Chapter 2 Discrete-Time Markov Models (Part iv)

Andrey Markov (1856-1922, Russian mathematician)

Section 2.7 First-Passage Time (hitting problem)

Example (Example 2.33)

Gambler's Ruin Problem (page 43).

Example

DTMC model : Simple random walk on $\{0, 1, 2, \ldots, N\}$ with absorbing boundary condition.

Hitting Problem

Definition

Let $\{X_n, n \geq 0\}$ be a DTMC on state space S, let $A \subset S$ be a subset of S.

- The **first passage time to hit** A is

$$T_A = \inf\{n \geq 0 : X_n \in A\}$$

  and $T_A = \infty$ if $\{n \geq 0 : X_n \in A\} = \emptyset$.

- The expectation of T_A is called the **mean first passage time**, denoted as [a]

$$h_A(i) = \mathrm{E}(T_A | X_0 = i).$$

- *Where* the process X hits the set A for the first time? It is the location of X at the time T_A:

$$X_{T_A}$$

  which is a random variable taking values in A.

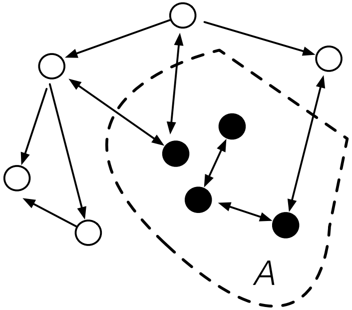
- The **hitting probability** is the distribution of X_{T_A}, denoted by

$$g_j(i) = \Pr(X_{T_A} = j | X_0 = i), \forall i \in S, j \in A.$$

[a] The textbook uses the notation m instead of h. Please use h in this course.

- $T_A \in \{0, 1, 2, \cdots, \} \cup \{\infty\}$
- $h_A(i) = 0$ for all $i \in A$.
- Law of Total Probability gives

$$\sum_{j \in A} g_j(i) = \Pr(T_A < \infty | X_0 = i)$$



Equation for the mean first passage time $h_A(i)$

one-more-step analysis

For $i \in A$, then $T_A(i) = 0$, $h_A(i) = 0$. For all other $i \notin A$, by Markov property

$$
\begin{aligned}
h_A(i) &\triangleq \mathrm{E}(T_A | X_0 = i) = \sum_{j \in S} \mathrm{E}(T_A \cdot 1_{\{X_1 = j\}} | X_0 = i) \\
&= \sum_{j \in S} \mathrm{E}(T_A | X_0 = i, X_1 = j) \Pr(X_1 = j | X_0 = i) \\
&= \sum_{j \in S} (\mathrm{E}(T_A | X_0 = j) + 1) p_{ij} \\
&= (\sum_{j \notin A} h_A(j) p_{ij}) + 1
\end{aligned}
$$

Theorem (Thm 2.13)

The mean first passage time $h_A(i)$ satisfies the following linear equation

$$\sum_{j \in A^c} p_{ij} h_A(j) - h_A(i) = -1, \quad \forall i \in A^c$$

and the boundary condition $h_A(i) = 0, \forall i \in A$.

exercise

- Consider the DTMC on $S = \{1, 2, \dots, N\}$ with $p_{11} = 1 - a$ where $0 < a \le 1$. Let $A = \{2, 3, \dots, N\}$. Prove that $h_A(1) = 1/a$.

$$h_A(1) = p_{11}(h_A(1) + 1) + \sum_{j \in A} p_{1j}(h_A(j) + 1)$$

$$= p_{11}(h_A(1) + 1) + a = (1 - a)(h_A(1) + 1) + a.$$

So $h_A(1) = 1/a$.

Cost Model : Utility Functionals

The above "one-more-step analysis" technique can be applied to derive an equation for an expectation of the form

$$c_A(i) := \mathrm{E}\left[\sum_{t=0}^{T_A-1} f(X_t)|X_0 = i\right]$$

where $f : A^c \to \mathbb{R}$ is a given utility function*. For $i \in A$, we define $c_A(i) = 0$. For $i \notin A$, then

$$\begin{aligned} c_A(i) &= \sum_{j\in S} \mathrm{E}\left[\sum_{t=0}^{T_A-1} f(X_t)\middle|X_0 = i, X_1 = j\right] \Pr(X_1 = j|X_0 = i) \\ &= \sum_{j\in S} (c_A(j) + f(i))p_{ij} = (\sum_{j\in A^c} (c_A(j)p_{ij})) + f(i) \end{aligned}$$

$$\boxed{\sum_{j\in A^c} p_{ij}c_A(j) - c_A(i) = -f(i), \quad \forall i \in A^c}$$

*accumulated cost before hitting A

Example:

A lift stops at the ground, first, second and third floor. Initially, it is at the third floor $X_0 = 3$. Each time it moves to one of three available floors with equal probability $1/3$. When the lift runs upward, 3 unit of power energy is used to lift the cart for each floor height; while as the lift runs downward, it just consumes 1 unit of power energy to travel each floor. What is the average unit of total power energy the lift consumes before it stops at the ground floor?

solution: Define $c(i)$ be the expected unit of power before the lift stops at $A = \{0\}$ for $X_0 = i$. Then

$$c(i) = \sum_{j \in S} p_{ij}(d(i,j) + c(j))$$

where $d(i,j)$ is the cost from floor i to floor j: $d(i,j) = 3*(j-i)$ if $j > i$ and $d(i,j) = (i-j)$ if $j < i$. The boundary condition is $c(0) = 0$.

$$c(3) = 1/3*(1+c(2)) + 1/3*(2+c(1)) + 1/3*(3+c(0));$$

$$c(2) = 1/3*(3+c(3)) + 1/3*(1+c(1)) + 1/3*(2+c(0));$$

$$c(1) = 1/3*(6+c(3)) + 1/3*(3+c(2)) + 1/3*(1+c(0)).$$

So $c(3) = c(2) = 7$ and $c(1) = 8$. The answer is 7.

Hitting Probability $g_j(i), \quad j \in A$

For $i \in A$, then $X_{T_A} = X_0$, $g_j(i) = \delta_{ij}$ ($\triangleq 1$ if $j = i$; $\triangleq 0$ if $j \neq i$).
For $i \in A^c$, by Markov property,

$$\begin{aligned}
g_j(i) &= \Pr(X_{T_A} = j | X_0 = i) \\
&= \sum_{k \in S} \Pr(X_{T_A} = j | X_0 = i, X_1 = k) p_{ik} \\
&= \sum_{k \in S} \Pr(X_{T_A} = j | X_0 = k) p_{ik} \\
&= \sum_{k \in S} p_{ik} g_j(k)
\end{aligned}$$

The expected exit location: $\mathrm{E}(X_{T_A} | X_0 = i) = \sum_{j \in A} g_j(i) j$

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Theorem

The hitting probability $g_j(i) = \Pr(X_{T_A} = j | X_0 = i)$ *satisfies (for each* **fixed** $j \in A$*) the linear system,*

$$\sum_{k \in S} p_{ik} g_j(k) - g_j(i) = 0, \quad \forall i \in A^c$$

and the boundary condition $g_j(i) = \delta_{ij}, \forall i \in A$. *Furthermore, the expected hitting location is*

$$\mathrm{E}(X_{T_A} | X_0 = i) = \sum_{j \in A} j g_j(i)$$

Example

For a symmetric random work on $S = \{0, \ldots, N\}$. Let $A = \{0, N\}$. Show that $g_0(i) = (N - i)/N$ and $g_N(i) = i/N$. Then the expected value

$$\mathrm{E}[X_{T_A} | X_0 = i) = 0 \times g_0(i) + N \times g_N(i) = i = X_0.$$

The case that $T_A = \infty$

Note that

$$\begin{aligned} 1 &= \Pr(T_A = \infty | X_0 = i) + \Pr(T_A < \infty | X_0 = i) \\ &= \Pr(T_A = \infty | X_0 = i) + \Pr(X_{T_A} \in A | X_0 = i) \\ &= \Pr(T_A = \infty | X_0 = i) + \sum_{j \in A} g_j(i) \end{aligned}$$

Note that we may have $\Pr(T_A = \infty | X_0 = i) > 0$. Consider the transition matrix (state space $S = \{1,2,3\}$) $\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1. \end{bmatrix}$ and $A = \{1\}$ and $i = 2$. Then, $\Pr(T_A = \infty | X_0 = 2) = 0.5$. $\Pr(T_A = 1 | X_0 = 2) = 0.5$. State "1" and "3" are *absorbing states*.

First Return Time
— *when I go home ?*

The first *return* time

Assume $X_0 = i$. We know $T_i(i)$ is the first passage time to hit the state itself, so $T_i(i) = 0$. What is more useful is when the chain will come back to state i after it *leaves* i.

Definition

The **first return time** for a state $j \in S$ is defined by

$$T_j^{\mathbf{r}} = \inf\{n \geq 1 : X_n = j\} \quad \geq 1.$$

The expectation of $T_j^{\mathbf{r}}$ is called the **mean return time**:

$$h_j^{\mathbf{r}}(i) = \mathrm{E}\left[T_j^{\mathbf{r}} | X_0 = i\right] \quad \geq 1.$$

(we implicitly assume that $X_0 = i$ in this symbol $T_j^{\mathbf{r}}$.)
Recall $T_j = \inf\{n \geq 0 : X_n = j\}$. Then $T_j^{\mathbf{r}} = T_j > 0$ for all $j \neq i$, and it follows

$$\boxed{h_j^{\mathbf{r}}(i) = h_j(i). \quad \forall j \neq i}$$

All important conclusions come from the case $i = j$ where $h_i^{\mathbf{r}}(i)$ is unknown while $h_i(i) = 0$

1. Note that $T_j^{\mathbf{r}} \geq 1$ and $h_j^{\mathbf{r}}(i) \geq 1$ while $T_i = 0$ and $h_i(i) = 0$.
2. If $X_0 = i, X_1 = j$, then $T_j^{\mathbf{r}} = 1$.
3. If $X_0 = i$, then $T_i^{\mathbf{r}}$ or $h_i^{\mathbf{r}}(i)$ literally means the first *return* (or mean first) time from the state i and back to i.

In the next, we shall prove that

$$\boxed{h_i^{\mathbf{r}}(i) = \left(\sum_{k \neq i} h_i(k) p_{ik}\right) + 1, \quad \forall i \in S.} \tag{*}$$

Equations for the mean return time $h^{\mathbf{r}}_j(i)$:

Similar to deviation for $h_j(i)$, we have that for *any* i, j

$$
\begin{aligned}
h^{\mathbf{r}}_j(i) &\triangleq \mathrm{E}(T^{\mathbf{r}}_j | X_0 = i) \quad \because \text{ law of total prob.} \\
&= \sum_{k \in S, k \neq j} \mathrm{E}(T^{\mathbf{r}}_j \cdot 1_{\{X_1 = k\}} | X_0 = i) + \mathrm{E}(T^{\mathbf{r}}_j \cdot 1_{\{X_1 = j\}} | X_0 = i) \\
&= \sum_{k \neq j} \mathrm{E}(T^{\mathbf{r}}_j | X_0 = i, X_1 = k) \Pr(X_1 = k | X_0 = i) + \mathrm{E}(T^{\mathbf{r}}_j | X_0 = i, X_1 = j) p_{ij} \quad \because ② \\
&= \sum_{k \neq j} \Big(\mathrm{E}(T^{\mathbf{r}}_j | X_0 = k) + 1 \Big) p_{ik} + \mathbf{1} \times p_{ij} = \sum_{k \neq j} \Big(h^{\mathbf{r}}_j(k) + 1 \Big) p_{ik} + p_{ij} \\
&= \left(\sum_{k \neq j} h^{\mathbf{r}}_j(k) p_{ik} \right) + \sum_{j \in S} p_{ij} = \left(\sum_{k \neq j} h_j(k) p_{ik} \right) + 1
\end{aligned}
$$

This is the same equation as for $h_j(i)$ but **all** $i \in S, j \in S$:

$$
\boxed{\sum_{k \in S, k \neq j} p_{ik} h^{\mathbf{r}}_j(k) - h^{\mathbf{r}}_j(i) = -1, \quad \forall i \in S}
$$

W do not need the boundary condition here. For each fixed j, we have N unknowns $h^{\mathbf{r}}_j(i), i \in S$ with the above N equations.

return probability and recurrent state *

Definition

The **return probability** is the probability of return to state j in a finite time starting from state i:

$$\rho_{ij} \triangleq \Pr\left(T_j^{\mathbf{r}} < \infty | X_0 = i\right) = \Pr\left(\exists n \geq 1, \text{ such that } X_n = j | X_0 = i\right).$$

We focus on the return probability ρ_{ii} by letting $i = j$.

- A state i is said to be **recurrent** if (it always return in finite time)

$$\rho_{ii} = 1.$$

  Otherwise, it is called **transient** .
- A recurrent state i is called **positive recurrent** if (the mean return time is finite)

$$h_i^{\mathbf{r}}(i) = \mathrm{E}(T_i^{\mathbf{r}} | X_0 = i) < \infty$$

  and **null recurrent** if $h_i^{\mathbf{r}}(i) = \infty$.(it always return but the expectation is infinity.)

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* *optional* from this slide

Remark

- Compare with the "accessible" concept before. For any state i, i always communicate with itself. But it might not return itself almost surely (e.g., it is a transient state, $\rho_{ii} < 1$).
- If i and j are two different states, then $\rho_{ij} > 0$ *if and only if* i *can access* j.
- For any i, j, k, $\rho_{ij} \geq \rho_{ik}\rho_{kj}$.
- Note

  $$\Pr\left(T_i^{\mathbf{r}} = \infty | X_0 = i\right) = 1 - \rho_{ii}$$

  So if $\rho_{ii} < 1$ (transient state), then it follows that

  $$h_i^{\mathbf{r}}(i) = \mathrm{E}\left(T_i^{\mathbf{r}} | X_0 = i\right) = \infty \times (1 - \rho_{ii}) + \mathrm{E}\left(T_i^{\mathbf{r}} \times 1_{T^{\mathbf{r}} < \infty} | X_0 = i\right)\rho_{ii} \geq \infty.$$

  i.e., the non-vanishing probability of taking value ∞ will lead to the infinity of the expectation.
- However, if $\rho_{ii} = 1$ (recurret state), although

  $$T_i^{\mathbf{r}} < \infty, \quad a.s.$$

  it might occur that its expectation $h_i^{\mathbf{r}}(i)$ is infinite (for instance, $f(x) = 1/x < \infty$ for all $x > 0$, but $\int_0^1 f(x)dx = \infty$.),

Exercise:

- the two-state DTMC.

$$P = \begin{bmatrix} 1 & 0 \\ b & 1-b \end{bmatrix}$$

  $b > 0$. Show that state 2 is transient and state 1 is (positive) recurrent.
- A periodic DTMC on the ring $S = \{0, 1, \ldots, N\}$ is positive recurrent. (see `Homework`)

Occupancy time at a given state

Definition

The **number of returns** is the total number of steps the chain visits a give state during an infinitely long period

$$R_j \triangleq \sum_{t=1}^{\infty} 1_{\{X_t=j\}}, \quad \forall j \in S$$

Recall the occupancy matrix $\mathbf{M}^{(n)}$ and the theorem that $\mathbf{M}^{(n)} = \sum_{t=0}^{n} \mathbf{P}^t$. So,

$$\mathrm{E}[R_j|X_0=i] + \Pr(X_0=j|X_0=i) = \mathrm{E}\left[\sum_{t=0}^{\infty} 1_{\{X_t=j\}}|X_0=i\right] = \lim_{n\to\infty} m_{i,j}^{(n)} = \sum_{t=0}^{\infty} (\mathbf{P}^t)_{ij}$$

It follows that

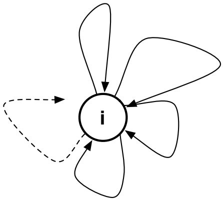
$$\boxed{\mathrm{E}[R_j|X_0=i] \quad = \quad \sum_{t=1}^{\infty} (\mathbf{P}^t)_{ij}}$$

Return probability ρ_{ii} and the finite number of returns

heuristic on return: — "once return, always return"

If a state i is recurrent, then its return probability $\rho_{ii} = 1$: it will always return to itself in finite time; after the first return, due to Markovian property, it will return again for second time, and so on. Thus, the number of return will be infinite ! i.e.,

$$R_i = +\infty$$



Theorem

$$\Pr(R_i = +\infty | X_0 = i) = \begin{cases} 1 & \textit{if } \rho_{ii} = 1; \\ 0 & \textit{if } \rho_{ii} < 1; \end{cases}$$

proof: Consider the Bernoulli trial that repeats an experiment with exactly two possible outcomes each time: "return" (w.p. ρ_{ii}) or "not return" (w.p. $1 - \rho_{ii}$). Then $\{R_i = m\}$ means exactly m returns followed by a "no return". So we have the following geometric distribution

$$\boxed{\Pr(R_i = m | X_0 = i) = (\rho_{ii})^m (1 - \rho_{ii}), \quad m \geq 0}$$

Then $\Pr(R_i < \infty | X_0 = i) = \sum_{m=0}^{\infty} \Pr(R_i = m | X_0 = i) = \begin{cases} 1 & \text{if } \rho_{ii} < 1; \\ 0 & \text{otherwise.} \end{cases}$

Note $\Pr(R_i = +\infty | X_0 = i) = 1 - \Pr(R_i < \infty | X_0 = i)$. The proof is completed.

The proof of the following theorem is left as an exercise by noting

$$\Pr(R_j = m | X_0 = i) = \rho_{ij} \times (\rho_{jj})^{m-1} \times (1 - \rho_{jj}), \quad m \geq 1$$

Theorem

$$E[R_j | X_0 = i] = \begin{cases} \frac{\rho_{ij}}{1-\rho_{jj}}, & \textit{if } \rho_{jj} < 1; \\ +\infty, & \textit{if } \rho_{jj} = 1. \end{cases}$$

From the Theorem, we have the equivalent conditions that a state is recurrent:

$$\boxed{\begin{aligned} & i \text{ is recurrent} \\ \Leftrightarrow & \rho_{ii} = 1 \\ \Leftrightarrow & \Pr(R_i = +\infty | X_0 = i) = 1 \\ \Leftrightarrow & \Pr(R_i < +\infty | X_0 = i) = 0 \\ \Leftrightarrow & E[R_i | X_0 = i] = \infty \\ \Leftrightarrow & \sum_{t=1}^{\infty} (\mathbf{P}^t)_{ii} = +\infty \end{aligned}}$$

Recurrent DTMC

Theorem

Assume a state i is recurrent (i.e., $\rho_{ii} = 1$), and $j \neq i$. If i can access j (i.e., $\rho_{ij} > 0$), then j is also recurrent (i.e., $\rho_{jj} = 1$), and $\rho_{ji} = 1$.

Proof: use $\sum_{t=0}^{\infty} (\mathbf{P}^t)_{ii} = +\infty$ and the CK equation to prove $\sum_{t=0}^{\infty} (\mathbf{P}^t)_{jj} = +\infty$. And consider the first time that i can reach j: $k \triangleq \inf\{t : (\mathbf{P}^t)_{ij} > 0\}$. The detail is left as exercise.

Exercise

If there exists a state j such that $\rho_{ij} > 0$ but $\rho_{ji} = 0$, then i is transient.

Proof: If i is recurrent, then $\rho_{ij} > 0$ implies i can access j; So by the Theorem, j should also be recurrent, and $\rho_{ji} = 1$: contradicting the condition.

Definition (recurrent chain)

An irreducible DTMC is called **recurrent** if any state is recurrent.

Existence of recurrent state for finite state DTMC

Assume that the state space S of a Markov chain is finite

Theorem

Any finite state DTMC has a recurrent state.

Proof: Assume $|S| = K$ and all states are transient, i.e., $\rho_{jj} < 1, \forall j$, then $C := \max_{i \in S, j \in S} \frac{\rho_{ij}}{1-\rho_{jj}}$ is finite. We then have the following contradiction:

$$KC \geq \sum_{j \in S} \frac{\rho_{ij}}{1-\rho_{jj}} = \sum_{j \in S} \mathrm{E}[R_j | X_0 = i] = \sum_{j \in S} \left(\sum_{t=1}^{\infty} (\mathbf{P}^t)_{ij} \right) = \sum_{t=1}^{\infty} \left(\sum_{j \in S} (\mathbf{P}^t)_{ij} \right) = \sum_{t=1}^{\infty} 1 = \infty$$

Property

Any recurrent state in a finite state DTMC is positive recurrent.

Example: Random work on $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ [†]

The transition matrix $\mathbf{P} = [p_{ij}]$ satisfies that $p_{i,i+1} = q$ and $p_{i,i-1} = p$. Now calculate $\sum_{t=1}^{\infty} (\mathbf{P}^t)_{ii}$. It is clear it always takes even steps to return. So $(\mathbf{P}^{2n+1})_{ii} = 0$ and

$$(\mathbf{P}^{2n})_{ii} = \binom{2n}{n} p^n q^n$$

So,

$$\sum_{t=1}^{\infty} (\mathbf{P}^t)_{ii} = \sum_{n=1}^{\infty} (\mathbf{P}^{2n})_{ii} = \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} p^n q^n = \frac{1}{\sqrt{1-4pq}} - 1.^*$$

If $p = q = 1/2$, then this series diverges and the random walk is recurrent; Otherwise, the state i is transient. Actually, we can directly calculate the return probability:

$$\rho_{ii} = \Pr(T_i^{\mathbf{r}} < \infty | X_0 = i) = 1 - \sqrt{1-4pq}\ ^* = 2\min(p,q)$$

which is equal to 1 (recurrent) iff $p = q = 1/2$. But when $p = q = 1/2$, the state is null recurrent! (because $\mathrm{E}[T_i^{\mathbf{r}} \cdot 1_{T_i^{\mathbf{r}} < \infty} | X_0 = i] = \frac{4pq}{|p-q|} \to \infty$ as $p = q$ *.)

*The proofs are too advanced and thus skipped. Refer to §4.4 of "Understanding ..."

†In multiple dimension $\mathbb{Z}^d$, the symmetric random walk is recurrent for $d = 1, 2$ but transient for $d \geq 3$.

Revisit the occupancy distribution

Now consider a recurrent and irreducible DTMC. Then any state is recurrent. Recall that

$$\mathrm{E}[R_j|X_0=i] = \mathrm{E}\left[\sum_{t=1}^{\infty} 1_{\{X_t=j\}}|X_0=i\right] = \sum_{t=1}^{\infty} (\mathbf{P}^t)_{ij}$$

and $T_i^{\mathbf{r}} = \inf\{n \geq 1 : X_n = i\}$. So,

$$X_t \neq i, \quad \forall\, t < T_i^{\mathbf{r}}$$

Intuition:

The dynamics looks like: the chain starts from i and returns i at time $T_i^{\mathbf{r}}$.
Due to the (strong) Markov property, the next cycle *statistically* follows the same pattern. Thus, we only look at the occupancy distribution within **one** time period $[0, T_i^{\mathbf{r}})$:

$$\widetilde{\nu}_i(j) := \mathrm{E}\left[\sum_{t=0}^{T_i^{\mathbf{r}}-1} 1_{\{X_t=j\}}|X_0=i,\right] \quad \forall\, j \in S$$

And we expect this is the same as the true occupancy ν after normalization.

Note $\widetilde{\nu}_i(i) = \mathrm{E}\left[1_{\{X_0=i\}}|X_0 = i\right] + 0 = 1$ and

$$\sum_{j\in S}\widetilde{\nu}_i(j) = \mathrm{E}\left[\sum_{t=0}^{T_i^{\mathbf{r}}-1}\sum_{j\in S}1_{\{X_t=j\}}|X_0 = i\right] = \mathrm{E}\left[\sum_{t=0}^{T_i^{\mathbf{r}}-1}1|X_0 = i\right] = \mathrm{E}\left[T_i^{\mathbf{r}}|X_0 = i\right] = h_i^{\mathbf{r}}(i)$$

is equal to the mean return time: finite if i is positive recurrent and $=\infty$ if i is null recurrent.
After the normalization, our intuition leads to the conjecture that

$$\nu_j = \frac{\widetilde{\nu}_i(j)}{\sum_{j\in S}\widetilde{\nu}_i(j)} = \frac{\widetilde{\nu}_i(j)}{h_i^{\mathbf{r}}(i)}, \quad \forall j \in S$$

In particular take $j = i$, then

$$\nu_i = \frac{\widetilde{\nu}_i(i)}{h_i^{\mathbf{r}}(i)} = \frac{1}{h_i^{\mathbf{r}}(i)}$$

Revisit the limiting distribution

Theorem

If a DTMC is irreducible, recurrent and aperiodic, then

$$\lim_{n\to\infty} \Pr(X_n = j | X_0 = i) = \frac{1}{h_j^{\mathbf{r}}(j)}, \quad i, j \in S$$

where $h_j^{\mathbf{r}}(j) = \mathrm{E}[T_j^{\mathbf{r}} | X_0 = j] \in [1, \infty]$
If furthermore, the chain is positive recurrent, then the above limit is the limiting distribution π_j *and all* $\pi_j, j \in S$ *is strictly positive since* $h_j^{\mathbf{r}}(j) < \infty$.

Corollary

For a finite state space, an irreducible, aperiodic DTMC is always positive recurrent and the above theorem guarantee the existence of limiting distribution which is constructed abve.

Homework

- For the two-state DTMC with $S = \{1, 2\}$ and $\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ where $a, b \in [0, 1]$. Let $T_j(i)$ be the first passage time of reaching the state j if $X_0 = i$. $h_j(i) = \mathrm{E}[T_j(i)|X_0 = i]$. (i): Calculate $h_1(1)$, $h_2(2)$ and $h_2(1)$ and $h_1(2)$. (ii): What is the distribution of $T_2(1)$? Calculate its expectation from this distribution. (iii) *: Let $T_j^{\mathbf{r}}(i)$ be the first return time and $h_j^{\mathbf{r}}(i)$ is its expectation. Show that if $b > 0$, then $h_1^{\mathbf{r}}(1) = 1 + \frac{a}{b}$, $h_1^{\mathbf{r}}(2) = \frac{1}{b}$. What is the distribution of $T_1^{\mathbf{r}}(1)$? What is the distribution of $T_2^{\mathbf{r}}(1)$? What is the return probability $\rho_{11} = \Pr(T_1^{\mathbf{r}}(1) < \infty | X_0 = 1)$? When do we have $\rho_{11} = 1$, i.e, the state 1 is recurrent ?
- For Gambler's Ruin Problem (Example 2.33, page 43), What is the expected dollars the gambler A has when the game stops? (hint for $p \neq q$ case: consider the difference $d_i \triangleq g(i+1) - g(i)$)
- Consider the symmetric random walk on $S = \{1, \ldots, N\}$ with periodic boundary on two ends 1 and N. Find the mean first passage time to N if starting from 1, i.e., $\mathrm{E}(T_{\{N\}}|X_0 = 1)$. Find the expected return time $h_1^{\mathbf{r}}(1) = \mathrm{E}\, T_1^{\mathbf{r}}(1)^*$.
- Textbook page 57: 2.42, 2.43.

*optional

optional

- Prove the following theorem*. If a state i is recurrent (i.e., $\rho_{ii} = 1$), and i can access j (i.e. $\rho_{ij} > 0$), then j is also recurrent(i.e., $\rho_{jj} = 1$), and $\rho_{ji} = 1$.
- The Department of Security has three floors, 0th(ground), 1st and 2nd floor. Each floor has three rooms: Room A, B and C from left to right. Let A_i be the room A at Floor i , $i = 0, 1, 2$. B_i and C_i are defined likewise. There is a door between two neighbouring rooms at the same level. It is also possible to break windows to climb upstair or downstair vertically into the room on the neighbouring floor. See figure for illustration.

A2	B2	C2
A1	B1	C1
A0	B0	C0

The only exit of this building is located in Room B_0. At initial time, there is a spy at Room C_2, unfortunately triggering the security alarm and without knowing which room is exit unless inside this room, he starts to run in panic randomly in this building. At each step, the spy, with equal probability, moves to one of the rooms that he can access through doors (taking 2 seconds) or by breaking windows (taking 15 seconds). On average, how much time does the spy need to escape this building?

*optional