H1.4 commutativity

$$(f*g)(x) = \int_{\mathbb{R}} f(x-x')\,g(x')\,dx' \quad , \quad y = x-x' \quad x' = x-y \quad dy = -dx$$

$$= -\int_{\mathbb{R}} f(y)\,g(x-y)\,(-dy)$$

$$= \int f(y)\,g(x-y)\,dy$$

$$= (g*f)(x) \qquad \square$$

differentiability

If $f \in C^0(\mathbb{R})$ and $g \in C^n(\mathbb{R})$, then $(f*g) \in C^n(\mathbb{R})$.
Hint: Show first that we have $(f*g)' = f*g'$.

$$(f*g)'(x) = (g*f)'(x) = \frac{d}{dx}\int_{\mathbb{R}} g(x-t)\,f(t)\,dt$$

$$= \int_{\mathbb{R}} \frac{d}{dx}\, g(x-t)\,f(t)\,dt$$

$$= \int_{\mathbb{R}} g'(x-t)\,f(t)\,dt$$

$$= (g'*f)(x) = (f*g')(x)$$

then

$$\left(f * \frac{d^{n+1}g}{dx^{n+1}}\right)(x) = \frac{d}{dx}\left(f * \frac{d^n g}{dx^n}\right)(x)$$

$$\Downarrow$$

if $g \in C^n(\mathbb{R})$ then $(f*g) \in C^n(\mathbb{R})$