Compulsory Assignment 1

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Problem 1: Use linear/quadratic functions for verification

a) Derive the equation for the first time step

The problem is formulated as follows

$$u^{''} + \omega^2 * U = F(t)$$

$$u(0) = T \text{ and } u^{'}(0) = V, \text{ for } t \in (0, T)$$

Function U gives the displacement from the equilibrium point and function F gives the source term. In order to fin numerical solution for such problem we should start with discretization of the equation. In operator notations it will look as follows:

$$[D_t D_t u + (\omega_2)u = f]^n$$

The equivalent "long" notations are the following:

$$\frac{u(n+1) - 2u(n) + u(n-1)}{dt^2} + \omega^2 u(n) = f$$

That can be written as:

$$u(n+1) = 2u(n) - (\omega * dt)^2 u(n) - u(n-1) + f dt^2$$

$$u(n+1) = (2 - (\omega * dt)^2) u(n) - u(n-1) + f dt^2$$

Now we can derive the equation for the first timestep $u(n=1)$. In order to do this we just need to discretize $u^{'}(0) = V$ as $[D_{2t} u = V]^0$.

$$u(-1) = u(1) - 2dtV \text{ and } u(1) = \frac{(2 - (\omega dt)^2)I + f dt^2 + 2dtV}{2}$$

b) Use the method of manufactured solutions

In this part we are using the method of manufactoring solutions for the verification purporses. The exact solution is chosen in a form of $u_e(x,t) = ct + d$ Initial conditions are used to find restrictions on c and d.

$$[D_t D_t(u)]_{(n)} = [D_t D_t(ct + d)]_{(n)} = c[D_t D_t(t)]_{(n)} + [D_t D_t(d)]_{(n)}$$

Let's consider this two parts of equation separately:

$$[D_t D_t(d)]_{(n)} = \frac{d(n+1) - 2d(n) + d(n-1)}{dt^2} = 0 \text{ as soos as d(n) is constant}$$

Another part is:

$$[D_t D_t(t)]_{(n)} = \frac{t(n+1) - 2t(n) + t(n-1)}{dt^2} = 0$$

As soon as

$$t(n+1) - t(n) = dt \text{ and } t(n-1) - t(n) = -dt$$

Now we can calculate R:

$$R = [D_t D_t(u)]_{(n)} - u(t(n))?f(t(n)) - u(t(n)) = 0 \text{ sinse } u^{''}(t) = 0$$

c) - e)

In this parts we are using symbolic python to calculate. Solution in code.

Exercise 21

a) and b)

Solution in the code.

c)

We consider now pure vertical motion of the elastic pendulum. Thi means that $x(t) = 0$.

$$y'' = -\frac{\beta(y-1)}{1-\beta}(1 - \frac{\beta}{|y-1|}) - \beta$$

For $(y-1) < 0$ we get:

$$y'' = -\frac{\beta}{1-\beta}(y - 1 + \beta) - \beta = -y\frac{\beta}{1-\beta}$$

Let's now consider initial conditions:

$$x(0) = 0, \ \Theta = 0, \ y(0) = 1 - (\epsilon + 1)cos(\Theta) = -\epsilon$$

The exact solution to the vertical motion is

$$y(t) = -\epsilon\cos(\sqrt{\frac{\beta}{1-\beta}}t), \ |\epsilon| < 1.$$

d)

Solution in code.