Divide and Conquer

Sorting & Inversions

Sorting Problem

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** The same set of n integers in ascending order.

How many sorting algorithms you know?

Insertion Sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 1:
  - Try **Insertion Sort:** fix the output one by one.
  - How fast is it?

1	10	5	26	3	4	16	4	2

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1	5	10						

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1	2	3	4	4	5	10	16	26

Insertion Sort

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- Plan 1:
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  - How fast is it?

How many number we should insert?

1	10	5	26	3	4	16	4	2

1	2	3	4	4	5	10	16	26

How long it takes to insert one number?

Insertion Sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 1:
  - Try **Insertion Sort:** fix the output one by one.
  - How fast is it?

n numbers

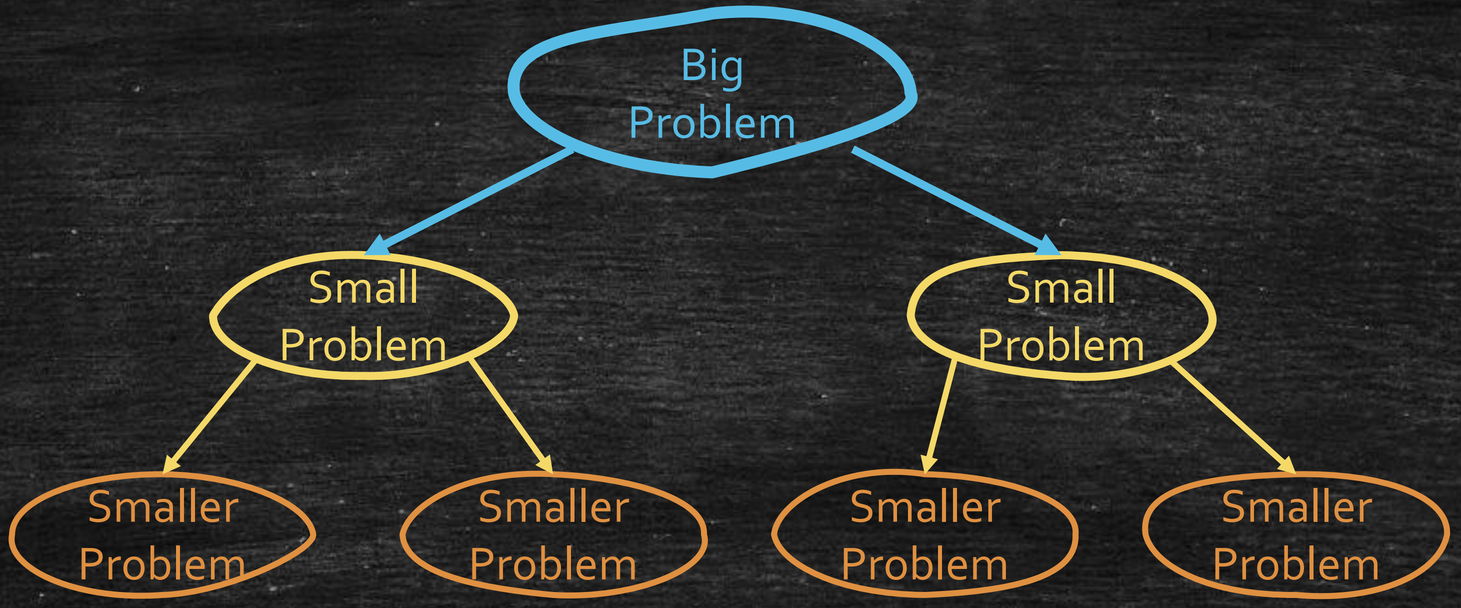
1	10	5	26	3	4	16	4	2

1	2	3	4	4	5	10	16	26

At most $2n$ time

Don't forget the correctness!

- Is it **correct**?
- How to **prove** it!
  - Using Induction?



Ok! Let’s move to divide and conquer!

Divide and Conquer

- Recall the divide and conquer
- Divide
  - Divide the problem into small size subproblems.
- Recurse
  - Solve the small size subproblems.
- Combine
  - Combine the output of small size subproblems to get the answer of the original problem.
- Basic solver
  - If the problem size is small enough, we should solve it directly.

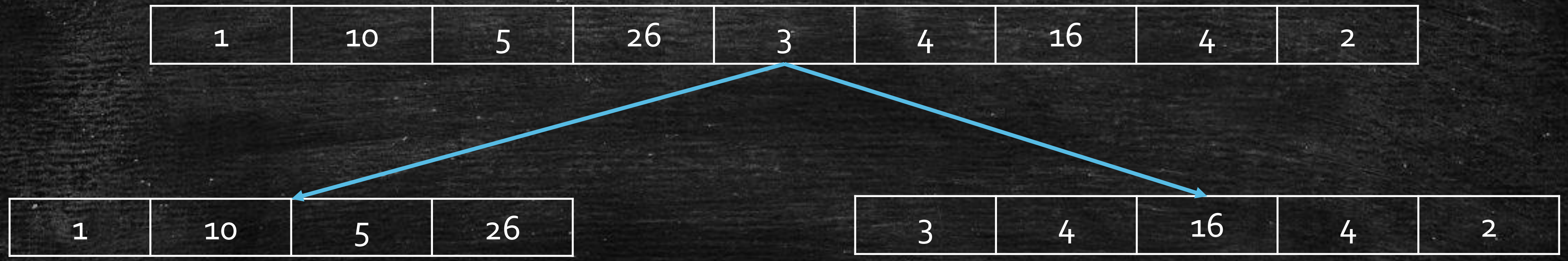
Merge sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 2: Divide and Conquer (Merge Sort)
  - Divide: Dive the input into two subsets:
    - $x_1, x_2, \ldots, x_{n/2}$; $x_{n/2+1}, x_{n/2+2}, \ldots, x_n$
  - Recurse: Sort two subsets (smaller size problems).
    - Let $y_1, y_2, \ldots, y_{n/2}$; $y_{n/2+1}, y_{n/2+2}, \ldots, y_n$ be the output (sorted list).
  - Combine: Merge to sorted list to one long sorted list.

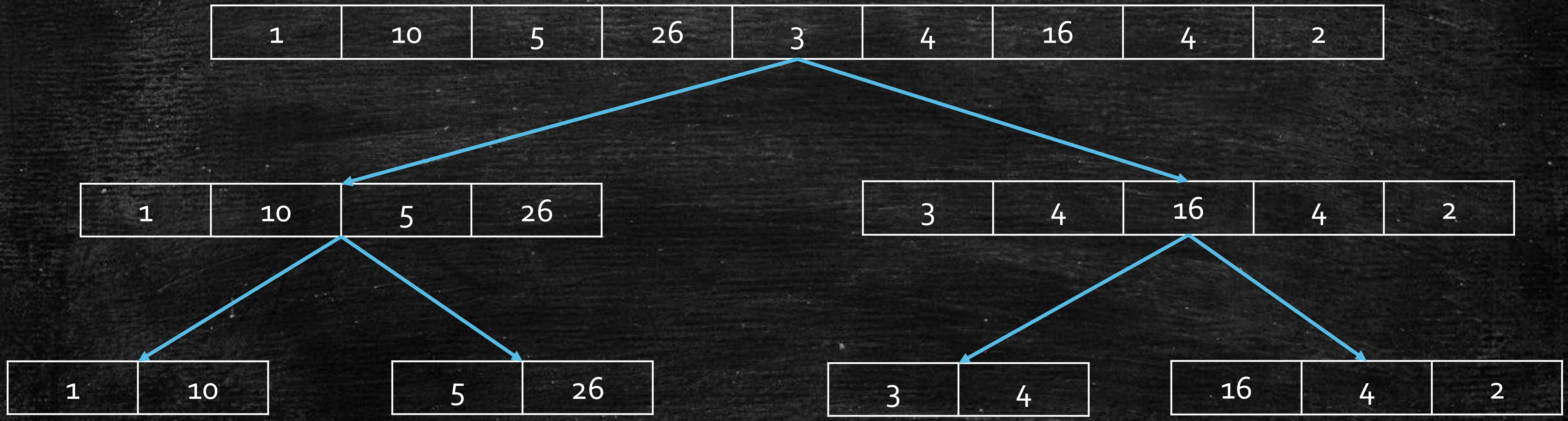
Merge Sort

1	10	5	26	3	4	16	4	2

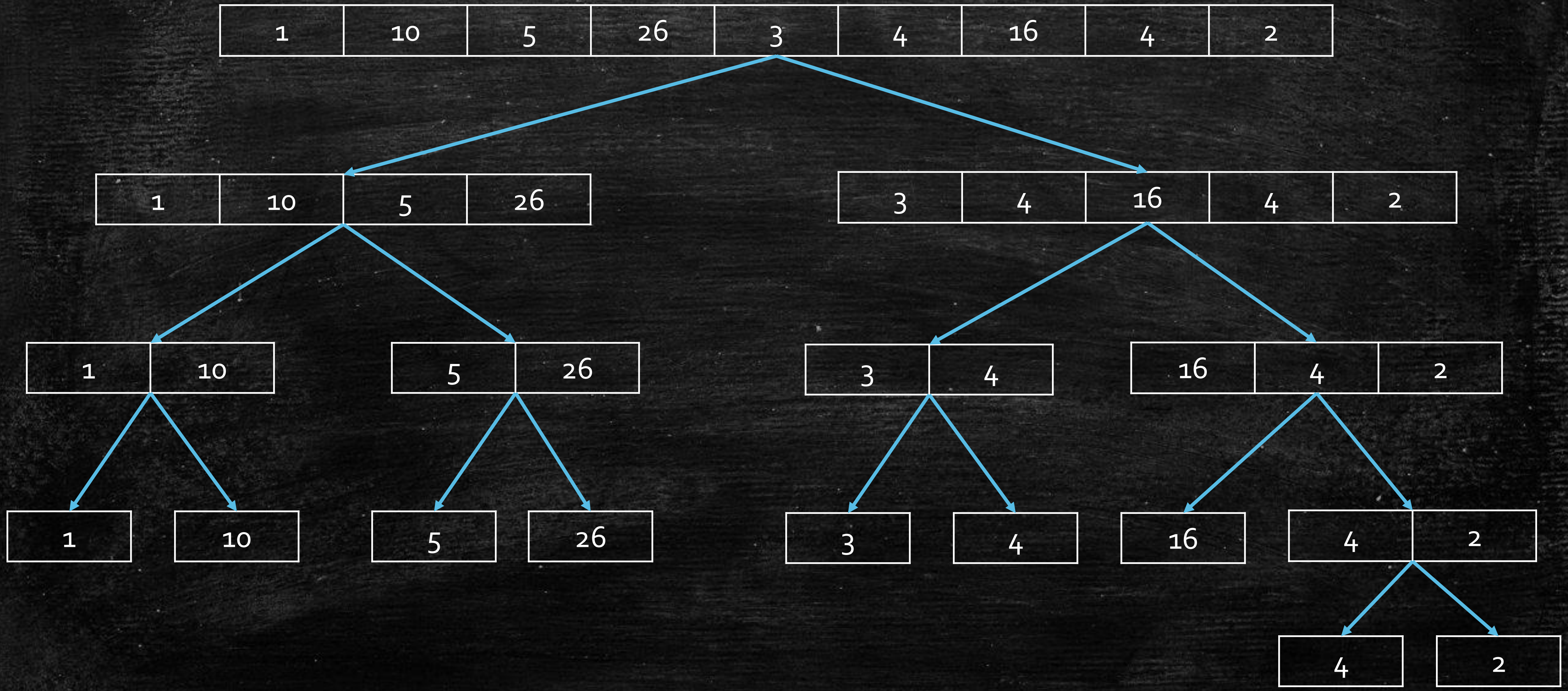
Merge Sort



Merge Sort



Merge Sort



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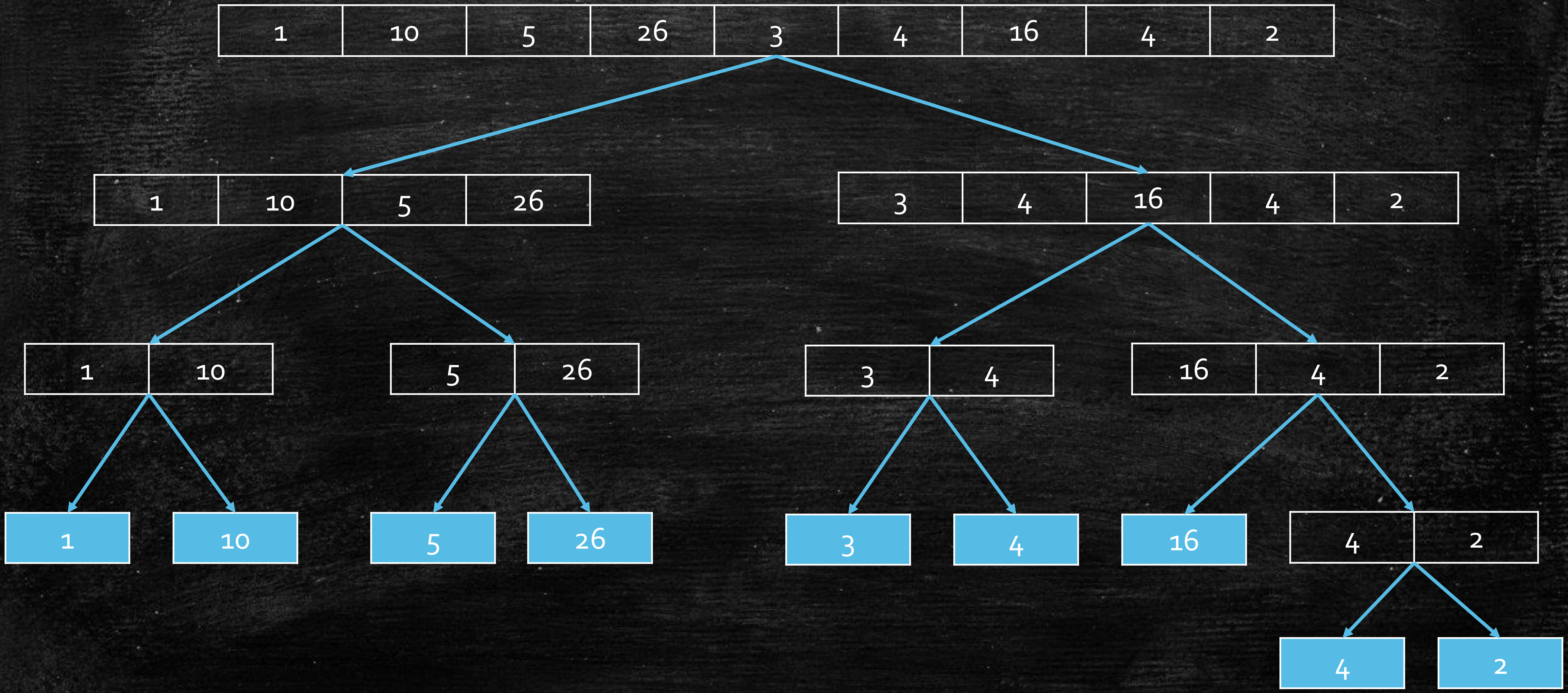
Merge Sort



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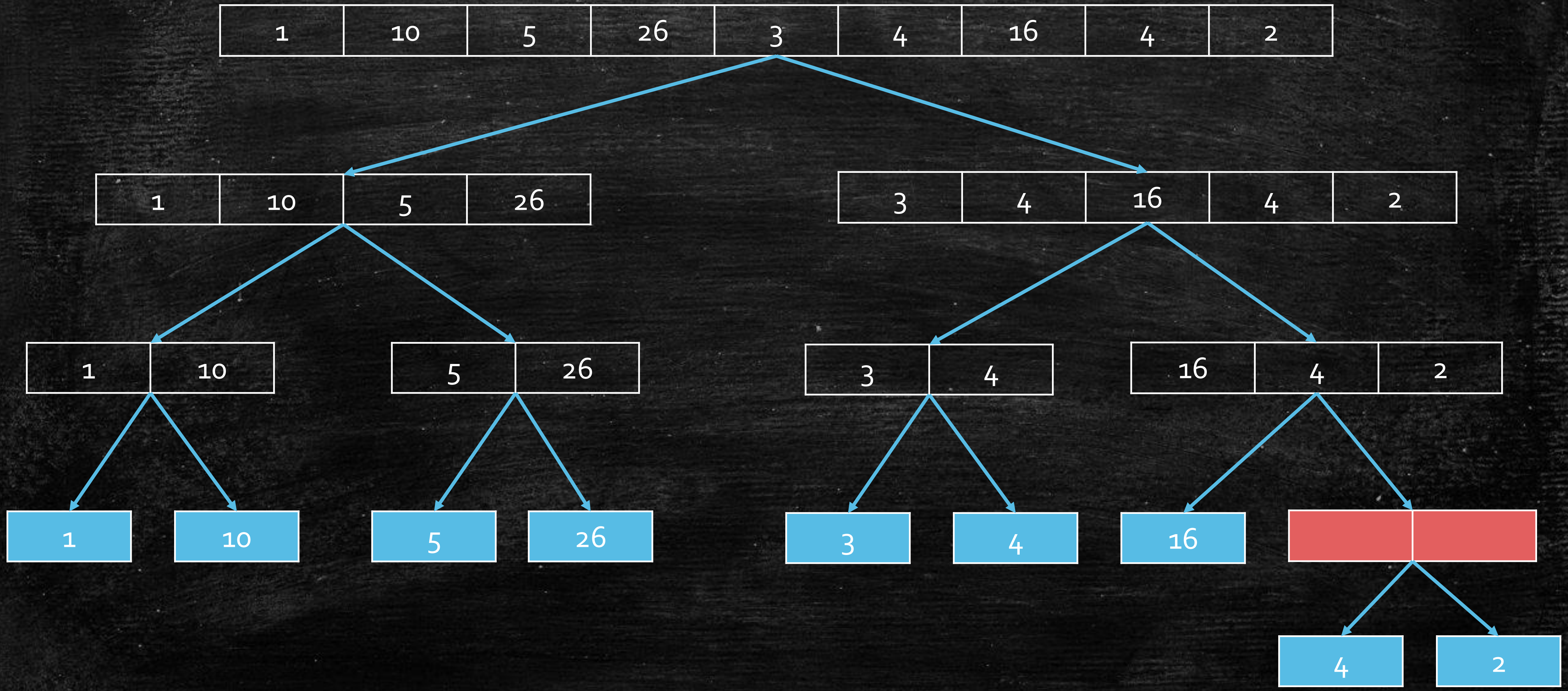
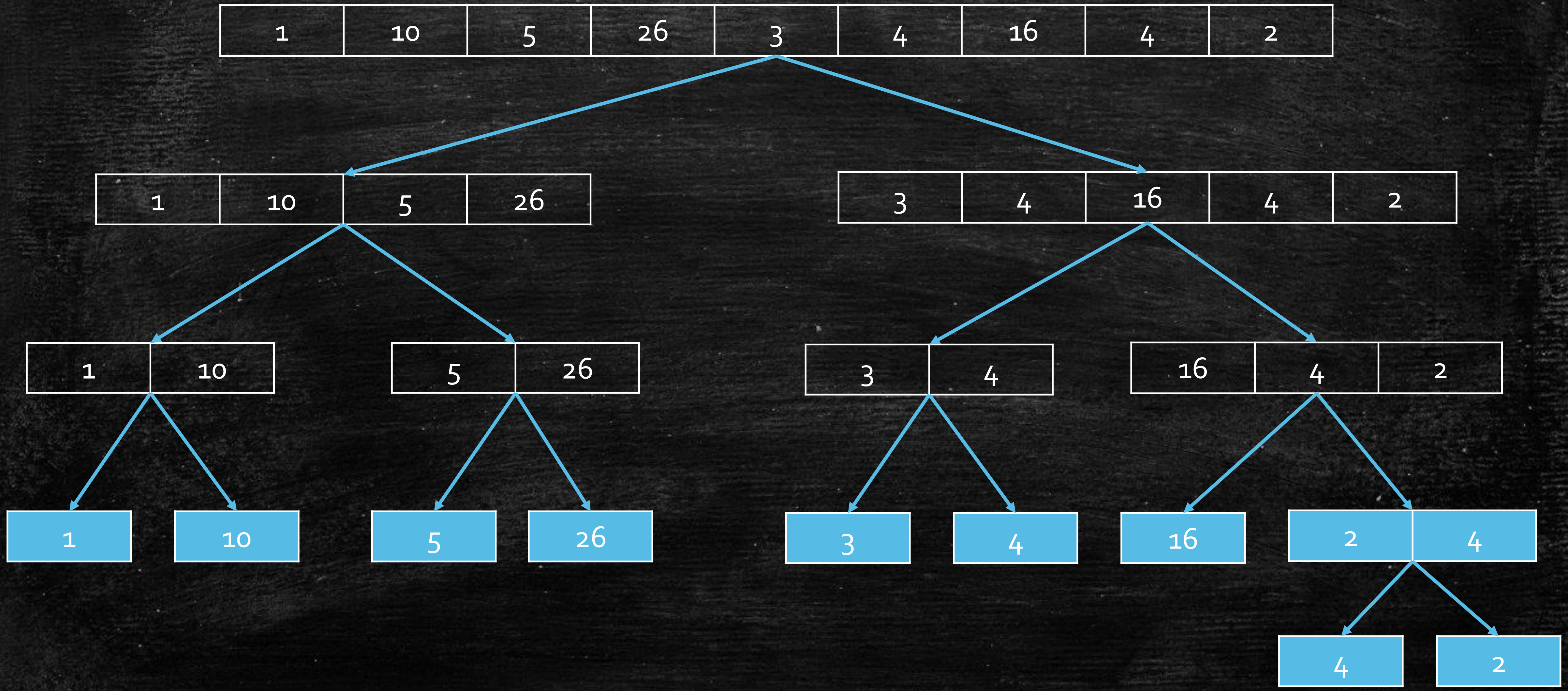
Merge Sort



Merge Sort



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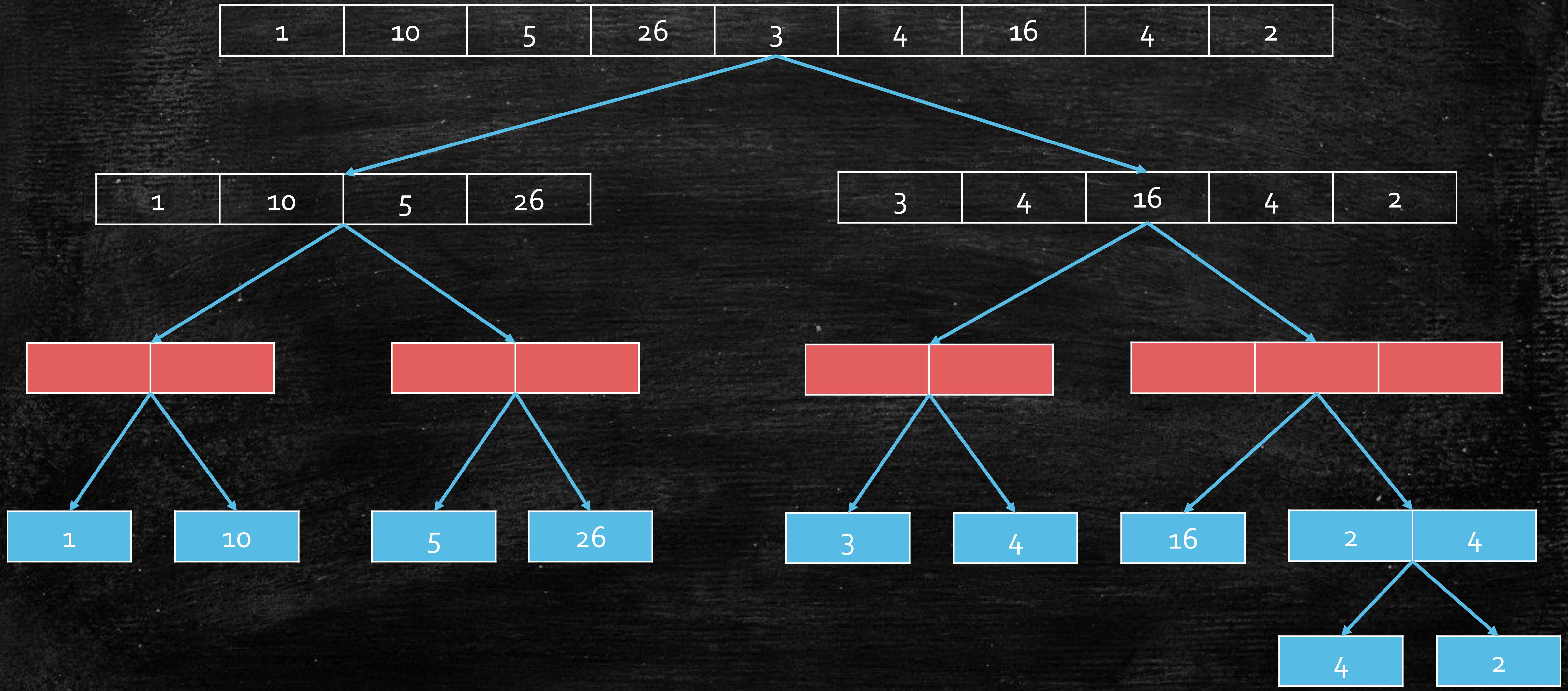

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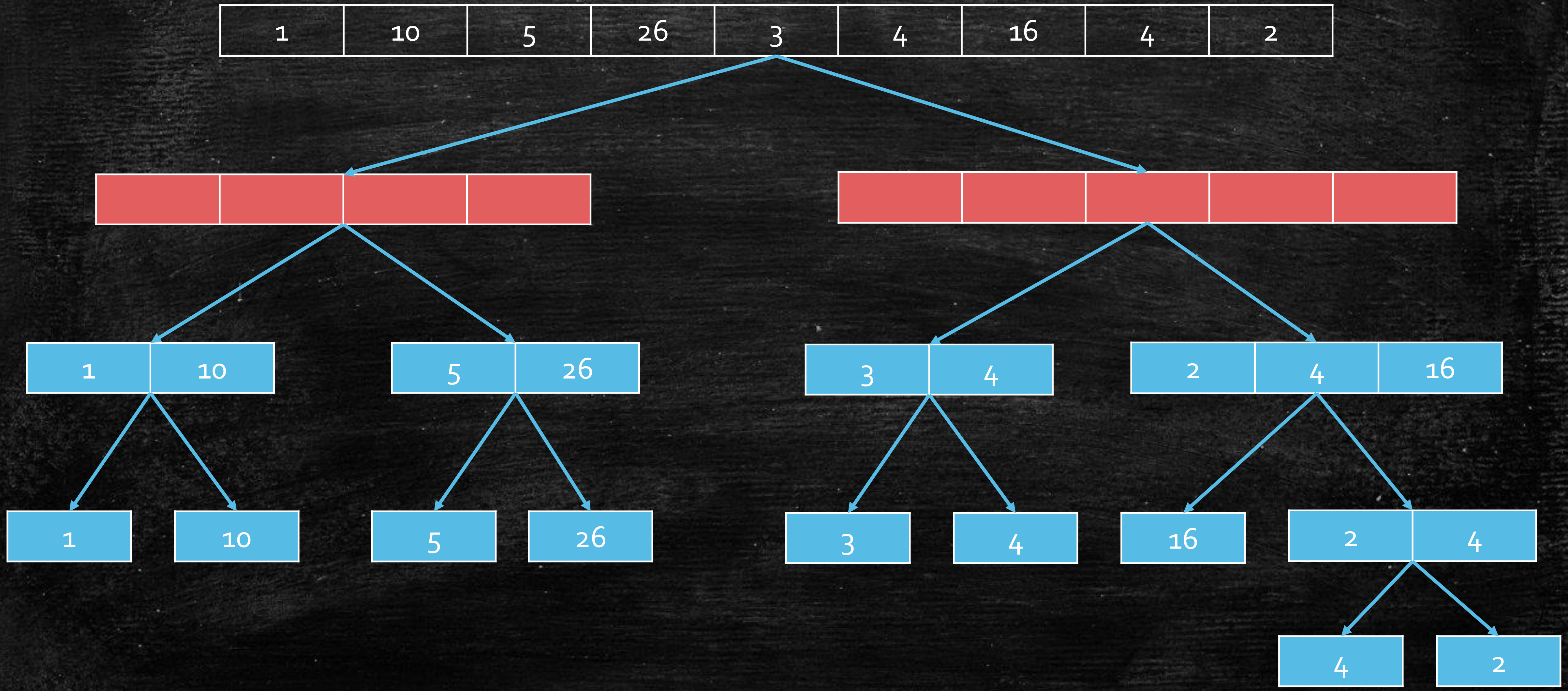

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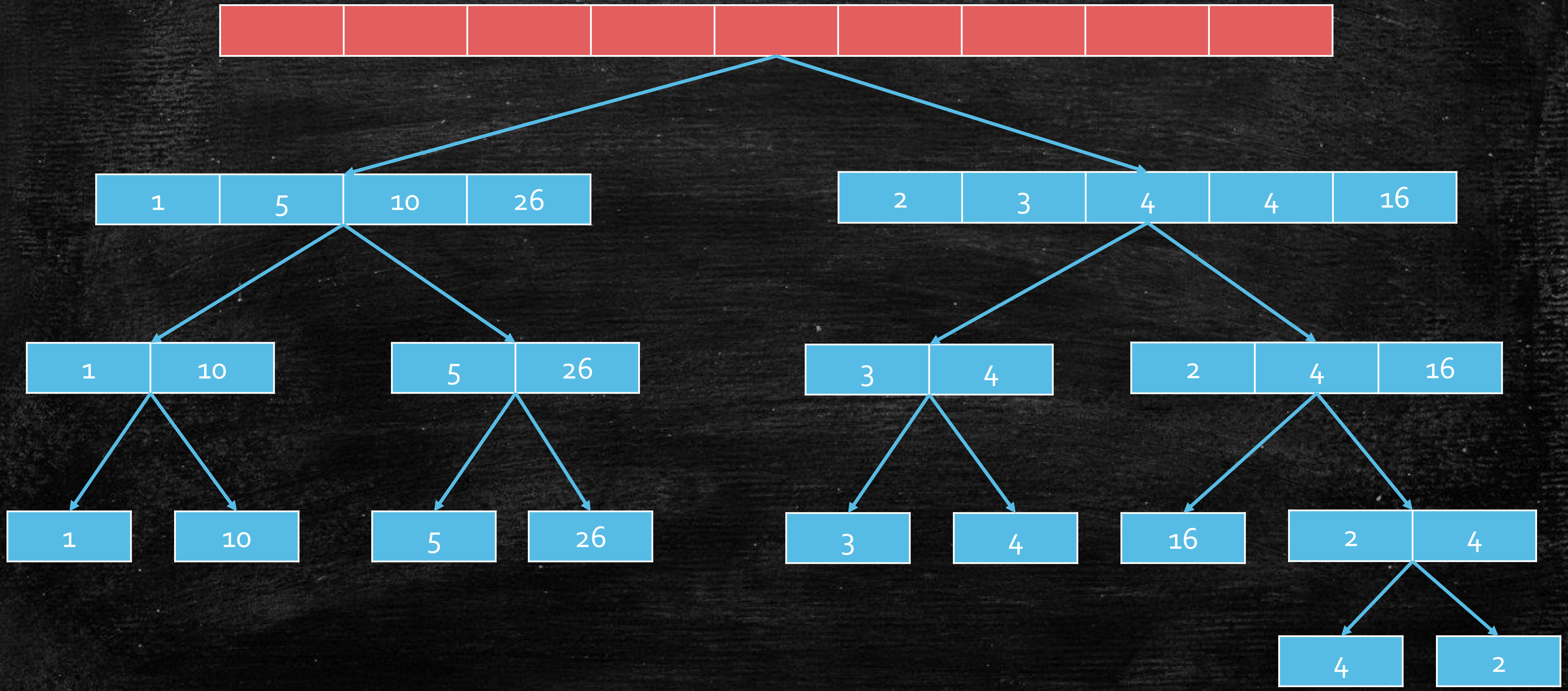
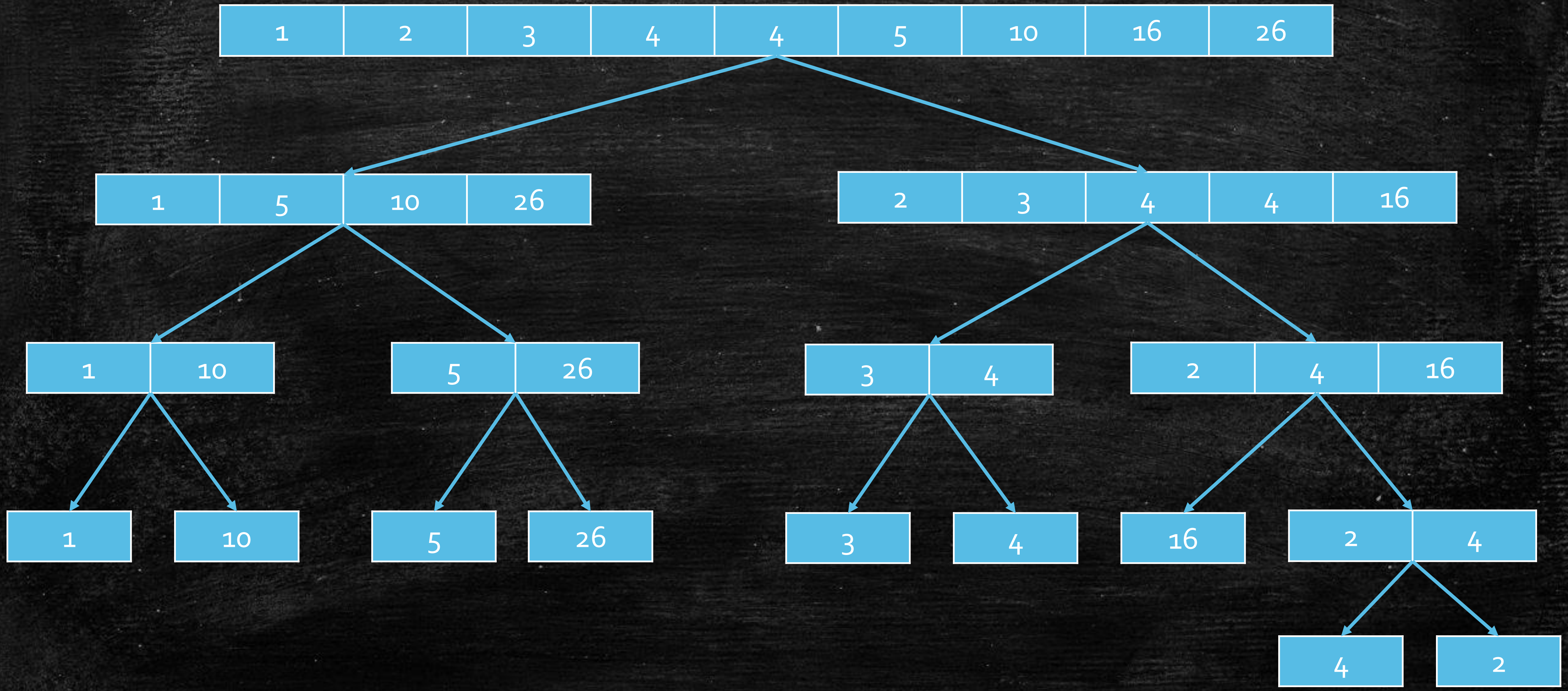


Merge Sort



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1 5 10 26
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5 26
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What is the next?

The remaining questions

- How to merge two sorted lists?
- How fast we can make it?

Merge two sorted lists

- **Input:** two sorted lists $A = a_1, a_2, \ldots, a_n$, $B = b_1, b_2, \ldots, b_m$
- **Output:** a sorted list C
- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C

Example

1	5	10	26

2	3	4	4	16

- Plan
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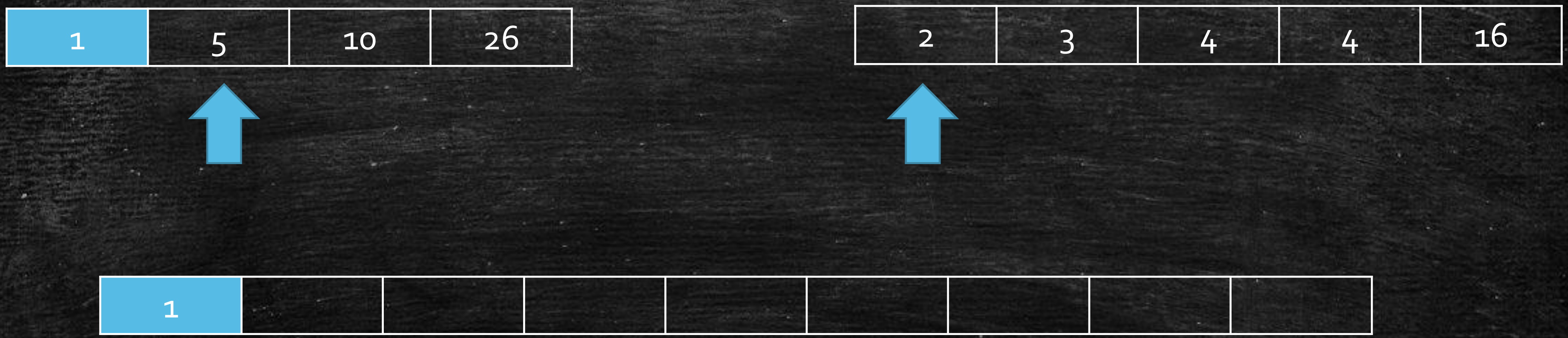

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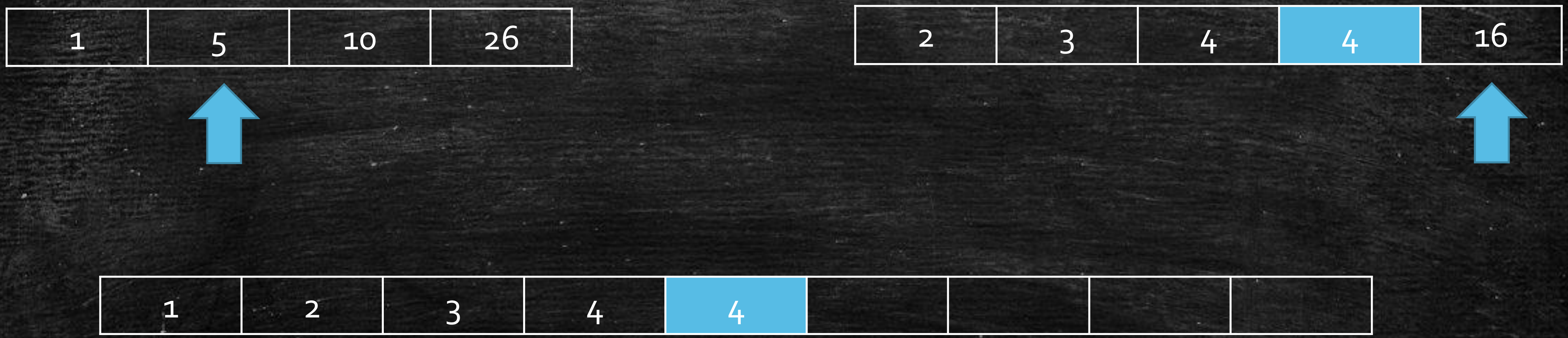

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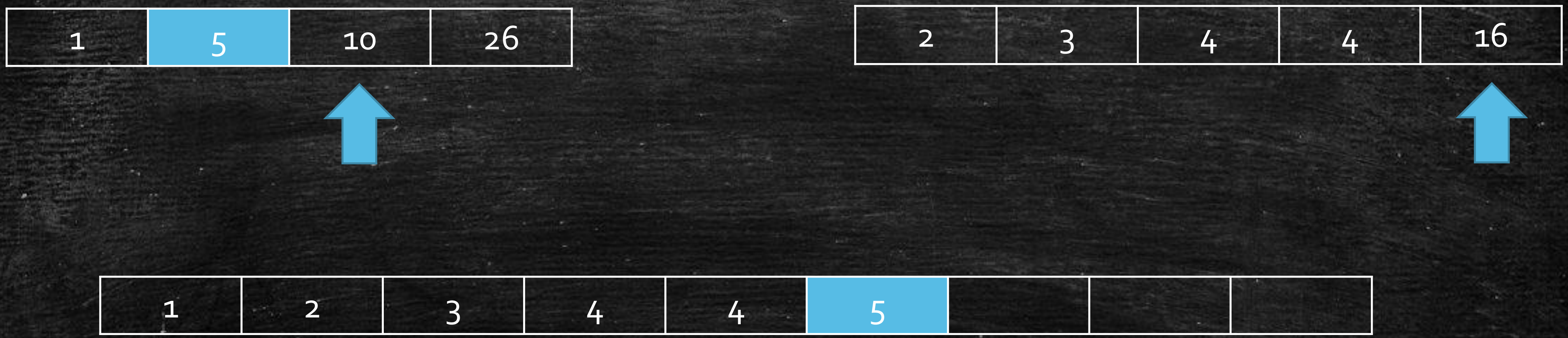

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Is it correct?

How fast is the algorithm?

- Plan
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  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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- Analysis 1
  - i at most move n times.
  - j at most move m times for one i move.
  - $O(nm)$?

How fast is the algorithm?

- Analysis 1
  - i at most move n times.
  - j at most move m times for one i movement.
  - $O(nm)$?
- Analysis 2
  - How many time it takes to output one number in C?
  - 1 number → 1 comparison!
  - Output $m + n$ numbers → $m + n$ comparisons!
  - A **linear time** algorithm!

Finally…

- What is the running time of merge sort?
  - Equip the linear time combining into merge sort.
  - Assume merge sort runs $T(n)$ time to sort a size n list.
  - What is $T(n)$?
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$ $\frac{n}{2} + \frac{n}{2}$

$$T(n) = T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = T\left(\frac{n}{2}\right) + O(n^2)$$

$$T(n) = 4T\left(\frac{n}{5}\right) + O(n^3)$$

An efficient tool to calculate these expressions

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n^4)$$

Master Theorem

- Master Theorem
  - If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$
  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

How to understand it?

Understand the parameters

- Master Theorem
  - If $T(n) = \underbrace{aT\left(\frac{n}{b}\right) + O(n^d)}$

Divide into a problems

  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

Understand the parameters

- Master Theorem
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    Subproblem size: n/b

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Understand the parameters

- Master Theorem
  - If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$

Combining cost: $O(n^d)$

Divide into a subproblems

Subproblem size: n/b

  - $$T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$$

Running time of merge sort

- Recall
  - Merge sort divides the problem to **two** $\boldsymbol{n/2}$**-size** problems.
  - Merging two $n/2$-size sorted lists takes $\boldsymbol{O(n)}$ times.
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$
- Use Master Theorem
  - If $T(n) = \boldsymbol{a}T\left(\frac{n}{\boldsymbol{b}}\right) + O\left(n^{\boldsymbol{d}}\right)$, then $T(n) = \begin{cases} O\left(n^d\right) & a < b^d \\ O\left(n^{\log_b a}\right) & a > b^d \\ O\left(n^d \log n\right) & a = b^d \end{cases}.$
  - $a = 2, b = 2, d = 1$
  - $T(n) = O(n \log n)$

Understand the formula & proof

Level 0 $\quad n$

Level 1 $\quad \frac{n}{b} \quad \frac{n}{b} \quad \frac{n}{b} \qquad a$ problems

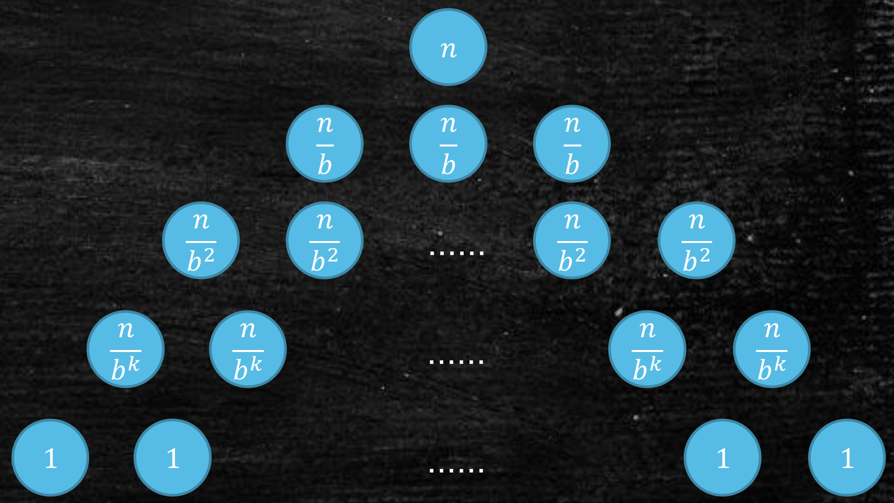
Level 2 $\quad \frac{n}{b^2} \quad \frac{n}{b^2} \quad \ldots\ldots \quad \frac{n}{b^2} \quad \frac{n}{b^2} \qquad a^2$ problems

Level k $\quad \frac{n}{b^k} \quad \frac{n}{b^k} \quad \ldots\ldots \quad \frac{n}{b^k} \quad \frac{n}{b^k} \qquad a^k$ problems

Level $\log_b n \quad 1 \quad 1 \quad \ldots\ldots \quad 1 \quad 1 \qquad a^{\log_b n}$ problems

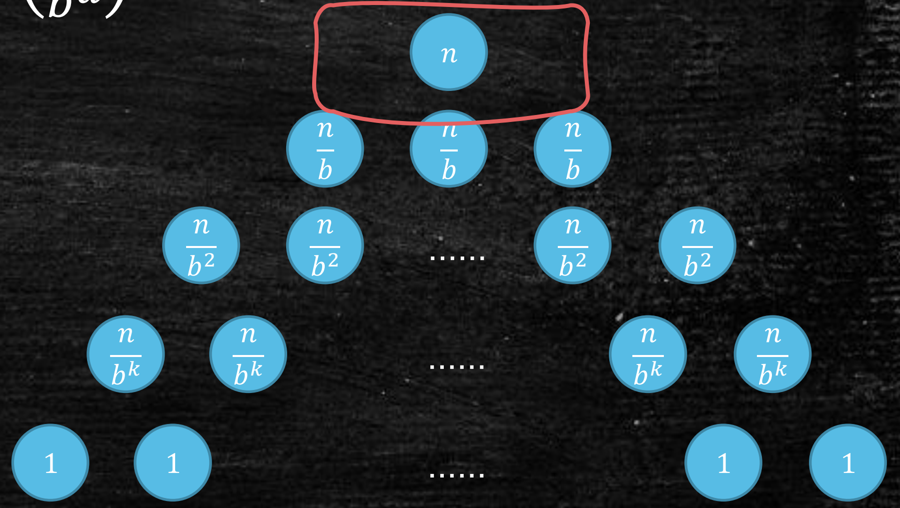
Understand the formula & proof

- The running time of solving all size-1 problem is
  - $a^{\log_b n} \cdot O(1) = O(n^{\log_b a})$
- The total running time is
  - $O(n^d) + a \cdot O\left(\left(\frac{n}{b}\right)^d\right) + \cdots + a^k \cdot O\left(\left(\frac{n}{b^k}\right)^d\right) + \cdots + a^{\log_b n} \cdot O(1)$
- Simplification
  - $O(n^d) \cdot (1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n})$



Case 1: $a < b^d$

- $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$
- $T(n) = O(n^d) \cdot (1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n})$
- $a < b^d \rightarrow \frac{a}{b^d} < 1$
- $T(n) = O(n^d)$



Case 2: $a > b^d$

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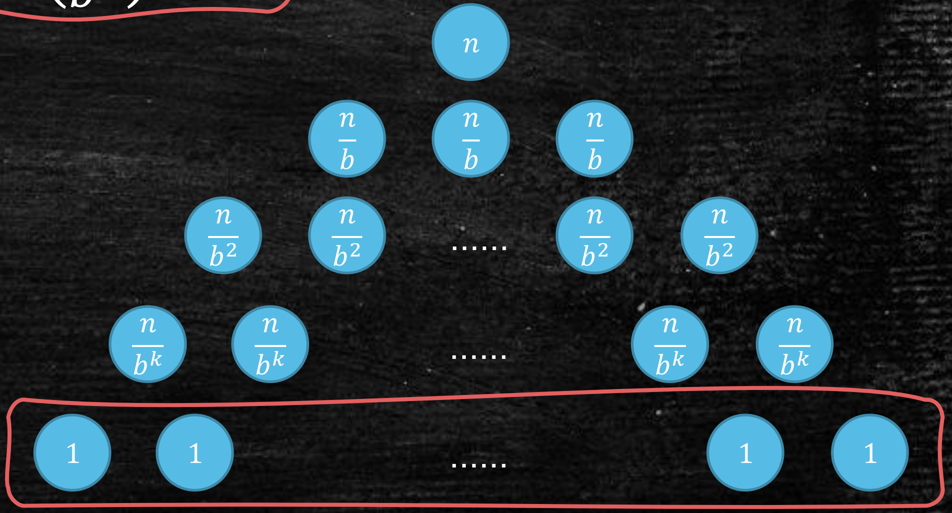
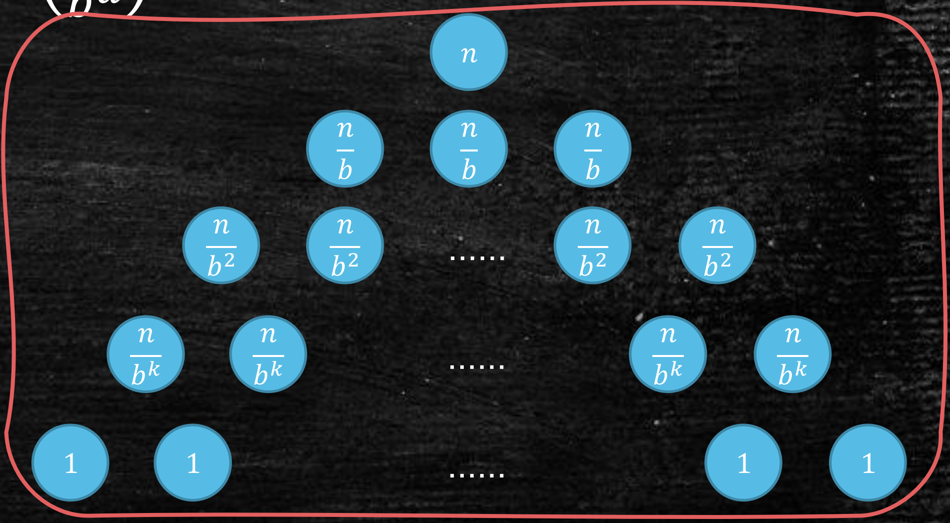
- $T(n) = O(n^d) \cdot \left(1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n}\right)$
- $a > b^d \rightarrow \frac{a}{b^d} > 1$
- The last term dominates the sum
- $T(n) = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^d \frac{a^{\log_b n}}{n^d}\right)$

  $= O(a^{\log_b n}) = O(n^{\log_b a})$



Case 3: $a = b^d$

- $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$
- $T(n) = O(n^d) \cdot (1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n})$
- $a = b^d \rightarrow \frac{a}{b^d} = 1$
- $T(n) = O(n^d \log_b n)$



Divide and Conquer

Counting Inversions

Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** number of **inversions**
- Application
  - You rank n songs.
  - Music site consults database to find people with **similar** tastes.
  - What is **similar**?
    - My rank: $1, 2, 3, 4, 5, \dots, n$
    - Your rank: $x_1, x_2, x_3, \dots, x_n$
    - Songs i, j are **inverted** if $i < j$ but $x_i > x_j$.
    - Similar metric: number of inversions.

Counting Inversions vs Merge Sort

Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** number of **inversions**
- Plan 1: Brute-force
  - $\boldsymbol{O(n^2)}$

Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** number of **inversions**
- Plan 2: Divide and Conquer (Merge Sort Style)
  - Divide: Dive the input into two subsets:
    - $x_1, x_2, \dots, x_{n/2}$; $x_{n/2+1}, x_{n/2+2}, \dots, x_n$
  - Recurse: count inversions in the two subsets.
    - Let c_1, c_2 be the two numbers.
  - Combine: Return the total number of inversions.
    - Count the inversions across subsets, to be c_3.
    - Return $c_1 + c_2 + c_3$.

Counting Inversions

1	10	5	26	3	4	16	4	2

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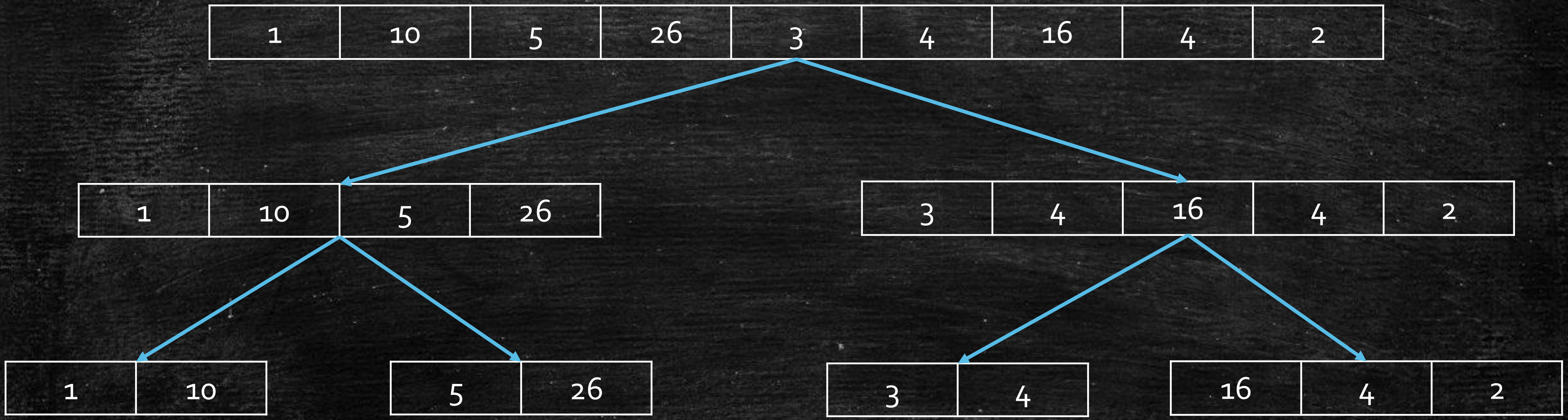
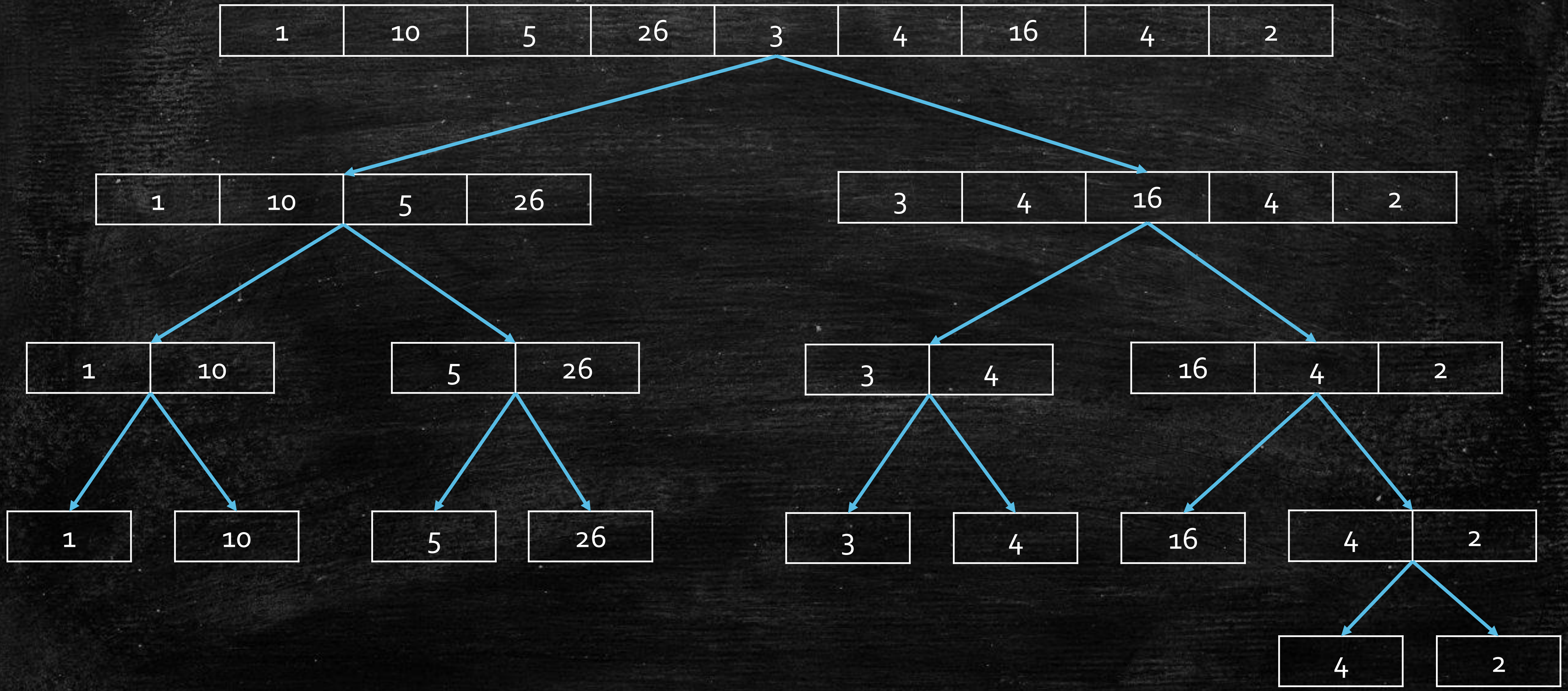
Counting Inversions



Counting Inversions



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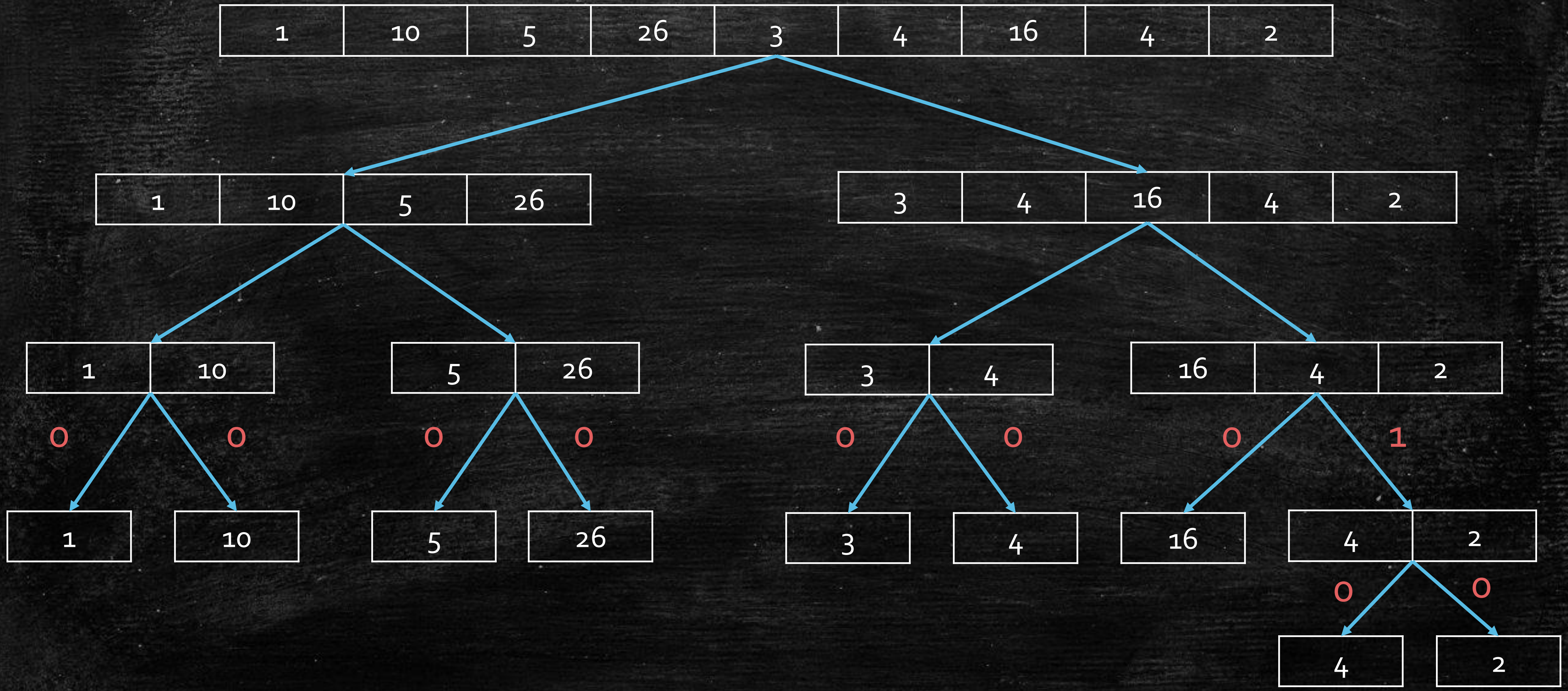

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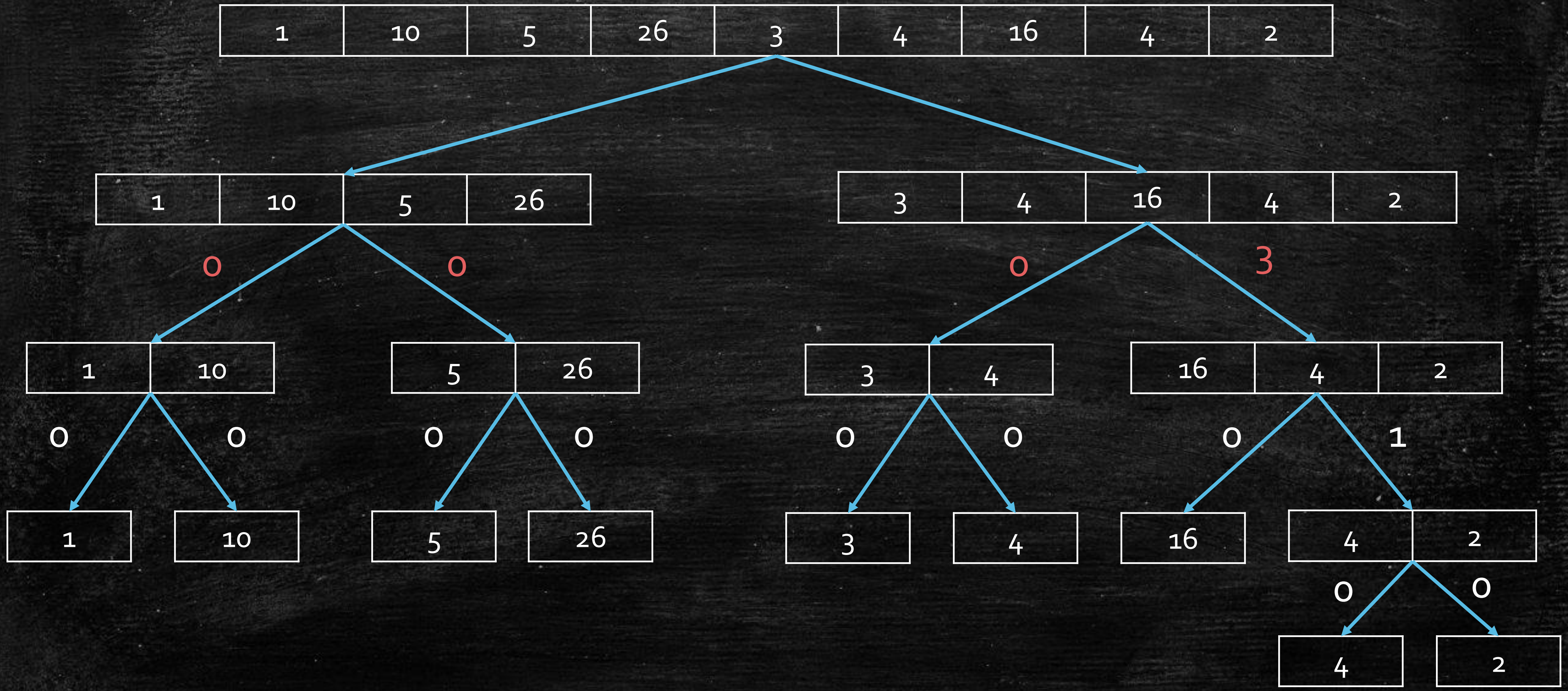

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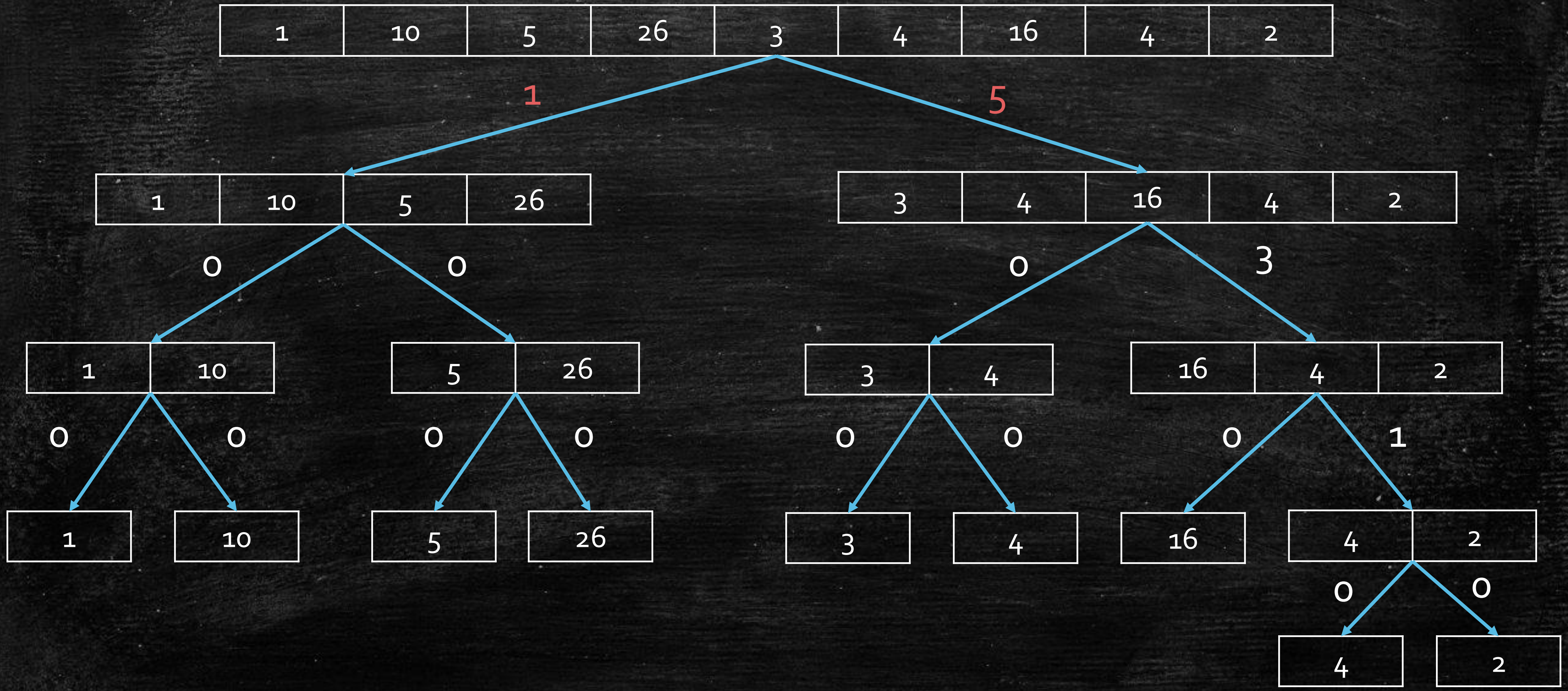

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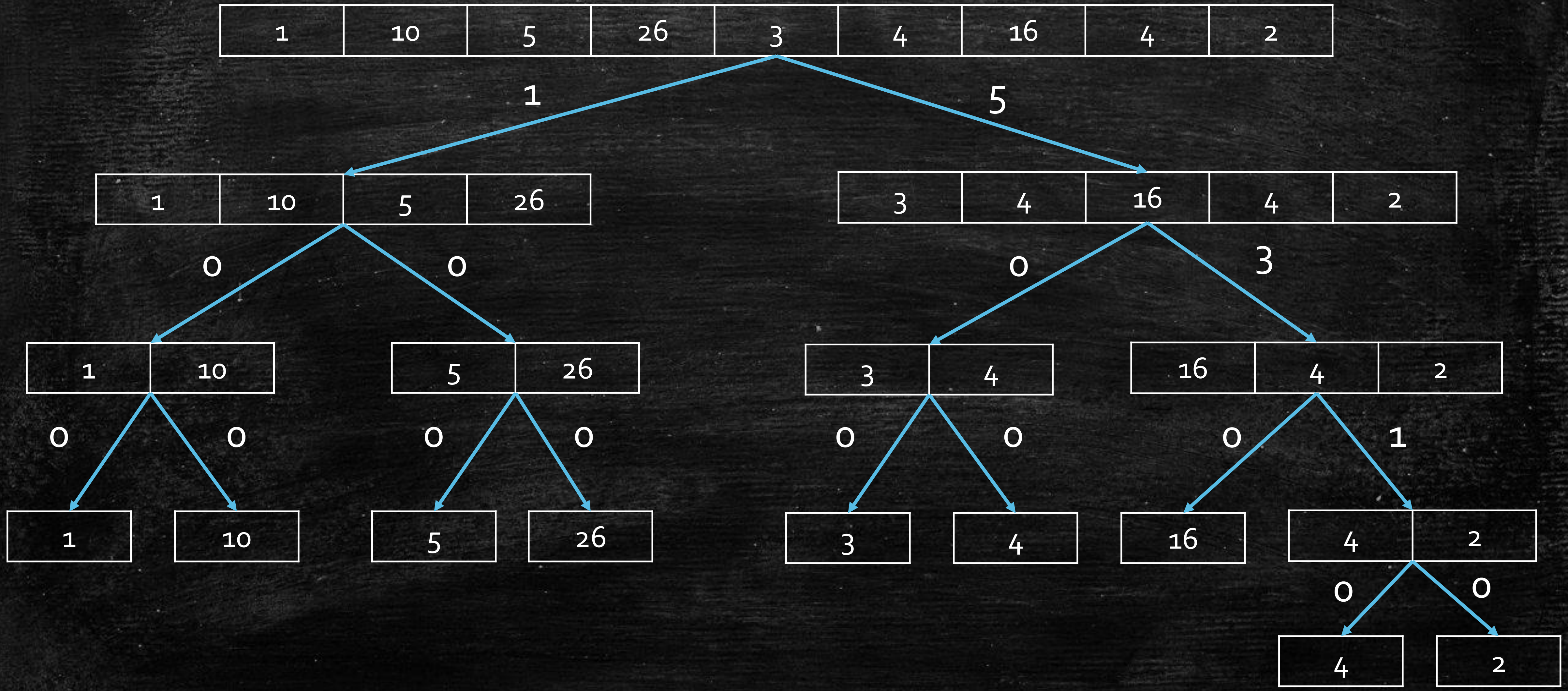

Count inversions across two lists

- **Input:** two lists $A = a_1, a_2, \dots, a_n$, $B = b_1, b_2, \dots, b_m$
- **Output:** number of inversions across two lists
- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 0

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 4

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 8

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 13

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 13

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

How long it takes?

- Analysis
  - Each a_i, scan the whole list B.
  - It takes $O(nm)$.
- How to improve?
  - It become easier when the two lists are sorted!
  - Why not do **merging** and **counting** together!

Merge & Count

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - If we move i to $i + 1$, then $c = c + j - 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C
  - If $i \le n$, $c = c + m \cdot (n - i + 1)$

$a_i > b_{j'}, \forall j' < j$

$a_{i'} > b_{j'}, \forall j' \le m, \forall i' \ge i$

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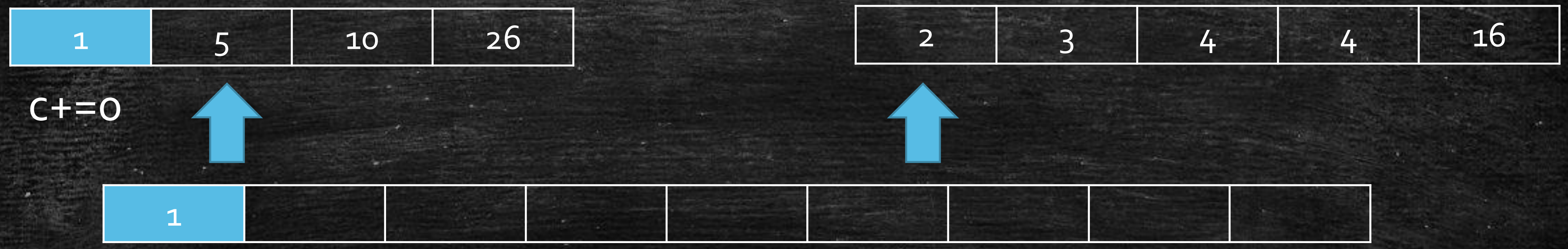

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Example

1	5	10	26

c+=0 c+=4 c+=4

2	3	4	4	16

1	2	3	4	4	5	10		

- Plan
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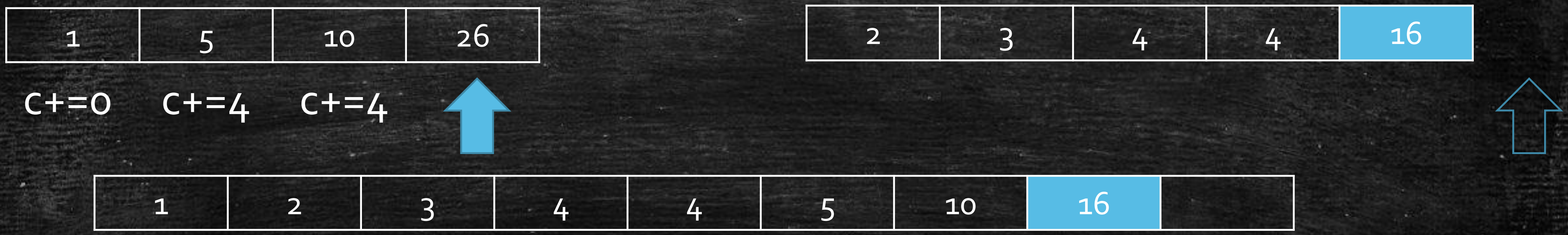

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Example

1	5	10	26

2	3	4	4	16

c+=0 c+=4 c+=4 c+=5

1	2	3	4	4	5	10	16	26

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How fast is it now?

- The same as Merging two sorted listed.
- Counting Inversion is as fast as Merge Sort.
- $T(n) = T\left(\frac{n}{2}\right) + O(n) = O(n \log n)$

Today's goal

- Learn Insertion Sort and Merge Sort
- Learn how to Count Inversions with Merge Sort
- Learn to prove the correctness of them
- Learn to analyze their running time
- Learn the Master Theorem