Time Series Analysis - Assignment 2
Introduction to ARCH and GARCH model

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April 23, 2023

Introduction

In this assignmnegt we're going to provide an overview of two time-series models — ARCH and GARCH. These models are also called volatility models. These statistical models are used to analyze and forecast time series data that exhibit volatility clustering and heteroscedasticity(i.e. the data with changing variance over time).

In a time series, heteroskedasticity refers to the phenomenon where the variance of the data changes over time. This is in contrast to homoskedasticity, where the variance remains constant over time.

ARCH models assume that the variance of a time series is a function of past squared residuals. This means that the variance of the series is not constant over time, but rather varies depending on the past values of the series. GARCH models build on the ARCH model by including additional parameters that allow for the analysis of more complex patterns in volatility.

Both ARCH and GARCH models are widely used in finance and economics for modeling the volatility of asset prices, exchange rates, and other financial variables. They have also found applications in other fields, such as engineering and environmental science, where modeling the variability of a process over time is important.

What is ARCH Model?

ARCH is an Autoregressive model with Conditional Heteroskedasticity.

The ARCH process introduced by Engle (1982) explicitly recognizes the difference between the unconditional and the conditional variance allowing the latter to change over time as a function of past errors. ARCH models were created in the context of econometric and finance problems having to do with the amount that investments or stocks increase (or decrease) per time period, so there's a tendency to describe them as models for that type of variable.

This is a model for the variance of a time series. ARCH models are used to describe a changing, possibly volatile variance. Although an ARCH model could possibly be used to describe a gradually increasing variance over time, most often it is used in situations in which there may be short periods of increased variation. (Gradually increasing variance connected to a gradually increasing mean level might be better handled by transforming the variable.)

ARCH model is exclusively used in the finance industry as many asset prices are conditional heteroskedastic. ARCH means Autoregressive Conditional Heteroskedasticity and GARCH means Generalized Autoregressive Conditional Heteroskedasticity.

ARCH(p) model is simply an AR(p) model applied to the variance of a time series.

ARCH(1)

To model a time series y_t using an ARCH process, let ϵ_t denote the error terms (return residuals, with respect to a mean process), i.e. the series terms which follows $N(0, \sigma_t^2)$. Then the ARCH(1) model for y_t will be specified as a mean process and a volatility process,

$$y_t = \mu_t + \epsilon_t$$
$$\epsilon_t = w_t \times \sigma_t$$
$$\sigma_t = \omega + \alpha_1 \epsilon_{t-1}^2$$

where μ_t is the conditional mean at time t, w_t is white noise - $w_t \overset{iid}{\sim} N(\mu = 0, \sigma^2 = 1)$; ω, α_1 are parameters of the model and $\omega > 0, \alpha_1 \geq 0$ to ensure that the conditional variance is positive. σ_{t-1}^2 is lagged square error.

Similarly, **ARCH(2):**

The volatility process is,

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2$$

Hence, **ARCH(P):**

$$y_t = \mu_t + \epsilon_t$$
$$\epsilon_t = w_t \times \sigma_t$$
$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2$$

p is the number of lag squared residual errors to include in the ARCH model. $i = (1, 2, 3, ..., p)$ tells us the number of logged periods of the square error.

What is GARCH model?

In most of the applications of the ARCH model a relatively long lag in the conditional variance is often called for, and this leads to the problem of negative variance and non-stationarity. To avoid this problem, generally a fixed lag structure is typically imposed. So it is necessary to extent the ARCH models to a new class of models allowing for a both long memory and much more flexible lag structure.

Bollerslev (1986, Journal of Econometrics) generalized Engle's ARCH model and introduced the GARCH or Generalized Autoregressive Conditional Heteroskedasticity model which is an extension of the ARCH model that incorporates a moving average component together with the autoregressive component. GARCH is the "ARMA equivalent" of ARCH, which only has an autoregressive component. GARCH models permit a wider range of behavior more persistent volatility. Introduction of moving average component allows to model the conditional change in variance over time and changes in the time-dependent variance.

GARCH model of order p,q:

GARCH(1,1):

Let the time series be $\{y_t\}$ and ϵ_t denote the error terms, $\epsilon_t \overset{iid}{\sim} N(0, \sigma_t^2)$; Here we are going to consider a single autoregressive lag and a single "moving average" lag. Then with a mean process and volatility process the model is given by the following:

$$y_t = \mu_t + \epsilon_t$$

$$\epsilon_t = w_t \times \sigma_t$$

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where, w_t is white noise - $w_t \overset{iid}{\sim} N(\mu = 0, \sigma^2 = 1)$

Hence, **GARCH(p,q):**

A time-series y_t with error terms $\epsilon_t [where, \epsilon_t \overset{iid}{\sim} (0, \sigma_t^2)]$ the GARCH(p,q) model is given by,

$$y_t = \mu_t + \epsilon_t$$

$$\epsilon_t = w_t \times \sigma_t$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

where p is the order of the ARCH terms ϵ^2, q is the order of the GARCH terms σ^2. α_i and β_j are parameters of the model.
$\omega > 0$, $\alpha_i \geq 0$, $i = 1, 2, ..., q$ and $\beta_j \geq 0$, $j = 1, 2, ..., p$ imposed to ensure that the conditional variances are positive.

GARCH(zoo):

The term "zoo" in GARCH(zoo) refers to the different variations and extensions of the GARCH model that have been developed over time. Each variation of the GARCH(zoo) model has its own set of assumptions and specifications for the conditional variance equation. The GARCH(zoo) includes various forms of GARCH models. Here we've given the volatility process of some such GARCH model variations.

- **IGARCH:** Integrated GARCH model, which allows for the possibility of long-term persistence in volatility.

$$\sigma_t^2 = \omega + \epsilon_{t-1}^2 + \sum_{i=2}^{p} \alpha_i(\epsilon_{t-i}^2 - \epsilon_{t-1}^2) + \sum_{j=1}^{q} \beta_j(\sigma_{t-j}^2 - \epsilon_{t-1}^2)$$

- **TGARCH:** Threshold GARCH model, which allows for different volatility dynamics in different market regimes.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i[(1-\gamma_i)\epsilon_{t-i}^+ - (1+\gamma_i)\epsilon_{t-i}^-] + \sum_{j=1}^{q} \beta_j\sigma_{t-j}^2$$

where, $\epsilon_{t-i}^+ = \epsilon_{t-i}$ if $\epsilon_{t-i} > 0$ and $\epsilon_{t-i}^+ = 0$ if $\epsilon_{t-i} \leq 0$. Likewise, $\epsilon_{t-i}^- = \epsilon_{t-i}$ if $\epsilon_{t-i} \leq 0$ and $\epsilon_{t-i}^- = 0$ if $\epsilon_{t-i} > 0$.

- **VGARCH:** Vector GARCH model is used to model and forecast the volatility of a system of two or more time series. The model allows for the modeling of the dynamic interdependencies between the time series, which is important in financial econometrics, where the volatility of one asset can affect the volatility of other assets.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i(\epsilon_{t-i} + \gamma_i)^2 + \sum_{j=1}^{q} \beta_j\sigma_{t-j}^2$$

e_t is a n-dimensional vector of residual errors at time t assumed to be normally distributed.

- **NGARCH:** Nonlinear GARCH model, which allows for nonlinear relationships between past returns and volatility.

$$\sigma_t^\delta = \omega \sum_{i=1}^{p} \alpha_i|\epsilon_{t-i}|^\delta + \sum_{j=1}^{q} \beta_j\sigma_{t-j}^\delta$$

- **EGARCH:** Exponential GARCH model, which allows for asymmetric responses to positive and negative shocks.

$$ln(\sigma_t^2) = \omega + \sum_{i=1}^{p}(\alpha_i\epsilon_{t-i} + \gamma_i\epsilon_{t-i}) + \sum_{j=1}^{q} \beta_j ln\sigma_{t-j}^2$$

There are also some other variations in GARCH(zoo) like **GJR-GARCH**(Glosten-Jagannathan-Runkle GARCH), **AGARCH**(Asymmetric GARCH), **log-GARCH** etc.

The **GARCH(zoo)** model is widely used in finance and economics for modeling asset returns, volatility, and risk management. It has also been applied in other fields, such as climatology and environmental sciences, to model temperature fluctuations and natural disasters. The flexibility of the GARCH(zoo) model makes it a powerful tool for analyzing time series data with complex and changing volatility patterns.

Application of ARCH/GARCH model:

- **Risk management:** ARCH/GARCH models are used to model and forecast financial risk, which is essential for risk management in finance. By modeling the volatility of financial returns, ARCH/GARCH models can help investors and financial institutions better understand the risks associated with various financial assets and portfolios.
- **Option pricing:** Option prices are heavily influenced by the volatility of the underlying asset. ARCH/GARCH models can be used to estimate the volatility of the underlying asset, which can then be used to price options more accurately.
- **Asset pricing:** ARCH/GARCH models can be used to model the conditional variance of asset returns, which can help explain the pricing of financial assets.
- **Portfolio optimization:** ARCH models can be used to estimate the volatility of different financial assets, which is an important input for portfolio optimization. By modeling the volatility of different assets, investors can construct portfolios that maximize expected returns while minimizing risk.
- **Forecasting:** ARCH/GARCH models can be used to forecast future volatility in financial time series data. This is important for financial planning and risk management purposes.
- **Econometrics:** ARCH/GARCH models are widely used in econometrics to model the volatility of macroeconomic variables, such as inflation and exchange rates.

Implementation of ARCH/GARCH Model

We've used Yahoo finance dataset to implement ARCH/GARCH model. This is a stock market's continuous data, it contains heteroskedasticity. So ARCH/GARCH model will be appropriate for this data. We're estimaating the volatility of stock returns through these models.
Now, let's visualize the data -

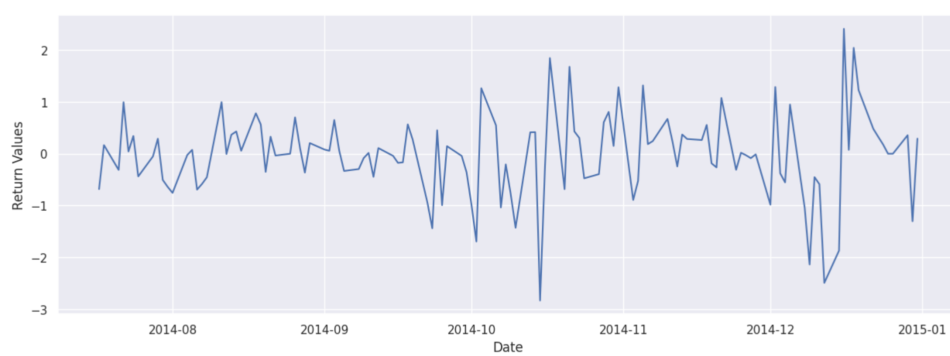

Figure 1: Plot of return values against the dates

From the graph of return values we can see that there is a high volatality in our data. First we've tried to fit the ARCH model in our data. So to estimate the ARCH component we've examined the A.I.C. and negative log-likelihood for several values of p and we've seen that for $p = 13$ we're getting our optimal ARCH model. Hence, we're fitting **ARCH(13)** model to forecast the returns.

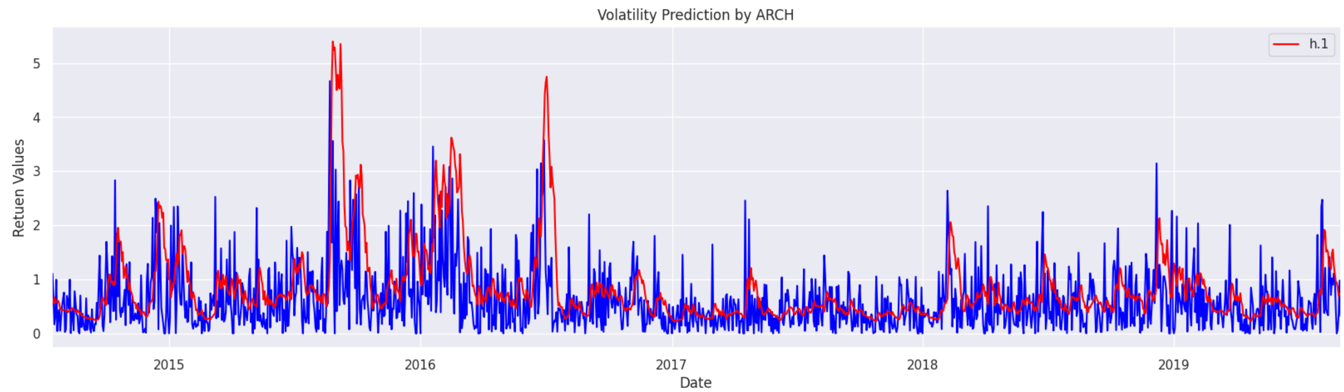

Figure 2: This the plot of ARCH prediction. Here the blue-line is the original return values and red-line is the fitted return values

Next we've moved to GARCH model. Here, we've estimated the order of GARCH model by performing grid-search using negative log-likelihood and A.I.C. We got our optimal $p = 2$ and optimal$q = 1$. Hence, we're fitting **GARCH(2,1)** model to forecast the returns.

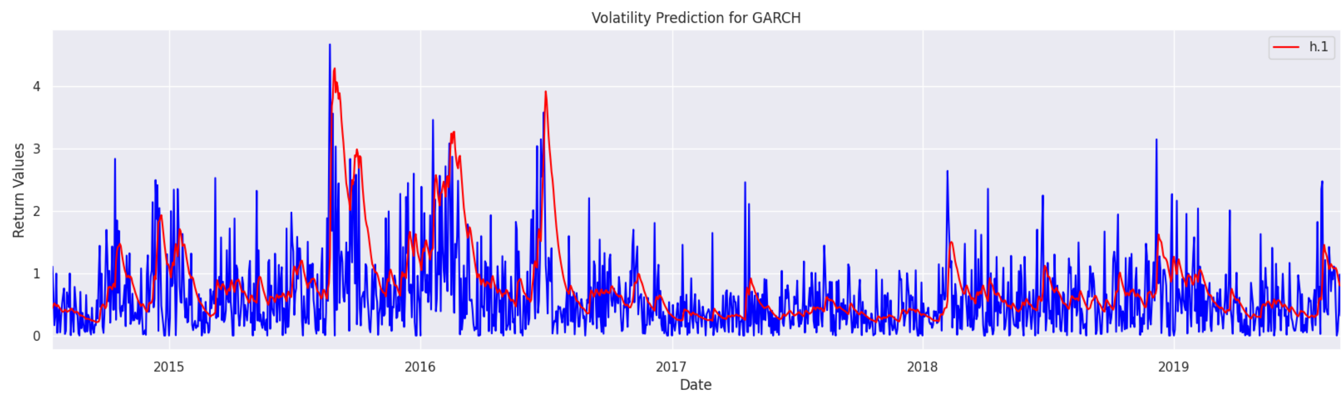

Figure 3: This the plot of GARCH prediction. Here the blue-line is the original return values and red-line is the fitted return values

Model Comparison:

Now let compare the accuracy of our time series models to determine which model performs best in terms of forecasting future values of our time series data. Here we've used Mean Absolute Error(MAE) and Root Mean Square(RMSE) to compare the accuracy of time series models:

Model	MAE	RMSE
ARCH	0.53126	0.21051
GARCH	0.51013	0.17720

Figure 4: MAE and RMSE metrics for comparing accuracy of ARCH and GARCH models

From the table we can see that both of the MAE and RMSE values are less in GARCH model compare to ARCH. So, we can conclude that GARCH model is performing better in forecasting future values of our dataset.