Solution Manual of MA4546

January 11, 2019

This solution manual includes more exercises than the assigned homework.

Chapter 1

Ex. 1 — A pairwise independent collection of random variables is a set of random variables any two of which are independent. Any collection of mutually independent random variables is pairwise independent, but some pairwise independent collections are not mutually independent. Consider the following example. Toss a fair coin two times. Define the following events

- A: head appears on the first toss
- B: head appears on the second toss
- C: both tosses yield the same outcome.

Are the events A,B and C mutually independent ? Are they pairwise independent?

**** **Ex. 2** — If p and q are two pdfs defined on $\mathbb{R}$, prove that the Kullback-Leibler divergence (or "relative entropy") of q from p, which is defined as

$$D(p\|q) = \int p(x) \log \frac{p(x)}{q(x)}$$

is always non-negative.

Ex. 3 — If two r.v.s X and Y are independent, then

$$p_{X|Y}(x|y) = p_X(x), \mathbb{E}(X|Y) = \mathbb{E}(X)$$

Ex. 4 — If $\{X_1, X_2, .., X_n\}$ is mutually independent, then $\mathrm{var}(\sum_i X_i) = \sum_i \mathrm{var}(X_i)$

Ex. 5 — Suppose that $X = (X_1, X_2)$ is a two dimensional Gaussian random variable with mean $\mu = (\mu_1, \mu_2)$ and the covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. What is the conditional pdf $p(x_1|x_2)$ of X_1 given $X_2 = x_2$? For what value of ρ, X_1 and X_2 are independent? Verify the variance decomposition theorem for this two Gaussian variables X_1 and X_2.

Ex. 6 — Find the moment-generating and characteristic functions for the following distributions: Bernoulli distribution $\mathsf{Bern}(p)$, Poisson distribution $\mathsf{Poi}(\lambda)$, exponential distribution $\mathsf{Exp}(\lambda)$, normal distribution $N(\mu, \sigma^2)$.

***** **Ex. 7 —** Show that

$$\mathrm{E}[|X - h(Y)|^2] = \min_{g \text{ is a function}} \mathrm{E}(|X - g(Y)|^2)$$

where the function $h(y)$ is the conditional expectation $h(y) = \mathrm{E}(X|Y = y)$.

Ex. 8 — Show that if (X_t) is a martingale, then its expectation $\mathrm{E}(X_t)$ is independent of time t.

**** **Ex. 9 —** For the random walk $(X_n = \sum_{i=1}^n Z_n)$ where $\Pr(Z_n = 1) = p$ and $\Pr(Z_n = -1) = q = 1 - p$, find the value of a positive number σ such that $Y_n := (X_n - \mu n)^2 - \sigma^2 n$ is a martingale, where $\mu = \mathrm{E}\, Z_n = p - q$.

Chapter 2 (part i)

Ex. 1 — Let X_n be the random walk on $S = \mathbb{Z}$ ($X_0 = 0$) with transition probability $\Pr(X_{n+1} = i+1 \mid X_n) = p$ and $\Pr(X_{n+1} = i-1 \mid X_n) = q = 1-p$. Calculate the mean $\mathrm{E}\, X_n$, the variance $\mathrm{var}\, X_n$ and the autocovariance $c(n,m) := \mathrm{E}[(X_n - \mathrm{E}\, X_n)(X_m - \mathrm{E}\, X_m)]$.

**** **Ex. 2** — Assume that $\{X_n\}$, $n = 1, 2, \cdots$, are iid $\{-1, 1\}$-valued random variables. $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$. For any $n \geq 1$, define new random variables

$$S_n = \sum_{i=0}^{n} X_i, \ Y_n = X_n + X_{n-1},$$

and the two-dim random vector $Z_n = \begin{bmatrix} X_{n-1} \\ X_n \end{bmatrix}$. Discuss if the stochastic processes $\{S_n\}$, $\{Y_n\}$, $\{Z_n\}$ are Markov chains. Why? Write the corresponding transition matrices for Markov chains.

Ex. 3 — (*textbook p51, 2.13*)Consider the following weather model. The weather normally behaves as in Example 2.3. However, when the cloudy spell lasts for two or more days, it continues to be cloudy for another day with probability .8 or turns rainy with probability .2. Develop a four-state DTMC model to describe this behaviour.

Ex. 4 — (*p51. 2.16*)A total of N balls are put in two urns, so that initially urn A has i balls and urn B has $N - i$ balls. At each step, one ball is chosen at random from the N balls. If it is from urn A, it is moved to urn B, and vice versa. Let X_n be the number of balls in urn A after n steps. Show that $\{X_n, n \geq 0\}$ is a DTMC, assuming that the successive random drawings of the balls are independent. Display the transition probability matrix of the DTMC.

Ex. 5 — (*p53,2.10*)Consider the weather model of Example 2.3. Compute the probability that once the weather becomes sunny, the sunny spell lasts for at least 3 days.

Ex. 6 — (*p53, 2.11*)Compute the expected length of a rainy spell in the weather model of Example 2.3.

Ex. 7 — (*p53-54, 2.15b*)Compute the occupancy matrix M(10) for the DTMCs with transition matrices as given below:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Chapter 2 (part ii, part iii)

Ex. 1 — Consider the two-state DTMC with the transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$a, b \in [0, 1]$. Discuss the reducibility, periodicity and limiting/stationary/occupancy distributions to classify the DTMC for different a, b.

Ex. 2 — Prove the transition matrix used by Metropolis $[p_{ij}]$ satisfies the detailed balance condition with π_i.

Ex. 3 — Define the matrix $\tilde{P}$ whose (i, j) entry is $\pi_i^{1/2} p_{ij} \pi_j^{-1/2}$, i.e.,

$$\tilde{P} := D^{1/2}\, P\, D^{-1/2},$$

where $D \triangleq \operatorname{diag}\{\pi_1, \pi_2, \ldots, \pi_N\}$ is the diagonal matrix with diagonal entry $\{\pi_i\}$. Verify that the detailed balance condition is equivalent to the symmetry of $\tilde{P}$:

$$\tilde{P}^{\mathsf{T}} = \tilde{P}.$$

Ex. 4 — (*p55, 2.19*)Classify the DTMCs with the transition matrices given in Computational Problem 2.15 as irreducible or reducible.

Ex. 5 — (*p55, 2.20*)Classify the irreducible DTMCs with the transition matrices given below as periodic or aperiodic:

Ex. 6 — (*p56, 2.21*)Compute a normalized solution to the balance equations for the DTMC in Computational Problem 2.20(a). When possible, compute:

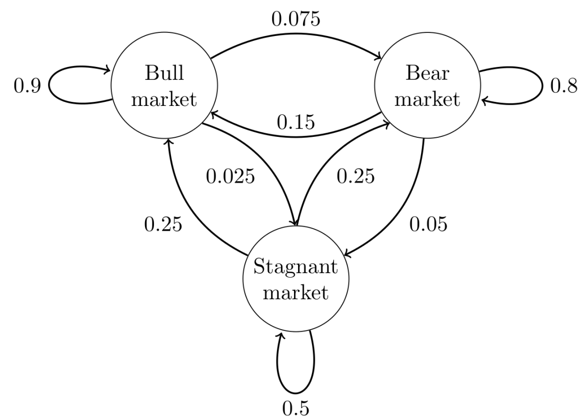
1.the limiting distribution;

2.the stationary distribution;

3.the occupancy distribution.

Ex. 7 — (*p57, 2.28*)Consider the weather model of Conceptual Problem 2.13. Compute the long-run fraction of days that are sunny.

** **Ex. 8** — We consider the annual movement of a stock market as a three-state DTMC with the transition diagram as shown in next page. Suppose that in the year of "Bull market" , "Stagnant market" and "Bear market", the return rates are 15%, 0% and -12% respectively. Suppose that the investment in money market is no risk at all with compound annualised interest rate 5%. As a long-time value investor, you prefer to investing in stock market or money market ?



Chapter 2 (part iv)

Ex. 1 — For the two-state DTMC with $S = \{1, 2\}$ and $\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ where $a, b \in [0, 1]$. Let $T_j(i)$ be the first passage time of reaching the state j if $X_0 = i$. $h_j(i) = \mathrm{E}[T_j(i)|X_0 = i]$. (i): Calculate $h_1(1)$, $h_2(2)$ and $h_2(1)$ and $h_1(2)$. (ii): What is the distribution of $T_2(1)$? Calculate its expectation by this distribution. (iii): Let $T^{\mathbf{r}}_j(i)$ be the first return time and $h^{\mathbf{r}}_j(i)$ is its expectation. Show that if $b > 0$, then $h^{\mathbf{r}}_1(1) = 1 + \frac{a}{b}$, $h^{\mathbf{r}}_1(2) = \frac{1}{b}$. What is the distribution of $T^{\mathbf{r}}_1(1)$? What is the distribution of $T^{\mathbf{r}}_2(1)$? What is the return probability $\rho_{11} = \Pr(T^{\mathbf{r}}_1(1) < \infty|X_0 = 1)$? When do we have $\rho_{11} = 1$, i.e, the state 1 is recurrent ?

Ex. 2 — For Gambler's Ruin Problem (Example 2.33, page 43), what is the expected dollars of gambler A when the game stops?

Ex. 3 — Consider the symmetric random walk on $S = \{1, \ldots, N\}$ with periodic boundary on two ends 1 and N. Find the mean first passage time to N if starting from 1, i.e., $\mathrm{E}(T_{\{N\}}|X_0 = 1)$. Find the expected return time $h^{\mathbf{r}}_1(1) = \mathrm{E}\, T^{\mathbf{r}}_1(1)$.

Ex. 4 — (*p57 2.42*)Compute the expected time to go from state 1 to 4 in the DTMCs of Computational Problems 2.20(a) and (c).

Ex. 5 — (*p57, 2.43*)Compute the expected time to go from state 1 to 4 in the DTMCs of Computational Problems 2.20(b) and (d).

***** **Ex. 6** — The Department of Security has three floors, 0th(ground), 1st and 2nd floor. Each floor has three rooms: Room A, B and C from left to right. Let A_i be the room A at Floor i , $i = 0, 1, 2$. B_i and C_i are defined likewise. There is a door between two neighbouring rooms at the same level. It is also possible to break windows to climb upstair or downstair vertically into the room on the neighbouring floor. See figure for illustration.

A2	B2	C2
A1	B1	C1
A0	B0	C0

The only exit of this building is located in Room B_0. At initial time, there is a spy at Room C_2, unfortunately triggering the security alarm and without knowing which room is exit unless inside this room, he starts to run in panic randomly in this building. At each step, the spy, with equal probability, moves to one of the rooms that he can access through doors (taking 2 seconds) or by breaking windows (taking 15 seconds). On average, how much time does the spy need to escape this building?

Chapter 3

Ex. 1 — (Minimum of Exponentials) $X = \min\{X_1, X_2, ..., X_k\}$, where $\{X_i\}$ are independent $\text{Exp}(\lambda_i)$ random variables, is an $\text{Exp}(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^{k} \lambda_i$$

Ex. 2 — (Sums of Poisson) Suppose $\{X_i, i = 1, 2, \ldots, n\}$ are independent random variables with $X_i \sim \text{Poi}(\lambda_i)$, $1 \le i \le n$. Let

$$Z_n = X_1 + X_2 + \cdots + X_n$$

Then Z_n is a $\text{Poi}(\lambda)$ random variable, where $\lambda = \sum_{i=1}^{n} \lambda_i$.

Ex. 3 — *(3.9, p79)*Let X be a $\text{Poi}(\lambda)$ random variable. Show that

$$\text{E}[X(X-1)(X-2)\ldots(X-k+1)] = \lambda^k, \quad k \ge 1$$

Ex. 4 — *(3.10, p79)*Let $X_1 \sim \text{Poi}(\lambda_1)$ and $X_2 \sim \text{Poi}(\lambda_2)$ be two independent random variables. Show that, given $X_1 + X_2 = k$,

$$X_i \sim \text{Bin}(k, \frac{\lambda_i}{\lambda_1 + \lambda_2}), i = 1, 2.$$

Ex. 5 — *(3.17, p79)*Let $\{N(t), t \ge 0\}$ be a $\text{PP}(\lambda)$. Let S_k be the time of occurrence of the kth event in this Poisson process. Show that, given $N(t) = 1$, S_1 is uniformly distributed over $[0, t]$.

Ex. 6 — *(3.19, p80)*Let $\{N(t), t \ge 0\}$ be a $\text{PP}(\lambda)$. Let $s, t \ge 0$. Compute the joint distribution

$$\Pr(N(s) = i, N(s+t) = j), \quad 0 \le i \le j < \infty$$

Ex. 7 — *(3.20, p80)*Let $\{N(t), t \ge 0\}$ be a $PP(\lambda)$. Show that

$$Cov(N(s), N(s+t)) = \lambda s, \quad s, t \ge 0$$

Ex. 8 — *(3.25, p80)*(3.25 p80) Let $\{C(t), t \geq 0\}$ be a CPP of Definition 3.6. Compute $Cov(C(t+s), C(s))$ for $s, t \geq 0$.

Ex. 9 — *(3.2, p80)*The lifetimes of two car batteries (brands A and B) are independent exponential random variables with means 12 hours and 10 hours, respectively. What is the probability that the Brand B batter outlasts the Brand A battery?

Ex. 10 — *(3.3, p80)*Suppose a machine has three independent components with $Exp(.1)$ lifetimes. Compute the expected lifetime of the machine if it needs all three components to function properly.

Ex. 11 — *(3.29, p83)*The number of cars visiting a national park forms a PP with rate 15 per hour. Each car has k occupants with probability p_k as given below:

$$p_1 = .2, p_2 = .3, p_3 = .3, p_4 = .1, p_5 = .05, p_6 = .05$$

Compute the mean and variance of the number of visitors to the park during a 10-hour window.

Ex. 12 — *(3.30, p83)*Now suppose the national park of Computational Problem 3.29 charges $4.00 per car plus $1.00 per occupant as the entry fee. Compute the mean and variance of the total fee collected during a 10-hour window.

Ex. 13 — Let N be a Poisson process with parameter λ. Let U_t denote the time of the first observation *after* time t. (In particular, $U_0 = S_1$.) Calculate the probability density function for U_t.

Ex. 14 — Let $N(t)$ be the number of arrival customers and assume $\{N(t)\}$ is a Poisson process with rate λ. S_n is the nth jump time. Let $t_i = i$ for $i = 1, 2, 3, 4$.

1. What is the expected number of arrival customers at time t_4?
2. What is the probably that there are no arrivals from t_1 to t_3?
3. What is the probably that there are two arrivals from t_1 to t_3 and one arrives before t_2 and the other arrives after t_2 ?
4. What is the probably that there are two arrivals from t_1 to t_4 and one arrives before t_3 and the other arrives after t_2 ?

5.$\mathbb{P}(N(t_3) = 5|N(t_1) = 1) =?$

6.$\mathbb{E}(N(t_2)N(t_1)) =?$

7.$\mathbb{E}[N(t_1)N(t_4)(N(t_3) - N(t_2))] =?$

8.$\mathrm{E}[N(t_2) \mid S_1 > t_1] =?$

9.$\mathbb{E}[N(t_2) \mid S_2 > t_1] =?$

Chapter 4 (Continuous-Time Markov Chain)

Ex. 1 — What is the transition semigroup $\mathbf{P}(t)$ and the generator Q of a Poisson process with rate λ. Verify they do satisfy forward Kolmogorov equation and $\mathbf{P}(t) = e^{tQ}$ by the definition of matrix exponential.

Ex. 2 — Consider the following time-homogeneous compound Poisson process with rate λ: $X_t = \sum_{i=0}^{N(t)} Z_i$ where Z_0=0 and for $i \geq 1$, $Z_i = \pm a$ with the equal probability 1/2. $N(t)$ is the $PP(\lambda)$. $a > 0$ is a constant. This process X_t has independent and stationary increment. What is the master equation and the generator Q of this process? What is $\mathrm{E}(X_t)$? For what condition on a and λ, the variance of X_t is equal to t? Under this condition, find $Cov(X_s, X_t)$.

Ex. 3 — Consider a mathematical professor wandering between three coffee shops with graphical structure

$$Ⓐ \underset{\lambda_1}{\overset{\mu_1}{\rightleftarrows}} Ⓑ \underset{\lambda_2}{\overset{\mu_2}{\rightleftarrows}} Ⓒ.$$

1. What is the generator Q matrix ?
2. What is the transition matrix for the embedded DTMC?
3. Given that the professor is at coffee shop B right now, what is the probability that he will next head to shop A, rather than C ?

Ex. 4 — *(4.1, p 138)*A weight of L tons is held up by K cables that share the load equally. When one of the cables breaks, the remaining unbroken cables share the entire load equally. When the last cable breaks, we have a failure. The failure rate of a cable subject to M tons of load is λM per year. The lifetimes of the K cables are independent of each other. Let $X(t)$ be the number of cables that are still unbroken at time t. Show that $\{X(t); t \geq 0\}$ is a CTMC, and find its rate matrix.

Ex. 5 — *(4.6, p 139)*A system consisting of two components is subject to a series of shocks that arrive according to a PP(λ). A shock can cause the failure of component 1 alone with probability p, component 2 alone with probability q, both components with probability r, or have no effect with probability $1 - p - q - r$: No repairs are possible. The system fails when both the components fail. Model the state of the system as a CTMC.

Ex. 6 — *(4.1, p 141)*Compute the transition probability matrix $\mathbf{P}(t)$ at

$t = 0.2$ for a CTMC on $S = \{1, 2, 3, 4, 5\}$ with the rate matrix

$$\begin{bmatrix} 0 & 4 & 4 & 0 & 0 \\ 5 & 0 & 5 & 5 & 0 \\ 5 & 5 & 0 & 4 & 4 \\ 0 & 5 & 5 & 0 & 4 \\ 0 & 0 & 5 & 5 & 0 \end{bmatrix}$$

Ex. 7 — *(4.4, p 141)*Consider the model in Conceptual Problem 4.1. Suppose the individual cables have a failure rate of .2 per year per ton of load. We need to build a system to support 18 tons of load, to be equally shared by the cables. If we use three cables, what is the probability that the system will last for more than 2 years?

Chapter 7 (Brownian Motion)

Ex. 1 — *(7.5, 7.6, p 276)*Let X be an $\mathcal{N}(0, \sigma^2)$ random variable. Show that the pdf of $|X|$ is given by

$$f(x) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

and $\mathrm{E}\,|X| = \sigma\sqrt{2/\pi}$.

Ex. 2 — *(7.20, p 277)*Let $\{X(t) : t \geq 0\}$ be a BM(μ, σ). Show that

$$\mathrm{E}(X(t) - \mu t | X(u) : 0 \leq u \leq s) = X(s) - \mu s, \quad 0 \leq s \leq t.$$

Ex. 3 — *(7.21, p 277)*Let $\{X(t) : t \geq 0\}$ be a BM(μ, σ) and $\theta = -\mu/2\sigma^2$. Show that $e^{\theta X}$ is a martingale, i.e., it satisfies

$$\mathrm{E}(e^{\theta X(t)} | X(u) : 0 \leq u \leq s) = e^{\theta X(s)}, \quad 0 \leq s \leq t.$$

Ex. 4 — *(Thm 7.4, p 251)*Show that the conditional distribution $X_1|X_2 = x_2$ is $\mathcal{N}(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), (1-\rho^2)\sigma_1^2)$ for the bivariant normal (X_1, X_2) defined in Example 7.2.

Ex. 5 — Calculate for the standard Brownian motion B:

$$\mathrm{E}[\exp(2B(1) + B(2))].$$

Ex. 6 — Show the following matrix is strictly positive definite:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

*** **Ex. 7** — Let $B(t)$ and $N(t)$ be the independent standard Brownian process and Poisson process with intensity λ, respectively. Find the mean and covariance of the process $X(t) = B(N(t))$.

*** **Ex. 8** — Define the time-integration of the Brownian motion

$$X(t) = \int_0^t B(s)ds.$$

To find the mean and variance of $X(t)$. One can consider the finite Riemann sum $X_n = \sum_{i=1}^n B(t_i)h$ where $t_i = hi, 1 \leq i \leq n$ and $h = t/n$, n is a large integer. Find the mean and variance of X_n and their limit when $n \to \infty$.