IPCV H3

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Problem 1 (Sampling Theorem) (2 points)

A music streaming service uses a sampling rate of 44.1 kHz for their audio. What is the bound on the highest frequency that can be represented? Justify your answer.

$44.1\,kHz = f_s$ - sampling rate

$f_s = 2W$, W - max frequency

$W = \frac{44.1}{2}\,kHz = 22.05\,kHz$

Sampling a higher frequency signal would lead to aliasing artifacts.

W is the highest frequency that can be represented and perfectly reconstructed

Problem 2 (Discrete Fourier Transform) (1+1+1+2+1 points)

Consider the following noisy discrete signal:

$$\boldsymbol{f} := (16, 12, 16, 12, 8, 4, 8, 4)^\top .$$

(a) Compute the Fourier coefficients $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$ and $\hat{f}_4$ of $\boldsymbol{f}$.

(b) Determine the remaining Fourier coefficients $\hat{f}_5$, $\hat{f}_6$ and $\hat{f}_7$ of $\boldsymbol{f}$ by using the properties of the DFT and the values computed in (a).

(c) Which coefficient corresponds to the highest frequency?

(d) Compute the signal which you obtain after removing the highest frequency of $\boldsymbol{f}$.

(e) What effects are achieved by eliminating the highest frequency?
How do you explain these effects?

$\sqrt{8} = 2\sqrt{2}$

a) $\hat{f}_0 = f^T b_0$ $\qquad b_p = \frac{1}{\sqrt{M}}\left(e^{\frac{i2\pi p 0}{M}}, \dots, e^{\frac{i2\pi p(M-1)}{M}}\right)$

$M=8$ $\quad b_0 = \frac{1}{\sqrt{8}} \cdot (1,1,1,1,1,1,1,1)$

$$b_1 = \frac{1}{\sqrt8}\left(e^{i\cdot0}, e^{\frac{i\pi 1}{4}}, e^{\frac{i\pi 2}{4}}, e^{\frac{i\pi 3}{4}}, e^{\frac{i\pi 4}{4}}, e^{\frac{i\pi 5}{4}}, e^{\frac{i\pi 6}{4}}, e^{\frac{i\pi 7}{4}}\right)$$

$$= \frac{1}{\sqrt8}\left(1, e^{\frac{i\pi}{4}}, i, e^{\frac{i\pi 3}{4}}, -1, e^{\frac{i\pi 5}{4}}, -i, e^{\frac{i\pi 7}{4}}\right) = \frac{1}{\sqrt8}\left(1, \frac{1}{\sqrt2}+\frac{i}{\sqrt2}, i, \frac{-1}{\sqrt2}+\frac{i}{\sqrt2}, -1, \frac{-1}{\sqrt2}-\frac{i}{\sqrt2}, -i, \frac{1}{\sqrt2}-\frac{i}{\sqrt2}\right)$$

$$b_2 = \frac{1}{\sqrt8}\left(e^{\frac{i\pi 0}{2}}, e^{\frac{i\pi}{2}}, e^{\frac{i\pi 2}{2}}, e^{\frac{i\pi 3}{2}}, e^{\frac{i\pi 4}{2}}, e^{\frac{i\pi 5}{2}}, e^{\frac{i\pi 6}{2}}, e^{\frac{i\pi 7}{2}}\right)$$

$$= \frac{1}{\sqrt8}(1, i, -1, -i, 1, i, -1, -i)$$

$18 \to \frac{2\pi}{4}$

$$b_3 = \frac{1}{\sqrt8}\left(e^{i\cdot0}, e^{\frac{i\pi 3}{4}}, e^{\frac{i\pi 3\cdot2}{4}}, e^{\frac{i\pi 3\cdot3}{4}}, e^{\frac{i\pi 3\cdot4}{4}}, e^{\frac{i\pi 3\cdot5}{4}}, e^{\frac{i\pi 3\cdot6}{4}}, e^{\frac{i\pi 3\cdot7}{4}}\right)$$

$$= \frac{1}{\sqrt8}\left(1, e^{\frac{i\pi 3}{4}}, -i, e^{\frac{i\pi 9}{4}} \to \frac{1\pi i}{4}, -1, e^{\frac{i\pi\cdot 7}{4}}, +i, e^{\frac{i\pi 5}{4}}\right) = \frac{1}{\sqrt8}\left(1, \frac{-1}{\sqrt2}+\frac{i}{\sqrt2}, -i, \frac{1}{\sqrt2}+\frac{i}{\sqrt2}, -1, \frac{1}{\sqrt2}-\frac{i}{\sqrt2}, +i, \frac{-1}{\sqrt2}-\frac{i}{\sqrt2}\right)$$

$$b_4 = \frac{1}{\sqrt8}\left(1, e^{i\pi}, e^{i\pi 2}, e^{i\pi 3}, e^{i\pi 4}, e^{i\pi 5}, e^{i\pi 6}, e^{i\pi 7}\right)$$

$$= \frac{1}{\sqrt8}(1,-1,1,-1,1,-1,1,-1)$$

$$f = (16,12,16,12,8,4,8,4)^T$$

$$\hat{f}_0 = \frac{1}{\sqrt8}(16+12+16+12+8+4+8+4) = \frac{80}{\sqrt8} = \frac{40}{\sqrt2} = 20\sqrt2$$

$$\hat{f}_1 = \frac{1}{\sqrt8}(16-16i-8+8i) + \frac{1}{\sqrt{16}}(12-12i-12-12i-4+4i-4+4i)$$

$$= \frac{\sqrt2}{4}(8-8i) + \frac{1}{4}(-16i) = 2\sqrt2 - 2\sqrt2 i - 4i$$

$$\hat{f}_2 = (16,12,16,12,8,4,8,4)^T \frac{1}{\sqrt{8}} (1, -i, -1, +i, 1, -i, -1, +i)$$

$$= \frac{1}{\sqrt{8}}(16 - 12i - 16 + 12i + 8 - 4i - 8 + 4i) = 0$$

$$\frac{1}{\sqrt{8}}\left(1, \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -i, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, +i, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$\hat{f}_3 = \frac{1}{\sqrt{8}}\left(16 + 16i - 8 - 8i\right) + \frac{1}{4}\left(-12 - 12i + 12 - 12i + 4 + 4i - 4 + 4i\right)$$

$$= \frac{\sqrt{2}}{4}(8 - 8i) + \frac{1}{4}(-16i) = 2\sqrt{2} + i2\sqrt{2} - 4i$$

$$\hat{f}_4 = (16,12,16,12,8,4,8,4)^T \frac{1}{\sqrt{8}}(1,-1,1,-1,1,-1,1,-1)$$

$$\frac{1}{\sqrt{8}}(16 - 12 + 16 - 12 + 8 - 4 + 8 - 4) = \frac{1}{\sqrt{8}}(8+8) = \frac{16}{\sqrt{8}} = 4\sqrt{2}$$

Validation from python script:

index	0		1		2		3	
	Re	Im	Re	Im	Re	Im	Re	Im
f	16	0	12	0	16	0	12	0
b_0	1	0	1	0	1	0	1	0
b_1	1	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	-1	$\frac{-1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
b_2	1	0	0	-1	-1	0	0	1
b_3	1	0	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0	1	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
b_4	1	0	-1	0	1	0	-1	0
b_5	1	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	-1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
b_6	1	0	0	1	-1	0	0	-1
b_7	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	1	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\hat{f}$	$20\sqrt{2}$	0	$2\sqrt{2}$	$-4-2\sqrt{2}$	0	0	$2\sqrt{2}$	$-4+2\sqrt{2}$
$\hat{f}_i b_i$								

index	4		5		6		7	
	Re	Im	Re	Im	Re	Im	Re	Im
f	8	0	4	0	8	0	4	0
b_0	1	0	1	0	1	0	1	0
b_1	-1	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
b_2	1	0	0	-1	-1	0	0	1
b_3	-1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	-1	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
b_4	1	0	-1	0	1	0	-1	0
b_5	-1	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	1	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$
b_6	1	0	0	1	-1	0	0	-1
b_7	-1	0	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0	-1	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$
$\hat{f}$	$4\sqrt{2}$	0	$2\sqrt{2}$	$4-2\sqrt{2}$	0	0	2	$4+2\sqrt{2}$
$\hat{f}_i b_i$								

b)

$$\hat{f}_5 = \mathrm{Re}(\hat{f}_3) - i\,\mathrm{Im}(\hat{f}_3) = 2\sqrt{2} + 4i - 2\sqrt{2}i$$

$$\hat{f}_6 = \mathrm{Re}(\hat{f}_2) - i\,\mathrm{Im}(\hat{f}_2) = 0$$

$$\hat{f}_7 = \mathrm{Re}(\hat{f}_1) - i\,\mathrm{Im}(\hat{f}_1) = 2\sqrt{2} + 2\sqrt{2}i + 4i$$

c) band 4 → nyquist frequency

d) $f = \sum_{i=0}^{7} \hat{f}_i b_i$

$\mathrm{Im}(f) = 0 \land \mathrm{Im}(b_4) = 0 \land \mathrm{Im}(\hat{f}_4) = 0$

$$f - f_4 = f - \hat{f}_4 b_4$$

$$= \begin{bmatrix} 16 - 4\sqrt{2} \\ 12 + 4\sqrt{2} \\ 16 - 4\sqrt{2} \\ 12 + 4\sqrt{2} \\ 8 - 4\sqrt{2} \\ 4 + 4\sqrt{2} \\ 8 - 4\sqrt{2} \\ 4 + 4\sqrt{2} \end{bmatrix} = 4 \begin{bmatrix} 4 - \sqrt{2} \\ 3 + \sqrt{2} \\ 4 - \sqrt{2} \\ 3 + \sqrt{2} \\ 2 - \sqrt{2} \\ 1 + \sqrt{2} \\ 2 - \sqrt{2} \\ 1 + \sqrt{2} \end{bmatrix}$$

Problem 3

a) $$h^1_p = \frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_{m-\frac{1}{2}} \exp\left(\frac{-2\pi i p m}{2M}\right) \cdot$$

$$\cdot \exp\left(\frac{-2\pi i p}{4M}\right) = \frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_{m-\frac{1}{2}} \cdot$$

$$\exp\left(\frac{-2\pi i p (m+\frac{1}{2})}{2M}\right) = \circledast$$

b) Because $\forall_{m=0,1,\dots,M-1}\ g_m = g_{2M-1-m} = f_m$

$$\circledast = \frac{1}{\sqrt{2M}} \sum_{m=0}^{M-1} f_{m-\frac{1}{2}} \left[\exp\left(\frac{-2\pi i p (m+\frac{1}{2})}{2M}\right) + \right.$$

$$\left. + \exp\left(\frac{-2\pi i p (2M-1-(m+\frac{1}{2}))}{2M}\right)\right]$$

c) $$\circledast = \frac{1}{\sqrt{2M}} \sum_{m=0}^{M-1} f_{M-\frac{1}{2}-m} \left[\exp\left(-\frac{2\pi i p}{2M}\left(M-\frac{1}{2}-m\right)\right) + \right.$$

$$\left. + \exp\left(-\frac{2\pi i p}{2M}\left(M-\frac{1}{2}+m\right)\right)\right] =$$

$$= \frac{1}{\sqrt{2M}} \sum_{m=0}^{M-1} f_{M-\frac{1}{2}-m} \left[\exp\left(\frac{-2\pi i p}{2M}\cdot\left(M-\frac{1}{2}\right)\right)\left(\right.\right.$$

$$\left.\left.\exp\left(-\frac{2\pi i p}{2M}\cdot m\right) + \exp\left(\frac{2\pi i p}{2M}\cdot m\right)\right)\right] =$$

$$= \frac{1}{\sqrt{2M}} \sum_{m=0}^{M-1} f_{M-\frac{1}{2}-m} \cdot \exp\left(\frac{2\pi i p}{2M}\cdot\left(M-\frac{1}{2}\right)\right) \cdot$$

$$\cdot \cos\left(\frac{2\pi p m}{2M}\right)$$

Subsequently, $\circledast$ is exactly

$$\tilde{f}_p = \sum_{m=0}^{M-1} f_m c_{p,M} \cos\left(\frac{\pi p (2m+1)}{2M}\right)$$

up to scaling constants.

Problem 4

a) Since $v_i^{k-1} = \dfrac{v_{2i}^k + v_{2i+1}^k}{2}$, $i=0,1,\dots 2^{k-1}-1$

$$v^2 = \left(\frac{7+11}{2}, \frac{4+8}{2}, \frac{5+0}{2}, \frac{8+8}{2}\right)^T = \left(9, 6, \frac{5}{2}, 8\right)^T$$

$$v^1 = \left(\frac{9+6}{2}, \frac{5+16}{4}\right)^T = \left(\frac{15}{2}, \frac{21}{4}\right)^T$$

$$v^0 = \frac{30+21}{8} = \frac{51}{8}$$

b) Since $w_i^k = v_i^k - v_{\lfloor \frac{i}{2} \rfloor}^{k-1}$, $i=0,1,\dots,2^k-1$

$$w^0 = \frac{51}{8}$$

$$w^1 = \left(\frac{15}{2} - \frac{51}{8}, \frac{21}{4} - \frac{51}{8}\right)^T = \left(\frac{9}{8}, -\frac{9}{8}\right)^T$$

$$w^2 = \left(9 - \frac{15}{2}, 6 - \frac{15}{2}, \frac{5}{2} - \frac{21}{4}, 8 - \frac{21}{4}\right)^T =$$

$$= \left(\frac{3}{2}, -\frac{3}{2}, -\frac{11}{4}, \frac{11}{4}\right)^T$$

$$w^3 = \left(7-9, 11-9, 4-6, 8-6, 5-\frac{5}{2}, 0-\frac{5}{2}, 8-8, 8-8\right)^T =$$

$$= \left(-2, 2, -2, 2, \frac{5}{2}, -\frac{5}{2}, 0, 0\right)^T$$

Problem 4

c) Since $f_i = v_i^3 = w_i^3 + v_{\lfloor \frac{i}{2} \rfloor}^2$ $i=0,1,\dots,8$

$$f = \left(-2+9, 2+9, -2+6, 2+6, \frac{5}{2}+\frac{5}{2}, -\frac{5}{2}+\frac{5}{2}, 0+8, 0+8\right)^T =$$

$$= (7, 11, 4, 8, 5, 0, 8, 8)^T$$

d) For each pair of w_{2i}^k, w_{2i+1}^k, $i=0,1,\dots,2^k-1$ we have $w_{2i}^k = -w_{2i+1}^k$. We could then only consider half of the values from each level, and compose them into a pyramid with a matrix of $2^k \times 2^{k-1}$ size and values

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \cdots & -1 \end{pmatrix}$$

without loss of information.

7.5

We used a gaussian filter, as it doesn't produce significant artifacts, as eg. the pillbox filter could.

In our experiments, the sigma of 4 px gave the best results.

We used the formula

$$G(u) = e^{(-2\pi^2\sigma^2 d^2)}$$

where the d is the distance from central point.

We used this formulation, as it reaches the value of 1 near the center - which results in no change of the lowest frequencies.

We performed the filtering only in the x direction, as the artifacts are vertical lines.

On both images, there were small artifacts on edges caused by border discontinuities.

Filtering using a pillbox filter would show some artifacts on edges of the building on the second image