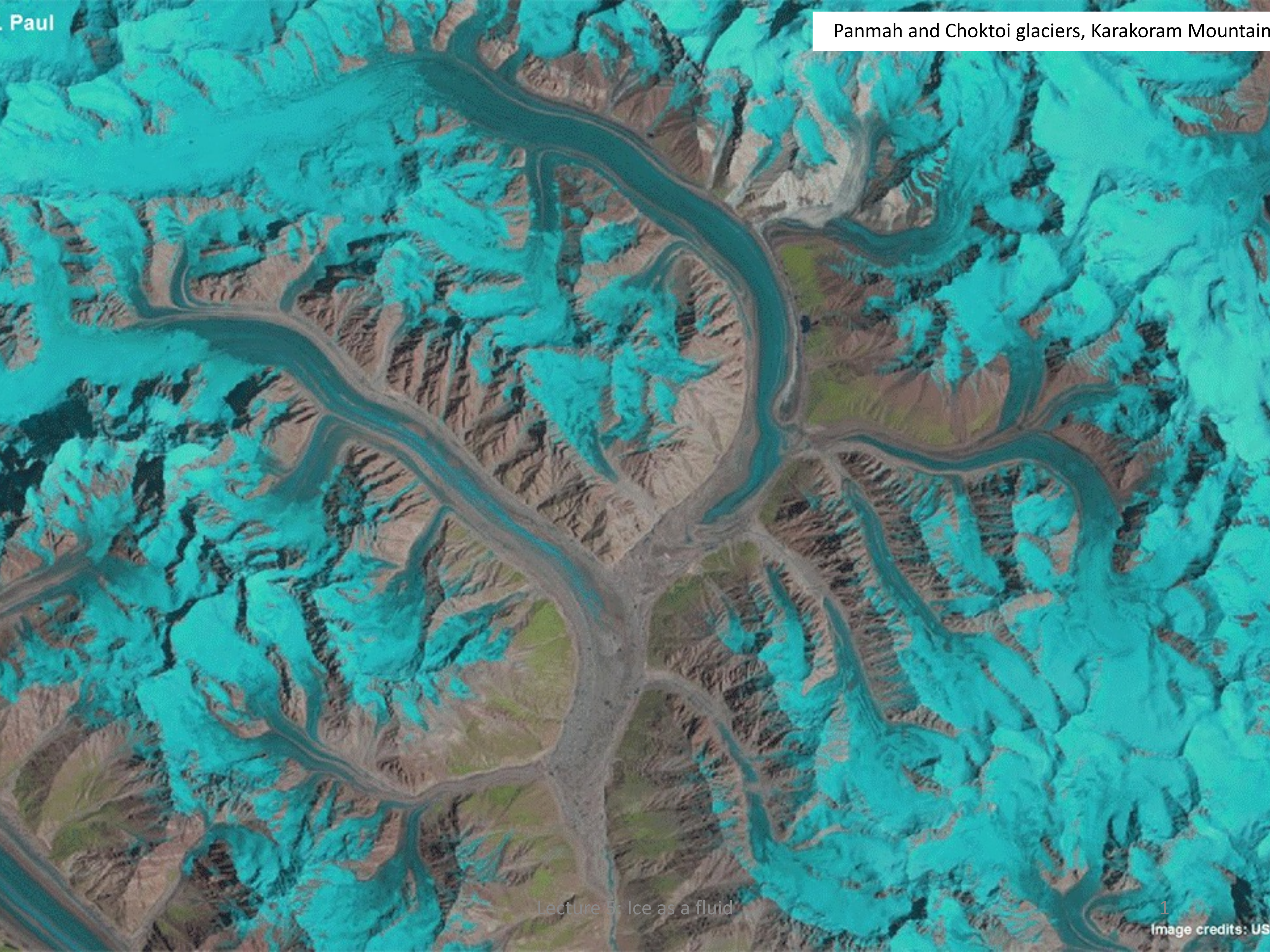


Panmah and Choktoi glaciers, Karakoram Mountain

Lecture 5:
Ice as a nonlinear fluid

Deformation
Deformation mechanisms
A basic flow law for ice
The stress *tensor*
Stress balance equations
Strain rates and velocity gradients.
Effective stress, deviatoric stress and Glen's flow law

Deformation

Elastic deformation

$$\epsilon = \frac{\tau}{E}$$

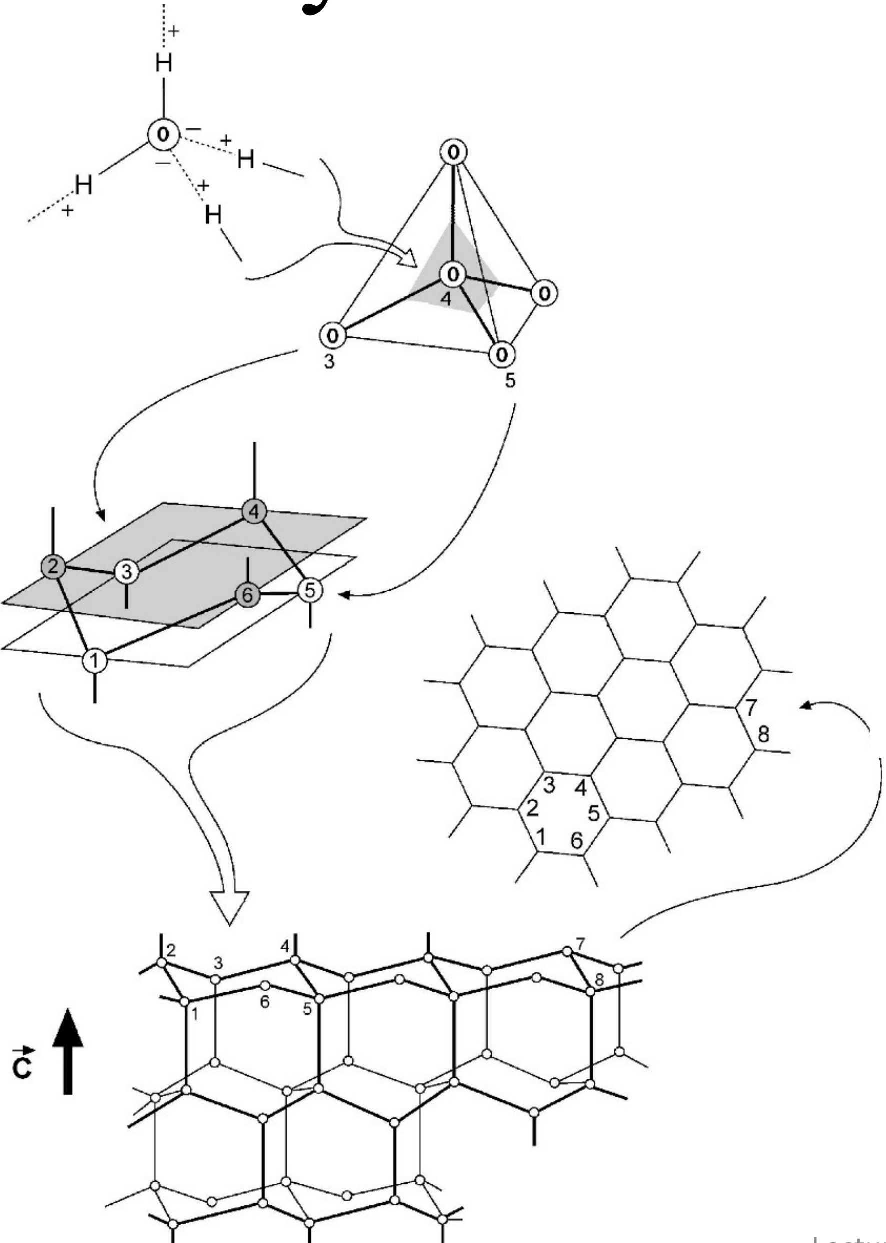
ϵ = strain [-]
τ = stress [Pa]
E = elastic modulus [Pa]

Viscous deformation

$$\dot{\epsilon} = \frac{\tau}{\mu}$$

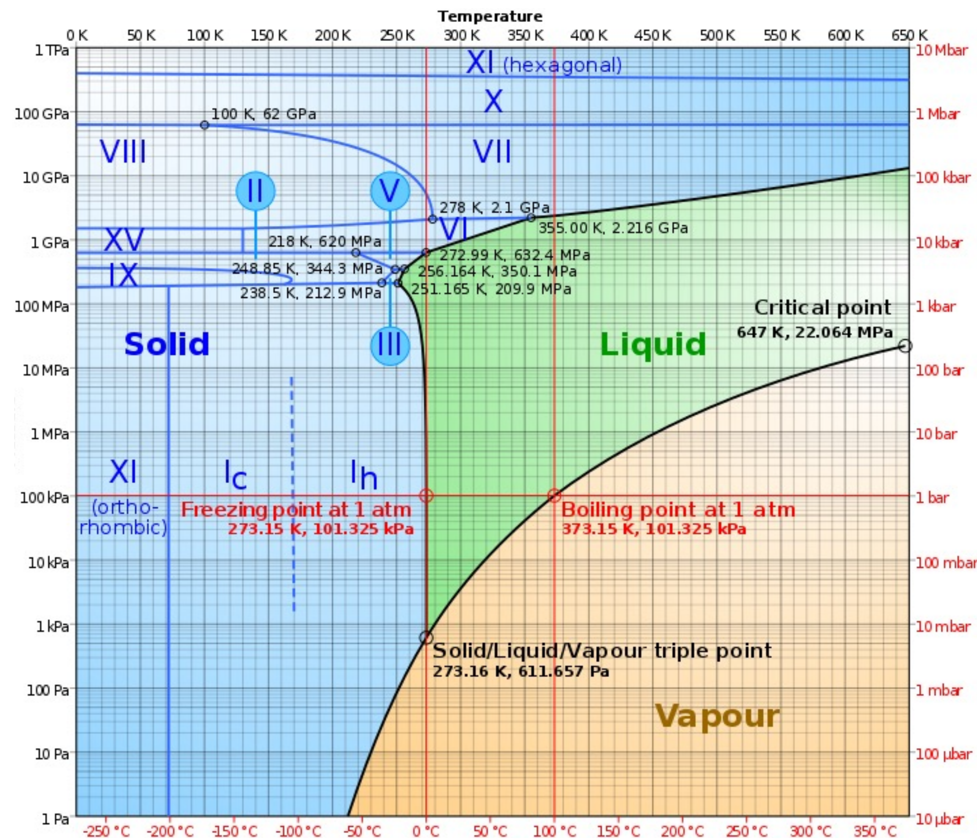
$\dot{\epsilon}$ = strain rate [1/s]
τ = stress [Pa]
μ = viscosity [Pa s]

Ice crystal structure



Temperature

Pressure

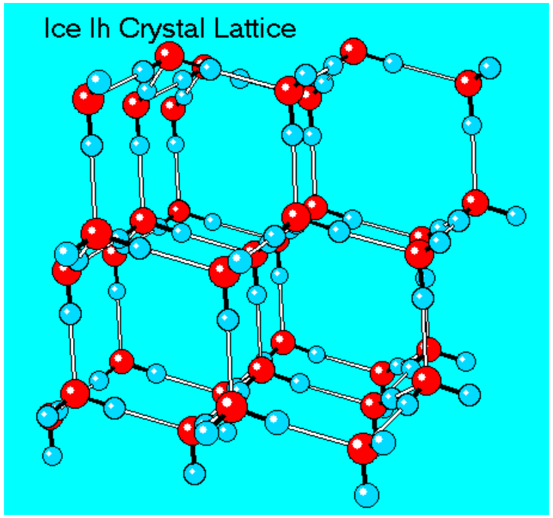


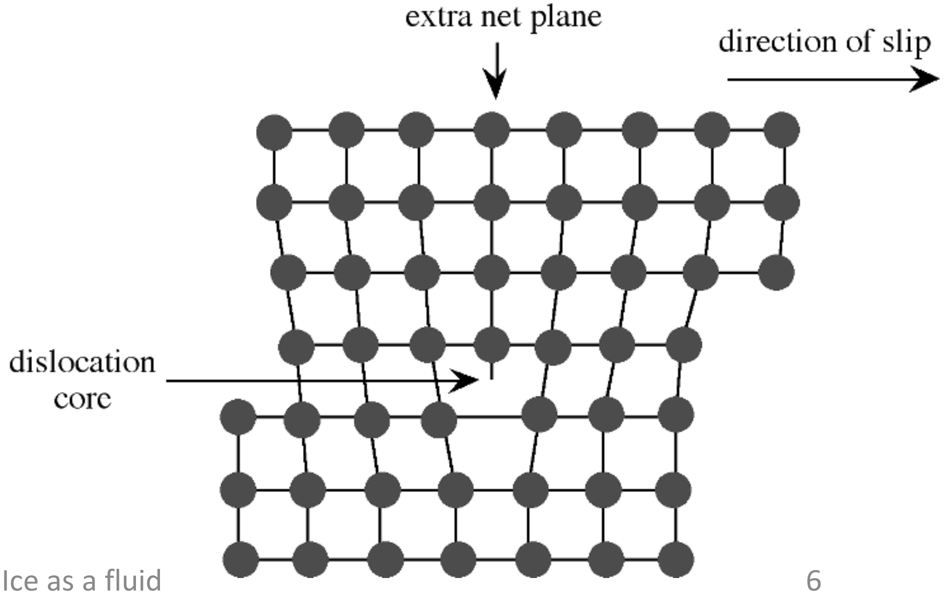
Crystals and Grains

Thin section from a Greenland ice core, viewed in polarized light, Center for ice and Climate, Neils Bohr Institute

Theory: Deformation of single grains

- Dislocation creep
  - Dislocations in lattice move through the crystal.
  - Highly anisotropic
- Depends sensitively on temperature.

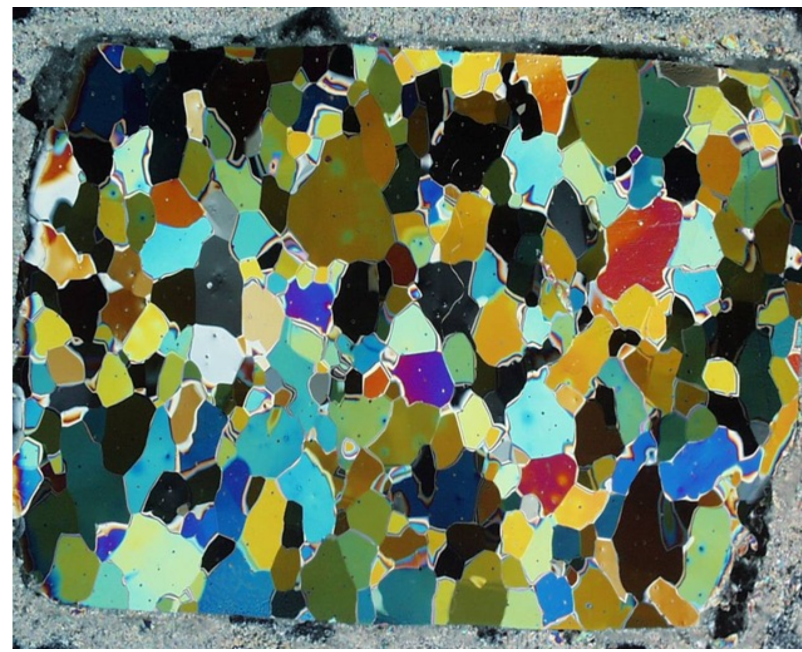




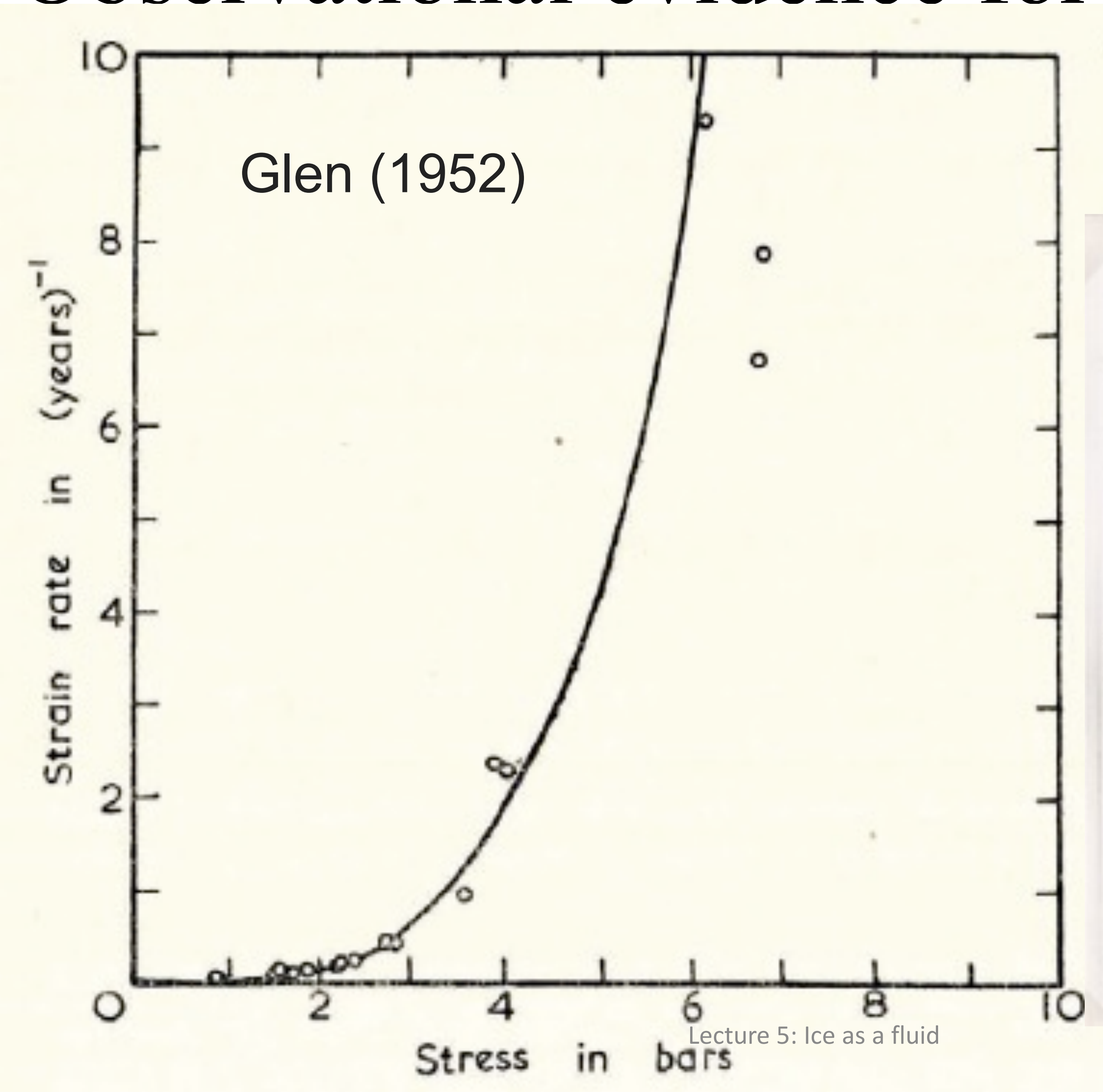
Theory: deformation of polycrystalline ice

- Grain orientation is non-uniform.
- Individual grains deform via dislocation creep ($n \approx$ 3-4).
- Neighboring grains deform at very different rates.
- "Accommodated" by grain boundary mechanisms, such as grain boundary sliding ($n \approx 1.8$).

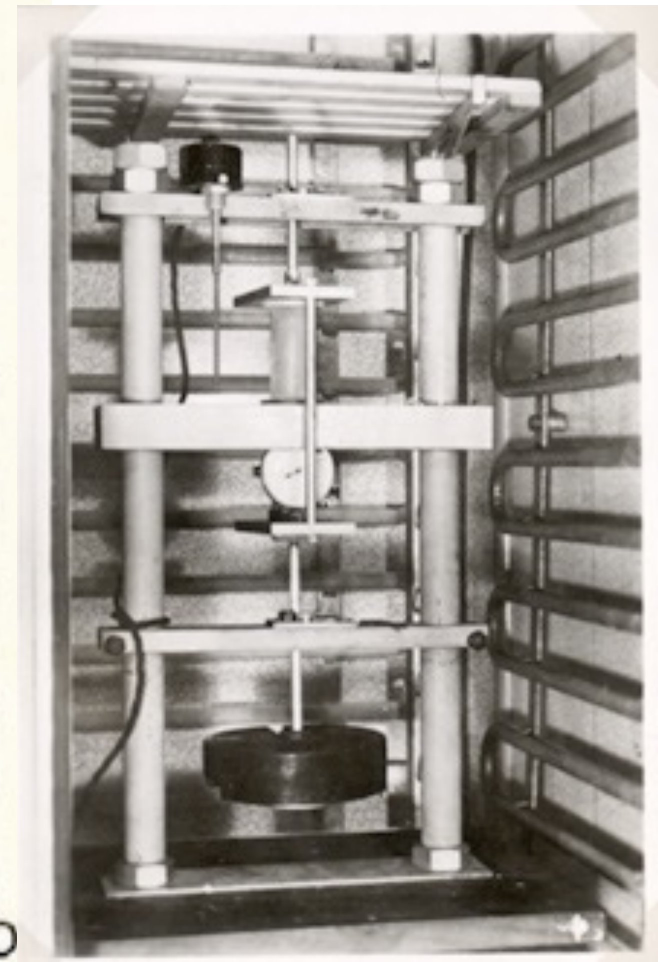
$$\dot{\epsilon} = A\tau^n$$



Observational evidence for n > 1



$$\dot{\epsilon} = A\tau^n$$



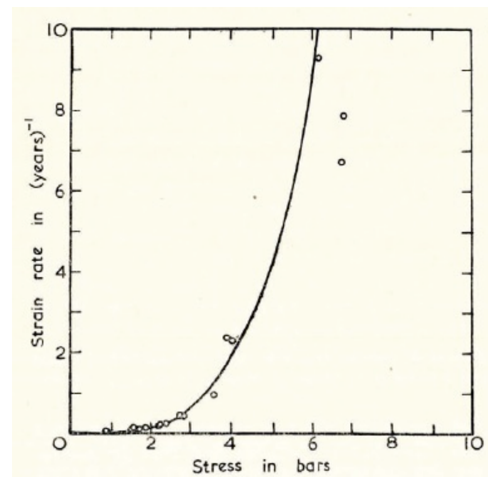
Observational evidence for n > 1

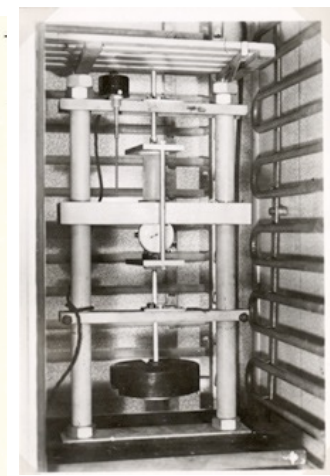
- Lab experiments explore the different stress and strain regimes with varying success.
- Glen (1952) is the canonical experiment.

--> Glen's flow law

- Field observations:
  - Ice shelf spreading rates
  - Borehole closure rates
  - Vertical velocity profiles

The consensus:
$n = 3$ is a good approximation.



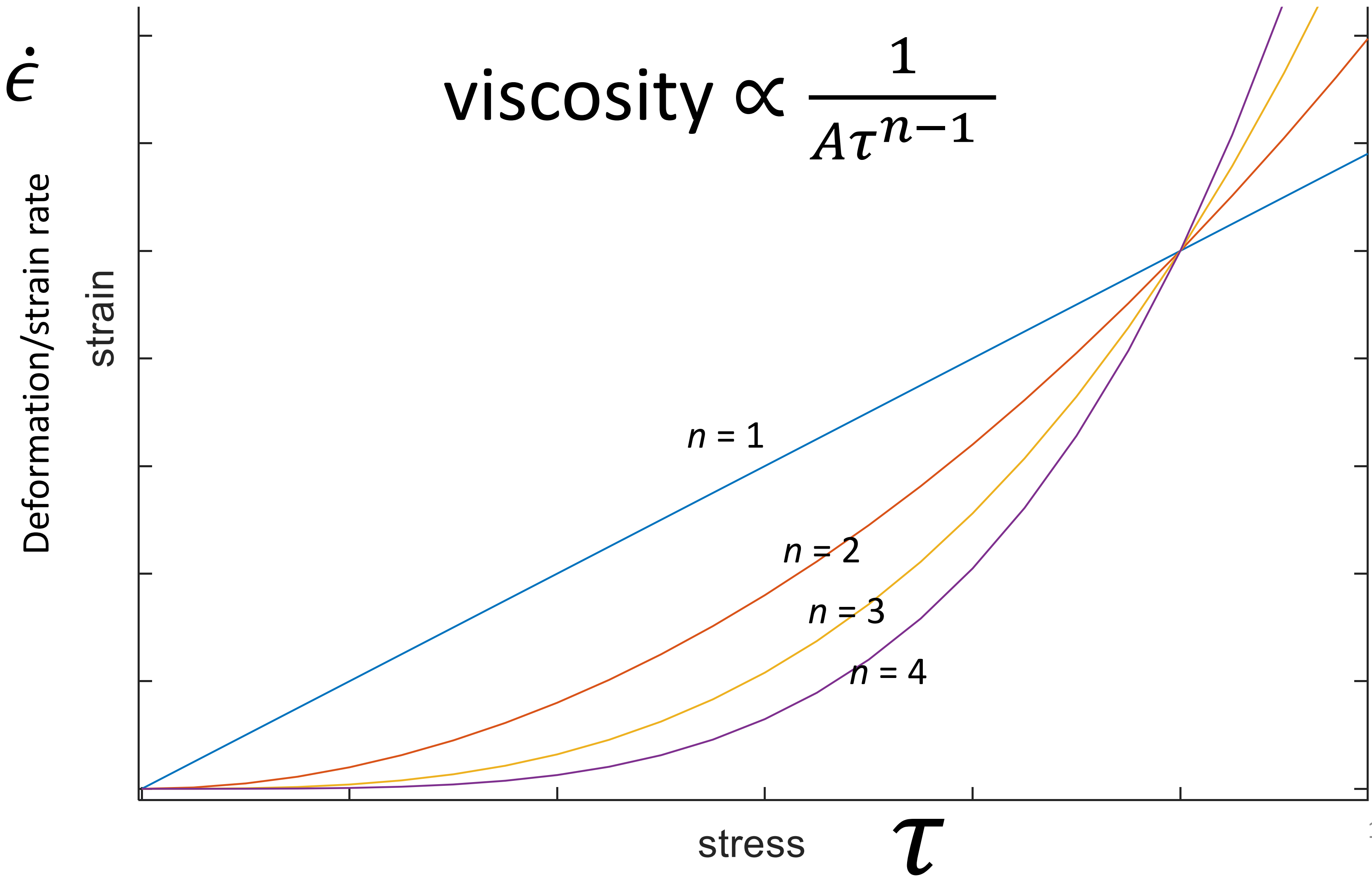


Glen (1952)

$$\dot{\epsilon} = A\tau^n$$

$$\dot{\epsilon} = \frac{\tau}{\mu}$$

$$\dot{\epsilon} = A\tau^n$$



Glen's flow law

$$\dot{\epsilon} = A\tau^n$$

$$\text{viscosity} \propto \frac{1}{A\tau^{n-1}}$$

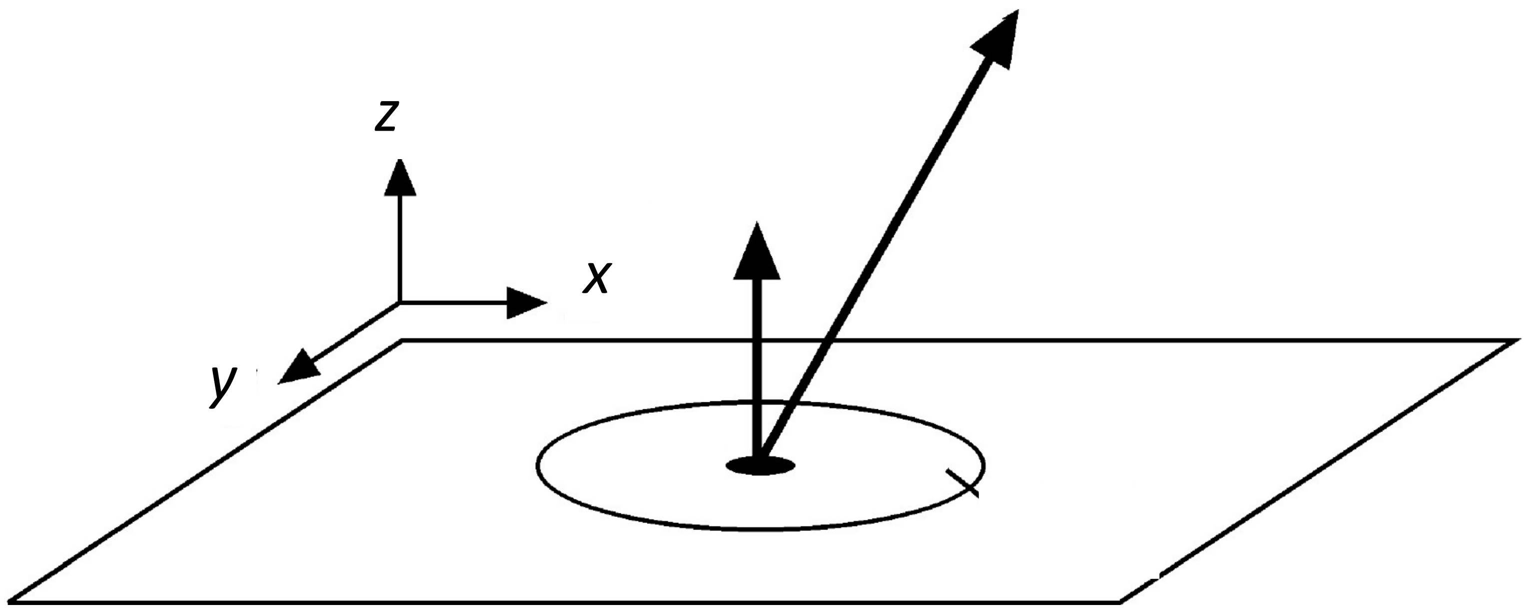
- A varies over several orders of magnitude in glaciers and ice sheets.

- A increases with:

  Temperature
  Pressure
  Water content
  Impurities
  Grain size
  Preferred crystal orientation fabric.

Tightly coupled properties
--> difficult to untangle the effects of each.

Stress



Stress: force per unit area [units: Pa, i.e. N m^{-2}]

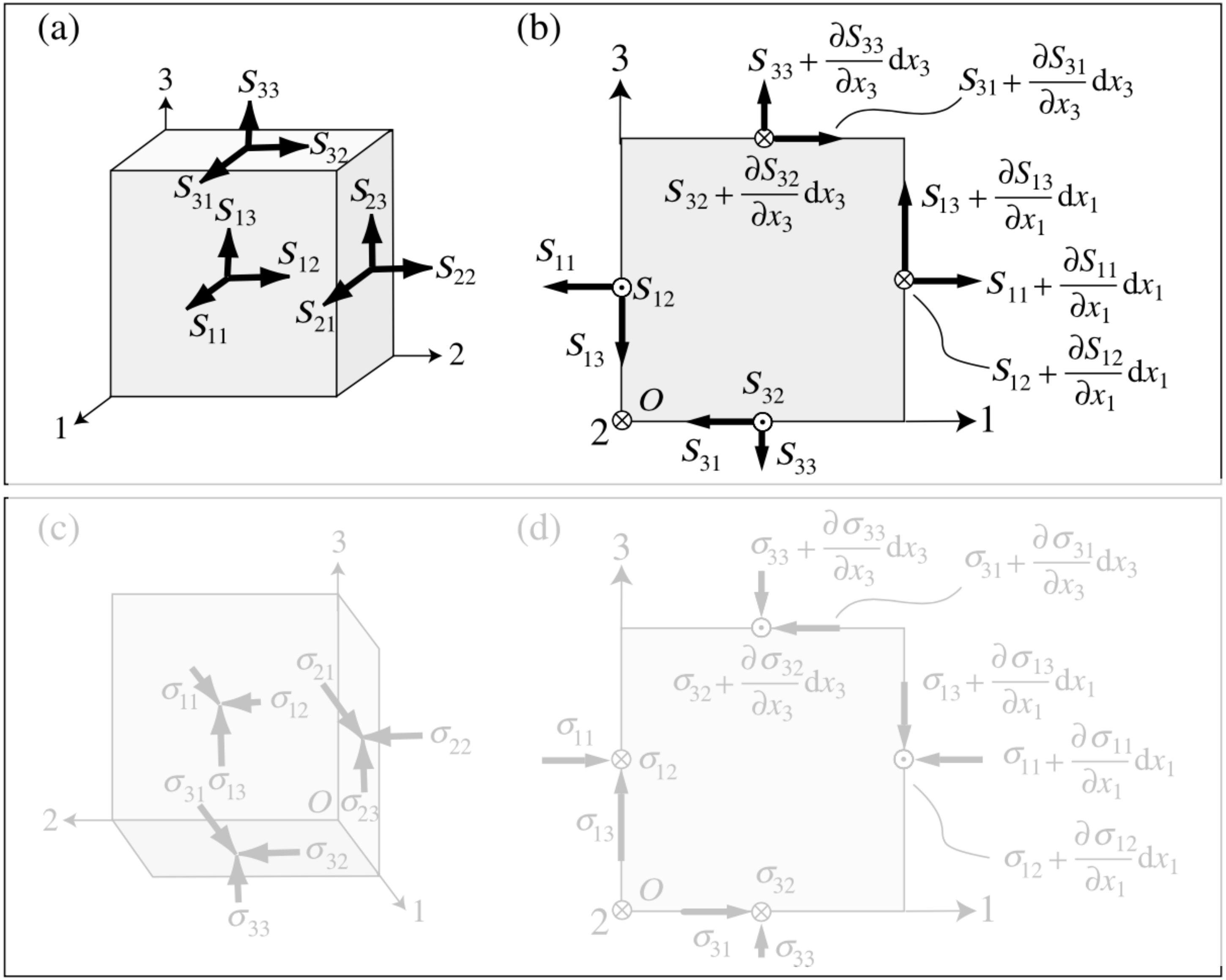
Has components perpendicular to this plane

$$\sigma_{zz}$$

and in the plane

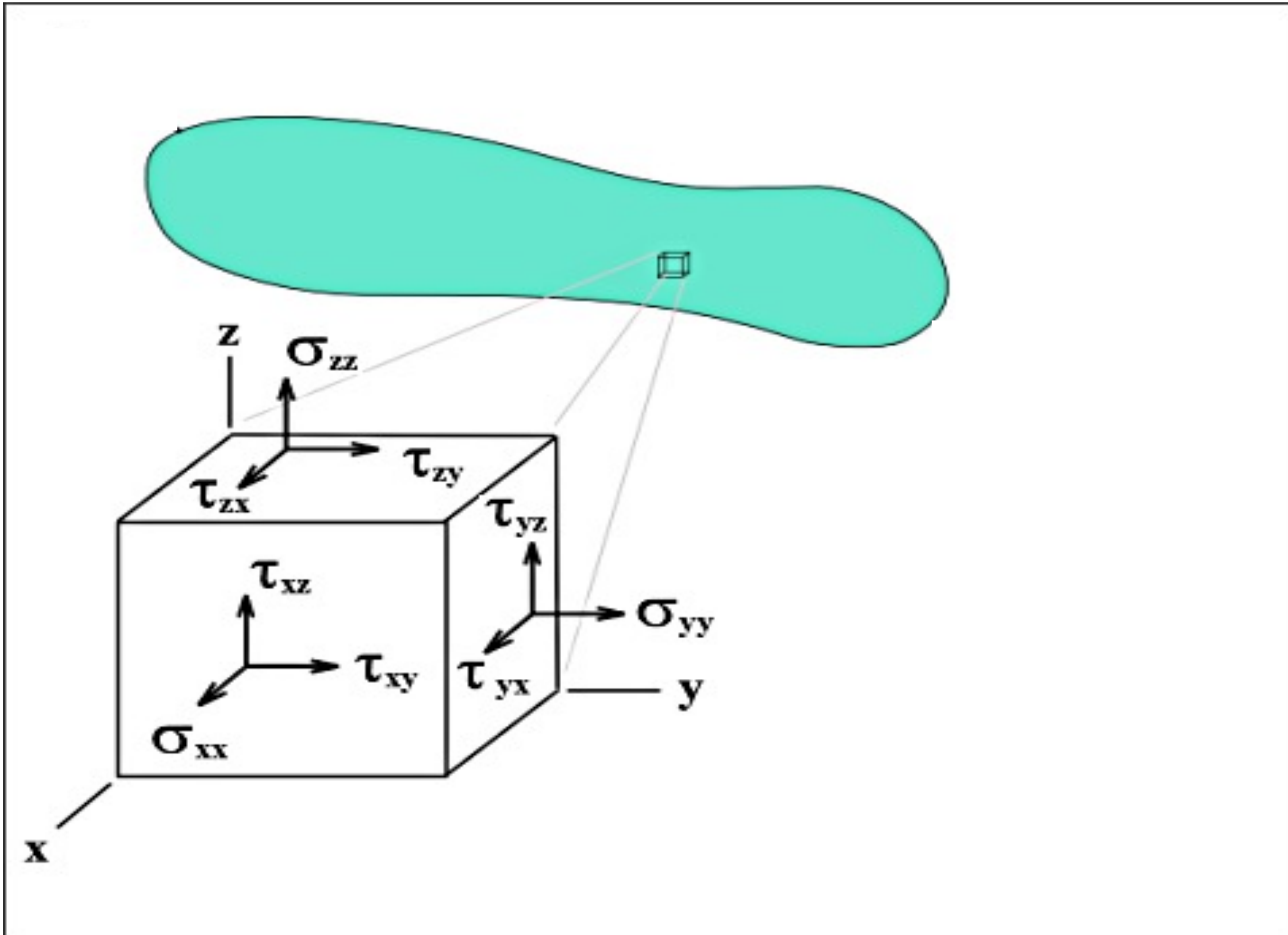
$$\tau_{zy} \quad \& \quad \tau_{zx}$$

Sign convention in glaciology:

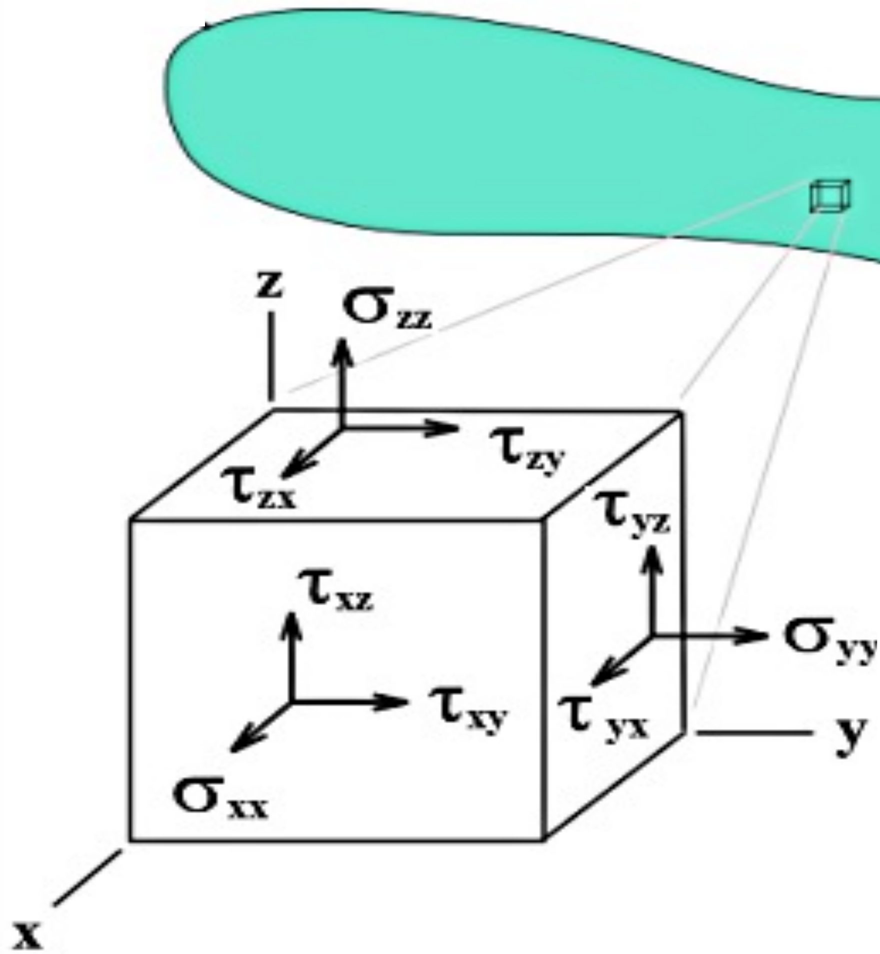


http://www.terrapub.co.jp/e-library/yamaji/pdf/059.pdf

Stress tensor



Stress tensor



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Stress-balance equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

x-direction

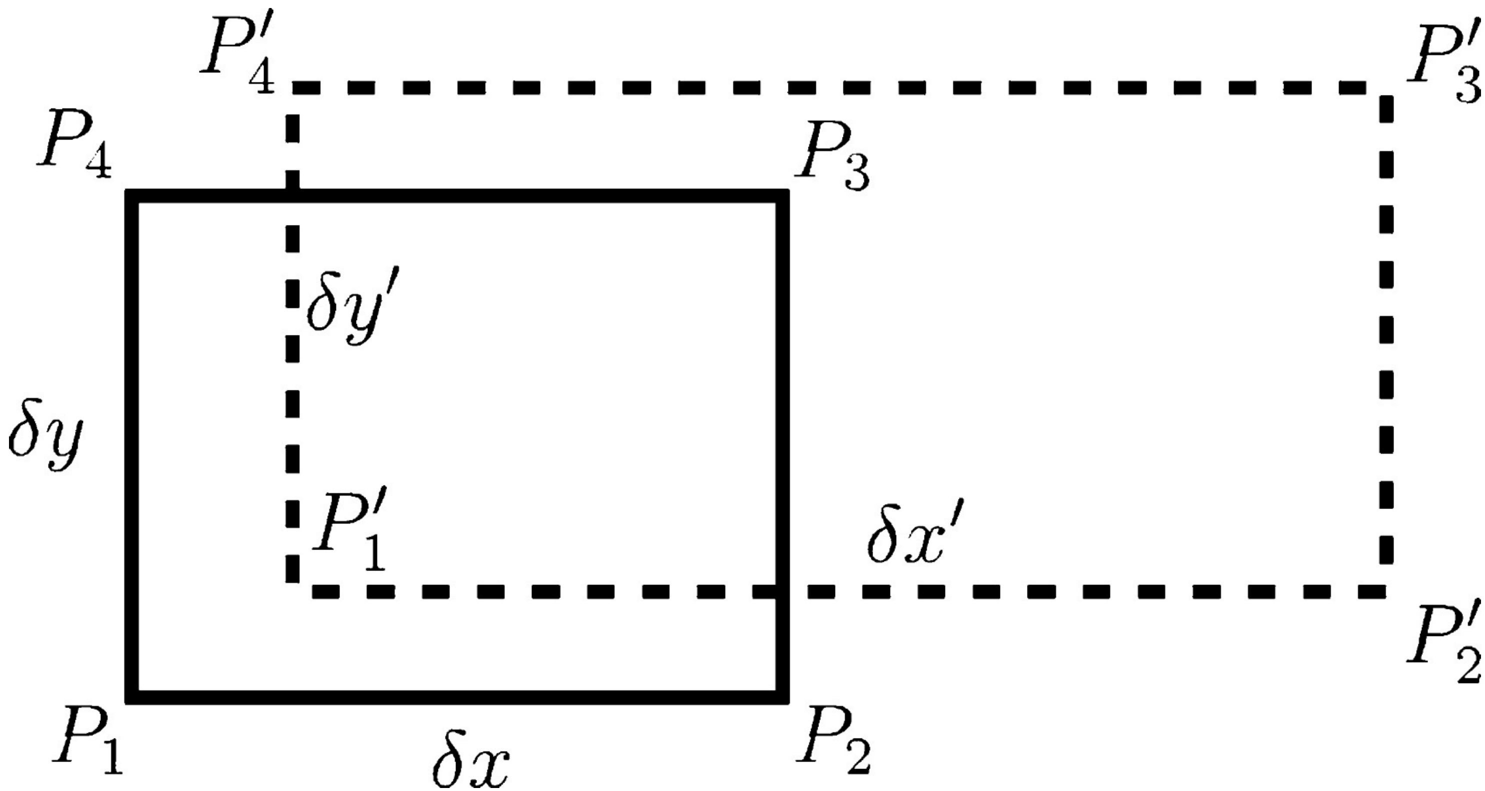
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

y-direction

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

z-direction

Normal strain and velocity gradients



Strain rate

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} \dot{\boldsymbol{\epsilon}}_{xx} & \dot{\boldsymbol{\epsilon}}_{xy} & \dot{\boldsymbol{\epsilon}}_{xz} \\ \dot{\boldsymbol{\epsilon}}_{yx} & \dot{\boldsymbol{\epsilon}}_{yy} & \dot{\boldsymbol{\epsilon}}_{yz} \\ \dot{\boldsymbol{\epsilon}}_{zx} & \dot{\boldsymbol{\epsilon}}_{zy} & \dot{\boldsymbol{\epsilon}}_{zz} \end{bmatrix}$$

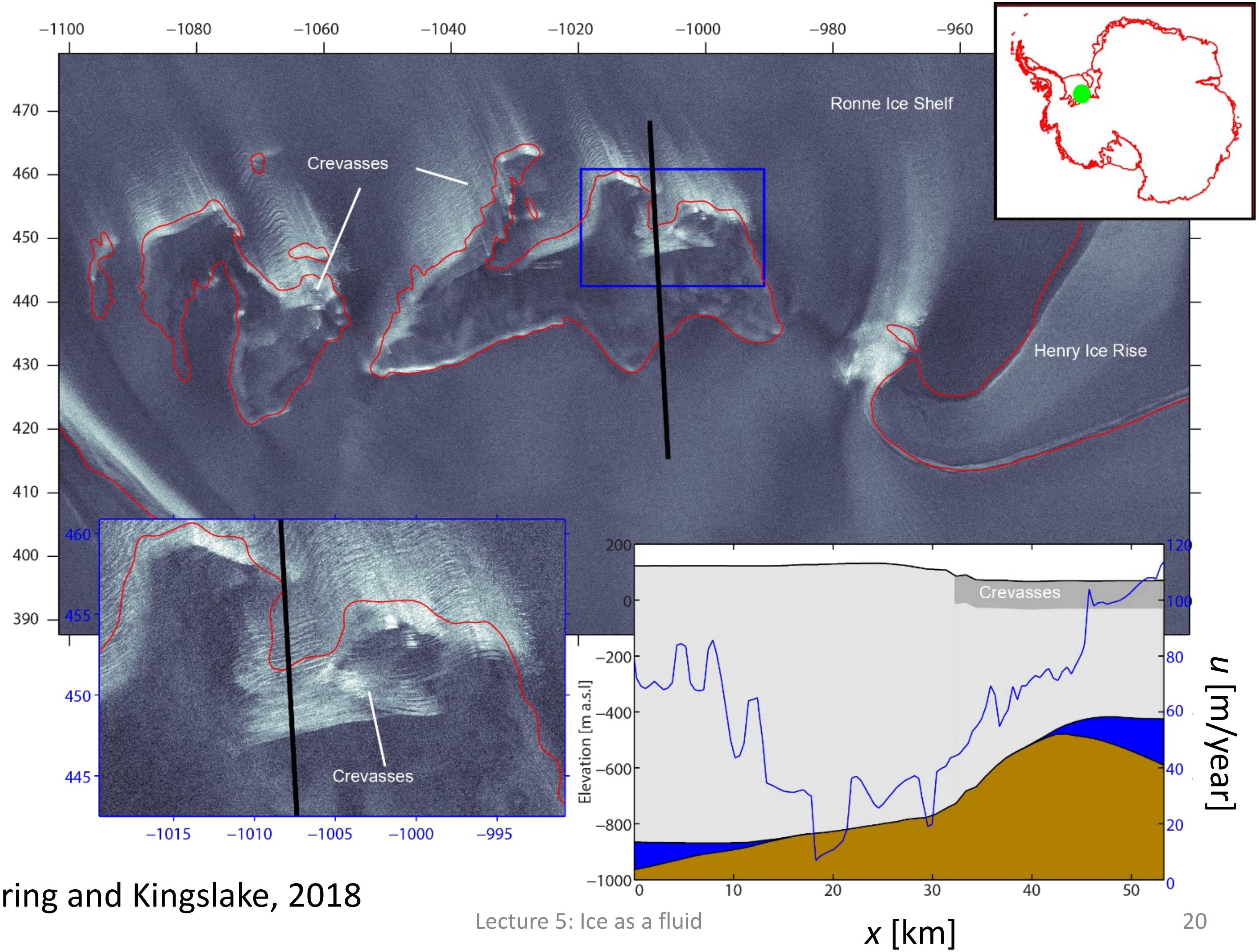
Strain rate

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

For example*:*

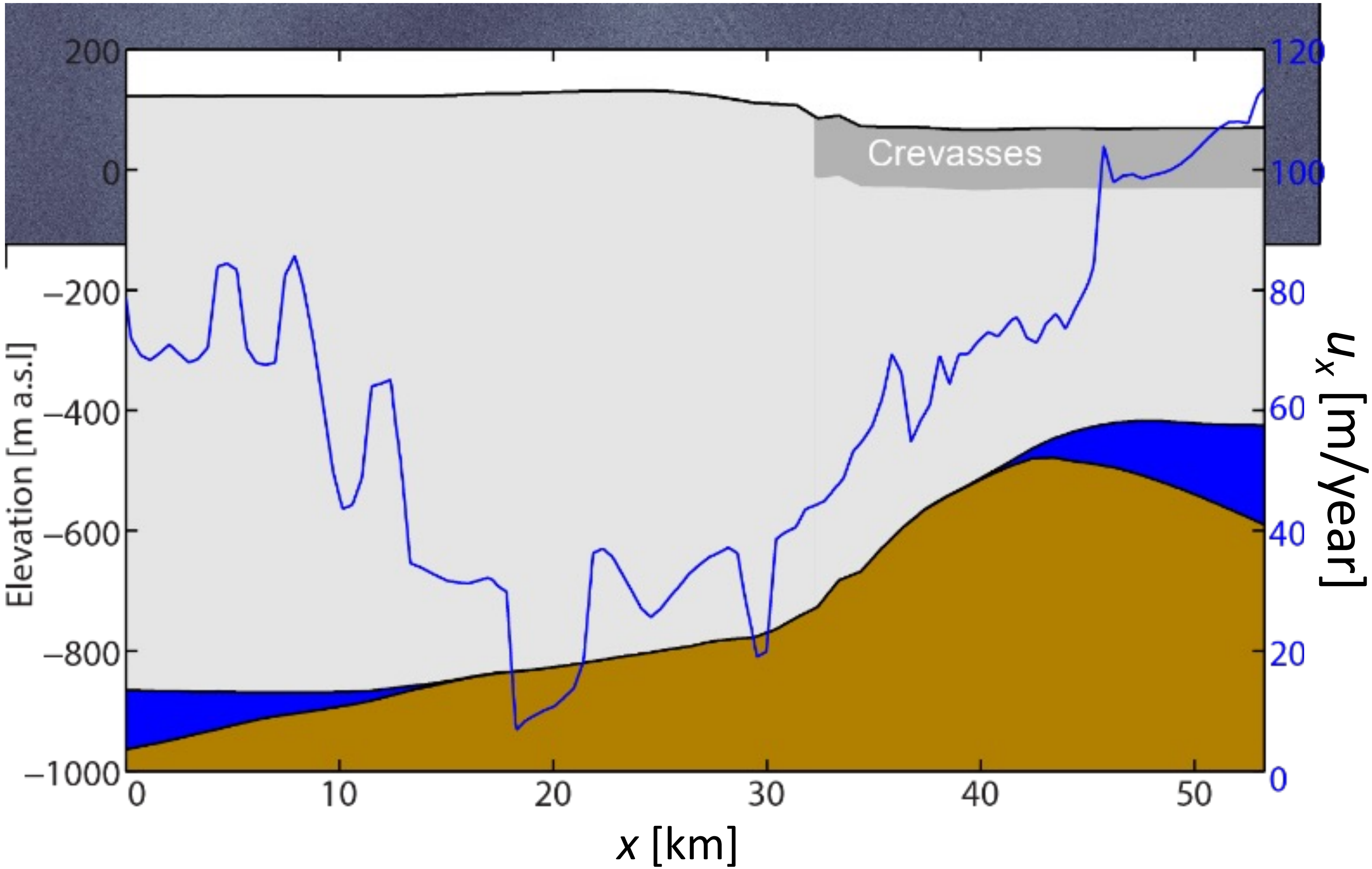
$$\dot{\epsilon}_{xx} = \frac{\partial u_x}{\partial x} \quad \text{or} \quad \dot{\epsilon}_{xz} = \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]$$

A velocity profile from WAIS



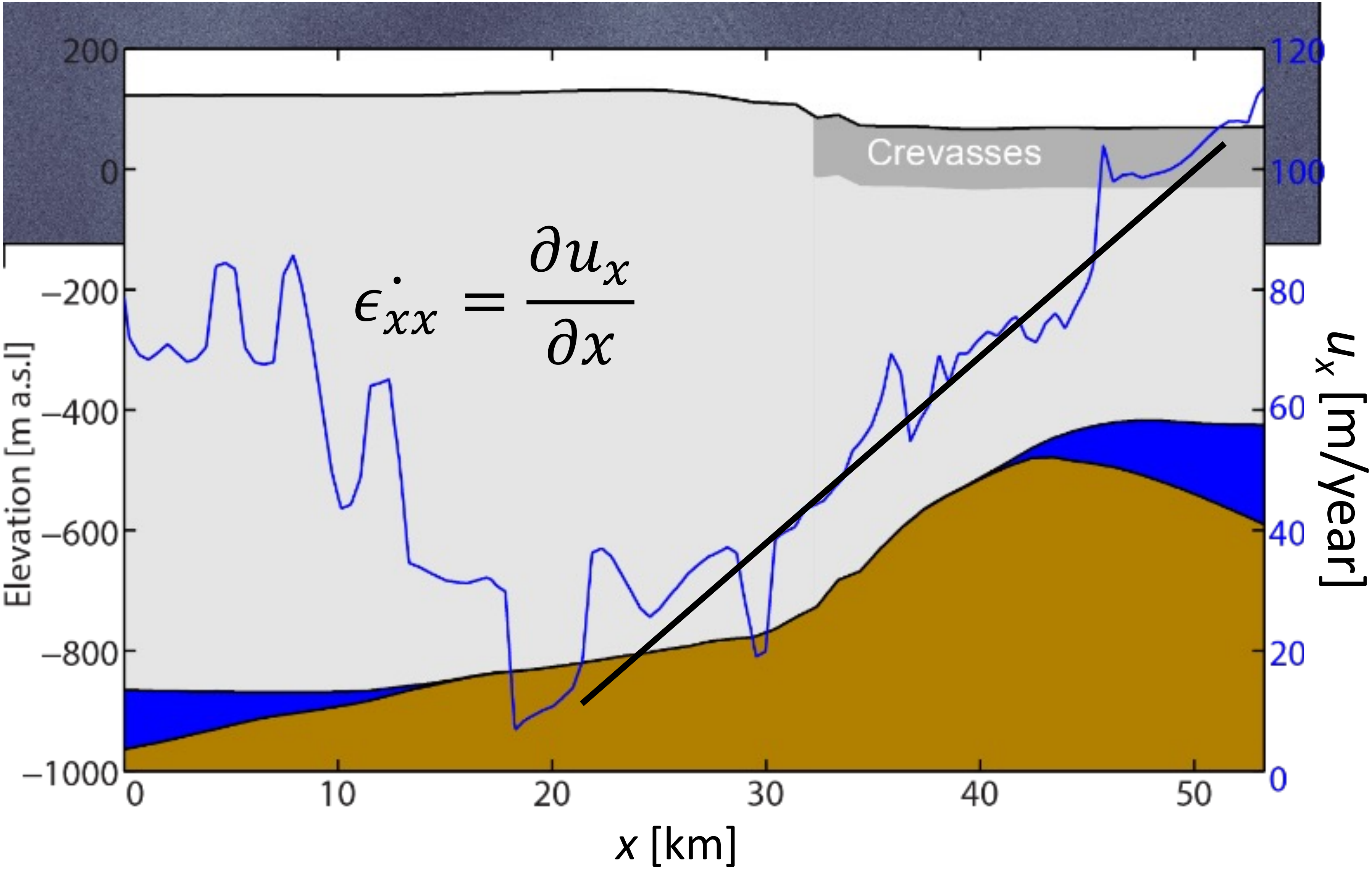
Wearing and Kingslake, 2018

A velocity profile from WAIS



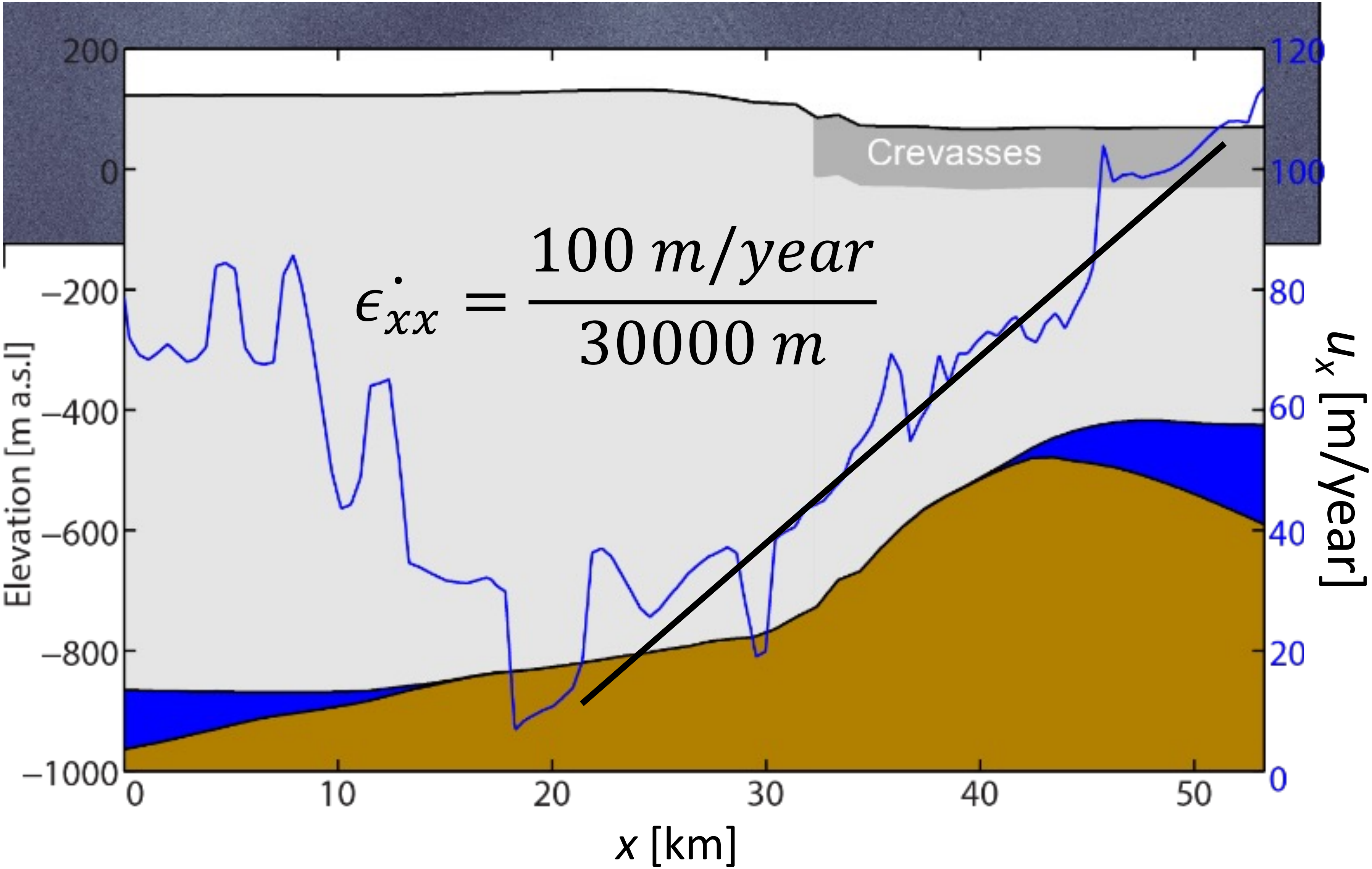
Wearing and Kingslake, 2018

A velocity profile from WAIS



Wearing and Kingslake, 2018

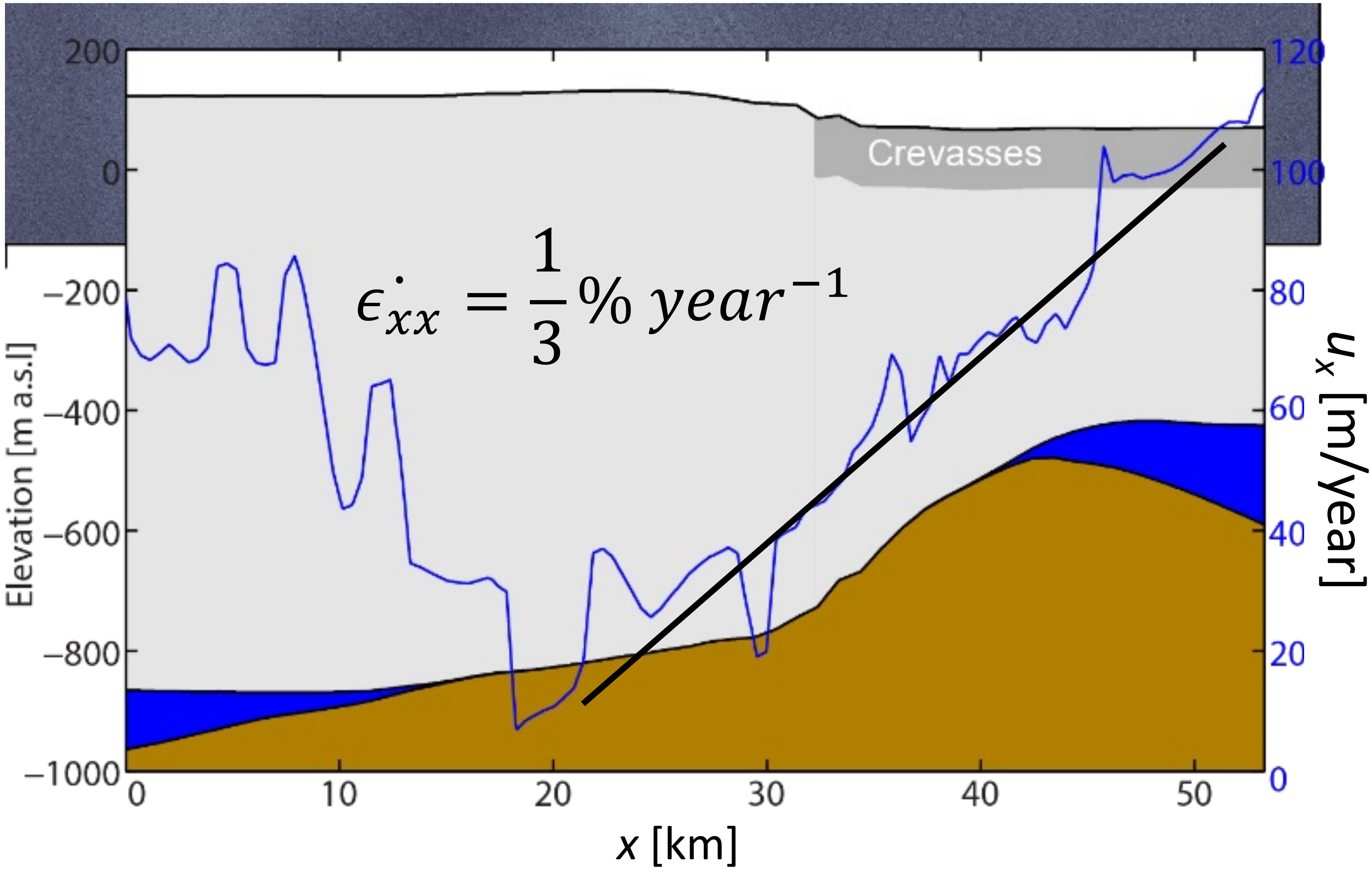
A velocity profile from WAIS



$$\dot{\epsilon}_{xx} = \frac{100\ m/year}{30000\ m}$$

Wearing and Kingslake, 2018

A velocity profile from WAIS



Wearing and Kingslake, 2018

- Break, or finish for the day

How to relate these more rigorous definitions of stress and strain?

Deviatoric stress

Because ice is incompressible, it is the 'deviatoric stress' that we care about.

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{\sigma_m}$$

Total stress

Deviatoric stress

- Mean normal stress
- Hydrostatic stress
- Isotropic stress
- Volume stress

$$\sigma_m = \frac{1}{3}[\sigma_{xx} + \sigma_{yy} + \sigma_{zz}]\boldsymbol{I}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_m \end{bmatrix}$$

Deviatoric stress is the total stress minus the mean normal stress.

Deviatoric stress

Because ice is incompressible, it is the 'deviatoric stress' that we care about.

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{\sigma_m}$$

Total stress

Deviatoric stress

- Mean normal stress
- Hydrostatic stress
- Isotropic stress
- Volume stress

$$\sigma_m = \frac{\mathbf{1}}{\mathbf{3}}[\sigma_{xx} + \sigma_{yy} + \sigma_{zz}]I$$

$$\boxed{\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}}$$

Effective stress

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

- The "*second invariant of the stress tensor*" is called the effective stress in glaciology.

  (we wont derive it here)

$$\tau_E^2 = \frac{1}{2}\left[\tau_{xx}^2 + \tau_{yy}^2 + \tau_{zz}^2\right] + \tau_{xz}^2 + \tau_{xy}^2 + \tau_{yz}^2$$

It is a scalar and is independent of the axes you use.

Back to Glen's flow law....

$$\dot{\epsilon}_{ij} = A\tau_E^{n-1}\tau_{ij}$$

$$\tau_E^2 = \frac{1}{2}\left[\tau_{xx}^2 + \tau_{yy}^2 + \tau_{zz}^2\right] + \tau_{xz}^2 + \tau_{xy}^2 + \tau_{yz}^2$$

We will be using this relation, the stress balance equations and the idea of deviatoric stress next week to derive a simple ice sheet model.

Summary

- Ice deforms through various mechanisms, dislocation creep in the basal plane dominates.
- Rheology is the relationship between stress and strain.
- Nonlinear rheology of ice is usually approximated by Glen's flow law with $n = 3$.
- Stress state is described by the "stress tensor".
- Effective stress is a measure of the amount of stress at a particular location.
- Strain is also a tensor. Its elements are spatial derivatives of the velocity field.

References

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Glen, J.W., 1952. Experiments on the deformation of ice. *Journal of Glaciology*, *2*(12), pp.111-114.

Wearing, M.G. and Kingslake, J., 2019. Holocene Formation of Henry Ice Rise, West Antarctica, Inferred from Ice-Penetrating Radar. *Journal of Geophysical Research: Earth Surface*, *124*(8), pp.2224-2240.