PHY407 Lab7

Genevieve Beauregard (1004556045)
Anna Shirkalina (1003334448)

Oct 2020

Question 1

a)

We are asked to implement the `Gauss-Seidel` method to calculate the electrostatic potential in the box with two charged plates of -1V and 1V respectively shown in figure 3 of the lab manual. We fix the potential to be zero at the walls of our box and to be +1, -1 at the charged plates. Once we calculate the potential we use the numpy gradiant method to obtain a stream plot of the electric field lines. Our results are shown in Figures 1 and 2.

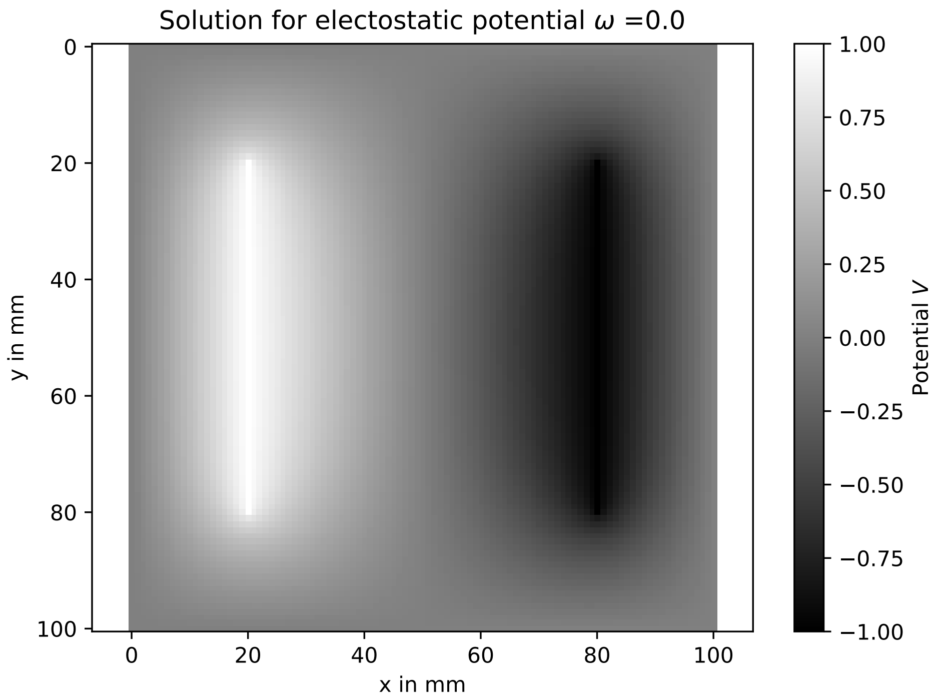

Figure 1: The solution to the electric potential using the Gauss-Seidel method, x and y are given in cm

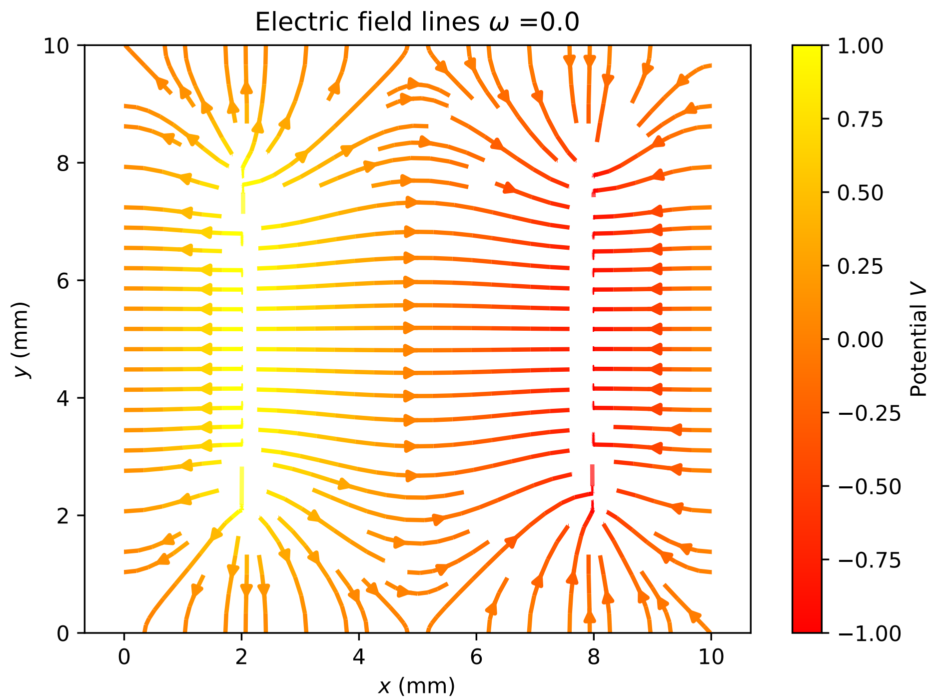

Figure 2: The stream plot of the electric field lines, x and y are given in cm

b)

We are now asked to modify our algorithm to apply over_relaxation. We try $\omega = 0.1, 0.4, 0.5$. We count the number of while loop iterations required for the algorithm before we have a changing potential less than our target of 1e-6. These results are displayed in table 1, where we show that an increase from no relaxation (w=0) to w=5 takes roughly half as many iterations and is twice as fast as a result. In figure 6 we notice that the field lines at the top of our positive plate and at the bottom of our negative plate appear to be behaving a bit incorrectly as a result of a loss of accuracy of our computational method.

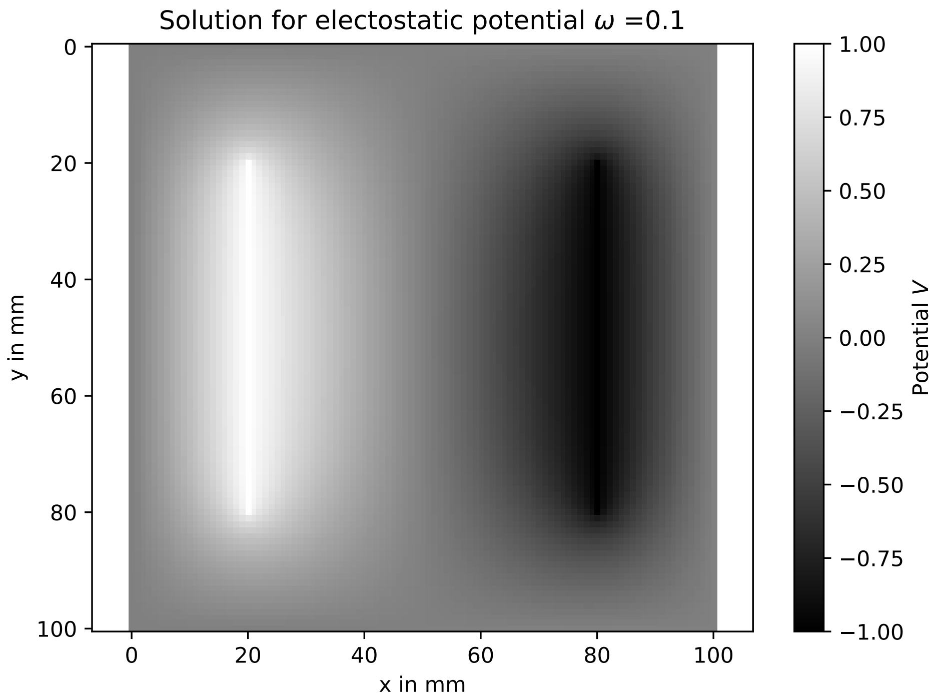

Figure 3: The solution to the electric potential using the Gauss-Seidel method, x and y are given in cm, with a relaxation parameter w=0.1

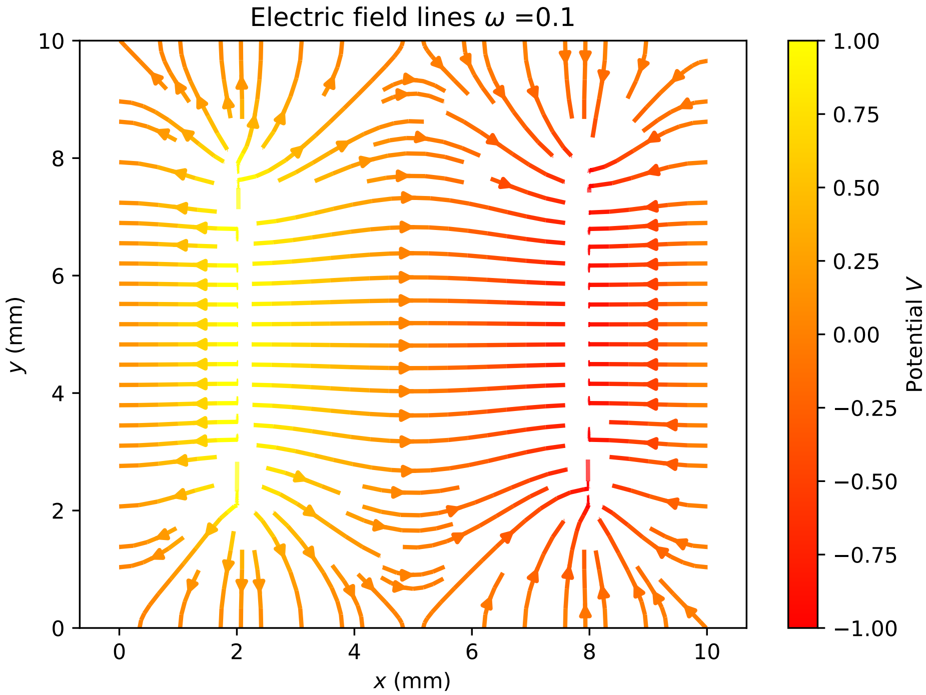

Figure 4: The stream plot of the electric field lines, x and y are given in cm, with a relaxation parameter w=0.1

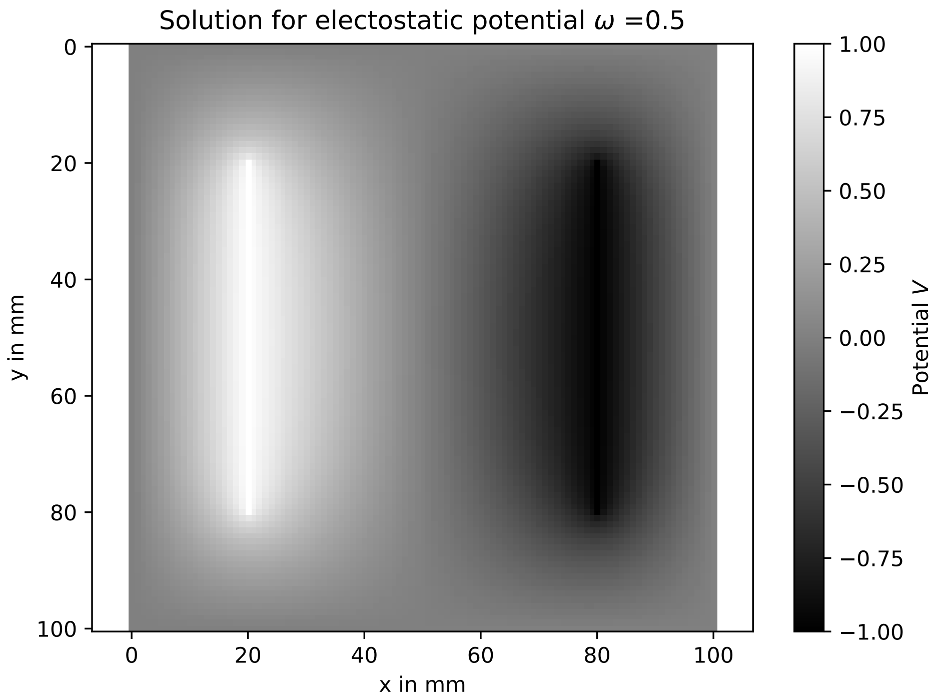

Figure 5: The solution to the electric potential using the Gauss-Seidel method, x and y are given in cm, with a relaxation parameter w=0.5

w	Number of while loop iterations	Run-time for calculation
0	2160	67.16 seconds
0.1	1946	62.05 seconds
0.4	1321	41.23 seconds
0.5	1118	33.99 seconds

Table 1: The timing of the while loop iterations in the Gauss-Seidel method

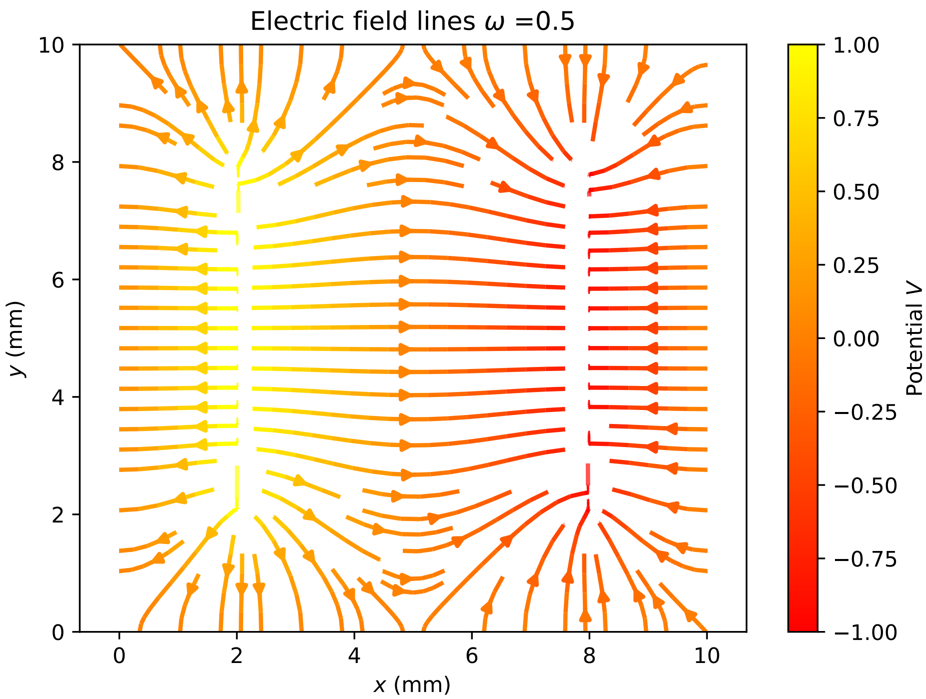

Figure 6: The stream plot of the electric field lines, x and y are given in cm, with a relaxation parameter w=0.5

Question 2

a)

We are asked to rearrange the following equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\mathrm{d}\eta}{\mathrm{d}x} \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2}$$

into Flux Conservative form

$$\frac{\partial u}{\partial t} = -\frac{\partial \vec{F}(\vec{u})}{\partial x} \tag{3}$$

where $\vec{u} = (u, \eta)$.

We are asked solve for $\vec{F}$. We rearrange Equation 1 and 2 to try to fit this form. From equation 1:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - g\frac{\partial u}{\partial x} \tag{4}$$

$$= -\frac{1}{2}\frac{\partial u^2}{\partial x} - g\frac{\partial u}{\partial x} \tag{5}$$

$$= -\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + gh\right) \tag{6}$$

From equation 2:

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(u(h)) \tag{7}$$

$$= -\frac{\partial}{\partial x}((\eta - \eta_b)u) \tag{8}$$

Thus, comparing these rearranged forms to Equation 3, we get:

$$\vec{F}(u, \eta) = \left[\frac{1}{2}u^2 + g\eta, (\eta - \eta_b)u\right] \tag{9}$$

Now we shall use the FTCS scheme in Equation 5 of the lab handout to discretize the 1D shallow water equations. For some spatial gap Δx, time gap Δt and a time step n, spatial step j, we get:

$$\vec{u}_j^{n+1} = \vec{u}_j^n - \frac{\Delta t}{2\Delta x}(\vec{F}_{j+1}^n - \vec{F}_{j-1}^n) \tag{10}$$

for $\vec{F}$ as defined in Equation 9, $\vec{u} = (u, \eta)$.

More explicitly we have:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}(0.5u_{j+1}^2 + g\eta_{j+1} - 0.5u_{j-1}^2 - g\eta_{j-1}) \tag{11}$$

b)

We are asked to 1D shallow wave system using the FTCS scheme. We want to solve 3 using the FTCS scheme we derived in in Equation 10.

However, we must deal with the boundaries of the 1-D grid. For the left of the grid (i.e. j = 0):

$$\frac{\partial F}{\partial x}\bigg|_0^n = \frac{1}{\Delta x}(F_1^n - F_0^n) \tag{12}$$

Using pretty much the same derivation as the handout, we can derive

$$\vec{u}_j^1 = \vec{u}_j^0 - \frac{\Delta t}{\Delta x}(\vec{F}_1^n - \vec{F}_0^n) \tag{13}$$

Likewise with the right of the grid, j = J:

$$\frac{\partial F}{\partial x}\bigg|_J^n = \frac{1}{\Delta x}(F_J^n - F_{J-1}^n) \tag{14}$$

$$\vec{u}_J^1 = \vec{u}_j^0 - \frac{\Delta t}{2\Delta x}(\vec{F}_1^n - \vec{F}_0^n) \tag{15}$$

We are instructed to use the initial conditions $u(x,0) = 0$ and $n(x,0) = H + Ae^{-(x-\mu)^2/\sigma^2} - \langle Ae^{-(x-\mu)^2/\sigma^2}\rangle$, i.e. a gaussian water wave.

Our values for the constants are $\eta_b = 0$, $H = 0.01$, $A = 0.002$, $\mu = 0.01$, $\sigma = 0.05$.

Our boundary conditions are $u(0,t) = u(L,t) = 0$.

We plot η in the spatial domain for $t = 0, 1, 4$ s.

Our plots can be seen in Fig 7, 8, 9.

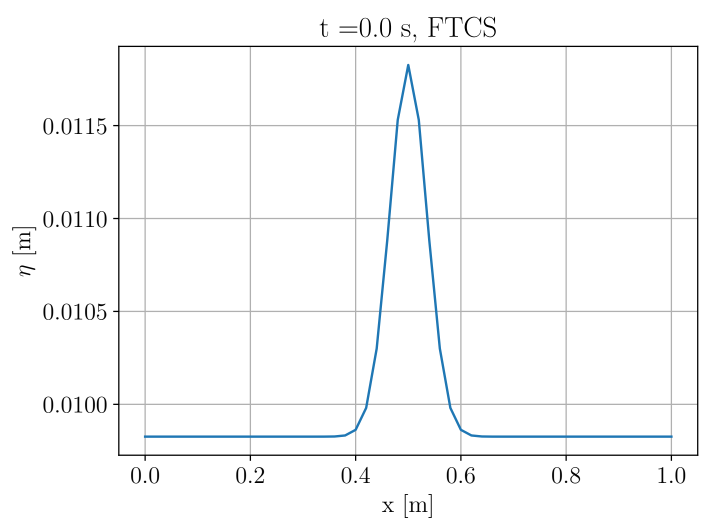

Figure 7: FTCS scheme Plot, for t = 0s.

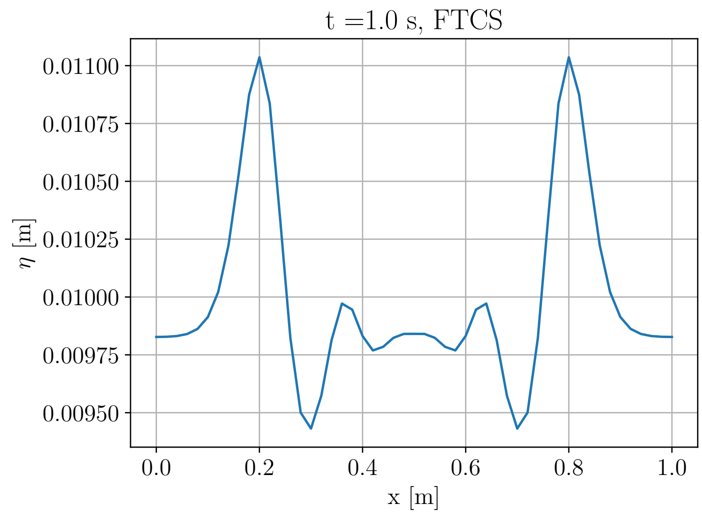

Figure 8: FTCS scheme Plot, for t = 1.0s.

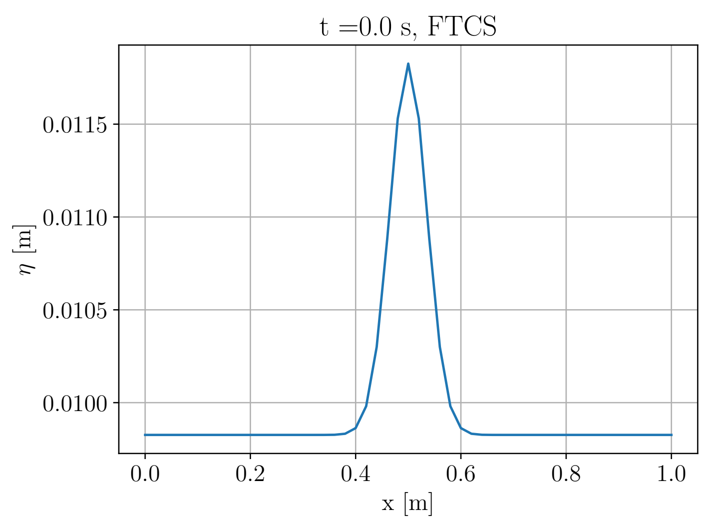

Figure 9: FTCS scheme Plot, for t = 4.0s.

c)

We are asked to do a von Neumann stability analysis for the 1D shallow water equations about $(u, \eta) = (0, H)$. We substitute this into Equations 1 and 2.

Doing this to the first equation:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{16}$$

We need to do some rearranging for equation 2 before substituting $(u, \eta) = (0, H)$ and $\eta_b = 0$:

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(u(\eta - \eta_b)) \tag{17}$$

$$= -(\eta - \eta_b)\frac{\partial u}{\partial x} - u\frac{\partial(\eta - \eta_b)}{\partial x} \tag{18}$$

$$= -(H - 0)\frac{\partial u}{\partial x} - (0)\frac{\partial(\eta - \eta_b)}{\partial x} \tag{19}$$

$$= -H\frac{\partial u}{\partial x} \tag{20}$$

For we implement the scheme we used to calculate the time partial derivative, the forward difference , and and the central difference for the partial with respect to x. From equation 16:

$$\frac{1}{\Delta t}[u(x, t + \Delta t) - u(x, t)] = -g\frac{1}{2\Delta x}[\eta(x + \Delta x, t) - \eta(x - \Delta x, t)] \tag{21}$$

From equation 20:

$$\frac{1}{\Delta t}[\eta(x, t + \Delta t) - \eta(x, t - \Delta t)] = -H\frac{1}{2\Delta x}[u(x + \Delta x, t) - u(x - \Delta x, t)] \tag{22}$$

Rearranging:

$$u(x, t + \Delta t) = u(x, t) - g\frac{\Delta t}{2\Delta x}[\eta(x + \Delta x, t) - \eta(x - \Delta x, t)] \tag{23}$$

$$\eta(x, t + \Delta t) = \eta(x, t) - H\frac{\Delta t}{2\Delta x}[u(x + \Delta x, t) - u(x - \Delta x, t)] \tag{24}$$

Now we substitute in the fourier series for these terms into the above rearranged froms; $u(x, t) = c_u(t)e^{ikx}$ and $\eta(x, t) = c_\eta(t)e^{ikx}$. We get:

$$c_u(t + \Delta t)e^{ikx} = c_u(t)e^{ikx} - g\frac{\Delta t}{2\Delta x}[c_\eta(t)e^{ik(x+\Delta x)} - c_\eta(t)e^{ik(x-\Delta x)}] \tag{25}$$

$$c_\eta(t + \Delta t)e^{ikx} = c_\eta(t)e^{ikx} - H\frac{\Delta t}{2\Delta x}[c_u(t)e^{ik(x+\Delta x)} - c_u(t)e^{ik(x-\Delta x)}] \tag{26}$$

We factor out the e^{ikx} term:

$$c_u(t + \Delta t) = c_u(t) - g\frac{\Delta t}{2\Delta x}[c_\eta(t)e^{-ik\Delta x} - c_\eta(t)e^{-ik\Delta x}] \tag{27}$$

$$c_\eta(t+\Delta t) = c_\eta(t) - H\frac{\Delta t}{2\Delta x}[c_u(t)e^{ik\Delta x} - c_u(t)e^{-ik\Delta x}] \tag{28}$$

Notice $e^{ik\Delta x} - e^{-ik\Delta} = 2i\sin(k\Delta x)$,

$$c_u(t+\Delta t) = c_u(t) - g\frac{\Delta t}{2\Delta x}2i\sin(k\Delta x)c_\eta(t) \tag{29}$$

$$c_\eta(t+\Delta t) = c_\eta(t) - H\frac{\Delta t}{2\Delta x}2i\sin(k\Delta x)c_u(t) \tag{30}$$

We can express these two sets of equations in vector form with $\mathbf{c}(t) = (c_u(t), c_\eta(t))$:

$$\mathbf{c}(t+\Delta t) = \mathbf{A}\mathbf{c}(t) \tag{31}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & -g\frac{\Delta t}{\Delta x}i\sin(k\Delta x) \\ -H\frac{\Delta t}{\Delta x}i\sin(k\Delta x) & 1 \end{pmatrix} \tag{32}$$

Calculating the eigenvalues (by taking $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$), we get:

$$(1-\lambda)^2 = -Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x) \tag{33}$$

$$1-\lambda = \pm i\sqrt{Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x)} \tag{34}$$

$$\lambda = 1 \mp i\sqrt{Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x)} \tag{35}$$

Taking the magnitude, we get:

$$\lambda^2 = 1^2 + \left(\sqrt{Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x)}\right)^2 \tag{36}$$

$$|\lambda| = \sqrt{1 + Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x)} \tag{37}$$

We can see that the magnitude of λ is always greater than or equal to unity as $Hg(\frac{\Delta t}{\Delta x})^2\sin^2(k\Delta x) \geq 0$. This means that we expect out scheme to be unstable over any long term time intervals. This is evident in our plots in Figures 8 and 9 as well. We can see that that the wave form at $t = 4$s is far more chaotic and unstable than that of $t = 1s$. Running the animation in our script would show the long term unstable behaviour of the scheme; the water waves η become increasingly higher unlike the physical system it represents.

Question 3

a)

We are asked to implement the Two-Step Lax-Wendroff scheme by calculating the half steps for η and u as given in equation 12 in the lab manual and using these half steps to obtain our η_{new} and u_{new} as given by equation 13. We notice that our plots and animations appear smoother and the increased stability decreases the dissipation of smaller waves from the main ones. Figures 10, 11, 12.

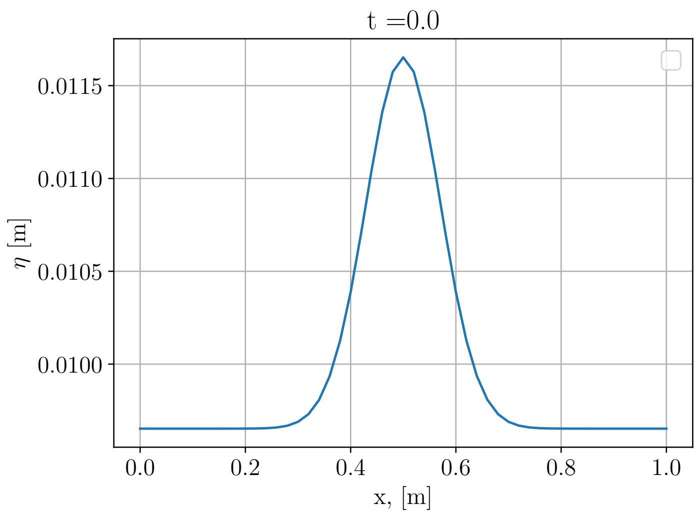

Figure 10: Lax-Wendroff scheme of wave simulation at t = 0s

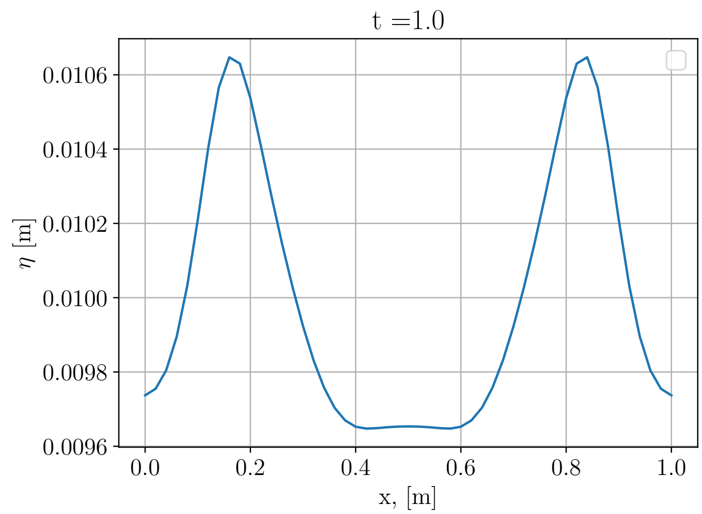

Figure 11: Lax-Wendroff scheme of wave simulation at t = 1s

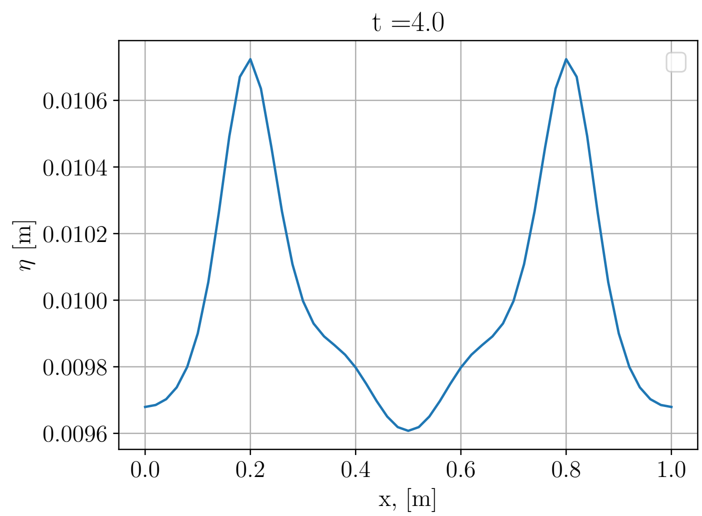

Figure 12: Lax-Wendroff scheme of wave simulation at t = 4s

b)

We are asked to modify our code to add a more complex bottom topography and observe as we initiate a tsunami at the left of our domain. The wave form travels to the right as a uniform peak until it hits the shallower water. As the depth of the water decreases our wave form gets taller and narrower and it's speed decreases.

We are asked to consider a bottom topography

$$\eta_b(x) = \frac{\eta_{bs}}{2}[1 + \tanh[(x - x_0)\alpha]$$

where $n_{bs} = H - 4 \times 10^{-4}$m, $\alpha = (1/8pi)\text{m}^{-1}$, $x_0 = 0.5$, and $H = 0.01$m.

For the space grid, we consider $J = 150$ spatial points in [0,1] and thus we use a $\Delta t = 0.01$m. We are also asked to set $\Delta t = 0.001$s, for at least 5 seconds (we set it to 5.001 seconds).

Our boundary conditions are the same as a).

Our initial conditions are the same as a), but we set the parameters $A = 2 \times 10^{-4}$m and $\sigma = 0.1$.

Figures 13, 14, 15, 16 has our plot of the free surface η at $t = 0.0, 1.0, 2.0, 4.0$ s, respectively. For a more physical context of η with respect to the coast η_b, we can refer to Fig 17 to 20, where we can see the trend of the wave becoming narrower and taller while decreasing in speed as it hits the coats.

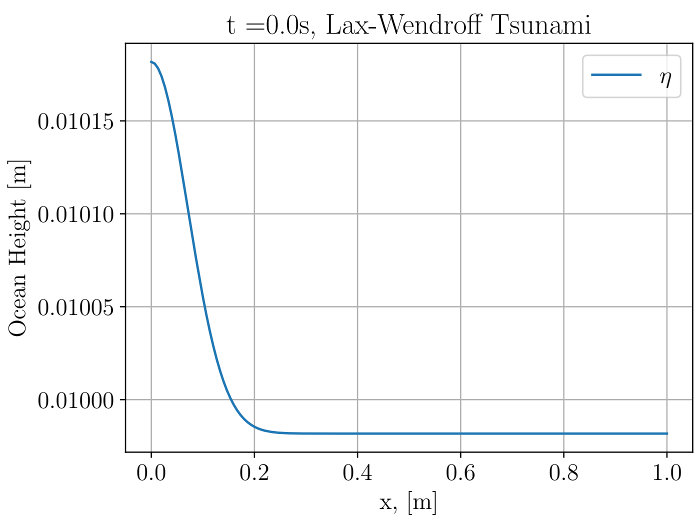

Figure 13: Lax-Wendroff scheme of tsunami simulation at t = 0s

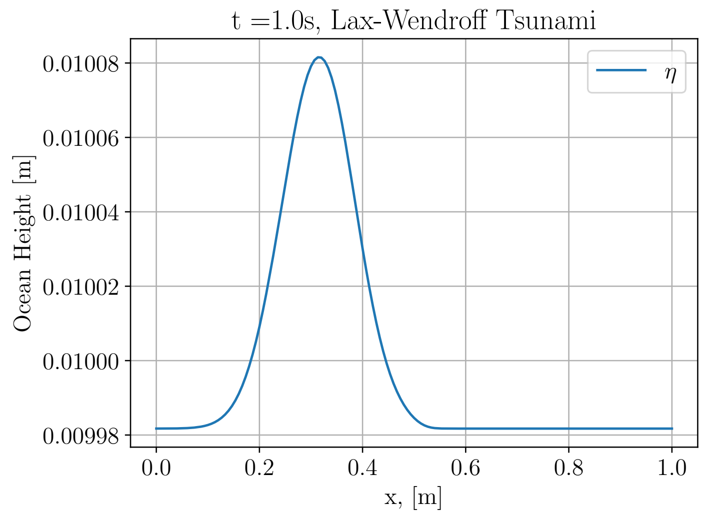

Figure 14: Lax-Wendroff scheme of tsunami simulation at t = 1s.

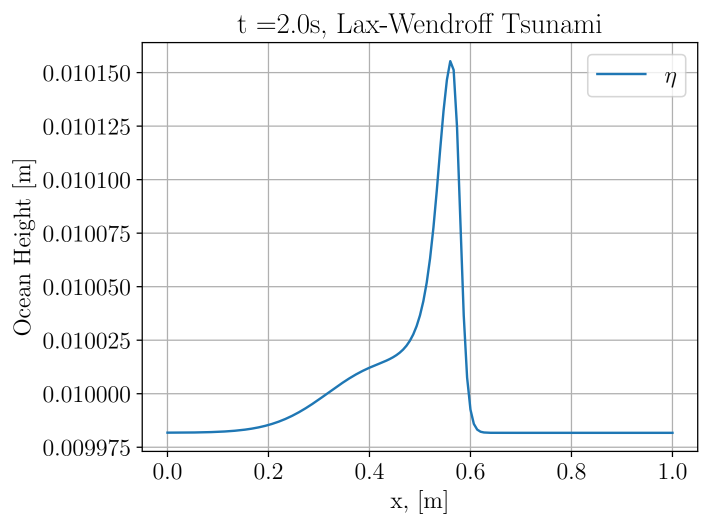

Figure 15: Lax-Wendroff scheme of tsunami simulation at t = 2s.

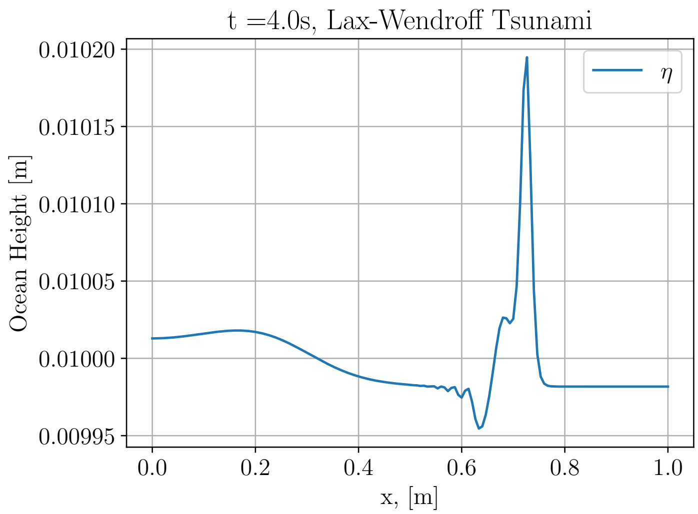

Figure 16: Lax-Wendroff scheme of tsunami simulation at t = 4s.

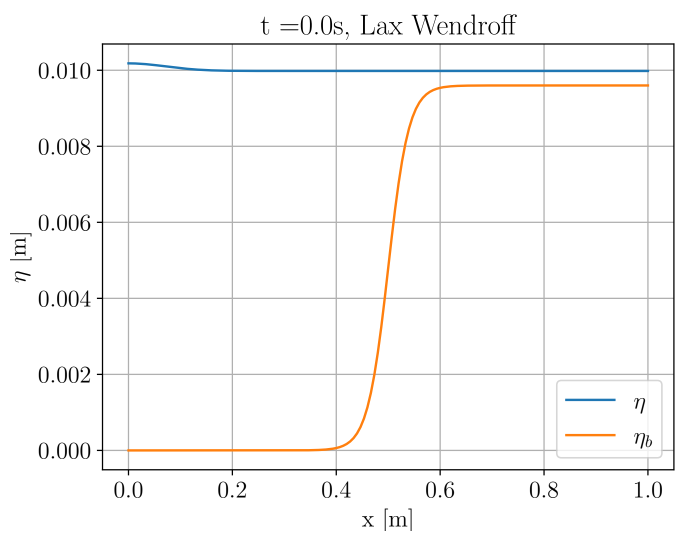

Figure 17: Lax-Wendroff scheme of tsunami simulation at t = 0s, with the coast η_b for context.

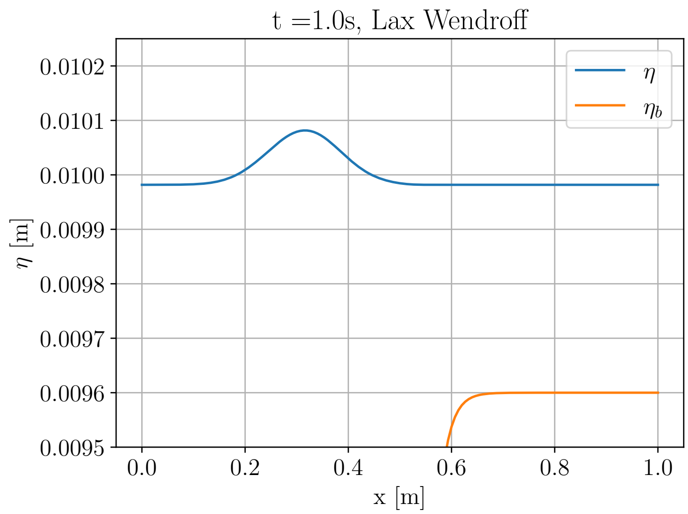

Figure 18: Lax-Wendroff scheme of tsunami simulation at t = 1s, with the coast η_b somewhat visible.

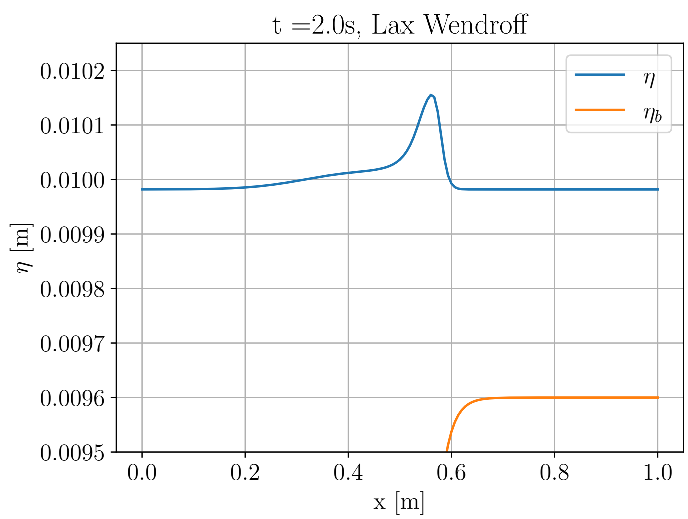

Figure 19: Lax-Wendroff scheme of tsunami simulation at t = 2s, with the coast η_b somewhat visible.

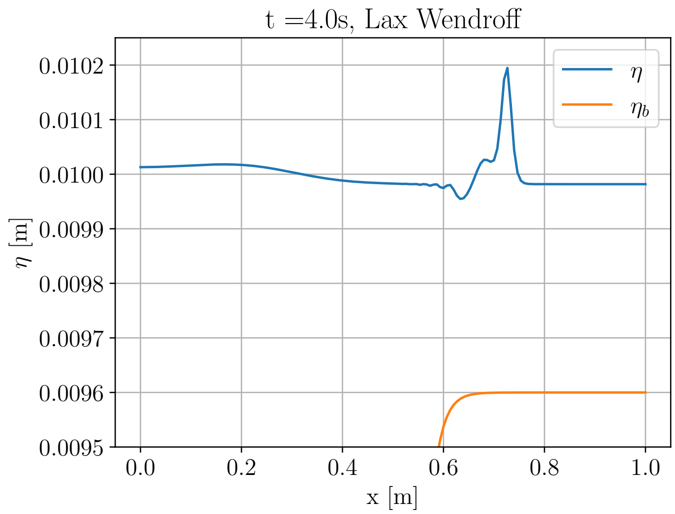

Figure 20: Lax-Wendroff scheme of tsunami simulation at t = 4s, with the coast η_b somewhat visible.