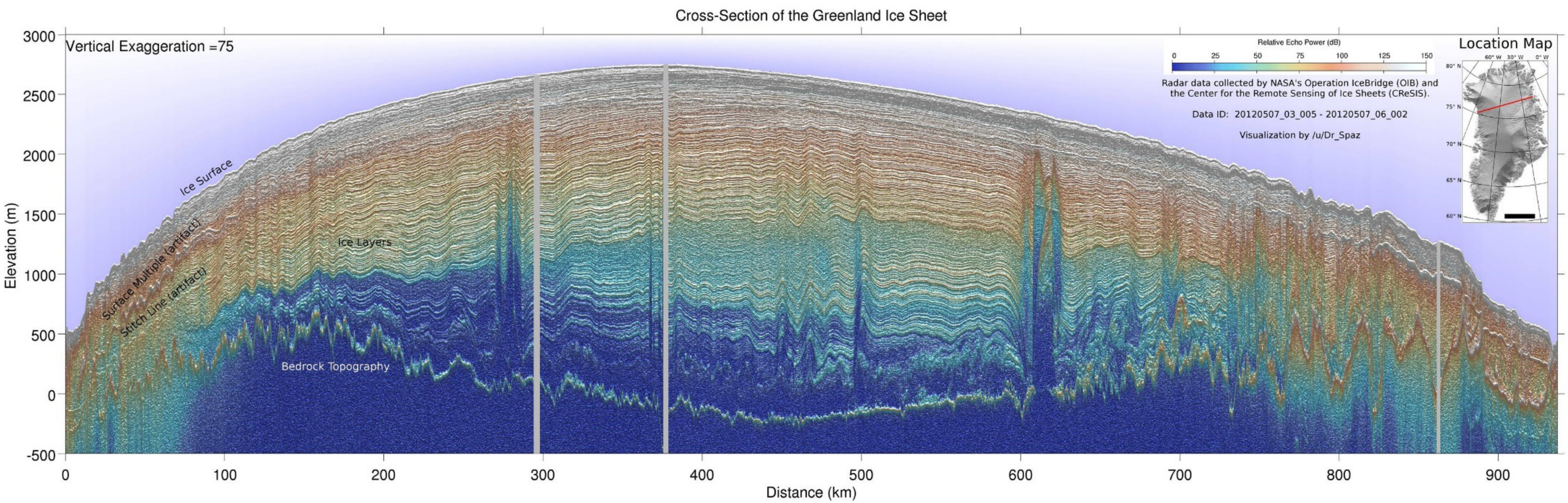

Cross-Section of the Greenland Ice Sheet
Vertical Exaggeration =75
Relative Echo Power (dB)
0 25 50 75 100 125 150
Radar data collected by NASA's Operation IceBridge (OIB) and the Center for the Remote Sensing of Ice Sheets (CReSIS).
Data ID: 20120507_03_005 - 20120507_06_002
Visualization by /u/Dr_Spaz
Location Map
Ice Surface
Surface Multiple (artifact)
Stitch Line (artifact)
Ice Layers
Bedrock Topography
Elevation (m)
3000 2500 2000 1500 1000 500 0 -500
0 100 200 300 400 500 600 700 800 900
Distance (km)

Lecture 6:
Ice sheet flow

Summary from last lecture

Stress balance equations

Strain rate

Our ice-sheet model:
1. Mass conservation

2. Stress balance

Summary

- Ice deforms through various mechanisms, dislocation creep in the basal plane dominates.
- Rheology is the relationship between stress and strain.
- Nonlinear rheology of ice is usually approximated by Glen's flow law with $n = 3$.
- Stress state is described by the "stress tensor".

Stress-balance equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

x-direction

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

y-direction

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

z-direction

Movie proof of this:
https://github.com/ldeo-glaciology/glaciology_practical/raw/main/derivation_movies/stress%20balance%20z.mov

The full derivation: https://ldeo-glaciology.github.io/glaciology-intro-book/sections/ice_flow/stress_balance_eqns.html

A hierarchy of ice sheet models based on which terms you neglect

- Shallow Ice Approximation
- Shallow Shelf Approximation
- 'Hybrid'
- 'Higher-order'
- 'Full-stokes'

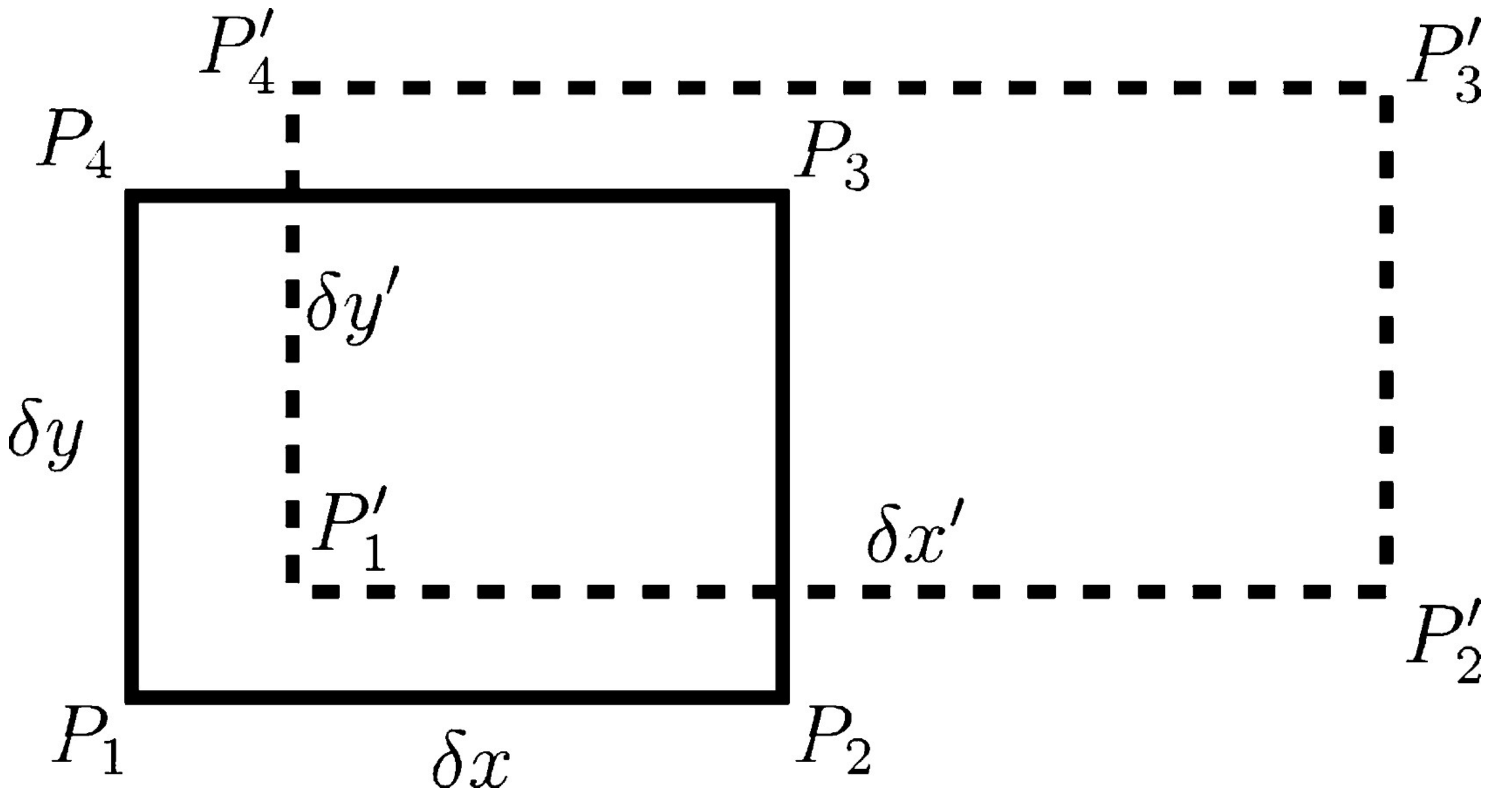
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

Hindmarsh (2004)

Normal strain and velocity gradients



Strain rate

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

$$\boldsymbol{u} = (u, v, w) = \left(u_x, u_y, u_z\right)$$

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} \dot{\boldsymbol{\epsilon}}_{xx} & \dot{\boldsymbol{\epsilon}}_{xy} & \dot{\boldsymbol{\epsilon}}_{xz} \\ \dot{\boldsymbol{\epsilon}}_{yx} & \dot{\boldsymbol{\epsilon}}_{yy} & \dot{\boldsymbol{\epsilon}}_{yz} \\ \dot{\boldsymbol{\epsilon}}_{zx} & \dot{\boldsymbol{\epsilon}}_{zy} & \dot{\boldsymbol{\epsilon}}_{zz} \end{bmatrix}$$

Strain rate

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

For example:

$$\boldsymbol{u} = (u, v, w) = (u_x, u_y, u_z)$$

$$\dot{\epsilon}_{xx} = \frac{\partial u_x}{\partial x}$$

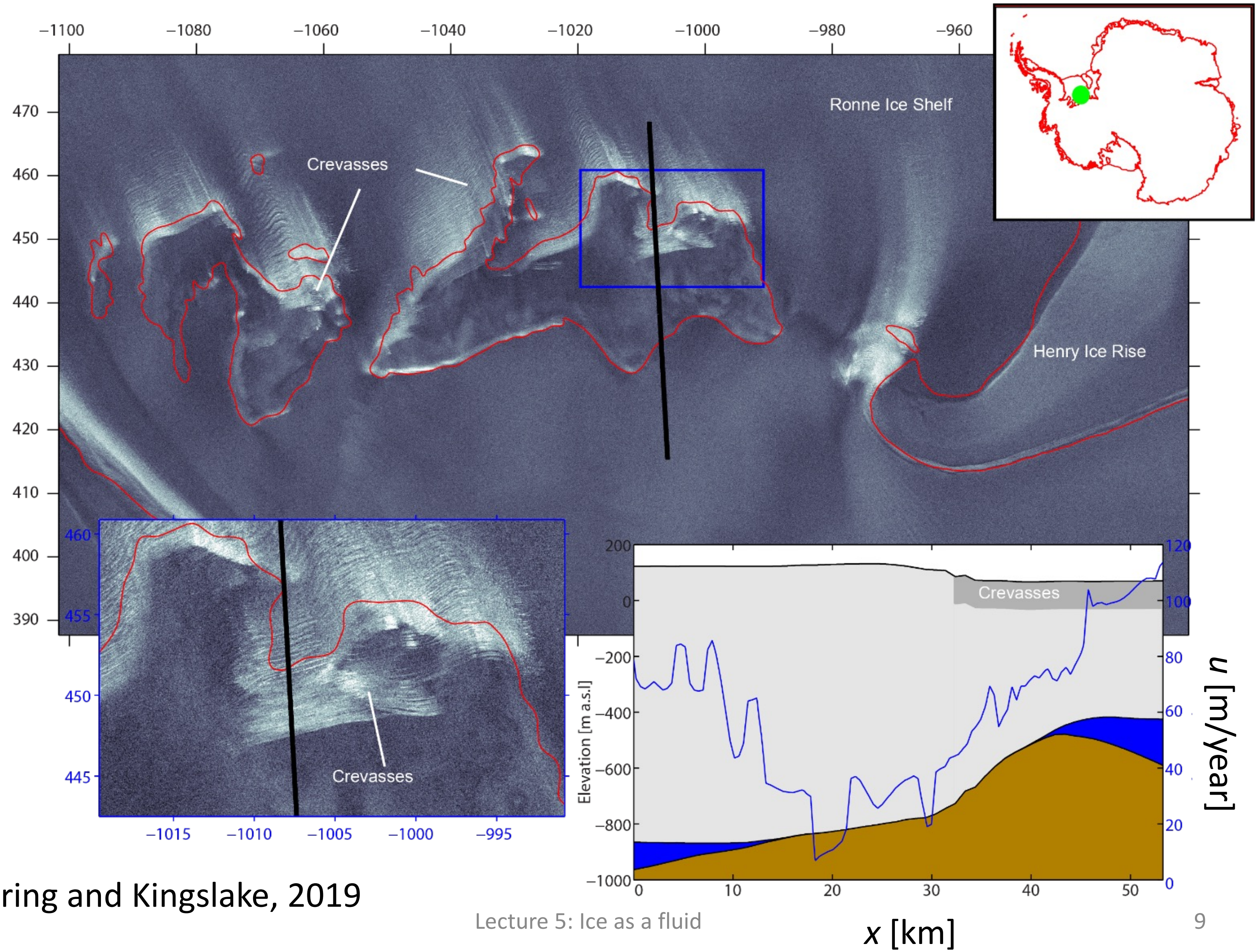
or

$$\dot{\epsilon}_{xz} = \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]$$

Movie proof of this:
https://github.com/ldeo-glaciology/glaciology_practical/raw/main/derivation_movies/strain%20and%20velocity%20gradients.mov

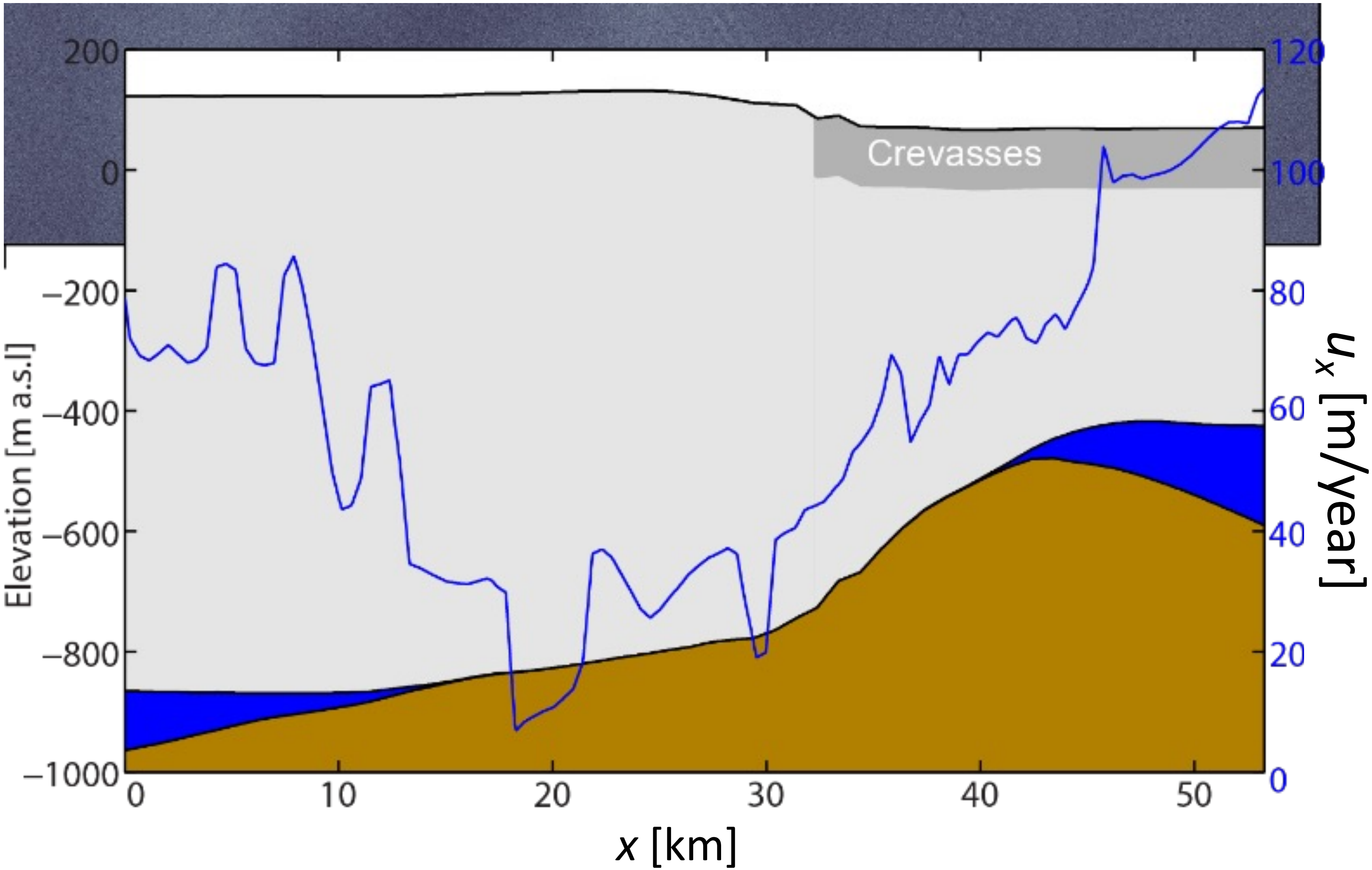
Text proof:
https://ldeo-glaciology.github.io/glaciology-intro-book/sections/ice_flow/strain_velocity.html

A velocity profile from WAIS



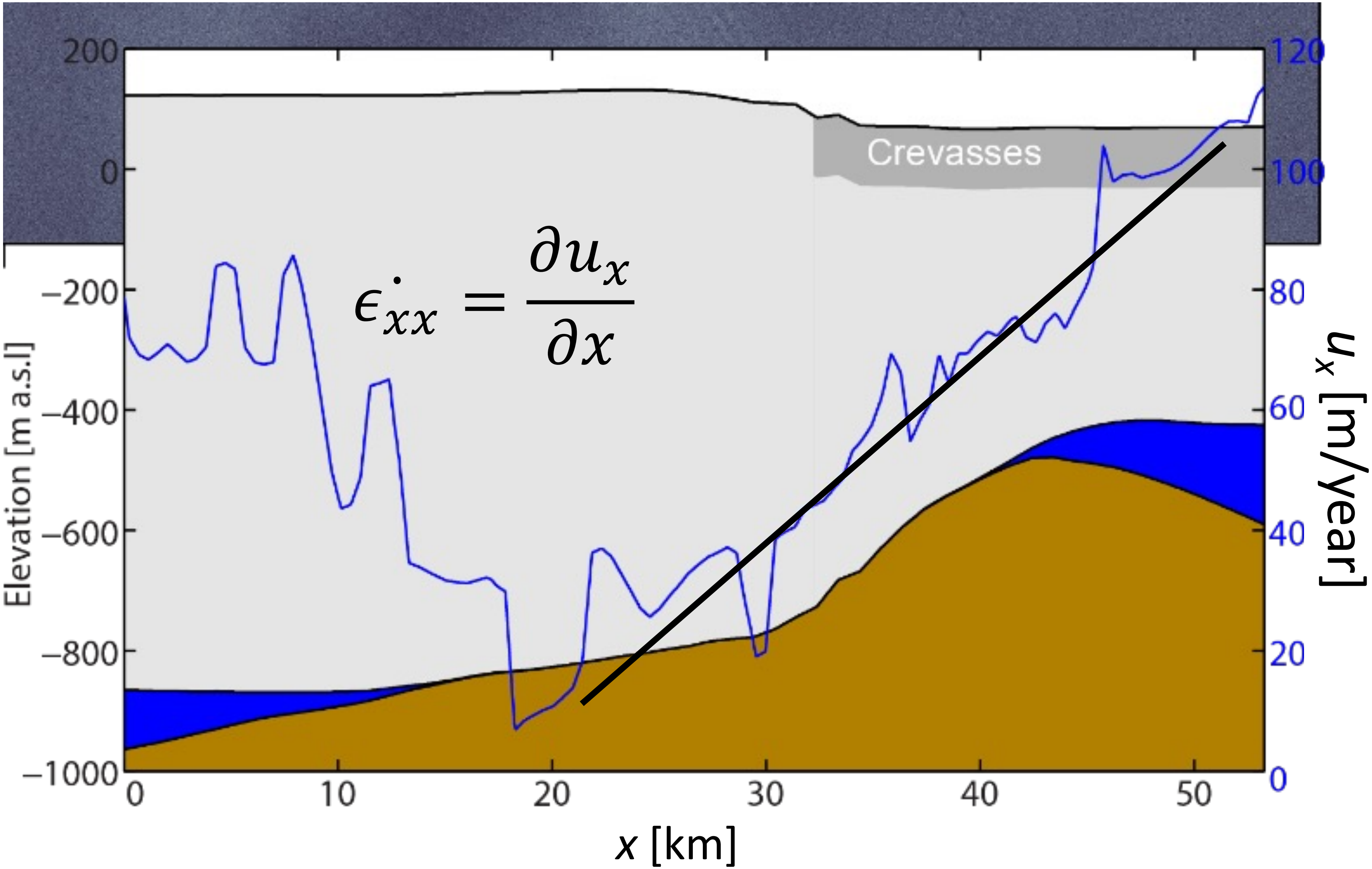
Wearing and Kingslake, 2019

A velocity profile from WAIS



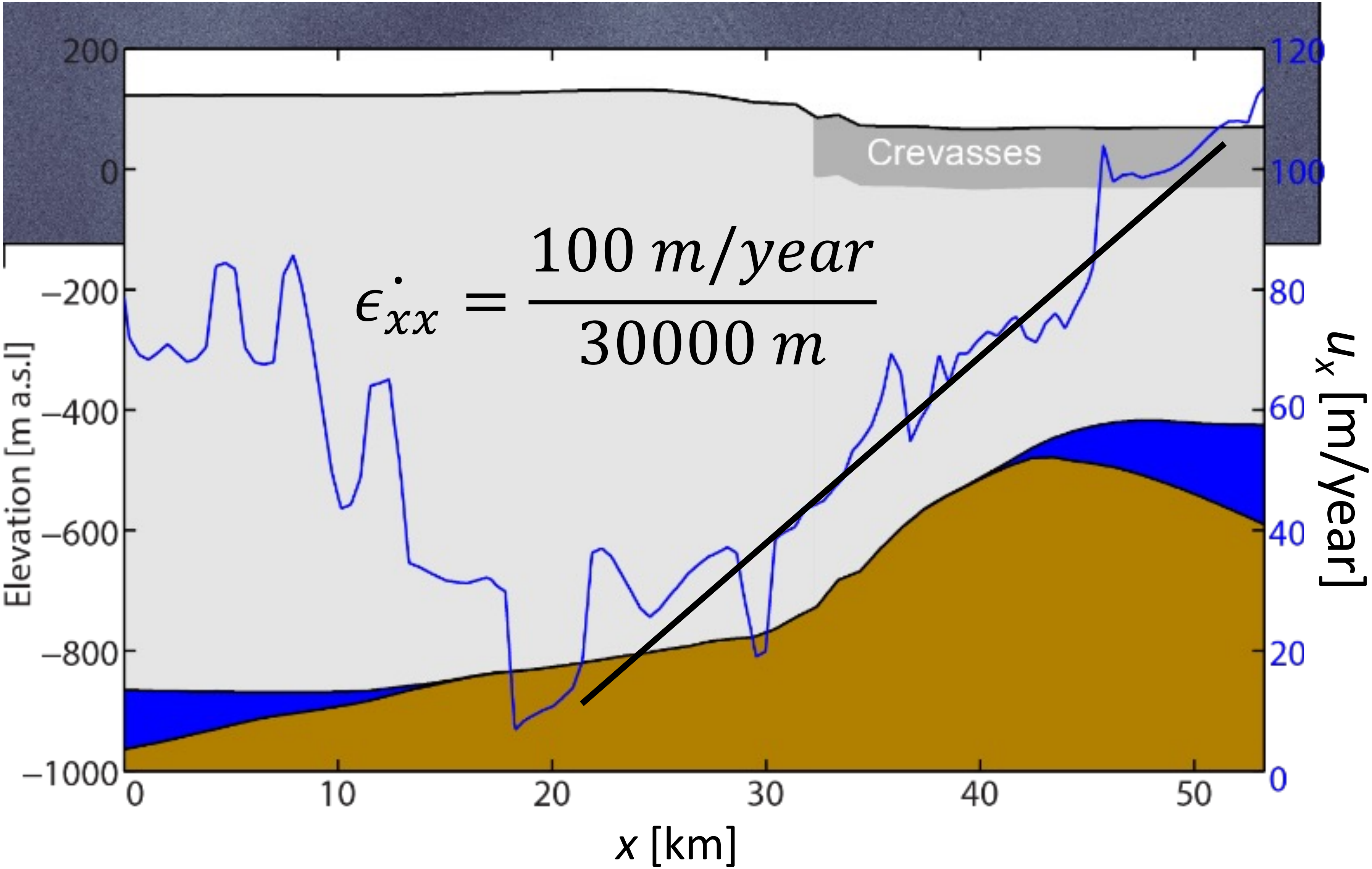
Wearing and Kingslake, 2019

A velocity profile from WAIS



Wearing and Kingslake, 2019

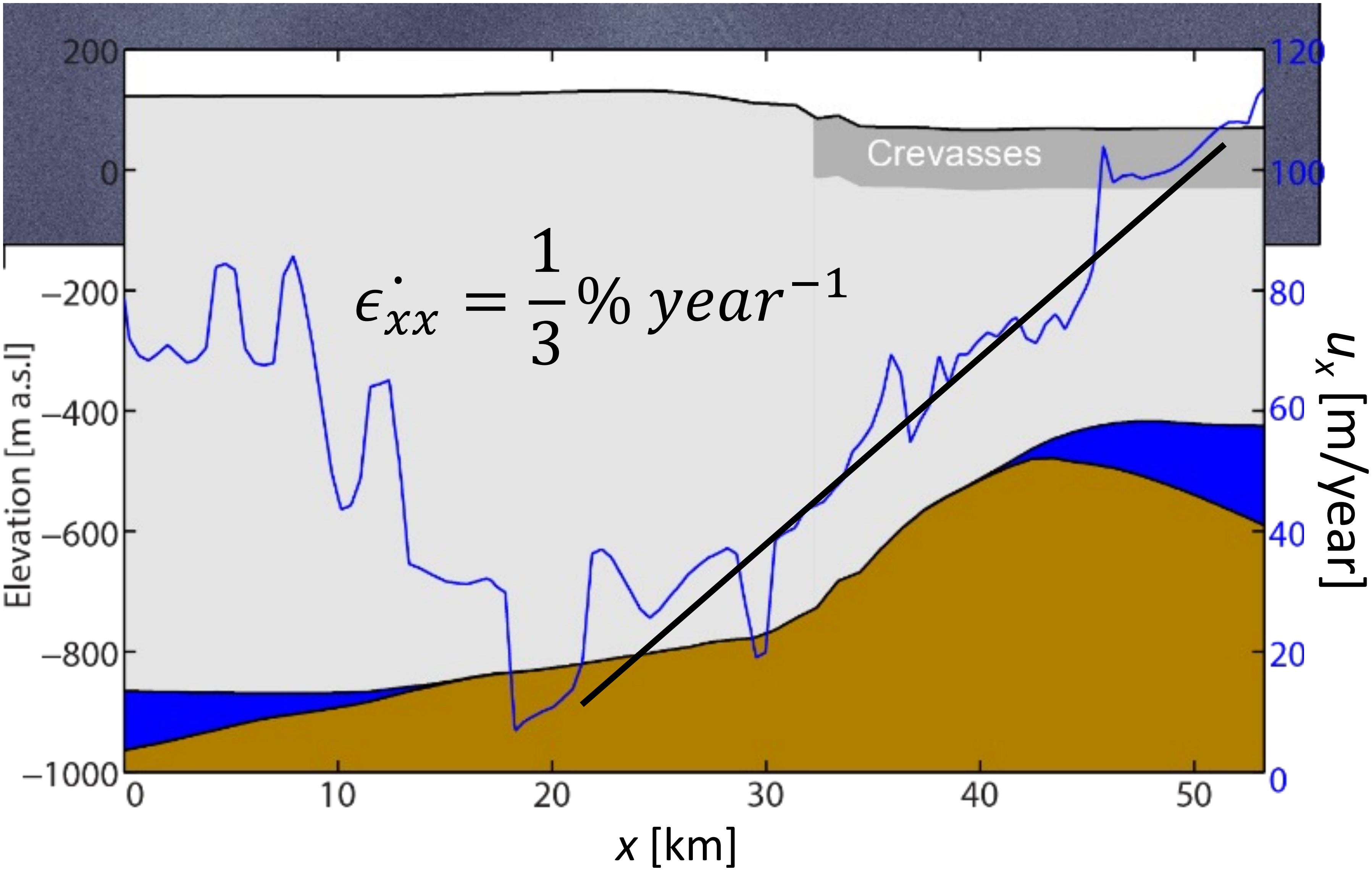
A velocity profile from WAIS



$$\dot{\epsilon}_{xx} = \frac{100\ m/year}{30000\ m}$$

Wearing and Kingslake, 2019

A velocity profile from WAIS



$$\dot{\epsilon_{xx}} = \frac{1}{3}\%\ year^{-1}$$

Wearing and Kingslake, 2019

Summary

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

- Stress balance equations
- Strain rates and velocity gradients
- ‘Glen’s flow law’
- Ice sheet models make various simplifications
- Our ice-sheet model will only consider τ_{zx}

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

$$\dot{\epsilon}_{zx} = A\tau_{zx}^{n}$$

Lecture 7: Ice sheet flow 2

Our ice-sheet model:

1. Mass conservation

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = \dot{b}_i$$

2. Stress balance

$$q = \frac{2A}{n+2}(\rho g \alpha)^n H^{n+2}$$

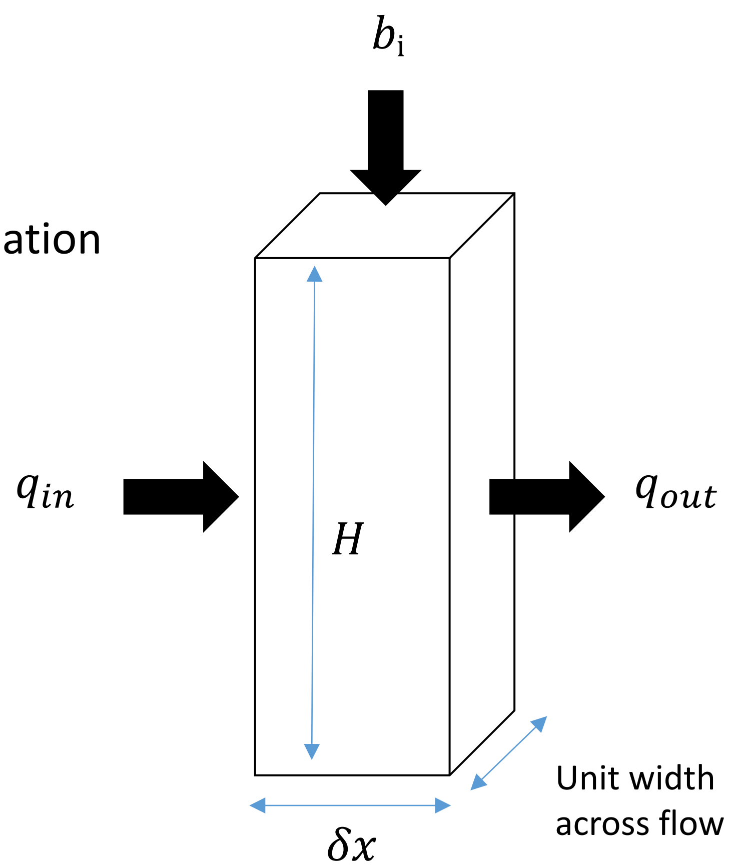
Mass conservation in a column of ice

ice thickness

volume flux
per unit width

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = \dot{b}_i}$$

net accumulation



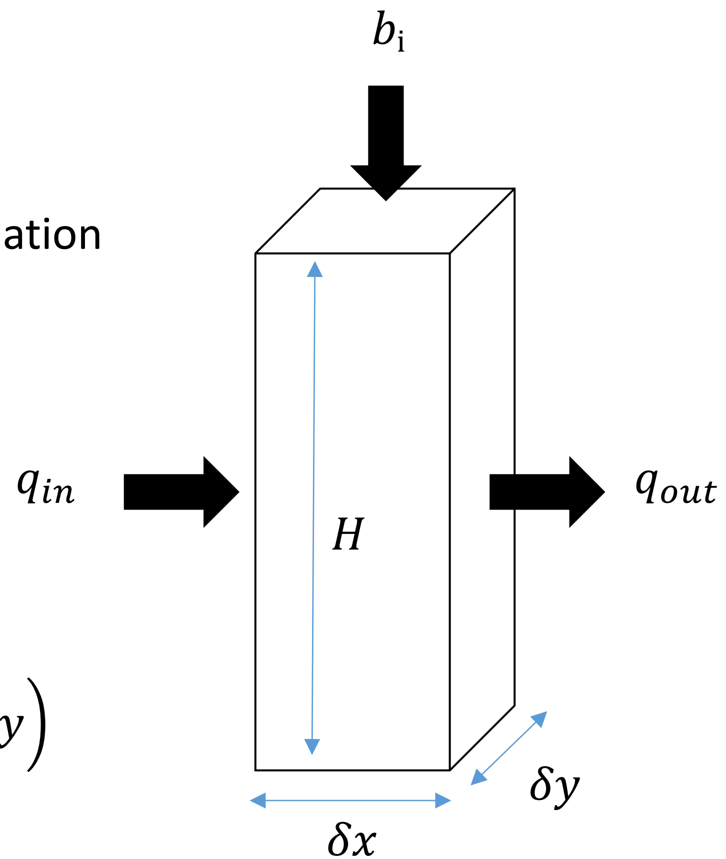
Mass conservation in a column of ice

ice thickness

volume flux
per unit width

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = \dot{b}_i}$$

net accumulation



$$\frac{\partial H}{\partial t}\delta x \delta y = b_i \delta x \delta y + \delta y q_{in} - \delta y q_{out}$$

$$\frac{\partial H}{\partial t}\delta x \delta y = b_i \delta x \delta y + \delta y q - \left(\delta y q + \frac{\partial q}{\partial x}\delta x \delta y\right)$$

$$\frac{\partial H}{\partial t} = b_i - \frac{\partial q}{\partial x}$$

Detailed derivation online here: https://ldeo-glaciology.github.io/glaciology-intro-book/sections/ice_flow/depth_integrated_mass_balance.html

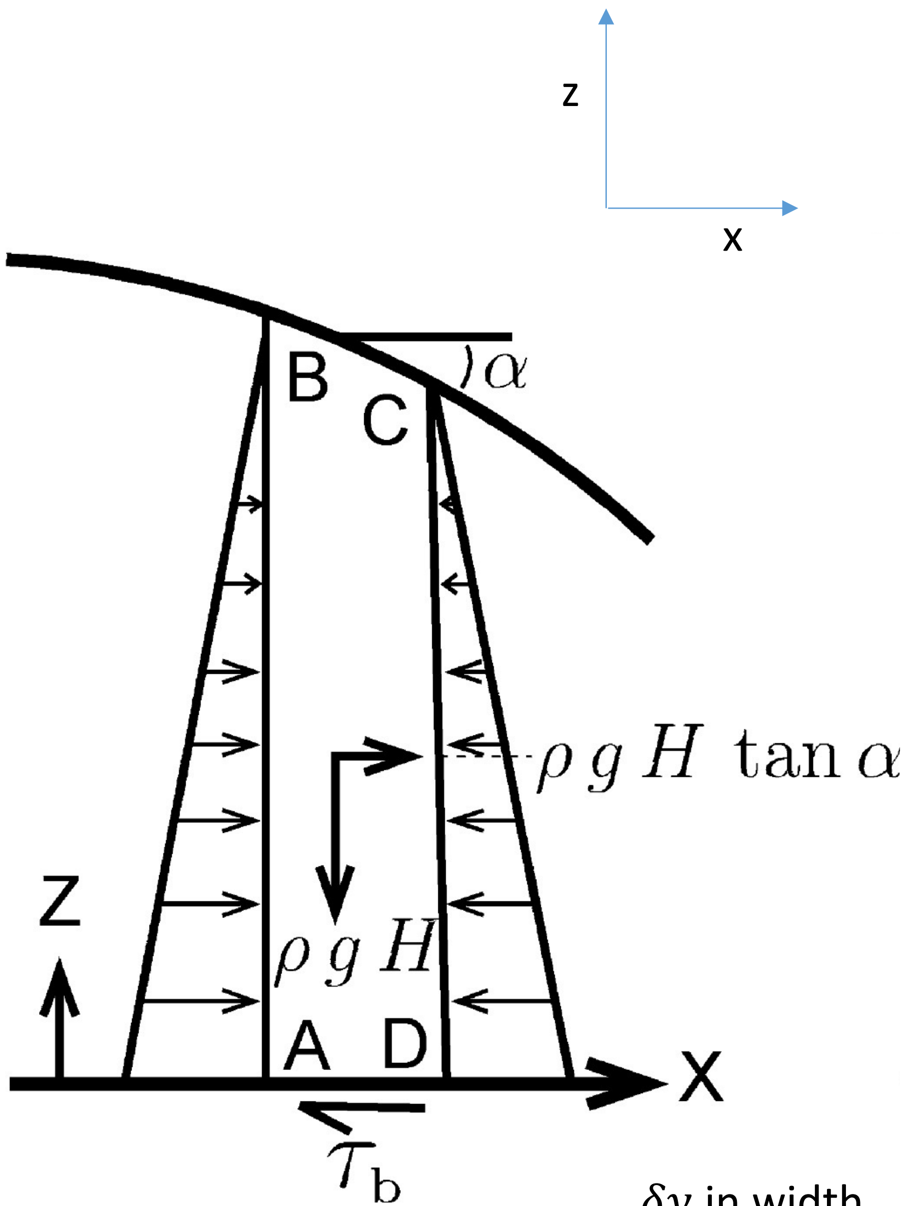
Stress balance

Plan:

- Derive ‘driving stress’
- Balance driving stress with viscous stresses in the ice
- Relate these viscous stresses to strain rate with Glen’s flow law
- Relate strain rates to gradient in velocity
- Integrate twice to get flux.

Driving stress

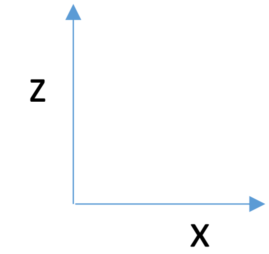
$$\tau_d = -\rho g H \frac{\partial H}{\partial x}$$



δy in width

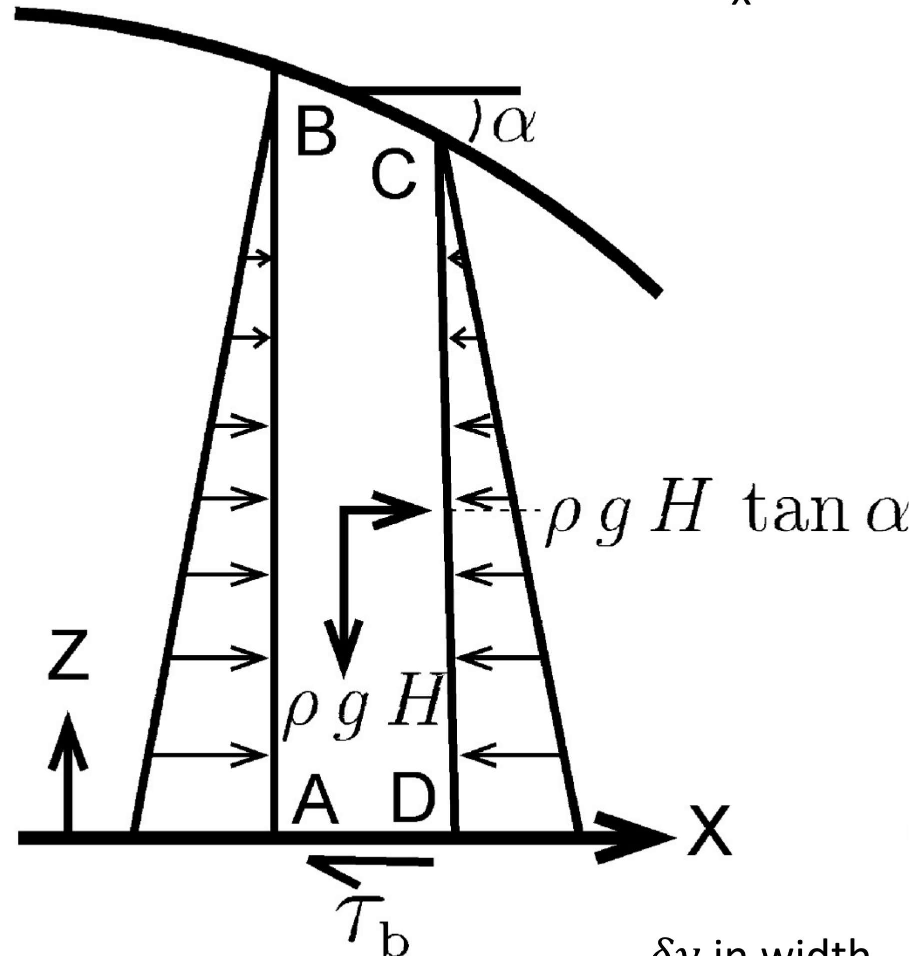
Driving stress

$$\tau_d = -\rho g H \frac{\partial H}{\partial x}$$



Total horizontal force acting in one direction integrated over the ice thickness

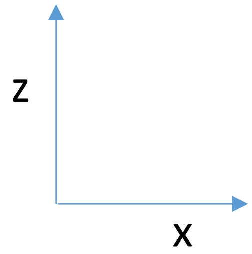
$$\int_0^H \delta y \rho g (H - z)\, dz = \frac{1}{2} \delta y \rho g H^2$$



δy in width

Driving stress

$$\tau_d = -\rho g H \frac{\partial H}{\partial x}$$



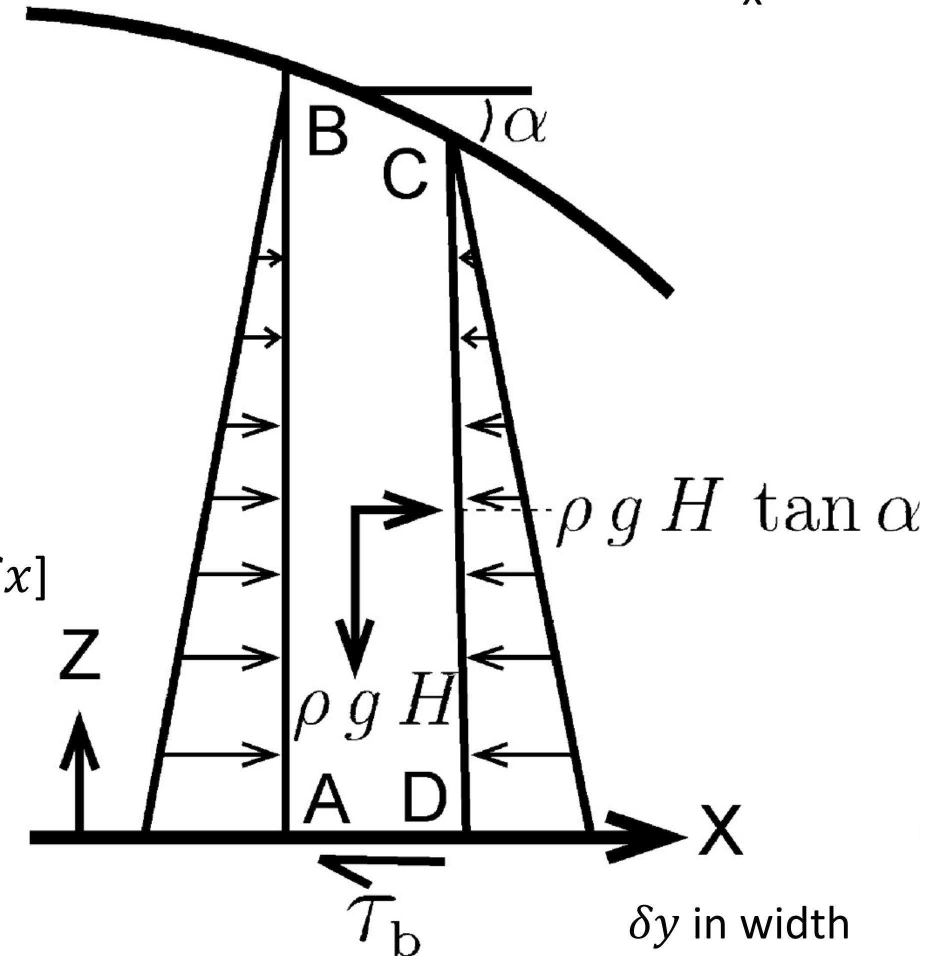
Total horizontal force acting in one direction integrated over the ice thickness

$$\int_0^H \delta y \rho g (H - z)\, dz = \frac{1}{2} \delta y \rho g H^2$$

net horizontal force

$$\frac{1}{2}\delta y \rho g H^2 - [\frac{1}{2}\delta y \rho g H^2 + \frac{\partial}{\partial x}\left(\frac{1}{2}\delta y \rho g H^2\right)\delta x]$$

$$= -\frac{\partial}{\partial x}\left(\frac{1}{2}\delta y \rho g H^2\right)\delta x$$



Driving force per unit area = driving stress

$$\tau_d = -\rho g H \frac{\partial H}{\partial x}$$

Driving stress

$$\tau_d = -\rho g H \frac{\partial H}{\partial x} \quad \longrightarrow \quad \tau_d = \rho g H \alpha$$

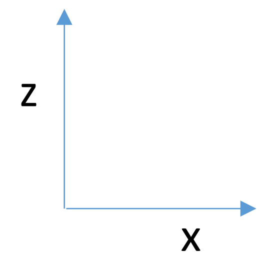
$$\alpha = -\frac{\partial H}{\partial x}$$

- 2-min break

Viscous stresses

We assume the driving stress is balanced by the viscous stresses in the ice.

And we assume that the driving stress is distributed linearly with depth.



driving stress

$$\tau_d = \rho g \alpha H$$

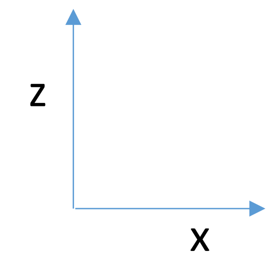
Proof of this: https://ldeo-glaciology.github.io/glaciology-intro-book/sections/ice_flow/sia_derivation.html#integrate-vertically

viscous stresses in the ice

$$\tau_{zx}(z) = \rho g \alpha H \left(1 - \frac{z}{H}\right)$$

Rheology

Next, we use Glen's flow law to relate these stresses to the shear strain rate ϵ_{zx}.



$$\tau_{zx}(z) = \rho g \alpha H \left(1 - \frac{z}{H}\right)$$

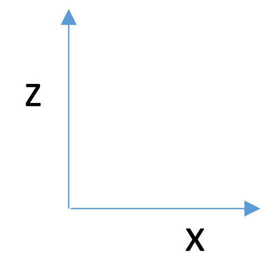
Glen's flow law

$$\boxed{\dot{\epsilon}_{zx} = A\tau_{zx}^{n}}$$

$$\dot{\epsilon}_{zx} = A(\rho g \alpha H)^{n} \left(1 - \frac{z}{H}\right)^{n}$$

Strain rate → velocity gradient

Now we use the definition of strain rate to relate this to the vertical gradient in horizontal velocity.



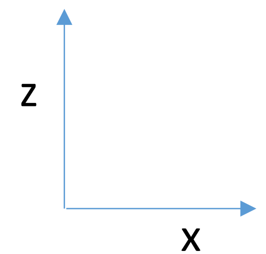
$$\dot{\epsilon}_{zx} = A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

Strain rate

$$\dot{\epsilon}_{zx} = \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$

Strain rate → velocity gradient

Now we use the definition of strain rate to relate this to the vertical gradient in horizontal velocity.



$$\dot{\epsilon}_{zx} = A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

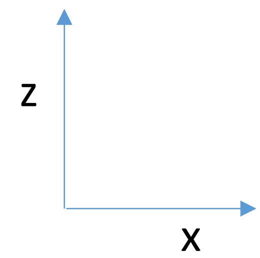
Strain rate

$$\dot{\epsilon}_{zx} = \frac{1}{2}\left[\cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}\right]$$

$$\dot{\epsilon}_{zx} = \frac{1}{2}\frac{\partial u}{\partial z}$$

Strain rate → velocity gradient

Now we use the definition of strain rate to relate this to the vertical gradient in horizontal velocity.



$$\dot{\epsilon}_{zx} = A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

$$\Downarrow$$

$$\frac{\partial u}{\partial z} = 2A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

Strain rate

$$\dot{\epsilon}_{zx} = \frac{1}{2}\left[\cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}\right]$$

$$\dot{\epsilon}_{zx} = \frac{1}{2}\frac{\partial u}{\partial z}$$

Horizontal velocity

We next integrate vertically to get the horizontal velocity as a function of depth.

$$\frac{\partial u}{\partial z} = 2A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

$$\zeta = 1 - \frac{z}{H}$$

Integrate by substitution

$$u(z) = 2A(\rho g \alpha H)^n \left(\frac{H}{n+1}\right)\left(1 - \left(1 - \frac{z}{H}\right)^{n+1}\right)$$

Ice flux

We next integrate vertically again to get ice flux *q*

$$\frac{\partial u}{\partial z} = 2A(\rho g \alpha H)^n \left(1 - \frac{z}{H}\right)^n$$

$$\zeta = 1 - \frac{z}{H}$$

Integrate by substitution

$$u(z) = 2A(\rho g \alpha H)^n \left(\frac{H}{n+1}\right)\left(1 - \left(1 - \frac{z}{H}\right)^{n+1}\right)$$

Integrate one more time to get depth-integrated volume-flux per unit width:

$$q = \frac{2A}{n+2}(\rho g \alpha)^n H^{n+2}$$

Full derivation here: https://ldeo-glaciology.github.io/glaciology-intro-book/sections/ice_flow/sia_derivation.html

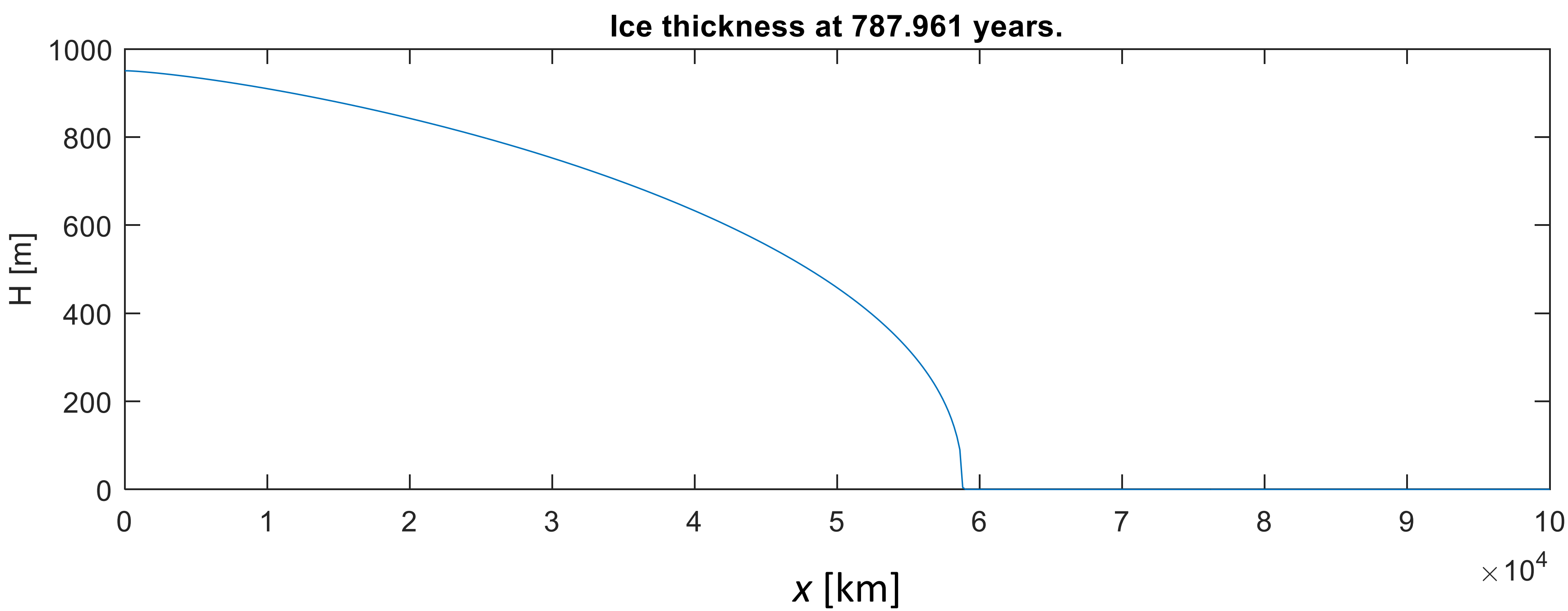
The complete model:

mass conservation:

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = \dot{b}_i$$

Depth-integrated volume-flux per unit width:

$$q = \frac{2A}{n+2}(\rho g \alpha)^n H^{n+2}$$



Flux strongly depends on H. Why?

$$q = \frac{2A}{n+2}(\rho g \alpha)^n H^{n+2}$$

$q \propto H^{n+2}$ With n = 3 that's H^5 !!!

n comes from the nonlinear rheology and the driving stress being proportional to *H*

+1 comes from the fact that thicker ice flows faster

+1 comes from the fact that thicker ice moves more ice for a given flow speed

Summary

- Driving stress drives ice flow. It is proportional to ice thickness and surface slope.
- Ice sheet models use depth-integrated mass conservation to evolve the surface up and down.

  --- a balance between flux coming in, flux going out and net specific accumulation.
- We derived a ’simple’ equations for the flux
- Together the mass conservation and flux equations make up our ice sheet model. s

For Tuesday

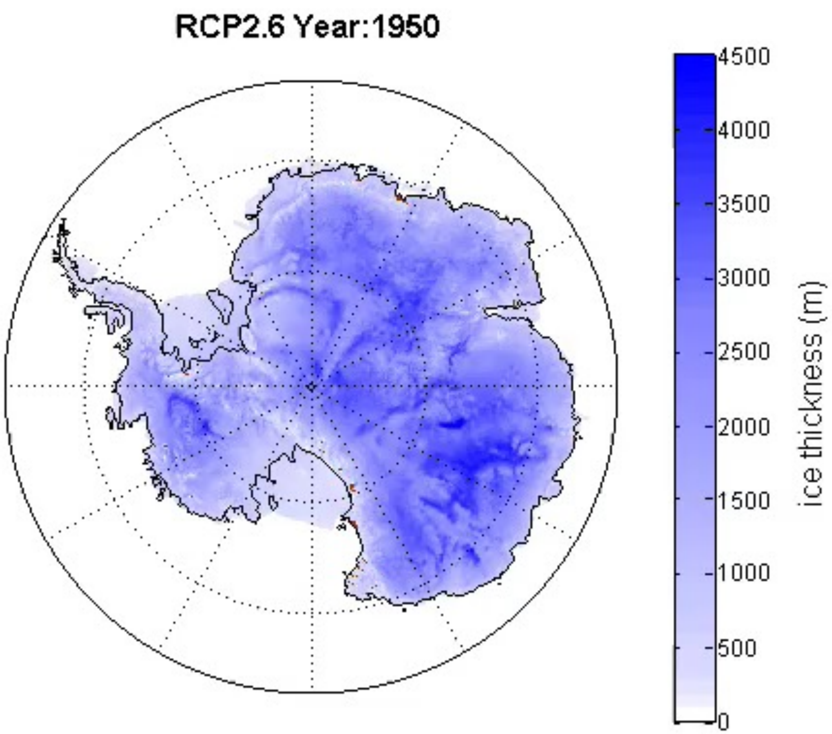
(1) go through these derivations and make sure you understand each step (you don’t have to memorize anything, just make sure you understand each step).

(2) revise the “finite-difference method” of solving differential equations. There are many online resources: e.g.

https://www.ljll.math.upmc.fr/frey/cours/UdC/ma691/ma691_ch6.pdf

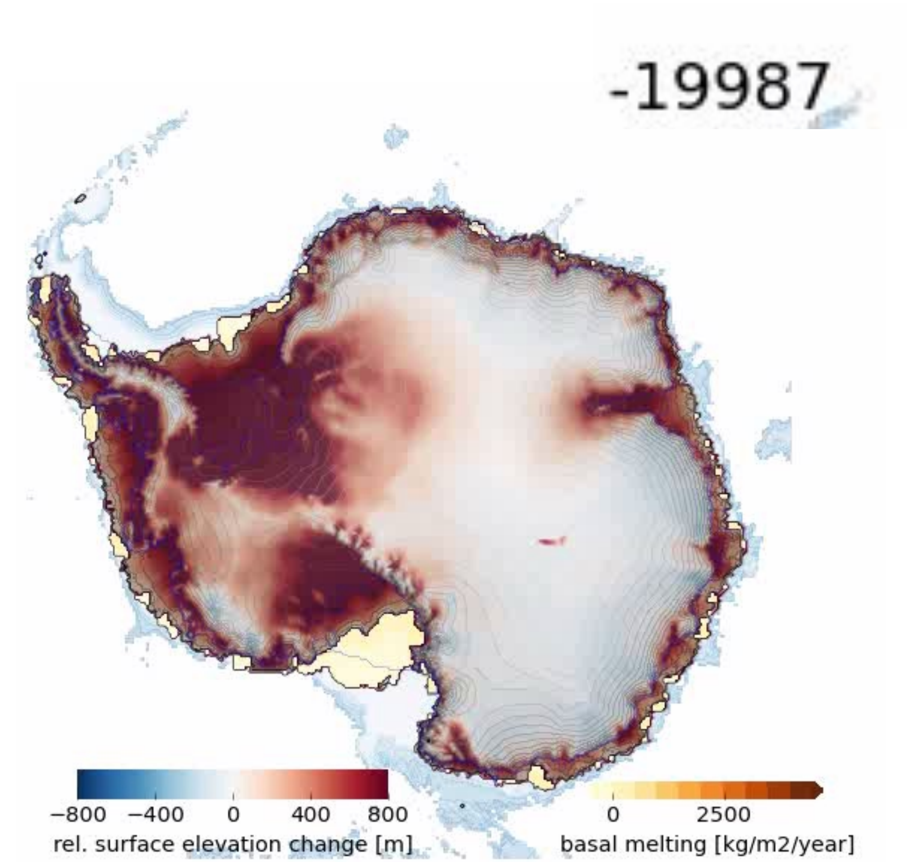
Two state-of-the-art hydrid models

Take into account GIA, ocean/atmosphere forcing, calving, surface mass balance, spatially-variable bed conditions.



DeConto and Pollard (2013)

Penn State University model



Kingslake et al. (*in prep.*)

Parallel Ice Sheet model (PISM)

References

- DeConto, Robert M., and David Pollard. "Contribution of Antarctica to past and future sea-level rise." *Nature* 531, no. 7596 (2016): 591.

- Hindmarsh, R.C.A., 2004. A numerical comparison of approximations to the Stokes equations used in ice sheet and glacier modeling. *Journal of Geophysical Research: Earth Surface*, *109*(F1).