Integral equation method for inverse boundary value problem for biharmonic equation

1 Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a doubly connected domain with a boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is an interior boundary and Γ_2 is an exterior one. And let Γ_1 be unknown.

$$\begin{cases} \Delta^2 u(x) = 0, \quad x \in \Omega, \\ u(x) = \dfrac{\partial u(x)}{\partial n} = 0, \quad x \in \Gamma_1, \\ \dfrac{\partial u(x)}{\partial n} = g(x), \quad x \in \Gamma_2, \\ Mu(x) = q(x), \quad x \in \Gamma_2, \end{cases} \tag{1}$$

$$u(x) = f(x), \quad x \in \Gamma_2, \tag{2}$$

where $\Delta^2 u(x) = \frac{\partial^4}{\partial x_1^4} u(x) + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} u(x) + \frac{\partial^4}{\partial x_2^4} u(x) = 0,\ u(x) = u(x_1, x_2),\ n = (n_1, n_2)$ - unit outward normal vector to Γ, $Mu = \nu \Delta u + (1-\nu)(u_{x_1x_1} u_1^2 + 2u_{x_1x_2} n_1 n_2 + u_{x_2x_2} u_2^2),\ \nu \in (0,1)$ and $g(x), q(x), f(x)$ - some given functions. Solving (1)-(2) consists of finding unknown Γ_1 for given boundary data.

Later, we will consider (1) as "field" equations and (2) as a "data" equation.

2 Some statements from potential theory

The fundamental solution to the biharmonic equation is given by

$$G(x,y) = \frac{1}{8\pi} |x-y|^2 \ln |x-y|, \quad x,\ y \in \mathbb{R}^2. \tag{3}$$

Consider such potentials with density φ defined on Γ:

$$V_1(\varphi)(x) = \int_\Gamma G(x,y)\varphi(y)d\sigma_y - \text{single-layer potential},$$

$$V_2(\varphi)(x) = \int_\Gamma \frac{\partial}{\partial n_y} G(x,y)\varphi(y)d\sigma_y - \text{double-layer potential},$$

The solution of direct boundary problem can be given by a combination of a single-layer and a double-layer potentials. Following theorem states the uniqueness of the solution of (1).

Theorem 1 *The solution of a direct boundary problem* (1) *is given by*

$$u(x) = \sum_{k=1}^{2} \int_{\Gamma_k} \left(G(x,y)\varphi_k(y) + \frac{\partial G(x,y)}{\partial n_y}\psi_k(y) \right) d\sigma_y + \omega(x), \quad x \in \Omega, \quad (4)$$

where $\omega(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$ $((\alpha_0, \alpha_1, \alpha_2) \in R^3), \varphi_k, \psi_k \in C(\Gamma_k), k = 1, 2,$ *and exists as a unique solution to the system of given integral equations*

$$\begin{cases} \sum_{k=1}^{2} \int_{\Gamma_k} \left(G(x,y)\varphi_k(y) + \frac{\partial G(x,y)}{\partial n_y}\psi_k(y) \right) d\sigma_y + \omega(x) = 0, \ x \in \Gamma_1, \\ \sum_{k=1}^{2} \int_{\Gamma_k} \left(\frac{\partial G(x,y)}{\partial n_x}\varphi_k(y) + \frac{\partial^2 G(x,y)}{\partial n_y \partial n_x}\psi_k(y) \right) d\sigma_y + \frac{\partial \omega(x)}{\partial n} = 0, \ x \in \Gamma_1, \\ \sum_{k=1}^{2} \int_{\Gamma_k} \left(\frac{\partial G(x,y)}{\partial n_x}\varphi_k(y) + \frac{\partial^2 G(x,y)}{\partial n_y \partial n_x}\psi_k(y) \right) d\sigma_y + \frac{\partial \omega(x)}{\partial n} = g(x), \ x \in \Gamma_2, \\ \sum_{k=1}^{2} \int_{\Gamma_k} \left(M_x G(x,y)\varphi_k(y) + \frac{\partial M_x G(x,y)}{\partial n_y}\psi_k(y) \right) d\sigma_y + M_x \omega(x) = q(x), \ x \in \Gamma_2, \\ \sum_{k=1}^{2} \int_{\Gamma_k} \varphi_k(y) d\sigma_y = A_0, \\ \sum_{k=1}^{2} \int_{\Gamma_k} (y_1 \varphi_k(y) + n_1(y)\psi_k(y)) d\sigma_y = A_1, \\ \sum_{k=1}^{2} \int_{\Gamma_k} (y_2 \varphi_k(y) + n_2(y)\psi_k(y)) d\sigma_y = A_2, \end{cases} \quad (5)$$

for given $(A_0, A_1, A_2) \in R^3$.

For equation (2) we have

$$\sum_{k=1}^{2}\int_{\Gamma_k}\left(G(x,y)\varphi_k(y)+\frac{\partial G(x,y)}{\partial n_y}\psi_k(y)\right)d\sigma_y+\omega(x)=f(x),\ x\in\Gamma_2, \quad (6)$$

Theorem 2 *The inverse boundary value problem* (1)-(2) *is equivalent to the system of integral equations* (5)-(6).

3 Algorithm for solving inverse boundary problem

The solving of boundary value problem (5)-(6) consists of following iterative process:

- By giving an initial value for Γ_1 we solve direct problem for subsystem (5) and find unknown densities.
- Then, we linearize "data" equation (6) and update the value for Γ_1 by solving linearized (6) for fixed densities, which are known from (5).

We assume that the curve Γ_1 is from star-like curves class. Thus, we define parametrization in polar coordinates given by $x_1(t)=\{r(t)c(t):t\in[0,2\pi]\}$, where $c(t)=(\cos(t),\sin(t))$ and $r:\mathbb{R}\to(0,\infty)$ is a 2π periodic function representing the radial distance from the origin.