Mathematical Method
MA-203
Solution Tutorial Sheet - 1

Q-1 :→ Evalute the Laplace transform of each function directly from the defining integral

(1). ~~fo~~ $f(t) = \cos^2 kt$

(2) $f(t) = e^{at}\cosh(kt)$

(3) $f(t) = t.\sin(kt)$.

Solⁿ :→

(1) $f(t) = \cos^2 kt$

taking Laplace on both side

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = L(\cos^2 kt)$$

$$= \int_0^\infty e^{-st}.\cos^2 kt\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{1+\cos 2kt}{2}\right]dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\,dt + \frac{1}{2}\int_0^\infty e^{-st}\cos 2kt.dt$$

$$= \frac{1}{2}\left[\frac{e^{-st}}{-s}\right]_0^\infty + \frac{1}{2}L.\left[\cos(2kt)\right]$$

$$= \frac{1}{-2s}\left[0-1\right] + \frac{1}{2}\left(\frac{s}{s^2+4k^2}\right)$$

$$\boxed{L(f(t)) = \frac{s^2+2k^2}{s(s^2+4k^2)}}$$

(ii) $f(t) = e^{at} \cosh(kt)$.

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} . f(t).dt$$

$$= \int_0^\infty e^{-st} . e^{at} \cosh kt \; dt$$

$$= \int_0^\infty e^{-(s-a).t} \cosh kt \; dt.$$

$\because$ $L[\cosh at] = \frac{s'}{s'^2 - a^2}$. , If $\int_0^\infty e^{-s't} \cosh at \; dt = \frac{s'}{s'^2 - a^2}$.

$$\boxed{L[f(t)] = \frac{(s-a)}{(s-a)^2 - k^2}}$$

Ans

(iii) $f(t) = t \sin kt$

$$L[f(t)] = \int_0^\infty e^{-st} (t . \sin kt) \, dt$$

$$= L(t \sin kt).$$

using pro. Derivative of Laplace transform

$[$ If $L\{F(t)\} = f(s)$. then $L\{t.F(t)\} = -f'(s).]$

$$= -[L[\sin kt]]'$$

$$= -\left[\frac{k}{k^2 + s^2}\right]'$$

$$\boxed{L[f(t)] = \frac{2ks}{(k^2 + s^2.)^2}}$$

Ans

Or.

Q-(1)→(i)

<u>Soln</u>:→ $f(t) = \cos^2 kt$

$$L(f(t)) = \int_0^\infty e^{-st} \cdot \cos^2 kt$$

$$= \int_0^\infty e^{-st} \left(\frac{1+\cos 2kt}{2}\right) dt$$

$$= \int_0^\infty \frac{e^{-st}}{2} dt + \int_0^\infty \frac{e^{-st}}{2} \cdot \cos 2kt \, dt$$

$$L[f(t)] = \frac{1}{2s} + \frac{1}{2} \int_0^\infty e^{-st} \cdot \cos 2kt \cdot dt$$

$$\cancel{= \frac{1}{2s} + \frac{1}{2} \left[\frac{e^{-st}}{-s} \cdot \cos 2kt \Big|_0^\infty - \int_0^\infty \sin 2kt \cdot (2k) \cdot \frac{e^{-st}}{-s} dt \right]}$$

Let

$$I = \frac{1}{2} \int_0^\infty e^{-st} \cos 2kt \, dt$$

$$= \frac{1}{2} \left[\frac{e^{-st}}{-s} \cos 2kt \Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (-\sin 2kt) \cdot 2k \cdot dt \right]$$

$$= \frac{1}{-2s} [0 - 1] - \frac{\cancel{2}k}{\cancel{2}s} \int_0^\infty e^{-st} \cdot \sin 2kt \, dt$$

$$= \frac{1}{2s} - \frac{k}{s} \left[\frac{e^{-st}}{-s} \cdot \sin 2kt \Big|_0^\infty - \int_0^\infty \cos 2kt \cdot 2k \cdot \frac{e^{-st}}{-s} \cdot dt \right]$$

$$= \frac{1}{2s} - \frac{k}{s} \left[0 - 0 + \frac{2k}{s} \cdot \int_0^\infty e^{-st} \cdot \cos 2kt \, dt \right]$$

$$I = \frac{1}{2s} - \frac{2k^2}{s^2} (2) \underbrace{\left(\frac{1}{2}\right) \cdot \int_0^\infty e^{-st} \cos(2kt) \, dt}_{I}$$

$$I = \frac{1}{2s} - \frac{4k^2}{s^2} \cdot I$$

$$\Rightarrow I\left[1 + \frac{4k^2}{s^2}\right] = \frac{1}{2s}.$$

$$\Rightarrow I\left[\frac{s^2 + 4k^2}{s^2}\right] = \frac{1}{2s}.$$

$$\Rightarrow I = \frac{s}{2(s^2 + 4k^2)}$$

Now $L[f(t)] = \frac{1}{2s} + I$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4k^2)}.$$

$$= \frac{1}{2}\left[\frac{s^2 + 4k^2 + s^2}{s(s^2+4k^2)}\right]$$

$$\boxed{L[f(t)] = \frac{s^2 + 2k^2}{s(s^2+4k^2)}}$$

formula – $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$.

Shifting property → If $L[F(t)] = f(p)$ then $L\{e^{at} F(t)\} = f(p-a)$.

Q-2 :→ If $L[f(t)] = \frac{e^{-1/s}}{s}$. then show that

$$L\{e^{-t} f(4t)\} = \frac{e^{-4/(s+1)}}{(s+1)}.$$

Proof :→ given that $L[f(t)] = \frac{e^{-1/s}}{s}$.

~~Let $L\{e^{-t} f(4t)\}$ =~~

Let, $L\{e^{-t} f(4t)\} = \int_0^\infty e^{-st} \cdot \underbrace{e^{-t} \cdot f(4t)}\, dt$

[using above property]

$$= \int_0^\infty e^{-(s+1)t} f(4t)\, dt$$

~~put~~ $4t = u \Rightarrow dt = \frac{du}{4}$

$$= \int_0^\infty e^{-(s+1)\frac{u}{4}} \cdot f(u)\, \frac{du}{4}$$

$$= \frac{1}{4} \int_0^\infty e^{-pu} \cdot f(u)\, du. \quad , \quad \text{put } \frac{s+1}{4} = p$$

$$= \frac{1}{4} \cdot L.f(u) \quad = \frac{1}{4} \cdot \frac{e^{-1/p}}{p}$$

~~$\frac{1}{4} \cdot \frac{1}{4} f(4t)$~~ $= \frac{1}{4} \cdot \frac{e^{-4/s+1}}{\left(\frac{s+1}{4}\right)} = \frac{e^{-4/s+1}}{s+1}$ H.P

~~$\frac{1}{4} \cdot \frac{1}{4} \cdot f$~~

Q-3 $\Rightarrow$ Evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$.

Soln !->

$$\because L\{e^{iat}\} = \frac{1}{p-ia} = \frac{p+ia}{p^2+a^2}$$

$$L[\cos at + i \sin at] = \frac{p}{p^2+a^2} + i \cdot \frac{a}{p^2+a^2}$$

equating real and Imaginary part

$$L[\sin at] = \frac{a}{p^2+a^2} \Rightarrow L[\sin t] = \frac{1}{1+p^2}$$

$$L[t^n . \sin at] = (-1)^n \frac{d^n}{dp^n} L.(\sin at)$$

$\because n=3, \; a=1$

$$L[t^3 . \sin t] = (-1)^3 \frac{d^3}{dp^3} L[\sin t]$$

$$= -\frac{d^3}{dp^3}\left[\frac{1}{1+p^2}\right]$$

$$= -\frac{d^2}{dp^2}\left[\frac{-2p}{(1+p^2)^2}\right]$$

$$= +2\frac{d}{dp}\left[\frac{1-3p^2}{(1+p^2)^3}\right]$$

$$L[t^3 . \sin t] = \frac{24p(1-p^2)}{(1+p^2)^4}$$

by $\int_0^\infty e^{-pt} t^3 \sin t \, dt = L[t^3 \sin t] = \frac{24p(1-p^2)}{(1+p^2)^4}$

put $p=1$

$$\Rightarrow \boxed{\int_0^\infty e^{-t} . t^3 \sin t \, dt = 0}$$

Q-4 :→ Find the Laplace transform of $f(t)$, where.

$$f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$$

expressing it in terms of Unit step function.

Solⁿ :→

first expressing $f(t)$ in terms of Unit step fⁿ.

$$f(t) = \cos t\,[u(t-0) - u(t-\pi)] + \cos 2t\,[u(t-\pi) - u(t-2\pi)] + \cos 3t\,[u(t-2\pi)]$$

$$= \cos t . u(t) + [\cos 2t - \cos t]\, u(t-\pi) + [\cos 3t - \cos 2t]\, u(t-2\pi)$$

taking Laplace Tran. on both sides.

$$L(f(t)) = L\{\cos t\, u(t)\} + L\{[\cos 2t - \cos t]\, u(t-\pi)\} + L\{[\cos 3t - \cos 2t]\, u(t-2\pi)\}$$

use formula

$$L\{f(t).\, u(t-a)\} = e^{-as}\, L\{f(t+a)\}$$

$$= e^{-0s}\, L[\cos(t+0)] + e^{-\pi s}\, L[\cos 2(t+\pi) - \cos(t+\pi)] + e^{-2\pi s}\, L[\cos 3(t+2\pi) - \cos 2(t+2\pi)]$$

$$= L[\cos t] + e^{-\pi s} L[\cos(2\pi + 2t) - \cos(t+\pi)]$$
$$+ e^{-2\pi s} L[\cos(6\pi + 3t) - \cos(4\pi + 2t)]$$

$$= \frac{s}{s^2+1} + e^{-\pi s} \cdot L[\cos 2t + \cos t] + e^{-2\pi s}[\cos 3t - \cos 2t]$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

Ans

* In general, $f(t+nT) = f(t)$ for all t, n is an integer and T is the period of the function

If $f(t)$ is a periodic function with period T

ie, $f(t+T) = f(t)$

then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$

Q-5 :→ Find L.T. of a periodic function $f(t)$ of period $2c$ given by $f(t) = \begin{cases} t, & 0<t<c \\ 2c-t, & c<t<2c \end{cases}$

Solⁿ :→ here $T = 2c$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2cs}} \int_0^{2c} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2cs}} \left[\underbrace{\int_0^c e^{-st} \cdot t\, dt}_{I_1} + \underbrace{\int_c^{2c} e^{-st} (2c-t) \cdot dt}_{I_2} \right]$$

Let $I_1 = \int_0^c e^{-st}\, t\, dt$

$$= \frac{e^{-st}}{-s} \cdot t \Big|_0^c - \int_0^c \frac{e^{-st}}{-s} \cdot 1 \cdot dt$$

$$= -\left[\frac{e^{-sc} \cdot c}{s} - 0\right] - \frac{e^{-st}}{s^2}\Big|_0^c = -\frac{c \cdot e^{-sc}}{s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2}$$

Let $I_2 = \int_c^{2c} e^{-st}.(2c-t)\,dt$

$$= 2c\int_c^{2c} e^{-st}\,dt - \int_c^{2c} t\cdot e^{-st}\,dt$$

$$= 2c.\left[\frac{e^{-st}}{-s}\right]_c^{2c} - \left[t.\frac{e^{-st}}{-s}\Big|_c^{2c} - \int_c^{2c} 1.\frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{-2c}{s}\left[e^{-2cs} - e^{-sc}\right] + \left[\frac{2ce^{-2cs}}{s} - \frac{c.e^{-sc}}{s}\right] + \frac{e^{-st}}{s^2}\Big|_c^{2c}$$

$$= -\frac{2c}{s}\left[\cancel{e^{-2cs}} - e^{-sc}\right] + \frac{1}{s}\left[\cancel{2ce^{-2cs}} - ce^{-sc}\right] + \frac{1}{s^2}\left[e^{-2cs} - e^{-sc}\right]$$

$$= \frac{2c}{s}e^{-sc} - \frac{c}{s}e^{-sc} + \frac{1}{s^2}\left[e^{-2cs} - e^{-sc}\right]$$

$$I_2 = \frac{c}{s}e^{-sc} + \frac{e^{-2cs}}{s^2} - \frac{e^{-sc}}{s^2}$$

Now

$$L\{f(t)\} = \frac{1}{1-e^{-2cs}}\left[\cancel{-\frac{c}{s}e^{-sc}} - \cancel{\frac{e^{-sc}}{s^2}} + \frac{1}{s^2} + \cancel{\frac{c}{s}e^{-sc}} + \cancel{\frac{e^{-2cs}}{s^2}} - \frac{e^{-sc}}{s^2}\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-2cs}}\left[\frac{1-e^{-sc}}{s^2}\right]}$$

$$\boxed{L\{f(t)\} = \left(\frac{1}{1+e^{cs}}\right)\left(\frac{1}{s^2}\right)}$$

Ans