Design of a parallel Schönflies motion generator

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1 Optimization problem

The parallel Schönflies motion generator should fulfill a pick and place motion described by following design specifications:

- The robot must be capable of producing a test cycle that is commonly accepted for SCARA systems, in at most 500 ms. The test cycle consists of:
  - 25-mm vertical displacement up
  - 300-mm horizontal displacement with a concomitant 90° turn
  - 25-mm vertical displacement down
  - 25-mm vertical displacement up
  - 300-mm horizontal displacement with a reversed 90° turn
  - 25-mm vertical displacement down
- The condition number κ is as small as possible, i.e., $\kappa < 10$.
- The mechanism should not be too bulky.

2 Type synthesis phase

1. In the case of parallel mechanisms, to fulfil the requirement stating that the robot should not be too bulky, we consider only non-redundant parallel mechanisms during the type synthesis phase to avoid redundant joints adding more weight to overall mechanism. This means that

$$R = \sum_{i=1}^{m} R^i = 0 \tag{1}$$

   And thus

$$\sum_{i=1}^{m} c^i = 6 + \Delta - \mathcal{F} \tag{2}$$

$$c^i = 6 - f^i \tag{3}$$

Where R is the redundant DOF of the robot, R^i is the redundant DOF of i-th leg of the robot; c^i is the order of wrench system of i-th leg of the robot, Δ is the over-constraint of the robot, $\mathcal{F}$ is the DOF of the robot, and f^i is the DOF of i-th leg of the robot.

2. To simplify the optimization problem, only 1-DOF prismatic joints and revolute joints are used (since all non 1-DOF joints can be represented by 1-DOF joints). We have chosen the 1-DOF joint equivalent of a 4-URU robot as the architecture for our robot.

3. An architecture equivalent to a 4-URU parallel robot is proposed. On each symmetrical leg of the robot, there are 5 revolute joints: the first and the fifth revolute joints are perpendicular to the base of the robot; the second, third and fourth revolute joints are parallel to each other, and are deviated from vertical direction by 45 degrees. Additionally, the first link (which is the link between the second and the third joint), the rotation axes of first and second joints all intersect at one point denoted as Ai; the first and second link (which is the link between the third and the fourth joint) all intersect with the rotation axis of the third joint at one point denoted as Bi; the second link, the rotation axes of fourth and fifth joints all intersect at one point denoted as Ci. Here i denotes the number of leg. See figure 2 for a visual representation of one leg within the robot.

3 Kinematic analysis

3.1 Inverse geometric model

First an absolute frame as shown in figure 1 is defined. The common centre of 1st and 2nd joints of each leg, denoted as point A_i (see figure 2), can be expressed in this fixed frame as:

$$\begin{cases} A_1 = \begin{bmatrix} r_B \\ 0 \\ h \end{bmatrix} \\ A_2 = \begin{bmatrix} 0 \\ r_B \\ h \end{bmatrix} \\ A_3 = \begin{bmatrix} -r_B \\ 0 \\ h \end{bmatrix} \\ A_4 = \begin{bmatrix} 0 \\ -r_B \\ h \end{bmatrix} \end{cases}$$

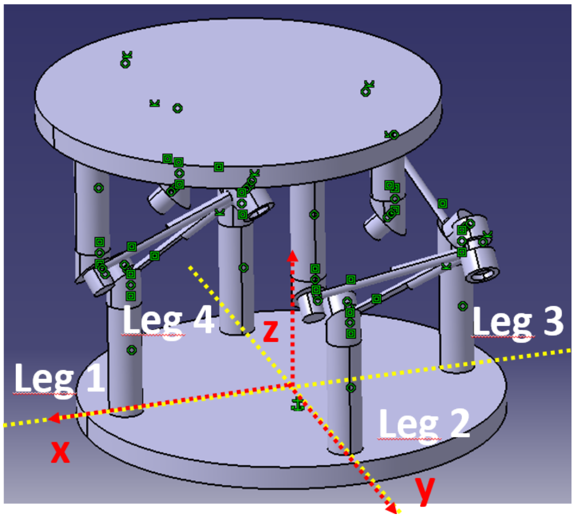

Figure 1: Absolute frame

For a given pose $P = \begin{bmatrix} x_p \\ y_p \\ z_p \\ \theta_p \end{bmatrix}$ of the moving platform expressed in base frame, the positions of the common centre between 4th and 5th joints of each leg,

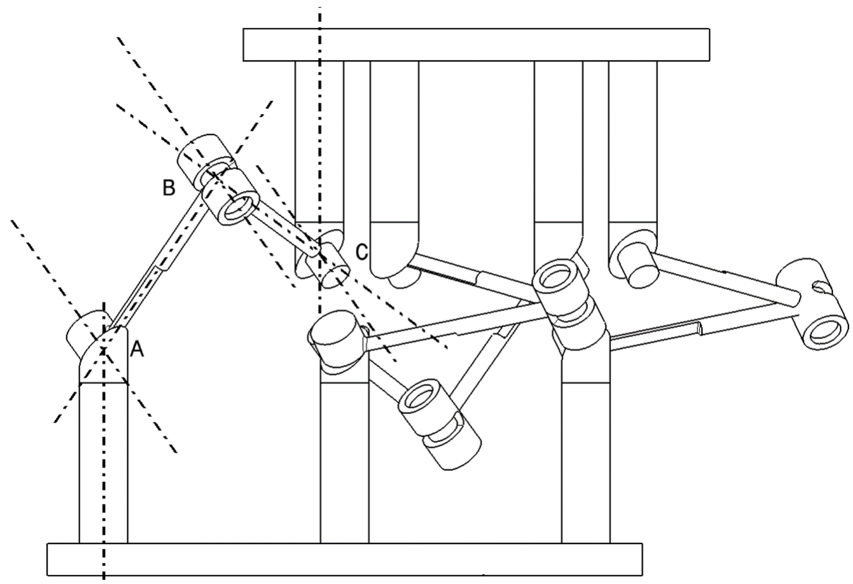

Figure 2: Joint frames

denoted as C_i in base frame can be uniquely find.

$$\begin{cases} C_1 = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + \begin{bmatrix} r_{MP}cos(\theta_p) \\ r_{MP}sin(\theta_p) \\ -h \end{bmatrix} \\ C_2 = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + \begin{bmatrix} -r_{MP}sin(\theta_p) \\ r_{MP}cos(\theta_p) \\ -h \end{bmatrix} \\ C_3 = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + \begin{bmatrix} -r_{MP}cos(\theta_p) \\ -r_{MP}sin(\theta_p) \\ -h \end{bmatrix} \\ C_4 = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + \begin{bmatrix} r_{MP}sin(\theta_p) \\ -r_{MP}cos(\theta_p) \\ -h \end{bmatrix} \end{cases}$$

In which r_{MP} is the distance between the centre of the moving platform and 5th joint of each leg. Point B (centre of joint 3 as defined in figure 2) lies on the intersection between 2 spheres, their radius equal to link lengths and their centre at centre of joint 2 and joint 4(5) as can be seen in figure 1. (assuming link lengths between AB and BC are the same):

$$\begin{cases}(x_{Bi}-x_{Ci})^2+(y_{Bi}-y_{Ci})^2=l_L^2\\(x_{Bi}-x_{Ai})^2+(y_{Bi}-y_{Ai})^2=l_L^2\end{cases}\tag{4}$$

Point B also lies on a plane formed by point A, B and C. The normal of this plane, which is pointing in the same direction as the rotational axis of the 2nd joint, can be expressed as (note that due the orientation of our absolute frame, the fixed angle between 1st joint and 2nd joint is smaller than zero, see figure 3):

$$\overrightarrow{n_{ABC_i}}=\begin{bmatrix}cos(\theta_{1i}) & -sin(\theta_{1i}) & 0\\ sin(\theta_{1i}) & cos(\theta_{1i}) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}cos(-\delta) & 0 & sin(-\delta)\\ 0 & 1 & 0\\ -sin(-\delta) & 0 & cos(-\delta)\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-sin(\delta)cos(\theta_{1i})\\-sin(\delta)sin(\theta_{1i})\\cos(\delta)\end{bmatrix}$$

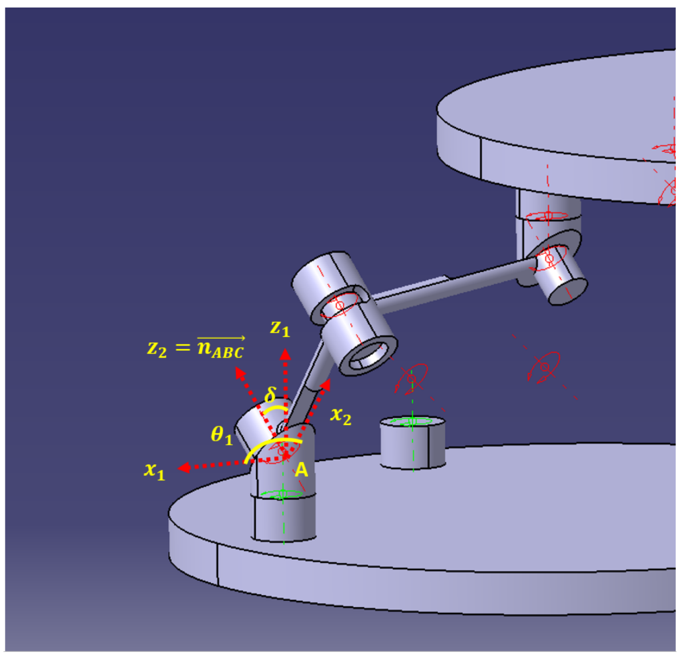

Figure 3: Fixed angle between 1st joint and 2nd joint

In which δ is the angle between the rotational axes of 1st and 2nd joints of each leg. In our design it is set to a fixed 45 degrees.

The plane ABC can be expressed as:

$$P_{ABC}: -(x-x_{Ai})sin(\delta)cos(\theta_{1i})-(y-y_{Ai})sin(\delta)sin(\theta_{1i})+(z-z_{Ai})cos(\delta)=0$$

or

$$P_{ABC} : -(x - x_{Ci})sin(\delta)cos(\theta_{1i}) - (y - y_{Ci})sin(\delta)sin(\theta_{1i}) + (z - z_{Ci})cos(\delta) = 0$$

Hence the coordinate of point B satisfies 4 equations listed below:

$$\begin{cases} -(x - x_{Ai})sin(\delta)cos(\theta_{1i}) - (y - y_{Ai})sin(\delta)sin(\theta_{1i}) + (z - z_{Ai})cos(\delta) = 0 \\ -(x - x_{Ci})sin(\delta)cos(\theta_{1i}) - (y - y_{Ci})sin(\delta)sin(\theta_{1i}) + (z - z_{Ci})cos(\delta) = 0 \\ (x - x_{Ai})^2 + (y - y_{Ai})^2 + (z - z_{Ai})^2 = l_L^2 \\ (x - x_{Ci})^2 + (y - y_{Ci})^2 + (z - z_{Ci})^2 = l_L^2 \end{cases}$$

The inverse geometric model can be solved. The symbolic solutions of actuated joint displacements in all four legs can be found in an attached .mat file named as "theta1_solutions.mat". There are in total 4 solutions for each leg, of which they are symmetric to each other in couples. This indicate 4 possible configurations of a single leg.

3.2 Screw analysis

The twist system of i-th leg is:

$$\mathcal{T}^i = span(\hat{\epsilon_{0i1}}, \hat{\epsilon_{0i2}}, \hat{\epsilon_{0i3}}, \hat{\epsilon_{0i4}}, \hat{\epsilon_{0i5}})$$

in which:

$$\begin{cases} \hat{\epsilon_{0i1}} = \begin{bmatrix} k \\ r_{Ai} \times k \end{bmatrix} \\ \hat{\epsilon_{0i2}} = \begin{bmatrix} m_i \\ r_{Ai} \times m_i \end{bmatrix} \\ \hat{\epsilon_{0i3}} = \begin{bmatrix} m_i \\ r_{Bi} \times m_i \end{bmatrix} \\ \hat{\epsilon_{0i4}} = \begin{bmatrix} m_i \\ r_{Ci} \times m_i \end{bmatrix} \\ \hat{\epsilon_{0i5}} = \begin{bmatrix} k \\ r_{Ci} \times k \end{bmatrix} \end{cases}$$

Here r denotes the coordinate vectors of any specific point in i-th leg of the robot with respect to the absolute frame depicted in figure 1.

The constraint wrench system of i-th leg is:

$$\mathcal{W}_c^i = span(\hat{\tau}_{\infty i}^c)$$

in which:

$$\hat{\tau}_{\infty i}^c = \begin{bmatrix} 0 \\ k \times m_i \end{bmatrix} = \begin{bmatrix} 0 \\ k \times \overrightarrow{n_{ABC_i}} \end{bmatrix}$$

The constraint wrench system of entire robot is:

$$\mathcal{W}_c = \bigoplus_{i=1}^{4} \mathcal{W}_c^i = span(\hat{\tau}_{\infty 1}^c, \hat{\tau}_{\infty 2}^c \hat{\tau}_{\infty 3}^c \hat{\tau}_{\infty 4}^c)$$

This is a 2-system.

The twist system of the entire robot is:

$$\mathcal{T} = \mathcal{W}_c^{\perp} = span(\hat{\epsilon_{\infty x}}, \hat{\epsilon_{\infty y}}, \hat{\epsilon_{\infty z}}, \hat{\epsilon_{0z}})$$

This robot produce 3T1R motion, as required.

The actuation wrench system of i-th leg is:

$$\mathcal{W}_a^i = span(\hat{\tau}_{0i}^a)$$

in which:

$$\hat{\tau}_{0i}^a = \begin{bmatrix} m_i \\ r_{235i} \times m_i \end{bmatrix}$$

Here r_{235i} is the coordinate of the intersection point between the rotational axis of the 5th joint and the plane formed by the rotational axes of 2nd and 3rd joint, as seen in figure 4. One can observe that $\hat{\tau}_{0i}^a$ is coplanar to all zero-pitch twists from revolute joints except for the one corresponding to the actuated joint. When the plane formed by the rotational axes of 2nd and 3rd joint on the same leg is vertical to the base, actuation wrench system for this particular leg is null.

Plane 23i can be defined by 2 vectors: $\overrightarrow{n_{ABC_l}}$ and $\overrightarrow{A_l B_l}$. $\overrightarrow{A_l B_l}$ can be calculated:

$$\overrightarrow{A_l B_l} = l_L \begin{bmatrix} cos(\theta_{2i}) & -sin(\theta_{2i}) & 0 \\ sin(\theta_{2i}) & cos(\theta_{2i}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(-\delta) & 0 & sin(-\delta) \\ 0 & 1 & 0 \\ -sin(-\delta) & 0 & cos(-\delta) \end{bmatrix} \begin{bmatrix} cos(\theta_{1i}) & -sin(\theta_{1i}) & 0 \\ sin(\theta_{1i}) & cos(\theta_{1i}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$= l_L \begin{bmatrix} cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}) \\ cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}) \\ sin(\delta)cos(\theta_{1i}) \end{bmatrix}$$

The normal of the plane 23i can be found by:

$$\overrightarrow{n_{23_l}} = \frac{\overrightarrow{A_l B_l}}{l_L} \times \overrightarrow{n_{ABC_l}}$$

$$= \begin{bmatrix} (cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))cos(\delta) + sin^2(\delta)sin(\theta_{1i})cos(\theta_{1i}) \\ -sin^2(\delta)cos^2(\theta_{1i}) - (cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))cos(\delta) \\ -(cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))sin(\delta)sin(\theta_{1i}) + (cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))sin(\delta)cos(\theta_{1i}) \end{bmatrix}$$

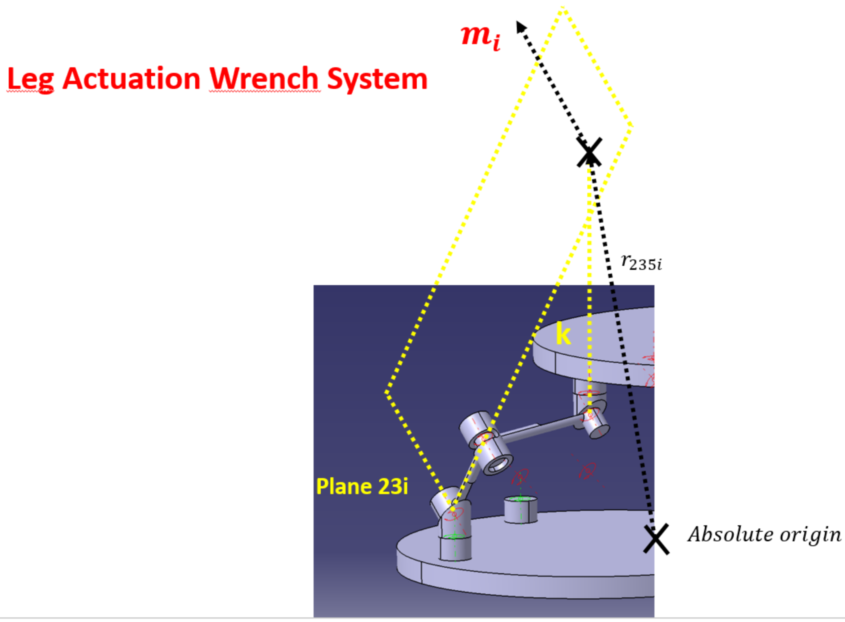

Figure 4: Leg actuation wrench system

Plane 23i is therefore:

$$\begin{aligned}P_{23i}: &(x - x_{Ai})[(cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))cos(\delta) + sin^2(\delta)sin(\theta_{1i})cos(\theta_{1i})]\\ &- (y - y_{Ai})[sin^2(\delta)cos^2(\theta_{1i})(cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))cos(\delta)]\\ &+ (z - z_{Ai})[-(cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))sin(\delta)sin(\theta_{1i})\\ &+ (cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))sin(\delta)cos(\theta_{1i})] = 0\end{aligned}$$

Therefore, $r_{235i} = \begin{bmatrix} r_{xi} \\ r_{yi} \\ r_{zi} \end{bmatrix}$ satisfies:

$$\begin{cases} (r_{xi} - x_{Ai})[(cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))cos(\delta) + sin^2(\delta)sin(\theta_{1i})cos(\theta_{1i})] \\ -(r_{yi} - y_{Ai})[sin^2(\delta)cos^2(\theta_{1i})(cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))cos(\delta)] \\ +(r_{zi} - z_{Ai})[-(cos(\delta)cos(\theta_{1i})cos(\theta_{2i}) - sin(\theta_{1i})sin(\theta_{2i}))sin(\delta)sin(\theta_{1i}) + (cos(\delta)cos(\theta_{1i})sin(\theta_{2i}) + sin(\theta_{1i})cos(\theta_{2i}))sin(\delta)cos(\theta_{1i})] = 0 \\ r_{xi} = x_{Ci} \\ r_{yi} = y_{Ci} \end{cases}$$

A solution of r_{235i} can be uniquely found.

The actuation wrench system of the entire robot is:

$$W_a = \oplus_{i=1}^{4} W_a^i = span(\hat{\tau}_{01}^a, \hat{\tau}_{02}^a, \hat{\tau}_{03}^a, \hat{\tau}_{04}^a)$$

The kinematic model of the entire robot in vector form is thus:

$$A \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \upsilon_x \\ \upsilon_y \\ \upsilon_z \end{bmatrix} = B \begin{bmatrix} \dot{\theta}_{11} \\ \dot{\theta}_{12} \\ \dot{\theta}_{13} \\ \dot{\theta}_{14} \end{bmatrix}$$

in which:

$$A = \begin{bmatrix} (r_{2351} \times m_1)^T & m_1^T \\ (r_{2352} \times m_2)^T & m_2^T \\ (r_{2353} \times m_3)^T & m_3^T \\ (r_{2354} \times m_4)^T & m_4^T \\ (k \times m_1)^T & 0_3^T \\ (k \times m_2)^T & 0_3^T \\ (k \times m_3)^T & 0_3^T \\ (k \times m_4)^T & 0_3^T \end{bmatrix}$$

$$\text{B=} \begin{bmatrix} (r_{2351} \times {m_1}^T k + {m_1}^T (r_{A1} \times k) & 0 & 0 & 0 \\ 0 & (r_{2352} \times {m_2}^T k + {m_2}^T (r_{A2} \times k) & 0 & 0 \\ 0 & 0 & (r_{2353} \times {m_3}^T k + {m_3}^T (r_{A3} \times k) & 0 \\ 0 & 0 & 0 & (r_{2351} \times {m_4}^T k + {m_4}^T (r_{A4} \times k) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Parallel singularity can be found when matrix A is rank deficient. For this robot, the condition is fulfilled iff any 2 rows within the first 4 rows of matrix A are linearly dependent. Geometrically, this means:

$$m_i \parallel m_j \text{ and } r_{235i} \parallel r_{235j} \text{ with } i, j = 1, 2, 3, 4 \text{ and } i \neq j$$

Serial singularity can be found when the determinant of the matrix formed by the first 4 rows of matrix B is zero. For this robot, the condition is fulfilled iff:

$$[(r_{235i} - r_{Ai}) \times m_1]^T k = 0 \text{ with } i = 1, 2, 3, 4$$

Geometrically, this means that serial singularity appears whenever the first link of any leg is perpendicular to the vertical direction in base frame.

3.3 Numeric model of constraints

The conditioning number can be found as:

$$\kappa(B^T A^{-T} A^{-1} B) = \frac{\sqrt{\lambda_{max}(B^T A^{-T} A^{-1} B)}}{\sqrt{\lambda_{min}(B^T A^{-T} A^{-1} B)}}$$

In our case, the forward Jacobian matrix is an 8-6 matrix, so the above equation is modifiedas:

$$\kappa(B^T(AA^T)^{-1}B) = \frac{\sqrt{\lambda_{max}(B^T(AA^T)^{-1}B)}}{\sqrt{\lambda_{min}(B^T(AA^T)^{-1}B)}}$$

The symbolic representation and solution of the generalized Jacobian matrix and its corresponding condition number can be found in attached .mat files called "generalized_J_solution.mat" and "condition_number_solution", respectively.
Note: Since the symbolic expressions of both the generalized Jacobian matrix and the condition number have exceeded the display limit of MATLAB software, no analytical conclusion can be drawn from them based on our current capabilities.
Our optimization problem is to minimize the link length between revolute joints (assuming all links share the same length), with respect to 2 constraints:

1. the joint displacements of actuated joints solved from 5 points on the desired trajectory using IGM should all be real, and
2. the condition number of the generalized Jacobian matrix for 5 poses on the desired trajectory should be less than 10.

To this end, we have in total 25 non-linear constraints, all of which are described in the attached .m script named as "nonlcon.m".

Note: The symbolic solution of condition number for this kinematic model has exceeded MATLAB's display limit. A numerical solution of the condition number can be found for a given set of robot parameters and moving platform pose, but optimization cannot initiate due to the lack of a processable symbolic expression of the constraint equation.

4 Readme for MATLAB code

1. Due to the long calculation time taken, the symbolic solutions for actuated joints in IGM, 3D coordinate of the third revolute joint IGM, forward and inverse Jacobian matrices can all be loaded from the .mat files attached in the folder.
2. To run the solvers for IGM and KM, first run the script named as "entire_IGM_solve", then run the script named as "KM_solver". Numerical solution for the generalised Jacobian matrix and the condition number can be found with specific parameters and moving platform poses, all of which can be modified in Line 72 to 76 in "KM_solver" under the matrix named as "num_values".
3. The fmincon is currently unusable due to reasons state above. However they can still be found in the folder as "main", "nonlcon" and "objfun". Descriptions on how the codes are constructed can be found inside the comment section of each script.