① Probability distribution

$$P(x) = \begin{cases} q & ; x=0 \\ 1-q & ; x=1 \end{cases}$$

This is a Bernoulli distribution & can be re-written as:-

$$P(x|q) = (1-q)^{x} \cdot q^{1-x}$$

Likelihood of observing n iid points together is given as:-

$$P(x_1|q) \cdot P(x_2|q) \cdot \dots \cdot P(x_n|q)$$

or

$$\prod_{i=1}^{i=n} P(x_i|q)$$

∴ The likelihood function in this case will be

$$\boxed{L = \prod_{i=1}^{i=n} (1-q)^{x_i} q^{1-x_i}} \quad \text{——— (A)}$$

② To find the maximum likelihood estimate of q, we log transform the likelihood L and then differentiate it equating to 0.

Taking $\log_e$ of (A) on both sides.

$$\ln L = \sum_{i=1}^{n} \ln (1-q)^{x_i} + \sum_{i=1}^{n} \ln q^{1-x_i}$$

$$\ln L = \ln(1-q)\sum_{i=1}^{n} x_i + \ln q \sum_{i=1}^{n}(1-x_i)$$

Differentiating wrt q.

$$\frac{1(-1)}{1-q}\sum_{i=1}^{n} x_i + \frac{1}{q}\sum_{i=1}^{n}(1-x_i)$$

$$\Rightarrow \quad \frac{\sum_{i=1}^{n}(1-x_i)}{q} - \frac{\sum_{i=1}^{n} x_i}{1-q}$$

To ~~fi~~ maximize, equating this to 0

$$(1-q)\sum_{i=1}^{n}(1-x_i) - q\sum_{i=1}^{n} x_i = 0$$

$$(1-q)\left[(n) - \sum_{i=1}^{n} x_i\right] - q\sum_{i=1}^{n} x_i = 0$$

$$n - nq - \sum_{i=1}^{n} x_i + q\sum_{i=1}^{n} x_i - q\sum_{i=1}^{n} x_i = 0$$

$$n - nq = \sum_{i=1}^{n} x_i$$

$$q = \frac{n - \sum_{i=1}^{n} x_i}{n} = 1 - \frac{\sum x_i}{n}$$

or $\boxed{q = 1 - \mu}$ where $\mu = \frac{\sum x_i}{n}$ = mean of observations

(3) For the given sample of 30 observations, we have

$$1 \rightarrow 17 \text{ times}$$
$$0 \rightarrow 13 \text{ times}$$

$$q = 1 - \mu$$

Mean of people ~~atter~~ buying the ticket $= \frac{17}{30}$

$$\Rightarrow \quad q = 1 - \frac{17}{30} = \frac{13}{30} = 0.4333$$

(4) If $E(\hat{q}) = q$, then MLE estimate is unbiased.

$$E(\hat{q}) = E\left(1 - \frac{\Sigma x_i}{n}\right)$$

$$= 1 - \frac{1}{n} \sum_{i=1}^{n} E(x_i)$$

$$= 1 - \frac{1}{n} \sum_{i=1} (1-q)$$

$$= 1 - \frac{n(1-q)}{n}$$

$$\boxed{E(\hat{q}) = q}$$