Example 1 : A random number, X, is generated using the probability density function given by

$$f(x)=\begin{cases}2x & , 0<x<1\\ 0 & , \text{otherwise}\end{cases}$$

$X \sim Beta(\alpha = 2, \beta = 1)$

$\mu_X = \frac{2}{3}$, $\sigma_X^2 = \frac{1}{18}$, $E[X^2] = \frac{1}{2}$

Then, given $X = x$, another random number, Y, is generated from the Uniform distribution over the interval $[0, x]$.

a) Find $P(X < b \mid Y = a)$, where $0 < a < b < 1$.

b) Find $P(X < b \mid Y > a)$, where $0 < a < b < 1$.

c) Find $Cov(X, Y)$.

$$\boxed{\begin{array}{l} Y \mid X = x \sim Uniform(0, x). \\ f(y \mid x) = \frac{1}{x}, \quad 0 < y < x, \quad 0 < x < 1. \\ E[Y \mid X = x] = \frac{x}{2}, \quad V[Y \mid X = x] = \frac{x^2}{12}, \\ E[Y^2 \mid X = x] = \frac{x^3}{3}. \end{array}}$$

a) Need $f(x \mid y = a)$.

$$f(x \mid y) = \frac{f(x, y)}{f_Y(y)}.$$

- $f(x, y) = f_X(x) f(y \mid x) = 2x \cdot \frac{1}{x} = 2, \quad 0 < y < x < 1.$
- $f_Y(y) = \int_y^1 2\, dx = 2(1-y), \quad 0 < y < 1.$

$Y \sim Beta(\alpha = 1, \beta = 2)$.

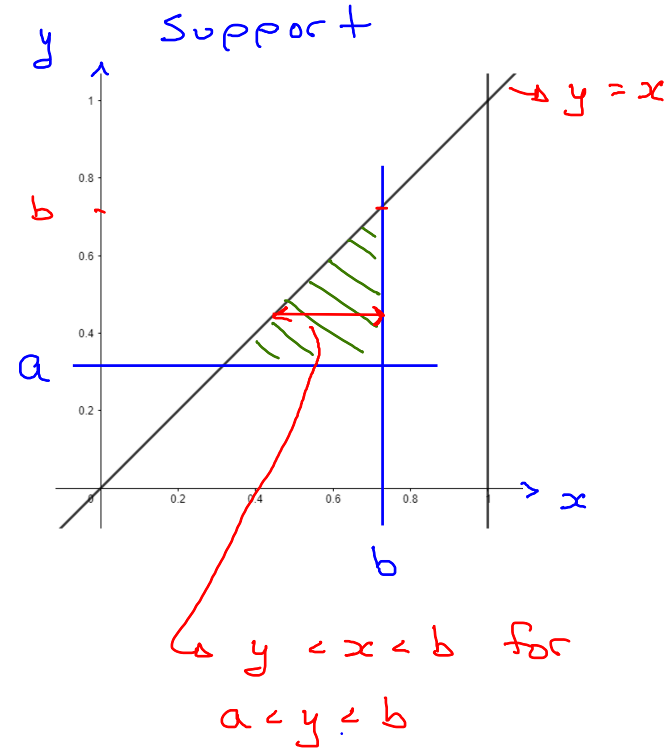
$\mu_Y = \frac{1}{3}$, $\sigma_Y^2 = \frac{1}{18}$, $E[Y^2] = \frac{1}{6}$.

- $f(x \mid y = a) = \frac{f(x, y = a)}{f_Y(a)} = \frac{2}{2(1-a)} = \frac{1}{1-a}, \quad 0 < a < x < 1.$
- $X \mid Y = a \sim Uniform(a, 1)$.

$$P(X < b \mid Y = a) = \int_a^b f(x \mid y = a)\, dx = \int_a^b \frac{1}{1-a}\, dx = \frac{b-a}{1-a},$$

$0 < a < b < 1.$

b) $P(X < b \mid Y > a) = \dfrac{P(X < b,\ Y > a)}{P(Y > a)}$.



• $P(Y > a) = \int_a^1 f_Y(y)\,dy = \int_a^1 2 - 2y\,dy$

$= 2y - y^2\Big]_a^1 = (1-a)^2$.

• $P(X < b,\ Y > a) = \int_a^b \int_y^b 2\,dx\,dy$.

OR $= \int_a^b \int_a^x 2\,dy\,dx = (b-a)^2$.

$$P(X < b \mid Y > a) = \frac{(b-a)^2}{(1-a)^2} = \left(\frac{b-a}{1-a}\right)^2, \quad 0 < a < b < 1.$$

c) $\mathrm{Cov}(X, Y) = E[XY] - \mu_X \mu_Y$

- $X \sim \mathrm{Beta}(\alpha = 2, \beta = 1)$, $\mu_X = 2/3$
- $Y \sim \mathrm{Beta}(\alpha = 1, \beta = 2)$, $\mu_Y = 1/3$

$$E[XY] = \int_0^1 \int_y^1 xy\, f(x, y)\, dx\, dy = \int_0^1 \int_y^1 2xy\, dx\, dy = \cdots = \frac{1}{4}$$

$$\mathrm{Cov}(X, Y) = \frac{1}{4} - \frac{2}{3}\left(\frac{1}{3}\right) = \frac{1}{36}.$$

OR

- $E[XY \mid X = x] = E[xY \mid X = x] = x\,E[Y \mid X = x] = x\left(\frac{x}{2}\right) = \frac{1}{2}x^2.$

$$\Rightarrow E[XY] = E[E[XY \mid X]] = E\left[\frac{1}{2}X^2\right] = \frac{1}{2}E[X^2] = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}.$$

OR

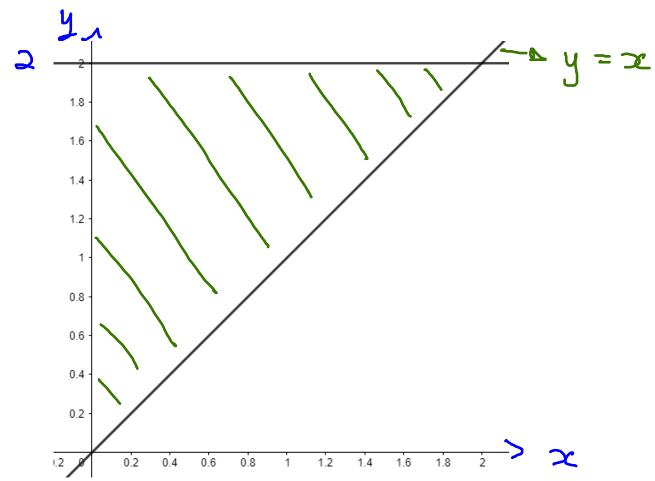
- $E[XY \mid Y = y] = y\,E[X \mid Y = y] = y\left(\frac{y+1}{2}\right) = \frac{1}{2}y^2 + \frac{1}{2}y.$

$$\Rightarrow E[XY] = E[E[XY \mid Y]] = E\left[\frac{1}{2}Y^2 + \frac{1}{2}Y\right]$$

$$= \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{1}{4}.$$

Example 2 : Suppose the joint density of X and Y is

$$f(x,y)=\begin{cases}\frac{3}{4}x & , 0<x<y<2 \\ 0 & , \text{otherwise}\end{cases}$$



$\mu_Y = 3/2$

$\sigma_Y^2 = 0.15$

$E[Y^2] = 2.4$

a) What is $V[X \mid Y=a]$, where $0<a<2$?

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}, \quad f_Y(y) = \int_0^y \frac{3}{4}x\,dx$$

$$= \frac{3}{8}y^2, \quad 0<y<2.$$

$$f(x \mid y=a) = \frac{\frac{3}{4}x}{\frac{3}{8}a^2} = \frac{2x}{a^2}, \quad 0<x<a, \quad 0<a<2.$$

$$E[X \mid Y=a] = \int_0^a x\, f(x \mid y=a)\,dx = \int_0^a \frac{2x^2}{a^2}\,dx = \frac{2}{3}a.$$

$$E[X^2 \mid Y=a] = \int_0^a \frac{2x^3}{a^2}\,dx = \frac{1}{2}a^2.$$

$$V[X \mid Y=a] = E[X^2 \mid Y=a] - (E[X \mid Y=a])^2 = \frac{1}{2}a^2 - \left(\frac{2}{3}a\right)^2 = \frac{a^2}{18}.$$

b) Find μ_X and σ_X^2:

$$\mu_X = E[E[X \mid Y]] = E[\tfrac{2}{3}Y] = \tfrac{2}{3}\mu_Y = \tfrac{2}{3}(\tfrac{3}{2}) = 1.$$

$$\sigma_X^2 = E[V[X \mid Y]] + V[E[X \mid Y]]$$

$$= E[\tfrac{Y^2}{18}] + V[\tfrac{2}{3}Y] = \tfrac{1}{18}E[Y^2] + \tfrac{4}{9}V[Y]$$

$$= \tfrac{1}{18}(2.4) + \tfrac{4}{9}(0.15) = 0.20.$$

OR

$$f_X(x) = \int_x^2 \tfrac{3}{4}x \, dy = \tfrac{3}{4}x(2-x) = \tfrac{3}{2}x - \tfrac{3}{4}x^2, \quad 0 < x < 2.$$

$$\mu_X = \int_0^2 \tfrac{3}{2}x^2 - \tfrac{3}{4}x^3 \, dx = \cdots = 1.$$

$$E[X^2] = \int_0^2 \tfrac{3}{2}x^3 - \tfrac{3}{4}x^4 \, dx = \cdots = 1.20.$$

$$\sigma_X^2 = 1.20 - 1^2 = 0.20.$$

Example 3 : Suppose the joint density of X and Y is

$$f(x,y)=\begin{cases} xy & , 0<y<2,\ \ 0<x<1 \\ 0 & \text{, otherwise} \end{cases}$$

What is $P(X<Y<2X)$?

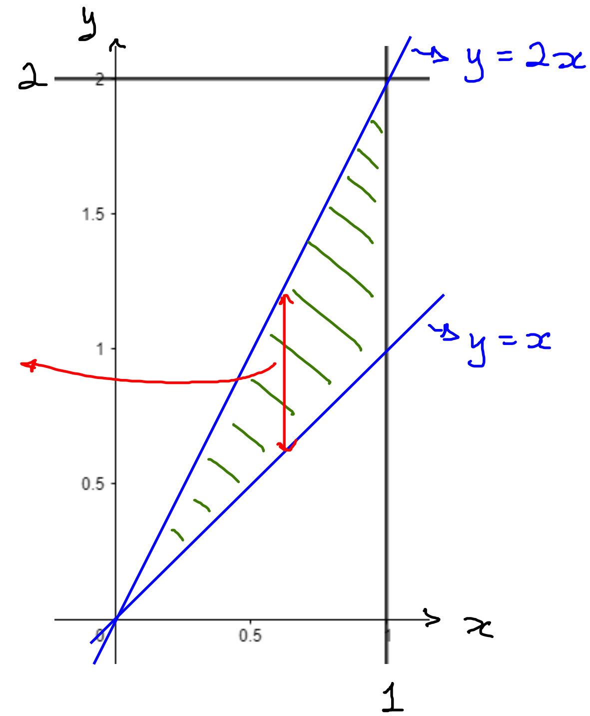
$$= \int_0^1 \int_x^{2x} xy \, dy \, dx$$

$$= \int_0^1 x \cdot \frac{y^2}{2}\Big]_x^{2x} dx = \int_0^1 \frac{3}{2} x^3 dx$$

$$= \frac{3x^4}{8}\Big]_0^1 = \frac{3}{8}.$$

$x < y < 2x$

for $0 < x < 1$



OR ($dx\,dy$): $\displaystyle\int_0^1 \int_{y/2}^{y} xy \, dx \, dy + \int_1^2 \int_{y/2}^{1} xy \, dx \, dy = \cdots = \frac{3}{8}.$

Example 4 : Let X and Y denote the lifetimes, in hours, for two components in an electronic system. Their joint desity is

$$f(x,y)=\begin{cases}\frac{1}{8}xe^{-\frac{x}{2}}e^{-\frac{y}{2}} & , x>0, y>0\\ 0 & \text{, else}\end{cases}$$

Let $Z=2X+4Y+50$. Find $E[Z]$ and $V[Z]$.

$\mu_Z = 2\mu_X + 4\mu_Y + 50$

$\sigma_Z^2 = 4\sigma_X^2 + 16\sigma_Y^2 + 16\underbrace{Cov(X,Y)}_{0}$

Note: • We can write $f(x,y)$ as $g(x)h(y)$.

• We have a rectangular support.

• X and Y are independent.

$f(x,y) = f_X(x) f_Y(y) = \left(c_1 x e^{-x/2}\right)\left(c_2 e^{-y/2}\right)$ where

$c_1 c_2 = \frac{1}{8}$.

• $X \sim \text{Gamma}(\alpha=2, \beta=2)$, $c_1 = \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{4}$

$\mu_X = \alpha\beta = 4$, $\sigma_X^2 = \alpha\beta^2 = 8$

• $Y \sim \text{Exp}(\beta=2)$, $c_2 = \frac{1}{\beta} = \frac{1}{2}$.

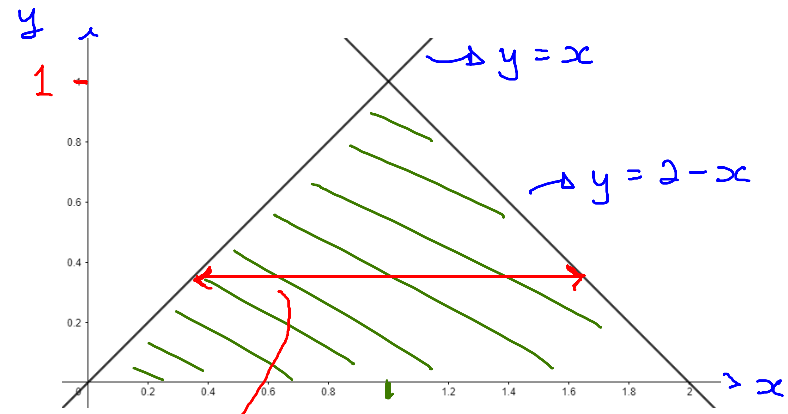
$\mu_Y = \beta = 2$, $\sigma_Y^2 = \beta^2 = 4$.

$\mu_Z = 2(4) + 4(2) + 50 = 66$

$\sigma_Z^2 = 4(8) + 16(4) = 96$.

Example 5 : Suppose X and Y are jointly continuous random variables with probability density function

$$f(x,y)=\begin{cases}3y, & 0<y<x,\ x+y<2\\ 0 & , \ else\end{cases}$$



$y < x < 2-y$ for $0 < y < 1$.

a) What is the expected value of C, where $C = XY$?

$$E[XY] = \int_0^1 \int_y^{2-y} 3xy^2 \, dx \, dy$$

$$= \int_0^1 \frac{3}{2} x^2 y^2 \Big]_y^{2-y} dy$$

$$= \int_0^1 \frac{3}{2} y^2 \left((2-y)^2 - y^2\right) dy = \underbrace{\int_0^1 6y^2(1-y)\,dy}$$

$\rightarrow E[Beta(\alpha = 2, \beta = 2)]$

$$= \frac{\alpha}{\alpha+\beta} = \frac{1}{2}.$$

$$= \frac{6y^3}{3} - \frac{6}{4} y^4 \Big]_0^1 = 2 - 1.5 = \frac{1}{2}.$$

OR (dy dx)

$$E[XY] = \int_0^1 \int_0^x 3xy^2 \, dy \, dx + \int_1^2 \int_0^{2-x} 3xy^2 \, dy \, dx = \cdots = \frac{1}{2}.$$

b) Find $f_X(x)$ and $f_Y(y)$:

$$f_Y(y) = \int_y^{2-y} 3y\,dx = 3y(2-y-y) = 6y(1-y), \quad 0 < y < 1$$

$$Y \sim Beta(\alpha = 2, \beta = 2)$$

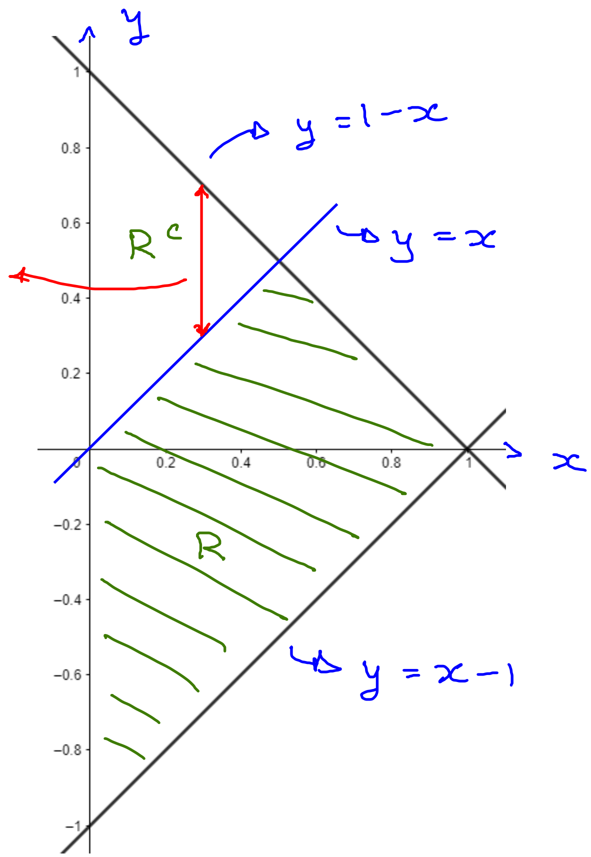
$$f_X(x) = \int 3y\,dy = \left.\frac{3y^2}{2}\right]$$

$$= \begin{cases} \left.\frac{3y^2}{2}\right]_0^{x} = \frac{3}{2}x^2, & 0 < x < 1 \\ \left.\frac{3y^2}{2}\right]_0^{2-x} = \frac{3}{2}(2-x)^2, & 1 < x < 2. \end{cases}$$

Example 6 : Suppose X and Y are jointly continuous random variables with probability density function

$$f(x,y)=\begin{cases}30xy^2, & x-1<y<1-x,\ 0<x<1\\ 0 & , else\end{cases}$$

a) Find $P(X>Y)$.



$$P(X>Y) = 1 - P(\underbrace{X<Y}_{R^c})$$

$$= 1 - \int_0^{0.5}\int_x^{1-x} 30xy^2\,dy\,dx$$

$$= \ldots = 1 - \frac{11}{32} = \frac{21}{32}.$$

OR (over R)

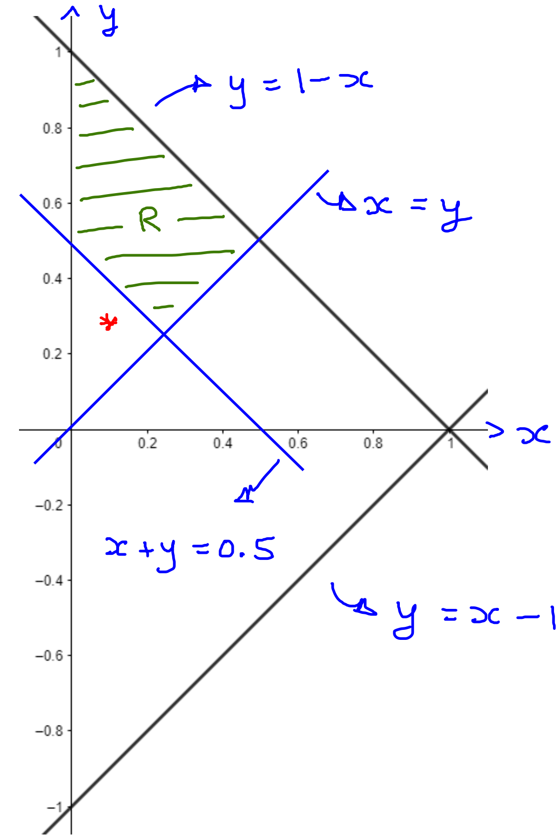
$$\bullet \int_{-1}^{0}\int_0^{y+1} 30xy^2\,dx\,dy + \int_0^{0.5}\int_y^{1-y} 30xy^2\,dx\,dy$$

$$= \frac{1}{2} + \frac{5}{32} = \frac{21}{32}$$

OR $\bullet \displaystyle\int_0^{0.5}\int_{x-1}^{x} 30xy^2\,dy\,dx + \int_{0.5}^{1}\int_{x-1}^{1-x} 30xy^2\,dy\,dx = \frac{15}{32} + \frac{3}{16} = \frac{21}{32}.$

Example 6 : Suppose X and Y are jointly continuous random variables with probability density function

$$f(x,y)=\begin{cases}30xy^2, & x-1<y<1-x,\ 0<x<1\\ 0 & ,\ else\end{cases}$$



b) Find $P(X+Y>0.5\,|\,X<Y)$. $= \dfrac{P(X+Y>0.50,\ X<Y)}{P(X<Y)}$ $P(X<Y) \to \dfrac{11}{32}$

$P(X+Y>0.50,\ X<Y) =$

$$1)\ \int_{0.25}^{0.50}\int_{0.5-y}^{y} 30xy^2\,dx\,dy + \int_{0.50}^{1}\int_{0}^{1-y} 30xy^2\,dx\,dy$$

$$= \frac{85}{1024} + \frac{1}{4} = \frac{341}{1024}$$

OR

$$2)\ \int_{0}^{0.25}\int_{0.5-x}^{1-x} 30xy^2\,dy\,dx + \int_{0.25}^{0.5}\int_{x}^{1-x} 30xy^2\,dy\,dx$$

$$= \frac{175}{1024} + \frac{166}{1024} = \frac{341}{1024}.$$

OR $\ 3)\ P(X<Y) - \displaystyle\int_{0}^{0.25}\int_{x}^{0.5-x} 30xy^2\,dy\,dx = \frac{11}{32} - \frac{11}{1024} = \frac{341}{1024}.$

$$P(X+Y>0.50\,|\,X<Y) = \frac{341/1024}{11/32} = \frac{31}{32} = 0.96875.$$

Example 6 : Suppose X and Y are jointly continuous random variables with probability density function

$$f(x,y)=\begin{cases}30xy^2, & x-1<y<1-x,\ 0<x<1\\ 0 & , else\end{cases}$$

c) Find the conditional densities of X and Y.

$$f_X(x) = 20x(1-x)^3, \quad 0<x<1$$

$$f_Y(y) = \begin{cases}15y^2(1+y)^2, & -1<y<0\\ 15y^2(1-y)^2, & 0<y<1\end{cases}$$

$$\cdot\ f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{30xy^2}{20x(1-x)^3} = \frac{3y^2}{2(1-x)^3}, \quad x-1<y<1-x, \text{ for } 0<x<1.$$

$$\cdot\ f(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases}\dfrac{30xy^2}{15y^2(1+y)^2} = \dfrac{2x}{(1+y)^2}, & 0<x<y+1 \text{ if } -1<y<0\\[2ex] \dfrac{30xy^2}{15y^2(1-y)^2} = \dfrac{2x}{(1-y)^2}, & 0<x<1-y \text{ if } 0<y<1.\end{cases}$$

$$= \frac{2x}{(1-|y|)^2}, \quad 0<x<1-|y|, \quad -1<y<1.$$

Example 7 : Consider random variables X and Y with joint density

$$f(x,y)=\frac{1}{y}e^{-(y+x/y)} \quad , x>0, y>0 \qquad = \frac{1}{y}e^{-y}e^{-x/y}$$

a) Find the marginal and conditional densities of X and Y.

b) Are X and Y independent? $\rightarrow$ Can't write $f(x,y)$ as $f_X(x)f_Y(y)$. No.

c) Find $P(X>Y)$.

$$\text{a)} \quad f_Y(y) = \int_0^\infty \frac{1}{y}e^{-y}e^{-x/y}\,dx = -e^{-y}e^{-x/y}\Big]_0^\infty = e^{-y} \quad , y>0.$$

$Y \sim Exp(\beta = 1)$.

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{y}e^{-y}e^{-x/y}}{e^{-y}} = \frac{1}{y}e^{-x/y} \quad , x>0, y>0.$$

$X|Y=y \sim Exp(\beta = E[X|Y=y] = y)$.

$$f_X(x) = \int_0^\infty \frac{1}{y}e^{-y}e^{-x/y}\,dy = \dots ?$$

$$\text{c)} \quad P(X>Y) = \int_0^\infty \int_y^\infty \frac{1}{y}e^{-y}e^{-x/y}\,dx\,dy = \int_0^\infty -e^{-y}e^{-x/y}\Big]_y^\infty dy$$

$$= \int_0^\infty e^{-y}e^{-1}\,dy = e^{-1}\int_0^\infty e^{-y}\,dy = e^{-1}.$$

Example 8 : Suppose the joint density of X and Y is

$$f(x,y)=\begin{cases} c & , x-3<y<x,\ 0<x<5, \\ 0 & , \text{otherwise} \end{cases}$$ where c is a constant.

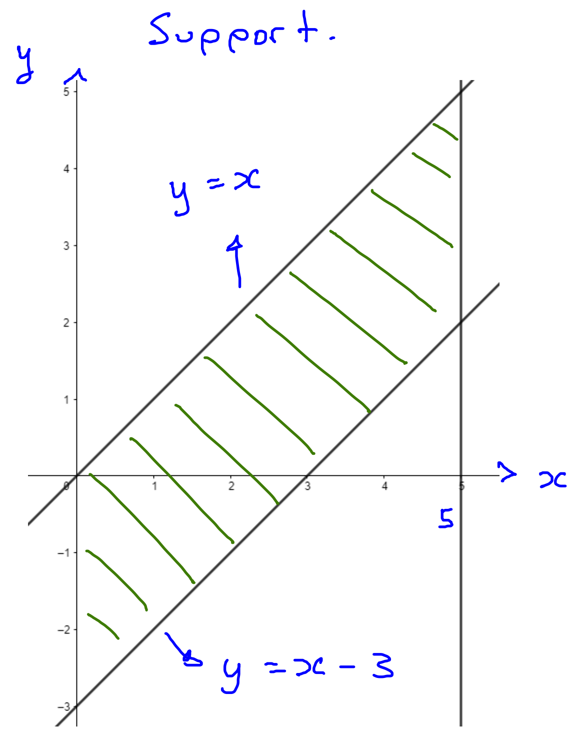
a) What is the value of c?

$$1 = \int_0^5 \int_{x-3}^{x} c \, dy \, dx = \cdots = 15c \quad , \quad c = \frac{1}{15} .$$

OR

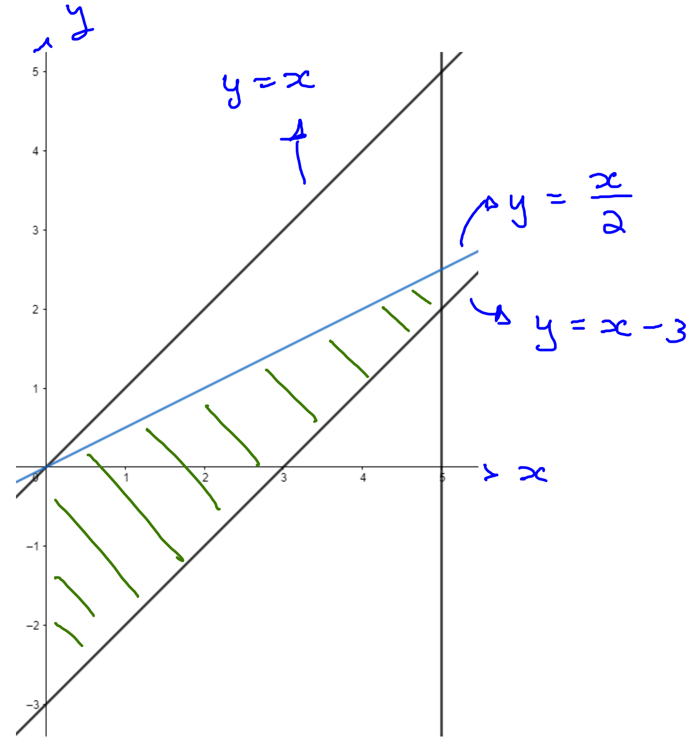
$$\int_{-3}^{0} \int_{0}^{y+3} c \, dx \, dy + \int_{0}^{2} \int_{y}^{y+3} c \, dx \, dy + \int_{2}^{5} \int_{y}^{5} c \, dx \, dy$$

$$= 15c .$$

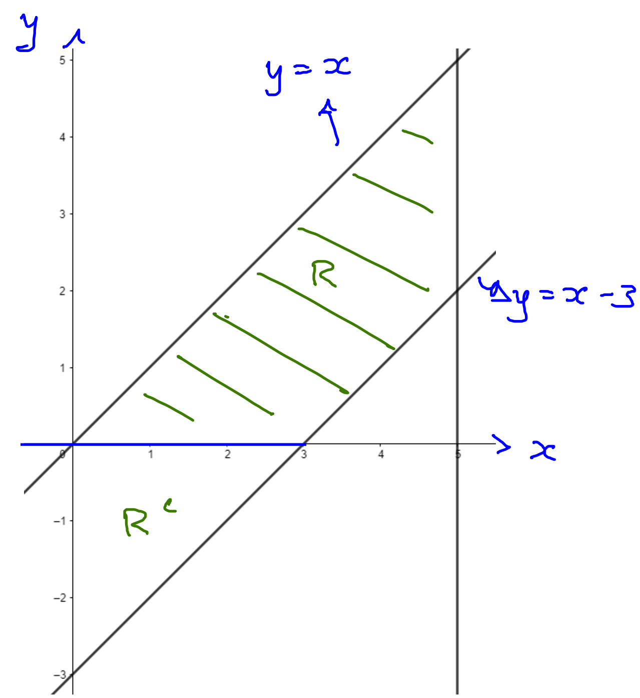


b) Find $P(Y < X/2)$.

$$= \int_0^5 \int_{x-3}^{x/2} \frac{1}{15}\, dy\, dx = \cdots = \frac{7}{12}.$$



c) Find $P(Y>0)$.



1) $= \int_0^3 \int_0^x \frac{1}{15}\,dy\,dx + \int_3^5 \int_{x-3}^{x} \frac{1}{15}\,dy\,dx$

$= \frac{3}{10} + \frac{2}{5} = \frac{7}{10}$.

OR

2) $= 1 - P(Y<0) = 1 - \int_0^3 \int_{x-3}^{0} \frac{1}{15}\,dy\,dx$

$= 1 - \frac{3}{10} = \frac{7}{10}$.

OR 3) $f_Y(y) = \int \frac{1}{15}\,dx = \frac{x}{15}\Big|$

$$= \begin{cases} \frac{x}{15}\Big]_0^{y+3} = \frac{y+3}{15}, & -3<y<0 \\ \frac{x}{15}\Big]_y^{y+3} = \frac{3}{15} = \frac{1}{5}, & 0<y<2 \\ \frac{x}{15}\Big]_y^{5} = \frac{5-y}{15}, & 2<y<5 \end{cases}$$

$P(Y>0) = 1 - P(Y<0) = 1 - \int_{-3}^{0} \frac{y+3}{15}\,dy = 1 - \frac{3}{10} = \frac{7}{10}$.

d) Find μ_Y and σ_Y^2.

e) Find $Cov(X,Y)$.

d) 1) Use $f_Y(y)$: $\mu_Y = E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

$$= \int_{-3}^{0} \frac{y^2+3y}{15}\,dy + \int_{0}^{2} \frac{y}{5}\,dy + \int_{2}^{5} \frac{5y-y^2}{15}\,dy = \cdots = 1.$$

$$\cdot\; E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \cdots = \frac{23}{6}, \quad \sigma_Y^2 = \frac{23}{6} - 1^2 = \frac{17}{6}.$$

2) Use $f(x,y)$: $E[Y] = \int_0^5 \int_{x-3}^{x} \frac{y}{15}\,dy\,dx = \cdots = 1.$

$$E[Y^2] = \int_0^5 \int_{x-3}^{x} \frac{y^2}{15}\,dy\,dx = \frac{23}{6}.$$

3) Consider conditioning on X first:

$$f_X(x) = \int_{x-3}^{x} \frac{1}{15}\,dy = \frac{1}{5}, \quad 0 < x < 5. \qquad X \sim \text{Uniform}(0,5)$$

$$\mu_X = 2.5, \quad \sigma_X^2 = \frac{25}{12},$$

$$E[X^2] = \frac{25}{3}.$$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/15}{1/5} = \frac{1}{3}, \quad x-3 < y < x \quad \text{where } 0 < x < 5.$$

$$Y|X=x \sim \text{Uniform}(x-3, x).$$

$$E[Y|X] = \frac{X-3+X}{2} = X - 1.5, \quad V[Y|X] = \frac{9}{12} = \frac{3}{4}.$$

$$E[Y] = E[E[Y|X]] = E[X-1.5] = 2.5 - 1.5 = 1.$$

$$V[Y] = E[V[Y|X]] + V[E[Y|X]]$$

$$= E[3/4] + V[X-1.5] = \frac{3}{4} + \sigma_X^2 = \frac{3}{4} + \frac{25}{12} = \frac{17}{6}.$$

e) $\mathrm{Cov}(X,Y) = E[XY] - \mu_X \mu_Y = E[XY] - 2.50$.

1) $E[XY] = \int_0^5 \int_{x-3}^{x} \frac{xy}{15}\, dy\, dx = \cdots = \frac{55}{12}$.

$\mathrm{Cov}(X,Y) = \frac{55}{12} - 2.50 = \frac{25}{12}$.

2) $E[XY \mid X = x] = E[xY \mid X = x] = x E[Y \mid X = x]$

$= x(x - 1.5) = x^2 - 1.5x$.

Then,

$E[XY] = E[E[XY \mid X]] = E[X^2 - 1.5X]$

$= E[X^2] - 1.5\mu_X$

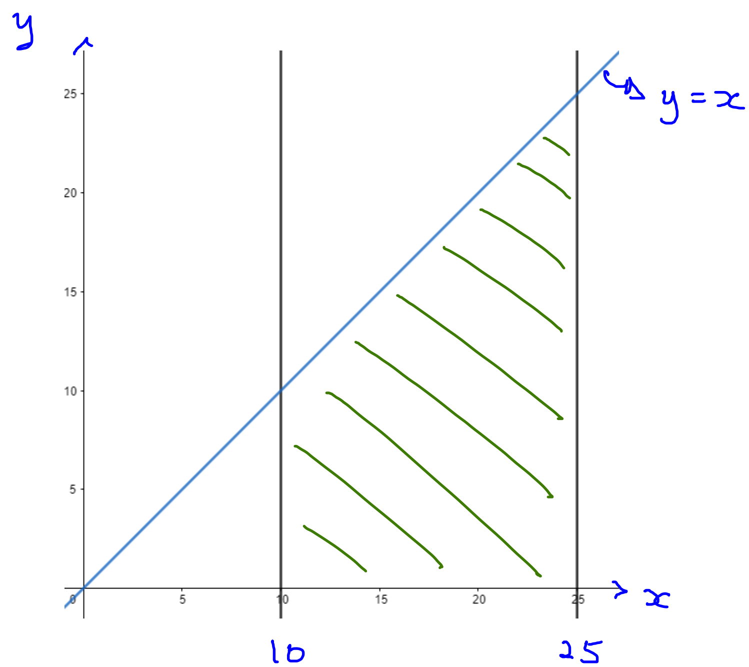
$= \frac{25}{3} - 1.5(2.5) = \frac{55}{12}$. ✓

Example 9 : Suppose the joint density of X and Y is

$$f(x,y)=\begin{cases}\dfrac{e^{-y/3x}}{45x} & , 10<x<25,\ y>0, \\ 0 & , \text{otherwise}\end{cases}.$$

a) Find $P(Y<X)$.

b) Find μ_Y and σ_Y^2.



a) $P(Y < X) = \displaystyle\int_{10}^{25}\int_{0}^{x} \frac{e^{-y/3x}}{45x}\,dy\,dx$

$= \cdots = 1 - e^{-1/3}.$

b) 1) <u>Use $f_Y(y)$</u>: $\quad f_Y(y) = \displaystyle\int_{10}^{25} \frac{e^{-y/3x}}{45x}\,dx = \cdots ?$

2) <u>Use $f(x,y)$</u>: $\quad \mu_Y = \displaystyle\int_{10}^{25}\int_{0}^{\infty} \frac{y\,e^{-y/3x}}{45x}\,dy\,dx$

$$= \int_{10}^{25} \frac{(3x)^2}{45x} \int_0^\infty \underbrace{\frac{y\,e^{-y/3x}}{(3x)^2}}_{\text{Density of a Gamma}(\alpha=2,\ \beta=3x).}\,dy\,dx$$

$$= \int_{10}^{25} \frac{x}{5}\,dx = \frac{x^2}{10}\Big]_{10}^{25} = \frac{625-100}{10} = 52.5.$$

$$E[Y^2] = \int_{10}^{25}\int_0^\infty \frac{y^2 e^{-y/3x}}{45x}\,dy\,dx = \cdots = 5850.$$

$$\sigma_Y^2 = 5850 - 52.5^2 = 3093.75.$$

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3) Condition on X first:

- $f_X(x) = \int_0^\infty \frac{e^{-y/3x}}{45x}\,dy = \frac{1}{15}\int_0^\infty \frac{e^{-y/3x}}{3x}\,dy = \frac{1}{15}, \quad 10 < x < 25.$

$X \sim \text{Uniform}(10, 25)$. $\mu_X = 17.5$, $\sigma_X^2 = 18.75$, $E[X^2] = 325$.

- $f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{3x} e^{-y/3x}, \quad y > 0, \quad 10 < x < 25.$

$Y | X = x \sim \text{Exp}(\beta = E[Y | X = x] = 3x)$.

- $E[Y | X] = 3X, \quad V[Y | X] = 9X^2$
- $\mu_Y = E[E[Y|X]] = E[3X] = 3(17.5) = 52.5$.
- $\sigma_Y^2 = E[V[Y|X]] + V[E[Y|X]] = E[9X^2] + V[3X]$

$= 9E[X^2] + 9\sigma_X^2 = 9(325) + 9(18.75) = 3093.75$.

Example 10: A miner is trapped in a cave with three possible exits.

- Exit 1 leads him to safety in 2 hours.
- Exit 2 leads him in a loop back to the cave after 3 hours.
- Exit 3 leads him in a loop back to the cave after 5 hours.

What is his expected time until safety? $\Rightarrow X =$ time until safety.

a) Case 1: Suppose that if the miner chooses a path that leads him back to the cave that he will choose his next path without any memory of his previous choices (maybe he hits his head upon returning and can't remember that he already attempted to escape!?!).

b) Case 2: Suppose the miner remembers the past, and thus won't choose a "bad" path more than once.

b)

Pick 1	Pick 2	X	$P(X=x)$
1	—	2	$1/3$
2	1	5	$(1/3)(1/2) = 1/6$
2	3	10	$(1/3)(1/2)(1/1) = 1/6$
3	1	7	$1/6$
3	2	10	$1/6$

$p_X(2) = \frac{1}{3}$, $p_X(5) = \frac{1}{6}$, $p_X(7) = 1/6$, $p_X(10) = 1/3$.

$$E[X] = \sum_{\text{all } x} x\, p_X(x) = \frac{2}{3} + \frac{5}{6} + \frac{7}{6} + \frac{10}{3} = 6.$$

$$E[X^2] = \sum_{\text{all } x} x^2 p_X(x) = \frac{4}{3} + \frac{25}{6} + \frac{49}{6} + \frac{100}{3} = 47.$$

$$V[X] = E[X^2] - \mu_X^2 = 47 - 6^2 = 11$$

a) Consider conditioning on pick 1.

Let Y = first exit pick, $p_Y(1) = p_Y(2) = p_Y(3) = \frac{1}{3}$.

Then,

$$X \mid Y = y \sim \begin{cases} 2, & \text{if } y = 1 \\ 3 + X, & \text{if } y = 2 \\ 5 + X, & \text{if } y = 3 \end{cases}$$

$$E[X \mid Y = y] = \begin{cases} 2, & \text{if } y = 1 \\ 3 + E[X], & \text{if } y = 2 \\ 5 + E[X], & \text{if } y = 3 \end{cases}$$

$$E[X] = E\{E[X \mid Y]\} = \sum_{y=1}^{3} E[X \mid Y = y]\, p_Y(y)$$

$$= 2\left(\tfrac{1}{3}\right) + (3 + E[X])\left(\tfrac{1}{3}\right) + (5 + E[X])\tfrac{1}{3}$$

$$= \frac{10}{3} + \frac{2}{3} E[X].$$

$\Rightarrow E[X] = 10.$

$$X^2 \mid Y = y \sim \begin{cases} 4, & \text{if } y = 1 \\ (3+X)^2, & \text{if } y = 2 \\ (5+X)^2, & \text{if } y = 3 \end{cases}$$

$$\begin{aligned} \underline{E[X^2]} &= E[E[X^2 \mid Y]] \\ &= \tfrac{1}{3}(4) + \tfrac{1}{3}E[(3+X)^2] + \tfrac{1}{3}E[(5+X)^2] \\ &= \tfrac{4}{3} + \tfrac{1}{3}(9 + 6\mu_X + E[X^2]) + \tfrac{1}{3}(25 + 10\mu_X + E[X^2]) \\ &= \underline{66 + \tfrac{2}{3}E[X^2]} \end{aligned}$$

$$E[X^2] = 198.$$

$$V[X] = E[X^2] - \mu_X^2 = 198 - 10^2 = 98.$$

OR : $V[X] = E[V[X|Y]] + V[E[X|Y]]$.

$$E[X|Y] = \begin{cases} 2, & \text{if } y=1 \\ 13, & \text{if } y=2 \\ 15, & \text{if } y=3 \end{cases}$$

$$V[E[X|Y]] = \frac{1}{3}(2^2+13^2+15^2) - 10^2 = 98/3$$

$$V[X|Y] = \begin{cases} 0, & \text{if } y=1 \\ V[X], & \text{if } y=2 \\ V[X], & \text{if } y=3 \end{cases}$$

$$E[V[X|Y]] = \frac{1}{3}(0 + V[X] + V[X]) = \frac{2}{3}V[X]$$

$$V[X] = \frac{2}{3}V[X] + \frac{98}{3}$$

$$\Rightarrow V[X] = 98.$$

Example 11 : The joint distribution for the lifetimes (in days) of two batteries (X and Y) is described by the following joint density function:

$$f(x,y)=\begin{cases}\frac{1}{3}e^{-\left(\frac{x}{3}+y\right)} & , x>0, y>0\\ 0 & , \text{otherwise}\end{cases}$$

$\Rightarrow$ Independent.

$X \sim Exp(3)$, $f_X(x) = \frac{1}{3}e^{-x/3}$, $x > 0$

$Y \sim Exp(1)$, $f_Y(y) = e^{-y}$, $y > 0$

Consider a device hooked up to both batteries, and suppose the device was just turned on with two brand new batteries.

a) If the device needs both batteries to function, what is the probability the device will last at least k days?

b) If the device needs at least one battery to function, what is the probability the device will last at least k days?

Let T = lifetime of the device in days. $P(T \geq k)$?

a) "$T \geq k$" = "$X \geq k \cap Y \geq k$".

$$P(T \geq k) = P(X \geq k, Y \geq k) = P(X \geq k)P(Y \geq k)$$

{ Recall: If $U \sim Exp(\beta)$, $F_U(u) = P(U \leq u) = 1 - e^{-u/\beta}$, $u \geq 0$. }

$$= (1-F_X(k))(1-F_Y(k)) = (e^{-k/3})(e^{-k}) = e^{-4k/3}.$$

b) "$T \geq k$" = "$X \geq k \cup Y \geq k$".

$$P(T \geq k) = P(X \geq k \cup Y \geq k)$$

$$= P(X \geq k) + P(Y \geq k) - P(X \geq k)P(Y \geq k)$$

$$= e^{-k/3} + e^{-k} - e^{-4k/3}.$$

$\Rightarrow$ $1 - P(X \leq k \cap Y \leq k) = 1 - (1-e^{-k/3})(1-e^{-k})$.