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INTEGRAL EQUATIONS FOR BIHARMONIC DATA COMPLETION

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Abstract. A boundary integral based method for the stable reconstruction of missing boundary data is presented for the biharmonic equation. The solution (displacement) is known throughout the boundary of an annular domain whilst the normal derivative and bending moment are specified only on the outer boundary curve. A recent iterative method is applied for the data completion solving mixed problems throughout the iterations. The solution to each mixed problem is represented as a biharmonic single-layer potential. Matching against the given boundary data, a system of boundary integrals is obtained to be solved for densities over the boundary. This system is discretised using the Nyström method. A direct approach is also given representing the solution of the ill-posed problem as a biharmonic single-layer potential and applying the similar techniques as for the mixed problems. Tikhonov regularization is employed for the solution of the corresponding discretised system. Numerical results are presented for several annular domains showing the efficiency of both data completion approaches.

1. Introduction.

Let u be a solution to the biharmonic equation

$$\Delta^2 u = 0 \quad \text{in } D \tag{1}$$

and suppose additionally that u satisfies the following conditions on the boundary,

$$u = f \quad \text{on } \Gamma, \qquad Nu = g \quad \text{on } \Gamma_2 \qquad \text{and} \qquad Mu = h \quad \text{on } \Gamma_2. \tag{2}$$

Here, D is an annular domain in $\mathbb{R}^2$ lying between the two simple closed non-intersecting curves Γ_2 and Γ_1, with Γ_1 contained in the bounded interior of Γ_2, and $\Gamma = \Gamma_1 \cup \Gamma_2$. The operators on the boundary are given by

$$\begin{aligned} Nu &= \frac{\partial u}{\partial n}, \\ Mu &= \nu \Delta u + (1-\nu)\left(u_{x_1x_1} n_1^2 + 2u_{x_1x_2} n_1 n_2 + u_{x_2x_2} n_2^2\right), \end{aligned} \tag{3}$$

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where $n = (n_1, n_2)$ is the outward unit normal to Γ and $0 < \nu < 1$. In plate theory, M is the normal bending moment and ν the Poisson ratio. We assume that data is given such that there exists a classical or weak solution to (1)–(2).

In [9], it is shown that (1)–(2) is ill-posed and that there exists at most one solution. Further in [9], an iterative method is proposed and analysed for the biharmonic data completion. At each iteration step, mixed boundary value problems are solved for (1) updating functions on the boundary curves. It is mentioned at the end of [9] that a single-layer approach can be used to solve the mixed problems numerically.

We shall undertake the laborious task of deriving and implementing a numerical solution strategy based on integral equations for solving (1) with mixed conditions. Moreover, it will be shown how to directly solve the biharmonic data completion problem using integral equations, that is without iterations. This in total constitute the novelty of the presented work. Related inverse problems for the biharmonic equation are mentioned in the introduction to [9]; for history and applications of the biharmonic equation, see [19].

For the Cauchy problem of the Laplace equation, boundary integral equations in combination with iteration schemes have been effective, see for example [2, 17]. A direct integral approach with no iterations is given in [7] using ideas of [5]. We follow those works and extend the techniques to (1)–(2). The solution to the mixed problems in the iterative scheme is represented as a biharmonic single-layer potential with densities to be determined. Matching the given data, a system of boundary integral equations is obtained for the densities. This system is discretised using the Nyström method. We also investigate representing the solution to (1)–(2) directly as a single-layer potential following ideas in [7].

Layer potentials is a classical field with abundance of work (for direct problems), and we can therefore only give some guidance to where results and further references can be found for the biharmonic equation. The biharmonic single-layer potential that we use is studied for example in [10, 14, 20], see further [4, 11, 23]. Layer potential based methods for ill-posed Cauchy problems for other elliptic equations and the heat and wave equation are presented in [1, 3, 8, 13, 18, 21, 22].

For the outline of the work, in Section 2, the iterative method of [9] is briefly surveyed. In Section 3, the potential representation is given of the solution to the mixed biharmonic problems needed in the iterative procedure. Matching the boundary data renders a system of boundary integral equations for the densities, which is discretised using the Nyström method; details are in Section 4. In Section 5, a direct approach with no iterations is outlined. Properties of the obtained system of integral equations are shown in Theorem 5.1. Numerical examples are presented in Section 6, for both data completion approaches, showing that accurate reconstructions can be obtained. Some conclusions are stated in Section 7.

2. **An iterative method for (1)–(2).** The iterative procedure given in [9] for the stable solution to (1)–(2) runs as follows:

- Choose an arbitrary initial approximation ζ_0 on the inner boundary part Γ_1.
- The first approximation u_0 of the solution u is obtained by solving (1) supplied with the boundary conditions

$$u_0 = f \quad \text{on } \Gamma, \qquad M u_0 = \zeta_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad N u_0 = g \quad \text{on } \Gamma_2.$$

- Next, v_0 is constructed by solving (1) with the boundary conditions changed to

$$v_0 = 0 \quad \text{on } \Gamma, \qquad Nv_0 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mv_0 = h - Mu_0 \quad \text{on } \Gamma_2.$$

- Given that u_{k-1} and v_{k-1} are known, the approximation u_k is determined from (1) with

$$u_k = f \quad \text{on } \Gamma, \qquad Mu_k = \zeta_k \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_k = g \quad \text{on } \Gamma_2.$$

  Here,

$$\zeta_k = \zeta_{k-1} + \gamma M v_{k-1}|_{\Gamma_1},$$

  where $\gamma > 0$ is a relaxation parameter.
- Then v_k is determined from (1) with boundary conditions

$$v_k = 0 \quad \text{on } \Gamma, \qquad Nv_k = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mv_k = h - Mu_k \quad \text{on } \Gamma_2.$$

The iterations continues repeating the last two steps until a suitable stopping rule has been satisfied. In the case of noisy data, a discrepancy principle can be applied to cease the iterations. Note that at each step, data is updated with data of the same kind (for example the normal bending moment is updated with a bending moment from a previous step). This is appealing from a physical point of view.

We recall from [9],

Theorem 2.1. *Let $f \in H^{3/2}(\Gamma_2)$, $g \in H^{1/2}(\Gamma_2)$ and $h \in H^{-1/2}(\Gamma_2)$. Assume that problem (1)–(2) has a solution u. Let the relaxation parameter γ satisfy $0 < \gamma < 2/(\|T\|\|K\|)$, and let u_k be the k–th approximation in the given algorithm. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^2(D)} = 0$$

for any initial function $\zeta_0 \in H^{-1/2}(\Gamma_1)$.

From the proof given in [9], it can be seen that the procedure is of Landweber type. Moreover, as a special case when the relaxation parameter $\gamma = 1$, the alternating method [15] is obtained.

3. A single-layer approach for mixed biharmonic problems.

We consider the mixed problem:

$$\Delta^2 u = 0 \quad \text{in } D, \tag{4}$$

$$u = f \quad \text{on } \Gamma, \qquad Nu = g_2 \quad \text{on } \Gamma_2 \qquad \text{and} \qquad Mu = h_1 \quad \text{on } \Gamma_1, \tag{5}$$

with the boundary operators N and M as in (3). This mixed biharmonic problem has a unique solution $u \in H^2(D)$ for $f \in H^{3/2}(\Gamma)$, $h \in H^{-1/2}(\Gamma_1)$ and $g \in H^{1/2}(\Gamma_2)$ that depends continuously on the data (see for example [9]).

The fundamental solution for the biharmonic equation in $\mathbb{R}^2$ is

$$G(x,y) = \frac{1}{8\pi}|x-y|^2 \ln|x-y|, \tag{6}$$

and satisfies

$$\Delta_x^2 G(x,y) = \delta(x-y), \quad \text{in } \mathbb{R}^2,$$

with δ the Dirac delta function. We consider the single-layer potential for the biharmonic equation,

$$v(x) = \int_\Gamma \Big(G(x,y)\varphi(y) + N_y G(x,y)\psi(y) \Big)\, ds(y), \quad x \in D. \tag{7}$$

This single-layer potential has the following continuity and jump properties (see [10, Chapt. 8] and [14, Chapt. 2.4 and 10.4.4]),

$$v(x) = \int_\Gamma \Big(G(x,y)\varphi(y) + N_y G(x,y)\psi(y) \Big)\, ds(y), \quad x \in \Gamma,$$

$$N_x v(x) = \int_\Gamma \Big(N_x G(x,y)\varphi(y) + N_x N_y G(x,y)\psi(y) \Big)\, ds(y), \quad x \in \Gamma,$$

$$M_x v(x) = \mp \frac{1}{2}\psi(x) + \int_\Gamma \Big(M_x G(x,y)\varphi(y) + M_x N_y G(x,y)\psi(y) \Big)\, ds(y), \quad x \in \Gamma.$$

For the kernels in these relations, we have the representations (see [14, Chapt. 10.4.4])

$$\begin{aligned}
N_y G(x,y) &= -\frac{1}{8\pi} n(y)\cdot(x-y)(1+2\ln|x-y|),\\
N_x G(x,y) &= \frac{1}{8\pi} n(x)\cdot(x-y)(1+2\ln|x-y|),\\
N_x N_y G(x,y) &= -\frac{1}{8\pi}\left(2\frac{n(x)\cdot(x-y)n(y)\cdot(x-y)}{|x-y|^2} + n(x)\cdot n(y)(1+2\ln|x-y|)\right),\\
M_x G(x,y) &= \frac{1+3\nu}{8\pi} + \frac{(1-\nu)(n(x)\cdot(x-y))^2}{4\pi|x-y|^2} + \frac{(1+\nu)\ln|x-y|^2}{8\pi},\\
M_x N_y G(x,y) &= \frac{1-\nu}{2\pi}\left(\frac{(n(x)\cdot(x-y))^2 n(y)\cdot(x-y)}{|x-y|^4} - \frac{n(x)\cdot n(y) n(x)\cdot(x-y)}{|x-y|^2} \right)\\
&\quad - \frac{(1+\nu)n(y)\cdot(x-y)}{4\pi|x-y|^2}.
\end{aligned}$$

Matching the representation (7) against the data (5) and taking into account the above continuity and jump properties, a system of boundary integral equations is obtained to be solved for the densities φ and ψ. To have a unique solution to this system, (7) has to be slightly modified. According to [10, Chapt. 8.4] the following holds.

Theorem 3.1. *The solution u of the mixed problem (4)–(5) can be represented as*

$$u(x) = \sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x), \quad x \in D, \tag{8}$$

with the elements $w(x) = a_0 + a_1 x_1 + a_2 x_2$ ($(a_0,a_1,a_2) \in \mathbb{R}^3$), $\varphi_k, \psi_k \in C(\Gamma_k)$, $k = 1,2$, being the unique solution of the system of integral equations consisting of

$$\begin{cases}
\displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x)\\
\qquad = f(x),\ x \in \Gamma_\ell,\ \ell = 1,2,\\
\displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(N_x G(x,y)\varphi_k(y) + N_x N_y G(x,y)\psi_k(y) \Big)\, ds(y) + Nw(x)\\
\qquad = g_2(x),\ x \in \Gamma_2,\\
\displaystyle -\frac{1}{2}\psi_1(x) + \sum_{k=1}^{2} \int_{\Gamma_k} \Big(M_x G(x,y)\varphi_k(y) + M_x N_y G(x,y)\psi_k(y) \Big)\, ds(y)\\
\qquad = h_1(x),\ x \in \Gamma_1,
\end{cases} \tag{9}$$

together with

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \varphi_k(y)\, ds(y) = A_0, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} (y_1\varphi_k(y) + n_1(y))\psi_k(y))\, ds(y) = A_1, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} (y_2\varphi_k(y) + n_2(y))\psi_k(y))\, ds(y) = A_2, \end{cases} \tag{10}$$

for a given triplet $(A_0, A_1, A_2) \in \mathbb{R}^3$.

Sobolev spaces for the boundary data are given in [10, Chapt. 8.4]. Note that the constants A_0, A_1 and A_2 can be chosen arbitrarily, but the solution of (9) depends on the choice of them. As is shown in [10, Ex. 8.2] it is not in general possible to put the constants a_0, a_1, a_2, A_0, A_1 and A_2 simultaneously equal to zero.

For the sake of completeness, we give the similar result for the second mixed problem used in the iterative method,

$$\Delta^2 v = 0 \quad \text{in } D, \tag{11}$$

$$v = f \quad \text{on } \Gamma, \qquad Nv = g_1 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mv = h_2 \quad \text{on } \Gamma_2. \tag{12}$$

From [10, Chapt. 8.4] we have,

Theorem 3.2. *The solution v of the mixed problem (11)–(12) is given by*

$$v(x) = \sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x), \quad x \in D, \tag{13}$$

with the elements $w(x) = a_0 + a_1 x_1 + a_2 x_2$ ($(a_0, a_1, a_2) \in \mathbb{R}^3$), and $\varphi_k, \psi_k \in C(\Gamma_k)$, $k = 1, 2$, being the unique solution of the system of integral equations consisting of

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x) \\ \qquad\qquad = f_\ell(x),\ x \in \Gamma_\ell,\ \ell = 1, 2, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(N_x G(x,y)\varphi_k(y) + N_x N_y G(x,y)\psi_k(y) \Big)\, ds(y) + Nw(x) \\ \qquad\qquad = g_1(x),\ x \in \Gamma_1, \\ \displaystyle\frac{1}{2}\psi_2(x) + \sum_{k=1}^{2} \int_{\Gamma_k} \Big(M_x G(x,y)\varphi_k(y) + M_x N_y G(x,y)\psi_k(y) \Big)\, ds(y) \\ \qquad\qquad = h_2(x),\ x \in \Gamma_2, \end{cases} \tag{14}$$

together with (10) for a given triplet $(A_0, A_1, A_2) \in \mathbb{R}^3$.

We shall outline in the next section how to discretise (9) and (14) taking into account the singularities in the kernels, together with discretisation of (10).

4. **Discretisation of the systems (9), (10) and (14).** We assume that the boundary curves Γ_ℓ, $\ell = 1, 2$, are sufficiently smooth and given by a parametric representation

$$\Gamma_\ell = \{x_\ell(s) = (x_{1\ell}(s), x_{2\ell}(s)) : \ s \in [0, 2\pi]\}.$$

The system (9) can then be written in parametric form, and is expressed as

$$\begin{cases} \dfrac{1}{2\pi} \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} (H_{\ell k}(s,\sigma)\varphi_k(\sigma) + \tilde{H}_{\ell k}(s,\sigma)\psi_k(\sigma))\, d\sigma \\ + w(x_\ell(s)) = f(x_\ell(s)), \ \ell = 1, 2, \\ \dfrac{1}{2\pi} \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} (L_{2k}(s,\sigma)\varphi_k(\sigma) + \tilde{L}_{2k}(s,\sigma)\psi_k(\sigma))\, d\sigma \\ + Nw(x_2(s)) = g_2(x_2(s)), \\ -\dfrac{1}{2|x_1'(s)|}\psi_1(s) + \dfrac{1}{2\pi} \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} (Q_{1k}(s,\sigma)\varphi_k(\sigma) + \tilde{Q}_{1k}(s,\sigma)\psi_k(\sigma))\, d\sigma \\ = h_1(x_1(s)), \end{cases} \tag{15}$$

with $s \in [0, 2\pi]$, together with the parametric form of (10),

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} \varphi_k(\sigma)\, d\sigma = A_0, \\ \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} (x_{1k}(\sigma)\varphi_k(\sigma) + n_1(x_k(\sigma)))\psi_k(\sigma))\, d\sigma = A_1, \\ \displaystyle\sum_{k=1}^{2} \int_0^{2\pi} (x_{2k}(\sigma)\varphi_k(\sigma) + n_2(x_k(\sigma)))\psi_k(\sigma))\, d\sigma = A_2. \end{cases} \tag{16}$$

We use the notation $\varphi_\ell(s) := \varphi_k(x_\ell(s))|x_\ell'(s)|$ and $\psi_\ell(s) := \psi_\ell(x_\ell(s))|x_\ell'(s)|$ for the densities, and for the kernels,

$$\begin{aligned} H_{\ell k}(s,\sigma) &= G(x_\ell(s), x_k(\sigma)), \quad &\tilde{H}_{\ell k}(s,\sigma) &= N_y G(x_\ell(s), x_k(\sigma)), \\ L_{\ell k}(s,\sigma) &= N_x G(x_\ell(s), x_k(\sigma)), \quad &\tilde{L}_{\ell k}(s,\sigma) &= N_x N_y G(x_\ell(s), x_k(\sigma)), \\ Q_{\ell k}(s,\sigma) &= M_x G(x_\ell(s), x_k(\sigma)), \quad &\tilde{Q}_{\ell k}(s,\sigma) &= M_x N_y G(x_\ell(s), x_k(\sigma)). \end{aligned} \tag{17}$$

These kernels are all continuous functions, but their derivatives do contain a logarithmic singularity. It is advantageous to make these singularities explicit when applying quadratures for the numerical solution of the system (15). We therefore rewrite some of the kernels,

$$
\begin{aligned}
H_{\ell\ell}(s,\sigma) &= H_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right) + H_{\ell\ell}^{(2)}(s,\sigma),\\
\tilde{H}_{\ell\ell}(s,\sigma) &= \tilde{H}_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right) + \tilde{H}_{\ell\ell}^{(2)}(s,\sigma),\\
L_{\ell\ell}(s,\sigma) &= L_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right) + L_{\ell\ell}^{(2)}(s,\sigma),\\
\tilde{L}_{\ell\ell}(s,\sigma) &= \tilde{L}_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right) + \tilde{L}_{\ell\ell}^{(2)}(s,\sigma),\\
Q_{\ell\ell}(s,\sigma) &= Q_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right) + Q_{\ell\ell}^{(2)}(s,\sigma),
\end{aligned}
\tag{18}
$$

where

$$
H_{\ell\ell}^{(1)}(s,\sigma) = \frac{1}{8}|x_\ell(s) - x_\ell(\sigma)|^2, \quad \tilde{H}_{\ell\ell}^{(1)}(s,\sigma) = -\frac{1}{4}n(x_\ell(\sigma))\cdot(x_\ell(s) - x_\ell(\sigma)),
$$

$$
L_{\ell\ell}^{(1)}(s,\sigma) = \frac{1}{4}n(x_\ell(s))\cdot(x_\ell(s) - x_\ell(\sigma)), \quad \tilde{L}_{\ell\ell}^{(1)}(s,\sigma) = -\frac{1}{4}n(x_\ell(s))\cdot n(x_\ell(\sigma))
$$

and

$$
Q_{\ell\ell}^{(1)}(s,\sigma) = \frac{1+\nu}{4}.
$$

Moreover, we have the following representation

$$
H_{\ell\ell}^{(2)}(s,\sigma) = H_{\ell\ell}(s,\sigma) - H_{\ell\ell}^{(1)}(s,\sigma)\ln\left(\frac{4}{e}\sin\frac{s-\sigma}{2}\right).
$$

The other kernels can be written analogously. The diagonal terms are in general zero apart from

$$
Q_{\ell\ell}^{(2)}(s,s) = \frac{1+3\nu}{4} + \frac{1+\nu}{4}\ln(e|x_\ell'(s)|^2),
$$

and

$$
\tilde{Q}_{\ell\ell}(s,s) = \frac{1-3\nu}{4}\,\frac{n(x_\ell(s))\cdot x_\ell''(s)}{|x_\ell'(s)|^2}.
$$

We can then, taking into account (18), apply the Nyström method to (15)–(16) based on the following trigonometric quadrature rules [16],

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma &\approx \frac{1}{2m}\sum_{k=0}^{2m-1} f(s_k),\\
\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)d\sigma &\approx \sum_{k=0}^{2m-1} R_k(s)\,f(s_k)
\end{aligned}
\tag{19}
$$

with mesh points

$$
s_k = kh, \quad k = 0,\dots,2m-1, \quad h = \pi/m, \tag{20}
$$

and the weight functions

$$
R_k(s) = -\frac{1}{2m}\left(1 + 2\sum_{j=1}^{m-1}\frac{1}{j}\cos j(s-s_k) - \frac{1}{m}\cos m(s-s_k)\right). \tag{21}
$$

As a result, we obtain the linear system consisting of

$$
(22)\qquad \begin{cases}
\displaystyle\sum_{j=0}^{2m-1}\Big([H_{\ell\ell}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}H_{\ell\ell}^{(2)}(s_i,s_j)]\varphi_{\ell j}+\frac{1}{2m}H_{\ell,3-\ell}(s_i,s_j)\varphi_{3-\ell,j}\\
\displaystyle +[\tilde{H}_{\ell\ell}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}\tilde{H}_{\ell\ell}^{(2)}(s_i,s_j)]\psi_{\ell j}+\frac{1}{2m}\tilde{H}_{\ell,3-\ell}(s_i,s_j)\psi_{3-\ell,j}\Big)\\
+w_{\ell i}=f_{\ell i},\ \ell=1,2,\\
\displaystyle\sum_{j=0}^{2m-1}\Big(\frac{1}{2m}L_{21}(s_i,s_j)\varphi_{1j}+[L_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}L_{22}^{(2)}(s_i,s_j)]\varphi_{2j}\\
\displaystyle +\frac{1}{2m}\tilde{L}_{21}(s_i,s_j)\psi_{1j}+[\tilde{L}_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}\tilde{L}_{22}^{(2)}(s_i,s_j)]\psi_{2j}\Big)\\
+Nw_{2i}=g_{2i},\\
\displaystyle\sum_{j=0}^{2m-1}\Big([Q_{11}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}Q_{11}^{(2)}(s_i,s_j)]\varphi_{1j}+\frac{1}{2m}Q_{12}(s_i,s_j)\varphi_{2j}\\
\displaystyle +\frac{1}{2m}(\tilde{Q}_{11}(s_i,s_j)\psi_{1j}+\tilde{Q}_{12}(s_i,s_j)\psi_{2j})\Big)-\frac{\psi_{1i}}{2|x_1'(s_i)|}=h_{1i},
\end{cases}
$$

together with

$$
(23)\qquad \begin{cases}
\displaystyle h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}\varphi_{kj}=A_0,\\
\displaystyle h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}(x_{1k}(s_j)\varphi_{kj}+n_1(x_k(s_j))\psi_{kj})=A_1,\\
\displaystyle h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}(x_{2k}(s_j)\varphi_{kj}+n_2(x_k(s_j)))\psi_{kj}=A_2,
\end{cases}
$$

for $i=0,\dots,2m-1$, with $f_{\ell i}=f_\ell(x_\ell(s_i))$, $g_{2i}=g_2(x_2(s_i))$, $h_{1i}=h_1(x_1(s_i))$, $w_{\ell i}=w(x_\ell(s_i))$, $Nw_{2i}=Nw(x_2(s_i))$ and $R_j=R_j(0)$, to be solved for the values $\varphi_{kj}\approx\varphi_k(s_j)$, $\psi_{kj}\approx\psi_k(s_j)$, $k=1,2$, $j=0,\dots,2m-1$.

Combining the representation (8) with the jump relation from Section 3 for the bending moment Mu, give on Γ_2

$$
Mu(x)=\frac{1}{2}\psi_2(x)+\sum_{k=1}^{2}\int_{\Gamma_k}\Big(M_xG(x,y)\varphi_k(y)+M_xN_yG(x,y)\psi_k(y)\Big)\,ds(y),\quad x\in\Gamma_2.
$$

Therefore, using (17)–(18) and the above quadratures, the sought numerical values of Mu on Γ_2 can be calculated as

$$
(24)\qquad
\begin{aligned}
M\tilde{u}(x_2(s_i))=\sum_{j=0}^{2m-1}\Big(&\frac{1}{2m}Q_{21}(s_i,s_j)\varphi_{1j}+[Q_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}Q_{22}^{(2)}(s_i,s_j)]\varphi_{2j}\\
&+\frac{1}{2m}\sum_{k=1}^{2}\tilde{Q}_{2k}(s_i,s_j)\psi_{kj}\Big)+\frac{\psi_{2i}}{2|x_2'(s_i)|}.
\end{aligned}
$$

A similar expression can be derived for the normal derivative Nu on Γ_2. The numerical solution of the system (14) can be realized analogously, and the representation (13) can be used to find the requested data on the boundary.

The quadratures and discretisation strategies applied are all well-studied. Thus, in principle, a full error analysis can be carried out following, for example, [16, Chapt. 12]. However, this is better done in a separate work, and instead we present some numerical examples in Section 6 that highlights that the method has the expected properties, in particular exponential convergence.

5. A "direct" integral equation approach for the Cauchy problem (1)–(2). As an alternative to the iterative regularizing method introduced above, it is possible to use a representation of the form (8) and match not against the data of a mixed problem but the original data (2). This renders an ill-posed system of integral equations.

We thus search for the solution of (1)–(2) as a (modified) single-layer potential,

$$u(x) = \sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x), \quad x \in D, \tag{25}$$

where $w(x) = a_0 + a_1 x_1 + a_2 x_2$ and $(a_0, a_1, a_2) \in \mathbb{R}^3$, $\varphi_k, \psi_k \in C(\Gamma_k)$, $k = 1, 2$, together solve the system of integral equations consisting of

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(G(x,y)\varphi_k(y) + N_y G(x,y)\psi_k(y) \Big)\, ds(y) + w(x) \\ = f(x),\ x \in \Gamma_\ell,\ \ell = 1, 2, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \Big(N_x G(x,y)\varphi_k(y) + N_x N_y G(x,y)\psi_k(y) \Big)\, ds(y) + Nw(x) \\ = g(x),\ x \in \Gamma_2, \\ \displaystyle\frac{1}{2}\psi_2(x) + \sum_{k=1}^{2} \int_{\Gamma_k} \Big(M_x G(x,y)\varphi_k(y) + M_x N_y G(x,y)\psi_k(y) \Big)\, ds(y) \\ = h(x),\ x \in \Gamma_2, \end{cases} \tag{26}$$

and

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} \varphi_k(y)\, ds(y) = A_0, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} (y_1 \varphi_k(y) + n_1(y))\psi_k(y))\, ds(y) = A_1, \\ \displaystyle\sum_{k=1}^{2} \int_{\Gamma_k} (y_2 \varphi_k(y) + n_2(y))\psi_k(y))\, ds(y) = A_2, \end{cases} \tag{27}$$

for a given triplet $(A_0, A_1, A_2) \in \mathbb{R}^3$.

To analyse this system, we introduce an operator formulation. Let

$$K(x,y) = \tag{28}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & y_1 & n_1(y) \\ 0 & 0 & 0 & y_2 & n_2(y) \\ 1 & x_1 & x_2 & G(x,y) & N_yG(x,y) \\ 0 & n_1(x) & n_2(x) & N_xG(x,y) & N_xN_yG(x,y) \\ 0 & 0 & 0 & M_xG(x,y) & \frac{1}{2}\delta(x-y) + M_xN_yG(x,y) \end{pmatrix}$$

when $x \in \Gamma_2$ and

$$K(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & x_1 & x_2 & G(x,y) & N_yG(x,y) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

when $x \in \Gamma_1$. Furthermore, put

$$\bar{\zeta}(x) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \varphi(x) \\ \psi(x) \end{pmatrix} \tag{30}$$

with the right-hand side

$$\bar{f}(x) = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ f(x) \\ g(x) \\ h(x) \end{pmatrix} \text{ for } x \in \Gamma_2 \quad \text{and} \quad \bar{f}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \\ 0 \\ 0 \end{pmatrix} \text{ for } x \in \Gamma_1. \tag{31}$$

Define the operator

$$[A\bar{\zeta}](x) = \int_\Gamma K(x,y)\bar{\zeta}(y)\, ds(y), \quad x \in \Gamma. \tag{32}$$

Here, it is assumed without loss of generality that $\int_\Gamma ds(y) = 1$, and we made the trivial extension of g and h to Γ by putting $g = h = 0$ on Γ_1. The operator A is considered as a mapping

$$A : \mathbb{R}^3 \times L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}^3 \times L^2(\Gamma) \times L^2(\Gamma_2) \times L^2(\Gamma_2).$$

The system (26)–(27) can be written

$$[A\bar{\zeta}](x) = \bar{f}(x), \quad x \in \Gamma. \tag{33}$$

We show properties of the mapping A following [5, 6].

Theorem 5.1. *The operator A corresponding to the system (26)–(27) and defined in (32) is injective and has dense range.*

Proof. We start with injectivity. Let therefore $A\bar{\zeta} = 0$. Using the components of this element $\bar{\zeta}$, denoted as in (30), define u by (25); the restriction of φ to Γ_k generates φ_k and similarly ψ_k is obtained. Then, from the assumption that $A\bar{\zeta} = 0$ and since the system (26)–(27) can be written as (33), u has the data (2) being zero. Due to the uniqueness of a solution to (1)–(2) shown in [9], u is therefore identically zero in $\bar{D}$. In particular, u and its normal derivative are zero on the boundary Γ. To show that u being identically zero implies that $\bar{\zeta}$ is zero, we need an additional operator.

Let the matrix $\tilde{K}$ consist of the first five rows of K from (28), and define

$$[\tilde{A}\bar{\zeta}](x) = \int_\Gamma \tilde{K}(x,y)\bar{\zeta}(y)\,ds(y), \quad x \in \Gamma. \tag{34}$$

Then from [10, Thm. 8.6], $\tilde{A}$ is an isomorphism for the biharmonic equation (1) supplied with Dirichlet and Neumann data on Γ. Hence, since this data is zero for u, the constants and densities in the element $\bar{\zeta}$ are all identically zero. Thus, $\bar{\zeta} = 0$ and we have shown injectivity of A.

To prove that A has a dense range, we show that the adjoint operator, A^*, is injective. Formally, A^* is obtained by

$$[A^*\bar{f}](y) = \int_\Gamma K^T(x,y)\bar{f}(x)\,ds(x), \quad y \in \Gamma, \tag{35}$$

with notation as in (31). The transpose of K from (28)–(29) needed in (35) is

$$K^T(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & n_1(x) & 0 \\ 0 & 0 & 0 & x_2 & n_2(x) & 0 \\ 1 & y_1 & y_2 & G(x,y) & N_xG(x,y) & M_xG(x,y) \\ 0 & n_1(y) & n_2(y) & N_yG(x,y) & N_xN_yG(x,y) & \frac{1}{2}\delta(x-y) + M_xN_yG(x,y) \end{pmatrix}, \tag{36}$$

when $x \in \Gamma_2$ and

$$K^T(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & G(x,y) & 0 & 0 \\ 0 & 0 & 0 & N_yG(x,y) & 0 & 0 \end{pmatrix}, \tag{37}$$

when $x \in \Gamma_1$.

Assume then that $A^*\bar{f} = 0$, with $\bar{f}$ as in (31). Define the layer potential

$$\begin{aligned} v(y) = &\int_{\Gamma_2} \Big(G(x,y)f(x) + N_xG(x,y)g(x)\Big)\,ds(x) \\ &+ \int_{\Gamma_1} G(x,y)f(x)\,ds(x) + \int_{\Gamma_2} M_xG(x,y)h(x)\,ds(x) + w(y), \end{aligned} \tag{38}$$

for $y \in \mathbb{R}^2 \setminus \Gamma$, and $w(y) = A_0 + A_1y_1 + A_2y_2$. The element v is a well-defined solution to the biharmonic equation in the domain D. Due to the assumption $A^*\bar{f} = 0$, it follows using the last two rows of K^T in (36)–(37), that v has zero Dirichlet data and normal derivative on the boundary Γ of D. Hence, v is identically zero in D.

Since v is analytic, v is zero also in a region containing Γ_2 in its interior (for an explicit extension formula for the biharmonic equation with zero Dirichlet and Neumann condition, see [12, Theorem 1]). We now apply the normal derivative N_y in (38). This derivative has a jump across Γ_2 due to the integral involving h, according to the jump relations stated in [14, Chapt. 2.4.1]. This and since v is identically zero, imply $h = 0$.

Then v in (38) with the third integral removed is a modified biharmonic single-layer potential over Γ_2. This potential generates an isomorphism (generated as in the first part of the proof, see (34), but using the first five columns of K^T from (36)) for the biharmonic equation with Dirichlet and Neumann data specified on Γ, see [10, Thm. 8.6]. Using that and since the data on Γ is zero, it follows that the remaining constants and densities in (38) are identically zero, that is $\bar{f} = 0$. Hence, A^* is injective implying that A has dense range. $\square$

5.1. **Discretisation of (26)—(27).** Discretization of the system (26)–(27) using the Nyström method is similar to what has been presented in Section 4 for the mixed problems. Taking into account (17)–(18) and the quadratures (19) with mesh points (20) and weights (21), lead to the system of linear equations

$$
\begin{cases}
\sum_{j=0}^{2m-1}\Big([H_{\ell\ell}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}H_{\ell\ell}^{(2)}(s_i,s_j)]\varphi_{\ell j}+\frac{1}{2m}H_{\ell,3-\ell}(s_i,s_j)\varphi_{3-\ell,j}\\
[\tilde{H}_{\ell\ell}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}\tilde{H}_{\ell\ell}^{(2)}(s_i,s_j)]\psi_{\ell j}+\frac{1}{2m}\tilde{H}_{\ell,3-\ell}(s_i,s_j)\psi_{3-\ell,j}\Big)\\
+w_{\ell i}=f_{\ell i},\ \ell=1,2,\\
\sum_{j=0}^{2m-1}\Big(\frac{1}{2m}L_{21}(s_i,s_j)\varphi_{1j}+[L_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}L_{22}^{(2)}(s_i,s_j)]\varphi_{2j}\\
+\frac{1}{2m}\tilde{L}_{21}(s_i,s_j)\psi_{1j}+[\tilde{L}_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}\tilde{L}_{22}^{(2)}(s_i,s_j)]\psi_{2j}\Big)\\
+Nw_{2i}=g(x_2(s_i)),\\
\sum_{j=0}^{2m-1}\Big(\frac{1}{2m}Q_{21}(s_i,s_j)\varphi_{1j}+[Q_{22}^{(1)}(s_i,s_j)R_{|i-j|}+\frac{1}{2m}Q_{22}^{(2)}(s_i,s_j)]\varphi_{2j}\\
+\frac{1}{2m}(\tilde{Q}_{21}(s_i,s_j)\psi_{1j}+\tilde{Q}_{22}(s_i,s_j)\psi_{2j})\Big)+\frac{\psi_{2i}}{2|x_2'(s_i)|}=h(x_2(s_i)),
\end{cases}
\tag{39}
$$

$$
\begin{cases}
h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}\varphi_{kj}=A_0,\\
h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}(x_{1k}(s_j)\varphi_{kj}+n_1(x_k(s_j))\psi_{kj})=A_1,\\
h\sum_{k=1}^{2}\sum_{j=0}^{2m-1}(x_{2k}(s_j)\varphi_{kj}+n_2(x_k(s_j)))\psi_{kj}=A_2,
\end{cases}
\tag{40}
$$

for $i = 0, \dots, 2m-1$, to be solved for $\varphi_{kj} \approx \varphi_k(s_j)$, $\psi_{kj} \approx \psi_k(s_j)$, $k = 1, 2$, $j = 0, \dots, 2m-1$. As above, $f_{\ell i} = f_\ell(x_\ell(s_i))$ and $R_j = R_j(0)$. The system (39)–(40) has a high-condition number since (1)–(2) is ill-posed, and therefore Tikhonov regularization is incorporated. Denoting the matrix corresponding to this system by A and the right-hand side by F, the regularization means solving

$$(A^*A + \alpha I)\phi_\alpha = A^*F, \tag{41}$$

where A^* is the adjoint (transpose) of A, with $\alpha > 0$ a regularization parameter to be chosen appropriately.

6. **Numerical results.** We present some numerical results of the data completion in (1)–(2) for both the iterative procedure and the direct layer approach. In the examples, the Poisson ratio is taken as $\nu = 0.5$. For the iterative method, it is important to have a good solver for the mixed problems. Thus, in the first example, numerical solution of mixed problems are investigated.

Ex. 1: The outer boundary curve Γ_2 is chosen as a circle with centre at the origin and radius 2, and the inner boundary Γ_1 is kite-shaped with parametrization

$$\Gamma_1 = \{x_1(s) = (\cos s + 0.4\cos 2s,\, \sin s) : s \in [0, 2\pi]\}.$$

We consider the source point $z_1 = (3,\,0)$ and use as boundary data the restriction of the corresponding fundamental solution $G(x,z)$ and the values $N_xG(x,z)$ and $M_xG(x,z)$. In the systems (9) and (14) supplemented with (10) corresponding to the two mixed problems, we take $A_0 = A_1 = A_2 = 1$. Table 1 contains the discrete L_2 relative errors for the normal bending moment Mu on Γ_2 for the mixed problem (4)–(5) and correspondingly Mv on Γ_1 for (11)–(12), that is

$$e_{2M}(\Gamma_\ell) = \frac{\left(\displaystyle\sum_{i=0}^{2m-1} \left(Mu_{ex}(x_\ell(s_i)) - M\tilde{u}(x_\ell(s_i))\right)^2\right)^{1/2}}{\left(\displaystyle\sum_{i=0}^{2m-1} \left(Mu_{ex}(x_\ell(s_i))\right)^2\right)^{1/2}}, \quad \ell = 1, 2$$

and

$$\tilde{e}_{2M}(\Gamma_1) = \frac{\left(\displaystyle\sum_{i=0}^{2m-1} \left(Mv_{ex}(x_1(s_i)) - M\tilde{v}(x_1(s_i))\right)^2\right)^{1/2}}{\left(\displaystyle\sum_{i=0}^{2m-1} \left(Mv_{ex}(x_1(s_i))\right)^2\right)^{1/2}}.$$

The element u_{ex} and v_{ex} are the respectively exact solution and $\tilde{u}$ and $\tilde{v}$ the corresponding numerical ones. Here, m is the parameter for the number of collocation points on each boundary curve, see further (20). The sub-index $2M$ indicates that it is the L_2 error of the bending moment M. To calculate Mu on the boundary Γ_2 the expression (24) is used, and similarly Mv on Γ_1 is generated.

The expected exponential convergence order is clearly exhibited in Table 1. We have tested our approach for the mixed problems on other domains and data with the similar results. The quadratures applied are well-established with error estimates both in spaces of smooth functions and in Sobolev spaces. Thus, no real surprise is to be expected. We therefore move on to the biharmonic data completion problem.

Ex. 2: We consider again an arbitrary source point $z_1 \in \mathbb{R}^2 \setminus D$, and construct the needed boundary data functions

$$f(x) = G(x, z_1),\ x \in \Gamma, \quad g(x) = NG(x, z_1), \quad h(x) = MG(x, z_1), \quad x \in \Gamma_2. \tag{42}$$

m	$e_{2M}(\Gamma_2)$	$\tilde{e}_{2M}(\Gamma_1)$
8	0.012728889	0.0080868877
16	0.000480837	0.0000485082
32	0.000000700	0.0000000002
64	0.0	0.0

TABLE 1. The relative L_2 error for Mu on Γ_2 with u solving (4)–(5), and for Mv on Γ_1 with v solving (11)–(12), for the source point $z_1 = (3,\, 0)$ and the setup of Ex. 1.

The exact solution corresponding to (42) is u_{ex} given by (6) and the computed one is denoted by $\tilde{u}$. Thus, we can compare the exact and the numerically calculated data on the inner boundary Γ_1. We choose the boundary curves as in the first example. The initial guess ζ_0 that starts the procedure is taken to be zero.

The discrete relative L_2 errors of the reconstructions for the data completion problem (1)–(2) on Γ_1 with the proposed iterative approach, that is

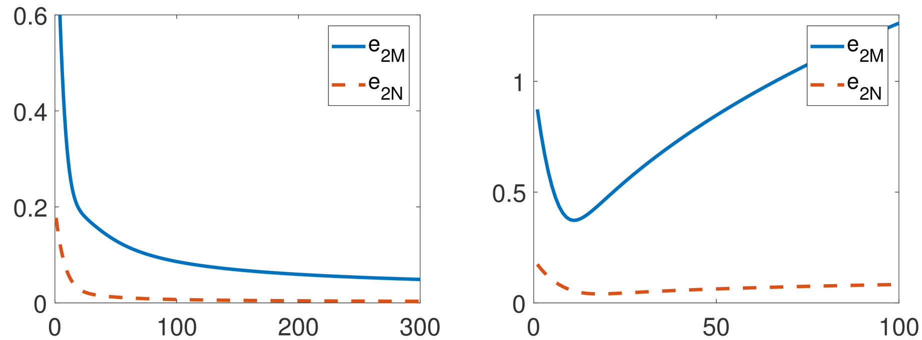
$$e_{2N} = \frac{\left(\displaystyle\sum_{i=0}^{2m-1} (Nu_{ex}(x_1(s_i)) - N\tilde{u}(x_1(s_i))^2\right)^{1/2}}{\left(\displaystyle\sum_{i=0}^{2m-1} (Nu_{ex}(x_1(s_i)))^2\right)^{1/2}},$$

are presented in Fig. 1. The sub-indices $2M$ and $2N$ means the (discrete) L_2 norm of the normal bending moment (M) respectively the normal derivative (N). In generating the collocation (grid) points $m = 32$ was chosen (see further (20)), and in the iterations the relaxation parameter is $\gamma = 0.5$, and the source point is $z_1 = (3, 0)$.

For noisy data, random pointwise errors are added to the corresponding boundary function, with the percentage given in terms of the L_2-norm. The minimal values obtained for the errors are $e_{2N} = 0.00328$ and $e_{2M} = 0.04886$ for exact data after 300 iterations, and $e_{2N} = 0.06675$ and $e_{2M} = 0.44542$ for 3% noise in the data after 18 and 11 iterations, respectively. It is natural that the reconstruction of the normal bending moment Mu is less accurate than the normal derivative Nu, since Mu contains higher derivatives. It is an advantage with the integral approach that also boundary data with derivatives can be completed accurately; it is due to the exact expressions for Mu and Nu that can be derived and discretised on the boundary for the mixed problems.

Table 2 contains the corresponding errors obtained for the data completion (1)–(2) using the "direct" single-layer approach from Section 5, with α being the Tikhonov regularization parameter in (41). The choice of the constants A_0, A_1 and A_2 in (27) is as above. These constants can be chosen freely in both methods.

Comparing with the results of the iterative scheme, for exact data the layer approach is more accurate. However, for noisy data, the iterations produce more accurate reconstructions of the normal derivative Nu and bending moment Mu. This is mainly due to the high condition number of the discretised system (39) making it hard to regularize, whilst in the iterations well-posed mixed problems and well-conditioned systems are solved.



a) Exact data b) 3% noise in the data g

FIGURE 1. Reconstruction on the inner boundary Γ_1 in Ex. 2

	$\delta = 0\%$			$\delta = 3\%$		
M	α	e_{2M}	e_{2N}	α	e_{2M}	e_{2N}
8	1E−05	4.0752E−02	4.7541E−03	1E−02	0.23812	0.02612
16	1E−07	8.1073E−03	2.5088E−04	1E−02	0.23534	0.02959
32	1E−10	3.4936E−05	9.9830E−07	1E−03	0.23824	0.03871
64	1E−10	3.4553E−05	9.8739E−07	1E−02	0.23239	0.02678

TABLE 2. The errors in the second example calculated on Γ_1 for the "direct" single-layer approach with exact ($\delta = 0\%$) and noisy data ($\delta = 3\%$).

The reconstructions of Nu and Mu corresponding to Fig. 1 and Table 2 follow the respectively exact solution well as indicated by the reported error levels. Plots of the reconstructions versus exact solutions are of the form similar to what have been reported in for example [2, 6, 7], and figures of the approximations are therefore left out.

7. **Conclusion.**

A boundary data completion problem has been investigated for the biharmonic equation in planar annular domains. The missing normal derivative and bending moment on the inner boundary are reconstructed from known deflection on the boundary and the normal derivative and bending moment on the outer boundary curve. An iterative method was employed solving mixed problems at each iteration step. An efficient solver for these mixed problems based on a single-layer representation was developed. Numerical experiments included showed that accurate and stable numerical solutions can be obtained. As an alternative, a direct single-layer approach was developed for data completion without iterations. With exact data the direct layer approach produces reconstructions with higher accuracy than the iterative procedure, whilst for noisy data the situation is reversed.

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