Introduction to Stochastic Process

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Chapter 4: Continuous-Time Markov Chain

- occupancy time
- limiting behaviour
- first-passage time

Occupancy Times

Let $\mathbf{P}(t) = [P_{ij}(t)]$ be the transition matrix of a CTMC $\{X(t) : t \in \mathbf{R}_+\}$. The occupancy time of state j is defined as

$$m_{ij}(T) = \mathbb{E}\left(\int_0^T 1_{\{X(t)=j\}} \, \mathrm{d}t \middle| X(0) = i \right),$$

which is the expected amount of time the CTMC spends in state j during the interval $[0, T]$. We compute

$$\begin{aligned} m_{ij}(T) &= \int_0^T \mathbb{E}\left(1_{\{X(t)=j\}} \middle| X(0) = i \right) \mathrm{d}t = \int_0^T \mathbb{P}\left(X(t) = j \middle| X(0) = i \right) \, \mathrm{d}t \\ &= \int_0^T P_{ij}(t) \, \mathrm{d}t. \end{aligned}$$

Theorem (Thm 4.4)

Define **occupancy matrix** $M(T) := [m_{ij}(T)]$, *then*

$$M(T) = \int_0^T \mathbf{P}(t) \, \mathrm{d}t$$

Example: Two-State CTMC

Example

Consider a two-state CTMC with the infinitesimal generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix},$$

with $\alpha, \beta \geq 0$. Find $M(T)$.

We already obtained

$$P(t) = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{1}{\alpha+\beta} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} \mathrm{e}^{-(\alpha+\beta)t}.$$

Then using Thm 4.4, we get

$$\begin{aligned} M(T) &= \int_0^T \mathbf{P}(t)\ \mathrm{d}t \\ &= \frac{T}{\alpha+\beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{1}{(\alpha+\beta)^2} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} \left(1 - \mathrm{e}^{-(\alpha+\beta)T}\right). \end{aligned}$$

Limiting behavior

Limiting behavior

Definition

The limiting distribution of CTMC $\{X(t) : t \in \mathbf{R}_+\}$ with transition semigroup $\{\mathbf{P}(t) = [P_{ij}(t)]\}$ is $\pi = [\pi_j]$ where

$$\pi_j := \lim_{t \to \infty} P_{ij}(t)$$

provided the above limit exists and is independent i, i.e., the limit $\lim_{t\to\infty} \mathbf{P}(t)$ have the same row vectors, which is π.

Definition

CTMC $\{X(t) : t \in \mathbf{R}_+\}$ is **irreducible** if its embedded DTMC $\{Z_n : n = 0, 1, \ldots\}$ is irreducible.

Balance Equation

Theorem (Thm 4.6,4.7,4.8)

- *An irreducible CTMC* $\{X(t) : t \in \mathbf{R}_+\}$ *with (strictly positive) rate matrix* $R = [r_{ij}]$ *has a unique limiting distribution* π.
- π *is the solution to the balance equation and the normalizing equation*

$$\boxed{\pi_j r_j = \sum_i \pi_i r_{i,j}, \quad \sum_i \pi_i = 1.}$$

- π *is also the stationary distribution of* X.
- π *is also the occupancy distribution of* X, *i.e.,* $\lim_{T\to\infty} \dfrac{m_{ij}(T)}{T} = \pi_j$.

The balance equation can be rewritten as

$$\boxed{\pi Q = 0}.$$

By Kolmogorov equation, the distribution $\pi(t) := \pi(0)\mathbf{P}(t)$ of $X(t)$ satisfies

$$\pi'(t) = \pi(0)\mathbf{P}'(t) = \pi(0)\mathbf{P}(t)Q = \pi(t)Q.$$

So $\pi(t) \equiv \pi(0)$ if and only if $\pi(0)Q = 0$.

Example(Example 4.22): Two-State CTMC

Consider a two-state CTMC with the infinitesimal generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}, \quad \alpha, \beta \geq 0.$$

It is irreducible. Hence it has a unique limiting distribution $[\pi_0, \pi_1]$. Solving simultaneously the two balance equations

$$\alpha\pi_0 = \beta\pi_1,$$
$$\beta\pi_1 = \alpha\pi_0$$

and the normalizing equation

$$\pi_0 + \pi_1 = 1$$

yields

$$\pi_0 = \frac{\beta}{\alpha+\beta}, \quad \pi_1 = \frac{\alpha}{\alpha+\beta}.$$

Note that this is also the stationary distribution and the occupancy distribution of the CTMC.

Example: Birth and Death Process

Consider the birth and death process on $\{0, 1, ..., N\}$ with the generator

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_2 & -\lambda_2 - \mu_2 & \lambda_2 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \ddots & \ddots & \mu_{N-1} & -\lambda_{N-1} - \mu_{N-1} & \lambda_{N-1} \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \mu_N & -\mu_N \end{bmatrix}.$$

The balance equation reads

$$\begin{cases} 0 = -\lambda_0 \pi_0 + \mu_1 \pi_1, \\ 0 = \lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2, \\ \vdots \\ 0 = \lambda_{j-1}\pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1}\pi_{j+1}, \\ \vdots \\ 0 = \lambda_{N-1}\pi_{N-1} - \mu_N \pi_N. \end{cases}$$

Example: Birth and Death Process

$0 = -\lambda_0\pi_0 + \mu_1\pi_1$ gives $\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$,

$0 = \lambda_0\pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2\pi_2$ gives $\pi_2 = -\frac{\lambda_0}{\mu_2}\pi_0 + \frac{\lambda_1+\mu_1}{\mu_2}\pi_1 = -\frac{\lambda_0}{\mu_2}\pi_0 + \frac{\lambda_1+\mu_1}{\mu_2}\frac{\lambda_0}{\mu_1}\pi_0 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0$,

$\vdots$

$0 = \lambda_{j-1}\pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1}\pi_{j+1}$ gives

$\pi_{j+1} = -\frac{\lambda_{j-1}}{\mu_{j+1}}\pi_{j-1} + \frac{\lambda_j+\mu_j}{\mu_{j+1}}\pi_j = -\frac{\lambda_{j-1}}{\mu_{j+1}}\frac{\lambda_{j-2}\cdots\lambda_0}{\mu_{j-1}\cdots\mu_1}\pi_0 + \frac{\lambda_j+\mu_j}{\mu_{j+1}}\frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1}\pi_0 = \frac{\lambda_j\cdots\lambda_0}{\mu_{j+1}\cdots\mu_1}\pi_0$,

$\vdots$

$0 = \lambda_{N-1}\pi_{N-1} - \mu_N\pi_N$ gives $\pi_N = \frac{\lambda_{N-1}}{\mu_N}\pi_{N-1} = \frac{\lambda_{N-1}}{\mu_N}\frac{\lambda_{N-2}\cdots\lambda_0}{\mu_{N-1}\cdots\mu_1}\pi_0 = \frac{\lambda_{N-1}\cdots\lambda_0}{\mu_N\cdots\mu_1}\pi_0$.

Then the normalizing equation

$$1 = \sum_{i=0}^{N}\pi_i = \pi_0\sum_{i=0}^{N}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}^{\dagger} \Longrightarrow \pi_0 = \frac{1}{\sum_{i=0}^{N}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}},$$

and so

$$\pi_j = \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1\sum_{i=0}^{N}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}}, \quad j = 0, \cdots, N.$$

† The conventions $\lambda_{i-1}\cdots\lambda_0 = \prod_{j=0}^{i-1}\lambda_j = 0$ for $i = 0$ and $\mu_i\cdots\mu_1 = \prod_{j=1}^{i}\mu_j = 0$ for $i = 0$ are adopted.

Expected Total Cost

Assume that whenever the CTMC is in state i, it incurs costs at rate $c(i)$.

- The cost rate at time t is $c\big(X(t)\big)$.
- The total cost up to time T is given by $\int_0^T c\big(X(t)\big)\mathrm{d}t$.
- The expected total cost up to T, starting from state i, is given by

$$g(i,T) = \mathbb{E}\left(\int_0^T c\big(X(t)\big)\ \mathrm{d}t \,\middle|\, X(0) = i \right).$$

$$\begin{aligned}
g(i,T) &= \int_0^T \mathbb{E}\left(c\big(X(t)\big) \,\middle|\, X(0) = i \right) \mathrm{d}t \\
&= \int_0^T \sum_{j=1}^{N} c(j) \mathbb{P}\left(X(t) = j \,\middle|\, X(0) = i \right) \mathrm{d}t \\
&= \sum_{j=1}^{N} c(j) \int_0^T P_{ij}(t)\ \mathrm{d}t = \sum_{j=1}^{N} c(j) m_{i,j}(T).
\end{aligned}$$

Theorem (Thm 4.9)

Write $c = \begin{bmatrix} c(1) \\ \vdots \\ c(N) \end{bmatrix}$ *and* $g(T) = \begin{bmatrix} g(1,T) \\ \vdots \\ g(N,T) \end{bmatrix}$ *as column vectors, then the expected total cost is*

$$g(T) = M(T)c$$

where $M(T)$ *is the occupancy matrix.*

Example(Example 4.28): Two-State CTMC

Example

Consider a two-state CTMC with the infinitesimal generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}, \quad \alpha, \beta \geq 0.$$

Set $c(0) = -B$, $c(1) = A$ where A and B are two positive numbers. Find the expected total cost $g(T)$.

We already computed

$$M(T) = \frac{T}{\alpha+\beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{1}{(\alpha+\beta)^2} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} \left(1 - \mathrm{e}^{-(\alpha+\beta)T}\right).$$

Then by Thm 4.9, we have

$$\begin{aligned} g(T) = M(T)c &= \left(\frac{T}{\alpha+\beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{1 - \mathrm{e}^{-(\alpha+\beta)T}}{(\alpha+\beta)^2} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} \right) \begin{bmatrix} -B \\ A \end{bmatrix} \\ &= \frac{\left(\alpha A - \beta B\right) T}{\alpha+\beta} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{B+A}{(\alpha+\beta)^2} \left(1 - \mathrm{e}^{-(\alpha+\beta)T}\right) \begin{bmatrix} -\alpha \\ \beta \end{bmatrix}. \end{aligned}$$

Long-run Cost Rates

- The long-run cost rate is defined as

$$g(i) = \lim_{T\to\infty} \frac{g(i,T)}{T}.$$

Theorem (Thm 4.10)

Suppose $\{X(t) : t \in \mathbf{R}_+\}$ *is an irreducible CTMC with limiting distribution* $\pi = [\pi_1, \cdots, \pi_N]$. *Then*

$$g = g(i) = \sum_{j=1}^{N} \pi_j c(j), \quad 1 \le i \le N.$$

$$\begin{aligned} g(i) &= \lim_{T\to\infty} \frac{g(i,T)}{T} = \lim_{T\to\infty} \frac{\sum_{j=1}^{N} m_{i,j}(T)c(j)}{T} \\ &= \sum_{j=1}^{N} c(j) \lim_{T\to\infty} \frac{m_{i,j}(T)}{T} = \sum_{j=1}^{N} c(j)\pi_j. \end{aligned}$$

Example(Example 4.30): Two-State CTMC

Example

Consider a two-state CTMC with the infinitesimal generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}, \quad \alpha, \beta \geq 0.$$

Set $c(0) = -B$, $c(1) = A$. Find the long-run cost rate g.

We already computed

$$\pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}.$$

Then by Thm 4.10, we get

$$g = -B\pi_0 + A\pi_1 = \frac{A\alpha - B\beta}{\alpha + \beta}.$$

First-Passage Times

Definition

The expected first-passage time to a subset $A \subset S$ is defined as $h_A(i) = \mathbb{E}(T_A | X(0) = i)$, where $T_A = \inf\{t \geq 0 : X(t) \in A\}$ is the first-passage time into a set A.

Start from time $t = 0$ now. Let τ be on the time of the first jump of X. Conditioning on the state $X(\tau)$ of the CTMC (i.e., the embedded chain Z_1), we perform the following "one-more-step analysis",

$$\begin{aligned}
h_A(i) = \mathbb{E}(T_A | X(0) = i) &= \sum_j \mathbb{E}(T_A | X(0) = i, X(\tau) = j)\mathbb{P}(X(\tau) = j | X(0) = i) \\
&= \sum_j \left[\mathbb{E}(\tau | X(0) = i) + \mathbb{E}(T_A | X(0) = j)\right]\mathbb{P}(X(\tau) = j | X(0) = i) \\
&= \sum_j \left(\frac{1}{r_i} + h_A(j)\right)\frac{r_{i,j}}{r_i} = \frac{1}{r_i^2}\sum_j r_{i,j} + \frac{1}{r_i}\sum_j r_{i,j} h_A(j) = \frac{1}{r_i} + \frac{1}{r_i}\sum_{j \notin A} r_{i,j} h_A(j).
\end{aligned}$$

Here we use the facts that the expectation of the first jump time $\mathbb{E}(\tau | X(0) = i) = \frac{1}{r_i}$ and transition probability of the embedded DTMC $p_{i,j} = \frac{r_{i,j}}{r_i}$, as well as $r_i = \sum_j r_{ij}$.

First-passage time Formula

Theorem (Thm 4.11)

The mean first passage times $\{h_A(i) : i \notin A\}$ *satisfy*

$$r_i h_A(i) = 1 + \sum_{j \notin A} r_{i,j} h_A(j), \quad i \notin A,$$

and the boundary condition $h_A(i) = 0$ *for* $i \in A$.

Example (Example 4.32)

Consider a CTMC with the infinitesimal generator

$$Q = \begin{bmatrix} -5 & 2 & 3 & 0 \\ 4 & -6 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 1 & 0 & 3 & -4 \end{bmatrix}.$$

Compute the expected time to reach state 4.

By Thm 4.11, we have

$$\begin{cases} 5h_{\{4\}}(1) = 1 + 2h_{\{4\}}(2) + 3h_{\{4\}}(3), \\ 6h_{\{4\}}(2) = 1 + 4h_{\{4\}}(1) + 2h_{\{4\}}(3), \\ 4h_{\{4\}}(1) = 1 + 2h_{\{4\}}(2). \end{cases}$$

Solving simultaneously, we get

$$h_{\{4\}}(1) = \frac{14}{11} \approx 1.2727, \quad h_{\{4\}}(2) = \frac{29}{22} \approx 1.3182, \quad h_{\{4\}}(3) \approx \frac{10}{11} = 0.9091.$$

Example (Birth and Death Process)

Consider the birth and death process on $\{0, 1, ..., N\}$ with the generator

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \mu & -\lambda-\mu & \lambda & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \ddots & \ddots & \mu & -\lambda-\mu & \lambda \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \mu & -\mu \end{bmatrix}.$$

Compute the expected time to reach states 0 and N.

By Thm 4.11, we have the boundary conditions $h_{\{0,N\}}(0) = h_{\{0,N\}}(N) = 0$ and

$$h_{\{0,N\}}(i) = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} h_{\{0,N\}}(i-1) + \frac{\lambda}{\lambda+\mu} h_{\{0,N\}}(i+1), \quad 1 \le i \le N-1.$$

We can verify that the solution is given by

$$h_{\{0,N\}}(i) = \begin{cases} \frac{1}{\mu-\lambda}\left(i - N\frac{1-(\mu/\lambda)^i}{1-(\mu/\lambda)^N}\right), & \text{if} \quad \mu \neq \lambda, \\ \frac{1}{2\mu} i(N-i), & \text{if} \quad \mu = \lambda, \end{cases} \quad i = 0, \cdots, N.$$

Homework

- TEXTBOOK: Page 142: 4.11, 4.20, 4.22, 4.32
- A workshop has five machines and one repairman. Each machine functions until it fails at an exponentially distributed random time with parameter 0.20 per hour. On the other hand, it takes a exponentially distributed random time with parameter (rate) 0.50 per hour to repair a given machine. We assume that the machines behave independently of one another, and that (a) up to five machines can operate at any given time, (b) at most one can be under repair at any time. Compute the proportion of time the repairman is idle in the long run.

Homework

- A system consists of two machines and two repairmen. Each machine can work until failure at an exponentially distributed random time with parameter 0.2. A failed machine can be repaired only be a single repairman, within an exponentially distributed random time with parameter 0.25. We model the number $X(t)$ of working machines at time $t \in \mathbf{R}_+$ as a continuous-time Markov process.

  1. Complete the missing entries in the generator matrix
  $$Q = \begin{bmatrix} \square & 0.5 & \square \\ 0.2 & \square & \square \\ 0 & \square & -0.4 \end{bmatrix}.$$
  2. Calculate the long-run probability distribution (π_0, π_1, π_2) of $X(t)$.
  3. Compute the average number of working machines in the long run.
  4. Given that a working machine can produce 100 units every hour, how many units can the system produce per hour in the long run?