Freéchet derivative.

Let D be a doubly connected domain in $\mathbb{R}^2$ with smooth boundary Γ of class C^2. We assume that Γ consists of two disjoints curves Γ_1 and Γ_2, meaning $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$, such that Γ_1 is contained in the interior of Γ_2.

We consider the integral operator

$$(A\phi)(x) = \int_{\Gamma_1} \phi(y)\Phi(x, y)\, ds(y), \quad x \in \Gamma_2. \tag{0.1}$$

The exterior curve has the parametric representation

$$\Gamma_2 = \{x_2(s) = (x_{12}(s), x_{22}(s)), s \in [0, 2\pi]\}.$$

For simplicity, we consider starlike interior curve

$$\Gamma_1 = \{x_1(s) = r(s)(\cos s, \sin s) : \ s \in [0, 2\pi]\}, \tag{0.2}$$

where $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin.

The parametrized operator has the form

$$(A\phi)(s) = \int_0^{2\pi} \phi(\sigma)\Phi(x_2(s), x_1(\sigma))d\sigma.$$

Let q is the radial function of the perturbed interior boundary.

The Fréchet derivative $\mathcal{A}$ of the integral operator A with respect to x_1 as a linear operator on q. The integral operator $\mathcal{A}$ can be obtained by formal differentiation of the kernel Φ with respect to x_1

$$\mathcal{A}[\phi, r; q](s) = \int_0^{2\pi} q(\sigma)\phi(\sigma)\Psi(s, \sigma)d\sigma$$

with kernel

$$\Psi(s, \sigma) = \operatorname{grad}_{x_1} \Phi(x_2(s), x_1(\sigma)) \cdot (\cos\sigma, \sin\sigma).$$

For example, let

$$\Phi(x, y) = \ln \frac{1}{|x - y|}, \quad x \neq y.$$

Then

$$\Psi(s, \sigma) = \frac{(x_2(s) - x_1(\sigma)) \cdot (\cos\sigma, \sin\sigma)}{|x_2(s) - x_1(\sigma)|^2}.$$