Algorithm Design and Analysis (Fall 2021)
Assignment 5

1. (0 points) Given an $n \times n$ matrix A whose entries are either 0 or 1, you are allowed to choose a set of entries with value 1 and modify their value to 0. Design an efficient algorithm to decide if we can make A invertible (A is invertible if the determinant of A is not 0) by the above-mentioned modification. Prove the correctness of your algorithm and analyze its time complexity. Notice that your algorithm only needs to output "yes" or "no" and does not need to find the minimum number of modified entries. (Hint: Prove that this is possible if and only if there exist n entries with value 1 such that no two entries are in the same row and no two entries are in the same column.)

2. (25 points) Consider the maximum flow problem $(G = (V, E), s, t, c)$ on graphs where the capacities for all edges are 1: $c(e) = 1$ for each $e \in E$. You can assume there is no pair of anti-parallel edges: for each pair of vertices $u, v \in V$, we cannot have both $(u, v) \in E$ and $(v, u) \in E$. You can also assume every vertex is reachable from s.
   (a) (13 points) Prove that Dinic's algorithm runs in $O(|E|^{3/2})$ time.
   (b) (12 points) Prove that Dinic's algorithm runs in $O(|V|^{2/3} \cdot |E|)$ time. (Hint: Let f be the flow after $2|V|^{2/3}$ iterations of the algorithm. Let D_i be the set of vertices at distance i from s in the residual network G^f. Prove that there exists i such that $|D_i \cup D_{i+1}| \leq |V|^{1/3}$.)

3. (20 points) Given an undirected and unweighted graph $G = (V, E)$, a *vertex cover* is a subset S of vertices such that it contains at least one endpoint of every edge, and an *independent set* is a subset T of vertices such that $(u, v) \notin E$ for any $u, v \in T$. Notice that V is a vertex cover, and $\emptyset$, or any set containing a single vertex, is an independent set. A *minimum vertex cover* is a vertex cover containing a minimum number of vertices, and a *maximum independent set* is an independent set containing a maximum number of vertices.
   (a) (5 points) Prove that S is a vertex cover if and only if $V \setminus S$ is an independent set.
   (b) (5 points) Prove that S is a minimum vertex cover if and only if $V \setminus S$ is an maximum independent set.
   (c) (10 points) Design an efficient algorithm to find a minimum vertex cover, or a maximum independent set, on *bipartite graphs*. Prove the correctness of your algorithm and analyze its time complexity.

4. (20 points) Covert the following linear program to its standard form, and compute its dual program.

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 7x_2 + x_3 \\ \text{subject to} \quad & x_1 - x_3 = 7 \\ & 3x_1 + x_2 \geq 24 \\ & x_2 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

5. (35 points) In this question, we will prove König-Egerváry Theorem, which states that, in any bipartite graph, the size of the maximum matching equals to the size of the minimum vertex cover. Let $G = (V, E)$ be a bipartite graph.

   (a) (5 points) Explain that the following is an LP-relaxation for the maximum matching problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e: e=(u,v)} x_e \leq 1 & (\forall v \in V) \\ & x_e \geq 0 & (\forall e \in E) \end{aligned}$$

   (b) (5 points) Write down the dual of the above linear program, and justify that the dual program is an LP-relaxation to the minimum vertex cover problem.

   (c) (10 points) Show by induction that the *incident matrix* of a bipartite graph is totally unimodular. (Given an undirected graph $G = (V, E)$, the incident matrix A is a $|V| \times |E|$ zero-one matrix where $a_{ij} = 1$ if and only if the i-th vertex and the j-th edge are incident.)

   (d) (10 points) Use results in (a), (b) and (c) to prove König-Egerváry Theorem.

   (e) (5 points) Give a counterexample to show that the claim in König-Egerváry Theorem fails if the graph is not bipartite.

6. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.