1. Evaluate $\lim\limits_{x\to\infty,\, y\to 2} \dfrac{xy+5}{x^2+2y^2}$. (May 2014)

Solution:

$$\lim_{x\to\infty,\, y\to 2} \frac{xy+5}{x^2+2y^2} = \lim_{x\to\infty}\left\{\lim_{y\to 2}\left[\frac{xy+5}{x^2+2y^2}\right]\right\} = \lim_{x\to\infty}\left[\frac{2x+5}{x^2+8}\right] = \lim_{x\to\infty}\left[\frac{x\left(2+\frac{5}{x}\right)}{x^2\left(1+\frac{8}{x^2}\right)}\right] = 0.$$

2. If $u = \dfrac{x+y}{xy}$, find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$. (Apr 2015)

Solution: $u = \dfrac{x+y}{xy} = \dfrac{x}{xy} + \dfrac{y}{xy} = \dfrac{1}{y} + \dfrac{1}{x}$

$\therefore\ \dfrac{\partial u}{\partial x} = -\dfrac{1}{x^2}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{1}{y^2}$.

3. If $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$, find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. (May 2012, Nov 2014)

Solution: $\dfrac{\partial u}{\partial x} = \dfrac{1}{y} - \dfrac{z}{x^2};\ \dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2} + \dfrac{1}{z};\ \dfrac{\partial u}{\partial z} = -\dfrac{y}{z^2} + \dfrac{1}{x}.$

$\therefore\ x\dfrac{\partial u}{\partial x} = \dfrac{x}{y} - \dfrac{z}{x};\ y\dfrac{\partial u}{\partial y} = -\dfrac{x}{y} + \dfrac{y}{z};\ z\dfrac{\partial u}{\partial z} = \dfrac{z}{x} - \dfrac{y}{z}\quad \therefore\ x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

4. If $x = r\cos\theta$ and $y = r\sin\theta$ then find $\dfrac{\partial r}{\partial x}$. (Jan. 2018)

Solution: $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$

Therefore $\dfrac{\partial r}{\partial x} = \dfrac{1}{2\sqrt{x^2+y^2}}.2x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}.$

5. If $z = e^{ax+by} f(ax-by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$.

Solution: $\dfrac{\partial z}{\partial x} = ae^{ax+by}f'(ax-by) + ae^{ax+by}f(ax-by),\ \dfrac{\partial z}{\partial y} = -be^{ax+by}f'(ax-by) + be^{ax+by}f(ax-by)$

$b\dfrac{\partial z}{\partial x} = ab[e^{ax+by}][f'(ax-by) + f(ax-by)],\quad a\dfrac{\partial z}{\partial y} = -ab[e^{ax+by}][f'(ax-by) - f(ax-by)]$

$$\therefore\ b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abe^{ax+by}f = 2abz.$$

6. If $u = x^y$ show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. (Jan 2012)

Solution: $u = x^y = e^{y\log x},\ \dfrac{\partial u}{\partial x} = e^{y\log x}\dfrac{y}{x},\ \dfrac{\partial u}{\partial y} = e^{y\log x}\log x.$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = e^{y\log x}\left(\frac{1}{x}\right) + e^{y\log x}\frac{y}{x}\log x$$

$$u_{xy} = \frac{x^y}{x} + x^y\frac{y}{x}\log x = \frac{x^y}{x}\left[1+y\log x\right] = x^{y-1}\left[1+y\log x\right] \qquad ----(1)$$

$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = e^{y\log x}\left(\frac{1}{x}\right) + \frac{y}{x}e^{y\log x}\log x$$

$$= \frac{e^{y\log x}}{x}\left[1+y\log x\right] = \frac{x^y}{x}\left[1+y\log x\right] = x^{y-1}\left[1+y\log x\right] \qquad ----(2)$$

From (1) and (2), we get $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

8. If $u=\sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$. (Jan 2010)

Solution: Let $\sin u=\frac{x^3-y^3}{x+y}$ $\quad\therefore$ sin u is a homogenous function of degree 2.

$\therefore$ By Euler's theorem, $x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=2\sin u\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=2\sin u$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$$

9. If $z=\log(x^2+xy+y^2)$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2$. (May 2009, June 2010)

Solution: Here z is not a homogeneous function but $e^z=x^2+xy+y^2$ is a homogeneous function of degree 2 in x, y.

By Euler's theorem we have $x\frac{\partial(e^z)}{\partial x}+y\frac{\partial(e^z)}{\partial y}=2e^z\Rightarrow xe^z\frac{\partial z}{\partial x}+ye^z\frac{\partial z}{\partial y}=2e^z\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2.$

10. Using Euler's Theorem, given $u(x,y)$is a homogeneous function of degree n, prove that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=n(n-1)u.$ (Jan 2009)

Solution: By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu----(1).$

Differentiating (1) partially w.r.to x we get $x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$

$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}----(2);$

Differentiating (1) partially w.r.to y we get

$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$

$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}----(3);$

Multiplying (2) by x & (3) by y & adding we get

$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n(n-1)u.$

11. Given $u(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)$. Find the value of $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}$. (May 2010)

Solution: $u(tx,ty)=(tx)^2\tan^{-1}\left(\frac{y}{x}\right)=t^2u(x,y)$. u is a homogeneous function of degree 2 in x and y and it is differentiable partially twice.

$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=2(2-1)u=2u.$

12. Find $\frac{du}{dt}$ if $u=\sin\left(\frac{x}{y}\right)$ where $x=e^t, y=t^2$. (Nov 2010, Jan 2010)

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}=\frac{1}{y}\cos\left(\frac{x}{y}\right)e^t+\left(\frac{-x}{y^2}\right)\cos\left(\frac{x}{y}\right)2t=\frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)-\frac{2e^t}{t^3}\cos\left(\frac{e^t}{t^2}\right).$

13. If $u=x^2+y^2$ where $x=at^2, y=2at$, find $\frac{du}{dt}$. (May 2011, May2015, May 2017)

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}$

$$=(2x)(2at)+(2y)(2a)=4xat+4ay$$

$$=4\left(at^2\right)at+4a(2at)=4a^2t^3+8a^2t.$$

14. Find $\frac{du}{dt}$, if $u = \frac{x}{y}$ where $x = e^t, y = \log t$. (May 2017)

Solution:
$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}\\ &= \frac{1}{y}.e^t + \left(\frac{-x}{y^2}\right).\frac{1}{t}\\ &= \frac{e^t}{\log t} - \frac{e^t}{(\log t)^2.(t)}\\ &= \frac{e^t}{\log t}\left[1 - \frac{1}{t.\log t}\right].\end{aligned}$$

15. Find $\frac{du}{dt}$ in terms of t, if $u = x^3 + y^3$ where $x = at^2, y = 2at$. (April/May 2019)

Solution:
$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}\\ &= (3x^2)(2at) + (3y^2)(2a)\\ &= 3(a^2t^4)(2at) + 6a(4a^2t^2)\\ &= 6a^3t^5 + 24a^3t^2.\end{aligned}$$

16. If $x^y + y^x = c$ then find $\frac{dy}{dx}$. (Jan 14, May 14, Jan 16, Nov/Dec. 2018)
Solution: Given $x^y + y^x = c \Rightarrow x^y + y^x - c = 0$.

Let $f(x,y) = x^y + y^x - c$, $\quad \frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-\left(yx^{y-1} + y^x \log y\right)}{\left(x^y \log x + xy^{x-1}\right)}$.

17. Find $\frac{dy}{dx}$ when $y \sin x = x \cos y$. (June 2010)
Solution: Given $y \sin x = x \cos y \Rightarrow y \sin x - x \cos y = 0$.

Let $f(x,y) = y \sin x - x \cos y \quad \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{(y \cos x - \cos y)}{(\sin x + x \sin y)}$.

18. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (Jan 2010,May/June 2012)

Solution:
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix} = -\frac{y^3}{2x^3} - \frac{y}{x}\left(\frac{x^2-y^2}{2x^2}\right) = -\frac{y^3}{2x^3} - \frac{x^2y}{2x^3} + \frac{y^3}{2x^3} = -\frac{y}{2x}.$$

19. If $x = u^2 - v^2,\ y = 2uv$, find the Jacobian of x and y with respect to u and v. (A/M 2011, Jan 2012, A/M 2019)

Solution: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$.

20. Find $\dfrac{\partial(r,\theta)}{\partial(x,y)}$, if $x = r\cos\theta,\ y = r\sin\theta$ (May/June 2013 Jan2014, N/D 14, Jan2016,M/J 2016)

Solution: $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

w.k.t $\dfrac{\partial(x,y)}{\partial(r,\theta)}\dfrac{\partial(r,\theta)}{\partial(x,y)} = 1 \Rightarrow (r)\dfrac{\partial(r,\theta)}{\partial(x,y)} = 1 \Rightarrow \dfrac{\partial(r,\theta)}{\partial(x,y)} = \dfrac{1}{r}$.

21. If $u = \dfrac{y^2}{x},\ v = \dfrac{x^2}{y}$, evaluate $\dfrac{\partial(x,y)}{\partial(u,v)}$. (Jan 2011)

Solution: $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{y^2}{x^2} & \dfrac{2y}{x} \\ \dfrac{2x}{y} & -\dfrac{x^2}{y^2} \end{vmatrix} = -3$

By property, we have $\dfrac{\partial(u,v)}{\partial(x,y)}.\dfrac{\partial(x,y)}{\partial(u,v)} = 1, \qquad \therefore \dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{3}$.

22. If $x = uv$ and $y = \dfrac{u}{v}$, then find $\dfrac{\partial(x,y)}{\partial(u,v)}$. (Jan 2018)

Solution: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \dfrac{1}{v} & -\dfrac{u}{v^2} \end{vmatrix} = -\dfrac{2u}{v}$.

23. If $u = \dfrac{yz}{x},\ v = \dfrac{zx}{y},\ w = \dfrac{xy}{z}$, find $\dfrac{\partial\,(u,v,w)}{\partial\,(x,y,z)}$. (M/J 2011, Jan 2013, M/J 2014, Jan 2014)

Solution : $\dfrac{\partial\,(u,v,w)}{\partial\,(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\dfrac{yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ \dfrac{z}{y} & -\dfrac{xz}{y^2} & \dfrac{x}{y} \\ \dfrac{y}{z} & \dfrac{x}{z} & -\dfrac{yx}{z^2} \end{vmatrix} = \dfrac{1}{x^2y^2z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$

$= \dfrac{(xy)(yz)(xz)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4.$

24. Show that $u = 2x - y + 3z,\ v = 2x - y - z,\ w = 2x - y + z$ are functionally dependent. (A/M2011)

Solution: $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = -4 + 4 + 3(0) = 0$. Hence u, v, w are functionally dependent.

25. State the properties of Jacobians. (N/D 2018)

Solution: (i) If $u = u(x,y)$ and $v = v(x,y)$ then $\dfrac{\partial(u,v)}{\partial(x,y)}.\dfrac{\partial(x,y)}{\partial(u,v)} = 1$.

(ii) If $u = u(r,s), v = v(r,s)$ and $r = r(x,y), s = s(x,y)$, then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)}.\dfrac{\partial(r,s)}{\partial(x,y)}$.

26. Expand $e^x\log(1+y)$ in powers of x and y upto terms of second degree. (A/M 2008, Jan 2014)

Solution: $f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + ...$$

$$f(x,y) = e^x \log(1+y), \quad Here\, a = 0, b = 0\ f(0,0) = 0$$

$$\frac{\partial f}{\partial x} = e^x \log(1+y), f_x(0,0) = 0; \qquad \frac{\partial^2 f}{\partial x^2} = e^x \log(1+y); f_{xx}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{1+y}, f_y(0,0) = 1; \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-e^x}{(1+y)^2}; f_{yy}(0,0) = -1$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^x}{1+y}; f_{xy}(0,0) = 1$$

Taylor's series is

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + ..$$

$$= y + xy - \frac{y^2}{2} + ...$$

27. Expand $\sin xy$ in powers of $(x-1)$ and $(y - \frac{\pi}{2})$ upto 2^{nd} degree terms. (N/D 2014)

Solution: $f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + ...$$

Here $a = 1, \quad b = \frac{\pi}{2}$

$$f(x,y) = \sin xy, f(1,\frac{\pi}{2}) = 1; \qquad f_x(x,y) = y\cos xy : f_x(1,\frac{\pi}{2}) = 0$$

$$f_{xx}(x,y) = -y^2 \sin xy, f_{xx}(1,\frac{\pi}{2}) = \frac{-\pi^2}{4}; \qquad f_y(x,y) = x\cos xy, f_y(1,\frac{\pi}{2}) = 0$$

$$f_{yy}(x,y) = -x^2 \sin xy, f_{yy}(1,\frac{\pi}{2}) = -1; \qquad f_{xy}(x,y) = -xy\sin xy + \cos xy : f_{xy}(1,\frac{\pi}{2}) = -\frac{\pi}{2}$$

$$f(x,y) = 1 + (x-1)\times 0 + (y-\frac{\pi}{2})\times 0 + \frac{1}{2!}\left[(x-1)^2(\frac{-\pi^2}{4}) + 2(x-1)(y-\frac{\pi}{2})\times(-\frac{\pi}{2}) + (y-\frac{\pi}{2})^2(-1)\right] +$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\frac{\pi}{2}) - \frac{1}{2}(y-\frac{\pi}{2})^2 + ...$$

28. State the necessary conditions for maxima and minima of $f(x,y)$. (A/M 15)

Solution: The necessary conditions for $f(x,y)$ to have a maximum or a minimum at (a,b) are $f_x(a,b) = 0, \ f_y(a,b) = 0$.

29. A flat circular plate is heated so that temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$. Find the coldest point on the plate. (May/June2005)

Solution: $u = x^2 + 2y^2 - x;\ p = u_x = 2x - 1;\ q = u_y = 4y$

$$u_x = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}; u_y = 0 \Rightarrow 4y = 0 \Rightarrow y = 0$$

$$r = u_{xx} = 2;\ t = u_{yy} = 4;\ s = u_{xy} = 0, \qquad rt - s^2 > 0$$

u is minimum at $\left(\frac{1}{2}, 0\right)$. The coldest point is $\left(\frac{1}{2}, 0\right)$.