Descent into Chaos

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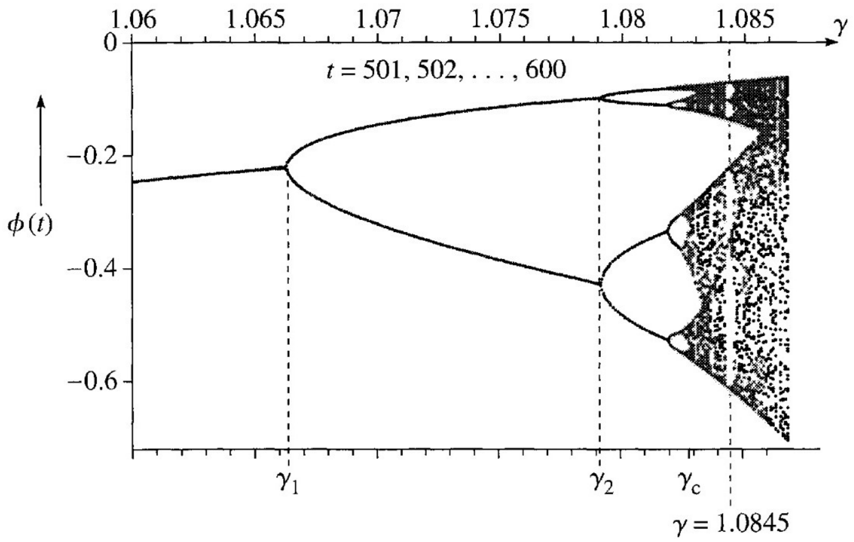
The damped, driven pendulum is a standard example of a highly nonlinear differential equation exhibiting features commonly studied in Chaos Theory. Here we assume the pendulum is driven by a harmonic force, so that the equation of motion can be written as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\omega_0^2 sin\phi - 2\beta\frac{\mathrm{d}\phi}{\mathrm{d}t} + \gamma\omega_0^2 cos(\omega t) \tag{1}$$

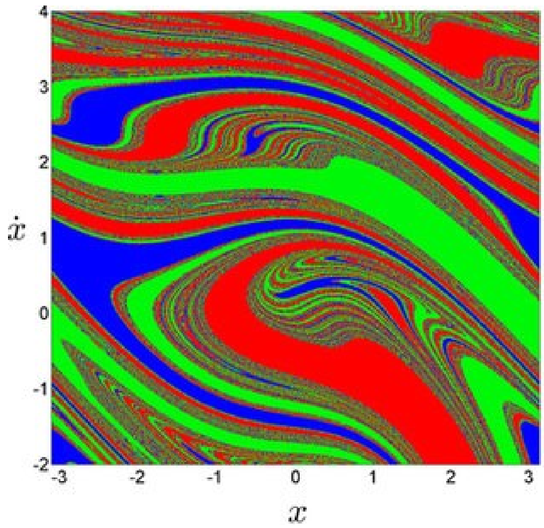
where $\phi(t)$ is the angle from the vertical, and the mass has been divided out and absorbed into the coefficients. Our goal will be to examine the behaviour of the pendulum for different parameters. Any ODE solver can be used for this problem, although one should choose an accurate one. This problem will require simulations that may take an hour or more.

a) Reproduce the bifurcation diagram below from Chapter 12 of "Classical Mechanics" by John R. Taylor. The parameters are $\omega = 2\pi$, $\omega_0 = 1.5\omega$, $\beta = \omega_0/4$, $\phi(0) = -\pi/2$ and $\frac{\mathrm{d}\phi}{\mathrm{d}t}(0) = 0$. This can be done as follows: At each γ, solve the ODE between times 0 and 600. With exception of regions at high γ where "chaos" kicks in, you will find that after some transients ϕ undergoes n-cycles of motion: 1 cycle below γ_1, 2 cycles between γ_1 and γ_2 etc. This particular bifurcation diagram was found by plotting $\phi(501), \phi(502), ..., \phi(600)$, as the periods in this example are $\Delta t = 1$. If you visually find more points per γ than in the figure, then your ODE solver is not accurate enough.



b) A useful way to represent the motion of the pendulum is in phase space, where the x-axis is $\phi(t)$ and the y-axis is $\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)$. Plot the orbits in phase space for three cases: 1) $\gamma = 1.08$, 2) $\gamma = 1.0845$ and 3) A single choice of $1.0829 < \gamma < 1.085$. For the first two cases, remove the part of the orbit corresponding the transient, and confirm the presence of four and six periods respectively. For the final case, you should find that the orbits have random behaviour - this is the footprint of chaos! In these regions there are no cycles, and the pendulums motion is highly sensitive to the initial conditions.

c) If we change the initial conditions slightly when we are not in a regime of chaos, then we will find that the steady state behavior doesn't change. If we change the initial conditions a little more, we may sometimes find ourselves with a different steady state solution. These different solutions in in the phase space of initial conditions are called basins of attractions. Here our goal is to reproduce the following figure in [1] with the parameters $w = w_0 = 1$, $\beta = 0.1$, $\gamma = 1.66$ (ϕ is denoted as x). Red, Green and Blue represent three distinct steady state solutions. The resolution of this image is 1000x1000, but since we just want to confirm the general features, 300x300 should be sufficient. The total time and the period should be appropriately chosen.



[1] A. Daza, A. Wagemakers, M. Sanjuán, and J. Yorke, Scientific Reports 5, 16579 (2015).