H2.1

b) $F\left[f(x-a)\right](u) = e^{-i2\pi ua} \cdot F[f](u)$

$$F\left[f(x-a)\right](u) = \int_{-\infty}^{\infty} f(x-a) e^{-i2\pi ux} dx$$

$b = x-a$
$db = dx$
$x = b+a$

$$= \int_{-\infty}^{\infty} f(b) \cdot e^{-2i\pi u(b+a)} db$$

$$= \int_{-\infty}^{\infty} f(b) \cdot e^{-2i\pi ub} \cdot e^{-2i\pi ua} db$$

$$= e^{-2i\pi ua} \cdot \int_{-\infty}^{\infty} f(b) \cdot e^{-2i\pi ub} db$$

$$= e^{-2i\pi ua} F[f](u) \quad \square$$

c) $F\left[f(x) \cdot e^{-i2\pi u_0 x}\right](u) = F[f](u+u_0)$

$$F\left[f(x) \cdot e^{-i2\pi u_0 x}\right](u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi u_0 x} \cdot e^{-i2\pi ux} dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi(u_0+u)x} dx = F[f](u+u_0)$$

$\square$

#2.2

$$f(x) = \begin{cases} 0, & (x \le -3), \\ \frac{x^2+6x+9}{16}, & (-3 < x \le -1), \\ \frac{6-2x^2}{16}, & (-1 < x \le 1), \\ \frac{x^2-6x+9}{16}, & (1 < x \le 3), \\ 0, & (x > 3) . \end{cases}$$

$$h = \begin{cases} \frac{1}{2} & (-1 \leq x \leq 1) \\ 0 & \text{else} \end{cases}$$

$$f = h * h * h$$

$$F[f] = F[h * h * h] = F[h] \cdot F[h] \cdot F[h]$$

$$F[h](u) = \int_{-\infty}^{\infty} h(x)\, e^{-i2\pi u x}\, dx$$

$$(e^{2x})' = 2e^{2x}$$

$$= \int_{-1}^{1} \frac{1}{2} e^{-i2\pi u x}\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-2i\pi u x}\, dx$$

$$= \frac{1}{2} \left[\frac{-1}{i2\pi u} e^{-i2\pi u} \right]_{-1}^{1}$$

$$= \frac{-1}{4i\pi u} \left(e^{-i2\pi u} - e^{i2\pi u} \right)$$

$$= \frac{1}{4i\pi u} \left(e^{i2\pi u} - e^{-i2\pi u} \right) \quad , \quad \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

$$\phi = 2\pi u$$

$$= \frac{2i}{4i\pi u} \cdot \sin(2\pi u)$$

$$F[h](u) = \frac{\sin(2\pi u)}{2\pi u} = \operatorname{sinc}(u)$$

$$F[f] = (F[h])^3 = \operatorname{sinc}^3$$

$$F[f](u) = \operatorname{sinc}^3(u)$$

H 2,4

b) with low subsampling ratios (up to 4) the effect is barely noticable. With subsampling ratio of 8 and higher, the chroma blocks become visible. It shows how human perception is much more sensitive to luminosity changes, than the chroma.

c) Subsampling of S reduces the Cb/Cr array h/w size S times

For RGB / $S=1$

$$3 \cdot 8 \cdot h \cdot w = 24 \cdot h \cdot w$$

$S=2$

$$8 \cdot h \cdot w + 2 \cdot 8 \cdot \frac{h}{2} \cdot \frac{w}{2} = (8+4) \cdot h \cdot w, \quad 12 \text{ bits per pixel}$$

$S=4$

$$8 \cdot h \cdot w + 2 \cdot 8 \cdot \frac{h}{4} \cdot \frac{w}{4} = (8+1) h \cdot w, \quad 9 \text{ bits per pixel}$$

$S=8$

$$8 \cdot h w + 2 \cdot 8 \cdot \frac{h}{8} \cdot \frac{w}{8} = (8+\frac{1}{4}) h \cdot w, \quad 8{,}25 \text{ bits per pixel}$$

Problem 1 Antoni Kowalczuk Bartłomiej Pogoziński

a) $F[af(x)+bg(x)](u) =$

$$= \int_{\mathbb{R}} [af(x)+bg(x)] e^{-i2\pi ux} dx =$$

$$= \int_{\mathbb{R}} af(x)e^{-2i\pi ux} + bg(x)e^{-2i\pi ux} dx =$$

$$= a\int_{\mathbb{R}} f(x)e^{-2i\pi ux} dx + b\int_{\mathbb{R}} g(x)e^{-2i\pi ux} dx =$$

$$= aF[f\cancel{(x)}]u + bF[g\cancel{(x)}]u$$

Problem 1

d) $F[f(x\cdot a)](u) = \int_{\mathbb{R}} f(ax) e^{-2\pi i u x} dx = ⊛$

$$\Leftarrow \begin{cases} b = ax \\ db = a\,dx \\ dx = \frac{db}{a} \\ x = \frac{b}{a} \end{cases}$$

$$⊛ = \int_{\mathbb{R}} f(b) \cdot \frac{1}{a} \cdot e^{-2\pi i \frac{u}{a} x} db =$$

$$= \frac{1}{a} \int_{\mathbb{R}} f(b) \cdot e^{-2\pi i \frac{u}{a} x} db = \frac{1}{a} F[f]\left(\frac{u}{a}\right)$$

f) $[f(x) e^{-2\pi i u x}]' = f'(x) e^{-2\pi i u x} - 2\pi i u x \cdot$

$\cdot f(x) e^{-2\pi i u x} \quad / \int_{\mathbb{R}}$

$$\int_{\mathbb{R}} [f(x) e^{-2\pi i u x}]' dx = \int_{\mathbb{R}} f'(x) e^{-2\pi i u x} dx -$$

$$- \cancel{}\, 2\pi i u x \int_{\mathbb{R}} f(x) e^{-2\pi i u x} dx$$

$$f(x) e^{-2\pi i u x} = F[f'](u) - 2\pi i u F[f](u)$$

From Hint: LHS $\longrightarrow 0$

$$0 = F[f'](u) - 2\pi i u F[f](u)$$

$$F[f'](u) = 2\pi i u F[f](u)$$

Problem 1

e) $F[(f*g)(x)](\omega) = \int_{\mathbb{R}} (f*g)(x) e^{-2\pi i \omega x} dx =$

$= \int_{\mathbb{R}^2} f(x-x') g(x') dx' \, e^{-2\pi i \omega x} dx =$

Spatial shift

$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x-x') e^{-2\pi i \omega x} dx\right] g(x') dx' \overset{\text{Spatial shift}}{=}$

$= \int_{\mathbb{R}} e^{-2\pi i \omega x'} \left[\int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx\right] g(x') dx' =$

$= F[f](\omega) \cdot \int_{\mathbb{R}} g(x') e^{-2\pi i \omega x'} dx' =$

$= F[f](\omega) \cdot F[g](\omega)$

Problem 3

a) $F[f](u) = \int_{-\infty}^{+\infty} f(x)\overline{b_u(x)}\,dx =$

$= \int_0^{+\infty} f(x)\overline{b_u(x)}\,dx + \int_{-\infty}^{0} f(x)\overline{b_u(x)}\,dx =$

$= \int_0^{+\infty} f(x)\overline{b_u(x)}\,dx + \int_0^{+\infty} f(-x)\overline{b_u(-x)}\,dx = \circledast$

$\overline{b_u(x)} = \overline{\cos(2\pi ux) + i\sin(2\pi ux)} =$

$= \cos(2\pi ux) - i\sin(2\pi ux)$

$\overline{b_u(-x)} = \cos(-2\pi ux) - i\sin(-2\pi ux) =$

$= \underset{\text{even}}{\cos(2\pi ux)} + i\underset{\text{odd}}{\sin(2\pi ux)} = b_u(x)$

Therefore

$\circledast = \int_0^{+\infty} f(x)\overline{b_u(x)}\,dx + \int_0^{+\infty} f(-x)\, b_u(x)\,dx$

$f = f_e + f_o \Rightarrow F[f] = F[f_e] + F[f_o]$

$F[f_e](u) = \int_0^{+\infty} f_e(x)\overline{b_u(x)}\,dx + \int_0^{+\infty} f_e(-x)\, b_u(x)\,dx =$

$= \int_0^{+\infty} f_e(x)[\overline{b_u(x)} + b_u(x)]\,dx = 2\int_0^{+\infty} f_e(x) Re[b_u(x)]dx$

$F[f_o](u) = \int_0^{+\infty} f_o(x)\overline{b_u(x)}\,dx + \int_0^{+\infty} f_o(-x)\, b_u(x)\,dx =$

$= \int_0^{+\infty} f_o(x)[\overline{b_u(x)} - b_u(x)]\,dx = 2i\int_0^{+\infty} f_o(x) Im[b_u(x)]$

1) $\int_0^{+\infty} f_e(x) Re[b_u(x)]\,dx \in \mathbb{R} \to \mathbb{R}$

2) $\int_0^{+\infty} f_o(x) Im[b_u(x)]\,dx \in \mathbb{R} \to \mathbb{R}$ $\Big\} \Rightarrow$

$\Rightarrow Re[F[f](u)] = 1)$

$Im[F[f](u)] = 2)$

b) It's a frequency in the imaginary domain