Episode

$O(nlogn)$ Closest Pair and Mathematical Induction

Closest Pair: $O(n \log n)$

- Solution 1: From 黄相阁 and 李竞翔
  - Use Merge Sort idea to maintain the sorted-by-y lists.
  - In the combine step, we also merge two sorted lists.
- Solution 2: From 王煌基
  - Sort by x and Sort by y at the beginning.

Solution 1

Function ClosestPair(S)

- Sort the points (by the x-coordinate) and draw all the vertical lines.
- Divide:
  ~~1. Sort the points (by the x-coordinate).~~
  ~~2. Draw such a **vertical line** ℓ that~~ each side has $n/2$ points.
- Recurse
  3. Find the closest pair in each side, let δ_L, δ_R be the distance.
- Combine
  4. Let $\delta = \min\{\delta_L, \delta_R\}$ and S' be the set of points at most δ from ℓ.
  ~~5. Sort S' by the y-coordinate.~~
  6. Merge two sorted-by-y point lists
  7. For each $a \in S'$, check 7 b above a inside S', find the closest pair.
  8. Return the closest pair among step 3 and 6.

Solution 2

Function ClosestPair(S)

- Sort the points (by the x-coordinate) and draw all the vertical lines.
- Sort the points (by the y-coordinate).
- Divide:
  1. ~~Sort the points (by the x-coordinate).~~
  2. ~~Draw such a **vertical line** ℓ that each side has $n/2$ points.~~
  3. Loop the sorted-by-y point lists and make two sorted-by-y sublists.
- Recurse
  3. Find the closest pair in each side, let δ_L, δ_R be the distance.
- Combine
  4. Let $\delta = \min\{\delta_L, \delta_R\}$ and S' be the set of points at most δ from ℓ.
  5. The list is sorted by y.
  6. For each $a \in S'$, check 7 b above a inside S', find the closest pair.
  7. Return the closest pair among step 3 and 6.

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n\log n)$$

- Solution 1: Explore

- $T(n) = 2T\left(\frac{n}{2}\right) + O(n\log n) \le 4\mathrm{T}\left(\frac{\mathrm{n}}{4}\right) + C \cdot n\log n\ + 2C \cdot \frac{n}{2}\log\frac{n}{2}$

$$= 2^{\log n}\mathrm{T}(1) + \mathrm{C}\cdot\left(n\log n + n\log\frac{n}{2} + n\log\frac{n}{4} + \cdots\right)$$

$$= nT(1) + Cn\log^2 n\ - Cn(\log 1 + \log 2 + \log 4 + \cdots \log n)$$

$$= O(n\log^2 n)$$

$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n)$

- Solution 2: Induction
- Base
  - $T(1) = O(n \log^2 n)$
- Induction
  - Assume $T(n) = O(n \log^2 n)$ for all $n < N$
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n) = 2O\left(\frac{n}{2} \log^2 \frac{n}{2}\right) + O(n \log n)$

    $= O(n \log^2 n) + O(n \log n) = O(n \log^2 n)$
- Is it Correct?

Discussion

- $T(n) = O(n)$
- There exists a constant C,n_0, such that $T(n) \leq Cn$ for all $n > n_0$.
- What happens when we do induction?
- $T(n) = T(n-1) + O(n)$
- $T(1) \leq Cn$
- $T(2) \leq 2Cn$
- …
- $T(n) \leq nCn$
- At some moment, kC is not a constant.

Be careful when we use induction!

$T(n) = 2T\left(\frac{n}{2}\right) + O(n\log n)$

- Solution 2: Induction
- Prove: $T(n) \le Bn\log^2 n$
- Base
  - $T(1) \le Bn\log^2 n$
- Induction
  - Assume $T(k) = Bk\log^2 k$ for all $\mathrm{k} < n$
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n\log n) = Bn\log^2\frac{n}{2} + C \cdot n\log n$

    $\le Bn\log n\left(\log\frac{n}{2} + \frac{C}{B}\right)$

    $\le Bn\log n\log n$ if $C < B$
- Guess $B > C$!

An Interesting Induction!

Proof: All the flower is red.

- Observation: I see a red flower today.
- Lemma: All the flowers are the same color.
- Base case:
  - One flower is the same color.
- Induction
  - Assume all k flowers are the same color, for all $k < n$.
  - Remove the first flower, all the other $n - 1$ flowers are the same color.
  - Remove the last flower, all the other $n - 1$ flowers are the same color.
  - The first flower and the last flower are the same color.
  - All the n flowers are the same color.

What’s wrong?

Wrong!

We can not go through $n = 2$ from $n = 1$.