Algorithm Design and Analysis (Fall 2021)
Assignment 6

1. (50 Points) Consider the following variant of the scheduling problem. We have n jobs with size (time required for completion) $p_1, \ldots, p_n \in \mathbb{Z}^+$. Instead of fixing the number of machines m and minimizing the makespan, we fix the makespan and minimize the number of machines used. That is, you can decide how many (identical) machines to use, but each machine can be operated for at most $T \in \mathbb{Z}^+$ units of time. Assume $p_i \leq T$ for each $i = 1, \ldots, n$. Your objective is to minimize the number of machines used, while completing all the jobs.

   (a) (20 Points) Show that this minimization problem is NP-hard.

   (b) (20 Points) Consider the following local search algorithm. Initialize the solution where n machines are used such that each machine handles a single job. Do the following update to the solution until no more update is possible: if there are two machines such that the total size of the jobs on the two machines is less than T, update the solution by using only one machine to complete all these jobs (instead of using two machines). Show that this algorithm gives a 2-approximation.

   (c) (10 Points) Provide a tight example showing that the algorithm in (b) can do 1.5-approximation at best.

2. (50 Points) Choose *any one* of the following questions. (You are encouraged to solve as many the remaining questions as possible "in your mind".)

   (a) Given an undirected graph $G = (V, E)$ with $n = |V|$, decide if G contains a clique with size exactly $n/2$. Prove that this problem is NP-complete.

   (b) Given an undirected graph $G = (V, E)$, the *3-coloring* problem asks if there is a way to color all the vertices by using three colors, say, red, blue and green, such that every two adjacent vertices have different colors. Prove that 3-coloring is NP-complete.

   (c) Given two undirected graphs G and H, decide if H is a subgraph of G. Prove that this problem is NP-complete.

   (d) Given an undirected graph $G = (V, E)$ and an integer k, decide if G has a spanning tree with maximum degree at most k. Prove that this problem is NP-complete.

   (e) Given a ground set $U = \{1, \ldots, n\}$ and a collection of its subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, the *exact cover* problem asks if we can find a subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{T}} S = U$ and $S_i \cap S_j = \emptyset$ for any $S_i, S_j \in \mathcal{T}$. Prove that exact cover is NP-complete.

   (f) Given a collection of integers (can be negative), decide if there is a subcollection with sum exactly 0. Prove that this problem is NP-complete.

   (g) In an undirected graph $G = (V, E)$, each vertex can be colored either black or white. After an initial color configuration, a vertex will become black if all its neighbors are black, and the updates go on and on until no update is possible. (Notice that once a vertex is black, it will be black forever.) Now, you are given an initial configuration where all vertices are white, and you need to change k vertices from white to black such that all vertices will eventually become black after updates. Prove that it is NP-complete to decide if this is possible.

3. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.