Exam

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May 2019

Matrix Formulation of the *M/M/*1 Queue

The *M/M/*1 queue with service rate μ and arrival rate λ has the following infinite infinitesimal generator:

$$Q = \begin{pmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

In the notation used for Markov chains, we have $\pi Q = 0$ (with $\pi_i = p_i$ for all i) and it is obvious that $-\lambda\pi_0 + \mu\pi_1 = 0$, i.e., that $\pi_1 = (\lambda/\mu)\pi_0$. In general, we have

$$\lambda\pi_{i-1} - (\lambda+\mu)\pi_i + \mu\pi_{i+1} = 0,$$

from which, by induction, we may derive

$$\pi_{i+1} = ((\lambda+\mu)/\mu)\pi_i - (\lambda/\mu)\pi_{i-1} = (\lambda/\mu)\pi_i.$$

Thus, once π_0 is known, the remaining values π_i, $i = 1, 2, \ldots$, may be determined recursively just as before. For the *M/M/*1 queue it has already been shown that the probability that the system is empty is given by $\pi_0 = (1 - \lambda/\mu)$.

Observe that the coefficient matrix is tridiagonal. and that once p_0 is known, the solution is just a forward elimination procedure. However, there is no computational advantage to be gained by setting up and solving the matrix equation, rather than using the previously developed recursive relations. We show this formulation at this time because it will become useful in other, more complex, cases when the matrix is Hessenberg, rather than tridiagonal.

11.2.2 Performance Measures

We now turn our attention to computing various performance measures concerning the *M/M/*1 queue, such as mean number in system, mean queue length, and so on.

Mean Number in System

Let N be the random variable that describes the number of customers in the system at steady state, and let $L = E[N]$. Then

$$L = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\rho\sum_{n=0}^{\infty} n\rho^{n-1}. \tag{11.4}$$

If we assume that the system is stable, then $\rho < 1$ and

$$\sum_{n=0}^{\infty} n\rho^{n-1} = \frac{\partial}{\partial\rho}\left[\sum_{n=0}^{\infty}\rho^n\right] = \frac{\partial}{\partial\rho}\left[\frac{1}{1-\rho}\right] = \frac{1}{(1-\rho)^2}. \tag{11.5}$$

It now follows from Equation (11.4) that the mean number of customers in the *M/M/*1 queue is given by

$$L = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}.$$

Variance of Number in System

To compute the variance of the number of customers in an *M/M/*1 queue, we use the formula

$$\text{Var}[N] = E[N^2] - E[N]^2.$$

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\end{pmatrix}.
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It now follows from Equation (11.4) that the mean number of customers in the $M/M1$ queue is given by

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Variance of Number in System
To compute the variance of the customers in an $M/M1$ queue, we use the formula

$$\mathrm{Var}[N] = E[N^2] - E[N]^2.$$

LaTeX code

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\documentclass{report}
\usepackage[utf8]{inputenc}

\usepackage[paperheight=305mm,paperwidth=212mm,margin=30mm,heightrounded]{geometry}
\usepackage{nopageno}
\usepackage{amsmath}
\title{Exam}
\author{Anna Bogachova}
\date{May 2019}

\begin{document}
\thispagestyle{empty}
\maketitle

% \includegraphics[width=\textwidth]{photo.jpg}

\pagebreak


\noindent \textbf{406}   \hspace{1cm}\textbf{ Elementary Queueing Theory}

\noindent\rule{15cm}{0.4pt}
\vspace*{20px}

\noindent \textbf{Matrix Formulation of the \textit{M/M/}1 Queue}

\noindent The \textit{M/M/}1 queue with service rate $\mu$
 and arrival rate $\lambda$ has the following infinite infinitesimal generator:

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    & \mu  & -(\lambda + \mu) & \lambda &  &  \\
   &  & \mu  & -(\lambda + \mu) & \lambda &  \\
   &  &  & \ddots & \ddots &  & \ddots
 \end{pmatrix}    .
 \end{center}

\hfill

 \noindent In the notation used for Markov chains, we have $\pi Q = 0$
 (with $\pi_i = p_i$ for all $i$) and it is obvious that -$\lambda\pi_0 +
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 In general, we have

$$\lambda\pi_{i-1} - (\lambda + \mu)\pi_i + \mu\pi_{i+1} = 0,$$

\noindent from which, by induction, we may derive

$$\pi_{i+1} = ((\lambda + \mu)/\mu)\pi_i - (\lambda/\mu)\pi_{i-1} = (\lambda/\mu)\pi_i.$$

\noindent Thus, once $\pi_0$ is known, the remaining values $\pi_i$, i = 1,2, . . . ,
may be determined recursively just as before. For the \textit{M/M/}1 queue it
has already been shown that the probability that the system is empty is given
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```latex
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\noindent\textbf{Mean Number in System}

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number of customers in the system at steady state, and let  $L = E[N]$. Then


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= (1 - p)\sum_{n=0}^{\infty} np^n
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\begin{equation}\sum_{n=0}^{\infty} np^{n-1}
= \frac{\partial}{\partial p}\Bigg[\sum_{n=0}^{\infty}p^n\Bigg]
= \frac{\partial}{\partial p}\Bigg[\frac{1}{1 - p}\Bigg]
=  \frac{1}{(1 - p)^2} .\tag{11.5}\end{equation}

\noindent It now follows from Equation (11.4) that the mean number
of customers in the \textit{M/M}1 queue is given by

$$L = (1 - p)\frac{p}{(1 - p)^2}
= \frac{p}{1 - p}
= \frac{\lambda}{\mu - \lambda}.$$

\noindent\textbf{Variance of Number in System}

\noindent To compute the variance of the customers in an \textit{M/M}1 queue,
we use the formula

\begin{center}
    Var$[N] = E[N^2] - E[N]^2$.
\end{center}
\end{document}
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