CS 6375 Spring 2020

Problem Set 1

Netid: SAS19003
Name: Shubham Sanghavi
Email: shubham.sanghavi@utdallas.edu

Warm-Up : Subgradients and More

1. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1], if$ $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$. Using this definition, show that

   (a) $f(x) = wf_1(x)$ is a convex function for $x \in \mathbb{R}^n$ whenever $f_1 : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $w \geq 0$

   Ans:

$$
\begin{aligned}
& \lambda f(x) + (1-\lambda)f(y) \\
= \; & \lambda w f_1(x) + (1-\lambda) w f_1(y) \quad [\because f(x) = wf_1(x)\ ] \\
= \; & w(\lambda f_1(x) + (1-\lambda) f_1(y)) \\
\geq \; & w f_1(\lambda x + (1-\lambda)y) \\
& \qquad [\because w \geq 0 \qquad \text{and} \\
& \qquad \because f_1 \text{ is a convex function} \\
& \qquad \qquad \iff \lambda f_1(x) + (1-\lambda) f_1(y) \geq f_1(\lambda x + (1-\lambda)y)\ ] \\
= \; & f(\lambda x + (1-\lambda)y) \\
\therefore \; & \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y \\
\therefore \; & f \text{ is a convex function}
\end{aligned}
$$

   (b) $f(x) = f_1(x) + f_2(x)$ is a convex function for $x \in \mathbb{R}^n$ whenever $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex functions

   Ans:

$$
\begin{aligned}
& \lambda f(x) + (1-\lambda)f(y) \\
= \; & \lambda(f_1(x) + f_2(x)) + (1-\lambda)(f_1(y) + f_2(y)) \\
= \; & \lambda f_1(x) + (1-\lambda) f_1(y) + \lambda f_2(x) + (1-\lambda) f_2(y) \\
\geq \; & f_1(\lambda x + (1-\lambda)y) + f_2(\lambda x + (1-\lambda)y) \\
& \qquad [\because f_1 \text{ and } f_2 \text{ are convex functions }] \\
= \; & f(\lambda x + (1-\lambda)y) \\
\therefore \; & \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y \\
\therefore \; & f \text{ is a convex function}
\end{aligned}
$$

(c) $f(x) = max\{f_1(x), f_2(x)\}$ is a convex function for $x \in \mathbb{R}^n$ whenever $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex functions

Ans:

$$\begin{array}{ll}
 & \lambda f(x) + (1-\lambda) f(y) \\
= & \lambda\ max\{f_1(x), f_2(x)\} + (1-\lambda)\ max\{f_1(y), f_2(y)\} \\
\geq & \lambda f_1(x) + (1-\lambda) f_1(y) \qquad \text{and} \qquad \geq \quad \lambda f_2(x) + (1-\lambda) f_2(y) \\
 & \quad [\ \because\ max\{f_1(x), f_2(x)\} \geq f_1(x)\ ] \qquad [\ \because\ max\{f_1(x), f_2(x)\} \geq f_2(x)\ ] \\
\geq & f_1(\lambda x + (1-\lambda) y) \qquad \text{and} \qquad \geq \quad f_2(\lambda x + (1-\lambda) y) \\
 & \quad [\ \because\ f_1 \text{ is a convex function }] \qquad [\ \because\ f_2 \text{ is a convex function }] \\
\Leftrightarrow & \lambda f(x) + (1-\lambda) f(y) \geq max\{f_1(\lambda x + (1-\lambda) y), f_2(\lambda x + (1-\lambda) y)\} \\
\Leftrightarrow & \lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \\
\therefore & f \text{ is a convex function}
\end{array}$$