MA4546: Introduction to Stochastic Process

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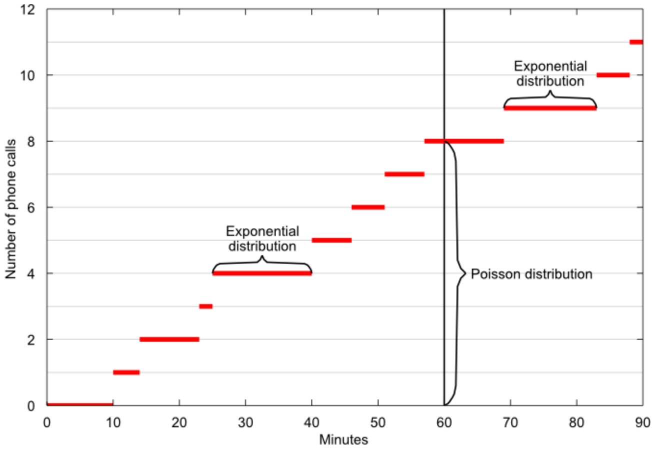
Chapter 3 Poisson Process



Everything about PP is shown in this figure.

Outline of this chapter *

- Exponential random variable(review)
- Poisson random variable(review)
- Poisson process
- Compound Poisson process

*Section 3.4 and 3.5 in textbook are not covered in class. All other materials in textbook for this chapter are required

Exponential random variable: $Exp(\lambda)$

A r.v. $X \geq 0$ with parameter $\lambda > 0$ with the following pdf

$$p(x) = ?, x \geq 0.$$

The cdf is

$$F(x) = \Pr(X \leq x) = ?, x \geq 0.$$

The expectation and variance is

$$\mathrm{E}(X) = ?, \quad \mathrm{var}(X) = ?$$

Exponential random variable: $Exp(\lambda)$

Definition: A r.v. $X \geq 0$ with parameter $\lambda > 0$ with the following pdf

$$p(x) = \lambda e^{-\lambda x}, x \geq 0.$$

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$$F(x) = \Pr(X \leq x) = 1 - e^{-\lambda x}, x \geq 0.$$

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$$\mathrm{E}(X) = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \frac{1}{\lambda^2}$$

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Example 3.1 (page 60): the life time of machine is $Exp(\lambda)$

Characteristic function and moment-generating function

For a scalar random variable X, the **characteristic function** is defined as the expected value of $\mathrm{E}[e^{\mathbf{i}tX}]$, where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit, and $t \in \mathbb{R}$ is the argument of the characteristic function. If $X \sim Exp(\lambda)$, then

$$\varphi(t) := \mathrm{E}[e^{\mathbf{i}tX}] = (1 - \mathbf{i}t\lambda^{-1})^{-1}, \quad t \in \mathbb{R}.$$

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The **moment-generating function** of $X \sim Exp(\lambda)$

$$M(\theta) := \mathrm{E}[e^{\theta X}] = (1 - \theta\lambda^{-1})^{-1}, \quad -\infty < \theta < \lambda.$$

Memoryless Property (Thm 3.1)

Theorem

Let X be a continuous random variable taking values in $[0,+\infty)$. It has the memoryless property, i.e.,

$$\Pr(X > t+s | X > s) = \Pr(X > t), \quad \forall s,t > 0,$$

if and only if it is an $Exp(\lambda)$ for some $\lambda > 0$.

Proof: page 61 (required).

Poisson random variable$\sim Poi(\lambda)$

A r.v. $X \in S = 0, 1, 2, 3, \cdots$ with parameter $\lambda > 0$ with the following pdf *

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*it is called "pmf" (probability mass function) in textbook for discrete random variable

Poisson random variable~ $Poi(\lambda)$

Definition: A r.v. $X \in S = \{0, 1, 2, 3, \cdots\}$ with parameter $\lambda > 0$ with the pdf

$$p_k = \Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

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$$\mathrm{E}(X) = \lambda, \quad \mathrm{var}(X) = \lambda$$

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Example 3.4 (page 69): Poisson distribution as the small probability limit of binomial distribution.

Moment-generating function

The moment-generating function of $X \sim Poi(\lambda)$ is

$$\boxed{M(\theta) = \mathrm{E}[e^{\theta X}] = \exp\Big(\lambda(e^{\theta} - 1)\Big)}, \quad \theta \in \mathbb{R}.$$

Proof.

$$\begin{aligned} M(\theta) &= \sum_{k=0}^{\infty} e^{\theta k} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{\theta}\lambda)^k}{k!} \\ &= e^{-\lambda} \exp(e^{\theta}\lambda) \\ &= \exp(e^{\theta}\lambda - \lambda). \end{aligned}$$

□

The characteristic function is

$$\varphi(t) := \mathrm{E}[e^{\mathbf{i}tX}] = \exp\Big(\lambda(e^{\mathbf{i}t} - 1)\Big)$$

Exercise

(optional) Suppose that $(Y_j)_{1 \le j \le J}$ *are independent Poisson r.v.s with* $Y_j \sim Poi(\lambda_j)$ *and* $\left\{a_j\right\}$ *are* J *real numbers. Let* $X := \sum_{j=1}^{J} a_j Y_j$ *be the weighted average of* y_j*. Show that for any* $\theta \in \mathbb{R}$*, the Laplace transform of* X *(i.e., the moment-generating function) is given by*

$$\log \mathrm{E}\left(e^{\theta X}\right) = \sum_{j=1}^{J} \left(e^{\theta a_j} - 1\right) \lambda_j$$

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$$\log \mathrm{E}\left(e^{\theta X}\right) = \sum_{j=1}^{J}\left(e^{\theta a_j} - 1\right)\lambda_j$$

Proof:

$$\mathrm{E}\left(e^{\theta X}\right) = \mathrm{E}\left(e^{\theta \sum_{j=1}^J y_j a_j}\right) = \mathrm{E}\left(\Pi_{j=1}^J e^{\theta y_j a_j}\right) = \Pi_{j=1}^J \mathrm{E}\left(e^{\theta y_j a_j}\right)$$

$$\log\left(\mathrm{E}\left(e^{\theta X}\right)\right) = \sum_{j=1}^{J}\log\left(\mathrm{E}\left(e^{\theta y_j a_j}\right)\right) = \sum_{j=1}^{J}\log\left(\sum_{k=0}^{\infty} e^{-\lambda_j}\frac{\lambda_j^k}{k!}\cdot e^{\theta k a_j}\right)$$

$$= \sum_{j=1}^{J}\log\left(e^{-\lambda_j}\exp\left(\lambda_j e^{\theta a_j}\right)\right) = \sum_{j=1}^{J}\left(-\lambda_j + \lambda_j e^{\theta a_j}\right)$$

Poisson Process

Background of Poisson process

- The transitions from state to state are triggered by a stream of events that occur at the sequence of times

$$0 = S_0 < S_1 < S_2 < \cdots < S_k < \cdots$$

examples: arrivals to a queuing system; failures in a manufacturing system; biological stimuli in a neural system,

* $N(t-) = \lim_{h \downarrow 0} N(t-h)$ is the left-limit. $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

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- The successive inter-event time

$$T_k := S_k - S_{k-1}$$

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  - Assume the inter-event time $\{T_k\}$ is iid $Exp(\lambda)$.
  - The state space is $\mathbb{Z}$. The change of state is just to add "+1" to the old value: that is

$$N(S_k) := N(S_k-)^* + 1.$$

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  - Assume the inter-event time $\{T_k\}$ is iid $Exp(\lambda)$.
  - The state space is $\mathbb{Z}$. The change of state is just to add "$+1$" to the old value: that is

$$N(S_k) := N(S_k-)^* + 1.$$

  - $N(t)$ is thus the number of events arrived up to time t, i.e., within $(0, t]$. by assuming $N_0 = 0$.

---

* $N(t-) = \lim_{h \downarrow 0} N(t-h)$ is the left-limit. $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

Equivalently, the Poisson process$\{N(t) : t \in \mathbb{R}_+)\}$ is defined as

$$N(t) = \begin{cases} 0 & \text{if } 0 \le t < S_1 \\ 1 & \text{if } S_1 \le t < S_2 \\ 2 & \text{if } S_2 \le t < S_3 \\ \vdots & \\ k & \text{if } S_k \le t < S_{k+1} \\ \vdots & \end{cases}$$

So, we have that

$$N(t) = \sum_{k=0}^{\infty} k \cdot 1_{[S_k, S_{k+1})}(t) = \sum_{k=1}^{\infty} 1_{[S_k, +\infty)}(t)$$

and $\{N(t) \ge k\} \iff \{S_k \le t\}$.

A model of pure birth process

Think of Poisson process as a special *counting process* : count "the number of customers to a queuing system up to time t", or a *pure birth process* (no death): the species gives a descendant with a rate λ.

T_n is the time span between arrivals of two successive customers.

*the process satisfying this condition is in general called *renewal process*.

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- Why T_n is $Exp(\lambda)$? A "microscopic" model (short time asymptotics):

Let h be a small time step. *Assume* during $(t, t+h]$, there is a probability "λh" that one new customer will arrive, and "$1-\lambda h$" probability that no new customer will arrive and this arrival event is **independent** of the past information up to time t. Then during time $(0, t)$ the probability of no arrival of new customer is

$$\Pr(T_1 > t) \approx (1-\lambda h)^{[t/h]} \to e^{-\lambda t}, \quad \text{as} \quad h \to 0$$

where $[t/h]$ is the integer part of t/h. This implies that T_1 is $Exp(\lambda)$.

This model is called a **pure birth process.** We have shown here the pure birth process leads to an exponential distributed inter-arrival time T_i.

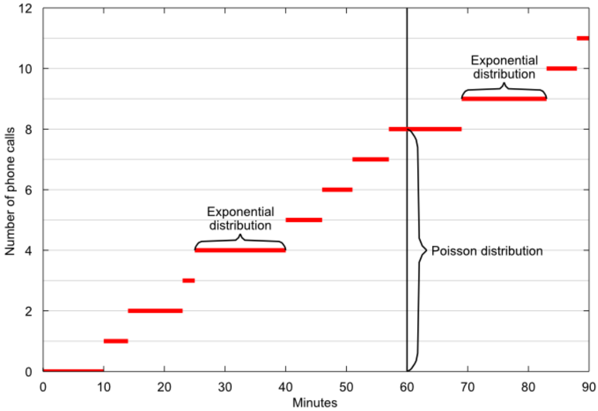
*the process satisfying this condition is in general called *renewal process*.

Poisson process: $N(t)$

λ is called **rate** or **intensity** = the prob. of exactly one arrival per unit time.

$$\boxed{\lambda = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr(\text{ one arrival in } (t, t+\delta t] \text{ })}$$

Denote the corresponding Poisson process $N(t)$ as $PP(\lambda)$.



Why has the name "Poisson" process?

Theorem (Thm 3.6, 3.7 (p71))

Let $\{N(t)\}$ *be a* $PP(\lambda)$.

- *It is a Markov process: that is*
$$\Pr(N(t+s)=k|N(s)=j, N(u), 0\le u\le s) = \Pr(N(t+s)=k|N(s)=j)$$
- *For a given* t, $N(t)\sim Poi(\lambda t)$. *That is*
$$\Pr(N(t)=k) = e^{-\lambda t}\frac{\lambda^k t^k}{k!}$$

sketch of proof in textbook

Proof.

First, note that $\{N(t) \geq k\} = \{S_k \leq t\}$ because

- If $N(t) \geq k$, then there exists $n \geq k$ such that $S_n \leq t$. Noting by definition that $S_n \geq S_k$ for all $n \geq k$. So, we have $S_k \leq t$.
- If $S_k \leq t$, then $k \in \{n \geq 0 : S_n \leq t\}$ and $k \leq N(t)$ by definition.

Second, $\Pr(N(t) \geq k) = \Pr(S_k \leq t) = \Pr(\sum_{i=0}^{k} T_i \leq t)$. The sum of k iid r.v.s $\{T_i\} \sim Exp(\lambda)$ follows the Erlang distribution with CDF $F(x) = 1 - \sum_{r=0}^{k-1} e^{-\lambda x}(\lambda x)^r / r!$ (Eqn. (3.9), p62, textbook).
Then the conclusion follows after some calculations for $\Pr(N(t) = k) = \Pr(N(t) \geq k) - \Pr(N(t) \geq k+1)$ $\square$

An alternative (and easy) proof *

this proof can be generalized to inhomogeneous Poisson process where λ is a function of time

We start from the pure birth process and prove the theorem (only part (ii)) by computing the moment-generating function of $N(t)$:

$$\mathscr{G}(t,\theta) := \mathrm{E}[e^{\theta N(t)}]$$

Note that $Poi(t\lambda)$'s momentum generating function is $M_t(\theta) = \exp\left(t\lambda(e^{\theta} - 1)\right)$. So we just need to show $\mathscr{G}(t,\theta) = M_t(\theta)$, equivalently $\log \mathscr{G} = t\lambda(e^{\theta} - 1)$.

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$$N(s,t) := N(t) - N(s)$$

Then $\dot{\mathscr{G}}(t,\theta) = \lim_{h\to 0} \frac{\mathscr{G}(t+h,\theta) - \mathscr{G}(t,\theta)}{h} = \lim_{h\to 0} h^{-1}\,\mathrm{E}\left[e^{\theta(N(0,t)+N(t,t+h))} - e^{\theta N(0,t)}\right]$

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Note that by the independence between $(0,t]$ and $(t,t+h]$, we have

$$\begin{aligned}
&\mathrm{E}\left[e^{\theta(N(0,t)+N(t,t+h))} - e^{\theta N(0,t)}\right] = \mathrm{E}\left[e^{\theta N(0,t)}\left(e^{\theta N(t,t+h)} - 1\right)\right] \\
=&\,\mathrm{E}\left[e^{\theta N(0,t)}\right]\mathrm{E}\left[e^{\theta N(t,t+h)} - 1\right] = \mathscr{G}(t,\theta)\left[(e^{\theta}-1)h\lambda + (1-1)\times(1-\lambda h) + O(h^2)\right]
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\end{aligned}$$

So, we have ODE $\dot{\mathscr{G}}(t,\theta) = \mathscr{G}(t,\theta)(e^{\theta}-1)\lambda$, i.e., $\frac{d}{dt}\log\mathscr{G} = \lambda(e^{\theta}-1)$. Since $\mathscr{G}(0,\theta)=1$, the conclusion is obtained.

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Properties of Poisson process

Let $\{N(t)\}$ be a $PP(\lambda)$.

1. It is a Markov process.

*This property is abbreviated as "LLRC" and also popularly called Càdlàg in French language, meaning "continue à droite, limite à gauche".

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4. $\{N(t) \geq k\} = \{S_k \leq t\}, t \in \mathbb{R}_+, k = 1, 2, \ldots$).

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5. Reconstruction of the jump time : $S_n = \inf\{t \in \mathbb{R}_+ : N_t = n\}, n \geq 1.$

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6. $N(t+s) - N(t)$ means the number of arrival events during time period $(t, t+s]$.
7. **Independence of increments**: for all $0 \leq t_0 < t_1 < \cdots < t_n$ and $n \geq 1$, the increments $N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}}$ over the disjoint time intervals $(t_0, t_1], (t_1, t_2], \ldots, (t_{n-2}, t_{n-1}], (t_{n-1}, t_n]$ are mutually independent random variables. In particular, $N_t = N_t - N_0$ is independent of $N_{t+s} - N_t$ for $\forall s, t > 0$.

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Let $\{N(t)\}$ be a $PP(\lambda)$.

1. It is a Markov process.
2. With probability 1, the sample path $N(t)$ is right-continuous with left limits *
3. $N(t) = \max\{n \geq 0 : S_n \leq t\}, \quad t \geq 0.$
4. $\{N(t) \geq k\} = \{S_k \leq t\}, t \in \mathbb{R}_+, k = 1, 2, \ldots .$).
5. Reconstruction of the jump time : $S_n = \inf\{t \in \mathbb{R}_+ : N_t = n\}, n \geq 1.$
6. $N(t+s) - N(t)$ means the number of arrival events during time period $(t, t+s]$.
7. **Independence of increments**: for all $0 \leq t_0 < t_1 < \cdots < t_n$ and $n \geq 1$, the increments $N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}}$ over the disjoint time intervals $(t_0, t_1], (t_1, t_2], \ldots, (t_{n-2}, t_{n-1}], (t_{n-1}, t_n]$ are mutually independent random variables. In particular, $N_t = N_t - N_0$ is independent of $N_{t+s} - N_t$ for $\forall s, t > 0$.
8. **Stationarity of increments**: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leq s \leq t$. This is

$$\Pr(N_{t+h} - N_{s+h} = k) = \Pr(N_t - N_s = k)$$

*This property is abbreviated as "LLRC" and also popularly called Càdlàg in French language, meaning "continue à droite, limite à gauche".

Review of the increment for DTMC random walk (assignment 1)

For the above independent and stationary increments, the Random Walk $\{X_n\} = \sum_{i=0}^{n} Z_i$ also has these two properties:

- $X_{n+k} - X_n$ is independent from $X_{n+k+m} - X_{n+k+l}$ for $m > l > 0$.
- $X_{n+k} - X_n$ and $X_{n+m+k} - X_{n+m}$ have the same distribution as $\sum_{i=1}^{k} Z_i$

Review of the increment for DTMC random walk (assignment 1)

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Recall the assignment of calculating the auto-covariance function

$$\begin{aligned}
\operatorname{cov}(X_{m+n}, X_m) &= \mathrm{E}\left[\left((X_m - m\mu) + \sum_{i=m+1}^{m+n}(Z_i - \mu)\right)(X_m - m\mu)\right] \\
&= \operatorname{var}(X_m) + \sum_{i=m+1}^{m+n} \mathrm{E}\left[(Z_i - \mu)(X_m - m\mu)\right] \\
&= \operatorname{var}(X_m) + \sum_{i=m+1}^{m+n} \mathrm{E}(Z_i - \mu)\cdot \mathrm{E}(X_m - m\mu) \\
&= \operatorname{var}(X_m) \\
&= 4mpq
\end{aligned}$$

Here we used the independent increment property ! We shall see this application again for Poisson process.

corollary

1. $\mathrm{E}(N(t)) = \lambda t$, $\mathrm{var}(N(t)) = \lambda t$, $\mathrm{E}(N(t))^2 = \lambda t + \lambda^2 t^2$.
2. stationary increment
$$N(t+s) - N(s) \sim N(t) - N(0) \sim Poi(\lambda t)$$
3. covariance function
$$\mathrm{cov}(N(t+s), N(s)) = \mathrm{E}\left[(N(t+s) - (t+s)\lambda) \cdot (N(s) - s\lambda)\right] = \lambda s$$
$$\mathrm{cov}(N(t), N(s)) = \lambda(s \wedge t)$$

Exercise

$N(t) \sim PP(\lambda)$. S_n is the nth jump time. Let $t_i = i$ for $i = 1,2,3,4$.

1. What is the expected number of arrival customers at time t_4?
2. What is the probably that there are no arrivals from t_1 to t_3?
3. What is the probably that there are two arrivals from t_1 to t_3, and for these two events, one arrives before t_2 and the other arrives after t_2 ?
4. What is the probably that there are two arrivals from t_1 to t_4 among which one arrives before t_3 and the other arrives after t_2 ?
5. $\Pr(N(t_3) = 5 | N(t_1) = 1) =?$
6. $\mathrm{E}(N(t_2)N(t_1)) =?$
7. $\mathrm{E}[N(t_2) | S_1 > t_1] =?$

Exercise

1. $\mathrm{E}(N(t_4)) = \lambda t_4 = 4\lambda$
2. $\Pr(N(t_3) - N(t_1) = 0) = \Pr(N(t_3 - t_1) = 0) = e^{-\lambda(t_3 - t_1)} = e^{-2\lambda}$.

Exercise

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3. $$\begin{aligned} &\Pr(N(t_2) - N(t_1) = 1, N(t_3) - N(t_2) = 1) \\ =&\Pr(N(t_2) - N(t_1) = 1)\Pr(N(t_3) - N(t_2) = 1) \\ =&e^{-\lambda(t_2 - t_1)}\lambda(t_2 - t_1)e^{-\lambda(t_3 - t_2)}\lambda(t_3 - t_2) = \lambda^2 e^{-2\lambda} \end{aligned}$$

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4. We need discuss all possible cases in the following table. The answer is the sum of the last columns : $3\lambda^2 e^{-2\lambda} + \lambda^2 e^{-\lambda}/2$

$(t_1, t_2]$	$(t_2, t_3]$	$(t_3, t_4]$	prob
1	1	0	$\lambda^2 e^{-2\lambda}$
0	2	0	$\lambda^2 e^{-\lambda}/2$
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Exercise

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6. $\mathrm{E}(N(t_2)N(t_1)) = \mathrm{E}[(N(t_2) - N(t_1))N(t_1)] + \mathrm{E}(N(t_1)^2) = \mathrm{E}(N(t_2) - N(t_1))\,\mathrm{E}(N(t_1)) + \mathrm{E}(N(t_1)^2) = 2\lambda^2 + \lambda$

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7. $\mathrm{E}[N(t_2) | S_1 > t_1] = \mathrm{E}[N(t_2) | N(t_1) < 1] = \mathrm{E}[N(t_2) | N(t_1) = 0] = \mathrm{E}[N(t_2 - t_1)] = \lambda$

Conditional distribution of jump times *

Theorem

Fix T and condition the PP $N(t)$ to have K jumps in this interval $[0, T]$. Then, regardless of the Poisson rate λ, the jump times $\{S_k : k = 1, \cdots, K\}$ are distributed as K i.i.d. uniform $[0, T]$ r.v.s. (actually as their order statistics since you need to rearrange these uniform r.v.s in increasing order.) That is

$$(S_1, S_2, \cdots, S_K) \sim (X_{(1)}, \cdots, (X_{(K)})$$

where $X_{(1)} \leq \cdots \leq X_{(K)}$ is the **order statistics** *of $\{X_k : k = 1, \cdots, K\}$ [a] and X_k are i.i.d. uniform r.v.s in $[0, T]$.*

[a] $X_{(i)}$ is the i-th smallest value among $(X_k : k = 1, \cdots, K)$.

*Theorem 2.3.1 p.67 in *Stochastic Processes* by Ross, S.M. Wiley, 2nd Edition,1996.

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Note that the joint pdf of the uniform order statistics is

$$f(s_1, s_2, \cdots, s_K) \equiv \frac{K!}{T^K}.$$

For $K = 1$, the proof is an exercise.

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Proof.

We shall compute the conditional density function of $S_1, S_2, \cdots, S_K$ given that $N(T) = K$. Let $0 < t_1 < t_2 \cdots < t_K \leq T$ and let h_i be small enough so that $(t_i - h_i/2, t_i + h_i/2]$ has no overlap for any $i = 1, \cdots, K$. Now

$$\Pr(S_i \in (t_i - h_i/2, t_i + h_i/2], i = 1, 2, \cdots, K \mid N(t) = K)$$

$$= \frac{1}{\Pr(N(t) = K)} \times \Pr(\text{exactly one event in } (t_i - h_i/2, t_i + h_i/2],, i = 1, 2, \cdots, K \text{ and no events elsewhere })$$

$$= \frac{\Pi_{i=1}^{K} (\lambda h_i e^{-\lambda h_i}) \times e^{-\lambda (T - \sum_i h_i)}}{e^{-\lambda T} (\lambda T)^K / K!}$$

$$= \frac{K!}{T^K} \Pi_{i=1}^{K} h_i$$

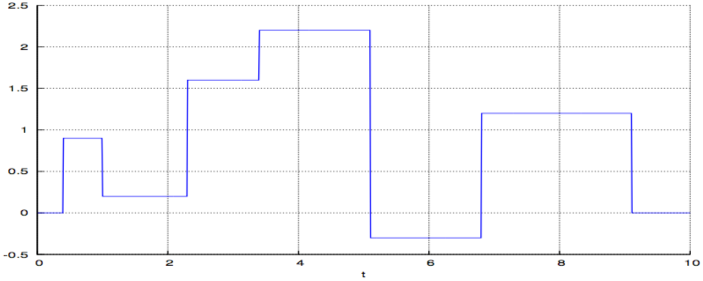
By letting $h_i \to 0$, we obtain that the conditional *density* is

$$f(t_1, \cdots, t_K) = \frac{K!}{T^K}, \quad 0 < t_1 < \cdots < t_K < T.$$

□

Compound Poisson process(cPP)

At each arrival time S_n, the jump size is also random.



A sample path of compound PP.

Compound Poisson process(cPP)

Definition

At each jump time, the jump size is from iid r.v. Z_n following the distribution ν. And $\{Z_n\}$ is also independent of the PP. Define

$$C(t) = C(0) + \sum_{n=1}^{N(t)} Z_n = C(0) + \sum_{k=1}^{\infty} Z_k 1_{\{S_k \leq t\}}$$

($\sum_{n=1}^{0} := 0$). We assume $C(0) = 0$. $Z_n \equiv 1$ gives the standard Poisson process.

$$C(t) = \begin{cases} 0 & \text{if } 0 \leq t < S_1 \\ Z_1 & \text{if } S_1 \leq t < S_2 \\ Z_1 + Z_2 & \text{if } S_2 \leq t < S_3 \\ \vdots & \\ \sum_{n=1}^{k} Z_n & \text{if } S_k \leq t < S_{k+1} \\ \vdots & \end{cases}$$

Characteristic function of compound Poisson process

$Z_1, Z_2, \cdots, Z_n$, are iid r.v. on $\mathbb{R}$ with distribution $\nu(z)dz$. Denote the characteristic function of Z_1 as

$$g(\alpha) := \mathrm{E}\, e^{\mathbf{i}\alpha Z_1} = \int_{\mathbb{R}} e^{\mathbf{i}z\alpha}\nu(z)dz, \quad \mathbf{i} := \sqrt{-1}.$$

Then we can compute the characteristic function of the increment $C_t - C_s$ for any $0 \leq s < t$ as follows.

Theorem

$$\mathrm{E}\left[\exp(i\alpha(C_t - C_s)\right] = \exp\left(\lambda(t-s)\int_{\mathbb{R}}(e^{iz\alpha} - 1)\nu(z)dz\right) = e^{\lambda(t-s)(g(\alpha)-1)}, \forall \alpha \in \mathbb{R}$$

Proof.

(step 1): definition; (step 2): law of total prob. and indpt. of N and Z; (step 3): iid of Z_k and indpt. increment of N.

$$\mathrm{E}\left[\exp(i\alpha(C_t - C_s)\right] = \mathrm{E}\left[\exp\left(i\alpha \sum_{k=N_s+1}^{N_t} Z_k\right)\right] \quad \because \text{ law. of total. prob.}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathrm{E}\left[\exp\left(i\alpha \sum_{k=m+1}^{m+n} Z_k\right)\right] \Pr(N_s = m, N_t - N_s = n)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathrm{E}\left[\exp\left(i\alpha \sum_{k=1}^{n} Z_k\right)\right] \Pr(N_s = m)\Pr(N_t - N_s = n)$$

$$= \sum_{n=0}^{\infty} \mathrm{E}\left[\exp\left(i\alpha \sum_{k=1}^{n} Z_k\right)\right] \Pr(N_t - N_s = n)\left(\sum_{m=0}^{\infty} \Pr(N_s = m)\right)$$

$$= \sum_{n=0}^{\infty} \left(\mathrm{E}\left[\exp\left(i\alpha Z_1\right)\right]\right)^n \times e^{-\lambda(t-s)} \frac{\lambda^n (t-s)^n}{n!} \times 1$$

$$= \exp\left(\lambda(t-s)\,\mathrm{E}\left[\exp\left(i\alpha Z_1\right)\right]\right) e^{-\lambda(t-s)}$$

$$= \exp\left(\lambda(t-s)\left(\mathrm{E}[\exp\left(i\alpha Z_1\right)] - 1\right)\right)$$

$$= \exp\left(\lambda(t-s)\int_{\mathbb{R}} (e^{iz\alpha} - 1)\nu(z)dz\right)$$

Examples

- If $Z(\omega) \equiv 1$, then C_t is just the standard Poisson process, $PP(\lambda)$. Now $\nu(z) = \delta(z-1)$ and $\mathrm{E}\, Z_1 = \mathrm{E}\, Z_1^2 = 1$, $g(\alpha) = e^{\mathbf{i}\alpha}$. We calculate the characteristic function by the above formula and obtain that

$$\mathrm{E}\left[\exp(i\alpha(C_t - C_s)\right] = \exp\Big(\lambda(t-s)(e^{i\alpha} - 1)\Big).$$

  Actually, if a continuous-time stochastic process has the same characteristic function as this, then it has the same distribution as the Poisson process $PP(\lambda)$.

- If (Z_i) is a sequence of Bernoulli trials with parameter p: $\Pr(Z_1 = 1) = p$ and $\Pr(Z_1 = 0) = q$. Show that the compound Poisson process is actually a Poisson process with rate $p\lambda$.

Examples

- If $Z(\omega) \equiv 1$, then C_t is just the standard Poisson process, $PP(\lambda)$. Now $\nu(z) = \delta(z-1)$ and $\mathrm{E}\, Z_1 = \mathrm{E}\, Z_1^2 = 1$, $g(\alpha) = e^{\mathbf{i}\alpha}$. We calculate the characteristic function by the above formula and obtain that

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  Actually, if a continuous-time stochastic process has the same characteristic function as this, then it has the same distribution as the Poisson process $PP(\lambda)$.

- If (Z_i) is a sequence of Bernoulli trials with parameter p: $\Pr(Z_1 = 1) = p$ and $\Pr(Z_1 = 0) = q$. Show that the compound Poisson process is actually a Poisson process with rate $p\lambda$.
  We calculate $g(\alpha) = e^{\mathbf{i}\alpha} p + (1-p)$. Then

$$\mathrm{E}\left[\exp(i\alpha(C_t - C_s)\right] = \exp\left(\lambda(t-s)(e^{\mathbf{i}\alpha}p - p)\right) = \exp\left(p\lambda(t-s)(e^{\mathbf{i}\alpha}-1)\right)$$

  This means that (C_t) is a $PP(p\lambda)$.

Applications

We can derive the mean and variance of $C(t)$ from the characteristic function. (Thm 3.11 in textbook uses alternative proof by considering the iid random sum of r.v.s) Set $s=0$ and $C_0=0$ and note that $g'(0)=\mathbf{i}\,\mathrm{E}\,Z_1$ and $g''(0)=-\mathrm{E}(Z_1)^2$

- $$\mathrm{E}[C_t] = -\mathbf{i}\frac{d}{d\alpha}\mathrm{E}[e^{\mathbf{i}\alpha C_t}]|_{\alpha=0} = -\mathbf{i}\lambda t g'(0) = \lambda t\int_{\mathbb{R}} z\nu(z)dz = \lambda t\cdot\mathrm{E}\,Z_1$$
  So, after subtracted by the compensator $\lambda(\mathrm{E}\,Z_1)t$, the new process $C_t-(\lambda\cdot\mathrm{E}\,Z_1)t$ is called "compensated" compound Poisson process, because is has zero mean (actually it is a martingale).
- $$\mathrm{E}[C_t^2] = -\frac{d^2}{d\alpha^2}\mathrm{E}[e^{\mathbf{i}\alpha C_t}]|_{\alpha=0} = -\lambda t(g''(0)+\lambda t(g'(0))^2) = \lambda t\big(\mathrm{E}\,Z_1^2+\lambda t(\mathrm{E}\,Z_1)^2\big)$$
- The variance is thus $\mathrm{var}[C_t]=\lambda t\cdot\mathrm{E}\,Z_1^2$.
- (C_t) has stationary increment because the characteristic function for the increment C_t-C_s only depends on the difference $t-s$.
- (C_t) has independent increment because (verify by yourself)
  $$\mathrm{E}\left[\Pi_{k=1}^n e^{i\alpha_k(C_{t_k}-C_{t_{k-1}})}\right] = \Pi_{k=1}^n\,\mathrm{E}\left[e^{i\alpha_k(C_{t_k}-C_{t_{k-1}})}\right]$$

- The covariance function

$$
\begin{aligned}
&\operatorname{cov}(C_{t+s}, C_s) \\
=&\,\mathrm{E}[(C_{t+s} - \mathrm{E}\, C_{t+s})(C_s - \mathrm{E}\, C_s)] \\
=&\,\mathrm{E}[C_{t+s} C_s] - (\mathrm{E}\, C_{t+s} \cdot \mathrm{E}\, C_s) \\
=&\,\mathrm{E}[(C_{t+s} - C_s) C_s] + \mathrm{E}(C_s)^2 - (\mathrm{E}\, C_{t+s} \cdot \mathrm{E}\, C_s) \\
=&\,\mathrm{E}[(C_{t+s} - C_s)] \cdot \mathrm{E}[C_s] + \mathrm{E}(C_s)^2 - \mathrm{E}\, C_{t+s} \cdot \mathrm{E}\, C_s \\
=&\,-(\mathrm{E}\, C_s)^2 + \mathrm{E}(C_s)^2 \\
=&\,\operatorname{var}(C_s) \\
=&\,(\lambda \cdot \mathrm{E}\, Z_1^2) s
\end{aligned}
$$

$$
\therefore \quad \operatorname{cov}(C_t, C_s) = (\lambda \cdot \mathrm{E}\, Z_1^2)(s \wedge t).
$$

- Markov property : (C_t) is a continuous-time Markov process.
- Restaurant Arrival Process (Example 3.9)

Homework

- Prove Thm3.3, Thm 3.5 in textbook (do not read the proof in textbook first. try independently)
- Exercise 3.9, 3.10, 3.17, 3.19, 3.20, 3.25 (CONCEPTUAL PROBLEMS, page 79)
- Exercise 3.2, 3.3, 3.29, 3.30 (COMPUTATIONAL PROBLEMS, page 80)
- Let N be a Poisson process with parameter λ. Let U_t denote the time of the first jump *after* time t. (In particular, $U_0 = S_1$.) Calculate the probability density function of U_t.

- Let $N(t)$ be the number of arrival customers and assume $\{N(t)\}$ is a Poisson processwith rate λ. S_n is the nth jump time. Let $t_i = i$ for $i = 1, 2, 3, 4$.
    1. What is the expected number of arrival customers at time t_4?
    2. What is the probably that there are no arrivals from t_1 to t_3?
    3. What is the probably that there are two arrivals from t_1 to t_3 and one arrives before t_2 and the other arrives after t_2 ?
    4. $\Pr(N(t_3) = 5 | N(t_1) = 1) = ?$
    5. $\mathrm{E}(N(t_2)N(t_1)) = ?$
    6. $\mathrm{E}[N(t_1)N(t_4)(N(t_3) - N(t_2))] = ?$
    7. $\mathrm{E}[N(t_2) | S_1 > t_1]$
    8. $\mathrm{E}[N(t_2) | S_2 > t_1]$