STA-6106- Statistical Computing 1 FALL 2018 -> Quiz 1

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Solution – Problem 1:

We can use a while loop here, which is used when it is not known ahead of time how many loop iterations are needed. So, if we wish to calculate the partial sums of the harmonic series while loop will be beneficial.

$$\sum_{j=1}^{n} 1/j \qquad \sum_{j=1}^{\infty} 1/j = \infty,$$

The following program will produce the first value n such that;

$$\sum_{j=1}^{n} 1/j > 5.$$

n = 1 ## Don't forget

x = 0 ## to initialize

while (x < 5)

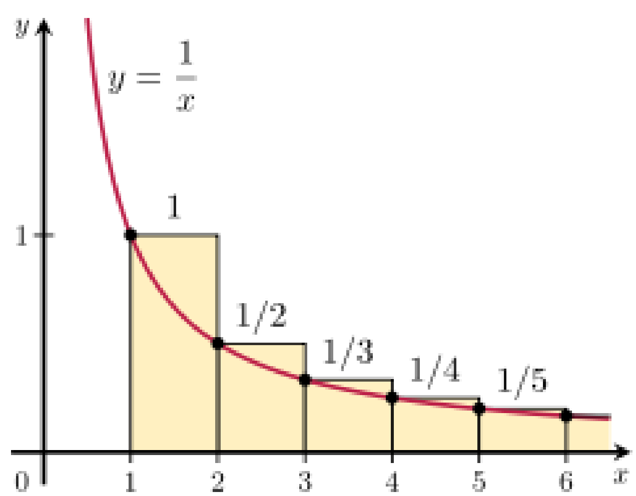
{

x = x + 1/n ## Note that the order of these two statements

n = n+1 ## in the block is important.

}

It is a fact that the harmonic series diverges when n tend towards infinity. This property can be depicted by a graph shown below:



[Src: Wikipedia]

The harmonic series diverges to infinity. Its partial sums form a monotone sequence increasing without bound. The integral estimates are justified geometrically. When Combined, they give

$$\mathbf{\ln(n + 1) < Hn < 1 + \ln n, n > 1.}$$

Therefore, Hn tend to infinity at the same rate as ln n, which is slow. For instance, the sum of the first million terms is H1000000 < 6 ln 10 + 1 ≈ 14.8. Consider now the differences δn = Hn − ln n. Since ln(1 + 1 n) < Hn − ln n < 1, n > 1, we conclude that every δn is a positive number not exceeding 1.

So δn > 0 are monotone decreasing. By the Monotone Sequence Theorem, δn must converge as n → ∞. The limit γ = limn→∞ δn = limn→∞ (Hn − ln n) is called the Euler constant (Euler, 1735), its value is about γ ≈ .5772. Thus, for large n, we have a convenient approximate equality.

$$\mathbf{Hn \approx \ln n + \gamma.}$$

It is not known to this day whether γ is rational or irrational.

So, eventually the partial sum must exceed any given constant. Also we know that Euler discovered this beautiful property of *harmonic number* Hn that

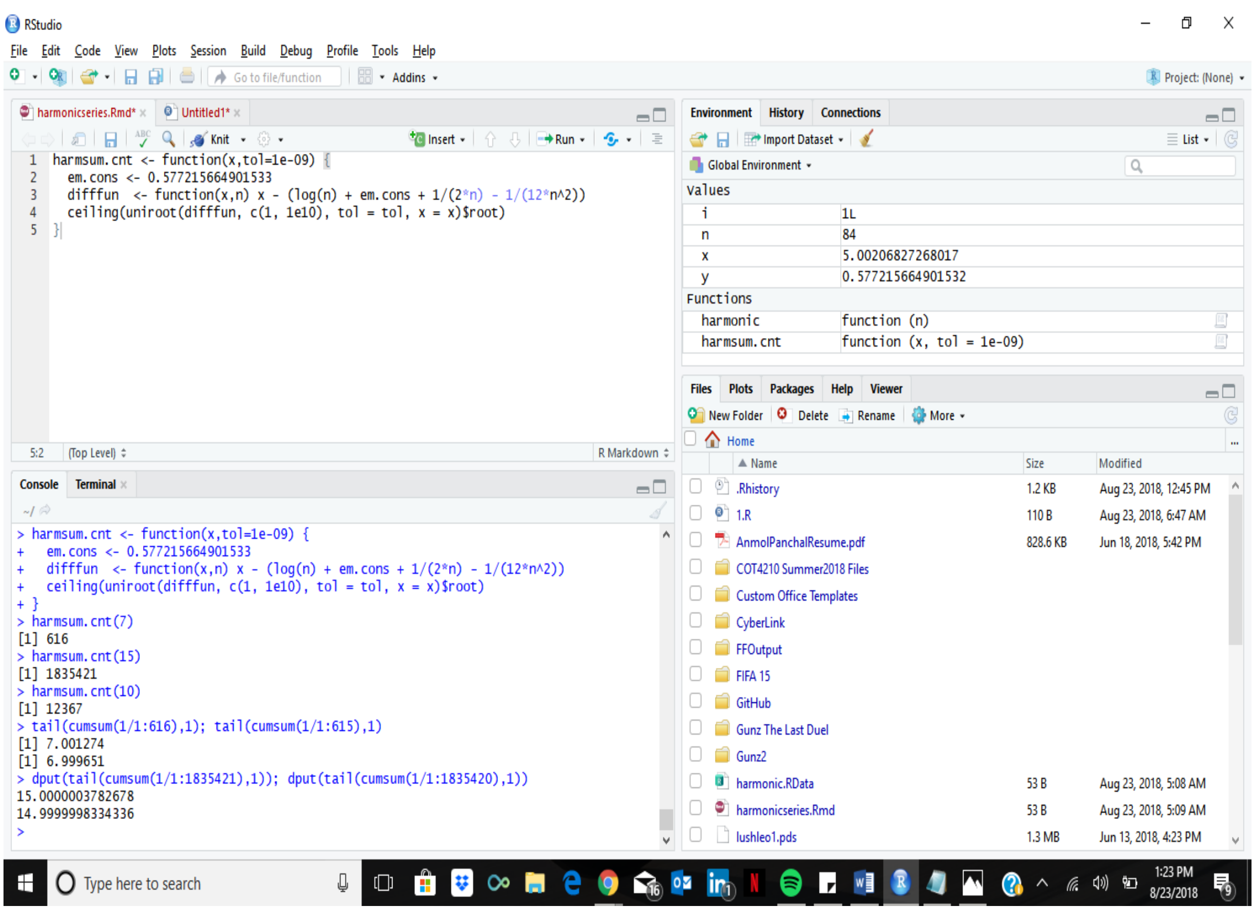
Lim n→∞ (Hn− ln n) = γ ≈ 0 . 5 7 7 2 1 5 6 6 4 9 0 1 5 3 2 8 6 0 6 0 6 5 1 2 0

where γ is called the **Euler-Mascheroni** constant.

So, we can use the following Approximation:

Hn = ln (n) + γ + 1/2n -1/12n*n

So if we write R function to show this problem it would be as shown in figure below:



Code (in console): -

harmsum.cnt <- function(x,tol=1e-09) {

em.cons <- 0.577215664901533

difffun <- function(x,n) x - (log(n) + em.cons + 1/(2*n) - 1/(12*n^2))

ceiling(uniroot(difffun, c(1, 1e10), tol = tol, x = x)$root)

}

```
harmsum.cnt <- function(x,tol=1e-09) {
+   em.cons <- 0.577215664901533
+   difffun  <- function(x,n) x - (log(n) + em.cons + 1/(2*n) - 1/(12*n^2))
+   ceiling(uniroot(difffun, c(1, 1e10), tol = tol, x = x)$root)
+ }
> harmsum.cnt(7)
[1] 616
> harmsum.cnt(15)
[1] 1835421
> harmsum.cnt(10)
[1] 12367
> tail(cumsum(1/1:616),1); tail(cumsum(1/1:615),1)
[1] 7.001274
[1] 6.999651
> dput(tail(cumsum(1/1:1835421),1)); dput(tail(cumsum(1/1:1835420),1))
15.0000003782678
14.9999998334336
```

OR

harmsum <- function(n){

x <- log(n) + (-digamma(1)) + 1/(2*n) - 1/(12*n^2)

return (x)

}

```
> harmsum(10)
[1] 2.928967
> harmsum(616)
[1] 7.001274
```

Solution – Problem 2:

```
Type 'q()' to quit R.

> print( sum( 1/seq(1000)^2 ) )
[1] 1.643935
> print( sum( 1/seq(1000000)^2 ) )
[1] 1.644933
> print( sum( 1/seq(10000)^2 ) )
[1] 1.644834
> print( sum( 1/seq(1000000)^2 ) )
[1] 1.644933
> source('~/.active-rstudio-document', echo=TRUE)

> print( sum( 1/seq(100000000)^2 ) )
[1] 1.644934
> print( sum( 1/seq(500000000)^2 ) )
[1] 1.644934
>
```

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

The sum of the series is approximately equal to 1.644934. And above executed code also compliments the results and show that value successfully matches the actual approximate value evaluated Euler who found the exact sum to be π2/6 and announced this discovery in 1735.

Euler's original derivation of the value π2/6 essentially extended observations about finite polynomials and assumed that these same properties hold true for infinite series.

Of course, Euler's original reasoning requires justification (100 years later, Karl Weierstrass proved that Euler's representation of the sine function as an infinite product is valid, by the Weierstrass factorization theorem), but even without justification, by simply obtaining the correct value, he was able to verify it numerically against partial sums of the series. The agreement he observed gave him enough confidence to announce his result to the mathematical community.

To follow Euler's argument, recall the Taylor series expansion of the sine function

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Dividing through by x, we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Using the Weierstrass factorization theorem, it can also be shown that the left-hand side is the product of linear factors given by its roots, just as we do for finite polynomials (which Euler assumed as a heuristic for expanding an infinite degree polynomial in terms of its roots, but is in general not always true for general P(x).

$$\frac{\sin x}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$
$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

multiply out this product and collect all the x^2 terms;

$$-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

But from the original infinite series expansion of sin x/x, the coefficient of x2 is −1/3! −

1/6. These two coefficients must be equal; thus

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Multiplying both sides of this equation by −π2 gives the sum of the reciprocals of the positive square integers.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

This method of calculating zeta (2) is detailed in expository fashion most notably in Havil's Gamma book which details many zeta function and logarithm-related series and integrals, as well as a historical perspective, related to the Euler gamma constant. When tends towards inifinity indeed converge. When compared the values it give the same answer as Euler equivalent to pi^2/6 as shown below.

```
> print( sum( 1/seq(500000000)^2 ) )

[1] 1.644934
> print (pi^2/6)
[1] 1.644934
> print( sum( 1/seq(100000000)^2 ) )
[1] 1.644934
```

Now for the second part of the problem related to convergence of the infinite series, the infinite sum is defined as mean of the limit of the sum of first n terms as n approaches infinitely.

$$s_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}$$

Now Multiplying Sn by 2 gives;

$$2s_n = \frac{2}{2}+\frac{2}{4}+\frac{2}{8}+\frac{2}{16}+\cdots+\frac{2}{2^n} = 1+\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}\right] = 1+\left[s_n-\frac{1}{2^n}\right].$$

Now Subtracting Sn from both the sides gives;

$$s_n = 1 - \frac{1}{2^n}.$$

Now as n approaches infinitely , Sn tends to 1.