geq 4) = 1 - 0.981 = 0.019$ Ans

Q- In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Calculate the approximate number of packets containing no defective blades in an assignment of 10,000 packets. (Given $e^{-0.02} = 0.9802$)

Sol^n Let p be the probability of defective blades.

$\therefore \boxed{p = 0.002}$ Here $n = 10$ $\quad \therefore \lambda = np = 10 \times 0.002$

$\Rightarrow \boxed{\lambda = 0.02}$

No defective blades $\Rightarrow$ for $\boxed{r = 0}$

$P(0) = \dfrac{\lambda^0 e^{-0.02}}{\lfloor 0} = e^{-0.02} = 0.9802$

No of packets containing no defective blades

$= 10{,}000 \times 0.9802$

$= 9802$ Ans

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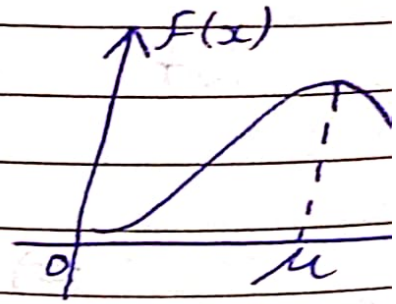
★ Normal Distribution

It is a continuous distribution

The graph of the normal distribution is the normal curve.

It is bell-shaped and symmetrical about x.

The mean, mode and median of this distribution coincide.



The probability density func^n of the normal distribution is given by,

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

where, μ = mean , σ = Standard Deviation

This PDF has following properties:

(i) $f(x) \geq 0$

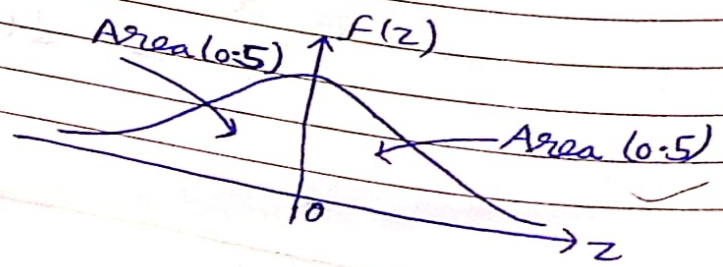
(ii) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

area under the normal curve above x-axis is 1.

Standard Form of Normal Distribution

For standard form, $\boxed{z = \frac{x-\mu}{\sigma}}$ with $\mu = 0$ & $\sigma = 1$

$$\therefore f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



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- Mean of Normal Distribution

$$\bar{x} = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

Let $\frac{x-\mu}{\sqrt{2}\sigma} = t$ or $x = \sqrt{2}\sigma t + \mu$

$\Rightarrow \frac{dx}{\sqrt{2}\sigma} = dt$ or $dx = \sqrt{2}\sigma \cdot dt$

$$\Rightarrow \bar{x} = \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu) \frac{1}{\cancel{\sigma}\cancel{\sqrt{2\pi}}} e^{-t^2} (\cancel{\sqrt{2}}\cancel{\sigma} \cdot dt)$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{2}\sigma}{\sqrt{\pi}} t e^{-t^2} \cdot dt + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{\pi}} e^{-t^2} \cdot dt$$

$$= \underbrace{\frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} \cdot dt}_{\text{odd}} + \underbrace{\frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt}_{\text{even}}$$

$$= 0 + \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\, dt$$

Let $t^2 = z \Rightarrow 2t \cdot dt = dz$

$\Rightarrow dt = \frac{dz}{2t} = \frac{dz}{2\sqrt{z}}$

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$$\Rightarrow \bar{x} = \frac{\cancel{2}\mu}{\sqrt{\pi}} \int_0^\infty e^{-z} \cdot \frac{dz}{\cancel{2}\sqrt{z}}$$

$$\Rightarrow \bar{x} = \frac{\mu}{\sqrt{\pi}} \int_0^\infty z^{-1/2} e^{-z} \cdot dz$$

Gamma func^n

- $\int_0^\infty x^{n-1} e^{-x} \cdot dx = \Gamma n$
- $\Gamma \frac{1}{2} = \sqrt{\pi}$

$$\Rightarrow \bar{x} = \frac{\mu}{\sqrt{\pi}} \int_0^\infty z^{\frac{1}{2}-1} e^{-z} \cdot dz$$

from gamma func^n

$$\Rightarrow \bar{x} = \frac{\mu}{\sqrt{\pi}} \Gamma \frac{1}{2} = \frac{\mu}{\cancel{\sqrt{\pi}}} \times \cancel{\sqrt{\pi}}$$

$$\Rightarrow \boxed{\bar{x} = \mu}$$

Variance of Normal Distribution

first calculate $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$

Let $\frac{x-\mu}{\sqrt{2}\sigma} = t$ or $x = \sqrt{2}\sigma t + \mu$

$$\Rightarrow \frac{dx}{\sqrt{2}\sigma} = dt \quad \text{or} \quad dx = \sqrt{2}\sigma\, dt$$

$$\Rightarrow E(X^2) = \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)^2 \cdot \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}\pi} e^{-t^2} \cdot \cancel{\sqrt{2}}\cancel{\sigma}\, dt$$

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$$= \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)^2 \frac{e^{-t^2}}{\sqrt{\pi}} \, dt$$

$$= \int_{-\infty}^{\infty} \frac{2\sigma^2 t^2}{\sqrt{\pi}} e^{-t^2} dt + \int_{-\infty}^{\infty} \frac{\mu^2}{\sqrt{\pi}} e^{-t^2} dt + \int_{-\infty}^{\infty} \frac{2\sqrt{2}\,\sigma t \cdot \mu}{\sqrt{\pi}} \cdot e^{-t^2} dt$$

$$= \underbrace{\frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt}_{\text{even}} + \underbrace{\frac{\mu^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt}_{\text{even}} + \underbrace{\frac{2\sqrt{2}\sigma\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} \cdot dt}_{\text{odd}}$$

P_1 $\qquad$ P_2 $\qquad$ P_3

on solving P_1

$$P_1 = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} \cdot dt = \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^2 \cdot e^{-t^2} dt$$

Let $t^2 = z \Rightarrow 2t \cdot dt = dz$

$$\Rightarrow dt = \frac{dz}{2t} = \frac{dz}{2\sqrt{z}}$$

$$\Rightarrow P_1 = \frac{\cancel{4}^{2}\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z e^{-z} \frac{dz}{\cancel{2}\sqrt{z}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^{1/2} e^{-z} \cdot dz$$

By gamma func^n

$$\Rightarrow P_1 = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow P_1 = \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{1}{2} \times \sqrt{\pi}$$

$$\Rightarrow \boxed{P_1 = \sigma^2}$$

on solving P_2 we get, $\boxed{P_2 = \mu^2}$

$$\therefore E(X^2) = P_1 + P_2 + P_3$$

$$= \sigma^2 + \mu^2 + 0$$

$$\Rightarrow E(X^2) = \sigma^2 + \mu^2$$

Hence, Variance $= E(X^2) - \bar{x}^2 = \sigma^2 + \mu^2 - \mu^2$

or $\boxed{\text{Variance} = \sigma^2}$

Hence, Standard deviation $= \sqrt{\text{variance}} = \sigma$

Q- A sample of 100 dry battery cells tested to find the length of life produced the following results:

$\bar{x} = 12$ hours, $\sigma = 3$ hours

Assuming the data to the normally distributed, what percentage of battery cells are expected to have life more than 15 hours (ii) less than 6 hours (iii) between 10 and 14 hours.

(i) $z = \frac{x - \mu}{\sigma}$ Here, $x = 15$

$$\Rightarrow z = \frac{15 - 12}{3} \Rightarrow \boxed{z = 1}$$

$P(x>15) = P(z>1) = 0.5 -$ Area b/w $z=0$ and $z=1$

$= 0.5 - 0.3413 = 0.1587$

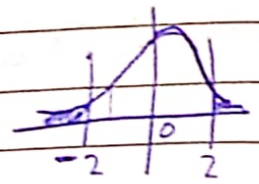
for % $= 0.1587 \times 100 = 15.87\%$

(ii) $P(x<6)$

when $x=6 \Rightarrow z = \frac{6-12}{3} = -2$

$P(x<6) = P(z<-2)$

$= P(z>2)$



$= 0.5 -$ Area b/w $z=0$ & $z=2$

$= 0.5 - 0.4772 = 0.0228$ Ans

for % $= 100 \times 0.0228 = 2.28\%$

(iii) When $x=10$, $z = \frac{10-12}{3} = \frac{-2}{3} = -0.67$

When $x=14$, $z = \frac{14-12}{3} = \frac{2}{3} = 0.67$

$P(10<x<14) = P(-0.67<z<0.67)$

$= 2 \times$ Area b/w $z=0$ and $z=0.67$

$= 2 \times 0.2485 = 0.4970$

for % $= 0.4970 \times 100 = 49.7\%$

Q- Assume mean height of soldiers to be 68·22 inches with a variance 10·8 inches square. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall. Given that the area under the standard normal curve b/w $z=0$ and $z=0.35$ is 0·1368 and b/w $z=0$ and $z=1.15$ is 0·3746.

Soln

$\bar{x} = 68.22$ inches, $\sigma^2 = 10.8$ inches

$\Rightarrow \sigma = \sqrt{10.8} = 3.286$

$x = 6 \times 12 = 72$ inches

$$z = \frac{x-\mu}{\sigma} = \frac{72-68.22}{3.286} \Rightarrow \boxed{z = 1.15}$$

$P(x > 72) = P(z > 1.15) = 0.5 -$ Area b/w $z=0$ and $z = 1.15$

$= 0.5 - 0.3746 = 0.1254$

No. of soldiers $= 0.1254 \times 1000 = 125.4$

$\simeq 126$ Ans

Q- The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are insured, how many pairs would be expected to need replacement after 12 months? (Given $P(z \geq 2) = 0.0228$ and $z = \frac{x-\mu}{\sigma}$)

Soln $\bar{x} = 8$ months $\sigma = 2$ months

$x = 12$ months

$$z = \frac{x-\mu}{\sigma} = \frac{12-8}{2} = \frac{4}{2} \Rightarrow \boxed{z=2}$$ (ii)

$$P(x>12) = P(z>2) \cancel{= 0.5} = 0.0228$$

Number of shoes work above 12 months
$= 0.0228 \times 5000$
$= 114$ pairs

Number of shoes that require replacement after 12 months $= 5000 - 114$
$= 4886$ Prs (iii)

Q- The daily wages of 1000 workers are distributed around a mean of Rs. 140 and with a standard deviation of Rs. 10. Estimate the number of workers whose daily wages will be:

(i) between 140 and Rs. 144

(ii) less than Rs. 126 iii) more than Rs. 160

Sol:

$\bar{x} = 140, \quad \sigma = 10$

(i) $z = \frac{x-\mu}{\sigma}$ $\quad z_1 = \frac{140-140}{10}$ $\quad z_2 = \frac{144-140}{10}$

$\Rightarrow \boxed{z_1 = 0}$ $\quad \boxed{z_2 = 0.4}$

$P(140 < x < 144) = P(0 < z < 0.4)$
$= 0.1554$

No. of workers $= 0.1554 \times 1000 = 155.4$
≈ 155

) $z = \frac{x-\mu}{\sigma} = \frac{126-140}{10} = -1.4$

$P(x < 126) = P(z < -1.4) = P(z > 1.4)$

$= 0.5 -$ Area b/w $z=0$ & $z=1.4$

$= 0.5 - 0.4192 = 0.0808$

No. of workers $= 0.0808 \times 1000$

$= 80.8 \simeq 81$ workers

) $z = \frac{x-\mu}{\sigma} = \frac{160-140}{10} \Rightarrow \boxed{z=2}$

$P(x > 160) = P(z > 2) = 0.5 -$ Area b/w $z=0$ & $z=2$

$= 0.5 - 0.4772$

$= 0.0228$

No. of workers $= 0.0228 \times 1000 = 22.8$

$\simeq 23$ workers

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★ Population / Universe

- The aggregate of objects under study is called population or universe.

- It is a collection of individuals or collection of individuals attributes or collection of results of operations which can be numerically specified.

- Universe can be finite or infinite.

→ **Finite & Infinite Universe**

- A universe containing a finite number of individuals or members is called a finite universe.

  e.g. → Universe of the weights of students in a particular class

- A universe containing infinite number of members is called as infinite universe.

  e.g. → Universe of pressures at various points in atmosphere.

★ Sampling

→ A part selected from the population is called a sample

→ The process of selection of a sample is called sampling.

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* Parameters and Statistics

* The statistical constants of a population such as mean (μ), standard deviation (σ) are called Parameters

* The mean ($\bar{x}$), standard deviation (s) of a sample are known as Statistics.

	Population Parameter	Sample Parameter
Symbols Used	Population Size = N	Sample Size = n
	Mean = μ	Mean = $\bar{x}$
	S.D. = σ	S.D. = s
	Proportion = p	Proportion = $\tilde{p}$

* Hypothesis and Testing a Hypothesis

- Making an assumption about a population parameter is called Hypothesis.

→ Null Hypothesis → The null hypothesis represents a theory that has been put forward either because it is believed to be true or because it is used as a basis for an argument and has not been proven.

- It is denoted by H_0.

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→ Alternative Hypothesis

* Any hypothesis which is complementary to the null hypothesis (H_0) is called an Alternative hypothesis.

- It is denoted by H_1.

→ Testing of Hypothesis

- Collecting sample data
- Calculating sample statistic
- Evaluate / Test the hypothesis using the sample statistic.
  (To know about the correctness of hypothesized population parameter based on the difference between the hypothesized parameter value and the sample value)

★ Test of Significance

- The tests which enables us to define (decide) whether to accept or reject the null hypothesis is called the test of significance.

- The null hypothesis is rejected if the difference between the sample values and the population values are so large

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t-Test - I used for Population mean and sample mean difference

Students t-distribution (t-Test - I)

- The t-test is used when sample size [redacted] $n \leq 30$ and the population standard deviation is unknown.

- t-statistic is defined as:

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n}}} \qquad \text{where } s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}}$$

where $\bar{x}$ is mean of sample
μ is population mean, n is sample size
and s is sample standard deviation.

Q- A machine which produces mica insulating washers for use in electric device to turn out washers having a thickness of 10 mm. A sample of 10 washers has an average thickness 9.52 mm with a standard deviation of 0.6 mm. Find out t?

Ans

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n}}} \qquad \mu = 10\,mm, \quad \bar{x} = 9.52\,mm$$
$$s = 0.6\,mm \qquad n = 10$$

$$t = \frac{9.52 - 10}{0.6/\sqrt{10}} \Rightarrow \boxed{t = -2.529} \text{ Ans}$$

Q- A sample of 20 items has mean 42 and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.

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Sol^n Here, $n = 20$, $\bar{x} = 42$, $s = 5$ & $\mu = 45$

$H_0 : \boxed{\mu = \bar{x}}$

$H_1 : \boxed{\mu \neq \bar{x}}$

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{42 - 45}{5/\sqrt{20}} = -2.68$$

$$\boxed{|t|_{cal} = 2.68}$$

sample size

$|t|_{tab}$ at degree of freedom $= (n-1)$

$= (20-1) = 19$

$\Rightarrow \boxed{|t|_{tab} = 2.09}$

$|t|_{cal} > |t|_{tab}$

$\therefore$ H_0 is rejected.

Q- The 9 items of a sample have the following values:

45, 47, 50, 52, 48, 47, 49, 53, 51

Does the mean of these values differ significantly from the assumed mean 47.5?

Ans-

$$\bar{x} = \frac{45 + 47 + 50 + 52 + 48 + 47 + 49 + 53 + 51}{9}$$

$\Rightarrow \bar{x} = 49.1$

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$\boxed{\mu = 47.5}$

$H_0 : \boxed{\bar{x} = \mu}$

x	45	47	50	52	48	47	49	53	51
$x-\bar{x}$	-4.1	-2.1	0.9	2.9	-1.1	-2.1	-0.1	3.9	1.9
$(x-\bar{x})^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\Sigma(x-\bar{x})^2 = 54.89$$

$$S = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}} = \sqrt{\frac{54.89}{9-1}} = \sqrt{\frac{54.89}{8}}$$

$$\Rightarrow \boxed{S = 2.619}$$

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}} = \frac{49.1-47.5}{2.619/\sqrt{9}} \Rightarrow \boxed{t = 1.83}$$

$$\boxed{|t|_{cal} = 1.83}$$

Degree of freedom $= n-1 = 9-1 = 8$

$$|t|_{tab} = 2.31$$

As, we can see $|t|_{cal} < |t|_{tab}$

H_0 is accepted.

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Q- A manufacturer intends that his electric bulbs have a life of 1000 hours. He tests a sample of 20 bulbs, drawn at random from a batch and discovers that the mean life of the sample bulbs is 990 hours with a S.D. of 22 hours. Does this signify that the batch is not up to the standard.

[Given: Table value of t at 1% level of significance with 19 degree of freedom is 2.539]

(t - Test - II)

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(Test for differences of means of two samples)

This test is used to test whether the two samples $x_1, x_2, --- x_n$ and $y_1, y_2, --- y_n$ of sizes n_1 and n_2 have been drawn from two normal population with mean μ_1 and μ_2 respectively are same.

The test performed under the assumption that the two population variance are equal $(\sigma_1 = \sigma_2 = \sigma)$

Let $\bar{x}$ and $\bar{y}$ be the means of the two samples:

then $$\boxed{t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}}$$

where $$\sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

s_1 & s_2 are samples S.D.

and $\sigma \Rightarrow$ population S.D.

If s_1 and s_2 are not given,

$$\sigma^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

- Two samples of sodium vapour bulbs were tested for length of life and the following results were got:

	Size	Sample Mean	Sample S.D.
e I	8	1234 Hrs	36 Hrs
pe II	7	1036 Hrs	40 Hrs

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Q. Is the difference in the means significant to generalise that type I is superior to type II regarding length of life?

Sol^n

H_0: Two means are equal $(\bar{x} = \bar{y})$

Ans-

Here, $n_1 = 8$ $n_2 = 7$ $S_1 = 36$ $S_2 = 40$

$$t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\sigma^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}$$

$$\Rightarrow \sigma^2 = \frac{(8)(36)^2 + 7(40)^2}{8 + 7 - 2}$$

$$= \frac{1234 - 1036}{40.73\sqrt{\frac{1}{8} + \frac{1}{7}}}$$

$$\Rightarrow \boxed{\sigma = 40.73}$$

$$\Rightarrow t = 18.14$$

$$\boxed{|t|_{cal} = 18.14}$$

$|t|_{tab}$ at $\gamma = n_1 + n_2 - 2 = 8 + 7 - 2 = 13$

$$\Rightarrow |t|_{tab} = 2.16$$

$$|t|_{cal} > |t|_{tab}$$

Hence, H_0 is rejected.

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2- Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of two populations are same at 5% level.

H_0: The means of two populations are same. $(\mu_1 = \mu_2)$

H_1: The means of two populations are not same.

$$\sigma = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2}}$$

$$\Rightarrow \sigma = 4.772$$

$$t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{4.772\sqrt{\frac{1}{10} + \frac{1}{14}}} \Rightarrow t = 0.8604$$

$$\boxed{|t|_{cal} = 0.8604}$$

$|t|_{tab}$ at $(n_1 + n_2 - 2) = 10 + 14 - 2 = 22$

$$\Rightarrow |t|_{tab} = 2.07$$

$$|t|_{cal} < |t|_{tab}$$

Hence, H_0 is accepted i.e. the means of two populations are same

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F-Test (used to check difference b/w variance of two populations)

- In t-test, to find the difference of two means of two samples, drawn from two different population, we assumed that the two population have equal variance.

- The aim of F-test is to determine whether the two samples drawn from the normal population having the same variance.

- Hence, before applying t-test for the significance of the difference of two means, we have to first apply F-test for determining the equality of population variance.

F-Statistics

F-test statistics is defined as:

$$\boxed{F = \frac{\sigma_1^2}{\sigma_2^2}}$$

where $\sigma_1^2 > \sigma_2^2$

Note → If $\sigma_2^2 > \sigma_1^2$ calculate

$$\boxed{F = \frac{\sigma_2^2}{\sigma_1^2}}$$

where $\sigma_1^2 = \frac{n_1 \cdot S_1^2}{n_1 - 1}$ and $\sigma_2^2 = \frac{n_2 \cdot S_2^2}{n_2 - 1}$

σ_1^2 and $\sigma_2^2 \Rightarrow$ population variance and S_1 & S_2 are S.D. of sample with sizes n_1 & n_2

Tabular value of F is calculated for degree of freedom $(n_1 - 1), (n_2 - 1)$

Two random samples reveal the following data:

Sample No.	Size	Mean	Variance
I	16	440	40
II	25	460	42

Test whether the samples come from the same normal population.

F test $H_0: \sigma_1^2 = \sigma_2^2$

$$\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{16 \times (40)}{16-1} \Rightarrow \sigma_1^2 = 42.67$$

$$\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{25 \times 42}{25-1} \Rightarrow \sigma_2^2 = 43.75$$

$$|F|_{cal} = \frac{\sigma_2^2}{\sigma_1^2} = \frac{43.75}{42.67} = 1.02$$

$|F|_{tab}$ at $(n_1 - 1), (n_2 - 1)$

$\Rightarrow 15, 24 \Rightarrow |F|_{tab} = 2.11$

$|F|_{cal} < |F|_{tab} \Rightarrow$ Hence, H_0 is accepted

$$\boxed{\sigma_1^2 = \sigma_2^2}$$

test II $H_0: \bar{x} = \bar{y}$

$$t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{440 - 460}{6.5825\sqrt{\frac{1}{16} + \frac{1}{25}}}$$

$$\Rightarrow t = -9.49$$

$$\boxed{|t|_{cal} = 9.49}$$

$$\sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$\cancel{\sigma^2 = \frac{440 \times (40) + 460 \times (42)}{40 + 42 - 2}}$$

$$\cancel{\Rightarrow \sigma^2 = 461.5}$$

$$\cancel{\Rightarrow \sigma =}$$

$$\sigma^2 = \frac{16 \times 40 + 25(42)}{16 + 25 - 2}$$

$$\Rightarrow \sigma^2 = 43.33$$

$$\Rightarrow \boxed{\sigma = 6.5825}$$

$$|t|_{tab} \; (\gamma = n_1 + n_2 - 2)$$

$$(\gamma = 16 + 25 - 2)$$

$$(\gamma = 39)$$

$$\Rightarrow \boxed{|t|_{tab} = 1.96}$$

$|t|_{cal} > |t|_{tab} \Rightarrow$ Hence, H_0 is rejected

$\Rightarrow$ The samples doesn't come from the same normal population.

Q- Two random samples drawn from 2 normal populations are as follows:

A	17	27	18	25	27	29	13	17
B	16	16	20	27	26	25	21	

Test whether the samples are drawn from the same normal population.

F-test $\quad H_0 : \sigma_1^2 = \sigma_2^2$

B
16
16
20
27
26
25
21

$\Sigma X_2 = 151$

$$\bar{X}_1 = \frac{\Sigma X_1}{n_1} \Rightarrow \bar{X}_1 = \frac{173}{8} \Rightarrow \boxed{\bar{X}_1 = 21.625}$$

$$\bar{X}_2 = \frac{\Sigma X_2}{n_2} \Rightarrow \bar{X}_2 = \frac{151}{7} \Rightarrow \boxed{\bar{X}_2 = 21.571}$$

X_1	$X_1 - \bar{X}_1$	$(X_1 - \bar{X}_1)^2$	X_2	$X_2 - \bar{X}_2$	$(X_2 - \bar{X}_2)^2$
17	-4.625	21.39	16	-5.571	31.04
27	5.38	28.89	16	-5.57	31.04
18	-3.625	13.14	20	-1.571	2.47
25	3.38	11.39	27	5.431	29.47
27	5.38	28.89	26	4.43	19.61
29	7.38	54.39	25	3.429	11.756
13	-8.625	74.39	21	-0.571	0.36
17	-4.625	21.39			
		$\Sigma(X_1 - \bar{X}_1)^2 = 253.88$			$\Sigma(X_2 - \bar{X}_2)^2 = 125.71$

$$\sigma_1^2 = \frac{\Sigma(X_1 - \bar{X}_1)^2}{n_1 - 1} = \frac{253.88}{8-1} \Rightarrow \sigma_1^2 = 36.26$$

$$\sigma_2^2 = \frac{\Sigma(X_2 - \bar{X}_2)^2}{n_2 - 1} = \frac{125.71}{7-1} \Rightarrow \sigma_2^2 = 20.95$$

$$F = \frac{\sigma_1^2}{\sigma_2^2} = \frac{36.26}{20.95} = 1.73$$

$$\boxed{|F|_{cal} = 1.73}$$

$|F|_{tab}$ at $(n_1-1),(n_2-1) \Rightarrow |F|_{tab}$ at $7, 6$

$$|F|_{tab} = 4.21$$

$|F|_{cal} < |F|_{tab} \Rightarrow H_0$ is accepted

<u>t-test</u>

$H_0: \bar{x} = \bar{y}$

$$\sigma = \sqrt{\frac{\Sigma(X_1-\bar{X}_1)^2 + \Sigma(X_2-\bar{X}_2)^2}{n_1+n_2-2}}$$

$$= \sqrt{\frac{253.88 + 125.71}{8+7-2}} \Rightarrow \sigma = 5.40$$

$$t = \frac{\bar{x}-\bar{y}}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{21.625 - 21.571}{5.40\sqrt{\frac{1}{8}+\frac{1}{7}}}$$

$$\Rightarrow t = 0.02$$

$$\boxed{|t|_{cal} = 0.02}$$

$|t|_{tab}$ at $(\gamma = n_1 + n_2 - 2)$
$(\gamma = 8 + 7 - 2)$
$(\gamma = 13)$

$$\boxed{|t|_{tab} = 2.16}$$

$|t|_{cal} < |t|_{tab} \Rightarrow$ Hence, H_0 is accepted
$\Rightarrow$ i.e. $\bar{x} = \bar{y}$

Hence, the samples are drawn from the same normal population

Chi-Square Test

(as a Goodness of Fit)

The quantity (χ^2) describes the magnitude of discrepancy between theory and observations.

If O_i $(i = 1, 2, 3 - - - - n)$ is a set of observed (experimental) frequencies and $E_i (i = 1, 2, 3 - - - n)$ is the corresponding set of expected (theoretical) frequencies then

$$\boxed{\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}}$$

(degree of freedom $= n - 1$)

- In experiment on pea breeding, the following frequencies of seeds are obtained:

Round & Yellow	Wrinkled and Yellow	Round & Green	Wrinkled & Green	Total
315	101	108	32	556

Theory predicts that the frequencies should be in proportions 9:3:: 3:1. Examine the correspondence between theory and experiment.

Ans- Theoretical frequencies

$E_A = \frac{9}{16} \times 556 = 312.75$

$E_B = \frac{3}{16} \times 556 = 104.25$

$E_C = \frac{3}{16} \times 556 = 104.25$

$E_D = \frac{1}{16} \times 556 = 34.75$

No: There is no difference b/w theory and experiment

Ans-

	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
A	315	312.75	2.25	5.0625	0.0161
	101	104.25	-3.25	10.56	0.1013
	108	104.25	3.75	14.0625	0.1348
	32	34.75	-2.75	7.5625	0.2175
					$\Sigma = 0.47$

Note ⇒ $\sum O_i = \sum E_i$ (check karne ke liye ki sahi hai ya nahi)

$$\therefore \quad \chi^2 = \frac{\sum (O_i - E_i)^2}{E_i} = 0.47$$

χ^2_{tab} at $(n-1)$

$\Rightarrow \chi^2_{tab}$ at $(4-1)$

$\Rightarrow \chi^2_{tab}$ at 3

$\chi^2_{tab} = 7.815$

$\boxed{\chi^2_{cal} < \chi^2_{tab}}$ $\Rightarrow$ H_0 is accepted

A die is thrown 276 times and the results of these throws are given below:

No. appeared on the die	1	2	3	4	5	6
Frequency	40	32	29	59	57	59

Test the whether die is biased or not.

H_0: Die is not biased

$$E_i = \frac{276}{6} = 46$$

O_i	40	32	29	59	57	59	
E_i	46	46	46	46	46	46	
$\frac{(O_i-E_i)^2}{E_i}$	36	196	289	169	121	169	980

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$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i} \right] = \frac{980}{46} = 21.30$$

$$\boxed{\chi^2_{cal} = 21.30}$$

χ^2_{tab} at $(n-1) \Rightarrow \chi^2_{tab}$ at 5

$$\boxed{\chi^2_{tab} = 11.07}$$

Hence, H_0 is rejected.

Chi-Square Test as a Test of Independence

- With the help of χ^2 test, we find whether two attributes are associated or not.
- We select Null Hypothesis, as there is no association or two attributes are independent
- If $\chi^2_{cal} < \chi^2_{tab}$, then H_0 is accepted, else H_0 is rejected or we conclude that two attributes are dependent on each other

$$\boxed{\chi^2_{cal} = \sum \left[\frac{(O_i - E_i)^2}{E_i} \right]}$$

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What are expected frequency of 2X2 contigency table

a	b	(a+b)
c	d	(c+d)
(a+c)	(b+d)	a+b+c+d

$$E_a = \frac{(a+c)(a+b)}{a+b+c+d} \qquad E_b = \frac{(b+d)(a+b)}{a+b+c+d}$$

$$E_c = \frac{(c+d)(a+c)}{a+b+c+d} \qquad E_d = \frac{(b+d)(c+d)}{(a+b+c+d)}$$

Ex

Q- From the following table, regarding the color of eyes of father and son, test if the color of son's eyes is associated with that of father

		Eye color of son.	
		Light	Not Light
Eye color of father	Light	471	51
	Not Light	148	230

Sol^n

	Light	Not Light	Total
Light	471	51	522
Not Light	148	230	378
Total	619	281	900

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$$E(471) = \frac{619 \times 512}{900} = 359.02$$

$$E(51) = \frac{281 \times 522}{900} = 162.98$$

$$E(148) = \frac{619 \times 378}{900} = 259.98$$

$$E(230) = \frac{281 \times 378}{900} = 118.02$$

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2 / E_i$
471	359.02	111.98	34.93
51	162.98	-111.98	76.94
148	259.98	-111.98	48.23
230	118.02	111.98	106.25

$$\sum \frac{(O_i - E_i)^2}{E_i} = 266.35$$

χ^2_{tab} at degree of freedom

$(n_1 - 1)(n_2 - 1)$

$\Rightarrow (2-1)(2-1)$

$= 1$

$n_1 \times n_2$

size of given matrix

$$\chi^2_{tab} = 3.841$$

$\chi^2_{cal} > \chi^2_{tab} \Rightarrow$ Hence, H_0 is rejected i.e. eyes color of Father and Son are associated.

Q- To test the effectiveness of inoculation against cholera, the following table was obtained:

	Attacked	Not Attacked	Total
Inoculated	30	160	190
Un-Inoculated	140	460	600
Total	170	620	790

Use χ^2 test to defend or repute the statement that the inoculation prevents attacks from cholera.

Ans H_0: There is no association between inoculation & cholera

$$E(30) = \frac{190 \times 170}{790} = 40.89$$

$$E(160) = \frac{190 \times 620}{790} = 149.1$$

$$E(140) = \frac{600 \times 170}{790} = 129.11$$

$$E(460) = \frac{600 \times 620}{790} = 470.89$$

O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2 / E_i$
30	40.89	-10.89	2.90
160	149.11	10.89	0.80
140	129.11	10.89	0.92
460	470.89	-10.89	0.25
			$\Sigma = 4.87$

$$\chi^2_{cal} = 4.87$$

χ^2_{tab} at $(n_1 - 1)(n_2 - 1)$

χ^2_{tab} at 1

$$\Rightarrow \chi^2_{tab} = 3.841$$

$$\chi^2_{cal} > \chi^2_{tab}$$

Hence, H_0 is rejected.

$\therefore$ The inoculation prevents attacks from ~~Chole~~ ~~Cholera~~ Cholera.

ANOVA (Analysis of Variance)

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ANOVA is a statistical technique which can be used to make comparisons among more than two groups.

In this, we calculate the variance of total population, variance between the groups and the variance within the groups.

Then we examine whether there is any significant difference between the calculated variances.

Anova Table This table shows all the required parameter and the computation of F-ratio required for ANOVA.

Source of Variation	Sum of Squares	Degree of Freedom	Mean Sum of Squares	Variance Ratio or F-ratio
Between Samples				
Within Samples				

It is desired to compare three hospitals w.r.t the number of deaths per month. A sample of death records were selected from the records of each hospital and the number of deaths was as given below.

From these data, suggest a difference in the

No. of deaths per month among three hospitals

Hospitals		
A	B	C
3	6	7
4	3	3
3	3	4
5	4	6
0	4	5

Soln

Null Hypothesis H_0: There is no difference in the number of deaths per month among three hospitals.

Step I Grand Total

$\Sigma A = 3+4+3+5+0 = 15$

$\Sigma B = 6+3+3+4+4 = 20$

$\Sigma C = 7+3+4+6+5 = 25$

$$\boxed{G.T. = \Sigma A + \Sigma B + \Sigma C}$$

$G.T. = 15+20+25 = 60$

Step II Correction factor

$$\boxed{C.F. = \frac{(G.T.)^2}{n}} = \frac{(60)^2}{15} = 240$$

Total no. of sample items

Step III. Sum of squares of samples

$\Sigma A^2 = 3^2 + 4^2 + 3^2 + 5^2 + 0 = 59$

$\Sigma B^2 = 6^2 + 3^2 + 3^2 + 4^2 + 4^2 = 86$

$\Sigma C^2 = 7^2 + 3^2 + 4^2 + 6^2 + 5^2 = 135$

Total sum of squares $= \Sigma A^2 + \Sigma B^2 + \Sigma C^2 - C.F.$

$= 59 + 86 + 135 - 240 = 40$

Step IV Sum of squares between samples

$= \frac{(\Sigma A)^2}{n_1} + \frac{(\Sigma B)^2}{n_2} + \frac{(\Sigma C)^2}{n_3} - C.F.$

$= \frac{(15)^2}{5} + \frac{(20)^2}{5} + \frac{(25)^2}{5} - 240 = 10$

Step V Sum of squares within samples

= Total sum of squares − Sum of squares b/w samples

$= 40 - 10 = 30$

Step VI Degree of freedom

$n \rightarrow$ no. of samples

$K \rightarrow$ no. of hospitals / groups

between samples $\rightarrow$ d.f. $= (k-1) = 2$

within samples $\rightarrow$ d.f. $= (n-k) = 12$

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Source of Variation	Sum of Squares	Degree of freedom	Mean sum of squares	F-ratio
• b/w samples	10	2	$\frac{10}{2} = 5$	$F = \frac{5}{2.5} = 2$
• within samples	30	12	$\frac{30}{12} = 2.5$	

$F_{2,12}$ (tab) $= 3.89$

$\boxed{F_{cal} < F_{tab}}$ Hence, H_0 is accepted.

Therefore, there is no difference in the number of deaths per month among three hospitals

**Q- A manufacturing company purchased three new machines of different makes and wishes to determine whether one of the machine is faster than the others is producing a certain output. Five hourly production figures are observed at random from each machine and results are given below.

Observations	A_1	A_2	A_3
1	25	31	24
2	30	39	30
3	36	38	28
4	38	42	25
5	31	35	28

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Use Anova and determine whether the machines are significantly different in their mean speed.

(Given $F_{2,13}$ at 5% level = 3.89)

H_0: The machines are not significantly different in their mean speed

- Grand Total

$$\Sigma A = 25+30+36+38+31 = 160$$
$$\Sigma B = 31+39+38+42+35 = 185$$
$$\Sigma C = 24+30+28+25+28 = 135$$

$$G.T. = \Sigma A + \Sigma B + \Sigma C = 160+185+135 = 480$$

$$C.F. = \frac{(G.T.)^2}{n} = \frac{(480)^2}{15} = 15360$$

Sum of Squares of samples

$$= \Sigma A_1^2 + \Sigma A_2^2 + \Sigma A_3^2 - C.F.$$

$$= 5226 + 6915 + 3669 - 15360 = 450$$

Sum of Square between samples

$$= \frac{(\Sigma A_1)^2}{n_1} + \frac{(\Sigma A_2)^2}{n_2} + \frac{(\Sigma A_3)^2}{n_3} - C.F.$$

$$= \frac{(160)^2}{5} + \frac{(185)^2}{5} + \frac{(135)^2}{5} - 15360$$

$$= 250$$

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- Sum of squares within samples
= Total sum of squares - sum of b/w

$= 450 - 250 = 200$

- Degree of freedom

$n \rightarrow$ no. of samples, $k \rightarrow$ no. of

b/w the samples $= k - 1 = 3 - 1$

within the samples $= \overset{n-k}{15 - 3} = 1$

Source of variation	Sum of squares	Degree of freedom	Mean sum of squares
b/w samples	250	2	$\frac{250}{2} = 125$
within samples	200	12	$\frac{200}{12} = 16.67$

$F_{(2,12)}$ (tab) $= 3.89$

As, $\boxed{F_{cal} > F_{tab}} \Rightarrow$ Hence, H_0 is

$\therefore$ The machines have significant d
b/w in their mean speed.

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Statistical Quality Control

- A quality control system performs inspection, testing and analysis to ensure that the quality of the products is as per quality standards.

- The quality control system is called Statistical quality control when statistical techniques are employed to control, improve and maintain quality or to solve quality problems.

★ Control Chart

- A control chart is a graphical representation of the collected information.

- It detects the variation in processing and warns if there is any departure from the specified tolerance limits.

→ Different Control charts

- Control chart for variables ⇒ $\bar{X}$ and R chart

- Control chart for attributes ⇒ P, np, C chart

★ X' chart

- CL (control line) = $\bar{\bar{x}}$, when tolerance limits are not given

  μ, when tolerance limits are given

where $\boxed{\mu = \frac{1}{2}\left[LCL + UCL\right]}$

(Lower control limit) $LCL = \bar{\bar{x}} - A_2 \bar{R}$

(Upper control limit) $UCL = \bar{\bar{x}} + A_2 \bar{R}$

where A_2 is constant and depend on sample size

★ R- chart

Central line (CL) $= \bar{R}$

$LCL = D_3 \bar{R}$ and $UCL = D_4 \bar{R}$

where D_3 and D_4 depend on sample size

→ Process Capability $\boxed{6\sigma = \frac{\bar{R}}{d_2}}$ where σ is standard deviation

Q- The following are the mean lengths and ranges of lengths of a finished product from 10 samples each of size 5.
The specification limits for length are 200 ± 5 cm
Construct $\bar{X}$ and R chart and examine whether the process is under control and state your recommendations.

Sample No.	1	2	3	4	5	6	7	8	9	10
Mean ($\bar{X}$)	201	198	202	200	203	204	199	196	199	201
Range (R)	5	0	7	3	3	7	2	8	5	6

Assume for $n = 5$, $A_2 = 0.58$, $D_4 = 2.11$, $D_3 = 0$

- $\bar{X}$ chart Since the tolerance limits are given

$$\mu = \frac{1}{2}\left[LCL + UCL\right] = \frac{1}{2}\left[195 + 205\right]$$

$$\Rightarrow \boxed{\mu = 200}$$

$$CL = \mu = 200$$

$$UCL_{\bar{x}} = \bar{\bar{x}} + A_2\bar{R} = \mu + A_2\bar{R} \quad \text{and} \quad LCL_{\bar{x}} = \bar{\bar{x}} + A_2\bar{R} = \mu + A_2\bar{R}$$

where $\bar{R} = \dfrac{5+0+7+3+3+7+2+8+5+6}{10} = 4.6$

Hence, $UCL_{\bar{x}} = 200 + 4.6(0.58) = 202.668$

$LCL_{\bar{x}} = 200 - 4.6(0.58) = 197.332$

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- For R- chart $CL = \bar{R} = 4.6$

$UCL_R = D_4 \bar{R} = 2.11 \times 4.6 = 9.706$

$LCL_R = D_3 \bar{R} = 0 \times 4.6 = 0$

→ By observation, we can conclude that, all points lie within the control limits on R- chart. Hence the process is under control.

→ But for $\bar{X}$ chart, sample 5 and sample 6 are lie over the $UCL_{\bar{x}}$ and sample 8 lies below the $LCL_{\bar{x}}$. Hence, the process is not in statistical control.

Q- Determine the control limits for $\bar{X}$ and R charts if $\Sigma\bar{X} = 357.50$, $\Sigma R = 9.90$, number of subgroups = 20. It is given that $A_2 = 0.18$ $D_3 = 0.41$, $D_4 = 1.59$ and $d_2 = 3.736$.

Also, find the process capability.

Ans

$$\bar{\bar{X}} = \frac{\Sigma \bar{X}}{20} = \frac{357.50}{20} \Rightarrow \bar{\bar{X}} = 17.875$$

$$\bar{R} = \frac{\Sigma R}{20} = \frac{9.90}{20} \Rightarrow \bar{R} = 0.495$$

$$UCL_{\bar{x}} = \bar{\bar{x}} + A_2 \bar{R} = 17.875 + 0.18 \times 0.495$$

$$\Rightarrow UCL_{\bar{x}} = 17.9641 \text{ Ohm}$$

$$LCL_{\bar{x}} = \bar{\bar{x}} - A_2 \bar{R} = 17.875 - 0.18 \times 0.495$$

$$\Rightarrow LCL_{\bar{x}} = 17.7859 \text{ Ohm}$$

$$UCL_R = D_4 \bar{R} = 1.59 \times 0.495 = 0.78705$$

$$LCL_R = D_3 \bar{R} = 0.41 \times 0.495 = 0.20295$$

$$\sigma = \frac{\bar{R}}{d_2} = \frac{0.495}{3.736} = 0.1325$$

Process Capability $\Rightarrow$ $6\sigma = 6 \times 0.1325$

$= 0.795$ Ohm

p chart

A control chart based on fraction defective $\bar{p}$ is called p-chart

$$\bar{p} = \frac{\text{No. of defective articles found in inspection}}{\text{Total no. of articles inspected}}$$

<u>Control limits</u> (3σ limits)

- $\boxed{CL = \bar{p}}$

- $UCL_p = \bar{p} + 3\sigma_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

- $LCL_p = \bar{p} - 3\sigma_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

where n is sample size.

★ <u>np chart</u>

* When sample size is variable, p-chart is used.

* For constant sample size, np chart is used.

$\boxed{CL_{np} = n\bar{p}}$ where $\bar{p} = \frac{\Sigma np}{\Sigma n}$

$UCL_{np} = n\bar{p} + 3\sigma_{np} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$

$LCL_{np} = n\bar{p} - 3\sigma_{np} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$

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Defect → Any characteristics that does not meet the specifications is a defect.

Defective → An item is called defective if it contains atleast one defect.

* The np chart applies to the number of defectives in subgroup of constant size.

* The C chart applies to the number of defects in a subgroup of constant size.

* C chart

$$\boxed{CL = \bar{c}}$$

where,

$$\boxed{UCL_c = \bar{c} + 3\sqrt{\bar{c}}}$$

$$\boxed{LCL_c = \bar{c} - 3\sqrt{\bar{c}}}$$

$\bar{c} = \frac{\text{No. of defects in all samples}}{\text{Total no. of samples}}$

Q- In a blade manufacturing factory, 1000 blades are examined daily. Draw the np chart for the following table and examine whether the process is under control?

Date	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No. of defective blades	9	10	12	8	7	15	10	12	10	8	7	13	14	15	16

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Sol:

→ 1000 blades inspected in one day

→ Blades inspected in 15 days = 15 × 1000 = 15,000

$$\bar{p} = \frac{\text{No. of blades defective}}{\text{Total no. of articles inspected}} = \frac{166}{15000}$$

$$\Rightarrow \boxed{\bar{p} = 0.011}$$

Here, $n = 1000$

$$CL_{np} = n\bar{p} = 1000 \times 0.011 = 11$$

$$UCL_{np} = n\bar{p} + 3\sigma_{np} = n\bar{p} + \sqrt{n\bar{p}(1-\bar{p})}$$

$$\Rightarrow UCL = 11 + 3\sqrt{11(1-0.011)} = 20.89$$

$$LCL_{np} = n\bar{p} - 3\sigma_{np} = n\bar{p} - \sqrt{n\bar{p}(1-\bar{p})}$$

$$\Rightarrow LCL = 11 - 3\sqrt{11(1-0.011)}$$

$$= 1.105$$

→ The process is under control.

Q- In a factory producing spark plug, the number of defectives found in inspection of 20 lots of 100 each is given below:

Lot No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of defectives	5	10	12	8	6	4	6	3	3	5	4	7	8	3	3	4	5	8	6	10

Construct p-chart and state whether the process is in statistical control.

Soln Total number of defective articles =

$5+10+12+8+6+4+6+3+3+5+4+7+8+3+3+4+5+8+6+10 = 120$

Total articles $= 20 \times 100 = 2000$

$\bar{p} = \frac{120}{2000} = 0.06$ Here $n = 100$

$\boxed{CL = 0.06}$

$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

$= 0.06 + 3\sqrt{\frac{0.06(1-0.06)}{100}}$

$\Rightarrow UCL = 0.131$

$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.06 - 3\sqrt{\frac{0.06(1-0.06)}{100}} = -0.011$

$\simeq 0$

By plotting graph, we will see no defective fraction passes LCL or UCL, therefore the process is under control.

$\bar{X}$ chart for mean) (R chart for Range)

(p & np chart for number of defectives) (C chart for no. of defects)

Q- The following table shows the number of missing rivets observed at the time of the inspection of 12 aircrafts. Find the ~~[illegible]~~ control limit for the number of defects chart and comment on the state of control.

Aircraft No.	1	2	3	4	5	6	7	8	9	10	11	12
No. of missing rivets	7	15	13	18	10	14	13	10	20	11	22	15

Sol: Total no. of defects $= 7+15+13+18+10+14+13+10+20+11+22+15$

$= 168$

Total no. of samples $= 12$

$$\bar{C} = \frac{168}{12} = 14$$

$$UCL = \bar{C} + 3\sqrt{\bar{C}} = 14 + 3\sqrt{14} = 25.23$$

$$LCL = \bar{C} - 3\sqrt{\bar{C}} = 14 - 3\sqrt{14} = 2.775$$

→ By plotting graph we see all values are higher than LCL and lower than UCL

Hence, the process is under control.

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