7 TMD Design and Optimization

This chapter underscores the iterative process between theory, simulation, and empirical testing, ensuring the TMD design's efficacy and applicability in enhancing the medical device's performance by mitigating the negative effects of periodic forces. The chapter is structured into three main sections, each addressing a key phase in the TMD development process: analytical modeling for calculating optimal TMD parameters, digital simulation of the tuned system (primary system + TMD), and experimental analysis of the entire (tuned) system to validate the digital and mathematical models.

7.1 Analytical Model & Optimal Parameters TMD

This section delves into the analytical modeling required to determine the optimal parameters for a TMD. Optimal parameters for the TMD are calculated using established methods, ensuring precise tuning to the primary system's vibrational characteristics. This analytical foundation is essential for subsequent digital simulations and practical implementations, aiming to enhance the performance and stability of the medical device by effectively mitigating unwanted vibrations.

7.1.1 Analytical Calculations for FRFs

To effectively analyze and understand the dynamic behavior of system under study, it is beneficial to represent it as a single degree of freedom system (Figure 48). Additionally, by integrating a tuned mass damper, the system transitions into a two degrees of freedom system (Figure 49), allowing for a more detailed analysis of vibration control. This approach captures the core dynamics while maintaining analytical simplicity and clarity.

The resulting transfer functions and FRFs provide insight into how the system responds to various frequencies, guiding the design and tuning of the TMD. All calculations in this section were done similarly to Ketterer (2020).

SDOF System

The analytical modeling process begins with formulating the system's equations of motion. For an SDOF system, this involves identifying key parameters: mass (***m***), stiffness (***k***), and damping (***c***). These parameters are crucial for characterizing the system's response to vibrational inputs.

Simplifying a system to an SDOF model involves focusing on the primary mode of interest. This simplification is particularly useful for effective vibration absorption using a TMD. The steps include:

- Identifying the dominant mode: From the modal analysis of the mounted plate, mode shape #2 was identified as the key target for vibration mitigation (see Chapter 6). For the medical device representation model, mode shape #1 and #2 are the targets (see Chapter 3).
- Defining system parameters: ***m, k*** and ***c*** values for these mode shapes are extracted to define the equivalent SDOF system.

- Creating the SDOF model: The complex physical system is simplified into an equivalent SDOF model, which includes the primary system's mass, a spring (stiffness), and a damper.
- Deriving solutions: Using the SDOF model to derive analytical solutions for the system's response to external forces, solving the equation of motion under different loading conditions.

Figure 48 illustrates an SDOF system, which is represented by a spring-damper-mass system. In this figure, $\boldsymbol{k_1}$ is stiffness, $\boldsymbol{c_1}$ is damping, $\boldsymbol{m_1}$ – mass, $\boldsymbol{x_1}$ – displacement and $\boldsymbol{F_1}$ – external force.

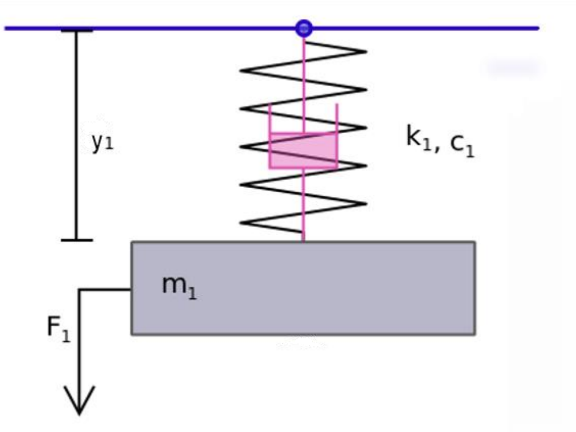

Figure 48. SDOF System

Equation of motion for this system is expressed as shown in **Eq. 5**:

$$m_1 Y_1(t) + k_1 \ddot{Y}_1(t) + c_1 \dot{Y}_1(t) = F_1(t)$$

where:

- $Y_1(t)$ is the displacement over time,
- $\dot{Y}_1(t)\ and\ \ddot{Y}_1(t)$ are the velocity and acceleration, respectively,
- $F_1(t)$ is the external force applied.

To facilitate solving this differential equation, the Laplace transform is applied. The Laplace transform converts the differential equation from the time domain to the s-domain, making it easier to handle algebraically.

Laplace transform:

$$m_1 s^2 y_1(s) + k_1 s y_1(s) + c_1 y_1(s) = F_1(s)$$

Rearranging and solving for y_1 in terms of F_1:

$$y_1 s^2 = \frac{F_1}{m_1} - \frac{k_1}{m_1} y_1 - \frac{c_1}{m_1} y_1 s$$

$$y_1(s^2 + \frac{c_1}{m_1}s + \frac{k_1}{m_1}) = \frac{F_1}{m_1}$$

$$\frac{y_1}{F_1} = \frac{1}{m_1} \cdot \frac{1}{s^2 + \frac{c_1}{m_1}s + \frac{k_1}{m_1}}$$

$$\frac{y_1}{F_1} = \frac{1}{m_1} \cdot \frac{1}{\frac{1}{m_1}(m_1 s^2 + c_1 s + k_1)}$$

$$\frac{y_1}{F_1} = \frac{1}{m_1 s^2 + c_1 s + k_1} \quad (12)$$

Eq.12 represents the transfer function in the Laplace domain.

Next, to express the transfer function in the frequency domain, we substitute $s = j\omega$,where ω is the angular frequency and j is the imaginary unit. This substitution helps in understanding how the system responds to sinusoidal inputs at different frequencies.

Transfer function in frequency domain:

Substituting:

$s = j\omega,$

$s^2 = -\omega^2$

$$\frac{y_1}{F_1} = \frac{1}{-\omega^2 m_1 + c_1 j\omega + k_1}$$

The resulting transfer function G is:

$$\boxed{G = \frac{\ddot{y}_1}{F_1} = \left|\frac{-\omega^2}{-\omega^2 m_1 + c_1 j\omega + k_1}\right|} \quad (13)$$

This transfer function **(Eq.13)** characterizes the frequency response of the SDOF system. It shows how the output acceleration relates to the input force across different frequencies. With the calculated **Eq. 13** we can plot a FRF of an SDOF system in terms of acceleration over force. For this calculation, the input parameters include the system´s mass, stiffness, and damping. Additionally, the angular frequency range ω should be determined.

Figure 49 illustrates the FRF of a primary system, constructed in Python (see Appendix II) using the calculated **Eq. 13.**

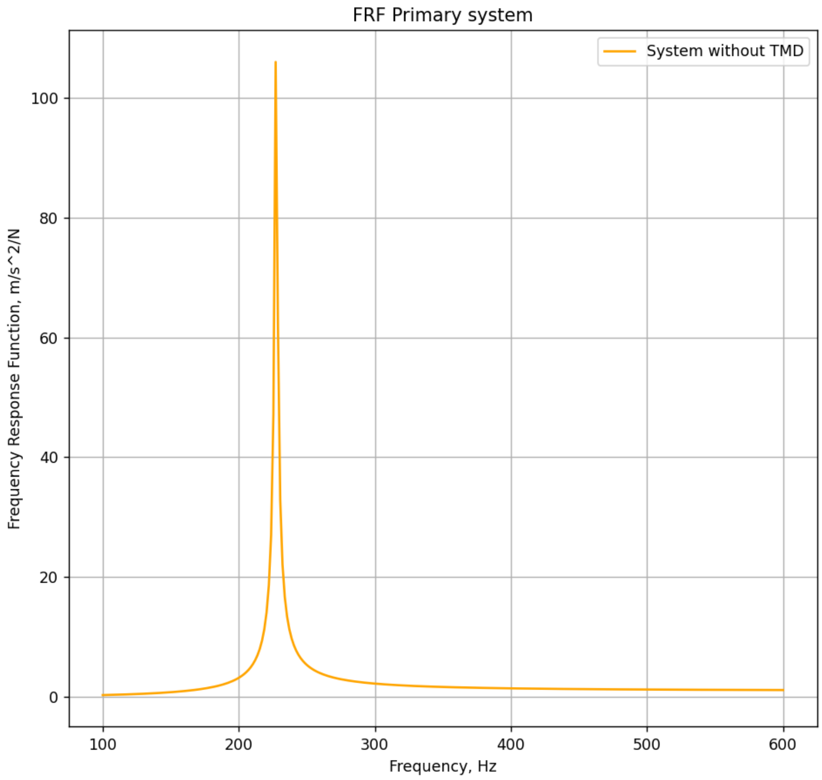

Figure 49. FRF of a SDOF Built in Python

For this example, shown in Figure 49, the primary system is a mounted plate system with the following inputs:

$m_1 = 0{,}95\ kg$

$f_1 = 227{,}2\ Hz$ (Natural frequency)

$f\ = from\ 100\ to\ 600\ Hz$ (Range of frequencies to analyze)

$\zeta_1 = 0{,}0041$ (Damping ratio)

The data analysis was conducted using a custom script provided in Appendix I. **Eq. 13** requires $k_1\ and\ c_1$ as inputs. Therefore, **Eq. 14** and **Eq.15** were implemented in the Python script to calculate these parameters:

$$k = {\omega_n}^2 \cdot m \tag{14}$$

$$c = 2\omega_n\,\zeta m \tag{15}$$

Since the input includes only frequencies in Hz, the angular frequency ω should be calculated using **Eq.16.**

$$\omega = 2\pi f \tag{16}$$

With these parameters, the FRF of the SDOF system can be plotted over the desired frequency range using Python. The Python script calculates the FRF by substituting the values of $m_1, k_1, c_1 and\ \omega$ into **Eq.13**. This visualization helps us understand the system's dynamic response and guides the design and tuning of the TMD.

The transfer function derived above helps in predicting how the system will respond to different frequencies of excitation. This is particularly important when designing a TMD, as the TMD needs to be tuned to the specific frequencies that the system is most susceptible to. By analyzing the transfer function, we can identify these critical frequencies and design the TMD to effectively mitigate the vibrations at these frequencies.

2DOF System

By integrating a TMD, the system transitions from an SDOF to a 2DOF model (Figure 50). The TMD system includes an additional mass, spring, and damper, which interact with the primary system to absorb and mitigate vibrations more effectively.

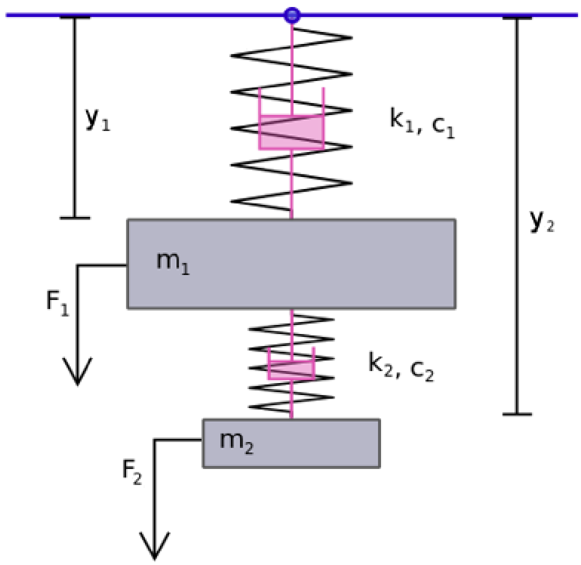

Figure 50. 2DOF System

Equation of motion for a 2DOF system, incorporating the TMD:

$$\begin{cases} I)\ m_1\ddot{Y}_1(t) + k_1Y_1(t) + c_1\dot{Y}_1(t) + k_2Y_1(t) + c_2\dot{Y}_1(t) - k_2Y_2(t) - c_2\dot{Y}_2(t) = F_1(t) \\ II)\ m_2\ddot{Y}_2(t) + k_2Y_2(t) + c_2\dot{Y}_2(t) - k_2Y_1(t) - c_2\dot{Y}_1(t) = F_2(t) \end{cases} \quad (17)$$

where:

- $Y_1(t)\ and\ Y_2(t)$ are the displacements of the primary system and the TMD, respectively,
- $m_1\ and\ m_2$ are the masses of the primary system and the TMD,

- $c_1 \; and \; c_2$ are the damping coefficients,
- $k_1 \; and \; k_2$ are the stiffness values.

Equation I) The first equation represents the motion of the primary system, including the effects of the TMD. The terms involving k_2 and c_2 represent the interaction between the primary system and the TMD.

Equation II): The second equation represents the motion of the TMD itself. The TMD's behavior is influenced by the primary system through the terms $-k_2 Y_1(t)$ and $-c_2 \dot{Y}_1(t)$.

$$\begin{cases} I)\ \ddot{Y}_1 m_1 = F_1 - k_1 Y_1 - c_1 \dot{Y}_1 - k_2 Y_1 - c_2 \dot{Y}_1 + k_2 Y_2 + c_2 \dot{Y}_2 \\ II)\ \ddot{Y}_2 m_2 = F_2 - k_2 Y_2 - c_2 Y_2 + k_2 \dot{Y}_1 + c_2 \dot{Y}_1 \end{cases}$$

$$\begin{cases} I)\ \ddot{Y}_1 = -\frac{k_1}{m_1} Y_1 - \frac{c_1}{m_1} \dot{Y}_1 + \frac{F_1}{m_1} - \frac{k_2}{m_1} Y_1 + \frac{k_2}{m_1} Y_2 - \frac{c_2}{m_1} \dot{Y}_1 + \frac{c_2}{m_1} \dot{Y}_2 \\ II)\ \ddot{Y}_2 = \frac{k_2}{m_2} Y_1 - \frac{k_2}{m_2} Y_2 + \frac{c_2}{m_2} \dot{Y}_1 - \frac{c_2}{m_2} \dot{Y}_2 + \frac{F_2}{m_2} \end{cases}$$

$X_1 = Y_1 \qquad \dot{X}_1 = \dot{Y}_1 = X_2$

$X_2 = \dot{Y}_1$

$X_3 = Y_2 \qquad \dot{X}_3 = \dot{Y}_2 = X_4$

$X_4 = \dot{Y}_2$

$$\begin{cases} I)\ \dot{X}_2 = -\frac{k_1}{m_1} X_1 - \frac{c_1}{m_1} X_2 + \frac{F_1}{m_1} - \frac{k_2}{m_1} X_1 + \frac{k_2}{m_1} X_3 - \frac{c_2}{m_1} X_2 + \frac{c_2}{m_1} X_4 \\ II) \dot{X}_4 = \frac{k_2}{m_2} X_1 - \frac{k_2}{m_2} X_3 + \frac{c_2}{m_2} X_2 - \frac{c_2}{m_2} X_4 + \frac{F_2}{m_2} \end{cases}$$

The state space representation reformulates the equations of motion into a first-order system of differential equations, making it easier to analyze the dynamic behavior of the system.

State space:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} - \frac{k_2}{m_1} & -\frac{c_1}{m_1} - \frac{c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ F_1 \\ 0 \\ F_2 \end{pmatrix}$$

Laplace transform:

$$\begin{cases} I)\ y_1 s^2 = -\frac{k_1}{m_1} y_1 - \frac{c_1}{m_1} y_1 s + \frac{F_1}{m_1} - \frac{k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2 - \frac{c_2}{m_1} y_1 s + \frac{c_2}{m_1} y_2 \\ II)\ y_2 s^2 = \frac{c_2}{m_2} y_1 - \frac{c_2}{m_2} y_2 + \frac{d_2}{m_2} y_1 s - \frac{d_2}{m_2} y_2 s + \frac{F_2}{m_2} \end{cases}$$

$$\begin{cases} I)\ y_1(s^2+\left(\frac{c_1}{m_1}+\frac{c_2}{m_1}\right)s+(\frac{k_1}{m_1}+\frac{k_2}{m_1}) = y_2\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right)+\frac{F_1}{m_1} \\ II)\ y_2(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2}) = y_1(\frac{c_2}{m_2}s+\frac{k_2}{m_2})+\frac{F_2}{m_2} \end{cases}$$

$$\begin{cases} I)\ y_1(s^2+\left(\frac{c_1}{m_1}+\frac{c_2}{m_1}\right)s+(\frac{k_1}{m_1}+\frac{k_2}{m_1}) = y_2\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right)+\frac{F_1}{m_1} \\ II)\ y_2 = \dfrac{y_1(\frac{c_2}{m_2}s+\frac{k_2}{m_2})+\frac{F_2}{m_2}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})} \end{cases}$$

$\underset{II}{y_2} \rightarrow I.$ This leads to a single equation that can be solved to find the transfer function:

$$y_1(s^2+\left(\frac{c_1}{m_1}+\frac{c_2}{m_1}\right)s+(\frac{k_1}{m_1}+\frac{k_2}{m_1}) = \frac{y_1(\frac{c_2}{m_2}s+\frac{k_2}{m_2})+\frac{F_2}{m_2}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right)+\frac{F_1}{m_1}$$

No external force acting on the TMD, therefore $F_2 = 0 \quad => \quad \frac{F_2}{m_2} = 0.$

$$y_1(s^2+\left(\frac{c_1}{m_1}+\frac{c_2}{m_1}\right)s+(\frac{k_1}{m_1}+\frac{k_2}{m_1}) - y_1\frac{\frac{c_2}{m_2}s+\frac{k_2}{m_2}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1}{m_1}+\frac{(c_1+c_2)s}{m_1}+\frac{k_1+k_2}{m_1}-\frac{\frac{c_2}{m_2}s+\frac{k_2}{m_2}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right)) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1+(c_1+c_2)s+k_1+k_2}{m_1}-\frac{(\frac{c_2}{m_2}s+\frac{k_2}{m_2})\left(\frac{k_2}{m_1}+\frac{c_2}{m_1}s\right)}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1+(c_1+c_2)s+k_1+k_2}{m_1}-\frac{\frac{(c_2s+k_2)(k_2+c_2s)}{m_2m_1}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1+(c_1+c_2)s+k_1+k_2}{m_1}-\frac{\frac{(k_2+c_2s)^2}{m_2m_1}}{(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1+(c_1+c_2)s+k_1+k_2}{m_1}-\frac{(k_2+c_2s)^2}{m_2m_1(s^2+\frac{c_2}{m_2}s+\frac{k_2}{m_2})}) = \frac{F_1}{m_1}$$

$$y_1(\frac{s^2m_1 + (c_1 + c_2)s + k_1 + k_2}{m_1} - \frac{1}{m_1} \cdot \frac{(k_2 + c_2s)^2}{m_2s^2 + c_2s + k_2}) = \frac{F_1}{m_1}$$

$$\frac{y_1}{m_1}(s^2m_1 + (d_1 + d_2)s + c_1 + c_2 - \frac{(c_2 + d_2s)^2}{m_2s^2 + d_2s + c_2}) = \frac{F_1}{m_1}$$

$$\frac{y_1}{F_1} = \frac{1}{s^2m_1 + (c_1 + c_2)s + k_1 + k_2 - \frac{(k_2 + c_2s)^2}{m_2s^2 + c_2s + k_2}}$$

Transfer function in frequency domain:

$s = j\omega$

$s^2 = -\omega^2$

$$\frac{y_1}{F_1} = \frac{1}{-\omega^2m_1 + (c_1 + c_2)j\omega + k_1 + k_2 - \frac{(k_2 + c_2s)^2}{-\omega^2m_2 + c_2j\omega + k_2}}$$

$$\boxed{G = \frac{\ddot{y}_1}{F_1} = \frac{-\omega^2}{-\omega^2m_1 + (c_1 + c_2)j\omega + k_1 + k_2 + \frac{(c_2j\omega + k_2)^2}{\omega^2m_2 - c_2j\omega - k_2}}} \quad (18)$$

This transfer function G (**Eq. 18**) characterizes the frequency response of the 2DOF system. It shows how the output acceleration of the primary system relates to the input force across different frequencies.

With the calculated Equation 5 we can plot a FRF of a 2DOF-system in terms of acceleration over force. The inputs for that calculation should include the mass, stiffness, and damping of both the primary system and the TMD system. Additionally, the angular frequency range ω should be determined.

Currently, only the primary system's characteristics are known. The next step is to calculate the optimal parameters for the TMD system to effectively mitigate vibrations. This involves determining the optimal mass, stiffness, and damping for the TMD, which will be integrated into the 2DOF model to achieve the desired vibrational control.

7.1.2 Optimal Parameters for TMD

The primary goal is to tune the TMD so that its natural frequency matches that of the primary system's mode shape, thereby maximizing energy dissipation and minimizing vibrational amplitudes.

To determine the optimal parameters for the TMD, two methods are employed: Den Hartog's optimization method and Lin's optimization method.

- Den Hartog´s formulas for optimal TMD´s parameters are provided in Section 2.7 : **Eq.6** and **Eq.7.**
- Lin´s formulas for optimal TMD´s parameters are provided in Section 2.7 : **Eq.10** and **Eq.11.**

For the purposes of this research, a typical mass ratio ranges from 0,01 to 0,1 (Connor & Laflamme, 2014) is assumed. For each mass ratio, the optimal frequency ratio is calculated, followed by the natural frequency of the TMD. Subsequently, the optimal damping ratio of the TMD is determined, and the stiffness is calculated. Figure 51 represents the process of the optimal parameter calculation.

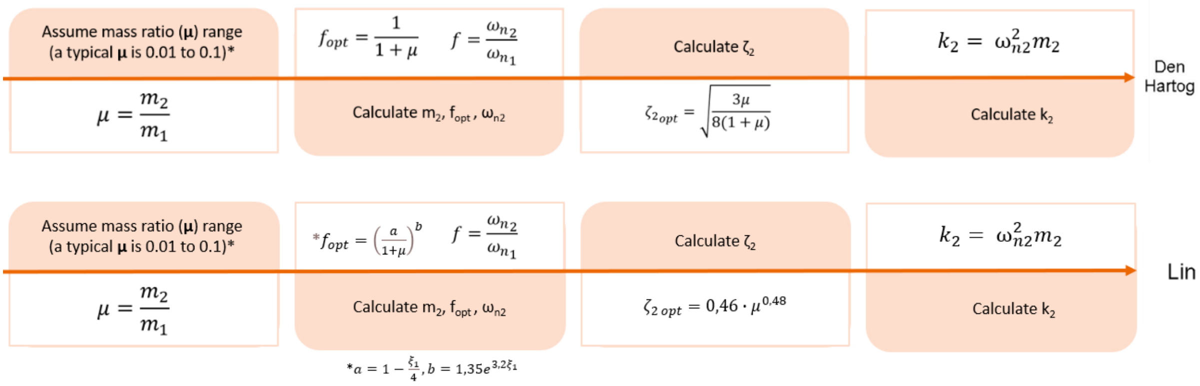

Figure 51. Process of the Optimal Parameter Calculations using Den Hartog´s (above) and Lin´s (below) Formulas

Using the calculated parameters, FRFs are plotted for each set of parameters (Figure 52). Parameter constraints should then be applied (if applicable). After applying these constraints, the set with the lowest peak amplitude should be chosen. These optimal parameters will be used for further simulation and implementation in the construction.

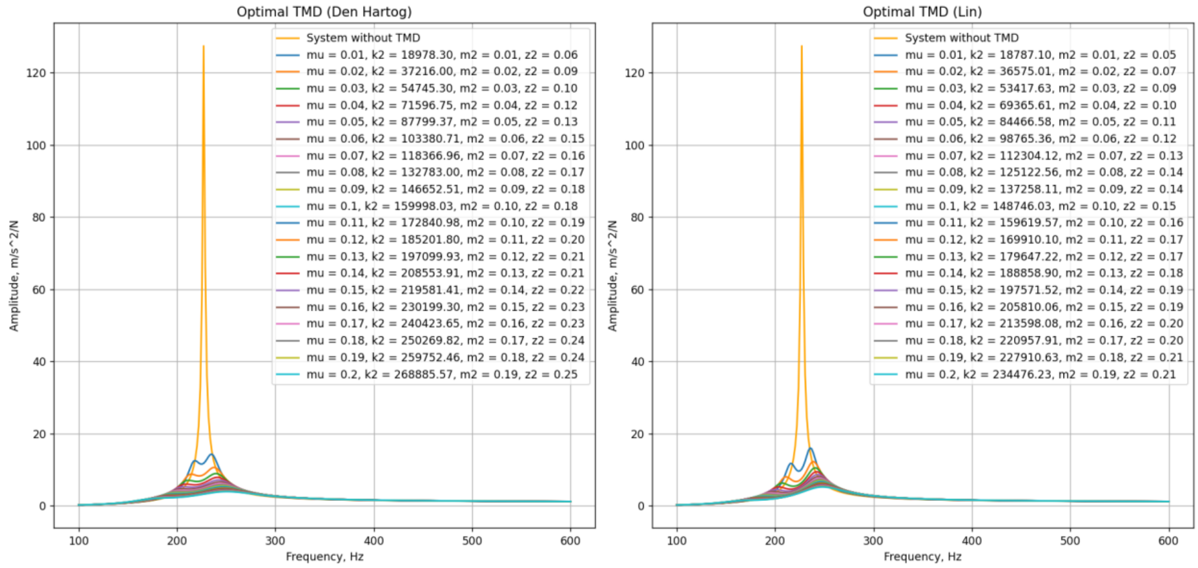

Figure 52. Result of TMD Optimal Parameters Calculation

The frequency response function of the primary system (in orange in Figure 52) was calculated above (Figure 50) and is now used to compare the response of the untuned system with the response of tuned system. For the untuned system, the inputs provided above are used. For the tuned system, the optimal parameters were calculated using Den Hartog's optimization method (left graph in Figure 52) and Lin's optimization method (right graph in Figure 52).

With these parameters, the FRF of the 2DOF system can be plotted over the desired frequency range using Python. The Python script calculates the FRFs by substituting the values of m_1, m_2, k_1, k_2, c_1, c_2 and ω into **Eq.18**.

By determining the optimal parameters using these methods, we can ensure that the TMD is tuned precisely to the primary system's vibrational characteristics, thereby enhancing its effectiveness in mitigating unwanted vibrations. This process forms the foundation for subsequent simulations and practical implementations, ensuring that the TMD performs optimally under real-world conditions.

7.2 Digital Simulation

Utilizing the validated model and optimal TMD parameters, this section involves simulating the tuned construction in NX to analyze the behavior and performance enhancements provided by the TMD.

As described in Chapter 5 , modeling starts with physical implementation. Figure 53 illustrates the tuned system implemented in the lab for EMA. This construction consists of the primary system (mounted plate) and the TMD system. For better structural stability, the construction is equipped with two tuned mass dampers in a parallel connection. Therefore, the calculated optimal stiffness and damping coefficient of the TMD should be divided by 2, representing parameters of a single TMD. Additionally, the analytically determined mass of the TMD is distributed among all components in the TMD system. For easier variation of the TMD system, the TMD system includes a damper plate that can easily be replaced with another one of different mass.