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Reproduction of "Fast and Accurate Matrix Completion via Truncated Nuclear Norm Regularization"

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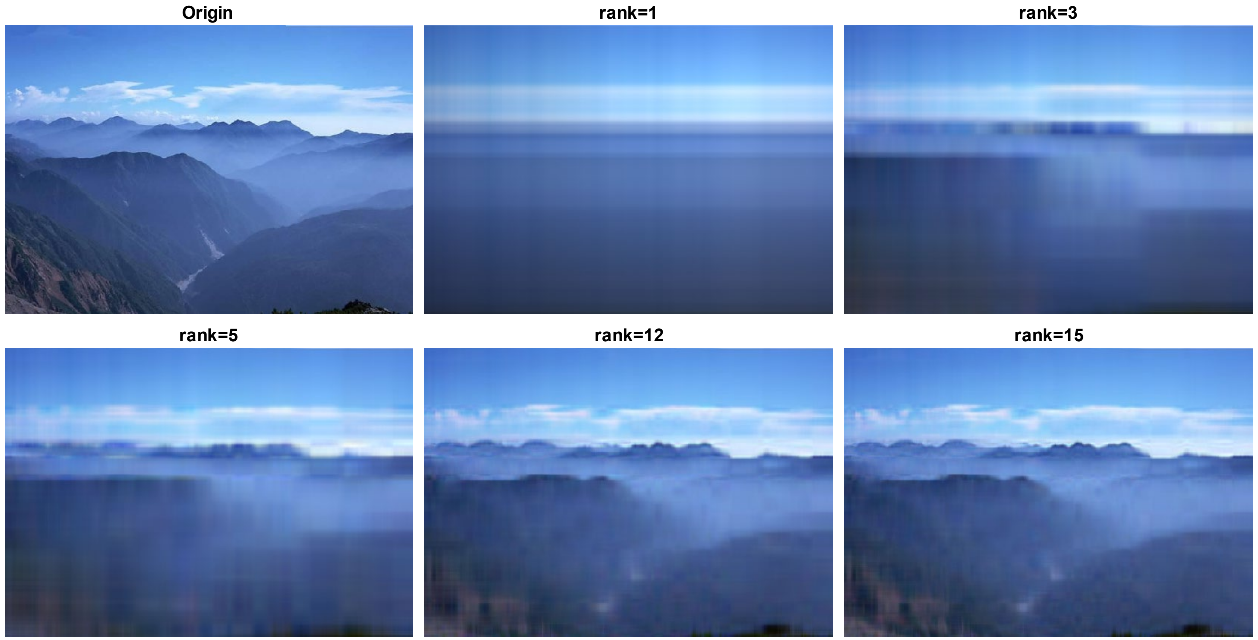
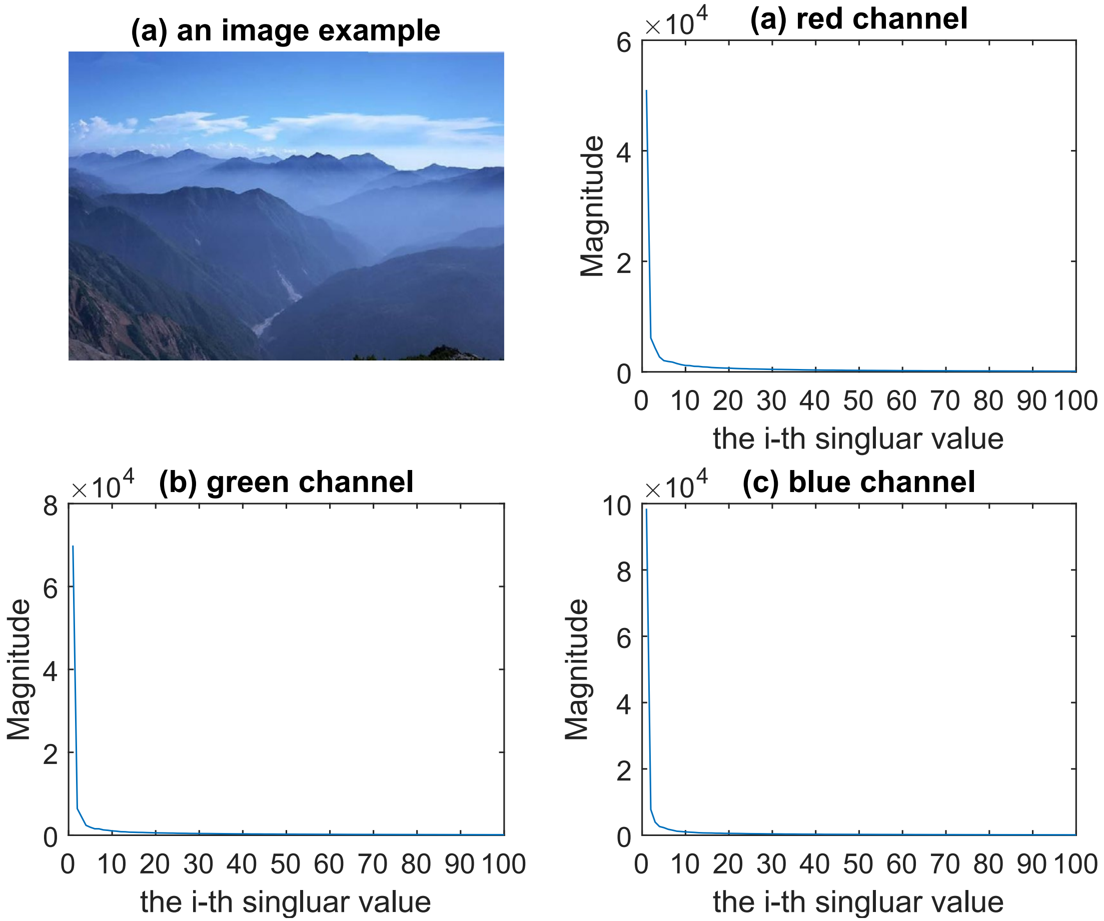
Contests

• 1. Background – rank

Using `svd` function decompose the image (Channels separately)

- The first few singular values are much larger than others
- For r > 20, r-th singular value σ_r close to 0



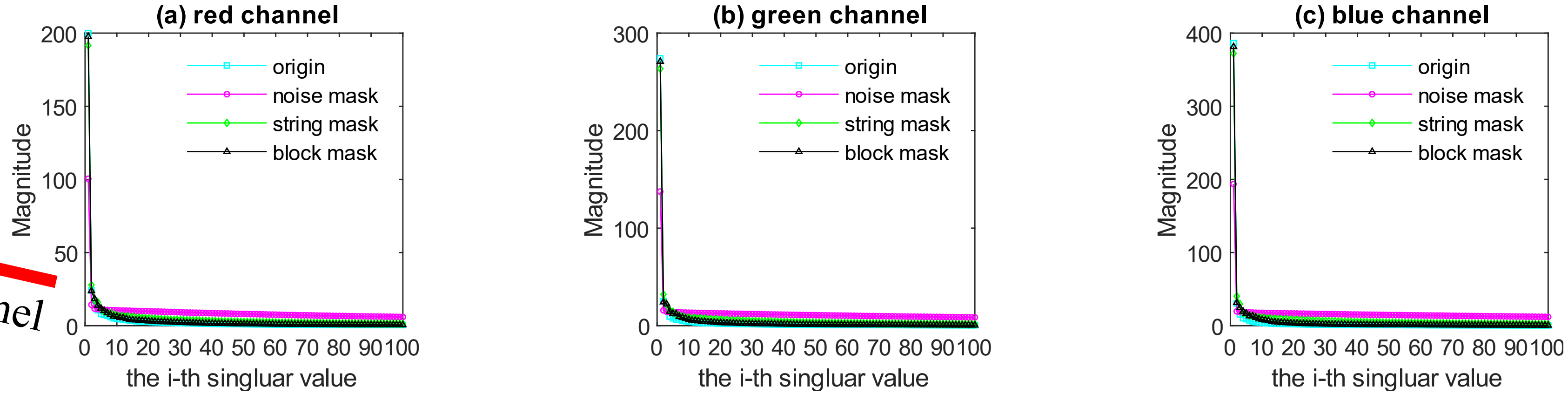


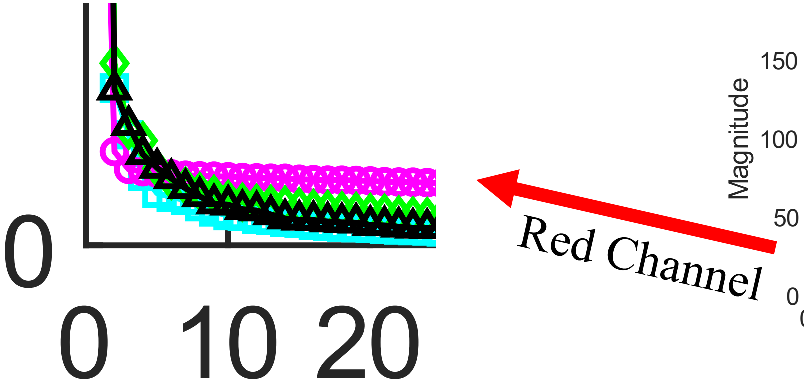
First Few values contain main information → Low rank approximate is reasonable → Truncated svd (svds)

• 1. Background – with mask

We mask the figure with three ways:

- Random mask (50% missing)
- String mask (add string)
- Block mask (add block)





$$
\begin{aligned}
\min_{\mathbf{X}} \quad & \operatorname{rank}(\mathbf{X}) \\
s.t. \quad & \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{M})
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min_{\mathbf{X}} \quad & \|\mathbf{X}\|_{*} \\
s.t. \quad & \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{M})
\end{aligned}
\ [1]
$$

$\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{X} \in \mathbb{R}^{m\times n}$, $\mathbf{\Omega}$ observed entries, Orthogonal projection operator $\mathcal{P}_{\Omega}$

[1] M. Fazel, “Matrix Rank Minimization with Applications,” PhD thesis, Stanford Univ., 2002.

• 2. Related Work

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- Singular Value Thresholding (SVT)[1]: Time complexity $\mathcal{O}(\frac{1}{N})$

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* + \alpha\|\mathbf{X}\|_F^2 \\ s.t. \quad & \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{M}) \end{aligned} \Rightarrow L(\mathbf{X},\mathbf{Y}) = \|\mathbf{X}\|_* + \alpha\|\mathbf{X}\|_F^2 + \langle \mathbf{Y}, \mathcal{P}_{\Omega}(\mathbf{M}-\mathbf{X})\rangle$$

- Singular Value Projection (SVP)[2]: Solve the rank minimization problem

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{P}_{\Omega}(\mathbf{X}) - P_{\Omega}(\mathbf{M})\|_F^2 \\ s.t. \quad & \mathrm{rank}(\mathbf{X}) \le r \end{aligned} \Rightarrow L(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{S}\in\mathbb{R}^{r\times r}} \|\mathcal{A}(\mathbf{X}) - \mathbf{y}\|_F^2 \Rightarrow \begin{aligned} \mathbf{Y}_{k+1} &= \mathbf{X}_k - \gamma_k \mathcal{A}^*(\mathcal{A}(\mathbf{X}_k) - y)) \\ \mathbf{X}_{k+1} &= \text{Trancated SVD}_r(\mathbf{Y}_{k+1}) \end{aligned}$$

- OptSpace[3]:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{P}_{\Omega}(\mathbf{X}) - P_{\Omega}(\mathbf{M})\|_F \\ s.t. \quad & \mathrm{rank}(\mathbf{X}) \le r \end{aligned} \Rightarrow L(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{S}\in\mathbb{R}^{r\times r}} L(\mathbf{X},\mathbf{Y},\mathbf{S}) = \frac{1}{2}\|\mathcal{P}_{\Omega}(\mathbf{M} - \mathbf{X}\mathbf{S}\mathbf{Y}^T)\|_F^2 + \frac{\lambda}{2}\|\mathcal{P}_{\Omega^c}(\mathbf{M} - \mathbf{X}\mathbf{S}\mathbf{Y}^T)\|_F^2$$

$\mathbf{Y} \in \mathbb{R}^{m\times n}$ is the Lagrange multiplier matrix, inner produce $\langle \mathbf{X}, \mathbf{Y}\rangle = \sum_{i,j} X_{ij}Y_{ij}$

[1] J.F. Cai, E.J. Candès, and Z. Shen, "A Singular Value Thresholding Algorithm for Matrix Completion," SIAM J. Optimization, vol. 20, pp. 1956-1982, 2010.
[2] P. Jain, R. Meka, and I. Dhillon, "Guaranteed Rank Minimization via Singular Value Projection," in Advances in Neural Information Processing Systems, 2010, vol. 23
[3] R. H. Keshavan and S. Oh, "A Gradient Descent Algorithm on the Grassman Manifold for Matrix Completion," Transportation Research Part C: Emerging Technologies, vol. 28, pp. 15–27, Mar. 2013.

• 3. Truncated Nuclear Norm Regularization (TNNR)

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- Truncated nuclear norm: $\|\mathbf{X}\|_r = \sum_{i=r+1}^{\min(m,n)} \sigma_i(\mathbf{X}) \ (\mathbf{X} \in \mathbb{R}^{m\times n})$

- By Von Neumann's trace inequality[1], we get $\mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) = \mathrm{Tr}(\mathbf{X}\mathbf{B}^T\mathbf{A}) \leq \sum_{i=1}^{r} \sigma_i(\mathbf{X})\sigma_i(\mathbf{B}^T\mathbf{A})$.

$$\mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) = \mathrm{Tr}(\mathbf{X}\mathbf{B}^T\mathbf{A}) \leq \sum_{i=1}^{m\times n} \sigma_i(\mathbf{X})\sigma_i(\mathbf{B}^T\mathbf{A}).$$

$$\begin{aligned}
&\sum_{i=1}^{\min(m,n)} \sigma_i(\mathbf{X})\sigma_i(\mathbf{B}^T\mathbf{A}) \\
&= \sum_{i=1}^{s} \sigma_i(\mathbf{X})\sigma_i(\mathbf{B}^T\mathbf{A}) + \sum_{i=s+1}^{\min(m,n)} \sigma_i(\mathbf{X})\sigma_i(\mathbf{B}^T\mathbf{A}) \\
&= \sum_{i=1}^{s} \sigma_i(\mathbf{X})\cdot 1 + \sum_{i=s+1}^{\min(m,n)} \sigma_i(\mathbf{X})\cdot 0 \\
&= \sum_{i=1}^{s} \sigma_i(\mathbf{X}) \leq \sum_{i=1}^{r} \sigma_i(\mathbf{X}) \quad (s \leq r \text{ and } \sigma_i(\mathbf{X}) \geq 0)
\end{aligned}$$

Suppose, $\mathbf{U}\Sigma\mathbf{V}^T$ is the singular value decomposition of $\mathbf{X}$, where $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_m) \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \in \mathbb{R}^{n\times n}$, $\mathbf{A} = (\mathbf{u}_1, \ldots, \mathbf{u}_m)^T$, $\mathbf{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)^T$.

$$\begin{aligned}
&\mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \\
&= \mathrm{Tr}((\mathbf{u}_1, \ldots, \mathbf{u}_m)^T \mathbf{X} (\mathbf{v}_1, \ldots, \mathbf{v}_m)) \\
&= \mathrm{Tr}(((\mathbf{u}_1, \ldots, \mathbf{u}_m)^T \mathbf{U})\Sigma(\mathbf{V}^T(\mathbf{v}_1, \ldots, \mathbf{v}_m))) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \mathrm{Tr}(diag(\sigma_1(\mathbf{X}), \ldots, \sigma_r(\mathbf{X}), 0, \ldots, 0)) \\
&= \sum_{i=1}^{r} \sigma_i(\mathbf{X})
\end{aligned}$$

[1] Von Neumann, J, "Some matrix-inequalities and metrization of matric-space," Tomsk Univ. Rev. 1, 286–300, 1937.

• 3. Truncated Nuclear Norm Regularization (TNNR)

$$\max_{\mathbf{A}\mathbf{A}^T=\mathbf{I},\mathbf{B}\mathbf{B}^T=\mathbf{I}} \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \le \sum_{i=1}^{r} \sigma_i(\mathbf{X}) \qquad \|\mathbf{X}\|_* - \max_{\mathbf{A}\mathbf{A}^T=\mathbf{I},\mathbf{B}\mathbf{B}^T=\mathbf{I}} \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) = \sum_{i=1}^{\min(m,n)} \sigma_i(\mathbf{X}) - \sum_{i=1}^{r} \sigma_i(\mathbf{X}) = \|\mathbf{X}\|_r$$

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_r \\ s.t. \quad & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M}) \end{aligned} \Rightarrow \begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* - \max_{\mathbf{A}\mathbf{A}^T=\mathbf{I},\mathbf{B}\mathbf{B}^T=\mathbf{I}} \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \\ s.t. \quad & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M}) \end{aligned} \Rightarrow \begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \\ s.t. \quad & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M}) \end{aligned}$$

Algorithm 1 The Proposed Two-Step Approach for Sovling (6)

Input: original incomplete data matrix $\mathbf{M}_\Omega$, where $\mathbf{\Omega}$ is the indices of the observed entries, tolerance ϵ_0

Initialization: $\mathbf{X}_1 = \mathcal{P}_\Omega(\mathbf{M})$.

repeat

step 1. Given $\mathbf{X}_l$, compute $\mathbf{U}_l, \Sigma_l, \mathbf{V}_l = \mathrm{svd}(\mathbf{X}_l)$,
where $\mathbf{U}_l = (\mathbf{u}_1, \ldots, \mathbf{u}_m) \in \mathbb{R}^{m\times m}$, $\mathbf{V}_l = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \in \mathbb{R}^{n\times n}$.
Compute $\mathbf{A}_l$ and $\mathbf{B}_l$, as $\mathbf{A}_l = (\mathbf{u}_1, \ldots, \mathbf{u}_m)^T$, $\mathbf{B}_l = (\mathbf{v}_1, \ldots, \mathbf{v}_m)^T$

step 2. Solve $\mathbf{X}_{l+1} = \arg\min_{\mathbf{X}} \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \quad \text{s.t.} \quad \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M})$

until $\|\mathbf{X}_{l+1} - \mathbf{X}_l\|_F \le \epsilon_0$.

return the recovered matrix.

• 4. Optimization ways – ADMM

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- Alternating Direction Method of Multipliers (ADMM)[1] is an algorithm that is intended to blend the decomposability of dual ascent with the superior convergence properties of the method of multipliers.

- If the optimization problem is separable, we can use ADMM to solve. The ADMM algorithm is suitable for solving the following 2-block (or N-block) convex optimization problems.

$$\min_{x,w} \quad f(\mathbf{x}) + g(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} = \mathbf{b}$$

The augmented lagrange function

$\mathbf{x} \in \mathbb{R}^n, \mathbf{w} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{p\times n}, \mathbf{B} \in \mathbb{R}^{p\times m}$,
$\mathbf{b} \in \mathbb{R}^p$ is convex set. f, g is convex function.

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = f(\mathbf{x}) + g(\mathbf{w}) + \mathbf{z}^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} - \mathbf{b}) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} - \mathbf{b}\|_2^2.$$

By dual ascent

$$(\mathbf{x}_{k+1}, \mathbf{w}_{k+1}) : = \min_{\mathbf{x},\mathbf{w}} \mathcal{L}(\mathbf{x}, \mathbf{z}_k, \mathbf{w})$$
$$\mathbf{z}_{k+1} : = \mathbf{z}_k + \beta(\mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{w}_{k+1} - \mathbf{b})$$

Change $(\mathbf{x}, \mathbf{w})$ joint optimization to separate alternate iterations and let $\mathbf{Y}_k = \frac{\mathbf{z}_k}{\beta}$

$$\mathbf{x}_{k+1} = \arg\min_{\mathbf{x}} \mathcal{L}_\beta(\mathbf{x}, \mathbf{y}_k, \mathbf{w}_k)$$
$$\mathbf{w}_{k+1} = \arg\min_{\mathbf{w}} \mathcal{L}_\beta(\mathbf{x}_{k+1}, \mathbf{y}_k, \mathbf{w})$$
$$\mathbf{y}_{k+1} = \mathbf{y}_k + (\mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{y}_{k+1} - \mathbf{b}).$$

[1] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers,” Foundations and Trends in Machine Learning, vol. 3, 2010.

• 4. Optimization ways – ADMM

- Rewrite the objective function:

$$\begin{array}{ll}\min\limits_{\mathbf{X}} & \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{B}^T) \\ \text{s.t.} & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M}).\end{array} \Rightarrow \begin{array}{ll}\min\limits_{\mathbf{X},\mathbf{W}} & \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}_l\mathbf{X}\mathbf{B}_l^T) \\ \text{s.t.} & \mathbf{X} = \mathbf{W},\ \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{M}).\end{array}$$

- The augmented Lagrange function:

$$L(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \beta) = \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}_l\mathbf{X}\mathbf{B}_l^T) + \frac{\beta}{2}\|\mathbf{X} - \mathbf{W}\|_F^2 + \mathrm{Tr}(\mathbf{Y}^T(\mathbf{X} - \mathbf{W}))$$

- Iteration function:

$$\begin{aligned}\mathbf{X}_{k+1} &= \arg\min_{\mathbf{X}} L(\mathbf{X}, \mathbf{Y}_k, \mathbf{W}_k, \beta) \\ &= \arg\min_{\mathbf{X}} \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}_l\mathbf{W}_k\mathbf{B}_l^T) + \frac{\beta}{2}\|\mathbf{X} - \mathbf{W}_k\|_F^2 + \mathrm{Tr}(\mathbf{Y}_k^T(\mathbf{X} - \mathbf{W}_k))\end{aligned}$$

- Singular Value Shrinkage operator[1]:

$$\mathcal{D}_\tau: \quad \mathcal{D}_\tau(\mathbf{X}) = \mathbf{U}\mathcal{D}_\tau(\Sigma)\mathbf{V}^T,\ \mathcal{D}_\tau(\Sigma) = \mathrm{diag}(\max\{\sigma_i - \tau, 0\}).$$

For each $\tau \geq 0$ and $\mathbf{Y} \in \mathbb{R}^{m\times n}$, we have

$$\mathcal{D}_\tau(\mathbf{X}) = \arg\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|_F^2 + \tau\|\mathbf{X}\|_*.$$

Algorithm 2 The Optimization using TNNR-ADMM

Input: $\mathbf{A}_l$, $\mathbf{B}_l$, $\mathbf{M}_\Omega$, and tolerance ϵ are given.
Initialization: $\mathbf{X}_1 = \mathbf{M}_\Omega$, $\mathbf{W}_1 = X_1$, $\mathbf{Y}_1 = \mathbf{X}_1$, and $\beta = 1$.
repeat
  step 1. $\mathbf{X}_{k+1} = \mathcal{D}_{\frac{1}{\rho}}(\mathbf{W}_k - \frac{1}{\rho}\mathbf{Y}_k)$.
  step 2. $\mathbf{W}_{k+1} = \mathbf{X}_{k+1} + \frac{1}{\beta}(\mathbf{A}_l^T\mathbf{B}_l + \mathbf{Y}_k)$.
  Fix values at observed entries, $\mathbf{W}_{k+1} = \mathbf{X}_{k+1} + \frac{1}{\beta}(\mathbf{A}_l^T\mathbf{B}_l + \mathbf{Y}_k)$.
  step 3. $\mathbf{Y}_{k+1} = \mathbf{Y}_k + \beta(\mathbf{X}_{k+1} - \mathbf{W}_{k+1})$.
until $\|\mathbf{X}_{k+1} - \mathbf{X}_k\|_F \leq \epsilon$.

[1] J.F. Cai, E.J. Candès, and Z. Shen, “A Singular Value Thresholding Algorithm for Matrix Completion,” SIAM J. Optimization, vol. 20, pp. 1956-1982, 2010.

• 4. Optimization ways -- APGL

- Accelerated Proximal Gradient Line (APGL), also called as iterative shrinkage-thresholding algorithm (FISTA)[1]. It is also a common approach to solve the 2-block convex optimization problems.

$$\min_{\mathbf{X}} \ g(\mathbf{X}) + f(\mathbf{X})$$

g is a closed, convex, possibly nondifferentiable function
f is a convex and differentiable function

- The APGL method constructs an approximation of $f(\mathbf{X})$ at a given point $\mathbf{Y}$ as

$$Q(\mathbf{X}, \mathbf{Y}) = f(\mathbf{Y}) + \langle \mathbf{X} - \mathbf{Y}, \nabla f(\mathbf{Y}) \rangle + \frac{1}{2t} \|\mathbf{X} - \mathbf{Y}\|_F^2 + g(\mathbf{X}).$$

- Above optimization problem by iteratively updating $\mathbf{X}$, $\mathbf{Y}$, and t.

$$\begin{aligned} \mathbf{X}_{k+1} &= \arg\min_{\mathbf{X}} Q(\mathbf{X}, \mathbf{Y}_k) \\ &= \arg\min_{\mathbf{X}} g(\mathbf{X}) + \frac{1}{2t} \|\mathbf{X} - (\mathbf{Y}_k - t_k \nabla f(\mathbf{Y}_k))\|_F^2 \end{aligned}$$

[1] A. Beck and M. Teboulle, "A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems," SIAM J. Imaging Sciences, vol. 2, no. 1, pp. 183-202, 2009.

4. Optimization ways -- APGL

- Relax the constrained problem:

$$\min \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}_l \mathbf{X} \mathbf{B}_l^T) + \frac{\lambda}{2}\|\mathbf{X}_\Omega - \mathbf{M}_\Omega\|_F^2$$

$$g(\mathbf{X}) = \|\mathbf{X}\|_* \quad f(\mathbf{X}) = -\mathrm{Tr}(\mathbf{A}_l \mathbf{X} \mathbf{B}_l^T) + \frac{\lambda}{2}\|\mathbf{X}_\Omega - \mathbf{M}_\Omega\|_F^2$$

- Iteration function:

$$\begin{aligned}\mathbf{X}_{k+1} &= \arg\min_{\mathbf{X}} \|\mathbf{X}\|_* + \frac{1}{2t}\|\mathbf{X} - (\mathbf{Y}_k - t_k \nabla f(\mathbf{Y}_k))\|_F^2 \\ &= \mathcal{D}_{t_k}(\mathbf{Y}_k - t_k \nabla f(\mathbf{Y}_k)) \\ &= \mathcal{D}_{t_k}(\mathbf{Y}_k - t_k(\mathbf{A}_l^T \mathbf{B}_l - \lambda(\mathcal{P}_\Omega(\mathbf{Y}_k) - \mathcal{P}_\Omega(\mathbf{M}))))\end{aligned}$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},$$

$$\mathbf{Y}_{k+1} = \mathbf{X}_{k+1} + \frac{t_k - 1}{t_{k+1}}(\mathbf{X}_{k+1} - \mathbf{X}_k)$$

Algorithm 3 The Optimization using TNNR-APGL

Input: $\mathbf{A}_l$, $\mathbf{B}_l$, $\mathbf{M}_\Omega$, and tolerance ϵ are given.
Initialization: $t_1 = 1$, $\mathbf{X}_1 = \mathbf{M}_\Omega$, $\mathbf{Y}_1 = \mathbf{X}_1$.
repeat
step 1. $\mathbf{X}_{k+1} = \mathcal{D}_{t_k}(\mathbf{Y}_k - t_k(\mathbf{A}_l^T \mathbf{B}_l - \lambda(\mathcal{P}_\Omega(\mathbf{Y}_k) - \mathcal{P}_\Omega(\mathbf{M}))))$.
step 2. $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$.
step 3. $\mathbf{Y}_{k+1} = \mathbf{X}_{k+1} + \frac{t_k - 1}{t_{k+1}}(\mathbf{X}_{k+1} - \mathbf{X}_k)$.
until $\|\mathbf{X}_{k+1} - \mathbf{X}_k\|_F \leq \epsilon$.

• 4. Optimization ways -- ADMMAP

- Problem of ADMM
  - The convergence of ADMM becomes slower with more constraints[1].
  - It is hard to choose an optimal penalty parameter in advance (improper penalty may lead to divergence).

- The authors purpose the approach named ADMM with adaptive penalty (TNNR-ADMMAP) to solve the problem.
  - New transform variant of ADMM: Deal with all the constraints together to accelerate the convergence
  - Adaptive update strategy for the penalty parameter[2]: further speed up the convergence

- Reformulate the objective function:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{A}_l \mathbf{X} \mathbf{B}_l^T) \\ \text{s.t.} \quad & \mathcal{A}(\mathbf{X}) + \mathcal{B}(\mathbf{M}) = \mathcal{C}, \end{aligned} \qquad \mathcal{A}(\mathbf{X}) = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & 0 \end{pmatrix}, \ \mathcal{B}(\mathbf{X}) = \begin{pmatrix} -\mathbf{W} & 0 \\ 0 & \mathcal{P}_\Omega(\mathbf{X}) \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{P}_\Omega(\mathbf{M}) \end{pmatrix}$$

[1] B. He, M. Tao, and X. Yuan, "Alternating Direction Method with Gaussian Back Substitution for Separable Convex Programming," SIAM J. Optimization, vol. 22, no. 2, pp. 313-340, 2012.
[2] Z. Lin, R. Liu, and Z. Su, "Linearized Alternating Direction Method with Adaptive Penalty for Low-Rank Representation," Proc. Advances in Neural Information Processing Systems, 2011.

• 4. Optimization ways -- ADMMAP

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- The augmented Lagrange function:

$$\mathcal{L}_{AP}(\mathbf{X},\mathbf{W},\mathbf{Y},\beta) = -\mathrm{Tr}(\mathbf{A}_l\mathbf{W}\mathbf{B}_l^T)+\langle\mathbf{Y},\mathcal{A}(\mathbf{X})+\mathcal{B}(\mathbf{W})-\mathcal{C}\rangle+\|\mathbf{X}\|_*+\frac{\beta}{2}\|\mathcal{A}(\mathbf{X})+\mathcal{B}(\mathbf{W})-\mathcal{C}\|_F^2$$

- Iteration function:

$$\begin{aligned}\mathbf{X}_{k+1} &= \arg\min_{\mathbf{X}} \mathcal{L}_{AP}(\mathbf{X},\mathbf{W}_k,\mathbf{Y}_k,\beta)\\ &= \arg\min_{\mathbf{X}} \frac{\beta}{2}\|\mathcal{A}(\mathbf{X})+\mathcal{B}(\mathbf{W}_k)-\mathcal{C}+\frac{1}{\beta}\mathbf{Y}_k\|_F^2+\|\mathbf{X}\|_*\end{aligned}$$

$$\mathbf{W}_{k+1} = \arg\min_{W} \mathcal{L}_{AP}(\mathbf{X}_{k+1},\mathbf{W},\mathbf{Y}_k,\beta)$$

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k+\beta[\mathcal{A}(\mathbf{X}_{k+1})+\mathcal{B}(\mathbf{W})-\mathcal{C}]$$

$$\beta_{k+1} = \min(\beta_{\max},\rho\beta_k) \qquad (54)$$

$$\rho = \begin{cases}\rho_0, & \text{if } \dfrac{\beta_k \max\{\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F\}}{\|\mathcal{C}\|_F} < \kappa\\ 1, & \text{otherwise.}\end{cases} \qquad (55)$$

Algorithm 4 Inner Optimization by ADMMAP

Input: $\mathbf{A}_l$, $\mathbf{B}_l$, $\mathbf{M}_\Omega$, and tolerance ϵ are given.

Initialization: $\mathbf{X}_1 = \mathbf{M}_\Omega$, $\mathbf{Y}_1 = \mathbf{X}_1$.

repeat

step 1. $\mathbf{X}_{k+1} = \mathcal{D}\left(\mathbf{W}_k - \frac{1}{\beta}(\mathbf{Y}_k)_{11}\right)$.

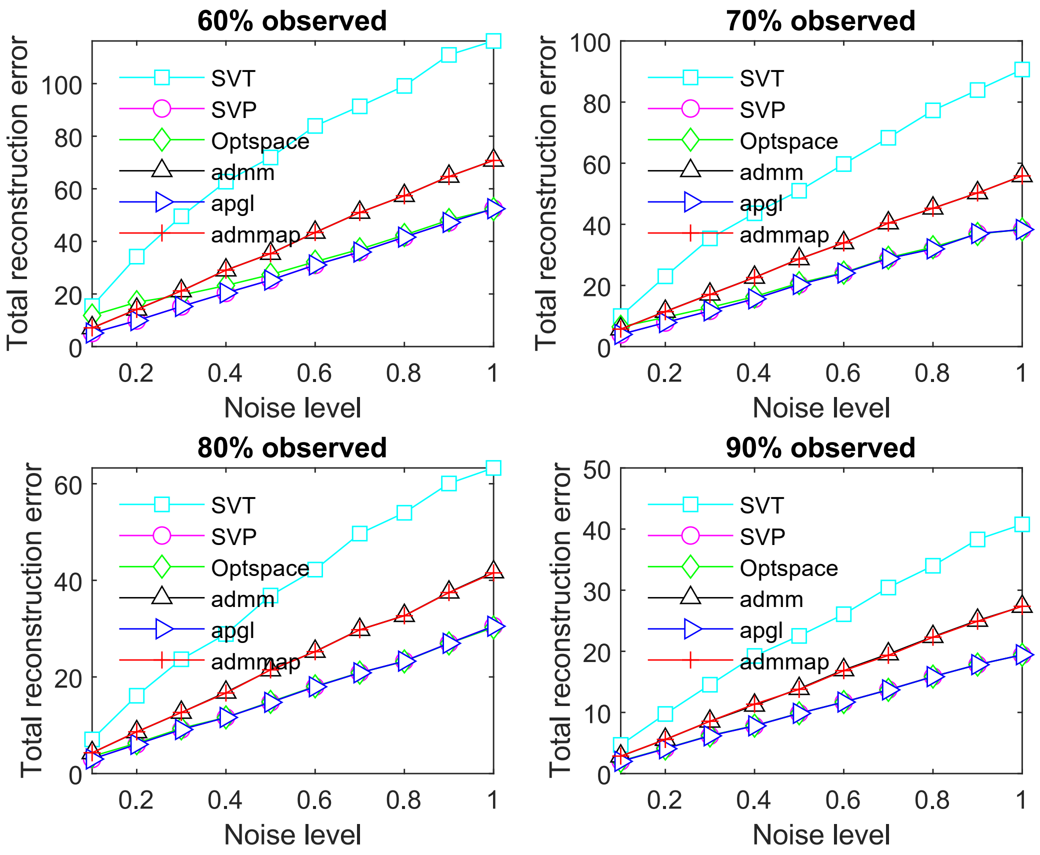
step 2. $\mathbf{W}_{k+1} = \frac{1}{2\beta}\mathcal{P}_\Omega[\beta(\mathbf{M}-\mathbf{X}_{k+1})-(\mathbf{A}_l^T\mathbf{B}_l+(\mathbf{Y}_k)_{11}+(\mathbf{Y}_k)_{22})]$ $+\mathbf{X}_{k+1}+\frac{1}{\beta}(\mathbf{A}_l^T\mathbf{B}_l+(\mathbf{Y}_k)_{11})$.

step 3. $\mathbf{Y}_{k+1} = \mathbf{Y}_k+\beta[\mathcal{A}(\mathbf{X}_{k+1})+\mathcal{B}(\mathbf{W})-\mathcal{C}]$.

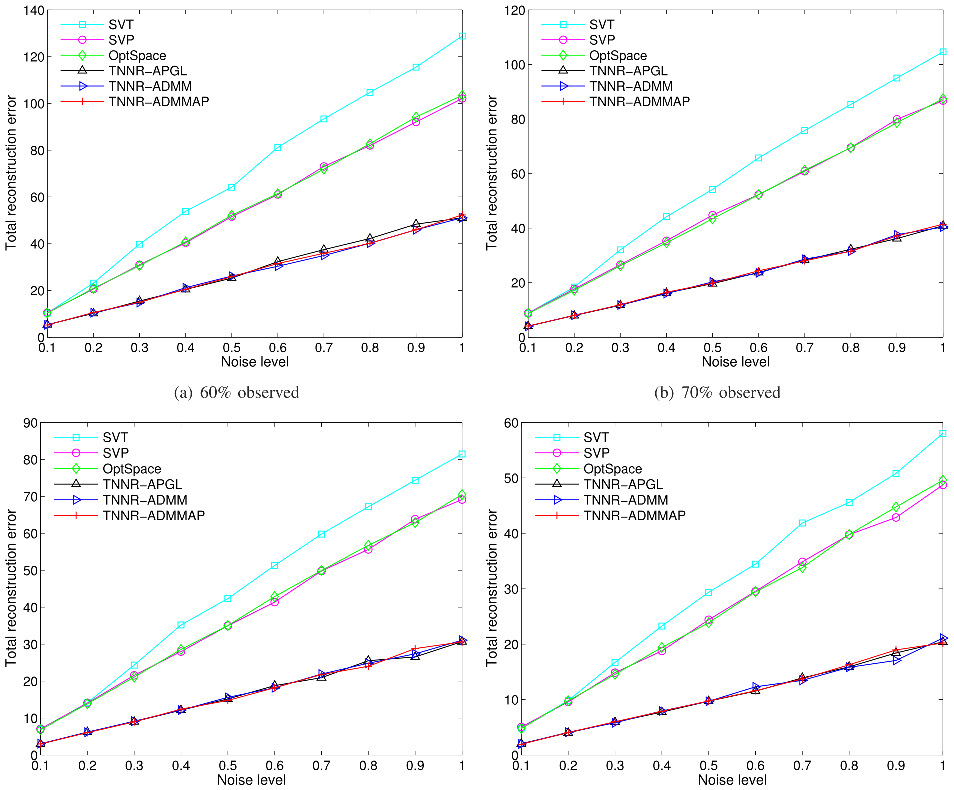
step 4. Update β_k by (54) and (55)

until $\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F \le \epsilon$.

• 5. Results – Reconstruction error



The reproduction of Fig. 2.



(a) 60% observed (b) 70% observed

(c) 80% observed (d) 90% observed

The original of Fig. 2.

Data Generation: $\mathbf{B} = \mathbf{M} + \sigma\mathbf{Z},\ B_{ij} = M_{ij} + \sigma Z_{ij},\ (i,j) \in \mathbf{\Omega}$

$\mathbf{M} = \mathbf{M}_L\mathbf{M}_R,\ \mathbf{M}_L \in \mathbb{R}^{m\times r_0},\ \mathbf{M}_R \in \mathbb{R}^{r_0\times n}$

$\mathbf{M}_L,\ \mathbf{M}_R,\ \mathbf{Z}$ is gaussian.

Metric:

Total reconstruction error: $\|\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X}_{sol} - \mathbf{X}_{full})\|_F$

• 5. Results – completion, PSNR, convergence

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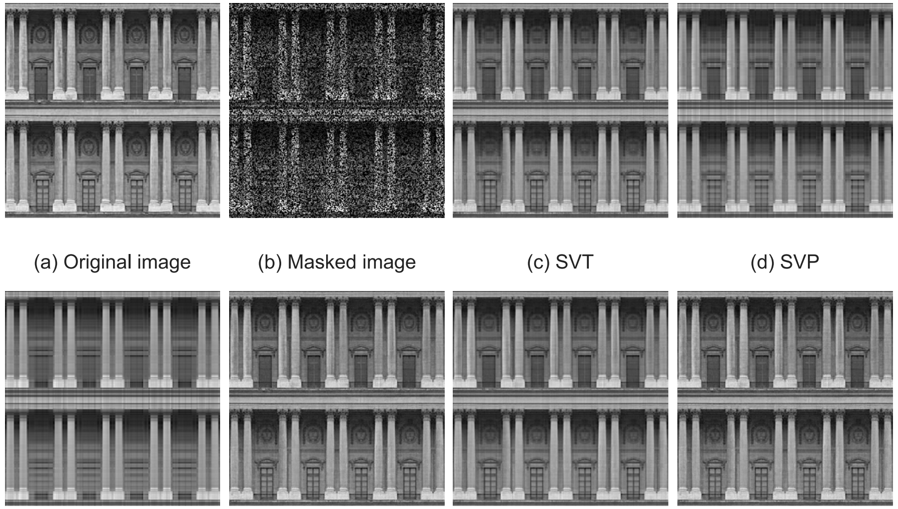
β	1e-3	r	5
$\beta_{\max}$	1e10	ϵ_0	1e-3
ρ_0	1.1	ϵ	1e-4
κ	1e-3	λ	1e-2
p_{SVT}	0.87	τ_{opt}	1e-3
δ_{2k}	0.2	$iter_{\max}$	100

Total number of missing pixels: T

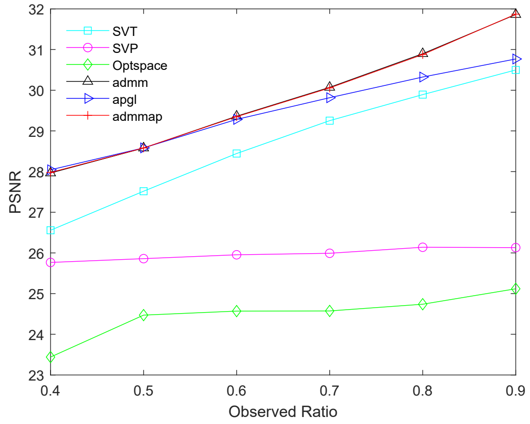
$$\text{MSE} = \frac{error_r^2 + error_g^2 + error_b^2}{3T}$$

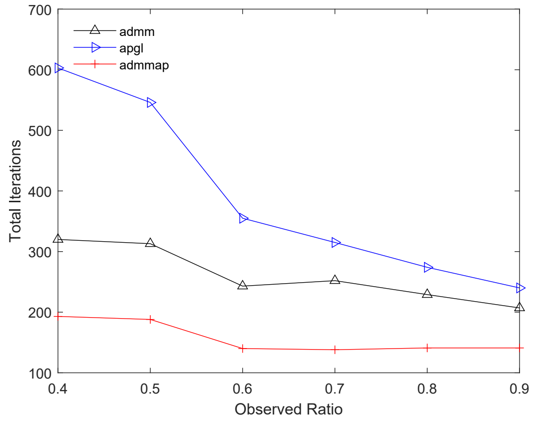
$$\text{PSNR} = 10 \times \log_{10}\left(\frac{255^2}{MSE}\right)$$



(a) Original image (b) Masked image (c) SVT (d) SVP

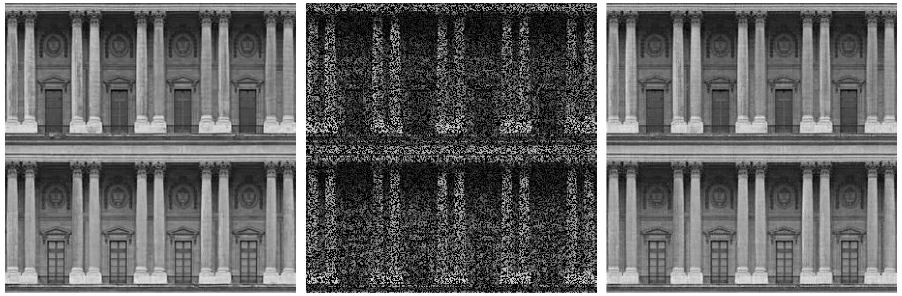
(e) Opt (f) tnnr-admm (g) tnnr-apgl (h) tnnr-admmap





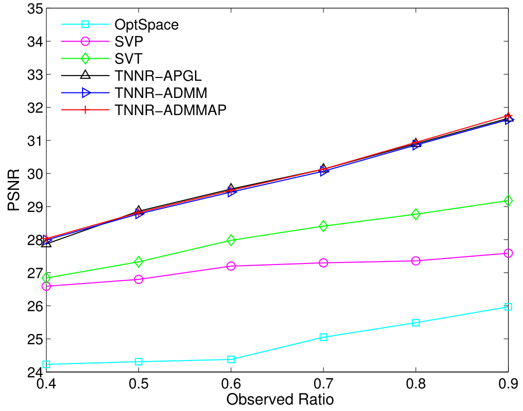
The reproduction of Fig. 4 (completion), Fig. 5 (PSNR), Fig. 6 (convergence).

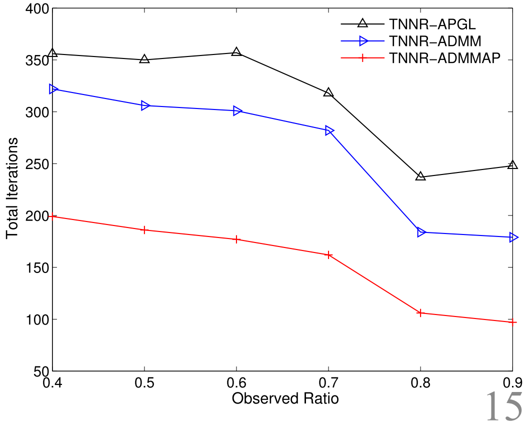
β	1e-3	r	1~20
$\beta_{\max}$	1e10	ϵ_0	1e-3
ρ_0	1.9	ϵ	1e-4
κ	1e-3	λ	0.06



(a) Original image (b) Masked image (c) Recovered image by TNNR-ADMMAP

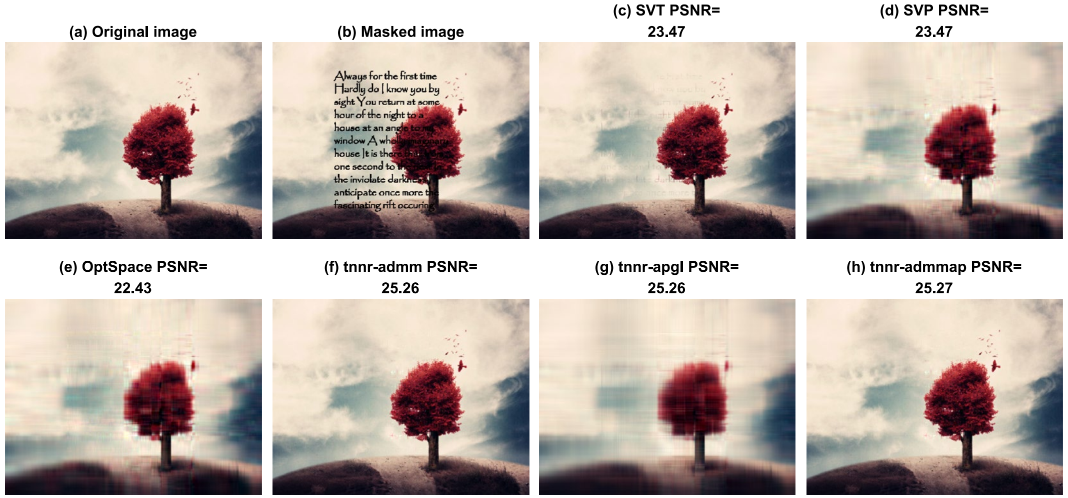
The original of Fig. 4, Fig. 5, Fig. 6.





• 5. Results – Eliminate String

β	1e-3	r	3~15
β_{max}	1e10	ϵ_0	1e-3
ρ_0	1.01	ϵ	1e-4
κ	1e-3	λ	1e-3
p_{SVT}	0.8	τ_{opt}	1e-3
δ_{2k}	0.2	$iter_{max}$	1e2/1e3



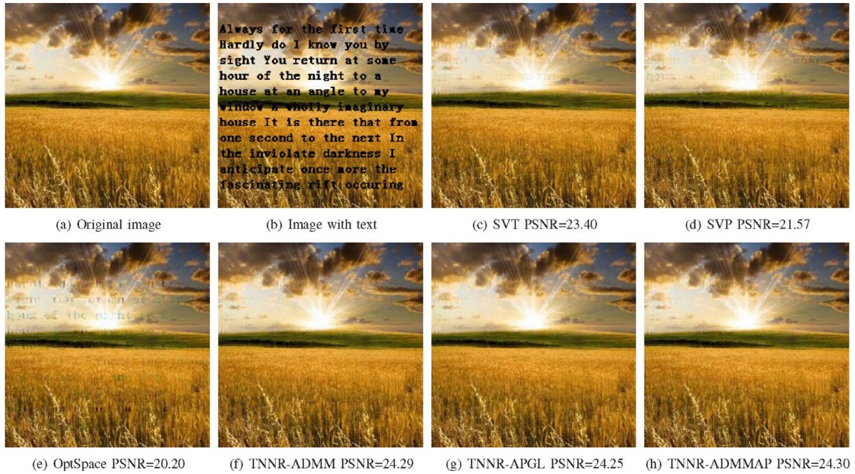
The reproduction of Fig. 7, Fig. 8.

Total number of missing pixels: T

$$\text{MSE} = \frac{error_r^2 + error_g^2 + error_b^2}{3T}$$

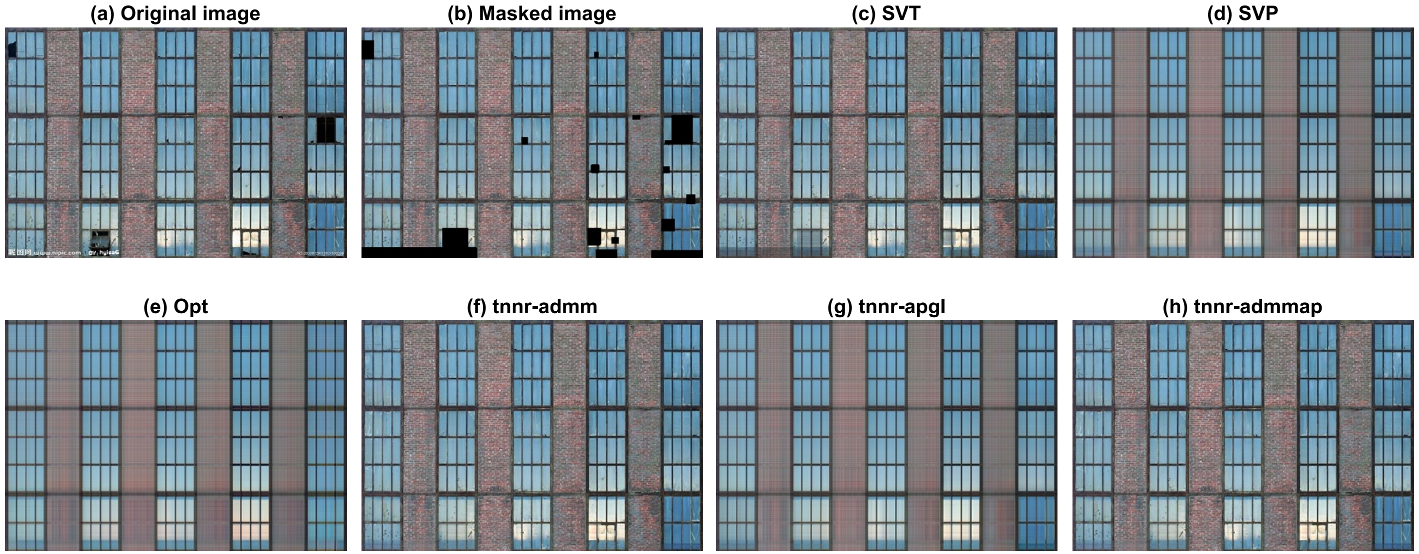
$$\text{PSNR} = 10 \times \log_{10}\left(\frac{255^2}{MSE}\right)$$

The original Parameter Setting is equal to Fig. 4-6.



The original of Fig. 7, Fig. 8.

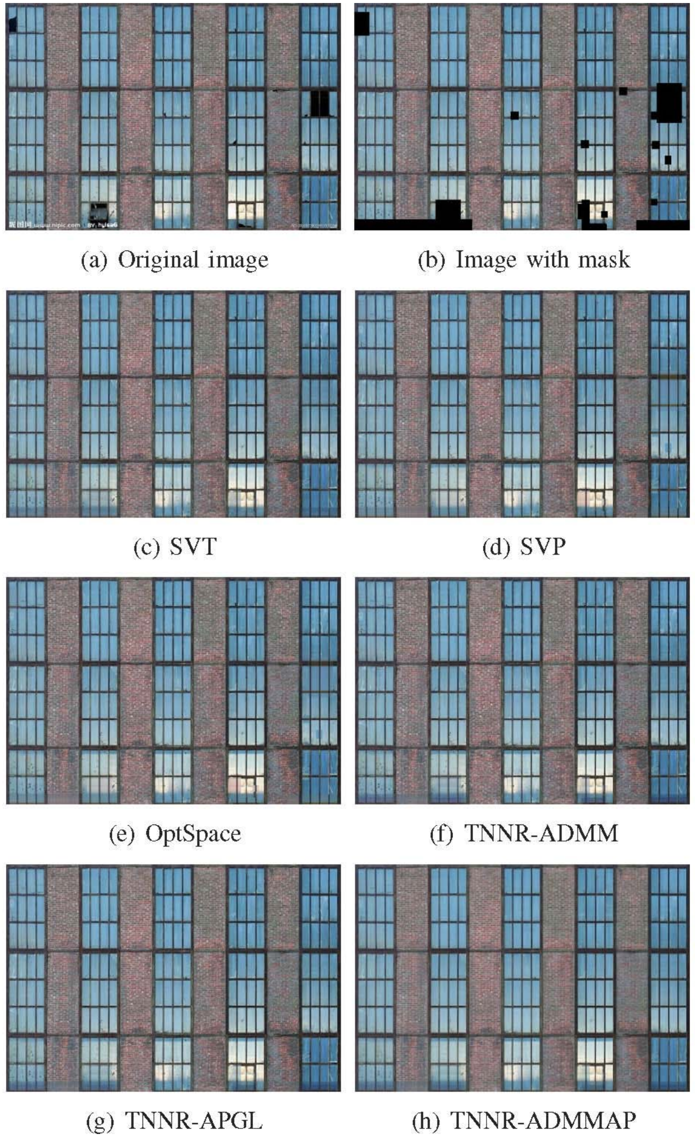
• 5. Results – Eliminate Blocks



The reproduction of Fig. 9.

β	1e-3	r	3	p_{SVT}	0.8
$\beta_{\max}$	1e10	ϵ_0	1e-3	τ_{opt}	1e-2
ρ_0	1.01	ϵ	1e-4	δ_{2k}	0.2
κ	1e-3	λ	1e-3	$iter_{\max}$	1e2/1e3

The original Parameter Setting is equal to Fig. 4-6.



The original of Fig. 9.

Source code available at: https://github.com/AnnLIU15/matrixTheory

Thanks for listening