Homework - 3

1a) Poisson Distribution for an iid Sample

$L(\theta|y) = p(y|\theta)$

$$= \prod_{i=1}^{5} \left(\frac{\theta^{y_i} \exp(-\theta)}{y_i!} \right)$$

$y_i = [3\ 5\ 1\ 4\ 4]$

~~$L(\theta)$~~

$L(\theta|y) = p(y|\theta)$

$$= \frac{\theta^3 \times \exp(-\theta)}{3!} \times \frac{\theta^5 \exp(-\theta)}{5!}$$

$\times \ldots$

$$= \frac{\theta^{17} \times \exp(-5\theta)}{3! \times 5! \times 1! \times 4! \times 4!}$$

1b) We know,

$$p(\theta|y) \propto p(y|\theta) \times p(\theta)$$

[Bayes Theorem]

~~$P(\theta)$ -~~

$$p(\theta) = ab^a \theta^{-(a+1)}, \quad a > 1, \theta \geq b$$

$$p(\theta|y) \propto \frac{\theta^{17} \times \exp(-5\theta)}{3! \times 5! \times 1! \times 4! \times 4!} \times ab^a \theta^{-(a+1)}$$

$$\propto \frac{\theta^{(16-a)} \times \exp(-5\theta) \times ab^a}{3! \times 5! \times 1! \times 4! \times 4!}$$

$$\propto \frac{\theta^{16-a} \times \exp(-5\theta) \times ab^a}{414720}$$

$6 \times 120 \times 24 \times 24$

$$\frac{\theta^{y} \exp(-\theta)}{y!} \times a b^{a} \theta^{-(a+1)}$$

$$= \frac{\theta^{y-a-1} \times \exp(-\theta) \times a b^{a}}{y!}$$

1c) By looking at posterior distribution, we can see a term of $\exp(-5\theta)$, which is not there in the prior pareto distribution. So, we can say that pareto distribution is not a conjugate prior for poisson likelihood.

1d) Plots stored in folder Figures.

Distribution of simulated theta samples
- plot of histogram of simulated theta

Alignment of true posterior and sampled posterior density - aligns both density.

1c)

$Prob(\theta \geq 5 | y) = 0.033$

$Prob(\theta \in [4, 5] | y] = 0.967$

$Prob(\theta \leq 3 | y) = 0$

as $b = 4$, and $\theta \geq 4$

~~In~~ 1f) Credible set - 80%

- [2 7]

All code in file 1d.R

2) a)

$Y \sim \text{Binomial}(n, \theta)$

$$P(Y|n,\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$P(n|m) = \frac{m^n}{n!} \exp(-m)$$

$$P(n|m,Y,\theta) \propto P(Y|n,\theta) \times P(n|m)$$

$$\propto \binom{n}{y} \theta^y (1-\theta)^{n-y} \times \frac{m^n}{n!} \times \exp(-m)$$

Normalizing Constant:

$$\Sigma P(n|m,Y,\theta) = \Sigma \binom{n}{y} \theta^y (1-\theta)^{n-y} \times \frac{m^n}{n!} \times \exp(-m)$$

$$= \Sigma \binom{n}{y} \frac{(m(1-\theta))^n}{n!} \times \exp(-m) \times \left(\frac{\theta}{1-\theta}\right)^{-y}$$

Posterior Distribution

$$p(n \mid m, y, \theta) = \frac{\binom{n}{y}\cancel{\left(\frac{\theta}{1-\theta}\right)^{y}} \times (1-\theta)^{n} \times \frac{m^{n}}{n!} \times \cancel{\exp(-m)}}{\sum \binom{n}{y}\left(\frac{m(1-\theta)}{n!}\right)^{n} \times \cancel{\left(\frac{\theta}{1-\theta}\right)^{y}} \times \cancel{\exp(-m)}}$$

$$= \frac{\binom{n}{y}\,\frac{(m(1-\theta))^{n}}{n!}}{\sum_{n} \binom{n}{y}\,\frac{(m(1-\theta))^{n}}{n!}}$$

2b)

$$p(u|y) \propto \frac{[m(1-\theta)]^{u}}{u!}$$

Let us choose a value of y,

Generate a value of u from a normal distribution with mean of 300 and sigma of 1.

If data is less than zero, set it to zero.

Compute posterior at that u, for m and θ,

Compute the ratio of posterior for new and old u.

Generate a random uniform sample, if it is less than accept prob, We accept the sample or reject it.

2c) 90% Credible set of n is $[3.101, 9.348]$

2d) 0.75

All code in the file Problem 2c.R