MSAS – Assignment #2: Modeling

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1 Questions

Question 1

1) List the stages of dynamic investigation and their meaning. 2) When going from the *real system* to the *physical model* a number of assumptions are made; report the most important ones along with their mathematical implications. 3) For each of the assumptions below, shortly state what sort of simplification may result: i) The gravity torque on a pendulum is taken proportional to the pendulum angle θ; ii) Only wind forces and gravity are assumed in studying the motion of an aircraft; iii) A temperature sensor is assumed to report the temperature exactly; iv) The pressure in a hydraulic actuator is assumed uniform throughout the chamber. 4) List the *effort* and *flow* variables for the domains treated and discuss their similarity.

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1. The stages of the dynamic investigation are the following:
   (a) modeling: specify the analysed system and imagine a simple physical model whose behavior will match sufficiently closely the behavior of the real system. Then derive a mathematical model to represent the physical model and finally write the differential equations of it;
   (b) analysis: by solving the differential equations study the dynamic behavior of the mathematical model;
   (c) synthesis: make design decisions: choose the physical parameters of the system and/or modify the system, so that it will behave as desired.
2. When going from the real system to the physical model the principal assumptions are:
   (a) to neglect small effects that would lead to needless complexity: this reduces the number of variables, of equations and then also the overall complexity of the equations of motion (i.e. differential equations);
   (b) the environment is independent from the physical system: this has the same results as the first assumption, so it makes the resulting mathematical expressions less complicated;
   (c) of lumping or replacing distributed characteristics by similar lumped characteristics: it leads to differential equations having a much simpler method of solution than the ones that would be required with distributed physical elements. Indeed, this assumption permits to not analyse partial, but ordinary differential equations, that are faster to solve since they require a lower computational burden;
   (d) of linearity, hence assume simple linear cause-and-effect relationships: where nonlinear ODEs are seldom amendable to analysis and they require a computational approach, the linear ODEs instead can be solved analytically. Moreover, the superposition principle applies and, in general, when a linear system is solved once, the solution holds for all the other cases. Also making the equations linear gives early, comprehensive insights into dynamics;
   (e) of constant parameters in time: this allows to have easier equations to solve, since they become differential equations with constant coefficients;

(f) to neglect uncertainty and noise: in the first stages of modeling this assumption permits to use a deterministic approach and to avoid statistical treatment, hence to proceed if all quantities are known exactly. Then this assumption is removed at the end of the first analysis and the statistical one is made.

3. For the following assumptions below, the simplifications made are specified:
   - the gravity torque on a pendulum is taken proportional to the pendulum angle θ: assumption of linearity and hence of small angles;
   - only wind forces and gravity are assumed in studying the motion of an aircraft: in this case the study is neglecting other small effects;
   - temperature sensor is assumed to report the temperature exactly: here the analysis is neglecting uncertainty and noise;
   - the pressure in a hydraulic actuator is assumed uniform throughout the chamber: in this case a distributed quantity is replaced by similar lumped characteristic.
4. The effort and flow variables for the treated domains are listed in Tab. 1. The efforts are

Table 1: Effort and flow variables

Domain	**Effort**	**Flow**
Mechanical translation	Force	Velocity (or position)
Mechanical rotation	Torque	Angular velocity (or angle)
Fluid	Pressure	Volumetric flow rate
Electrical	Voltage	Current
Thermal	Temperature	Heat flow

those variables that express the exertion that can be placed on a component; indeed, for each domain these variables have a similar meaning, i.e. of the parameter that force and lead the behaviour of the system. Analogously, also the flow variables are linked: they express the rate of change of a system variable.
Moreover, using these variables, for the domains it is possible to write similar equations; indeed, each dynamic system in the different domains contains a resistance, a capacitance and a inductance, as shown in Tab. 2.
The way in which the dynamic systems handle energy is also similar. The similarities

Table 2: Similarities between the domains

Domain	**Resistance**	**Capacitance**	**Inductance**
Mechanical	Damper	Spring	Mass or inertial
Fluid	Orifice	Accumulator	Long line with small cross sectional area
Electrical	Resistor	Capacitor	Inductance
Thermal	Heat transfer	Heat capacity (material)	⋆

⋆ the thermal systems have no equation that has the form of flow storage element

exist because the following characteristics hold:

(a) every effort is given by the product between the resistance and the flow;
(b) every flow is given by the product between the capacitance and the first derivative of the effort;
(c) every effort is also given by the product between the inductance and the first derivative of the flow.

Question 2

1) Briefly discuss the physical meaning of the bulk modulus; show how the *effective* bulk modulus is computed. 2) Under what circumstances the fluid resistance yields a linear relation between effort and flow variables? 3) Find the expression of the leakage through a thin annular gap starting from the balance between shear stress and pressure drop.

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1. The bulk modulus (β) is a measure of how resistant to compression a substance is. If β is a high number, than the fluid is slightly compressible; vice versa if β is low, this means that the fluid is fairly compressible. Indeed, it is defined as:

$$\frac{1}{\beta} = \frac{1}{\rho_0}\Big(\frac{\partial \rho}{\partial p}\Big)_{p_0,T_0} = -\frac{1}{V_0}\Big(\frac{\partial V}{\partial p}\Big)_{p_0,T_0} \tag{1}$$

where P_0, T_0, ρ_0, V_0 are respectively the initial values of the pressure, the temperature, the density and the volume of the substance. Hence β represents the ratio between the infinitesimal pressure and the infinitesimal volume: when the pressure increases, the volume decreases and their ratio is specified by β.
In particular, the effective bulk modulus (β_e) accounts for the presence of gas (β_g is its bulk modulus) and the deformation of container (β_c is its bulk modulus). It can be derived as follows, considering the effects that an applied external pressure has on the system and in particular on the rate of change of the volumes:

$$\Delta V_t = -\Delta V_l - \Delta V_g + \Delta V_c \tag{2}$$

where the values considered are respectively the difference of total volume (ΔV_t), of the liquid (ΔV_l), of the gas (ΔV_g) and of the container (ΔV_c).
For infinitesimal variations, Eq. 1 becomes:

$$\frac{1}{\beta_e} = \frac{\Delta V_t}{V_t \Delta p} \tag{3}$$

hence, substituting Eq. 2:

$$\frac{1}{\beta_e} = \frac{-\Delta V_l - \Delta V_g + \Delta V_c}{V_t \Delta p} \tag{4}$$

$$\rightarrow \frac{1}{\beta_e} = \frac{V_l}{V_t}\Big(-\frac{\Delta V_l}{V_l \Delta p}\Big) + \frac{V_g}{V_t}\Big(-\frac{\Delta V_g}{V_g \Delta p}\Big) + \Big(\frac{\Delta V_c}{V_t \Delta p}\Big) = \frac{V_l}{V_t}\frac{1}{\beta_l} + \frac{V_g}{V_t}\frac{1}{\beta_g} + \frac{1}{\beta_c} \tag{5}$$

Eq. 5 is written by using the definition of bulk modulus of Eq. 1.
Now the total volume is expressed as the sum between the volume of the liquid and the one of the gas:

$$V_t = V_l + V_g \tag{6}$$

and knowing that the bulk modulus of the liquid is of an higher order than the one of the gas (hence β_l » β_g), Eq. 5 can be approximated as reported in Eq. 8.

$$\frac{1}{\beta_e} = \frac{V_t - V_g}{V_t}\frac{1}{\beta_l} + \frac{V_g}{V_t}\frac{1}{\beta_g} + \frac{1}{\beta_c} = \frac{1}{\beta_l} + \frac{V_g}{V_t}\Big(\frac{1}{\beta_g} - \frac{1}{\beta_l}\Big) + \frac{1}{\beta_c} \tag{7}$$

$$\frac{1}{\beta_e} \simeq \frac{1}{\beta_l} + \frac{V_g}{V_t}\frac{1}{\beta_g} + \frac{1}{\beta_c} \tag{8}$$

2. The effort and flow variables (hence in this domain the pressure and the volumetric flow rate Q) are linked by a linear relation in case of head losses with a linear flow (9).

$$\Delta p = RQ \tag{9}$$

3. The balance between shear stress and pressure drop for a thin gap in the linear case and considering stationary conditions is given by:

$$2yh\Delta p = -2Lh\mu\frac{dv}{dy} \tag{10}$$

where h, L and y are the geometric quantities of the cube (e x L x h), v(y) is the velocity along x-direction, p is the pressure and μ the viscosity of the fluid. Therefore, the first term corresponds to the pressure forces and the second to the shear stress.
Integrating by parts Eq. 10, the velocity profile can be written as reported in Eq. 12.

$$\int_y^{\frac{e}{2}} -\frac{\Delta p}{L\mu} y\, dy = \int_{v(y)}^{v(\frac{e}{2})} dv \tag{11}$$

Since v($\frac{e}{2}$) = 0:

$$v(y) = \frac{\Delta p}{2L\mu}\Big(\frac{e^2}{4} - y^2\Big) \tag{12}$$

The total flow is obtained integrating the following equation from - $\frac{e}{2}$ to $\frac{e}{2}$:

$$dQ = v(y)h\, dy \rightarrow Q = \frac{he^3\Delta p}{12L\mu} \tag{13}$$

If the leakage through a thin annular gap is considered, then h corresponds to the width of the piston+cylinder system and it is equal to πD, where D is the diameter of the cylinder itself.
Hence in this case the total flow is expressed as:

$$Q = \frac{\pi De^3}{12L\mu}\Delta p = \frac{\pi Dg^3}{96L\mu}\Delta p \tag{14}$$

where g = 2e is the clearance.

Question 3

1) Derive from scratch the mathematical model for RC and RL circuits and express the system response in closed form. 2) Consider a real DC motor and a) sketch its physical model (list the assumptions made); b) derive its mathematical model; c) show how the motor constant depends on the physical parameters.

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1. With respect to the Fig. 1, the mathematical model for RC circuits is given by the following equations, using Kirchhoff's voltage law and the characteristic equation of the capacitors:

$$\begin{cases} i(t) = \dfrac{V_0 - V_c(t)}{R} & (15) \\ i(t) = C\dfrac{dV_c(t)}{dt} & (16) \end{cases}$$

where i is the current of the circuit. Hence, reorganising the equations, the non-homogeneous first order differential equation in V_C(t) that rules the model is given by Eq. 17:

$$\frac{dV_c(t)}{dt} + \frac{V_c(t)}{RC} = \frac{V_0}{RC} \tag{17}$$

The system response in closed form is given by Eq. 18:

$$V_c(t) = (V_c(0) - V_0)e^{-\frac{t}{\tau}} + V_0 \tag{18}$$

where τ is the time constant, equal to the product RC. In this formula, C is measured in Farads, R in Ohms [Ω] and τ in seconds.
Moreover, if the current i(t) (from Eq. 17) of the circuit has to be known, its expression is given by:

$$i(t) = C\frac{dV_c(t)}{dt} = -\frac{V_C(0) - V_0}{R}e^{-\frac{t}{\tau}} \tag{19}$$

From Eq. 18 it can be noticed that the the presence of a constant tension V_0 leads $V_C(t)$ to have a first exponential behaviour, from the initial condition $V_C(t = 0) = V_C(0)$, and then $V_C(t)$ tends to the value of V_0 itself. Hence $V_C(t) \rightarrow V_0$ as $t \rightarrow \infty$. Moreover, to solve the circuit the tension $V_C(0)$ has to be known.
Instead, the current i(t) (Eq. 19) as a function of time behaves in the opposite way: it starts from a finite value and then it tends to zero.

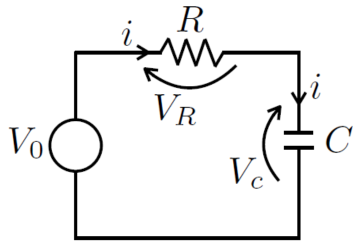


Figure 1: RC circuit

The mathematical model for RL circuits (Fig. 2) is given by the following equations, obtained through the application of Kirchhoff's voltage law and the characteristic equation of the inductors; here the inductor and the resistor are in series, whereas for the RC circuits the resistor is in series with the capacitor.

$$\begin{cases} i(t) = \dfrac{V_0 - V_L(t)}{R} & (20) \\ V_L(t) = L\dfrac{di(t)}{dt} & (21) \end{cases}$$

Hence the first order differential equation in i(t) that describes the circuit is:

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V_0}{L} \tag{22}$$

and the system response in closed form is given by Eq. 23:

$$i(t) = \Big(i(0) - \frac{V_0}{R}\Big)e^{-\frac{t}{\tau}} + \frac{V_0}{R} \tag{23}$$

In this case the time constant is equal to $\tau = \frac{L}{R}$, the steady-state of i(t) is given by $\frac{V_0}{R}$ and the initial condition of the current i(0) is needed to solve the circuit.
Moreover, the voltage across the inductor is given by Eq. 24

$$V_L(t) = L\frac{di(t)}{dt} = -\Big(i(0) - \frac{V_0}{R}\Big)Re^{-\frac{t}{\tau}} \tag{24}$$

Analogously to the RC circuit, the curve of the current i(t) starts from its initial value and then tends to $\frac{V_0}{R}$ for $t \rightarrow \infty$.

2. If a real DC motor is considered, the assumptions that have to be made are:

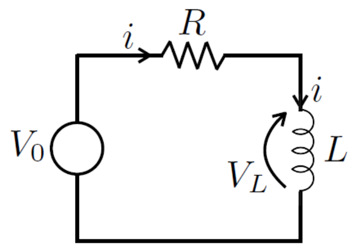


Figure 2: RL circuit

(a) the DC drives a shaft with negligibly small flexibility;
(b) the mechanical load is an inertia J (of the disk);
(c) b is the linear friction of the bearings;
(d) there is a disturbance load T_L.

The physical model is represented in Fig. 3, where the left side is the electrical and right side the mechanical parts. From the physical model the mathematical one can be obtained

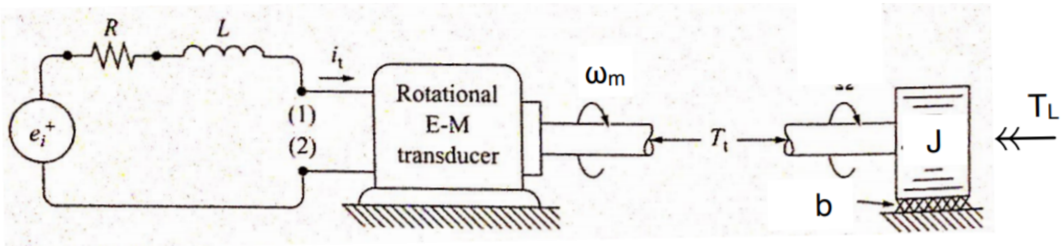


Figure 3: Physical model: real DC motor

as well; firstly, the electrical and the mechanical parts are studied individually, then their equations are linked in an unique system.
The equations of the electrical part are:

$$\begin{cases} e_1 - e_2 = Ri & (25) \\ L\dfrac{di(t)}{dt} = e_2 - e_m & (26) \end{cases}$$

where e is the voltage, R the resistor, i the current and L the inductance. The equation of the mechanical part is instead:

$$J\dot{w}_m + bw_m = T_m - T_L \tag{27}$$

where T_m is the motor torque and w_m the angular velocity.
The equations of motion of the system can now be found combining the electrical and mechanical elements, knowing that:

$$\begin{cases} T_m = Ki & (28) \\ e_m = Kw_m & (29) \end{cases}$$

where K is a constant of the motor. Then combining Eqs. 25, 26, 27, 28 and 29 the following system, that corresponds to the mathematical model, can be derived:

$$\begin{cases} L\dfrac{di}{dt} + Ri + Kw_m = e_1 & (30) \\ J\dot{w}_m + bw_m - Ki = -T_L & (31) \end{cases}$$

The physical model for the electrical part consists in representing the armature coil as a resistor and an inductance plus an additional voltage drop proportional to the angular velocity ω_m (Fig. 4). Moreover, through the armature coil the current i(t) flows. Hence the following equations can be written, specifying the links between the permanent-magnet and electrical part

$$F_e = ilB \tag{32}$$

where F_e is the electromagnetic force and B the magnetic fields, and l the length of the armature winding.
Since the torque due to F_e acts on each section of the armature winding, formed by N turns:

$$T_m = 2NF_e r \rightarrow T_m = 2NlBri \tag{33}$$

where r is the half-length (radius) of the armature winding. Now a constant K can be defined as K = 2NlBr, hence Eq. 33 becomes:

$$T_m = Ki \tag{34}$$

Analogously, from Fig. 4, also the equation that links the voltage of the motor with the magnetic field B can be written:

$$e_m = uBl = 2NBlrw_m = Kw_m \tag{35}$$

where u is the rotational velocity, that considers the N turns; moreover, two sections of each turn in the armature winding as to be accounted for, as in Eq. 33. In this equation, one can identify once again the constant K.
For this motor, the two 1^{st}-order differential equations (Eqs. 26 and 27) can be written as an unique 2^{nd}-order differential equation in ω_m:

$$LJ\ddot{\omega}_m + (Lb + RJ)\dot{\omega}_m + (Rb + K^2)\omega_m = Ke_1 - (RT_L + L\dot{T}_L) \tag{36}$$

$$\ddot{\omega}_m + 2\xi\omega_n\dot{\omega}_m + \omega_n^2\omega_m = \frac{K}{LJ}e_1 - \frac{R}{LJ}T_L - \frac{1}{J}\dot{T}_L \tag{37}$$

Here, the static gain (G_s), the natural frequency (ω_n) and the damping ratio (ξ) of Eq. 37 can be expressed as functions of K:

$$G_s = \frac{K}{Rb + K^2} \tag{38}$$

$$\omega_n = \sqrt{\frac{Rb + K^2}{LJ}} \tag{39}$$

$$\xi = \frac{Lb + RJ}{\sqrt{LJ(Rb + K^2)}} \tag{40}$$

Question 4

1) Write down the Fourier law and show how it is specialized in the case of conduction through a thin plate; discuss the concept of thermal resistance. 2) Report the equation for thermal radiation in case of a) black body and b) real body and discuss them.

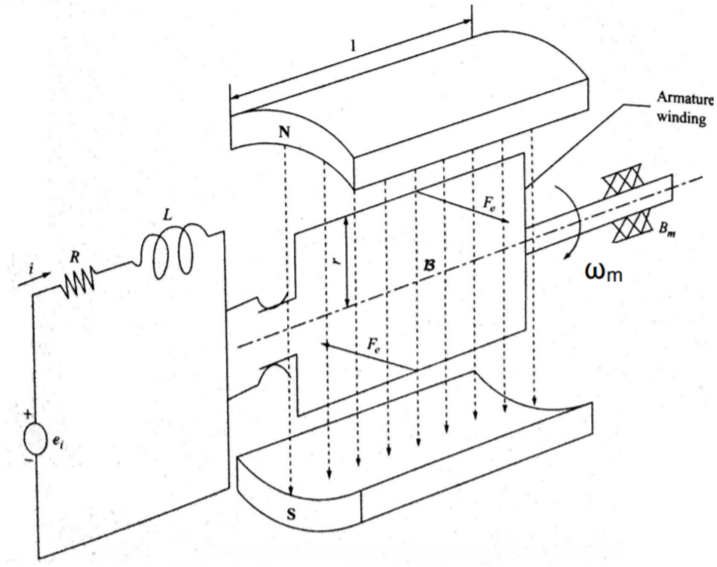


Figure 4: DC motor

1. The Fourier law Fourier's law of heat conduction shows how the heat flow Q_h is related to the temperature gradient $\frac{dT}{dx}$.
$$\frac{Q_h}{A} = -k_t \frac{dT}{dx} \tag{41}$$
where A is the cross-sectional area and k_t the thermal conductivity.
This law strongly depends on the geometry of the system. In particular, for the case of thin plate, the equation for a constant k_t and uniform Q_h becomes:
$$\int_0^L \frac{Q_h}{A}\, dx = -\int_{T_1}^{T_2} k_t(T)\, dT \tag{42}$$
$$\frac{Q_h}{A} L = -k_t(T_2 - T_1) \tag{43}$$
where in Eq. 43 L is the thickness and A the cross-sectional area.
The thermal resistance in case of conduction and thin plate is a measurement of a temperature difference by which a material resists a heat flow. Indeed, analogously to the electric case, the heat flow can be imagined as the current and the difference of temperature as the voltage; hence, the following equation (knowing Eq. 43) can be written:
$$\frac{\Delta T}{Q_h} = \frac{L}{k_t A} = R \tag{44}$$
where $\Delta T = T_1 - T_2$.

2. The equation for thermal radiation in case of ideal black body is given by Eq. 45:
$$\frac{Q_h}{A} = \sigma T^4 \tag{45}$$
where T is the temperature of the body and σ the Stefan-Boltzmann constant.
Eq. 46 instead is the equation for thermal radiation for a real body:
$$Q_h = F_e F_v A \sigma (T_H^4 - T_L^4) \tag{46}$$
where F_v is the viewing factor, that accounts for lost radiation, F_e is the emissivity factor, T_H and T_L are respectively the high and low temperatures of the body.

Eq. 46 considers that a real body has less emissivity than an ideal blackbody, which is a perfect emitter (or receiver) of thermal radiation; indeed, for a black body the emissivity factor is equal to 1. This equation also considers that the body exchange radiation with other objects (since it is not isolated), hence the neat heat transfer is proportional to the difference between the high and the low temperatures of the body, both to the power of four.
Moreover, also the term F_v is added, in order to avoid considering all the radiation emitted by a body, which may not entirely impinge another one.

2 Exercises

Exercise 1

A mass hangs from a movable support by a linear spring as shown in Figure 5. The system is initially in static equilibrium. At $t_0 = 0$, the support is given the motion $x(t) = x_0 u(t) \cos \omega_f t$. 1) Write down the mathematical model from first principles. 2) Using the data given in the figure caption, and guessing a value for the viscous friction coefficient b, compute the system response from t_0 to $t_f = 10$ s. 3) An accelerometer placed on m recorded samples at 10 Hz, which were saved in the file `samples.txt`; the samples are affected by measurement noise. Determine the value of b that allows retracing the experimental data.

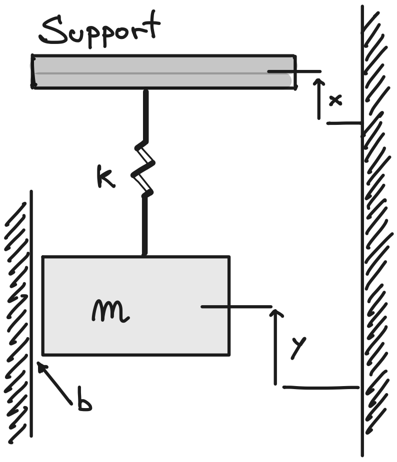


Figure 5: Physical model ($m = 3\,\mathrm{kg}$; $k = 100\,\mathrm{N/m}$; $x_0 = 0.2\,\mathrm{m}$; $\omega_f = 5\,\mathrm{s}^{-1}$; $u(t) = e^{-10t}$).

The mathematical model of the system is given by the following second order differential equation (Eq. 47).

$$m\ddot{y}(t) + b\dot{y}(t) + k[y(t) - x(t)] = 0 \tag{47}$$

In order to perform the integration and to lower the order of the equation, the following system is used:

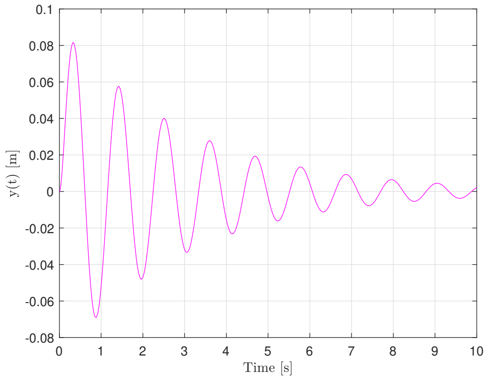
$$\begin{cases} v(t) = \dot{y}(t) & (48) \\ \dot{v}(t) = -\dfrac{b}{m}v(t) - \dfrac{k}{m}\Big(y(t) - x(t)\Big) & (49) \end{cases}$$

Where the initial condition for the displacement y(t) and the velocity v(t) are both zero.
With the viscous friction coefficient b equal to 2 $\frac{kg}{s}$, the system response is shown in Fig. 6a for the displacement and in Fig. 6b for the velocity.

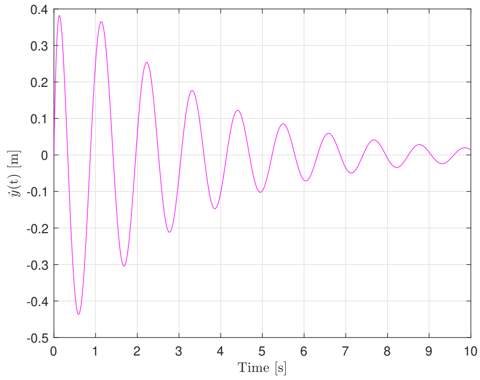
In order to retrace the experimental data the MATLAB function FMINCON has been used. Indeed, through this function the value of b that minimises the absolute difference between the experimental samples and the acceleration given by the model (Eq. 50) has been computed

$$\min \mid a - \ddot{y}(t) \mid \tag{50}$$

where a is the vector containing the experimental data.
The viscous friction coefficient that leads to the minimum difference is 3.0052. The plot of the model with respect to the experimental samples in shown in Fig. 7.



(a) Displacement y(t)



(b) Velocity $\dot{y}$(t)

Figure 6: System response

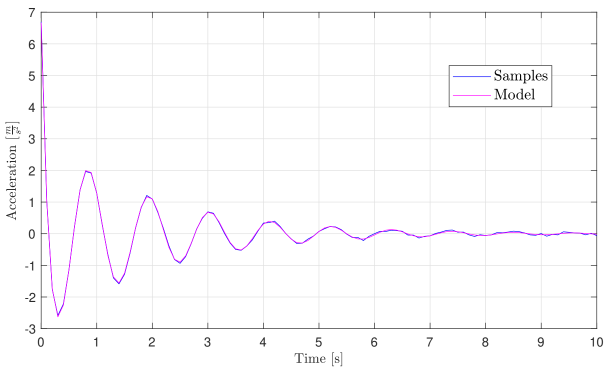


Figure 7: Experimental samples vs acceleration given by the model

Exercise 2

The hydraulic system in Figure 8 consists of an accumulator, a check valve, a distributor, an actuator, and a tank, plus delivery and return lines. At $t = t_{-\infty}$, the accumulator contains nitrogen only. To charge it, the nitrogen undergoes an isothermal transformation from $\{p_{N_2}(t_{-\infty}), V_{N_2}(t_{-\infty})\}$ to $\{p_{N_2}(t_0) = p_0, V_{N_2}(t_0) = V_0\}$, t_0 being the initial time. 1) Assuming incompressible fluid, adiabatic discharge of the accumulator, and no leakage in the actuator, write down a mathematical model that allows computing pressures and flow rates in the sections labeled. 2) Considering the distributor command z in Figure 8, carry out a simulation to show the system response in $[t_0, t_f]$. 3) Determine the time t_e that takes the piston to reach the maximum stroke, $x(t_e) = x_{\max}$, starting from $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$.

- Fluid: Skydrol, $\rho = 890$ kg/m^3.
- Accumulator: $V_{N_2}(t_{-\infty}) = 10$ dm^3, $p_{N_2}(t_{-\infty}) = 2.5$ MPa, $p_0 = 21$ MPa, adiabatic exponent $\gamma = 1.2$.
- Delivery: Coefficient of pressure drop[1] at accumulator outlet $k_A = 1.12$, coefficient of pressure drop across the check valve $k_{cv} = 2$, diameter of the delivery line $D_{23} = 18$ mm; Branch 2–3: Length $L_{23} = 2$ m, friction factor[2] $f_{23} = 0.032$.
- Distributor: Coefficient of pressure drop across the distributor $k_d = 12$, circular cross section, diameter $d_o = 5$ mm.
- Actuator: Diameter of the cylinder $D_c = 50$ mm, diameter of the rod $D_r = 22$ mm, mass of the piston $m = 2$ kg, maximum stroke $x_{\max} = 200$ mm; Load: $F(x) = F_0 + kx$, $F_0 = 1$ kN, $k = 120$ kN/m.
- Return: Diameter of the return line $D_r = 18$ mm; Branch 6–7: Length $L_{67} = 15$ m, friction factor $f_{67} = 0.035$.
- Tank: Pressure $p_T = 0.1$ MPa, initial volume $V_T(t_0) = 1$ dm^3, coefficient of pressure drop at tank inlet $k_T = 1.12$.
- Initial time: $t_0 = 0$, $x_0 = 0$, $v_0 = 0$; $t_1 = 1$ s, $t_2 = 1.5$ s; final time $t_f = 3$ s.

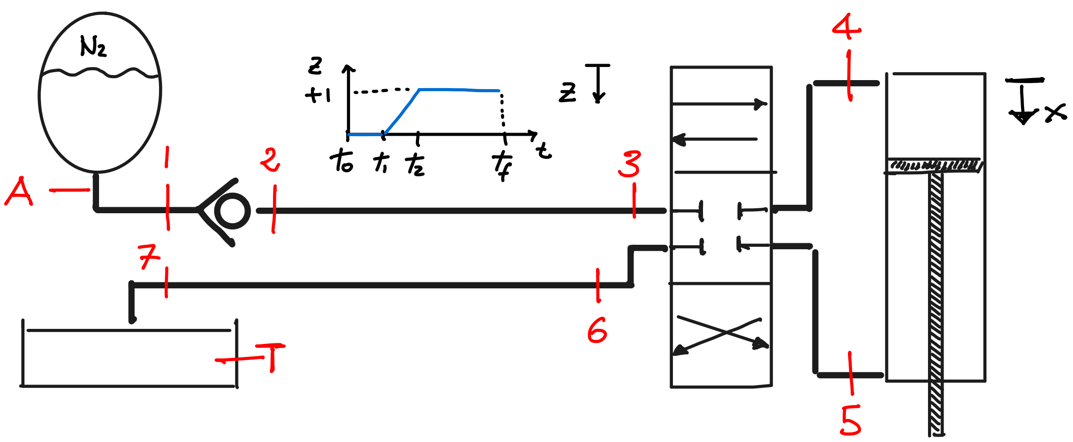


Figure 8: Hydraulic system physical model; assume any other missing data.

[1] $\Delta p = 1/2 k \rho v^2$.
[2] $\Delta p = f L/D 1/2 \rho v^2$.

The mathematical model that rules the system is given by the following system, which is formed by four differential equations:

$$\begin{cases} \dot{V}_{acc} = -Q_{acc} & (51)\\ \dot{V}_{tank} = -Q_7 & (52)\\ \dot{v} = \dfrac{P_4 A_c - P_5(A_C - A_P) - F}{m} & (53)\\ \dot{x} = v & (54) \end{cases}$$

So the system permits to evaluate the first derivatives of the accumulator and tank volumes, of the velocity and of the displacement of the actuator.
Moreover, the negative signs of the first two equations denote that the accumulator is being emptied, while the volume of the tank is increasing, as also shown in Fig. 9.

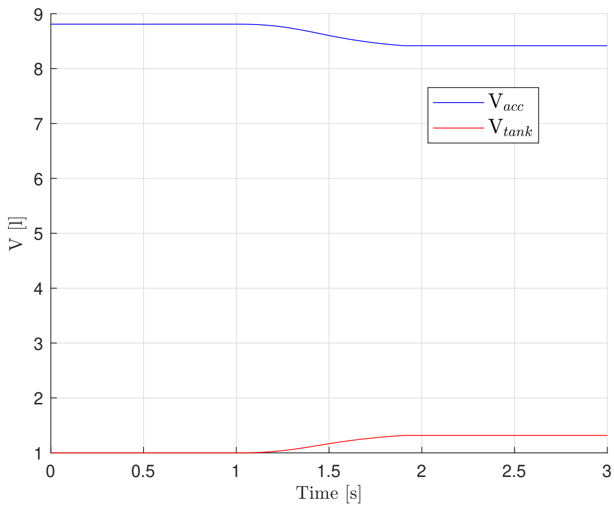


Figure 9: Volumes V(t)

The pressures of the delivery and return lines (Fig. 10) can be found by the following equations, once the volumetric flow rates are known as well as the pressure in the accumulator (P$_{acc}$):

$$\begin{cases} P_1 = P_{acc} - \dfrac{1}{2} k_A \rho \dfrac{Q_{acc} \mid Q_{acc} \mid}{A_{23}^2} & (55)\\ P_2 = P_1 - \dfrac{1}{2} k_{cv} \rho \dfrac{Q_1 \mid Q_1 \mid}{A_{23}^2} & (56)\\ P_3 = P_2 - \dfrac{1}{2} f_{23} \rho \dfrac{L_{23}}{D_{23}} v_{23} \mid v_{23} \mid & (57)\\ P_7 = P_T - \dfrac{1}{2} k_T \rho \dfrac{Q_7 \mid Q_7 \mid}{A_{67}^2} & (58)\\ P_6 = P_7 - \dfrac{1}{2} f_{67} \rho \dfrac{L_{67}}{D_{67}} v_{67} \mid v_{67} \mid & (59) \end{cases}$$

While the pressures of the actuator (Fig. 11) instead:

$$\begin{cases} P_4 = P_3 - \dfrac{1}{2} k_d \rho \dfrac{Q_4 \mid Q_4 \mid}{A_d^2} & (60)\\ P_5 = P_6 - \dfrac{1}{2} k_d \rho \dfrac{Q_5 \mid Q_5 \mid}{A_d^2} & (61) \end{cases}$$

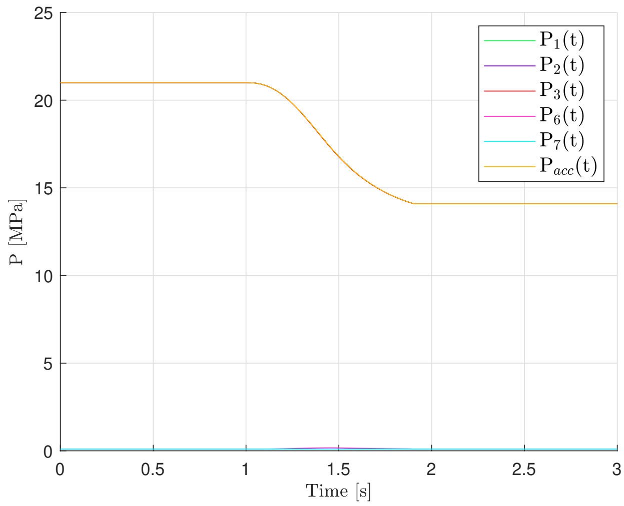


Figure 10: Pressures P(t)

In particular, the initial condition of P_5 has been set equal to the tank one, while the initial condition of P_4 is found by imposing the equilibrium in the actuator at t = 0. Moreover, the initial condition of the volume of the accumulator is given by the difference of the $V_{N_2,\text{inf}}$ and V_0; indeed, $V_{N_2,\text{inf}}$ is equal to the total volume and at t = 0 both the nitrogen and the fluid coexist in the accumulator.

The volumetric flow rates (Fig. 12), instead, are constants along their line, whether this is the

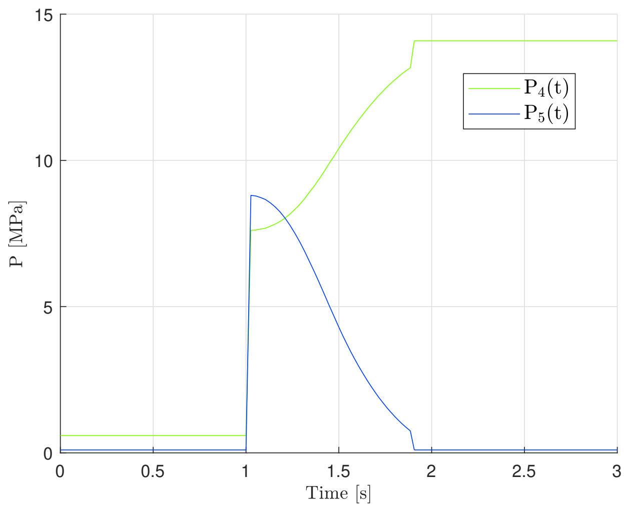


Figure 11: Pressures of the actuator

delivery or the return ones:

$$\begin{cases} Q_{acc} = Q_1 = Q_2 = Q_3 = Q_4 = vA_C & (62) \\ Q_5 = Q_6 = Q_7 = -v(A_C - A_P) & (63) \end{cases}$$

In particular, the volumetric flow rate of the return line in Fig. 12 is negative since the direction of the fluid is considered from the actuator to the tank.

The displacement and velocity of the actuator as functions of time are shown in Fig. 13.

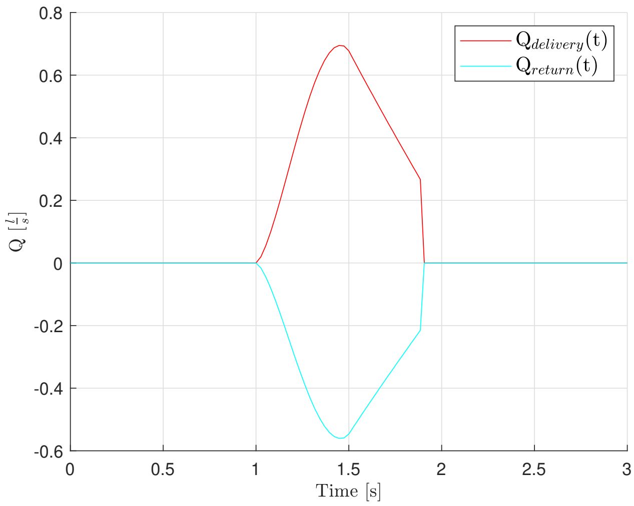


Figure 12: Volumetric flow rates Q(t)

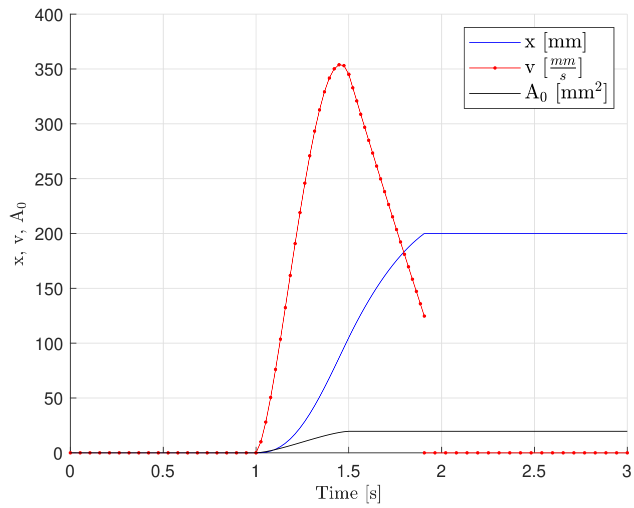


Figure 13: Stroke of the piston x(t)

In order to determine the time that takes the piston to reach the maximum stroke, an event function has been used. Since the maximum stroke is reached during the third integration (hence between t = 1.5 s and t = 3 s) this latter has been divided into two sub-integrations: the first from t_2 to t_e and the second from t_e to t_f. Indeed, once the maximum stroke is reached, the velocity is zero as well as the volumetric flow rates.
The time t_e is equal to 1.90654 s.

Exercise 3

Consider the ideal physical model network shown in Figure 14. The switch has been open for a long time. The capacitor is charged and has a voltage drop between its ends equal to 1 V. Then, at $t = 0$, the switch is closed. 1) Plot the subsequent time history of the voltage V_C across the capacitor. 2) Assume a voltage source characterized by $v(t) = \sin(2\pi f t)\arctan(t)$ having the positive terminal downward inserted in place of the switch. What is in this case the voltage history across the capacitor?

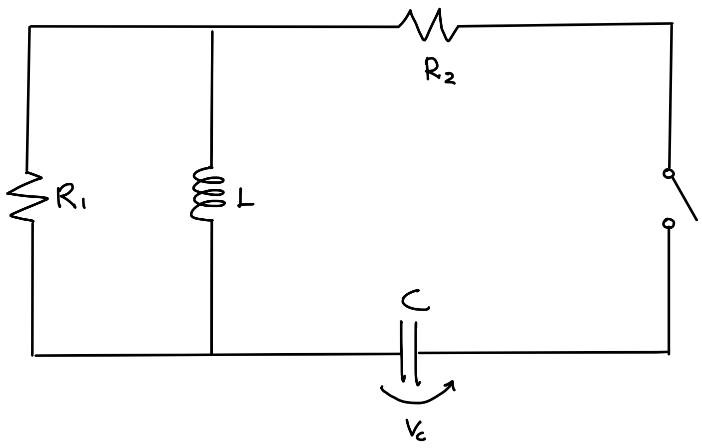


Figure 14: Circuit physical model ($R_1 = 1000\,\Omega$; $R_2 = 100\,\Omega$; $L = 1\,\text{mH}$; $C = 1\,\text{mF}$; $f = 5\,\text{Hz}$.)

The problem is modelled using the voltage across the capacitor as the unknown; the following second order equation (Eq. 64) describes how the network operates:

$$C\Big(1+\frac{R_2}{R_1}\Big)\ddot{V}_c(t) + \Big(\frac{R_2 C}{L} + \frac{1}{R_1}\Big)\dot{V}_c(t) + \frac{V_c(t)}{L} = 0 \tag{64}$$

From Eq. 64 the following system can be built, in order to lower the order of the equation and to perform the integration:

$$\begin{cases} C\Big(1+\frac{R_2}{R_1}\Big)\dot{z}(t) + \Big(\frac{R_2 C}{L} + \frac{1}{R_1}\Big)z(t) + \frac{V_c(t)}{L} = 0 & (65) \\ z(t) = \dot{V}_c(t) & (66) \end{cases}$$

Fig. 15 shows the results of the system formed by Eq. 65 and Eq. 66. If the voltage v(t) is considered as well, the second order equation becomes Eq. 67 and the system that permits to compute the integration is composed by Eq. 68 and Eq. 69.

$$C\Big(1+\frac{R_2}{R_1}\Big)\ddot{V}_c(t) + \Big(\frac{R_2 C}{L} + \frac{1}{R_1}\Big)\dot{V}_c(t) + \frac{V_c(t)}{L} = \frac{L}{R_1}\dot{v}(t) + v(t) \tag{67}$$

$$\begin{cases} C\Big(1+\frac{R_2}{R_1}\Big)\dot{z}(t) + \Big(\frac{R_2 C}{L} + \frac{1}{R_1}\Big)z(t) + \frac{V_c(t)}{L} = \frac{L}{R_1}\dot{v}(t) + v(t) & (68) \\ z(t) = \dot{V}_c(t) & (69) \end{cases}$$

The voltage history across the capacitor with v(t) is shown in Fig. 16.

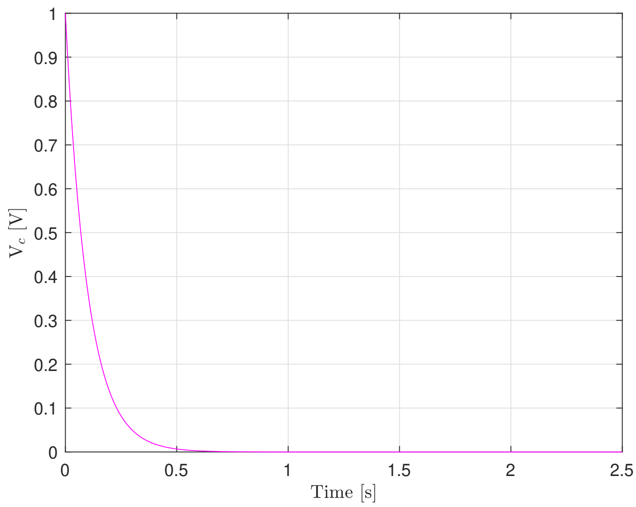


Figure 15: System response: V_c(t) without v(t)

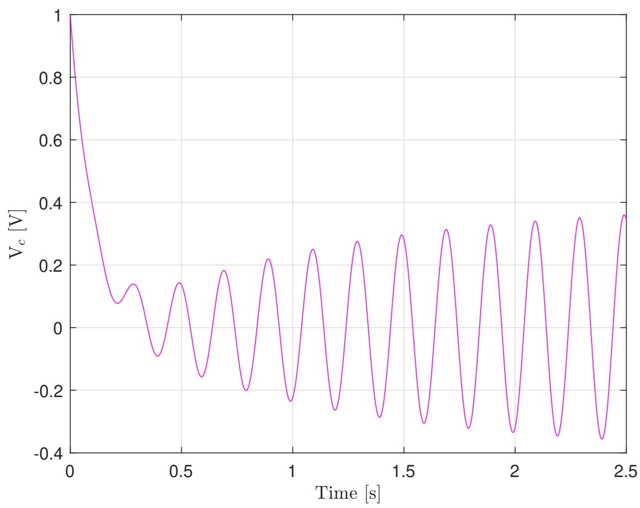


Figure 16: System response: V_c(t) with v(t)

Exercise 4

The rocket engine in Figure 17 is fired in laboratory conditions. With reference to Figure 17, the nozzle is made up of an inner lining (k_1), an inner layer having specific heat c_2 and high conductivity k_2, an insulating layer having specific heat c_4 and low conductivity k_4, and an outer coating (k_5). The interface between the conductor and the insulator layers has thermal conductivity k_3. 1) Select the materials of which the nozzle is made of[3], and therefore determine the values of k_i ($i = 1, \ldots, 5$), c_2, and c_4. Assign also the values of ℓ_i (i=1,. . . ,5), L, and A in Figure 17. 2) Derive a physical model and the associated mathematical model using one node per each of the five layers and considering that only the conductor and insulator layers have thermal capacitance. The inner wall temperature, T_i, as well as the outer wall temperature, T_o, are assigned. 3) Using the mathematical model at point 2), carry out a dynamic simulation to show the temperature profiles across the different sections. At initial time, $T_i(t_0) = T_o(t) = 20$ C°. When the rocket is fired, $T_i(t) = 1000$ C°, $t \in [t_1, t_f]$, following a ramp profile in $[t_0, t_1]$. Integrate the system using $t_1 = 1$ s and $t_f = 60$ s. 4) Repeat the simulation in point 3) using a mathematical model implementing two nodes for the conductor and insulator layers.

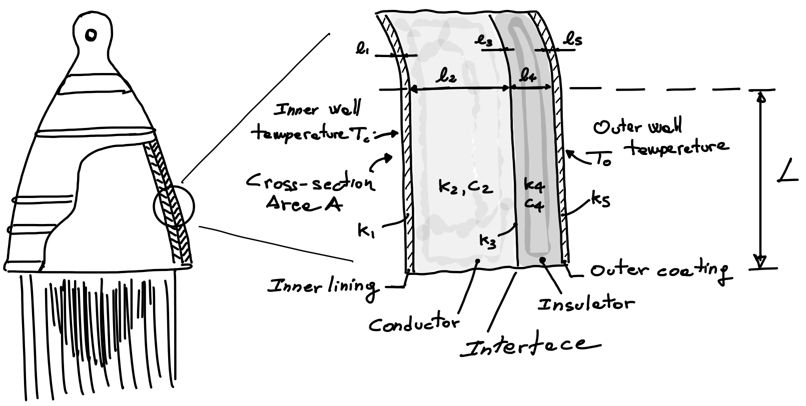


Figure 17: Real thermal system.

The area, length and diameter of the nozzle are reported Tab. 4, while the chosen materials and their characteristics in Tab. 3.

Table 3: Materials and their characteristics* [1] [2]

	Material	ρ [$\frac{kg}{m^3}$]	k [$\frac{W}{mK}$]	c [$\frac{J}{kgK}$]	R [Ω]	l [mm]
1	Inconel 718c		11.4			4.5
2	Copper - Chromium - Zirconium	8890	323.9	385		28
3					10^{-4}	
4	Coated Li-900 silica ceramics	2060	0.05	150		13
5	Aluminum		205			4.5

* the characteristics have been specified only when needed for the resolution of the problem.

The physical model of the system considering only one node per each of the five layers in reported in Fig. 18. As shown, each material has its own resistance, evaluated as reported in

[3]The interface layer is not made of a physically existing material, though it produces a thermal resistance. For this layer, the value of the thermal resistance R_3 can be direcly assumed, so avoiding to choose k_3 and ℓ_3.

Table 4: Dimensions of the nozzle [1]

Name	Value [m]
L	3.1000
D	0.2600
A	2.5321

Eq. 70, where the nodes have been placed in the middle of the layers.

$$R = \frac{l}{kA} \tag{70}$$

The mathematical model instead is given by the following system:

$$\begin{cases} \frac{1}{2}\rho_2 c_2 V_2 \dot{T}_2 = \frac{T_1}{R_1+R_2} + \frac{T_3}{R_2+R_3} - T_2\Big(\frac{1}{R_1+R_2} + \frac{1}{R_2+R_3}\Big) & (71) \\ \frac{1}{2}\rho_4 c_4 V_4 \dot{T}_4 = \frac{T_3}{R_3+R_4} + \frac{T_5}{R_4+R_5} - T_4\Big(\frac{1}{R_3+R_4} + \frac{1}{R_4+R_5}\Big) & (72) \\ T_1 = \frac{T_i(R_1+R_2)+T_2R_1}{2R_1+R_2} & (73) \\ T_3 = \frac{T_2(R_3+R_4)+T_4(R_2+R_3)}{R_2+2R_3+R_4} & (74) \\ T_5 = \frac{T_4R_5+T_o(R_4+R_5)}{R_4+2R_5} & (75) \end{cases}$$

where all the temperatures are functions of the time; the system is composed by two differential equations, that correspond to the temperatures of the layers which capacities are different from zero, and by three algebraic ones.
The plot that represents the temperature profiles in this first case is shown in Fig. 19.

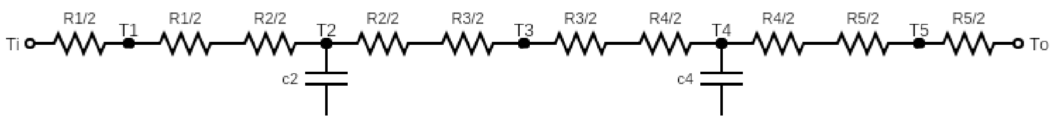


Figure 18: Circuit: single node case

If two nodes are considered for both the capacitor and the insulator layers, the physical model becomes as shown in Fig. 20 (the capacitance is now divided by two for each layer) and the

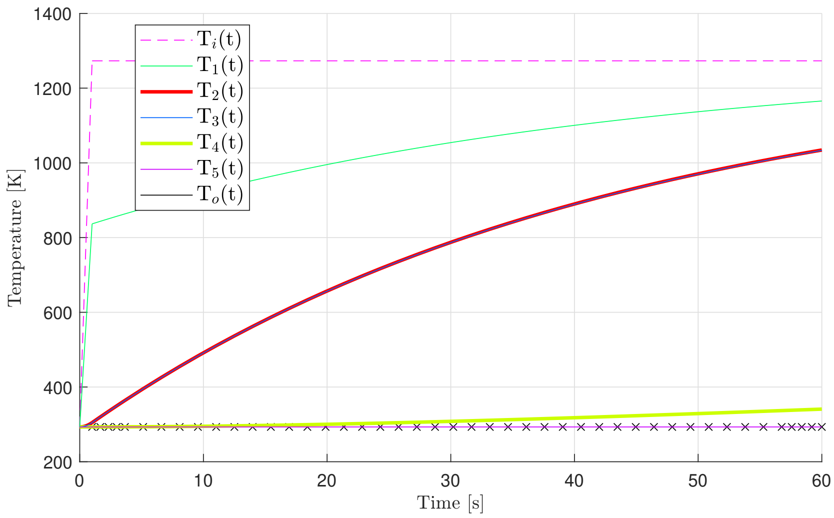


Figure 19: Temperature profiles: single node case

mathematical system is composed by four differential equations plus three algebraic ones:

$$
\begin{cases}
\frac{1}{2}\rho_2 c_2 V_2 \dot{T}_2^- = \frac{T_1 - T_2^-}{\frac{R_1}{2}+\frac{R_2}{3}} - \frac{T_2^- - T_2^+}{\frac{R_2}{3}} & (76)\\
\frac{1}{2}\rho_2 c_2 V_2 \dot{T}_2^+ = \frac{T_2^- - T_2^+}{\frac{R_2}{3}} - \frac{T_2^+ - T_3}{\frac{R_2}{3}+\frac{R_3}{2}} & (77)\\
\frac{1}{2}\rho_4 c_4 V_4 \dot{T}_4^- = \frac{T_3 - T_4^-}{\frac{R_3}{2}+\frac{R_4}{3}} - \frac{T_4^- - T_4^+}{\frac{R_4}{3}} & (78)\\
\frac{1}{2}\rho_4 c_4 V_4 \dot{T}_4^+ = \frac{T_4^- - T_4^+}{\frac{R_4}{3}} - \frac{T_4^+ - T_5}{\frac{R_4}{3}+\frac{R_5}{2}} & (79)\\
T_1 = \frac{T_i(3R_1+2R_2)+3R_1T_2^-}{6R_1+2R_2} & (80)\\
T_3 = \frac{T_4^-(2R_2+3R_3)+T_2^+(3R_3+2R_4)}{2R_2+6R_3+2R_4} & (81)\\
T_5 = \frac{T_o(2R_4+3R_5)+3R_5T_4^+}{2R_4+6R_5} & (82)
\end{cases}
$$

where all the temperatures are functions of the time. The initial conditions of the temperatures are all the same and equal to 293.15 K.

Figure 20: Circuit: double node case

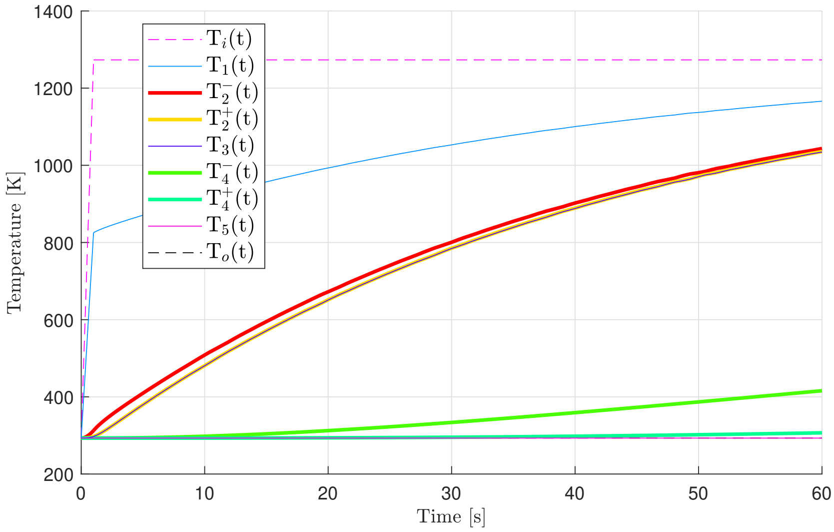


Figure 21: Temperature profiles: double node case

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[1] George P. Sutton, Oscar Biblarz *Rocket Propulsion Elements*. A Wiley-Interscience Publication - John Wiley Sons, Inc - Seventh Edition

[2] K. Keller, J. Antonenko, K.H. Weber *High-Temperature Insulations*. European Space Agency - Bulletin Nr. 80 - November 1994.
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